CONTINUOUS SELECTIONS OF SOLUTIONS TO PARAMETRIC VARIATIONAL INEQUALITIES*

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Abstract. This paper studies the existence of a (Lipschitz) continuous (single-valued) solution function of parametric variational inequalities under functional and constraint perturbations. At the most elementary level, this issue can be explained from classical parametric linear programming and its resolution by the parametric simplex method, which computes a solution trajectory of the problem when the objective coefficients and the right-hand sides of the constraints are parameterized by a single scalar parameter. The computed optimal solution vector (and not the optimal objective value) is a continuous piecewise affine function in the parameter when the objective coefficients are kept constant, whereas the computed solution vector can be discontinuous when the right-hand constraint coefficients are kept fixed and there is a basis change at a critical value of the parameter in the objective. We investigate this issue more broadly first in the context of an affine variational inequality (AVI) and obtain results that go beyond those pertaining to the lower semicontinuity of the solution map with joint vector perturbations; the latter property is closely tied to a stability theory of a parametric AVI and in particular to Robinson's seminal concept of strong regularity. Extensions to nonlinear variational inequalities are also investigated without requiring solution uniqueness (and therefore applicable to non-strongly regular problems). The role of solution uniqueness in this issue of continuous single-valued solution selection is further clarified.

1. Introduction.

This paper is motivated by a novel (ongoing) study on the existence of an appropriately defined equilibrium solution of a multi-leader multi-follower game [34] (there are many more references, but since this topic is not the focus of the paper, we provide just one reference of this game problem that leads to the present work). Such a game consists of a group of dominant players (the leaders) and a group of subordinate players (the followers) wherein each group of players is playing a noncooperative game among those in the same group. The studied approach is to convert this hierarchical game to a single-level noncooperative game by substituting out the followers' Nash equilibrium responses via a single-valued response function. For the resulting one-level game of the leaders to be tractable, it is essential that the followers' single-valued response function be continuous. Since in general, the followers' game can be formulated as a variational inequality (VI) [15], the question of when the solution set of a VI has a continuous selection of elements as a function of parameters (the leaders' strategy tuple in the hierarchical game context) needs to be answered. As noted by a referee, answers to this question are relevant to the study of extensions of the deterministic VI, such as the dynamic VI [8, 35], two-stage [7] and multi-stage stochastic VIs [39], and dynamic stochastic VIs [6]. These potential broad applications inspire us to undertake an independent investigation of the selection issue; nevertheless, their details, which easily involve technical issues of a stochastic nature for two-stage problems, are regrettably beyond the scope of this work.

Selection of a continuous single-valued function from a set-valued map is a fundamental problem in classical mathematics. The most celebrated result of this kind is due to Ernest Michael [29, Theorem 3.2] which states that "a lower semicontinuous multivalued map with nonempty convex closed values from a paracompact space into a Banach space has a continuous selection". There is actually a (less-known but much simpler to prove) converse to this result (see Proposition 2.2 in the cited reference) which asserts that if there exists a continuous selection at every pair in the graph of the set-valued map, then the map is lower semicontinuous. A related result in a finite-dimensional Euclidean space by Dommisch [12] states that a "Lipschtiz continuous set-valued map" (in the set-theoretic Hausdorff metric) into nonempty, convex, compact subsets has a Lipschitz continuous selection. In the context of the solution map of a finite-dimensional VI, the (global) Lipschitz continuity postulate of this (multivalued) solution map tends to be restrictive and abstract, except for special classes of problems. Indeed, there exist results for such problems that yield continuous solution functions (see the next section for a brief review). Furthermore, for the VI, there are two sets of studies that connect the lower semicontinuity of the map to solution single-valuedness. In this context, the three properties: lower

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semicontinuity, continuous selection, and solution uniqueness are therefore closely related. This paper has two major objectives: to clarify these properties for a parametric (monotone) variational inequality on a polyhedron with varying right-hand side and, more interestingly, to identify classes of such VIs for which the existence of a (Lipschitz) continuous single-valued solution selection does not require solution uniqueness. In this vein, the monograph [13] contains a wealth of results on continuity properties of solution mappings; our results supplement the results therein by focusing on the continuous selection of solutions to variational inequalities.

2. A Summary of the Literature. In order to more clearly explain the problem and summarize the existing results in the literature, we review some basic elements of a set-valued map $\Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ [2,3,13,38]. The domain of Γ, denoted dom Γ, the range of Φ, denoted ran Φ, and the graph of Φ, denoted gph Φ, are respectively, the sets:

$$\begin{aligned} \text{dom } \Gamma &\triangleq \{ \xi \in \mathbb{R}^n : \Gamma(\xi) \neq \emptyset \}, \\ \text{ran } \Gamma &\triangleq \bigcup_{\xi \in \text{dom } \Gamma} \Gamma(\xi), \\ \text{gph } \Gamma &\triangleq \{ (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m : \eta \in \Gamma(\xi) \}. \end{aligned}$$

Definition 2.1. A set-valued map $\Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with gph Γ is said to

• be *lower semicontinuous* at $\bar{\xi} \in$ dom Γ if for every open set $\mathcal{V}$ such that $\mathcal{V} \cap \Gamma(\bar{\xi}) \neq \emptyset$, there exists a neighborhood $\mathcal{U}$ of $\bar{\xi}$ such that $\mathcal{V} \cap \Gamma(\xi) \neq \emptyset$ for all $\xi \in \mathcal{U}$;

• be *lower semicontinuous* at the pair $(\bar{\xi}, \bar{\eta}) \in$ gph Γ if for every sequence $\{\xi^\nu\}$ converging to $\bar{\xi}$, there exists a sequence $\{\eta^\nu\}$ converging to $\bar{\eta}$ such that $\eta^\nu \in \Gamma(\xi^\nu)$ for all ν;

• be *lower semicontinuous* around the pair $(\bar{\xi}, \bar{\eta}) \in$ gph Γ if there exists a neighborhood $\mathcal{W}$ of this pair such that Γ is lower semicontinuous at every $(\xi, \eta) \in W \cap$ gph Γ;

• be *upper Lipschitz continuous* at $\bar{\xi} \in$ dom Γ if there exist a scalar $\text{Lip}_\uparrow > 0$ and a neighborhood $\mathcal{N}$ of $\bar{\xi}$ such that

$$\Gamma(\xi) \subseteq \Gamma(\bar{\xi}) + \text{Lip}_\uparrow \| \xi - \bar{\xi} \| \mathbb{B}, \quad \forall \xi \in \mathcal{N} \cap \text{dom } \Gamma,$$

where $\mathbb{B}$ is the unit ball in $\mathbb{R}^m$ (while the term "upper" is classical, the term "outer" is adopted in [13, Section 3D] from the perspective of modern variational analysis);

• be *lower Lipschitz continuous* at $\bar{\xi} \in$ dom Γ if there exist a scalar $\text{Lip}_\downarrow > 0$ and a neighborhood $\mathcal{N}$ of $\bar{\xi}$ such that

$$\Gamma(\bar{\xi}) \subseteq \Gamma(\xi) + \text{Lip}_\downarrow \| \xi - \bar{\xi} \| \mathbb{B}, \quad \forall \xi \in \mathcal{N} \cap \text{dom } \Gamma,$$

• be *Lipschitz continuous* on a domain $\mathcal{D}$ if there exist a scalar Lip > 0 such that

$$\Gamma(\xi') \subseteq \Gamma(\xi) + \text{Lip} \| \xi' - \xi \| \mathbb{B}, \quad \forall \xi', \xi \in \mathcal{D};$$

• have a *continuous (single-valued) selection* on a domain $\Xi \subseteq$ dom Γ if there exists a function $\gamma : \Xi \to \mathbb{R}^m$ such that (i) $\gamma(\xi) \in \Gamma(\xi)$ for all $\xi \in \Xi$ and (ii) γ is continuous on Ξ; i.e., for all $\bar{\xi} \in \Xi$, i.e., $\displaystyle\lim_{\xi(\in \Xi) \to \bar{\xi}} \gamma(\xi) = \gamma(\bar{\xi})$ (continuity is restricted to the domain of the function γ);

• have a *continuous (single-valued) selection* at the pair $(\bar{\xi}, \bar{\eta}) \in$ gph Γ if there exist an open neighborhood $\mathcal{U}$ of $\bar{\xi}$ and a function $\gamma : \mathcal{U} \to \mathbb{R}^m$ such that (i) $\gamma(\xi) \in \Gamma(\xi)$ for all $\xi \in \mathcal{U}$, (ii) $\gamma(\bar{\xi}) = \bar{\eta}$, and (iii) γ is continuous at $\bar{\xi}$ (the continuity requirement is restricted to the reference point $\bar{\xi}$ only);

• be a *polyhedral multifunction* if gph Γ is the union of finitely many polyhedra;

• be a *polyhedral convex multifunction* if gph Γ is a polyhedron (thus convex). □

A classical result of Robinson [36] asserts that a polyhedral multifunction is everywhere pointwise upper Lipschitz continuous on its domain; i.e., it is upper Lipschitz continuous at every point in its domain. It is also known that a polyhedral convex multifunction is Lipschitz continuous on its domain [13, Theorem 3C.3]; thus combining this result with Michael's selection theorem, it follows that a polyhedral convex multifunction must have a continuous selection. Nevertheless, this argument does not shed light on a constructive expression of such a selection function.

We summarize some fundamentals of finite-dimensional variational inequalities and the linear complementarity problems for which the reader can consult [11,17], respectively, for details. Given a closed convex set K in $\mathbb{R}^n$ and a continuous mapping F from $\mathbb{R}^n$ into itself, the VI defined by the pair (F,K) is to find a vector $\bar{x} \in K$ such that

$$(x - \bar{x})^\top F(\bar{x}) \, \geq \, 0, \quad \forall \, x \, \in \, K.$$

The solution set of this VI is denoted by $\mathrm{SOL}(F,K)$.

Definition 2.2. The mapping $F : K \to \mathbb{R}^n$ is

- *monotone* on K if
$$(x - y)^\top (F(x) - F(y)) \, \geq \, 0, \quad \forall \, x, y \, \in \, K;$$
- *strongly monotone* on K if there exists a scalar $\alpha > 0$ such that
$$(x - y)^\top (F(x) - F(y)) \, \geq \, \alpha \, \| \, x - y \, \|^2, \quad \forall \, x, y \, \in \, K;$$
- *co-coercive* on K if there exists a scalar $\beta > 0$ such that
$$(x - y)^\top (F(x) - F(y)) \, \geq \, \beta \, \| \, F(x) - F(y) \, \|^2, \quad \forall \, x, y \, \in \, K;$$
- *monotone-plus* on K if it is monotone on K and
$$[\, (x - y)^\top (F(x) - F(y)) \, = \, 0 \, \Rightarrow \, F(x) \, = \, F(y) \,], \quad \forall \, x, y \, \in \, K;$$
- *strongly monotone-composite* if a strongly monotone mapping $\widehat{G} : \mathbb{R}^\ell \to \mathbb{R}^\ell$, a matrix $\widehat{E} \in \mathbb{R}^{\ell \times n}$, and vectors $a \in \mathbb{R}^n$ and $e \in \mathbb{R}^\ell$ exist such that

$$F(x) \, = \, \widehat{E}^\top \widehat{G}(\widehat{E}x + e) + a, \quad \forall \, x \, \in \, K. \tag{2.1}$$

We refer the reader to [17, Section 2.3] for the relationships between the above classes of functions and their roles in the VI.

Definition 2.3. A set-valued map $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is

- *monotone* on K if
$$(x - y)^\top (u - v) \, \geq \, 0, \quad \text{for all pairs } (x,u) \text{ and } (y,v) \text{ in gph } \Phi,$$
- *strongly monotone* on K if there exists a scalar $\alpha > 0$ such that
$$(x - y)^\top (u - v) \, \geq \, \alpha \, \| \, x - y \, \|^2, \quad \text{for all pairs } (x,u) \text{ and } (y,v) \text{ in gph } \Phi.$$

Given a closed convex set K, if F is a monotone mapping on K, then $\mathrm{SOL}(F,K)$ is a convex set (possibly empty); thus in this case, if the VI (F,K) has a locally unique solution, then this solution is the unique element of $\mathrm{SOL}(F,K)$. If F is a continuous and strongly monotone mapping on K, then $\mathrm{SOL}(c+F,K)$ is a singleton for every $c \in \mathbb{R}^n$; moreover, if $x(c)$ denotes the single element of $\mathrm{SOL}(c+F,K)$, then $x(c)$ is a co-coercive, thus globally Lipschitz continuous, function of c [17, Proposition 2.3.11]. In particular, if $F : \mathbb{R}^n \to \mathbb{R}^n$ is strongly monotone on $\mathbb{R}^n$, then its inverse F^{-1} exists and is a globally Lipschitz continuous mapping on $\mathbb{R}^n$. If F is (single-valued, continuous, and) monotone-plus on K and if $\mathcal{S} \triangleq \mathrm{SOL}(F,K) \neq \emptyset$, then $F(\mathcal{S})$ is a singleton.

When K is the Cartesian product of finitely many sets of lower dimensions, say $K = \displaystyle\prod_{i=1}^{I} K^{\,i}$ for some positive integer I, where $K^{\,i} \subseteq \mathbb{R}^{n_i}$ is closed and convex, and $F = (F^{\,i})_{i=1}^I$, where each $F^{\,i} : \mathbb{R}^n \to \mathbb{R}^{n_i}$, is (continuous and) *uniformly P* on K, i.e., for some constant $\gamma > 0$, it holds that $\displaystyle\max_{1 \leq i \leq I} (x^i - y^i)^\top (F^i(x) - F^i(y)) \, \geq \, \gamma \, \|x - y\|^2$ for all x and y in K, then $\mathrm{SOL}(c+F,K)$ is a singleton for every $c \in \mathbb{R}^n$; moreover, this solution function is also co-coercive.

A solution $\bar{x} \in \text{SOL}(F, K)$ can be characterized by a "generalized equation" [37], which is the following inclusion:

$$0 \in F(\bar{x}) + \mathcal{N}(K; \bar{x}), \quad \text{where}$$

$$\mathcal{N}(K; \bar{x}) \triangleq \left\{ v \in \mathbb{R}^n \mid v^\top (x - \bar{x}) \leq 0, \ \forall x \in K \right\}$$

is the *normal cone* of K at $\bar{x}$. For a large part of our study, the VI is of the affine kind where $F(x) = q + Qx$ is an affine function with $q \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n \times n}$ being given and K is the polyhedron $P(b, A) \triangleq \{ x \in \mathbb{R}^n \mid Ax \leq b \}$ with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. We denote the solution set of the affine variational inequality (AVI) defined by the tuple (q, Q, b, A) by $\text{SOL}(q, Q, P(b, A))$. The AVI encompasses many important special cases; we mention two of them. When Q is symmetric, the AVI constitutes the stationarity conditions of the (indefinite) quadratic program:

$$\underset{x \in P(b,A)}{\textbf{minimize}} \ q^\top x + \tfrac{1}{2} x^\top Q x. \tag{2.2}$$

The AVI also generalizes the mixed linear complementarity problem (LCP), which is the problem:

$$\begin{aligned} 0 &\leq u \perp r + Ru + Vv \geq 0 \\ 0 &= s + Uu + Sv \end{aligned}$$

where $\perp$ is the perpendicularity notation which in this context denotes the complementary slackness between two vectors, R and S are square matrices, and all vectors and matrices are of appropriate dimensions. Polyhedral multifunctions are particularly relevant to the AVI. In particular, the set-valued map $(q, b) \mapsto \text{SOL}(q, Q, P(b, A))$ is a polyhedral multifunction. It is a useful fact that a single-valued polyhedral multifunction is a piecewise affine function; moreover if its domain is convex, then it must be Lipschitz continuous; for a sketch of a proof, see [17, Exercises 5.6.14] and also [38, Exercise 2.48] and [13, Corollary 3D.5]. Specializing the earlier-mentioned result for a strongly monotone VI, we see that if Q is positive definite (not necessarily symmetric), then for every fixed pair (b, A), the solution map $q \mapsto \text{SOL}(q, Q, P(b, A))$ of an AVI is single-valued and Lipschitz continuous on $\mathbb{R}^n$. When $P(b, A) = \mathbb{R}^n_+$, an analogous result holds for the LCP: $0 \leq z \perp q + Mz \geq 0$ with a P-matrix M (i.e., when all principle minors of M are positive). In [9], the P-matrix hypothesis was relaxed to the P_0-condition (i.e., when all principle minors of M are nonnegative) but an additional Z-property (i.e., all off-diagonal entries of M are nonpositive) and a feasibility assumption were needed. In the latter case, the LCP does not necessarily have a unique solution but its least-element solution [5,33] yields a Lipschitz solution selection as a function of the vector q in the domain of feasibility of the problem. This result can be extended to an "upper-bounded" LCP (i.e., the AVI with a rectangular defining set) via a least-element theory [32]. Inspired by the analysis in [9], we will subsequently extend the continuous least-element selection result to the LCP with M being a "hidden Z-matrix" [30,31]; see Proposition 4.1.

With results for the linear complementarity problem in the background [19] that are subsequently extended to the Lipschitz properties of polyhedral multifunctions [21], the reference [14] unifies many past results and focuses on a locally Lipschitz property of a set-valued map called the *Aubin property* [1]. Specialized to the (generally multi-valued) solution map $q \mapsto \text{SOL}(q, Q, P(\bar{b}, A))$ near a pair $(\bar{q}, \bar{x})$ with the triple $(\bar{b}, A, Q)$ being fixed (in particular the polyhedron $P(\bar{b}, A)$ and the solution $\bar{x} \in \text{SOL}(\bar{q}, Q, P(\bar{b}, A))$ are fixed), it is shown that the Aubin property is equivalent to the lower semicontinuity of the said solution map around $(\bar{q}, \bar{x})$, and is further equivalent to the local single-valuedness of this map around $(\bar{q}, \bar{x})$ as well as the strong regularity of the solution $\bar{x}$ of the AVI defined by the tuple $(\bar{q}, \bar{b}, A, Q)$. The latter is a renowned concept introduced by Robinson [22] in the theory of generalized equation. Another equivalent critical face condition in [14] is given a more detailed study in [24] from the perspective of polyhedral geometry and metric regularity. The later part of the reference [14] extends the equivalences for the AVI to the VI $(q + F(\bullet, c), K)$ with the pair (q, c) being the parameter, the set K being fixed, and $F(\bullet, c)$ satisfying a certain continuous differentiability condition with respect to the first argument.

In contrast to the above references where the lower semicontinuity is around a given pair $(\bar{q}, \bar{x})$, the monograph [25] provides an extensive study of the lower semicontinuity of the AVI solution map $(q, b) \mapsto \text{SOL}(q, Q, P(b, A))$ at a give pair of vectors $(\bar{q}, \bar{b})$ (so the defining polyhedron of the AVI is allowed to move parallel to the reference polyhedron $P(\bar{b}, A)$). The case of a symmetric matrix Q, i.e., for the QP (2.2), is studied in great details in several chapters, including properties of the solution map of the associated Karush-Kuhn-Tucker conditions:

$$\begin{array}{rcl} 0 & = & q + Qx + A^\top \lambda \\ 0 & \leq & \lambda \perp b - Ax \geq 0. \end{array} \tag{2.3}$$

The last Section 18.2 in the cited monograph is devoted to the lower semicontinuity of the solution map of the AVI at a given pair $(\bar{q}, \bar{b})$. In particular, combining Theorems 18.5 and 18.7 therein, we can prove the following result for a monotone AVI. For a generic (finite) set S, the notation $|S|$ denotes the cardinality of S.

THEOREM 2.4. Let Q be positive semidefinite. For a given pair $(\bar{q}, \bar{b})$, a necessary and sufficient condition for $\mathrm{SOL}(\bullet, Q, P(\bullet, A))$ to be lower semicontinuous at $(\bar{q}, \bar{b})$ is that there exists $\widehat{x}$ such that $A\widehat{x} < \bar{b}$ and $|\,\mathrm{SOL}(\bar{q}, Q, P(\bar{b}, A))\,| = 1$ $\square$

The proof of the sufficiency of the result is a consequence of a stability theory of parametric variational inequalities; see [20] and [17, Section 5.5]. Part of this proof is to show the nonemptiness of $\mathrm{SOL}(q, Q, P(b, A))$ for (q, b) near $(\bar{q}, \bar{b})$. With the existence of nearby solutions guaranteed and the solution uniqueness of the reference problem, the lower semicontinuity of the solution map follows readily from the classical Lipschitz property [40] of polyhedra as a function of the right-hand side; the latter property is closely related to the renowned Hoffman error bounds for systems of linear inequalities [23].

In summary, while there has been extensive research on the lower semicontinuity of the solution map of an (A)VI, the role of this property in the existence of a continuous selection of this map (according to Definition 2.1) has not been directly addressed. Although one could resort to Michael's seminal work [29] for the connections between these two properties, such a direct reference may be too restrictive and omits a lot of important details such as the role and/or necessity of solution uniqueness.

3. Our Contributions. A natural question (beyond the special least-element theory) raised by the cited results in the last section is whether a continuous selection of the solution map of an (A)VI can exist without the uniqueness requirement. A trivial situation that suggests this possibility is when q and Q are both zero; in this case the AVI reduces to just the feasibility of the polyhedron $P(b, A)$ and the existence of a continuous selection of feasible solutions (with A fixed) is immediate. When Q is positive definite, we have mentioned that $\mathrm{SOL}(q, Q, P(b, A))$ is a singleton if nonempty; moreover, this unique solution is a Lipschitz continuous, piecewise affine function of (q, b) on $\mathbb{R}^n \times \mathrm{dom}\, P(\bullet, A)$. As a unification of these two extreme cases, we are led to consider an AVI (q, b, A, Q) with Q being a positive semidefinite-plus (psd+) matrix for which we aim to establish the existence of a continuous solution function; we also want to gain a deeper understanding about the role of the unique solvability of the nonlinear problem. By definition, a matrix $Q \in \mathbb{R}^{n\times n}$ is psd+ if $v^\top Qv \geq 0$ for all $v \in \mathbb{R}^n$ (positive semidefiniteness) and the implication holds: $v^\top Qv = 0 \Rightarrow Qv = 0$ (the plus-property). Symmetric positive semidefinite matrices are psd+; so is a positive definite matrix; moreover, the class of psd+ matrices is closed under addition. A result of Luo-Tseng [27] shows that a matrix $Q \in \mathbb{R}^{n\times n}$ is psd+ if and only if there exist a nonsingular matrix $E \in \mathbb{R}^{n\times n}$ and a positive definite matrix $R \in \mathbb{R}^{\ell\times\ell}$ for some positive integer $\ell \in \{1, \cdots, n\}$ such that

$$Q = E^\top \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} E. \tag{3.1}$$

Thus with a psd+ matrix Q, the affine mapping $z \mapsto Qz + q$ is strongly monotone-composite. The four main contributions of our work are as follows:

• **Lipschizt selection for a special class of LCPs:** These LCPs are not necessarily monotone; yet they possess special solutions that can be shown to be continuous, thus Lipschitz continuous. This part of our work addresses comments from a referee about connections with results in the references [8, 9]; in particular, the main result herein, Proposition 4.1, is inspired by the reference [9] that pertains to LCPs with Z-matrices.

• **Lipschitz selection without uniqueness for AVIs:** We show that when Q is a psd+ matrix and q is fixed, a continuous single-valued selection $x(b) \in \mathrm{SOL}(q, Q, P(b, A))$ always exists for all b for which $\mathrm{SOL}(q, Q, P(b, A))$ is nonempty. It then follows that the derived solution function is Lipschitz continuous and piecewise affine on its domain. We also broaden the results for a fixed q to the family $q^0 + \mathrm{Range}\, Q$

where q belongs with q^0 being fixed but arbitrary. We remark that with Q being positive semidefinite, $\mathrm{SOL}(q, Q, P(b, A))$ is nonempty if and only if (see [17, Theorem 2.4.7]):

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} q \\ b \end{pmatrix} + \begin{bmatrix} Q & A^\top \\ -A & 0 \end{bmatrix} \begin{pmatrix} \mathbb{R}^n \\ \mathbb{R}^m_+ \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbb{R}^m_- \end{pmatrix}.$$

• **Nonlinear VI: Extension without uniqueness:** We extend the affine analysis to a nonlinear parametric VI $(F, P(b, A))$ with a strongly monotone-composite mapping F given by (2.1) with a restriction on the vector a but no restriction on e. While the final selection result is similar to the AVI, a key difference is the absence of the piecewise affine property of the solution function, due to the nonlinearity of F; so a different proof is needed. Parallel to the AVI, with F being monotone, $\mathrm{SOL}(F, P(b, A))$ is nonempty if and only if the set $\{ x \in P(b, A) \mid F(x)^\top (x - x^{\mathrm{ref}}) < 0 \}$ is bounded (possibly empty) for some $x^{\mathrm{ref}} \in P(b, A)$; see [17, Theorem 2.3.4].

• **Role of uniqueness:** For a given pair $(\bar{b}, \bar{c})$ for which $\mathrm{SOL}(F(\bullet, \bar{c}), P(\bar{b}, A)) \neq \emptyset$, where $F : \mathbb{R}^{n+k} \to \mathbb{R}^n$ is a bivariate function satisfying certain locally Lipschitz properties, and $F(\bullet, \bar{c})$ is monotone, we show that $|\mathrm{SOL}(F(\bullet, \bar{c}), P(\bar{b}, A))| = 1$ is a necessary condition for the existence of a continuous selection $x(b, c) \in \mathrm{SOL}(F(\bullet, c), P(b, A))$ at $(\bar{b}, \bar{c})$. Together with the converse, we obtain the equivalence of three properties: the existence of the continuous selection of solutions, the solution uniqueness of the base problem, and the pointwise lower semicontinuous of the solution map of a monotone VI under constraint and functional perturbations. These equivalences for the *internally parameterized* VI readily specialize to an *externally parameterized* VI, with the former referring to a bivariate function $F(\bullet, c)$ where the parameter c is embedded within it, and the latter referring to the special case where $F(x, c) = \widehat{F}(x) + c$ for some univariate mapping $\widehat{F} : \mathbb{R}^n \to \mathbb{R}^n$ with the parameter c appearing independently of the primary function $\widehat{F}$.

4. A Special Class of LCPs. In [28], Mangasarian introduced a class of matrices for which the LCP can be solved as a linear program (LP). Subsequently coined a hidden Z-matrix in [30,31], a matrix $M \in \mathbb{R}^{n \times n}$ belongs to this class if there exist Z-matrices X and Y and positive vectors $\bar{r}$ and $\bar{s}$ such that (a) $MX = Y$ and (b) $\bar{r}^\top X + \bar{s}^\top Y > 0$. [Much is known about such matrices [11, Section 3.11]; in particular, for a recent work on LCPs with hidden Z-matrices and their computational complexity, see [22].] An important property of such a matrix M is that the matrix X must be nonsingular; more importantly, the LCP (q, M), if feasible, must have a solution given by $\bar{x}(q) = X\bar{v}(q)$, where $\bar{v}(q)$ is the *unique* optimal solution of the linear program:

$$\underset{v \in \mathbb{R}^n}{\textbf{minimize}}\ p^\top v \ \ \textbf{subject to} \ \ q + Yv \geq 0 \ \text{ and } \ Xv \geq 0, \tag{4.1}$$

for any positive vector $p \in \mathbb{R}^n$. This solution $\bar{v}(q)$ is the least element of the feasible set of the above LP. Elements of this set are in one-to-one-correspondence (via the nonsingular relation $x = Xv$) with those of the feasible set of the LCP (q, M), which we denote $\mathrm{FEA}(M) \triangleq \{ q \mid \exists\, x \geq 0 \text{ satisfying } q + Mx \geq 0 \}$; the least-element property means that $\bar{v}(q) \leq v$ for all v feasible to (4.1). Proved in [10], these facts allow us to show that the solution function $\bar{x}(q)$ is Lipschitz continuous on $\mathrm{FEA}(M)$.

PROPOSITION 4.1. Let M be a hidden Z-matrix. Then there exists a solution function $\bar{x}(q)$ of the LCP (q, M) defined for every $q \in \mathrm{FEA}(M)$ such that $\bar{x}(\bullet)$ is Lipschitz continuous on $\mathrm{FEA}(M)$.

Proof. It suffices to show that the least-element solution function $\bar{v}(\bullet)$ is continuous on the relative interior of the domain of feasibility of the LP (4.1). By the least-element property of $\bar{v}(q)$, it can be shown that each component of the vector function $\bar{v}(\bullet)$ is convex on the feasible set of the LP (4.1), which itself is a polyhedron. Indeed, for any feasible vectors q^1 and q^2 and for any scalar $\tau \in (0, 1)$, the vector $\tau\bar{v}(q^1) + (1-\tau)\bar{v}(q^2)$ is feasible to the LP (4.1) for $\tau q^1 + (1-\tau)q^2$. Hence, $\bar{v}(\tau q^1 + (1-\tau)q^2) \leq \tau\bar{v}(q^1) + (1-\tau)\bar{v}(q^2)$, proving the claimed componentwise convexity of the solution function $\bar{v}(\bullet)$. (This convexity result was proved in [9, Theorem 2.1] for the case of the LCP with an “M-matrix”.) This is enough to yield the desired continuity of $\bar{v}(\bullet)$ on the relative domain of $\mathrm{FEA}(M)$, and thus that of $\bar{x}(\bullet) = X\bar{v}(\bullet)$. [Remark: Unlike the case of a Z-matrix, the components of $\bar{x}(\bullet)$ needs not be convex.] To argue the Lipschitz continuity on the entire domain $\mathrm{FEA}(M)$, we use the fact that $\bar{v}(q)$ is the unique optimal solution of (4.1), and also the fact that a single-valued polyhedral multifunction on a convex domain is piecewise affine and Lipschitz continuous. ∎

5. Lipschitz Selection Without Uniqueness for AVI. The cornerstone of our new AVI results is an equivalent formulation of the problem using the decomposition (3.1). By definition, a solution of the AVI

$(q, Q, P(b; A))$ is a vector $\bar{z}$ satisfying $A\bar{z} \le b$ such that

$$(\, z - \bar{z}\,)^\top (q + Q\bar{z}) \,\ge\, 0, \quad \forall\, z \,\in\, P(b, A).$$

Letting $\zeta \triangleq Ez$, $\bar{q} \triangleq (E^\top)^{-1} q$, and $\bar{A} \triangleq AE^{-1}$, we see that the above inequality is equivalent to:

$$(\, \zeta - \bar{\zeta})^\top \left(\bar{q} + \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \bar{\zeta} \right) \,\ge\, 0, \quad \forall\, \zeta \text{ satisfying } \bar{A}\zeta \,\le\, b, \tag{5.1}$$

for $\bar{\zeta} \triangleq E\bar{z}$. We partition the vectors ζ, $\bar{q}$ and matrix $\bar{A}$ in accordance with $\begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}$:

$$\zeta \,\triangleq\, \begin{pmatrix} u \,\in\, \mathbb{R}^\ell \\ w \,\in\, \mathbb{R}^{n-\ell} \end{pmatrix}, \quad \bar{q} \,\triangleq\, \begin{pmatrix} r \\ c \end{pmatrix}, \quad \text{and} \quad \bar{A} \,\triangleq\, [\; G \quad H \;]$$

so that (5.1) becomes

$$(\, u - \bar{u}\,)^\top (\, r + R\bar{u}\,) + c^\top (\, w - \bar{w}\,) \,\ge\, 0, \quad \forall\, (u, w) \text{ satisfying } Gu + Hw \le b. \tag{5.2}$$

5.1. The case of fixed q. The construction of the desired continuous selection in this case consists of two intermediate steps that are combined by a composition.

Step 1 (The LP component): We consider the linear program (LP) parameterized by the pair (b, u) with c fixed:

$$v_c(b, u) \,\triangleq\, \underset{w \in \mathbb{R}^{n-\ell}}{\textbf{minimum}}\; c^\top w \;\; \textbf{subject to} \;\; Hw \,\le\, b - Gu. \tag{5.3}$$

The following result summarizes some important properties of the value function v_c; the key property for our purpose is the existence of a continuous selection of optimal solutions of the problem on its domain of finiteness, which we denote $\mathcal{D}_c$; i.e., (b, u) belongs to $\mathcal{D}_c$ if $v_c(b, u)$ is finite; also let $\mathcal{S}_c(b, u)$ denote the optimal solution set of (5.3) for $(b, u) \in \mathcal{D}_c$

PROPOSITION 5.1. Let $c \in \mathbb{R}^{n-\ell}$ be such that the (constant) polyhedron

$$\Lambda_c \,=\, \left\{\, \lambda \,\in\, \mathbb{R}^m_+ \,:\, c + H^\top \lambda \,=\, 0 \,\right\}$$

is nonempty. The following statements hold for the parametric LP (5.3):

(a) The domain $\mathcal{D}_c$ is a polyhedron in $\mathbb{R}^{m+\ell}$, consisting of pairs (b, u) for which there exists w satisfying $Hw \le b - Gu$; moreover, the value function v_c is convex and piecewise affine on $\mathcal{D}_c$.

(b) A piecewise affine, thus Lipschitz continuous, function $w_c^{\text{opt}} : \mathcal{D}_c \to \mathbb{R}^{n-\ell}$ exists such that $w_c^{\text{opt}}(b, u)$ belongs to $\mathcal{S}_c(b, u)$ for all $(b, u) \in \mathcal{D}_c$.

Proof. Note that Λ_c is the feasible set of the dual of (5.3); thus statement (a) is an immediate consequence of LP duality. It therefore suffices to prove statement (b). Let $\mathcal{E}(\Lambda_c)$ be the (finite) set of extreme points of Λ_c. We then have

$$\mathcal{S}_c(b, u) \,=\, \left\{ w \,\in\, \mathbb{R}^{n-\ell} \mid Hw \,\le\, b - Gu \;\text{ and }\; c^\top w \,\le\, \max_{\lambda \in \mathcal{E}(\Lambda_c)} \left[-(\, b - Gu\,)^\top \lambda \right] \right\}.$$

Since $\mathcal{E}(\Lambda_c)$ is a finite set, the function $\chi_c : (b, u) \mapsto \chi_c(b, u) \,\triangleq\, \max_{\lambda \in \mathcal{E}(\Lambda_c)} \left[-(\, b - Gu\,)^\top \lambda \right]$ is a convex, piecewise affine function on $\mathbb{R}^{m+\ell}$. For every scalar α for which the polyhedron

$$\mathcal{W}_c(b, u; \alpha) \,\triangleq\, \left\{\, w \,\in\, \mathbb{R}^{n-\ell} \mid Hw \,\le\, b - Gu \;\text{ and }\; c^\top w \,\le\, \alpha \,\right\}$$

is nonempty, it has a minimum-norm vector, denoted $\widetilde{w}_c(b, u; \alpha)$, that is a piecewise affine, thus Lipschitz continuous function of the triplet $(b, u; \alpha) \in \mathbb{R}^{m+\ell+1}$; equivalently, $\widetilde{w}_c(b, u; \alpha)$ is the unique element of the polyhedron $\mathcal{W}_c(b,u;\alpha)$ that is closest to the origin in the Euclidean norm. Note that for $(b,u) \in \mathcal{D}_c$, we have $\mathcal{W}_c(b,u;v_c(b,u)) = \mathcal{S}_c(b,u) \neq \emptyset$. The desired solution function $w_c^{\rm opt}(b,u)$ is the composition of $\widetilde{w}_c(b,u;\bullet)$ with $\chi_c(b,u)$, both of which are piecewise affine. Since $w_c^{\rm opt}(\bullet,\bullet)$ is a piecewise affine function with the polyhedral domain $\mathcal{D}_c$, it is Lipschitz continuity there. $\square$

Remark. The existence of the continuous solution function $w_c^{\rm opt}(b,u)$ can be proved by the Lipschitz continuity of the solution mapping $\mathcal{S}_c$ [13, Exercise 3C.5] with the aid of Michael's selection theorem. The above proof offers a constructive expression of the claimed solution function via many possible choices of the vector $\widetilde{w}_c(b,u;\alpha)$ as a composition of the optimal solutions of strongly convex quadratic programs with a piecewise affine function. $\square$

Step 2 (The generalized variational inequality component): For a given $b \in \operatorname{dom} P(\bullet, A)$, let $\mathcal{U}_b$ be the set of vectors $u \in \mathbb{R}^\ell$ such that the pair $(b,u) \in \mathcal{D}_c$ and let $\partial_u v_c(b,\bullet)$ denote the subdifferential of the convex function $v_c(b,\bullet)$ when this subdifferential is well defined. Since by duality,

$$v_c(b,u) \,=\, \underset{\lambda \in \mathcal{E}(\Lambda_c)}{\textbf{maximum}} \left[-(\, b - Gu\,)^\top \lambda \right], \tag{5.4}$$

we deduce that when $\partial_u v_c(b,u)$ exists, we have

$$\partial_u v_c(b,u) \,=\, \underbrace{G^\top \widehat{\mathcal{E}}_c(b,u)}_{\text{denoted } \widehat{\partial}_u v_c(b,u)} \quad \subseteq\, G^\top[\text{convex hull of } \mathcal{E}(\Lambda_c)],$$

where, with $\mathcal{H}[\mathcal{E}(\Lambda_c)]$ denoting the convex hull of $\mathcal{E}(\Lambda_c)$,

$$\begin{aligned}
\widehat{\mathcal{E}}_c(b,u) \;&\triangleq\; \text{convex hull of } \underset{\lambda \in \mathcal{E}(\Lambda_c)}{\textbf{argmax}} \left[-(\, b - Gu\,)^\top \lambda \right] \\
&=\; \underset{\lambda \in \mathcal{H}[\mathcal{E}(\Lambda_c)]}{\textbf{argmax}} \left[-(\, b - Gu\,)^\top \lambda \right] \\
&\subseteq\; \underset{\lambda \in \Lambda_c}{\textbf{argmax}} \left[-(\, b - Gu\,)^\top \lambda \right], \quad \text{with equality not necessarily holding.}
\end{aligned}$$

Unlike $\partial_u v_c(b,u)$, which may fail to be well defined for certain $b \in \operatorname{dom} P(\bullet, A)$, $\widehat{\partial}_u v_c(b,u)$ is well defined for all pairs $(b,u) \in \mathbb{R}^{m+\ell}$. The result below shows that $\widehat{\partial}_u v_c(b,u)$ has all the properties of the subdifferential $\partial_u v_c(b,u)$.

PROPOSITION 5.2. The set-valued map $\widehat{\partial}_u v_c : \mathbb{R}^{m+\ell} \rightrightarrows \mathbb{R}^\ell$ has the following properties:

- it is nonempty-valued, convex-valued, and compact-valued; $\displaystyle\bigcup_{(b,u)\in\mathbb{R}^{m+\ell}} \widehat{\partial}_u v_c(b,u)$ is bounded;
- $\widehat{\partial}_u v_c$ is a polyhedral multifunction; thus it is everywhere pointwise upper Lipschitz continuous; moreover, $\widehat{\partial}_u v_c(b,\bullet)$ is monotone for all $b \in \mathbb{R}^m$;
- for all u and $\bar{u}$ in $\mathcal{U}_b$ and all $\eta \in \widehat{\partial}_u v_c(b,\bar{u})$,

$$v_c(b,u) \,\geq\, v_c(b,\bar{u}) + \eta^\top (\, u - \bar{u}).$$

Proof. Only the monotonicity of $\widehat{\partial}_u v_c(b,\bullet)$ requires proof. But this is obvious too because $\widehat{\partial}_u v_c(b,\bullet)$ is the subdifferential of the everywhere finite-valued convex function $\widehat{v}_c(b,\bullet)$, where

$$\widehat{v}_c(b,u) \,\triangleq\, \underset{\lambda \in \mathcal{H}[\mathcal{E}(\Lambda_c)]}{\textbf{maximum}} \left[-(\, b - Gu\,)^\top \lambda \right].$$

In turn, the everywhere finite-valuedness of $\widehat{v}_c$ is guaranteed by the compactness of $\mathcal{H}[\mathcal{E}(\Lambda_c)]$. $\square$

Define the set-valued map:

$$\Phi_c(b,u) \,\triangleq\, r + Ru + \widehat{\partial}_u v_c(b,u), \quad (b,u) \,\in\, \mathcal{D}_c. \tag{5.5}$$

For a given vector $b \in \mathbb{R}^m$ for which $\mathcal{U}_b \neq \emptyset$, we may consider the generalized variational inequality (GVI) defined by the pair $(\Phi_c(b, \bullet), \mathcal{U}_b)$ [4]; this problem is to find a pair $(\bar{u}, \bar{\eta})$ with $\bar{u} \in \mathcal{U}_b$ and $\bar{\eta} \in \widehat{\partial}_u v_c(b, \bar{u})$ such that

$$(u - \bar{u})^\top (r + R\bar{u} + \bar{\eta}) \geq 0 \quad \forall\, u \in \mathcal{U}_b. \tag{5.6}$$

Since R is a positive definite matrix and $\widehat{\partial}_u v(b, \bullet)$ is monotone, it follows that $\Phi_c(b, \bullet)$ is strongly monotone on $\mathcal{U}_b$. Hence the GVI $(\Phi_c(b, \bullet), \mathcal{U}_b)$ has a unique solution, which we denote $\bar{u}_c(b)$. The proof of existence can be found in [4]; the proof of uniqueness of the solution is standard as in the case of a point-valued strongly monotone VI; see e.g. [17, Theorem 2.3.3]. The derivation below aims to show that $\|\bar{u}_c(b)\|_2$ is bounded by a multiplicative factor of $\|b\|_2$.

Note that the set of vectors b for which $\mathcal{U}_b \neq \emptyset$ is equal to the set of vectors b for which $P(b, A) \neq \emptyset$. It follows from applying Hoffman's error bound for linear inequalities [23] to the origin that there exists a constant $\beta > 0$ only depending on A such that for all b for which $\mathcal{U}_b \neq \emptyset$, there exists $\widetilde{u} \in \mathcal{U}_b \neq \emptyset$ together with a certain $\widetilde{x} = (\widetilde{u}, \widetilde{w}) \in P(b, A)$ such that

$$\| \widetilde{u} \|_2 \leq \|\widetilde{x} - 0\|_2 \leq \beta \|[A0 - b]_+\|_2 \leq \beta \| b \|_2,$$

where the notation $[y]_+ \triangleq \max\{y, 0\}$ stands for the nonnegative part of y. Let $\widetilde{\eta} \in \widehat{\partial}_u v_c(b, \widetilde{u})$. Substituting this vector $\widetilde{u}$ into the inequality in (5.6), we deduce that, by the monotonicity of $\widehat{\partial}_u v_c(b, \bullet)$,

$$\begin{aligned} 0 &\leq (\widetilde{u} - \bar{u}_c(b))^\top (r + R\bar{u}_c(b) + \bar{\eta}) \\ &\leq (\widetilde{u} - \bar{u}_c(b))^\top [r + R(\bar{u}_c(b) - \widetilde{u}) + R\widetilde{u} + \widetilde{\eta}]. \end{aligned}$$

Letting $\lambda_{\min}(R) > 0$ be the smallest eigenvalue of the positive definite matrix R, we deduce

$$\| \bar{u}_c(b) - \widetilde{u} \|_2 \leq \lambda_{\min}(R)^{-1} \| r + R\widetilde{u} + \widetilde{\eta} \|_2,$$

which implies

$$\| \bar{u}_c(b) \|_2 \leq \beta \| b \|_2 + \lambda_{\min}(R)^{-1} [\| r + \widetilde{\eta} \|_2 + \beta \| b \|_2 \|R\|_2]$$

Since $\widehat{\partial}_u v_c(b, \bullet)$ is uniformly bounded, we deduce the existence of a constant $\beta' > 0$ such that

$$\| \bar{u}_c(b) \|_2 \leq \beta' [1 + \| b \|_2], \quad \forall\, b \in \operatorname{dom} P(\bullet, A). \tag{5.7}$$

Armed with this boundedness inequality, which implies in particular that $\bar{u}_c(b)$ is bounded on compact subsets of dom $P(\bullet, A)$, the next result shows that this solution map $\bar{u}_c$ is continuous on its domain.

Proposition 5.3. Let $\{b^k\}$ be a set of vectors converging to b^∞ such that $\mathcal{U}_{b_k} \neq \emptyset$ for all k. Then $\mathcal{U}_{b_\infty} \neq \emptyset$ and $\{\bar{u}_c(b^k)\}$ converges to $\bar{u}_c(b^\infty)$.

Proof. Since the set of vectors b for which $\mathcal{U}_b \neq \emptyset$ is equal to the domain of $P(\bullet, A)$, the nonemptiness of $\mathcal{U}_{b_\infty}$ follows readily. By (5.7), the sequence $\{\bar{u}_c(b^k)\}$ is bounded. Let $\widehat{u}$ be any one of its accumulation points. To show that $\widehat{u} = \bar{u}_c(b^\infty)$, we need to show that $\widehat{u} \in \mathcal{U}_{b^\infty}$ and there exists $\widehat{\eta} \in \widehat{\partial}_u v_c(b^\infty, \widehat{u})$ such that

$$(u - \widehat{u})^\top (r + R\widehat{u} + \widehat{\eta}) \geq 0, \quad \forall\, u \in \mathcal{U}_{b^\infty}.$$

Without loss of generality, we may assume that $\widehat{u}$ is the limit of the sequence $\{\bar{u}_c(b^k)\}$. For each k, there exists w^k such that $Hw^k \leq b^k - G\bar{u}_c(b^k)$. Since the right-hand vector converges to $b^\infty - G\widehat{u}$, it follows from Lipschitz property of $P(\bullet, H)$ that there exists $\widehat{w}$ (which does not need to be a limit point of $\{w^k\}$) satisfying $H\widehat{w} \leq b^\infty - G\widehat{u}$; thus $\widehat{u} \in \mathcal{U}_{b^\infty}$. Next, for any $u \in \mathcal{U}_{b^\infty}$, it follows from Hoffman's error bound for linear inequalities that there exists a sequence $\{u^k\}$ such that $u^k \in \mathcal{U}_{b^k}$ for all k and

$$\|u^k - u\|_2 \leq \beta \|[Au^k - b^\infty]_+\|_2 \leq \beta \|[Au^k - Au]_+\|_2,$$

where β is a certain constant depending only on A and the last inequality is due to $Au \leq b^\infty$. This implies u^k converges to u. For each k, there exists $\eta^k \in \widehat{\partial}_u v_c(b^k, \bar{u}_c(b^k))$ such that

$$(u^k - \bar{u}_c(b^k))^\top (r + R\bar{u}_c(b^k) + \eta^k) \geq 0.$$

With the sequence $\{\eta^k\}$ being bounded, let $\widehat{\eta}$ be an accumulation point of $\{\eta^k\}$. Since $\widehat{\partial}_u v_c$ is a closed map, it follows that $\widehat{\eta} \in \widehat{\partial}_u v_c(b^\infty, \widehat{u})$. $\square$

Final step (The composition): Define the composite function $\bar{w}_c(b) \triangleq w_c^{\mathrm{opt}}(b, \bullet) \circ \bar{u}_c(b)$, where $w_c^{\mathrm{opt}}(\bullet, \bullet)$ is the bivariate function identified in Proposition 5.1(b). Define the function

$$\bar{\zeta}_c : \mathrm{dom}\ P(\bullet, A) \to \mathbb{R}^n, \quad \text{with } \bar{\zeta}_c(b) \triangleq \begin{pmatrix} \bar{u}_c(b) \\ \bar{w}_c(b) \end{pmatrix} \text{ for } b \in \mathrm{dom}\ P(\bullet, A). \tag{5.8}$$

PROPOSITION 5.4. For every b for which $P(b, A) \neq \emptyset$, the vector $\bar{\zeta}_c(b)$ defined by (5.8) is a solution of the AVI (5.1).

Proof. We prove the assertion via the transformed inequality (5.2). For feasibility, we clearly have $G\bar{u}_c(b) + H\bar{w}_c(b) \leq b$. By (5.6), for some $\bar{\eta} \in \widehat{\partial}_u v_c(b, \bar{u}_c(b))$, we have, for all $u \in \mathcal{U}_b$,

$$\begin{aligned} 0 &\leq (u - \bar{u}_c(b))^\top (\, r + R\bar{u}_c(b) + \bar{\eta}\,) \\ &\leq (u - \bar{u}_c(b))^\top (\, r + R\bar{u}_c(b)\,) + v_c(b, u) - v_c(b, \bar{u}_c(b)) \end{aligned}$$

Let (u, w) be arbitrary satisfying $Gu + Hw \leq b$. Since $v_c(b, \bar{u}_c(b)) = c^\top \bar{w}_c(b)$, the second inequality in the above display yields

$$0 \leq (u - \bar{u}_c(b))^\top (\, r + R\bar{u}_c(b)\,) + c^\top (w - \bar{w}_c(b)),$$

which is the desired inequality (5.2). $\square$

Summarizing the above derivations, we have therefore proved the existence of a continuous selection of solutions to the AVI $(q, Q, P(b, A))$ with q fixed. Indeed, this solution is given by $\bar{z}_q(b) \triangleq E^{-1}\bar{\zeta}_c(b)$, where E is the nonsingular matrix that induces the decomposition (3.1). We claim that this solution function is piecewise affine in b; this is accomplished by the next lemma which shows the same property for $\bar{u}_c(b)$.

LEMMA 5.5. Let $\Phi : \mathbb{R}^{m+\ell} \rightrightarrows \mathbb{R}^\ell$ be a polyhedral multifunction and $\mathcal{P}$ a polyhedron in $\mathbb{R}^{m+\ell}$. Then with $\mathcal{U}(b) \triangleq \left\{ u \in \mathbb{R}^\ell \mid (b, u) \in \mathcal{P} \right\}$, the solution map $\mathcal{S} : b \mapsto \mathrm{SOL}(\Phi(b, \bullet), \mathcal{U}(b))$ is a polyhedral multifunction. Moreover, if $\mathcal{S}$ has a convex domain and is single-valued there, then it is Lipschitz continuous piecewise affine on dom $\mathcal{S}$.

Proof. It suffices to show that

$$\mathrm{gph}\ \mathcal{S} = \left\{ (b, u) \in \mathbb{R}^{m+\ell} \mid u \in \mathrm{SOL}(\Phi(b, \bullet), \mathcal{U}(b)) \right\}$$

is a union of finitely many polyhedra. We may write

$$\mathcal{P} \triangleq \{ (b, u) \mid Bb + Uu \leq a \}$$

for some matrices B and U and vector a of appropriate dimensions. In terms of the normal cone $\mathcal{N}(\mathcal{U}(b); u)$ of $\mathcal{U}(b)$ at its element $u \in \mathcal{U}(b)$, we have

$$u \in \mathrm{SOL}(\Phi(b, \bullet), \mathcal{U}(b)) \Leftrightarrow 0 \in \Phi(b, u) + \mathcal{N}(\mathcal{U}(b); u). \tag{5.9}$$

We claim that the set-valued map $\Upsilon : (b, u) \mapsto \mathcal{N}(\mathcal{U}(b), u)$ is a polyhedral multifunction. Indeed, a vector $\xi \in \Upsilon(b, u)$ if and only if $0 \in -\xi + \mathcal{N}(\mathcal{U}(b), u)$, or equivalently,

$$\begin{aligned} & u \in \underset{u'}{\mathbf{argmin}}\ -\xi^\top u' \ \ \mathbf{subject\ to}\ \ Uu' \leq a - Bb \\ \Longleftrightarrow \quad & \exists\, \lambda \text{ such that} \left\{ \begin{array}{lcl} \xi & = & U^\top \lambda \\ 0 & \leq & \lambda \perp a - Bb - Uu \geq 0. \end{array} \right. \end{aligned}$$

Hence $\xi \in \Upsilon(b, u)$ if and only if there exists λ such that the right-hand mixed complementarity conditions in the above display holds. This is enough to show that Υ is a polyhedral multifunction. Thus so are $\Phi + \Upsilon$ and its inverse $(\Phi + \Upsilon)^{-1}$. By the equivalence (5.9), it follows that gph $\mathcal{S} = (\Phi + \Upsilon)^{-1}(0)$ with the right-hand set being the union of finitely many polyhedra; thus $\mathcal{S}$ is a polyhedral multifunction. The second conclusion of the lemma therefore follows from the fact that a single-valued polyhedral multifunction on a convex domain must be a Lipschitz continuous piecewise affine function; see Proposition 4.2.2 in [17]. $\square$

Combining Lemma 5.5 with the previous analysis, the following theorem does not require a proof.

THEOREM 5.6. Let Q be psd+. For every fixed q such that the set $\Lambda_c \neq \emptyset$, there exists a piecewise affine, thus Lipschitz continuous, function $\bar{z}_q : \text{dom } P(\bullet, A) \to \mathbb{R}^n$ such that for every b for which $P(b, A)$ is nonempty, the vector $\bar{z}_q(b)$ is a solution of the AVI $(q, Q, P(b, A))$. $\square$

Remarks. The condition $\Lambda_c \neq \emptyset$ plays a hidden role ensuring in particular the solvability of the AVI $(q, Q, P(b, A))$. The piecewise affine property of both functions $\bar{u}_c$ and w_c^{opt} is very much responsible for the Lipschitz continuity of the resulting solution function $\bar{z}_q$. In the subsequent generalization to a nonlinear VI, the piecewise affine property of $\bar{u}_c$, and thus of the solution function, is lost, $\square$

5.2. A special family of q. Theorem 5.6 can be used to allow a varying q, based on the observation that the vector r does not need to be fixed in the above derivation. Indeed, instead of (5.5), we may define

$$\Phi_c(r, b, u) \triangleq r + Ru + \widehat{\partial}_u v_c(b, u), \quad (b, u) \in \mathcal{D}_c,$$

to signify the dependence of this function on r as well, and let $\bar{u}_c(r, b)$ be the unique solution of the GVI $(\Phi_c(r, b, \bullet), \mathcal{U}_b)$. Defining $\bar{z}_c(r, b) \triangleq E^{-1} \begin{pmatrix} \bar{u}_c(r, b) \\ \bar{w}_c(r, b) \end{pmatrix}$, where $\bar{w}_c(r, b) \triangleq w_c^{\text{opt}}(b, \bullet) \circ \bar{u}_c(r, b)$, we can show that the conclusion of Theorem 5.6 holds for this solution function. [Note, we write $\bar{z}_c(r, b)$ here instead of $\bar{z}_q(b)$ to emphasize that only the vector c is fixed that defines a family of vectors q that correspond to the same c.] Next, we apply a simple linear-algebraic maneuver to convert the pair (r, c) back in terms of the given vector q. We partition the matrix $E \triangleq \begin{bmatrix} C \\ D \end{bmatrix}$ in accordance with $\begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}$: thus $C \in \mathbb{R}^{\ell \times n}$ and $D \in \mathbb{R}^{(n-\ell) \times n}$ so that $Q = C^\top RC$. Notice that the matrix D can be chosen fairly arbitrarily. So we choose D so that its rows are linearly independent and each of these rows is perpendicular to all the rows of C, resulting in $DC^\top = 0$. We then have

$$q = E^\top \bar{q} = C^\top r + D^\top c. \tag{5.10}$$

Let $\mathcal{Q}$ be the null space of Q and $\mathcal{Q}^\perp$ be the orthogonal complement of $\mathcal{Q}$. Due to the plus-property of Q, one can deduce that $z \in \mathcal{Q}$ iff $z^\top Qz = (Cz)^\top R(Cz) = 0$, which amounts to $Cz = 0$ by the positive definiteness of R. What is more, because C and D have linearly independent rows and $DC^\top = 0$, it can be seen that $\text{span}(D^\top)$ is the orthogonal complement of $\text{span}(C^\top)$, where $\text{span}(\bullet)$ stands for the linear subspace spanned by the columns of the nominal matrix. Hence, $Cz = 0$ is further equivalent to $z \in \text{span}(D^\top)$. It follows that $\mathcal{Q} = \text{span}(D^\top)$ and $\mathcal{Q}^\perp = \text{span}(C^\top)$. As a result, $\Pi_{\mathcal{Q}}(q) = D^\top \left(DD^\top\right)^{-1} Dq$ and $\Pi_{\mathcal{Q}^\perp}(q) = C^\top \left(CC^\top\right)^{-1} Cq$, where $\Pi_\bullet(\bullet)$ denotes the Euclidean projection of a vector onto a closed convex set. Therefore, in view of (5.10) and the projection formulae, r and c are uniquely given by

$$r = (CC^\top)^{-1} Cq \quad \text{and} \quad c = (DD^\top)^{-1} Dq.$$

We have the following corollary of Theorem 5.6 for the AVI $(q, Q, P(b, A))$ in which the vector q varies in an affine subspace.

COROLLARY 5.7. Let Q be psd+. For every q^0 such that the set $\Lambda_{c^0} \neq \emptyset$, where $c^0 \triangleq (DD^\top)^{-1} Dq^0$, a piecewise affine, thus Lipschitz continuous, function $\bar{z}_{c^0} : \mathcal{Z} \triangleq (q^0 + \text{Range } Q) \times \text{dom } P(\bullet, A) \to \mathbb{R}^n$ exists such that for every $(q, b) \in \mathcal{Z}$, the image $\bar{z}_{c^0}(q, b) \in \text{SOL}(q, Q, P(b, A))$.

Proof. With $q = q^0 + Qu$ for some $u \in \mathbb{R}^n$, we have $Qu \in \mathcal{Q}^\perp$ because if $x \in \mathcal{Q}$, then $Qx = 0$, which implies $Q^\top x = 0$ by the psd+ property of Q; hence $x^\top Qu = 0$, proving $Qu \in \mathcal{Q}^\perp$. Therefore, $\Pi_{\mathcal{Q}}(Qu) = 0$. Hence

$$q = \Pi_{\mathcal{Q}}(q^0) + \left[\Pi_{\mathcal{Q}^\perp}(q^0) + \Pi_{\mathcal{Q}^\perp}(Qu) \right] = D^\top c^0 + C^\top r(q),$$

where $r(q) \triangleq (CC^\top)^{-1} Cq^0 + (CC^\top)^{-1} C(q - q^0)$. By defining $\bar{z}_{c^0}(q, b) \triangleq E^{-1} \begin{pmatrix} \bar{u}_{c^0}(r(q), b) \\ \bar{w}_{c^0}(r(q), b) \end{pmatrix}$, the stated properties of this function readily hold. $\square$

6. Nonlinear VI: Extension without Uniqueness. Consider the VI (F, K), where F is given by (2.1) with $a = a^0 + \widehat{E}^\top g$ (this is motivated by the form of the vector q in Corollary 5.7) and $K = P(b, A)$. Thus,

$$F(x) = \widehat{E}^\top \left[g + \widehat{G}(\widehat{E}x + e) \right] + a^0.$$

With F as given, the VI is equivalent to finding a pair $(\bar{x}, \bar{y}) \in Z$ such that

$$(y - \bar{y})^\top [\, g + \widehat{G}(y)\,] + (\, x - \bar{x}\,)^\top a^0 \, \geq \, 0, \quad \forall \, (\, x, y\,) \, \in \, Z, \tag{6.1}$$

where $Z \triangleq \left\{ (x, y) \in \mathbb{R}^{n+\ell} \mid Ax \leq b \text{ and } y - \widehat{E}x = e \right\}$. The form (6.1) is analogous to (5.2), with the pair (x, y) in (6.1) playing the role of (w, u) in (5.2) and the strongly monotone mapping $\widehat{G}$ in (6.1) replacing the positive definite matrix R in (5.2). Thus, we may follow the same three steps as before: first define an LP, followed by a GVI, and then the composition of the former two steps. Although the steps are the same, there are important differences in proving the desired property of the solution function due to the nonlinearity of the map $\widehat{G}$.

The LP: Similar to the value function $v_c(b, u)$ and the feasible region Λ_c of the dual linear program of this function, define, for a given vector $a^0 \in \mathbb{R}^n$:

$$\begin{aligned} v_{a^0}(b, e, y) \;&\triangleq\; \underset{x \in \mathbb{R}^n}{\textbf{minimum}} \; (a^0)^\top x \;\; \textbf{subject to} \;\; Ax \leq b \;\; \textbf{and} \;\; \widehat{E}x = y - e \\ &= \underset{(\lambda,\mu) \,\in\, \widehat{\Lambda}_{a^0}}{\textbf{maximum}} \; -b^\top \lambda + (\, y - e\,)^\top \mu, \end{aligned} \tag{6.2}$$

where

$$\widehat{\Lambda}_{a^0} \;\triangleq\; \left\{ (\lambda, \mu) \,\in\, \mathbb{R}^m_+ \times \mathbb{R}^\ell \mid a^0 + A^\top \lambda - \widehat{E}^\top \mu \,=\, 0 \right\},$$

assumed nonempty, depends only on the vector a^0 and the matrices A and $\widehat{E}$. Let $\mathcal{Y} : \mathbb{R}^{m+\ell} \rightrightarrows \mathbb{R}^\ell$ be the set-valued map with $\mathcal{Y}(b, e) \triangleq \widehat{E}P(b, A) + e$ for all pairs $(b, e) \in \mathbb{R}^{m+\ell}$. Note that $\operatorname{dom} \mathcal{Y} = \operatorname{dom} P(\bullet, A) \times \mathbb{R}^\ell$. As in Proposition 5.1, it can be shown that there exists a Lipschitz continuous, piecewise affine function $x^{\mathrm{opt}}_{a^0} : \operatorname{gph} \mathcal{Y} \to \mathbb{R}^n$ such that $x^{\mathrm{opt}}_{a^0}(b, e, y)$ is an optimal solution of the LP in (6.2) for all $y \in \mathcal{Y}(b, e)$.

Suppose that $\widehat{E}$ has linearly independent rows. Then the set $\mathcal{E}(\widehat{\Lambda}_{a^0})$ of extreme points of the polyhedron $\widehat{\Lambda}_a$ is nonempty. Define

$$\begin{aligned} \widehat{\mathcal{E}}_{a^0}(b, e, y) \quad &\triangleq \quad \underset{(\lambda,\mu) \,\in\, \mathcal{H}[\mathcal{E}(\widehat{\Lambda}_{a^0})]}{\textbf{argmax}} \; -b^\top \lambda + (\, y - e\,)^\top \mu, \quad \text{and} \\ \widehat{\partial}_y v_{a^0}(b, e, y) \quad &\triangleq \quad \left\{ \mu \,\in\, \mathbb{R}^\ell \mid \exists\, \lambda \text{ such that } (\lambda, \mu) \,\in\, \widehat{\mathcal{E}}_{a^0}(b, e, y) \right\}. \end{aligned}$$

These two sets play the same roles as $\widehat{\mathcal{E}}_c(b, u)$ and $\widehat{\partial}_u v_c(b, u)$ in the affine case, respectively. In particular, the set-valued map $\widehat{\partial}_y v_{a^0} : \mathbb{R}^{m+\ell+\ell} \rightrightarrows \mathbb{R}^\ell$ satisfies a set of analogous properties as $\widehat{\partial}_u v_c(b, u)$ described in Proposition 5.2.

The GVI: This is defined by the pair $(\Phi_{a^0}(g, b, e, \bullet), \mathcal{Y}(b, e))$, where

$$\Phi_{a^0}(g, b, e, y) \;\triangleq\; g + \widehat{G}(y) + \widehat{\partial}_y v_{a^0}(b, e, y).$$

Since $\Phi_{a^0}(b, e, g, \bullet)$ is a strongly monotone multifunction, the GVI $(\Phi_{a^0}(g, b, e, \bullet), \mathcal{Y}(b, e))$ has a unique solution, which we denote $\bar{y}_{a^0}(g, b, e)$, whenever $\mathcal{Y}(b, e) \neq \emptyset$. We next show that this solution is locally Lipschitz continuous on its domain, which is $\mathbb{R}^\ell \times \operatorname{dom} P(\bullet, A) \times \mathbb{R}^\ell$. This claim is the analog of Lemma 5.5 for a strongly monotone GVI with a non-polyhedral multifunction and with a changing defining set. The proof turns out to be a little involved. For this purpose, we summarize the properties of the set-valued mapping $\Gamma_{a^0} : (b, e, y) \mapsto \widehat{\partial}_y v_{a^0}(b, e, y) + \mathcal{N}(\mathcal{Y}(b, e); y) \subseteq \mathbb{R}^\ell$:

- $\operatorname{dom} \Gamma_{a^0} = \left\{ (b, e, y) \in \mathbb{R}^{m+2\ell} \mid y - e \,\in\, \widehat{E}P(b, A) \right\}$;
- Γ_{a^0} is a polyhedral multifunction on its domain;
- for each pair $(b, e) \in \operatorname{dom} P(\bullet, A) \times \mathbb{R}^\ell$, $\Gamma_{a^0}(b, e, \bullet)$ is monotone on $\mathcal{Y}(b, e)$; this is because $\widehat{\partial}_y v_{a^0}(b, e, \bullet)$ is so and $\mathcal{N}(\mathcal{Y}(b, e); \bullet)$ is the subdifferential of the (convex) indicator function of $\mathcal{Y}(b, e)$.

Combining the pair (b, e) into the vector $h \in \mathbb{R}^{\bar{m}}$, where $\bar{m} \triangleq m + \ell$, the GVI $(\Phi_{a^0}(g, h, \bullet), \mathcal{Y}(h))$ is equivalent to the inclusion:

$$0 \,\in\, g + \widehat{G}(y) + \Gamma_{a^0}(h, y), \tag{6.3}$$

whose unique solution, for a given pair (g,h), is the vector $\bar{y}_{a^0}(g,h)$.

LEMMA 6.1. Let $\widehat{G} : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be a Lipschitz continuous, strongly monotone mapping and $\widehat{E} \in \mathbb{R}^{\ell \times n}$ have full row rank. Let $a^0 \in A^\top \mathbb{R}^m_- + \text{Range}\, \widehat{E}^\top$ be arbitrary but fixed. The solution function $\bar{y}_{a^0}$ is Lipschitz continuous on $\mathbb{R}^\ell \times \text{dom}\, P(\bullet, A) \times \mathbb{R}^\ell$.

Proof. Throughout the proof, which is divided into 3 steps, we will drop the subscript a^0 in both $\bar{y}_{a^0}$ and Γ_{a^0}.

Step 1 (Continuity): By using the strong monotonicity of the mapping $\widehat{G}$ instead of the positive definiteness of the matrix R, and following the same argument as in the proof of Proposition 5.3, we can show that $\bar{y}$ is continuous at every $(\bar{g}, \bar{h})$ with $\mathcal{Y}(\bar{h}) \neq \emptyset$ i.e.,

$$\lim_{\substack{(g,h)\to(\bar{g},\bar{h})\\ h\,\in\,\text{dom}\mathcal{Y}}} y(g,h) = y(\bar{g},\bar{h}).$$

Step 2 (Pointwise upper Lipschitz continuity): Next we show the upper Lipschitz continuity of $\bar{y}$ at $(\bar{g}, \bar{h})$ for which $\mathcal{Y}(\bar{h}) \neq \emptyset$. For every pair (g,h) with $\mathcal{Y}(h) \neq \emptyset$, there exists a pair $(y(g,h), \eta(g,h))$ with $\eta(g,h) \in \Gamma(h, y(g,h))$ such that

$$0 \;=\; g + \widehat{G}(y(g,h)) + \eta(g,h).$$

Hence,

$$\lim_{\substack{(g,h)\to(\bar{g},\bar{h})\\ h\,\in\,\text{dom}\mathcal{Y}}} \eta(g,h) = \eta(\bar{g},\bar{h}).$$

Since Γ is a polyhedral multifunction, we can write

$$\text{gph}\;\Gamma \;=\; \bigcup_{i\in I} \mathcal{G}^{\,i}, \;\text{ where }\; \mathcal{G}^{\,i} \;\triangleq\; \left\{\, (h,y,\eta) \;\mid\; H^i h + Y^i y + \Sigma^{\,i} \eta \;\leq\; \alpha^i \,\right\}$$

for a finite index set I and a family of matrices $\{H^i, Y^i, \Sigma^{\,i}\}_{i\in I}$ and vectors $\{\alpha^i\}_{i\in I}$ of appropriate orders. By the Lipschitz property of $P(\bullet, \widehat{\Sigma}^{\,i})$, where $\widehat{\Sigma}^{\,i} \triangleq [\, Y^i \quad \Sigma^{\,i}\,]$, there exists a scalar $L_i > 0$ such that for $\widehat{\alpha}$ and $\widehat{\alpha}\,'$ in dom $P(\bullet, \widehat{\Sigma}^{\,i})$, we have

$$P(\widehat{\alpha}\,', \widehat{\Sigma}^{\,i}) \;\subseteq\; P(\widehat{\alpha}, \widehat{\Sigma}^{\,i}) + L_i \,\|\, \widehat{\alpha} - \widehat{\alpha}\,' \,\|_2 \,\mathbb{B},$$

where $\mathbb{B}$ is the Euclidean ball in the (y,η)-space.

Writing $(\bar{y}, \bar{\eta}) \triangleq (y(\bar{g},\bar{h}), \eta(\bar{g},\bar{h}))$, we let $\bar{I}$ be the set of indices $i \in I$ such that $(\bar{h}, \bar{y}, \bar{\eta}) \in \mathcal{G}^{\,i}$. For every pair (g,h) sufficiently close to $(\bar{g},\bar{h})$ and with $h \in \text{dom}\, \mathcal{Y}$, there exists an index $i \in \bar{I}$ such that $(h, y(g,h), \eta(g,h)) \in \mathcal{G}^{\,i}$. Let $(\widehat{y}(g,h), \widehat{\eta}(g,h))$ be the Euclidean projection of the pair $(y(g,h), \eta(g,h))$ onto the polyhedron $P(\alpha^i - H^{\,i}\bar{h}, \widehat{\Sigma}^{\,i})$. We then have

$$\|\, (y(g,h),\eta(g,h)) - (\widehat{y}(g,h),\widehat{\eta}(g,h)) \,\|_2 \;\leq\; L_i' \,\|\, h - \bar{h} \,\|_2, \quad \text{ where } \; L_i' \;\triangleq\; L_i \,\|\, H^i \,\|_2. \tag{6.4}$$

Thus, the triplet $(\bar{h}, \widehat{y}(g,h), \widehat{\eta}(g,h)) \in \mathcal{G}^{\,i}$, which implies $\widehat{\eta}(g,h) \in \Gamma(\bar{h}, \widehat{y}(g,h))$. By the monotonicity of $\Gamma(\bar{h}, \bullet)$, we have

$$(\, \bar{\eta} - \widehat{\eta}(g,h))^\top (\, \bar{y} - \widehat{y}(g,h)) \;\geq\; 0. \tag{6.5}$$

Moreover, (6.4) implies

$$\lim_{\substack{(g,h)\to(\bar{g},\bar{h})\\ h\,\in\,\text{dom}\mathcal{Y}}} (\widehat{y}(g,h), \widehat{\eta}(g,h)) \;=\; (\bar{y}, \bar{\eta}).$$

We have

$$\begin{aligned} 0 \;&=\; g + \widehat{G}(y(g,h)) + \eta(g,h) \;=\; g + \widehat{G}(y(g,h)) + \widehat{\eta}(g,h) + [\, \eta(g,h) - \widehat{\eta}(g,h)\,] \\ 0 \;&=\; \bar{g} + \widehat{G}(\bar{y}) + \bar{\eta}. \end{aligned}$$

Thus, subtracting, we obtain

$$\widehat{G}(\bar{y}) - \widehat{G}(y(g,h)) \;=\; g - \bar{g} + \widehat{\eta}(g,h) - \bar{\eta} + [\, \eta(g,h) - \widehat{\eta}(g,h)\,],$$

which yields, by (6.5),

$$(\bar{y} - \widehat{y}(g,h))^\top (\widehat{G}(\bar{y}) - \widehat{G}(y(g,h))) \leq (\bar{y} - \widehat{y}(g,h))^\top (g - \bar{g}) + (\bar{y} - \widehat{y}(g,h))^\top [\eta(g,h) - \widehat{\eta}(g,h)].$$

We have

$$\begin{aligned} \text{L.H.S.} &= (\bar{y} - y(g,h))^\top (\widehat{G}(\bar{y}) - \widehat{G}(y(g,h))) + (y(g,h) - \widehat{y}(g,h))^\top (\widehat{G}(\bar{y}) - \widehat{G}(y(g,h))) \\ &\geq \sigma_{\widehat{G}} \| \bar{y} - y(g,h) \|_2^2 - \left[\max_{1 \leq i \leq I} L_i' \right] \| h - \bar{h} \|_2 \, \mathrm{Lip}_{\widehat{G}} \| \bar{y} - y(g,h) \|_2, \end{aligned}$$

where $\sigma_{\widehat{G}} > 0$ and $\mathrm{Lip}_{\widehat{G}} > 0$ are the strong monotonicity and Lipschitz constants of $\widehat{G}$, respectively. Moreover,

$$\begin{aligned} \text{R.H.S.} &\leq (\bar{y} - y(g,h))^\top [\eta(g,h) - \widehat{\eta}(g,h)] + (y(g,h) - \widehat{y}(g,h))^\top [\eta(g,h) - \widehat{\eta}(g,h)] + \\ &\quad (\bar{y} - y(g,h))^\top (g - \bar{g}) + (y(g,h) - \widehat{y}(g,h))^\top (g - \bar{g}) \\ &\leq \left(\left[\max_{1 \leq i \leq I} L_i' \right] \| h - \bar{h} \|_2 \| + \| g - \bar{g} \|_2 \right) \| \bar{y} - y(g,h) \|_2 + \\ &\quad \left[\max_{1 \leq i \leq I} L_i' \right] \| h - \bar{h} \|_2 \| g - \bar{g} \|_2 + \left(\left[\max_{1 \leq i \leq I} L_i' \right] \| h - \bar{h} \|_2 \right)^2. \end{aligned}$$

Hence, with $\widehat{L} \triangleq \left[\max_{1 \leq i \leq I} L_i' \right] \max\left(1, \mathrm{Lip}_{\widehat{G}} \right)$, we deduce,

$$\sigma_{\widehat{G}} \| \bar{y} - y(g,h) \|_2^2 - 2 \left(\widehat{L} \| h - \bar{h} \|_2 + \| g - \bar{g} \|_2 \right) \| \bar{y} - y(g,h) \|_2 \leq \left(\widehat{L} \| h - \bar{h} \|_2 + \| g - \bar{g} \|_2 \right)^2,$$

which yields

$$\left(\| \bar{y} - y(g,h) \|_2 - \frac{1}{\sigma_{\widehat{G}}} \left(\widehat{L} \| h - \bar{h} \|_2 + \| g - \bar{g} \|_2 \right) \right)^2 \leq \left(\frac{1}{\sigma_{\widehat{G}}} + \frac{1}{\sigma_{\widehat{G}}^2} \right) \left(\widehat{L} \| h - \bar{h} \|_2 + \| g - \bar{g} \|_2 \right)^2$$

Consequently,

$$\| y(g,h) - y(\bar{g},\bar{h}) \|_2 \leq \left[\frac{1}{\sigma_{\widehat{G}}} + \left(\frac{1}{\sigma_{\widehat{G}}} + \frac{1}{\sigma_{\widehat{G}}^2} \right)^{1/2} \right] \left(\widehat{L} \| h - \bar{h} \|_2 + \| g - \bar{g} \|_2 \right).$$

In summary, we have shown that there exists a constant $\kappa > 0$ such that for every $(\bar{g}, \bar{h})$ with $\mathcal{Y}(\bar{h}) \neq \emptyset$, a neighborhood $\mathcal{N}$ of $(\bar{g}, \bar{h})$ exists such that for every $(g,h) \in \mathcal{N}$ with $\mathcal{Y}(h) \neq \emptyset$, it holds that

$$\| y(g,h) - y(\bar{g},\bar{h}) \|_2 \leq \kappa \left(\| h - \bar{h} \|_2 + \| g - \bar{g} \|_2 \right).$$

Step 3 (Lipschitz continuity): This follows from the following more general result: a single-valued, everywhere pointwise upper Lipschitz continuous mapping on a convex domain with a common Lipschitz constant $\kappa > 0$ is Lipschitz continuous on the domain. The proof is actually very standard; see e.g. that of Proposition 4.2.2 in [17] showing the Lipschitz continuity of a piecewise affine mapping. For completeness, we provide the details. Let $f : \mathrm{dom}\, f \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be such a (single-valued) mapping $\mathrm{dom}\, f$ being convex. Let $\kappa > 0$ be a scalar such that for every vector $x \in \mathrm{dom}\, f$, there exists an open neighborhood $\mathcal{N}(x)$ such that

$$\| f(y) - f(x) \| \leq \kappa \| y - x \|, \quad \forall\, y \in \mathcal{N}(x) \cap \mathrm{dom}\, f. \tag{6.6}$$

Let $\bar{x}$ and $\bar{y}$ two arbitrary vectors in $\mathrm{dom}\, f$ and consider the line segment

$$[\, \bar{x}, \bar{y} \,] \triangleq \{ \tau \bar{x} + (1 - \tau) \bar{y} \mid \tau \in [0,1] \},$$

joining $\bar{x}$ and $\bar{y}$. Since dom f is convex, this segment is a subset of dom f. The family of open neighborhoods $\mathcal{N}(z)$, for all $z \in [\bar{x}, \bar{y}]$, is an open covering of this compact segment. Thus, there exists a finite family of these neighborhoods that covers the segment. This implies that there exists a partition of the interval $[0, 1]$:

$$0 = \tau_0 < \tau_1 < \cdots < \tau_p = 1$$

for some positive integer p such that for all $r = 0, 1, \cdots, p-1$, $\mathcal{N}(x^r) \cap \mathcal{N}(x^{r+1}) \neq \emptyset$, where $x^r \triangleq \bar{x} + \tau_r(\bar{y} - \bar{x})$. Letting $y^r \in [\, x^r, x^{r+1} \,]$ be a common element of $\mathcal{N}(x^r)$ and $\mathcal{N}(x^{r+1})$, we can write

$$f(\bar{x}) - f(\bar{y}) = \sum_{r=0}^{p-1} [\, (f(x^r) - f(y^r)) + (f(y^r) - f(x^{r+1}))\,].$$

The inequality (6.6) easily yields $\| f(\bar{x}) - f(\bar{y}) \| \leq \tau \| \bar{x} - \bar{y} \|$. □

THEOREM 6.2. Let $\widehat{G} : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be a Lipschitz continuous, strongly monotone mapping and $\widehat{E} \in \mathbb{R}^{\ell \times n}$ have full row rank. Let $a^0 \in A^\top \mathbb{R}^m_- + \text{Range}\, \widehat{E}^\top$ be arbitrary but fixed. There exists a Lipschitz continuous function $\bar{x}_{a^0} : \mathcal{Z} \triangleq (a^0 + \text{Range}\, \widehat{E}^\top) \times \text{dom}\, P(\bullet, A) \times \mathbb{R}^\ell \to \mathbb{R}^n$ such that for every $(q, b, e) \in \mathcal{Z}$, the image $\bar{x}_{a^0}(q, b, e) \in \text{SOL}(q + \widehat{E}^\top \widehat{G}(\widehat{E} \bullet + e), P(b, A))$.

Proof. With $q \triangleq a^0 + \widehat{E}^\top g$, it suffices to define $\bar{x}_{a^0}(q, b, e) \triangleq x^{\text{opt}}_{a^0}(b, e, \bullet) \circ \bar{y}_{a^0}(g, b, e)$. □.

Remarks. As both vectors g and e are allowed to change, with $a \triangleq a^0 + \widehat{E}^\top g$, the function F given by (2.1) is parameterized both internally and externally. Another noteworthy remark is that while b is allowed to change, the polyhedrality of the defining set $P(b, A)$ in the VI is heavily responsible for the additional Lipschitz property of the solution function. It may be possible to relax this polyhedrality requirement to a finitely representable set defined by differentiable inequalities satisfying certain constraint qualifications, we choose to omit such a generalization in this work to avoid non-trivial technical complications. □

7. The Role of Solution Uniqueness. Theorem 5.6, Corollary 5.7, and Theorem 6.2 are selection results on the domain of solvability of the respective problems. Two key restrictions are imposed: the plus-property of the (monotone) mapping and the restriction of the parameter in the defining functions; furthermore the last theorem pertains to an internal parameterization of a nonlinear VI without any differentiability requirement. In this section, we remove the two functional restrictions while retaining the monotonicity of the function and the right-hand perturbation of the defining polyhedron. The main result is a succinct equivalence of three pointwise properties—lower semicontinuity, continuous selection, and solution uniqueness—for this class of nonlinear VIs. For the proof of the local solvability under functional and constraint perturbations, we employ degree theory in nonlinear analysis for which the reader can consult [18,26]; see also [17, Section 2.1.1] for a summary of some basic properties of the degree of a continuous mapping; in particular, the index of a continuous mapping at an isolated zero is the key concept employed. For a generic function $f : \mathbb{R}^n \to \mathbb{R}^m$, we denote $f'(\bar{x}, d) \triangleq \lim_{\tau \downarrow 0} \dfrac{f(\bar{x} + \tau d) - f(\bar{x})}{\tau}$ as the directional derivative of f at $\bar{x} \in \mathbb{R}^n$ along the direction $d \in \mathbb{R}^n$. In particular, for a bivariate function $F : \mathbb{R}^{n+k} \to \mathbb{R}^m$ and for a given $x \in \mathbb{R}^n$, we write $F(x, \bullet)'(\bar{y}; dy)$ for the directional derivative of the (univariate) function $F(x, \bullet)$ at $\bar{y} \in \mathbb{R}^k$ along the direction $dy \in \mathbb{R}^k$.

THEOREM 7.1. Let $F : \mathbb{R}^{n+k} \to \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$ be given. Let $(\bar{b}, \bar{c})$ be such that (a) there exists $\widehat{x}$ satisfying $A\widehat{x} < \bar{b}$ and (b) $F(\bullet, \bar{c})$ is monotone. Let $\bar{x} \in \text{SOL}(F(\bullet, \bar{c}), P(\bar{b}, A))$ be given. Suppose that

(c) there exist an open neighborhood $\mathcal{U}$ of $(\bar{b}, \bar{c})$ and an open neighborhood $\mathcal{V}$ of $\bar{x}$ and positive constants Lip_x and Lip_c such that

$$\begin{aligned} \sup_{x \in P(b,A) \cap \mathcal{V}} \| F(x, c) - F(x, \bar{c}) \| &\leq \text{Lip}_c \| (b, c) - (\bar{b}, \bar{c}) \|, \quad \forall (b, c) \in \mathcal{U} \cap (\text{dom}\, P(\bullet, A) \times \mathbb{R}^k) \\ \text{and} \quad \| F(x, \bar{c}) - F(x', \bar{c}) \| &\leq \text{Lip}_x \| x - x' \|, \quad \forall x, x' \in \mathcal{V}; \end{aligned}$$

(d) $\displaystyle \lim_{\substack{x \to \bar{x};\, \tau \downarrow 0 \\ x \in P(\bar{b};\, A)}} \frac{F(x, \bar{c} + \tau d) - F(x, \bar{c})}{\tau} = F(\bar{x}, \bullet)'(\bar{c}, d)$ exists for all $d \in \mathbb{R}^k$; and

(e) $F(\bar{x}, \bullet)'(\bar{c}, \bullet)$ is surjective on $\mathbb{R}^k$.

Then the following three statements are equivalent:

(A) the set-valued map $(b, c) \mapsto \mathrm{SOL}(F(\bullet, c), P(b, A))$ has a continuous (single-valued) selection at the pair $((\bar{b}, \bar{c}), \bar{x})$;

(B) $\mathrm{SOL}(F(\bullet, \bar{c}), P(\bar{b}, A)) = \{\bar{x}\}$;

(C) the set-valued map $(b, c) \mapsto \mathrm{SOL}(F(\bullet, c), P(b, A))$ is lower semicontinuous at $(\bar{b}, \bar{c})$.

Proof. (A) $\Rightarrow$ (B). Suppose that the set-valued map $(b, c) \mapsto \mathrm{SOL}(F(\bullet, c), P(b, A))$ has a continuous (single-valued) selection at $(\bar{b}, \bar{c}, \bar{x})$ but $\bar{\mathcal{S}} \triangleq \mathrm{SOL}(F(\bullet, \bar{c}), P(\bar{b}, A))$ contains a vector distinct from $\bar{x}$. Letting $x(b, c)$ be such a selection function for (b, c) sufficiently close to $(\bar{b}, \bar{c})$, we have $\lim_{(b,c)\to(\bar{b},\bar{c})} x(b, c) = \bar{x}$. Since $\bar{\mathcal{S}}$ has at least two distinct elements, there exists a vector $v \in \mathbb{R}^n$ such that $\infty \geq \max_{x\in\bar{S}} v^\top x > \min_{x\in\bar{S}} v^\top x \geq -\infty$. By the assumption (e), there exists a vector $d \in \mathbb{R}^k$ such that $F(\bar{x}, \bullet)'(\bar{c}, d) = v$. For $\tau > 0$ sufficiently small, write $x^\tau \triangleq x(\bar{b}, \bar{c} + \tau d)$. We then have $\lim_{\tau\downarrow 0} x^\tau = \bar{x}$; moreover, for any $\widehat{x} \in \bar{S}$,

$$(\widehat{x} - x^\tau)^\top F(x^\tau, \bar{c} + \tau d) \geq 0 \quad \text{and} \quad (x^\tau - \widehat{x})^\top F(\widehat{x}, \bar{c}) \geq 0.$$

Adding the two inequalities, we obtain

$$\begin{aligned} 0 &\leq (\widehat{x} - x^\tau)^\top [F(x^\tau, \bar{c} + \tau d) - F(x^\tau, \bar{c})] + (\widehat{x} - x^\tau)^\top [F(x^\tau, \bar{c}) - F(\widehat{x}, \bar{c})] \\ &\leq (\widehat{x} - x^\tau)^\top [F(x^\tau, \bar{c} + \tau d) - F(x^\tau, \bar{c})], \end{aligned}$$

where the second inequality is by the monotonicity of $F(\bullet, \bar{c})$. Dividing by $\tau > 0$ and letting $\tau \downarrow 0$, we deduce $(\widehat{x} - \bar{x})^\top v \geq 0$. Since $\widehat{x} \in \bar{S}$ is arbitrary, it follows that $\bar{x} \in \operatorname*{\mathbf{argmin}}_{x\in\bar{S}} v^\top x$. Similarly, letting $d' \in \mathbb{R}^k$ satisfy $F(\bar{x}, \bullet)'(\bar{c}, d') = -v$. we can show that $\bar{x} \in \operatorname*{\mathbf{argmax}}_{x\in\bar{S}} v^\top x$. This is a contradiction.

(B) $\Rightarrow$ (A) and (C). Suppose $\mathrm{SOL}(F(\bullet, \bar{c}), P(\bar{b}, A)) = \{\bar{x}\}$. We prove that (A) and (C) hold by proceeding as follows:

(i) first recall that a vector $\widetilde{x}$ is a solution of the VI $(F(\bullet, c), P(b, A))$ if and only if $\widetilde{x}$ is a zero of the "natural map" of the VI; i.e., the map $\Psi(\bullet; b, c) : x \mapsto x - \Pi_{P(b,A)}(x - F(x, c))$, where $\Pi_{P(b,A)}$ is the Euclidean projector onto the polyhedron $P(b, A)$;

(ii) note that the set $\{ x \in P(\bar{b}, A) \mid F(x, \bar{c})^\top (x - \bar{x}) < 0 \}$ is empty, by the monotonicity of $F(\bullet, \bar{c})$;

(iii) by the proof of [17, Proposition 2.2.3], it follows that the index of the map $\Psi(\bullet; \bar{b}, \bar{c})$ at its unique zero $\bar{x}$ is equal to one; and

(iv) by the existence of the vector $\widehat{x}$ satisfying $A\widehat{x} < \bar{b}$, the set-valued map $b \mapsto P(b, A)$ is continuous at $\bar{b}$.

With these steps and assumption (c) in place, all the assumptions of Proposition 5.4.1 in [17] are satisfied; this proposition then yields the existence of a neighborhood, which we may take to be the same as $\mathcal{U}$ in assumption (c) such that

- $\mathcal{S}_{\mathcal{N}}(b, c) \triangleq \mathrm{SOL}(F(\bullet, c), P(b, A)) \cap \mathcal{N} \neq \emptyset$ for all $(b, c) \in \mathcal{U}$, and
- $\lim_{(b,c)(\in\mathcal{U})\to(\bar{b},\bar{c})} \{ \| x(b, c) - \bar{x} \| : x(b, c) \in \mathcal{S}_{\mathcal{N}}(b, c) \} = 0.$

The conclusions easily produce the desired statements (A) and (C).

(C) $\Rightarrow$ (B). The proof is similar to that of (C) implying (A). Assume for contradiction that the set-valued map $(b, c) \mapsto \mathrm{SOL}(F(\bullet, c), P(b, A))$ is lower semicontinuous at $(\bar{b}, \bar{c})$ but $\bar{\mathcal{S}}$ has at least two distinct elements, say $\widetilde{x}^1$ and $\widetilde{x}^2$. Then there exists a vector $v \in \mathbb{R}^n$ such that $v^\top \widetilde{x}^1 > v^\top \widetilde{x}^2$. Let $d \in \mathbb{R}^k$ be such that $F(\bar{x}, \bullet)'(\bar{c}, d) = v$. Let $\{\varepsilon_k\}$ be an arbitrary sequence of positive scalars converging to zero. By the lower semicontinuity of the solution map, there exist a sequence $\{x^k\}$ converging to $\widetilde{x}^1$ such that $x^k \in \mathrm{SOL}(F(\bullet, \bar{c} + \varepsilon_k d), P(\bar{b}, A))$ for all k. By the same argument as before, we may deduce that $\widetilde{x}^1 \in \operatorname*{\mathbf{argmin}}_{x\in\bar{S}} v^\top x$. Similarly, we can also prove that

$\widetilde{x}^{\,2} \in \underset{x \in \bar{S}}{\mathbf{argmax}}\, v^\top x$, which is a contradiction. $\square$

We make several remarks about the above theorem. First, the Slater assumption (a) ensures in particular the nonemptiness of $P(b,A)$ for all b sufficiently close to $\bar{b}$. Second, the monotonicity of $F(\bullet,c)$ is assumed only at $\bar{c}$; in particular, the perturbed VI $(F(\bullet,c),P(b,A))$ does not need to be of the monotone kind. Third, assumption (c) imposes certain Lipschitz property of the bivariate function $F(x,c)$. Finally, the limit in (d) is a directional differentiability of $F(\bar{x};\bullet)$ at $\bar{c}$ allowing for variations in the first argument near the base solution $\bar{x}$. As a straightforward corollary of Theorem 7.1, we have the following result for the externally perturbed VI where the requirements (c), (d), and (e) hold trivially.

Corollary 7.2. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a monotone and Lipschitz continuous function on $P(\bar{b},A)$, where $A \in \mathbb{R}^{m\times n}$. Let $\bar{b} \in \mathbb{R}^m$ be such that there exists $\widehat{x}$ satisfying $A\widehat{x} < \bar{b}$. Then the following four statements are equivalent:

(A) $|\,\mathrm{SOL}(\bar{c}+F,P(\bar{b},A))\,| = 1$;

(B) the set-valued map $(b,c) \mapsto \mathrm{SOL}(c+F,P(b,A))$ has a continuous (single-valued) selection at the pair $((\bar{b},\bar{c}),\bar{x})$ for some $\bar{x} \in \mathrm{SOL}(\bar{c}+F,P(\bar{b},A))$;

(B$'$) the set-valued map $(b,c) \mapsto \mathrm{SOL}(c+F,P(b,A))$ has a continuous (single-valued) selection at the pair $((\bar{b},\bar{c}),x)$ for all $x \in \mathrm{SOL}(\bar{c}+F,P(\bar{b},A))$;

(C) the set-valued map $(b,c) \mapsto \mathrm{SOL}(c+F,P(b,A))$ is lower semicontinuous at $(\bar{b},\bar{c})$.

Furthermore, if F is additionally affine, then any one of the above statements is equivalent to

(D) the AVI $(\bar{c}+F,P(\bar{b},A))$ has a stable solution with respect to perturbations of the pair $(\bar{b},\bar{c})$; i.e., a solution $\bar{x} \in \mathrm{SOL}(\bar{c}+F,P(\bar{b},A))$ exists with the property that for any open neighborhood $\mathcal{N}$ of $\bar{x}$ satisfying $\mathrm{SOL}(\bar{c}+F,P(\bar{b},A)) \cap \mathrm{cl}\,\mathcal{N} = \{\bar{x}\}$, there exist two positive scalars ε and γ such that for all pairs (b,c) satisfying $\|(b,c)-(\bar{b},\bar{c})\| \leq \varepsilon$, the set $\mathcal{S}_{\mathcal{N}}(b,c) \triangleq \mathrm{SOL}(c+F,P(b,A)) \cap \mathcal{N} \neq \emptyset$ and

$$\sup\{\,\|\,x-\bar{x}\,\| \,:\, x \in \mathcal{S}_{\mathcal{N}}(b,c)\,\} \,\leq\, \gamma\,\|\,(b,c)-(\bar{b},\bar{c})\,\|.$$

Proof. Statement (D) implies (B) in general. Thus, it remains to show that statement (A) implies (D) if F is additionally affine. But this follows readily by the fact that the set-valued solution map $(b,c) \mapsto \mathrm{SOL}(c+F,P(b,A))$ is a polyhedral multifunction, thus everywhere pointwise upper Lipschitz continuous, for an affine F. $\square$

8. Conclusion. In this paper, we have studied the (single-valued) continuous selection of solutions of a monotone variational inequality under functional and constraint perturbations in three settings: (a) for the AVI solution map $b \mapsto \mathrm{SOL}(q,Q,P(b,A))$ with only perturbations in the constraint; (b) the AVI solution map $(b,u) \mapsto \mathrm{SOL}(c^0+Qu,Q,P(b,A))$, and (c) the VI solution map $(b,c) \mapsto \mathrm{SOL}(F(\bullet,c),P(b,A))$. With Q being a psd+ matrix, the first case sets the groundwork of the analysis that is extended to the second case; in these two cases, the existence of a Lipschitz continuous (single-valued) solution function is established on the domain of the problems. In the nonlinear case, the uniqueness of a solution of the problem corresponding to a base pair $(\bar{b},\bar{c})$ is shown to be equivalent to several pointwise properties the solution map of the parametric VI, with both internal and external parameterization. To conclude this paper, we note that throughout the study of the AVI, the matrices Q and A have been fixed. In general, the analysis of a varying pair (Q,A) belongs to the subject of "total sensitivity analysis" of variational inequalities that was studied in an unpublished manuscript [16] with limited results. In particular, with a varying psd (but not pd) matrix Q, we expect that the solution selection problem would be quite difficult to analyze, even when A is fixed. The difficulty would be compounded with a varying A. Nevertheless, with particular structural variations, it might be possible to obtain some extended results with varying (Q,A) for a singular Q. The investigation of such extensions is best left for a future study.

REFERENCES

[1] J.P. Aubin. Lipschitz behavior of solutions of convex minimization problems. *Mathematics of Operations Research* 9: 87–111 (1984).

[2] J.P. Aubin and H. Frankowska. *Set-Valued Analysis*. Modern Birkhäuser Classics (2009). [Originally published 1990.]
[3] C. Berge. *Topological Spaces*. The MacMillan Company (New York 1963).
[4] D. Chan and J.S. Pang. The generalized quasi-variational inequality problem. *Mathematics of Operations Research* 7: 211–222 (1982).
[5] R. Chandrasekaran. A special case of the complementary pivot problem. *Opsearch* 7: 263–268 (1970).
[6] X. Chen and J.L. Shen. Dynamic stochastic veriational inequalities and convergence of discrete approximations. *SIAM Journal on Optimization* 32: 2909–2937 (2022).
[7] X. Chen, T.K. Pong, and R.J.B. Wets. Two-stage stochastic veriational inequalities: an ERM-solution procedure. *Mathematical Programming* 165: 71–112 (2017).
[8] X. Chen and S. Xiang. Newton iterations in implicit time-stepping scheme for differential linear complementarity systems. *Mathematical Programming* 138: 59–606 (2013).
[9] X. Chen and S. Xiang. Implicit solution function of P_0 and Z matrix linear complementarity constraints. *Mathematical Programming* 128: 1–18 (2011).
[10] R.W. Cottle and J.S. Pang. On solving linear complementarity problems as linear programs. *Mathematical Programming Study* 7: 88–107 (1978).
[11] R.W. Cottle, J.S. Pang, and R.E. Stone. *The Linear Complementarity Problem*. Classics in Applied Mathematics, Volume 60. SIAM (Philadelphia 2009). [Originally published by Academic Press (Boston 1992).
[12] G. Dommisch. On the existence of Lipschitz-continous and differentiable selections for multifunctions. In J. Guddat *et al.* editors *Parametric Optimization and Related Topics II.* (Academic Press, Berlin 1987) pp. 60–73.
[13] A.L. Dontchev and R.T. Rockafellar. *Implicit Functions and Solution Mappings: A view from variational analysis*. Springer monographs in mathematics (Dordrecht 2009).
[14] A.L. Dontchev and R.T. Rockafellar. Characterizations of strong regularity for variational inequalities over polyhedral convex sets. *SIAM Journal on Optimization* 6: 1087–1105 (1996).
[15] F. Facchinei and J.S. Pang. Nash equilibria: the variational approach. In Daniel P. Palomar and Yonica C. Eldar (eds) *Convex Optimization in Signal Processing and Communications* pp. 443–493. Cambridge University Press (England 2010).
[16] F. Facchinei and J.S. Pang. Total stability of variational inequalities. Technical report 09-98, Dipartimento di Informatica e Sistemistica, Universit'a Degli Studi di Roma "La Sapienza" (September 1998).
[17] F. Facchinei and J.S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Springer Series in Operations Research (New York 2003).
[18] I. Fonseca and W. Gangbo. *Degree Theory in Analysis and Applications*. Oxford University Press (England 1995).
[19] M.S. Gowda. On the continuity of the solution map in linear complementarity problems. *SIAM Journal on Optimization* 2: 619–634 (1992).
[20] M.S. Gowda and J.S. Pang. Stability analysis of variational inequalities and nonlinear complementarity problems, via the mixed linear complementarity problem and degree theory. *Mathematics of Operations Research* 19(4): 831–879 (1994)
[21] M.S. Gowda and R. Sznajder. On the Lipschitzian properties of polyhedral multifunctions. *Mathematical Programming* 74: 267-–278 (1996).
[22] S. Han, X. Zhang, and J.S. Pang. On the number of pivots of Dantzig's simplex methods for linear and convex quadratic programs. *Operations Research Letters*. Revision under review (June 2023).
[23] A.J. Hoffman. Approximate solutions of systems of linear inequalities. *Journal of Research of the National Bureaur of Standards* 49: 263-265 (1952).
[24] A.D. Ioffe. On variational inequalities over polyhedral sets. *Mathematical Programming* 168(6): 261–278 (2018).
[25] G.M. Lee, N.N. Tam, and N.D. Yen. *Quadratic Programming and Affine Variational Inequalities: A Qualitative Study*. Springer (New York 2005).
[26] N.G. Lloyd. *Degree Theory*. Cambridge University Press (1978).
[27] Z.Q. Luo and P. Tseng. A decomposition property for a class of square matrices. *Applied Mathematics Letters* 4: 67–69 (1991).
[28] O.L. Mangasarian. Linear complementarity problems solvable by a single linear program. *Mathematical Programming* 10: 263–270 (1976).
[29] E. Michael. Continuous selections I. *Annals of Mathematics. Second Series* 63(2): 361–382 (1956).
[30] J.S. Pang. Hidden Z-matrices with positive principal minors, *Linear Algebra and Its Applications* 23: 201–215 (1979).
[31] J.S. Pang. On discovering hidden Z-matrices. In C.V. Coffman and G.J. Fix, editors: *Constructive Approaches to Mathematical Models*. Academic Press (New York 1979) pp. 231–241.
[32] J.S. Pang. On a class of least-element linear complementarity problems. *Mathematical Programming* 16: 111–126 (1979).
[33] J.S. Pang. *Least-Element Complementarity Theory*. Ph.D. thesis, Department of Operations Research. Stanford University (August 1976).
[34] J.S. Pang and M. Fukushima. Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games. *Computational Management Science* 2: 21–56 (2005). [Erratum: same journal 6: 373–375 (2009).]
[35] J.S. Pang and D.E. Stewart. Differential variational inequalities. *Mathematical Programming* 113: 345–424 (2008).
[36] S.M. Robinson. Some continuity properties of polyhedral multifunctions. *Mathematical Programming Study* 14: 206–214 (1981).
[37] S.M. Robinson. Strongly regular generalized equations. *Mathematics of Operations Research* 5: 43–62 (1980).
[38] R.T. Rockafellar and R.J.B. Wets. *Variational Analysis*. Third Printing. Grundlehren der mathematischen Wissenschaften, Volume 317. Springer-Verlag, (Berlin 2009).
[39] R.T. Rockafellar and R.J.B. Wets. Stochastic veriational inequalities: single-stage to multistage. *Mathematical Programming* 165: 331–360 (2017).
[40] D.W. Walkup and R.J.B. Wets. A Lipschitzian characterization of convex polyhedra. *Proceedings of the American Mathematical Society* 23: 167–173 (1969).