M-stationarity of Local Minimizers of MPCCs and Convergence of NCP-based Methods

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Abstract: This paper focuses on solving mathematical programs with complementarity constraints (MPCCs) without assuming MPCC-LICQ or lower level strict complementarity at a solution. We show that a local minimizer of an MPCC is "piecewise M-stationary" under MPCC-GCQ; furthermore, every weakly stationary point of an MPCC is B-stationary if MPCC-ACQ holds. For the Bounding Algorithm proposed in [22], which solves MPCCs via an NCP-based reformulation, we develop C- and M- stationarity of a limit point of the method by assuming only MPCC-GCQ. In particular, an inequality variant of this method offers an alternative viewpoint to understand the behavior of an algorithm when approaching a local minimizer of an MPCC which is not S-stationary. In addition, a few practical issues related to convergence to a non-strongly stationary solution are discussed, including a comparison between the behaviors of the NCP-based methods and of a typical regularization method, i.e., the REG method proposed in [19].

Keywords: MPCCs, B-stationarity, constraint qualification, duality, NCP

1 Introduction

We consider MPCCs of the form

$$\begin{aligned} \min \quad & f(z) \\ \text{s.t.} \quad & g(z) \le 0, \\ & h(z) = 0, \\ & 0 \le G_i(z) \perp H_i(z) \ge 0, \quad i = 1 \dots m, \end{aligned} \tag{1}$$

where $(f, g, h, G, H) : \mathbb{R}^n \to \mathbb{R}^{1+n_g+n_h+m+m}$ are differentiable functions. At a feasible point $\bar{z}$ of the MPCC, define the following index sets:

$$\begin{aligned} I_g(\bar{z}) &= \{i \,|\, g_i(\bar{z}) = 0\}, \\ \alpha(\bar{z}) &= \{i \,|\, G_i(\bar{z}) = 0, H_i(\bar{z}) > 0\}, \\ \gamma(\bar{z}) &= \{i \,|\, G_i(\bar{z}) > 0, H_i(\bar{z}) = 0\}, \\ \beta(\bar{z}) &= \{i \,|\, G_i(\bar{z}) = 0, H_i(\bar{z}) = 0\}. \end{aligned} \tag{2}$$

A feasible point $\bar{z}$ is weakly stationary, if there exist multipliers $\bar{\lambda} = (\bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$ with $\bar{\lambda}^g \geq 0$, such that

$$0 = \nabla f(\bar{z}) + \sum_{i \in I_g(\bar{z})} \bar{\lambda}_i^g \nabla g_i(\bar{z}) + \sum_{i=1}^{n_h} \bar{\lambda}_i^h \nabla h_i(\bar{z}) - \sum_{i \in \alpha(\bar{z}) \cup \beta(\bar{z})} \bar{\lambda}_i^G \nabla G_i(\bar{z}) - \sum_{i \in \gamma(\bar{z}) \cup \beta(\bar{z})} \bar{\lambda}_i^H \nabla H_i(\bar{z}). \tag{3}$$

Further, a weakly stationary point $\bar{z}$ is also

- S-stationary (strongly stationary), if $\bar{\lambda}_i^G, \bar{\lambda}_i^H \geq 0$ for all $i \in \beta(\bar{z})$;
- M-stationary, if either $\bar{\lambda}_i^G, \bar{\lambda}_i^H > 0$ or $\bar{\lambda}_i^G \bar{\lambda}_i^H = 0$ for all $i \in \beta(\bar{z})$;
- C-stationary, if $\bar{\lambda}_i^G \bar{\lambda}_i^H \geq 0$ for all $i \in \beta(\bar{z})$;
- A-Stationary, if either $\bar{\lambda}_i^G \geq 0$ or $\bar{\lambda}_i^H \geq 0$ for all $i \in \beta(\bar{z})$.

1.1 Local optimality and geometry simplification

A local minimizer $\bar{z}$ of MPCC (1) is called a B-stationary point, at which the following equivalent conditions hold

$$\nabla f(\bar{z})^T d \geq 0, \ \forall d \in \mathcal{T}(\bar{z}) \quad \Leftrightarrow \quad \nabla f(\bar{z}) \in \mathcal{T}(\bar{z})^*, \tag{4}$$

where $\mathcal{T}(\bar{z})$ is the tangent cone of the MPCC at the point $\bar{z}$, whose dual cone is denoted by $\mathcal{T}(\bar{z})^*$. Verifying these conditions directly is generally nontrivial. In practice, it is desirable to employ linearized cones to reconstruct the local optimality (4); constraint qualifications (CQs) play an important role in this task.

Standard linearization of $\mathcal{T}(\bar{z})$ can be carried out, by replacing the complementarity constraints $0 \leq G(z) \perp H(z) \geq 0$ with (see [4, Eqs. (10)-(11)])

$$G(z) \geq 0, \quad H(z) \geq 0, \quad G(z)^T H(z) = 0.$$

Then linearization of the MPCC constraints gives

$$\begin{aligned} \nabla g_i(\bar{z})^T d \leq 0, &\quad i = 1, \dots, n_g, \\ \nabla h_i(\bar{z})^T d = 0, &\quad i = 1, \dots, n_h, \\ \nabla G_i(\bar{z})^T d \geq 0, &\quad i = 1, \dots, m, \\ \nabla H_i(\bar{z})^T d \geq 0, &\quad i = 1, \dots, m, \\ H_i(\bar{z}) \nabla G_i(\bar{z})^T d + G_i(\bar{z}) \nabla H_i(\bar{z})^T d = 0, &\quad i = 1, \dots, m. \end{aligned}$$

Using the definition (2) of the index sets, we obtain the linearized tangent cone

$$\begin{aligned}
\mathcal{T}^{lin}(\bar{z}) = \{d\,|\nabla g_i(\bar{z})^T d \le 0, \quad & \forall i \in I_g(\bar{z}),\\
\nabla h_i(\bar{z})^T d = 0, \quad & \forall i = 1,\ldots,n_h,\\
\nabla G_i(\bar{z})^T d = 0, \quad & \forall i \in \alpha(\bar{z}),\\
\nabla H_i(\bar{z})^T d = 0, \quad & \forall i \in \gamma(\bar{z}),\\
\nabla G_i(\bar{z})^T d \ge 0,\ \nabla H_i(\bar{z})^T d \ge 0, \quad & \forall i \in \beta(\bar{z})\}.
\end{aligned}$$

Its dual cone is given by

$$\begin{aligned}
\mathcal{T}^{lin}(\bar{z})^* &= \{w\,|\,w^T d \ge 0, \quad \forall d \in \mathcal{T}^{lin}(\bar{z})\}\\
&= \{w\,|\,0 = w + \sum_{i\in I_g(\bar{z})} \bar{\lambda}_i^g \nabla g_i(\bar{z}) + \sum_{i=1}^{n_h} \bar{\lambda}_i^h \nabla h_i(\bar{z}) - \sum_{i\in\alpha(\bar{z})} \bar{\lambda}_i^G \nabla G_i(\bar{z}) - \sum_{i\in\gamma(\bar{z})} \bar{\lambda}_i^H \nabla H_i(\bar{z})\\
&\qquad - \sum_{i\in\beta(\bar{z})} \bar{\lambda}_i^G \nabla G_i(\bar{z}) - \sum_{i\in\beta(\bar{z})} \bar{\lambda}_i^H \nabla H_i(\bar{z});\\
&\qquad \bar{\lambda}_i^g \ge 0,\ \forall i \in I_g(\bar{z}); \quad \bar{\lambda}_i^G \ge 0, \bar{\lambda}_i^H \ge 0,\ \forall i \in \beta(\bar{z})\}.
\end{aligned}$$

By assuming $\mathcal{T}(\bar{z}) = \mathcal{T}^{lin}(\bar{z})$ or $\mathcal{T}(\bar{z})^* = \mathcal{T}^{lin}(\bar{z})^*$, i.e., assuming NLP-ACQ or NLP-GCQ at $\bar{z}$, respectively, the equivalence (4) can be rebuilt on the linearized cone. This converts the local optimality of MPCC (1) into the local optimality of the relaxed NLP

$$\begin{aligned}
\text{RNLP}: \quad \min \quad & f(z)\\
\text{s.t.} \quad & g(z) \le 0,\\
& h(z) = 0,\\
& G_i(z) = 0, \qquad i \in \alpha(\bar{z}),\\
& H_i(z) = 0, \qquad i \in \gamma(\bar{z}),\\
& G_i(z) \ge 0,\ H_i(z) \ge 0, \quad i \in \beta(\bar{z}).
\end{aligned} \tag{5}$$

Therefore, the assumptions NLP-ACQ and NLP-GCQ justify the KKT conditions for RNLP, i.e., the S-stationarity conditions, serving as the necessary conditions for $\bar{z}$ to be a local minimizer of the MPCC (see also [5, Theorem 4.1]).

Since NLP-CQs are usually too strong for MPCCs, several constraint qualifications have been proposed that are customized for complementarity constraints. In particular, MPCC-ACQ and MPCC-GCQ are apparently helpful in reconstructing the equivalence (4) with a linearized tangent cone. To be specific, MPCC-ACQ assumes $\mathcal{T}(\bar{z}) = \mathcal{T}^{lin}_{\text{MPCC}}(\bar{z})$, where the latter is the MPCC-linearized tangent cone at $\bar{z}$ and is defined by [4]

$$\begin{aligned}
\mathcal{T}^{lin}_{\text{MPCC}}(\bar{z}) = \{d\,|\nabla g_i(\bar{z})^T d \le 0, \quad & \forall i \in I_g(\bar{z}),\\
\nabla h_i(\bar{z})^T d = 0, \quad & \forall i = 1,\ldots,n_h,\\
\nabla G_i(\bar{z})^T d = 0, \quad & \forall i \in \alpha(\bar{z}),\\
\nabla H_i(\bar{z})^T d = 0, \quad & \forall i \in \gamma(\bar{z}),\\
\nabla G_i(\bar{z})^T d \ge 0, \quad & \forall i \in \beta(\bar{z}),\\
\nabla H_i(\bar{z})^T d \ge 0, \quad & \forall i \in \beta(\bar{z}),\\
(\nabla G_i(\bar{z})^T d) \cdot (\nabla H_i(\bar{z})^T d) = 0, \quad & \forall i \in \beta(\bar{z})\}.
\end{aligned}$$

Then the left side of (4) can be expressed as:

$$\nabla f(\bar{z})^T d \geq 0, \quad \forall d \in \mathcal{T}^{lin}_{\text{MPCC}}(\bar{z}), \tag{6}$$

and the equivalent right side becomes

$$\nabla f(\bar{z}) \in \mathcal{T}^{lin}_{\text{MPCC}}(\bar{z})^*. \tag{7}$$

On the other hand, MPCC-GCQ assumes $\mathcal{T}(\bar{z})^* = \mathcal{T}^{lin}_{\text{MPCC}}(\bar{z})^*$ [6], where the latter is described by

$$\mathcal{T}^{lin}_{\text{MPCC}}(\bar{z})^* = \{w \,|w^T d \geq 0, \; \forall d \in \mathcal{T}^{lin}_{\text{MPCC}}(\bar{z})\}.$$

(Further specification of this dual cone based on a *calmness* assumption can be found in [20, Section 2.3.2] and [6, Section 3].) Then the right side of (4) can be expressed by (7), and the equivalent left side becomes (6). Both reconstructions of the equivalence are implemented by simplifying the geometry of the problem while preserving the complementarity structure.

Note that the qualification by MPCC-GCQ is weaker and insures that the results built on this qualification are more broadly valid. In particular, MPCC-GCQ is implied by MPCC-ACQ, but the converse is in general not true. Their relations are analogous to the relations between NLP-GCQ and NLP-ACQ. Examples showing that NLP-GCQ and MPCC-GCQ have a better chance to be satisfied, even if NLP-ACQ and MPCC-ACQ do not hold, can be found in [20, Example 1.3] and [6, Example 2.1], respectively. Intuitively, the property that a dual cone, such as $\mathcal{T}^{lin}(\bar{z})^*$ or $\mathcal{T}^{lin}_{\text{MPCC}}(\bar{z})^*$, is always convex, even if the primal cone, such as $\mathcal{T}^{lin}(\bar{z})$ or $\mathcal{T}^{lin}_{\text{MPCC}}(\bar{z})$, is nonconvex, offers the opportunity for NLP-GCQ and MPCC-GCQ to hold more generally. Note that despite the fact that a primal cone is not necessarily equal to the closure of its convex hull, their dual cones are the same.

Flegel and Kanzow have established that under MPCC-GCQ, M-stationarity is a necessary first-order condition [6, Theorem 3.1]. In Section 2, we derive a property of "piecewise M-stationarity" at a local minimizer of MPCC (1), under MPCC-GCQ.

1.2 Degeneracy

To seek for a local minimizer of MPCC (1), many NLP-based schemes have been proposed. The original intention is to avoid dealing with the complementarity structure explicitly. In general, these schemes are designed to solve a sequence of regularized NLPs, yielding a sequence of stationary points $\{z^k\}$ which, hopefully, approximate solutions of MPCC (1). An important ingredient is to characterize conditions under which, as the regularization factor vanishes, a limit point of $\{z^k\}$ is a stationary point of the MPCC in one sense or another. For some representative work see [8, 12, 13, 15, 16, 19, 21], and comparison of some of the methods can be found in [11].

A difficulty in establishing stationarity of a limit point arises as the point is degenerate (on the lower level), namely, a sequence $\{z^k\} \to \bar{z}$ at which $\beta(\bar{z}) \neq \emptyset$. Fukushima and Pang studied the behavior of a sequence $\{z^k\}$ which is composed of KKT points of NLPs formulated by smoothing the MPCC with perturbed Fischer-Burmeister functions. The condition of *asymptotic weak nondegeneracy* is proposed, meaning that for every $i \in \beta(\bar{z})$, $G_i(z^k)$ and $H_i(z^k)$ approach zero in the same order of magnitude. Under this condition and the second-order necessary conditions at $\{z^k\}$ for the perturbed NLPs, together with MPCC-LICQ at $\bar{z}$, it has been proved that $\bar{z}$ is a B-stationary point of the MPCC [8, Theorem 3.1]. However, the condition of asymptotic weak nondegeneracy is hard to enforce in practice. Replacing this condition with upper level strict complementarity (ULSC), Scholtes recovered B-stationary of a limit point of a regularization scheme [19, Corollary 3.4]. Kadrani et al developed a regularization method whose limit points were shown to be M-stationary under MPCC-LICQ, without requiring asymptotic weak nondegeneracy or second-order conditions on $\{z^k\}$ (see [12]). The result was later proved valid under weaker assumption MPCC-CPLD (see [11]). Results under weaker assumptions also include, for example, that C-stationarity convergence of the method by Steffensen and Ulbrich under MPCC-CRCQ [21] and MPCC-CPLD [10], and M-stationarity convergence of the method by Kanzow and Schwartz under MPCC-CPLD [13].

In Section 3, we characterize conditions that guarantee a feasible point of MPCC (1) to be B-stationary, under MPCC-ACQ and MPCC-GCQ, respectively. In Section 4, we analyze convergence properties of the NCP-based methods we proposed in [22], without specifying a particular MPCC-CQ. In Section 5, we discuss some practical issues when approaching a solution of MPCC (1) which is not S-stationary.

2 M-stationarity of local minimizers of MPCCs

2.1 MPCC-GCQ and piecewise NLP-GCQ

Let $\bar{z}$ be a local minimizer of MPCC (1). Denote the set of partitions of $\beta(\bar{z})$ as $\mathcal{P}(\beta(\bar{z})) = \{(\beta_1, \beta_2) \,|\, \beta_1 \cap \beta_2 = \emptyset,\ \beta_1 \cup \beta_2 = \beta(\bar{z})\}$. A NLP problem defined on every partition $(\beta_1, \beta_2) \in \mathcal{P}(\beta(\bar{z}))$ is

$$\begin{array}{llll} \mathrm{NLP}_{(\beta_1,\beta_2)}: & \min & f(z) & \\ & \text{s.t.} & g(z) \le 0, & \\ & & h(z) = 0, & \\ & & G_i(z) = 0, & i \in \alpha(\bar{z}), \\ & & H_i(z) = 0, & i \in \gamma(\bar{z}), \\ & & G_i(z) = 0,\ H_i(z) \ge 0, & i \in \beta_1, \\ & & G_i(z) \ge 0,\ H_i(z) = 0, & i \in \beta_2. \end{array} \tag{8}$$

Lemma 2.1. *Let $\bar{z}$ be a local minimizer of MPCC (1). If MPCC-GCQ holds at $\bar{z}$, then for every $(\beta_1, \beta_2) \in \mathcal{P}(\beta(\bar{z}))$, NLP-GCQ holds at $\bar{z}$ for NLP$_{(\beta_1,\beta_2)}$. The converse is also true.*

Proof. Since $\bar{z}$ is a local minimizer of MPCC (1), we have from B-stationarity of $\bar{z}$ that

$$\nabla f(\bar{z}) \in \mathcal{T}(\bar{z})^*. \tag{9}$$

Given MPCC-GCQ at $\bar{z}$, we have

$$\mathcal{T}(\bar{z})^* = \mathcal{T}^{lin}_{\mathrm{MPCC}}(\bar{z})^* = \bigcap_{(\beta_1,\beta_2)\in\beta(\bar{z})} \mathcal{T}^{lin}_{(\beta_1,\beta_2)}(\bar{z})^*, \tag{10}$$

where $T^{lin}_{(\beta_1,\beta_2)}(\bar{z})$ is the linearized tangent of $\text{NLP}_{(\beta_1,\beta_2)}$ at $\bar{z}$ and is given by

$$\begin{aligned}
\mathcal{T}^{lin}_{(\beta_1,\beta_2)}(\bar{z}) = \{d \,|\, &\nabla g_i(\bar{z})^T d \le 0, && \forall i \in I_g(\bar{z}),\\
&\nabla h_i(\bar{z})^T d = 0, && \forall i = 1,\dots,n_h,\\
&\nabla G_i(\bar{z})^T d = 0, && \forall i \in \alpha(\bar{z}),\\
&\nabla H_i(\bar{z})^T d = 0, && \forall i \in \gamma(\bar{z}),\\
&\nabla G_i(\bar{z})^T d = 0,\ \nabla H_i(\bar{z})^T d \ge 0, && \forall i \in \beta_1,\\
&\nabla G_i(\bar{z})^T d \ge 0,\ \nabla H_i(\bar{z})^T d = 0, && \forall i \in \beta_2\}.
\end{aligned}$$

Then (9) and (10) together imply

$$\nabla f(\bar{z}) \in \mathcal{T}^{lin}_{(\beta_1,\beta_2)}(\bar{z})^*, \quad \forall (\beta_1,\beta_2) \in \mathcal{P}(\beta(\bar{z})). \tag{11}$$

On the other hand, $\bar{z}$ is also a local minimizer of $\text{NLP}_{(\beta_1,\beta_2)}$, for every $(\beta_1,\beta_2) \in \mathcal{P}(\beta(\bar{z}))$. Hence, we have [9, Lemma 4.3]

$$\nabla f(\bar{z}) \in \mathcal{T}_{(\beta_1,\beta_2)}(\bar{z})^*, \quad \forall (\beta_1,\beta_2) \in \mathcal{P}(\beta(\bar{z})), \tag{12}$$

where $\mathcal{T}_{(\beta_1,\beta_2)}(\bar{z})^*$ is the dual cone of the tangent cone of $\text{NLP}_{(\beta_1,\beta_2)}$ at $\bar{z}$. Combining (11) and (12) yields

$$\mathcal{T}_{(\beta_1,\beta_2)}(\bar{z})^* = \mathcal{T}^{lin}_{(\beta_1,\beta_2)}(\bar{z})^*, \quad \forall (\beta_1,\beta_2) \in \mathcal{P}(\beta(\bar{z})), \tag{13}$$

indicating that NLP-GCQ holds at $\bar{z}$ for every $\text{NLP}_{(\beta_1,\beta_2)}$ with $(\beta_1,\beta_2) \in \mathcal{P}(\beta(\bar{z}))$.

Conversely, suppose (13) holds. Then

$$\mathcal{T}(\bar{z})^* = \bigcap_{(\beta_1,\beta_2)\in\beta(\bar{z})} \mathcal{T}_{(\beta_1,\beta_2)}(\bar{z})^* = \bigcap_{(\beta_1,\beta_2)\in\beta(\bar{z})} \mathcal{T}^{lin}_{(\beta_1,\beta_2)}(\bar{z})^* = \mathcal{T}^{lin}_{\text{MPCC}}(\bar{z})^*,$$

indicating that MPCC-GCQ holds at $\bar{z}$. (Note that this part does not need $\bar{z}$ to be locally optimal.) This completes the proof. □

2.2 M-stationarity of local minimizers

Theorem 2.2. *Let $\bar{z}$ be a local minimizer of MPCC (1) at which MPCC-GCQ holds. Then for every $(\beta_1,\beta_2) \in \mathcal{P}(\beta(\bar{z}))$, there exist $\text{NLP}_{(\beta_1,\beta_2)}$ suitable multipliers at $\bar{z}$, that satisfy M-stationarity.*

Proof. Lemma 2.1 has shown that for every $(\beta_1,\beta_2) \in \mathcal{P}(\beta(\bar{z}))$, NLP-GCQ holds at $\bar{z}$ for $\text{NLP}_{(\beta_1,\beta_2)}$. This guarantees that KKT conditions are valid at the local minimizer $\bar{z}$ for each of these NLPs. Namely, for every $(\beta_1,\beta_2) \in \mathcal{P}(\beta(\bar{z}))$, the following system has solutions:

$$\begin{aligned}
0 = &\nabla f(\bar{z}) + \nabla g_I(\bar{z})\bar{\lambda}^g_I + \nabla h(\bar{z})\bar{\lambda}^h - \nabla G_\alpha(\bar{z})\bar{\lambda}^G_\alpha - \nabla H_\gamma(\bar{z})\bar{\lambda}^H_\gamma\\
&- \nabla G_{\beta_1}(\bar{z})\bar{\lambda}^G_{\beta_1} - \nabla H_{\beta_1}(\bar{z})\bar{\lambda}^H_{\beta_1} - \nabla G_{\beta_2}(\bar{z})\bar{\lambda}^G_{\beta_2} - \nabla H_{\beta_2}(\bar{z})\bar{\lambda}^H_{\beta_2},
\end{aligned} \tag{14a}$$

$$\bar{\lambda}^g_I \ge 0, \quad \bar{\lambda}^H_{\beta_1} \ge 0, \quad \bar{\lambda}^G_{\beta_2} \ge 0, \tag{14b}$$

where g_I denotes the constraints $\{g_i \,|\, \forall i \in I_g(\bar{z})\}$, and, similarly, $G_\alpha, H_\gamma, G_\beta$, and H_β denote the constraints related to the index sets $\alpha(\bar{z}), \gamma(\bar{z})$, and $\beta(\bar{z})$. Assume that there exists a partition $(\hat{\beta}_1, \hat{\beta}_2) \in \mathcal{P}(\beta(\bar{z}))$, such that the KKT multipliers at $\bar{z}$ for $\text{NLP}_{(\hat{\beta}_1,\hat{\beta}_2)}$ cannot satisfy M-stationarity. Then, with the partition $(\hat{\beta}_1, \hat{\beta}_2)$, all the solutions of (14) imply that $\bar{\lambda}^G_j \leq 0$ or $\bar{\lambda}^H_j \leq 0$, and at the same time $\bar{\lambda}^G_j \bar{\lambda}^H_j \neq 0$, for some indices $j \in \beta(\bar{z})$. The following derives contradictions.

One case is that $\bar{\lambda}^G_j, \bar{\lambda}^H_j < 0$ for some indices $j \in \beta(\bar{z})$. This case contradicts (14b), if either $j \in \hat{\beta}_1$ or $j \in \hat{\beta}_2$, indicating that $\bar{z}$ is not a local minimizer of $\text{NLP}_{(\hat{\beta}_1,\hat{\beta}_2)}$. The remaining case is that $\bar{\lambda}^G_j \bar{\lambda}^H_j < 0$ for some indices $j \in \beta(\bar{z})$. Without loss of generality, assume $\bar{\lambda}^G_j < 0, \bar{\lambda}^H_j > 0$, and $j \in \hat{\beta}_1$. Now consider the partition $(\hat{\beta}_1 \backslash \{j\}, \hat{\beta}_2 \cup \{j\}) \in \mathcal{P}(\beta(\bar{z}))$. For an arbitrary $d \in \mathcal{T}^{lin}_{(\hat{\beta}_1\backslash\{j\},\hat{\beta}_2\cup\{j\})}(\bar{z})$, we now have $\nabla G_j(\bar{z})^T d \geq 0$, and it follows from (14a) and $\bar{\lambda}^G_j < 0$ that

$$
\begin{aligned}
&\nabla f(\bar{z})^T d \\
=&- (\nabla g_I(\bar{z})\bar{\lambda}^g_I)^T d - (\nabla h(\bar{z})\bar{\lambda}^h)^T d + (\nabla G_\alpha(\bar{z})\bar{\lambda}^G_\alpha)^T d + (\nabla H_\gamma(\bar{z})\bar{\lambda}^H_\gamma)^T d \\
&+ (\nabla G_{\hat{\beta}_1\backslash\{j\}}(\bar{z})\bar{\lambda}^G_{\hat{\beta}_1\backslash\{j\}})^T d + (\nabla H_{\hat{\beta}_1\backslash\{j\}}(\bar{z})\bar{\lambda}^H_{\hat{\beta}_1\backslash\{j\}})^T d \\
&+ (\nabla G_{\hat{\beta}_2}(\bar{z})\bar{\lambda}^G_{\hat{\beta}_2})^T d + (\nabla H_{\hat{\beta}_2}(\bar{z})\bar{\lambda}^H_{\hat{\beta}_2})^T d \\
&+ (\nabla G_j(\bar{z})\bar{\lambda}^G_j)^T d + (\nabla H_j(\bar{z})\bar{\lambda}^H_j)^T d \\
=&\underbrace{-(\nabla g_I(\bar{z})\bar{\lambda}^g_I)^T d}_{\geq 0} + \underbrace{(\nabla H_{\hat{\beta}_1\backslash\{j\}}(\bar{z})\bar{\lambda}^H_{\hat{\beta}_1\backslash\{j\}})^T d}_{\geq 0} + \underbrace{(\nabla G_{\hat{\beta}_2}(\bar{z})\bar{\lambda}^G_{\hat{\beta}_2})^T d}_{\geq 0} + \underbrace{(\nabla G_j(\bar{z})\bar{\lambda}^G_j)^T d}_{\leq 0},
\end{aligned} \tag{15}
$$

which may not guarantee the nonnegativity of $\nabla f(\bar{z})^T d$, nor the optimality of $\bar{z}$ as a local minimizer of $\text{NLP}_{(\hat{\beta}_1\backslash\{j\},\hat{\beta}_2\cup\{j\})}$.

Therefore, the assumption is false. In essence, because $\bar{z}$ is a KKT point of $\text{NLP}_{(\beta_1,\beta_2)}$ for every partition $(\beta_1, \beta_2) \in \mathcal{P}(\beta(\bar{z}))$, "piecewise M-stationarity" is guaranteed. □

2.3 Example: GCQ failure

The following example illustrates that when MPCC-GCQ fails, a local minimizer of an MPCC is not necessarily M-stationary.

Consider the problem

$$
\begin{array}{lll}
\min & f(z) = (z_1 - 1)^2 + (z_2 + 1)^2 & \text{multipliers} \\
\text{s.t.} & z_2^2 \leq 0, & \lambda \\
& 0 \geq z_1 \perp z_2 \leq 0. & \sigma_1, \sigma_2
\end{array}
$$

It searches for the minimal distance between points (z_1, z_2) and $(1, -1)$, along the negative axis of z_1. The solution is $\bar{z} = (\bar{z}_1, \bar{z}_2) = (0, 0)$. The weak stationarity conditions at $\bar{z}$ are

$$
\begin{aligned}
0 &= \begin{bmatrix} 2(\bar{z}_1 - 1) \\ 2(\bar{z}_2 + 1) \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 2\bar{z}_2 \end{bmatrix} + \sigma_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sigma_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -2 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \sigma_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sigma_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\end{aligned} \tag{16}
$$

which give that

$$\sigma_1 = 2, \quad \sigma_2 = -2,$$

so that $\bar{z}$ is A- but not M- stationary.

In this example the biactive set $\beta(\bar{z}) = \{1\}$. For the partition $\hat{\beta}_1 = \{1\}$ and $\hat{\beta}_2 = \emptyset$, the corresponding $\mathrm{NLP}_{(\hat{\beta}_1,\hat{\beta}_2)}$ is given by

$$\begin{array}{lll} \min & f(z) = (z_1-1)^2 + (z_2+1)^2 & \text{multipliers} \\ \text{s.t.} & z_2^2 \le 0, & \lambda \\ & z_1 = 0,\ z_2 \le 0. & \sigma_1, \sigma_2 \end{array}$$

The KKT conditions at $\bar{z}$ for this NLP are the same as (16); however, $\sigma_2 = -2 < 0$ is not a suitable multiplier for the constraint $z_2 \le 0$. Hence, for $\mathrm{NLP}_{(\hat{\beta}_1,\hat{\beta}_2)}$, the local minimizer $\bar{z}$ is not a KKT point, indicating that NLP-GCQ fails at $\bar{z}$. It follows from Lemma 2.1 that MPCC-GCQ fails at $\bar{z}$ as well. In fact, we have the following cones:

$$\begin{aligned} \mathcal{T}(\bar{z}) &= \{d \in \mathbb{R}^2 \mid d_1 \le 0, d_2 = 0\}, \\ \mathcal{T}(\bar{z})^* &= \{w \in \mathbb{R}^2 \mid w^T d \ge 0, \forall d \in \mathcal{T}(\bar{z})\} = \{w \in \mathbb{R}^2 \mid w_1 \le 0, w_2 = R\}, \\ \mathcal{T}^{lin}_{\text{MPCC}}(\bar{z}) &= \left\{ d \in \mathbb{R}^2 \mid \begin{bmatrix} 0 \\ 2\bar{z}_2 \end{bmatrix}^T d \le 0,\ \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T d \le 0,\ \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T d \le 0,\ \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T d \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T d = 0 \right\} \\ &= \{d \in \mathbb{R}^2 \mid 0 \ge d_1 \perp d_2 \le 0\}, \\ \mathcal{T}^{lin}_{\text{MPCC}}(\bar{z})^* &= \{w \in \mathbb{R}^2 \mid w^T d \ge 0, \forall d \in \mathcal{T}^{lin}_{\text{MPCC}}(\bar{z})\} \\ &= \{w \in \mathbb{R}^2 \mid w_1 \le 0 \text{ and } w_2 \le 0\}, \end{aligned}$$

which validate that $\mathcal{T}(\bar{z})^* \neq \mathcal{T}^{lin}_{\text{MPCC}}(\bar{z})^*$, and MPCC-GCQ fails.

3 B-stationarity conditions

3.1 Conditions under MPCC-ACQ

Suppose that MPCC-ACQ holds at a feasible point $\bar{z}$ of MPCC (1). According to the condition (6), $\bar{z}$ is a local minimizer of the MPCC if and only if $d = 0$ solves the following linear program with equilibrium constraints (LPEC):

$$\begin{array}{ll} \min & \nabla f(\bar{z})^T d \\ \text{s.t.} & \nabla g_I(\bar{z})^T d \le 0, \\ & \nabla h(\bar{z})^T d = 0, \\ & \nabla G_\alpha(\bar{z})^T d = 0, \\ & \nabla H_\gamma(\bar{z})^T d = 0, \\ & 0 \le \nabla G_\beta(\bar{z})^T d \perp \nabla H_\beta(\bar{z})^T d \ge 0. \end{array} \tag{17}$$

The LPEC is a combination of classic linear programs each defined on a partition $(\beta_1, \beta_2) \in \mathcal{P}(\beta(\bar{z}))$ as follows:

$$
\begin{aligned}
\mathrm{LP}_{(\beta_1,\beta_2)}: \quad \min \quad & obj(d) = \nabla f(\bar{z})^T d \\
\text{s.t.} \quad & \nabla g_I(\bar{z})^T d \leq 0, \\
& \nabla h(\bar{z})^T d = 0, \\
& \nabla G_\alpha(\bar{z})^T d = 0, \\
& \nabla H_\gamma(\bar{z})^T d = 0, \\
& \nabla G_{\beta_1}(\bar{z})^T d = 0, \quad \nabla H_{\beta_1}(\bar{z})^T d \geq 0, \\
& \nabla G_{\beta_2}(\bar{z})^T d \geq 0, \quad \nabla H_{\beta_2}(\bar{z})^T d = 0.
\end{aligned}
\tag{18}
$$

The dual problem of (18) is given by

$$
\begin{aligned}
\mathrm{LP}^{dual}_{(\beta_1,\beta_2)}: \quad \max \quad & obj^{dual}(\eta) = \eta^T \cdot 0 && \text{(19a)} \\
\text{s.t.} \quad & \eta^g_I \geq 0, && \text{(19b)} \\
& \eta^h \ free, && \text{(19c)} \\
& \eta^G_\alpha \ free, && \text{(19d)} \\
& \eta^H_\gamma \ free, && \text{(19e)} \\
& \eta^G_{\beta_1} \ free, \quad \eta^H_{\beta_1} \geq 0, \\
& \eta^G_{\beta_2} \geq 0, \quad \eta^H_{\beta_2} \ free, && \text{(19f)} \\
& 0 = \nabla f(\bar{z}) + \eta^g_I \nabla g_I(\bar{z}) + \eta^h \nabla h(\bar{z}) - \eta^G_\alpha \nabla G_\alpha(\bar{z}) - \eta^H_\gamma \nabla H_\gamma(\bar{z}) \\
& \quad - \eta^G_{\beta_1} \nabla G_{\beta_1}(\bar{z}) - \eta^H_{\beta_1} \nabla H_{\beta_1}(\bar{z}) - \eta^G_{\beta_2} \nabla G_{\beta_2}(\bar{z}) - \eta^H_{\beta_2} \nabla H_{\beta_2}(\bar{z}). && \text{(19g)}
\end{aligned}
$$

The dual variables $\eta = (\eta^g_I, \eta^h, \eta^G_\alpha, \eta^H_\gamma, \eta^G_\beta, \eta^H_\beta)$ essentially play the role of the MPCC multipliers at $\bar{z}$. Duality theory characterizes the relations between the primal and the dual problems as follows.

(D1) If d is a feasible point of the primal problem (18) and η is a feasible point of the dual problem (19), then $obj^{dual}(\eta) \leq obj(d)$. [1, Theorem 4.3]

(D2) If the dual problem is infeasible, then either the primal problem is infeasible, or the optimal cost of the primal problem is $-\infty$. If the primal problem is infeasible, then either the dual problem is infeasible, or the optimal cost of the dual problem is ∞. [1, Corollary 4.1] [1, Table 4.2]

(D3) Let d and η be feasible points of the primal (18) and the dual (19), respectively, and suppose that $obj^{dual}(\eta) = obj(d)$. Then d and η are optimal solutions to the primal and the dual, respectively. [1, Corollary 4.2]

Theorem 3.1. *Suppose that MPCC (1) is solvable (namely, feasible and bounded below). If a point $\bar{z}$ is weakly stationary and MPCC-ACQ holds at $\bar{z}$, then $\bar{z}$ is B-stationary.*

Proof. Since $\bar{z}$ satisfies weak stationarity (3), it is a local minimizer of the tightened NLP

$$\begin{aligned} \text{TNLP}: \quad \min \quad & f(z) \\ \text{s.t.} \quad & g(z) \le 0, \\ & h(z) = 0, \\ & G_i(z) = 0, && i \in \alpha(\bar{z}), \\ & H_i(z) = 0, && i \in \gamma(\bar{z}), \\ & G_i(z) = 0,\ H_i(z) = 0, && i \in \beta(\bar{z}). \end{aligned} \tag{20}$$

It follows from Tucker's theorem of alternative (see also [17]) that the system

$$\begin{aligned} \nabla f(\bar{z})^T d &< 0, \\ \nabla g(\bar{z})^T d &\le 0, \\ \nabla h(\bar{z})^T d &= 0, \\ \nabla G_{\alpha\cup\beta}(\bar{z})^T d &= 0, \\ \nabla H_{\gamma\cup\beta}(\bar{z})^T d &= 0, \end{aligned}$$

has no solution. Hence, $d = 0$ solves the linear program

$$\begin{aligned} \min \quad & \nabla f(\bar{z})^T d \\ \text{s.t.} \quad & \nabla g(\bar{z})^T d \le 0, \\ & \nabla h(\bar{z})^T d = 0, \\ & \nabla G_{\alpha\cup\beta}(\bar{z})^T d = 0, \\ & \nabla H_{\gamma\cup\beta}(\bar{z})^T d = 0. \end{aligned} \tag{21}$$

Recall that under MPCC-ACQ, $\bar{z}$ is B-stationary if and only if $d = 0$ solves LPEC (17). Consider the linear programs (18) that comprise the LPEC. We know from (21) that for every $(\beta_1, \beta_2) \in \mathcal{P}(\beta(\bar{z}))$, the primal problem $\text{LP}_{(\beta_1,\beta_2)}$ has a feasible solution $d = 0$. In fact, every dual problem $\text{LP}^{dual}_{(\beta_1,\beta_2)}$ also has feasible solutions. Assume, for the purpose of deriving a contradiction, that there exists a partition $(\hat{\beta}_1, \hat{\beta}_2) \in \mathcal{P}(\beta(\bar{z}))$, such that the dual problem $\text{LP}^{dual}_{(\hat{\beta}_1,\hat{\beta}_2)}$ is infeasible. Then it follows from the result (D2) of duality theory that the primal problem $\text{LP}_{(\hat{\beta}_1,\hat{\beta}_2)}$ is either infeasible or unbounded below. Since $d = 0$ is feasible to the primal problem, it follows that the primal problem is unbounded below, namely,

$$\nabla f(\bar{z})^T d \to -\infty, \quad \exists d \in \mathcal{T}^{lin}_{(\hat{\beta}_1,\hat{\beta}_2)}(\bar{z}). \tag{22}$$

Since MPCC-ACQ holds at $\bar{z}$, we have

$$\mathcal{T}^{lin}_{(\hat{\beta}_1,\hat{\beta}_2)}(\bar{z}) \subseteq \mathcal{T}^{lin}_{\text{MPCC}}(\bar{z}) = \mathcal{T}(\bar{z}).$$

Hence,

$$\nabla f(\bar{z})^T d \to -\infty, \quad \exists d \in \mathcal{T}(\bar{z}), \tag{23}$$

namely, the MPCC is unbounded, which brings the desired contradiction.

Moreover, since the feasible solution $d = 0$ to the primal and any feasible solution η to the dual yield $obj(d) = obj^{dual}(\eta) = 0$, we have from the result (D3) of duality theory that $d = 0$ is an optimal solution to the primal problem $\text{LP}_{(\beta_1,\beta_2)}$, for every $(\beta_1, \beta_2) \in \mathcal{P}(\beta(\bar{z}))$. As a consequence, $d = 0$ solves the LPEC (17) and $\bar{z}$ is B-stationary. □

Example: *scholtes4*

This example illustrates that a weakly stationary point is also B-stationary, under the assumptions of Theorem 3.1.

The problem *scholtes4* from the MacMPEC collection [14] is given by

$$\begin{array}{lll} \min & z_1 + z_2 - z_3 & \text{multipliers} \\ \text{s.t.} & -4z_1 + z_3 \le 0, & \lambda_1 \\ & -4z_2 + z_3 \le 0, & \lambda_2 \\ & 0 \le z_1 \perp z_2 \ge 0. & \sigma_1, \sigma_2 \end{array}$$

Since the constraints are linear, MPCC-ACQ holds at every feasible point of the problem. Consider a weakly stationary point $\bar{z} = (\bar{z}_1, \bar{z}_2, \bar{z}_3)$ at which $\beta(\bar{z}) \neq \emptyset$, which is the case of interest. This gives that $\bar{z} = (0, 0, 0)$ and $\beta(\bar{z}) = \{1\}$.

To verify B-stationarity of $\bar{z}$, we check whether $\bar{z}$ is a local minimizer of $\text{NLP}_{(\beta_1,\beta_2)}$ for every $(\beta_1, \beta_2) \in \mathcal{P}(\beta(\bar{z}))$. Since $\bar{z}$ is weakly stationary, we have

$$0 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \lambda_1 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix} - \sigma_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \sigma_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

which implies

$$\lambda_1 + \lambda_2 = 1,$$
$$\sigma_1 + \sigma_2 = -2.$$

For the partitions $(\beta_1, \beta_2) = (\{1\}, \emptyset)$ and $(\beta_1, \beta_2) = (\emptyset, \{1\})$, since $(\sigma_1, \sigma_2) = (-2, 0)$ and $(\sigma_1, \sigma_2) = (0, -2)$, respectively, lead to suitable KKT multipliers for the corresponding NLPs, the point $\bar{z}$ is both B- and M- stationary.

3.2 Conditions under MPCC-GCQ

At first sight, requiring $d = 0$ to solve LPEC (17), i.e., requiring condition (6), may seem appropriate to solve MPCC (1) under MPCC-GCQ, in the sense that $\mathcal{T}(\bar{z}) \subseteq \mathcal{T}^{lin}_{\text{mpcc}}(\bar{z})$ always holds. Moreover, Lemma 2.1 shows that at a local minimizer $\bar{z}$ of the MPCC at which MPCC-GCQ holds, we have $\nabla f(\bar{z}) \in \mathcal{T}^{lin}_{(\beta_1,\beta_2)}(\bar{z})^*$ for every $(\beta_1, \beta_2) \in \mathcal{P}(\beta(\bar{z}))$; in other words, $\nabla f(\bar{z})^T d \ge 0$ for all $d \in \mathcal{T}^{lin}_{(\beta_1,\beta_2)}(\bar{z})$, where $(\beta_1, \beta_2) \in \mathcal{P}(\beta(\bar{z}))$. Hence, given MPCC-GCQ at $\bar{z}$, $\bar{z}$ is B-stationary if and only if $d = 0$ solves the LPEC.

On the other hand, with the MPCC-ACQ assumption is replaced by MPCC-GCQ, Theorem 3.1 does not hold any more. Recall that $d = 0$ is feasible for every $\text{LP}_{(\beta_1,\beta_2)}$ where $(\beta_1, \beta_2) \in \mathcal{P}(\beta(\bar{z}))$. To show that $d = 0$ is also optimal for every $\text{LP}_{(\beta_1,\beta_2)}$, every *dual problem* $\text{LP}^{dual}_{(\beta_1,\beta_2)}$ needs to be feasible. This is guaranteed under MPCC-ACQ by assuming that MPCC (1) is bounded below. Instead, an infeasible dual problem $\text{LP}^{dual}_{(\hat{\beta}_1,\hat{\beta}_2)}$ for a partition $(\hat{\beta}_1, \hat{\beta}_2) \in \mathcal{P}(\beta(\bar{z}))$, implies that the corresponding primal problem $\text{LP}_{(\hat{\beta}_1,\hat{\beta}_2)}$ is unbounded, thus contradicting the boundedness of the MPCC. However, we cannot exclude the possibility of an unbounded $\text{LP}_{(\hat{\beta}_1,\hat{\beta}_2)}$ in the same way under MPCC-GCQ. Because (22) does not imply (23) under MPCC-GCQ, the boundedness of the MPCC (1) does not imply the boundedness of $\text{LP}_{(\hat{\beta}_1,\hat{\beta}_2)}$. Thus, there might exist an infeasible $\text{LP}^{dual}_{(\hat{\beta}_1,\hat{\beta}_2)}$, so that $d=0$ is not optimal for $\text{LP}_{(\hat{\beta}_1,\hat{\beta}_2)}$. Therefore, the conditions of Theorem 3.1 are not sufficient to obtain B-stationarity of $\bar{z}$ under MPCC-GCQ.

Note that whenever the optimal cost of every $\text{LP}_{(\beta_1,\beta_2)}$ with $(\beta_1,\beta_2)\in\mathcal{P}(\beta(\bar{z}))$ is finite, $d=0$ solves LPEC (17). The following theorem gives a condition which verifies B-stationarity and guarantees validity of MPCC-GCQ.

Theorem 3.2. *Suppose that MPCC (1) is solvable (namely, feasible and bounded below). Let $\bar{z}$ be weakly stationary. Explore all partitions of LPEC (17); if every $LP_{(\beta_1,\beta_2)}$ with $(\beta_1,\beta_2)\in\mathcal{P}(\beta(\bar{z}))$ is bounded below, then $\bar{z}$ is B-stationary and, additionally, MPCC-GCQ holds at $\bar{z}$.*

Proof. According to the above analysis, the primal problem $\text{LP}_{(\beta_1,\beta_2)}$ with $(\beta_1,\beta_2)\in\mathcal{P}(\beta(\bar{z}))$ being bounded, means that the dual problem $\text{LP}^{dual}_{(\beta_1,\beta_2)}$ is feasible. Since the objective of the dual problem is zero and equal to the objective of the primal problem at its feasible solution $d=0$, it follows from the result (D3) of duality theory that $d=0$ is optimal to $\text{LP}_{(\beta_1,\beta_2)}$. Because this is the case for every partition $(\beta_1,\beta_2)\in\mathcal{P}(\beta(\bar{z}))$, then $d=0$ is optimal to LPEC (17) and $\bar{z}$ is B-stationary.

In addition, the optimality of $d=0$ for every partition indicates that $\nabla f(\bar{z})\in\mathcal{T}^{lin}_{(\beta_1,\beta_2)}(\bar{z})^*$ for every $(\beta_1,\beta_2)\in\mathcal{P}(\beta(\bar{z}))$. Since $\mathcal{T}^{lin}_{\text{MPCC}}(\bar{z})^*=\bigcap_{(\beta_1,\beta_2)\in\beta(\bar{z})}\mathcal{T}^{lin}_{(\beta_1,\beta_2)}(\bar{z})^*$, we have that $\nabla f(\bar{z})\in\mathcal{T}^{lin}_{\text{MPCC}}(\bar{z})^*$. On the other hand, it follows from B-stationarity of $\bar{z}$ that $\nabla f(\bar{z})\in\mathcal{T}(\bar{z})^*$. Therefore, $\mathcal{T}(\bar{z})^*=T^{lin}_{\text{MPCC}}(\bar{z})^*$, namely, MPCC-GCQ holds at $\bar{z}$. □

It is worth noting that whenever a dual problem $\text{LP}^{dual}_{(\beta_1,\beta_2)}$ is feasible, its solution provides KKT multipliers for $\text{NLP}_{(\beta_1,\beta_2)}$. This provides a bridge between optimality of $d=0$ for $\text{LP}_{(\beta_1,\beta_2)}$ and that $\bar{z}$ is a KKT point of $\text{NLP}_{(\beta_1,\beta_2)}$.

4 Convergence of NCP-based methods

4.1 Brief review of a bounding scheme

In [22] we proposed an algorithm to seek a solution of MPCC (1) by solving a sequence of NLP problems of the form

$$
\begin{array}{llll}
\text{BA}(\epsilon): & \min & f(z) & \text{multipliers} \\
 & \text{s.t.} & g(z)\le 0, & u^g \\
 & & h(z)=0, & u^h \\
 & & \Phi_i^\epsilon(z)+p_i=0, \quad i=1\dots m, & u_i^\Phi
\end{array}
\tag{24}
$$

where

$$
\Phi_i^\epsilon(z)=\frac{1}{2}\left(G_i(z)+H_i(z)-\sqrt{(G_i(z)-H_i(z))^2+\epsilon^2}\right) \tag{25}
$$

is a NCP function with a smoothing factor $\epsilon>0$, and the parameter p_i is adjusted adaptively (to take a value of zero or $\epsilon/2$). Define the Lagrangian for the program BA(ϵ) as

$$
\mathcal{L}(z,u)=f(z)+(u^g)^Tg(z)+(u^h)^Th(z)-(u^\Phi)^T(\Phi^\epsilon(z)+p). \tag{26}
$$

As $\epsilon \to 0$, a sequence of KKT points of BA(ϵ) tends to a limit point. Main results of this method are summarized below, and more details can be found in [22].

- *Feasibility:* The NCP function (25) is used to approximate the complementarity constraints in MPCC (1), and the largest difference between them is $\epsilon/2$ (see [22, Proposition 1.7]). When $\epsilon > 0$, then at every feasible point z of BA(ϵ) we have

$$\Phi_i^\epsilon(z)+p_i = 0 \quad \Leftrightarrow \quad G_i(z)+p_i \geq 0,\ H_i(z)+p_i \geq 0,\ (G_i(z)+p_i)(H_i(z)+p_i) = \frac{\epsilon^2}{4}, \tag{27}$$

  whose limit at $\epsilon = 0$ (thus $p_i = 0$) recovers the complementarity $0 \leq G_i(z) \perp H_i(z) \geq 0$.

- *Sensitivity and Bounding:* At a KKT point $z(p)$ of BA(ϵ), the sensitivities $\frac{\mathrm{d}f(z(p))}{\mathrm{d}p_i}$ ($i = 1 \ldots m$) are given by $-u_i^\Phi$, provided that NLP-LICQ and the second-order sufficient conditions (SOSC) hold at $z(p)$. This observation throws some light on the design of the Bounding Algorithm. We take advantage of the sensitivities at $z(p)$ to adjust the parameters p_i, with the aim of improving the objective at the subsequent solution of BA(ϵ), and thus yielding an efficient isolation of a solution to the MPCC. When $\epsilon > 0$ is sufficiently small, every KKT point of BA(ϵ) is an ϵ-approximate solution to the MPCC, which includes an $O(\epsilon^2)$ correction arising from the adjustment of the parameters p_i.

- *Convergence:* The following convergence results have been established for the Bounding Algorithm under MPCC-LICQ.

  (i) Suppose that MPCC-LICQ holds at a feasible point of the MPCC, then in a neighborhood of this point, NLP-LICQ holds at every feasible point of BA(ϵ), whenever $\epsilon > 0$ is sufficiently small.

  (ii) Suppose that a sequence of KKT points of BA(ϵ) tends to a limit point as $\epsilon \to 0$, at which MPCC-LICQ holds, then the limit point is C-stationary.

  (iii) In addition, suppose that the reduced Hessian of the Lagrangian at each of the KKT points of BA(ϵ) is bounded below when $\epsilon > 0$ is sufficiently small, then the limit point is M-stationary.

A natural question is how does the Bounding Algorithm behave in the absence of MPCC-LICQ. In this section, we develop C- and M- stationarity of a limit point of this method without assuming MPCC-LICQ. Further, we explore more convergence features by taking advantage of an inequality variant of BA(ϵ). We note that this variant is a modification of the Lin-Fukushima algorithm [16], which we call MLF.

4.2 Bounding Algorithm

Based on the formulation BA(ϵ), a Bounding Algorithm was proposed in [22] by noting that the sensitivities $\frac{\mathrm{d}f(z(p))}{\mathrm{d}p_i}$ ($i = 1 \ldots m$) are given by $-u_i^\Phi$. These can be exploited to adjust the parameters p_i to improve the objective $f(z(p))$. The main idea of the Bounding Algorithm is given below to facilitate the later analysis; more details of the algorithm can be found in [22].

Using $\epsilon > 0$ and arbitrary parameters $p_i \in [0, \epsilon/2]$ $(i = 1, \ldots, m)$ leads to a solution $z(p)$ of BA(ϵ). Note that some parameters $p_i^* \in [0, \epsilon/2]$ $(i = 1, \ldots, m)$ may correspond to a solution $z(p^*)$ of BA(ϵ), which coincides with a feasible point of MPCC (1). It is straightforward to show that

$$f(z(p^*)) = f(z(p)) + \frac{\mathrm{d}f(z(p))}{\mathrm{d}p}(p^* - p) + O(\|p^* - p\|^2).$$

Noting that the sensitivities $\frac{\mathrm{d}f(z(p))}{\mathrm{d}p}$ are given by $-u^\Phi$ from BA(ϵ), we have that

$$f(z(p)) - \frac{\epsilon}{2}\sum_{i=1}^{m}|u_i^\Phi(p)| - |O(\epsilon^2)| \le f(z^*) \le f(z(p)) + \frac{\epsilon}{2}\sum_{i=1}^{m}|u_i^\Phi(p)| + |O(\epsilon^2)|.$$

where z^* is the solution of MPCC (1). This relation interprets the approximation to a solution of the MPCC by the Bounding Algorithm.

- *Initialization:* Specify initial smoothing factor $\epsilon^0 > 0$, reducing factor $\kappa \in (0, 1)$, initial point z^0, solution tolerance $\epsilon_{\text{tol}} > 0$. Set initial parameters $p^0 \leftarrow 0$, counter $k \leftarrow 0$.

- *Main loop:* While $\epsilon^k \ge \epsilon_{\text{tol}}$, do the following.

  Step 1. Solve the program BA(ϵ^k) with parameters p^k, to obtain a stationary point z^k and multipliers $u^k = (u^{g,k}, u^{h,k}, u^{\Phi,k})$.

  Step 2. Approximate the upper bound of the MPCC with

  $$f^{up} = f(z^k) + \epsilon^k \sum_{i=1}^{m} |u_i^{\Phi,k}|.$$

  Step 3. Approximate the lower bound of the MPCC as follows. Define the index sets

  $$P_0 = \{i \,|\, p_i^k = 0 \text{ and } u_i^{\Phi,k} > 0\},$$
  $$P_\epsilon = \{i \,|\, p_i^k = \epsilon^k/2 \text{ and } u_i^{\Phi,k} < 0\}.$$

  Then the following settings would reduce $f(z^k)$.

  $$p_i^k \leftarrow \epsilon^k/2, \quad \forall i \in P_0,$$
  $$p_i^k \leftarrow 0, \qquad \forall i \in P_\epsilon.$$

  Improvement of the objective by adjusting p_i would approximately be

  $$f^{low} = f(z^k) - \epsilon^k \sum_{i \in P_0 \cup P_\epsilon} |u_i^{\Phi,k}|.$$

  Step 4. Update the parameters ϵ and p. Set $\epsilon^{k+1} \leftarrow \kappa\epsilon^k$, and

  $$p_i^{k+1} = \begin{cases} \epsilon^{k+1}/2, & i \in P_0; \\ 0, & i \in P_\epsilon; \\ \kappa p_i^k, & \text{otherwise.} \end{cases}$$

  Step 5. Set $k \leftarrow k + 1$ and go to *Step 1.*

Suppose the Bounding Algorithm generates a sequence $\{z^k\} \to \bar{z}$ as $\epsilon^k \to 0$. Without MPCC-LICQ at the limit point $\bar{z}$, NLP-LICQ is absent in a neighborhood of $\bar{z}$ for feasible points of BA(ϵ). However, the Bounding Algorithm is still helpful in seeking a solution of MPCC (1). View the parameters $-p_i$ as the right-hand sides of the constraints $\Phi_i^\epsilon(z)$, then the multipliers u_i^Φ indicate what perturbations of the constraints are desired to improve the objective. For example, if $p_i = 0$ and $u_i^\Phi < 0$, further reduction of the objective requires increasing the right-hand side of $\Phi_i^\epsilon(z)$, namely, requires p_i to be negative; this makes no sense because the difference between $\Phi_i^\epsilon(z)$ and the corresponding complementarity constraint is within $[0, \epsilon/2]$. In fact, this is usually an indication that S-stationary solutions (corresponding to solutions of RNLP) do not locate in the feasible region of BA(ϵ), nor in the feasible region of MPCC (1). This point will become clear later from the perspective of an inequality variant of BA(ϵ).

4.3 Derivatives of NCP function

It follows from (27) that

$$\begin{aligned}
\sqrt{(G_i(z) - H_i(z))^2 + \epsilon^2} &= \sqrt{((G_i(z) + p_i) - (H_i(z) + p_i))^2 + \epsilon^2} \\
&= \sqrt{(G_i(z) + p_i)^2 + (H_i(z) + p_i)^2 + 2(G_i(z) + p_i)(H_i(z) + p_i)} \\
&= |G_i(z) + H_i(z) + 2p_i| = G_i(z) + H_i(z) + 2p_i.
\end{aligned}$$

As shown in [22] the following derivatives can be derived at a point z satisfying $\Phi_i^\epsilon(z) + p_i = 0$:

$$\begin{aligned}
\nabla_G \Phi_i^\epsilon(z) &= \frac{H_i(z) + p_i}{G_i(z) + H_i(z) + 2p_i}, \\
\nabla_H \Phi_i^\epsilon(z) &= \frac{G_i(z) + p_i}{G_i(z) + H_i(z) + 2p_i}, \\
\nabla_{GG} \Phi_i^\epsilon(z) = \nabla_{HH} \Phi_i^\epsilon(z) &= \frac{-2(G_i(z) + p_i)(H_i(z) + p_i)}{(G_i(z) + H_i(z) + 2p_i)^3}, \\
\nabla_{GH} \Phi_i^\epsilon(z) = \nabla_{HG} \Phi_i^\epsilon(z) &= \frac{2(G_i(z) + p_i)(H_i(z) + p_i)}{(G_i(z) + H_i(z) + 2p_i)^3}.
\end{aligned} \tag{28}$$

4.4 C-stationarity

Let a sequence $\{z^k\} \to \bar{z}$ as $\epsilon^k \to 0$, where every z^k is a KKT point of BA(ϵ^k). The following proves C-stationarity of $\bar{z}$, without explicitly specifying a constraint qualification at $\bar{z}$. The reason is that assuming a particular MPCC-CQ at $\bar{z}$ amounts to enforcing a certain NLP-CQ at $\bar{z}$ or in its neighborhood. For example, MPCC-LICQ at $\bar{z}$ implies the presence of NLP-LICQ in a neighborhood of $\bar{z}$ for every feasible point of the regularized NLP problem [19, Lemma 2.1], and MPCC-MFCQ at $\bar{z}$ implies the presence of NLP-MFCQ at $\bar{z}$ for every NLP$_{(\beta_1,\beta_2)}$ with $(\beta_1, \beta_2) \in \mathcal{P}(\beta(\bar{z}))$ [4, Lemma 3.5]. Instead of requiring a particular constraint qualification at $\bar{z}$, we establish C-stationarity of $\bar{z}$ based on the local optimality of z^k for BA(ϵ^k), which is guaranteed by KKT conditions combined with classic constraint qualifications no weaker than NLP-GCQ at z^k.

Theorem 4.1. *For a sequence of positive scalars $\epsilon^k \to 0$, apply the Bounding Algorithm to BA(ϵ^k), such that the parameters p^k are updated whenever ϵ^k is updated. Assume this generates a sequence $\{z^k\} \to \bar{z}$, where every z^k is a KKT point of BA(ϵ^k). If NLP-GCQ holds at all z^k then $\bar{z}$ is a C-stationary point of MPCC (1).*

Proof. When $\epsilon^k > 0$, at every KKT point z^k of BA(ϵ^k), there exist multipliers $u^k = (u^{g,k}, u^{h,k}, u^{\Phi,k})$ with $u^{g,k} \geq 0$, such that

$$0 = \nabla f(z^k) + \sum_{i \in I_g(z^k)} u_i^{g,k} \nabla g_i(z^k) + \sum_{i=1}^{n_h} u_i^{h,k} \nabla h_i(z^k) - \sum_{i=1}^{m} u_i^{\Phi,k} \nabla \Phi_i^\epsilon(z^k), \tag{29}$$

where the gradient of Φ_i^ϵ is given by

$$\begin{aligned} \nabla \Phi_i^\epsilon(z) =& \nabla_G \Phi_i^\epsilon(z) \nabla G_i(z) + \nabla_H \Phi_i^\epsilon(z) \nabla H_i(z) \\ =& \frac{H_i(z) + p_i}{G_i(z) + H_i(z) + 2p_i} \nabla G_i(z) + \frac{G_i(z) + p_i}{G_i(z) + H_i(z) + 2p_i} \nabla H_i(z). \end{aligned}$$

At the limit $\epsilon = 0$, the function Φ_i^0 is in general not differentiable for $i \in \beta(\bar{z})$. However, since Φ_i^0 is *locally Lipschitz* [2, Section 1.2] near $\bar{z}$, the *generalized gradient* $\partial \Phi_i^0(\bar{z})$ is generated by a convex hull (see [2, Theorem 2.5.1] [3, Eq.(3.1.5)] [18, Lemma 1])

$$\begin{aligned} \partial \Phi_i^0(\bar{z}) &= \operatorname{conv} \left\{ \lim_{s^K \to \bar{z}} \nabla \Phi_i^0(s^K) \mid \nabla \Phi_i^0(s^K) \text{ exists} \right\} \\ &= \theta_i \nabla G_i(\bar{z}) + (1 - \theta_i) \nabla H_i(\bar{z}), \end{aligned}$$

where $\theta_i \in [0, 1]$. Therefore, as $\epsilon^k \to 0$, the gradients of functions Φ_i^ϵ are as follows:

$$\begin{aligned} &\nabla \Phi_i^\epsilon(z^k) \to \nabla G_i(\bar{z}), && i \in \alpha(\bar{z}), \\ &\nabla \Phi_i^\epsilon(z^k) \to \nabla H_i(\bar{z}), && i \in \gamma(\bar{z}), \\ &\nabla \Phi_i^\epsilon(z^k) \to \theta_i \nabla G_i(\bar{z}) + (1 - \theta_i) \nabla H_i(\bar{z}), && i \in \beta(\bar{z}). \end{aligned} \tag{30}$$

It follows from (29) and (30) that at a limit point $\bar{z}$, the multipliers u^k tend to $\bar{u} = (\bar{u}^g, \bar{u}^h, \bar{u}^\Phi)$ with $\bar{u}^g \geq 0$, such that

$$\begin{aligned} 0 =& \nabla f(\bar{z}) + \sum_{i \in I_g(z^k)} \bar{u}_i^g \nabla g_i(\bar{z}) + \sum_{i=1}^{n_h} \bar{u}_i^h \nabla h_i(\bar{z}) - \sum_{i=1}^{m} \bar{u}_i^\Phi \nabla \Phi_i^0(\bar{z}) \\ =& \nabla f(\bar{z}) + \sum_{i \in I_g(z^k)} \bar{u}_i^g \nabla g_i(\bar{z}) + \sum_{i=1}^{n_h} \bar{u}_i^h \nabla h_i(\bar{z}) \\ & - \sum_{i \in \alpha(\bar{z})} \bar{u}_i^\Phi \nabla G_i(\bar{z}) - \sum_{i \in \gamma(\bar{z})} \bar{u}_i^\Phi \nabla H_i(\bar{z}) - \sum_{i \in \beta(\bar{z})} \bar{u}_i^\Phi \left[\theta_i \nabla G_i(\bar{z}) + (1 - \theta_i) \nabla H_i(\bar{z}) \right], \end{aligned}$$

for some $\theta_i \in [0, 1]$. Thus $\bar{z}$ satisfies the weak stationarity conditions (3), with the MPCC multipliers given by

$$\begin{aligned}
&\bar{\lambda}^g = \bar{u}^g = \lim_{k\to\infty} u^{g,k},\\
&\bar{\lambda}^h = \bar{u}^h = \lim_{k\to\infty} u^{h,k},\\
&\bar{\lambda}_i^G = \begin{cases} \bar{u}_i^\Phi = \lim_{k\to\infty} u_i^{\Phi,k}, & i \in \alpha(\bar{z})\\ \bar{u}_i^\Phi \theta_i = \lim_{k\to\infty} u_i^{\Phi,k}\left[\dfrac{H_i(z^k)+p_i^k}{G_i(z^k)+H_i(z^k)+2p_i^k}\right], & i \in \beta(\bar{z}),\end{cases}\\
&\bar{\lambda}_i^H = \begin{cases} \bar{u}_i^\Phi = \lim_{k\to\infty} u_i^{\Phi,k}, & i \in \gamma(\bar{z})\\ \bar{u}_i^\Phi (1-\theta_i) = \lim_{k\to\infty} u_i^{\Phi,k}\left[\dfrac{G_i(z^k)+p_i^k}{G_i(z^k)+H_i(z^k)+2p_i^k}\right], & i \in \beta(\bar{z}).\end{cases}
\end{aligned} \tag{31}$$

Moreover, $\bar{z}$ is C-stationary because

$$\bar{\lambda}_i^G \cdot \bar{\lambda}_i^H = (\bar{u}_i^\Phi)^2 \theta_i (1-\theta_i) \geq 0, \quad \forall i \in \beta(\bar{z}). \tag{32}$$

□

4.5 M-stationarity

The property (32) allows for two possibilities. In the case $\bar{u}_i^\Phi \geq 0$ for all $i \in \beta(\bar{z})$, then $\bar{\lambda}_i^G, \bar{\lambda}_i^H \geq 0$ for all $i \in \beta(\bar{z})$, and the point $\bar{z}$ is S-stationary and obviously a local minimizer of the MPCC. In the other case, there exist indices $i \in \beta(\bar{z})$ such that $\bar{u}_i^\Phi < 0$. For these indices i, $\bar{\lambda}_i^G, \bar{\lambda}_i^H \leq 0$. If at the same time MPCC-LICQ fails at $\bar{z}$, it is still possible for $\bar{z}$ to be a local minimizer of the MPCC. As discussed in Section 3.1, if MPCC-ACQ holds at $\bar{z}$, then $\bar{z}$ is B-stationary. In the following, we establish M-stationarity of $\bar{z}$, in the circumstance that $\bar{u}_i^\Phi < 0$ for some indices $i \in \beta(\bar{z})$. Instead of specifying an MPCC-CQ at $\bar{z}$, the derivation is based on local optimality of z^k for BA(ϵ^k).

Let $\Omega \subseteq \beta(\bar{z})$ be a subset such that

$$\Omega = \{i \mid \bar{u}_i^\Phi < 0, i \in \beta(\bar{z})\}.$$

We introduce two assumptions at a KKT point z^k of BA(ϵ^k).

(A1) The set of gradients

$$\begin{aligned}
\mathcal{G}(z^k) = \{&\nabla g_i(z^k), && i \in I_g(\bar{z}),\\
&\nabla h_i(z^k), && i = 1, \dots n_h,\\
&\nabla G_i(z^k), && i \in \alpha(\bar{z}) \cup \beta(\bar{z}) \setminus \Omega,\\
&\nabla H_i(z^k), && i \in \gamma(\bar{z}) \cup \beta(\bar{z}) \setminus \Omega\}
\end{aligned}$$

satisfies $\text{rank}(\mathcal{G}(z^k)) \leq n - |\Omega|$ (where $|\Omega|$ denotes cardinality of the set).

(A2) The reduced Hessian of the Lagrangian (26) is bounded below in the sense that

$$d^T \nabla_{zz} \mathcal{L}(z^k, u^k) d > -\infty, \quad \forall d \in \mathcal{T}^{lin}_{\text{BA}}(z^k),$$

where

$$\begin{aligned} \mathcal{T}^{lin}_{\text{BA}}(z^k) = \{ d \,|\, & \nabla g_i(z^k)^T d \le 0, \quad \forall i \in I_g(z^k), \\ & \nabla h_i(z^k)^T d = 0, \quad \forall i = 1, \ldots, n_h, \\ & \nabla \Phi_i^\epsilon(z^k)^T d = 0, \quad \forall i = 1, \ldots, m \}. \end{aligned}$$

Theorem 4.2. *For a sequence of positive scalars $\epsilon^k \to 0$, apply the Bounding Algorithm to BA(ϵ^k), such that the parameters p^k are updated whenever ϵ^k is updated. Assume this generates a sequence $\{z^k\} \to \bar{z}$, where every z^k is KKT point of BA(ϵ^k). In addition to the assumptions of Theorem 4.1, suppose that the conditions (A1) and (A2) hold at every z^k when $\epsilon^k > 0$ and suitably small. Then $\bar{z}$ is an M-stationary point of MPCC (1).*

Proof. For the purpose of deriving a contradiction, assume that $\bar{z}$ is not M-stationary. Then, without loss of generality, it holds for all $j \in \Omega$ that

$$\begin{aligned} \bar{\lambda}_j^G &= \bar{u}_j^\Phi \theta_j < 0, \\ \bar{\lambda}_j^H &= \bar{u}_j^\Phi (1 - \theta_j) < 0. \end{aligned}$$

This implies that $0 < \theta_j < 1$ for all $j \in \Omega$.

Because of the assumption (A1), the vectors in $\mathcal{G}(z^k)$ lie in a proper subspace of $\mathbb{R}^n$, and there exists some nonzero vector $d^k \in \mathbb{R}^n$ such that

$$\begin{aligned} \nabla g_i(z^k)^T d^k &= 0, \quad i \in I_g(\bar{z}), \\ \nabla h_i(z^k)^T d^k &= 0, \quad i = 1, \ldots, n_h, \\ \nabla G_i(z^k)^T d^k &= 0, \quad i \in \alpha(\bar{z}) \cup \beta(\bar{z}) \setminus \Omega, \\ \nabla H_i(z^k)^T d^k &= 0, \quad i \in \gamma(\bar{z}) \cup \beta(\bar{z}) \setminus \Omega. \end{aligned} \tag{33}$$

This system together with

$$\begin{aligned} \nabla G_j(z^k)^T d^k &= \kappa_G = \nabla_H \Phi_j^\epsilon(z^k), \quad && j \in \Omega, \\ \nabla H_j(z^k)^T d^k &= \kappa_H = -\nabla_G \Phi_j^\epsilon(z^k), \quad && j \in \Omega, \end{aligned} \tag{34}$$

defines a vector $d^k \in \mathcal{T}^{lin}_{\text{BA}}(z^k)$. Note that the whole system (33)-(34) may have more than n equations; however, these equations are consistent because the right-hand sides of (34) are constructed to satisfy the linearized constraints

$$\nabla \Phi_j^\epsilon(z^k)^T d^k = \nabla_G \Phi_j^\epsilon(z^k) \nabla G_j(z^k)^T d^k + \nabla_H \Phi_j^\epsilon(z^k) \nabla H_j(z^k)^T d^k = 0.$$

Also note that the right-hand sides of (34) are bounded, because (28) and (31) indicate that they converge to $1 - \theta_j$ and $-\theta_j$, respectively.

For each $j \in \Omega$, the contribution of the constraint $\Phi_j^\epsilon(z^k) + p_j^k = 0$ to the reduced Hessian $(d^k)^T \nabla_{zz} \mathcal{L}(z^k, u^k) d^k$ is that

$$
\begin{aligned}
&-u_j^{\Phi,k}(d^k)^T \nabla_{zz} \Phi_{j0}^\epsilon(z^k) d^k \\
&= - u_j^{\Phi,k}(d^k)^T \big[\nabla_G \Phi_j^\epsilon(z^k) \nabla_{zz} G_j(z^k) + \nabla_H \Phi_j^\epsilon(z^k) \nabla_{zz} H_j(z^k) \\
&\quad + \nabla_{GG} \Phi_j^\epsilon(z^k) \nabla G_j(z^k) \nabla G_j(z^k)^T + \nabla_{GH} \Phi_j^\epsilon(z^k) \nabla G_j(z^k) \nabla H_j(z^k)^T \\
&\quad + \nabla_{HG} \Phi_j^\epsilon(z^k) \nabla H_j(z^k) \nabla G_j(z^k)^T + \nabla_{HH} \Phi_j^\epsilon(z^k) \nabla H_j(z^k) \nabla H_j(z^k)^T \big] d^k.
\end{aligned}
$$

Combining the definition of d^k and the derivatives of Φ_j^ϵ in (28), we have

$$
\begin{aligned}
&-u_j^{\Phi,k}(d^k)^T \nabla_{zz} \Phi_j^\epsilon(z^k) d^k \\
&= - u_j^{\Phi,k}(d^k)^T \nabla_G \Phi_j^\epsilon(z^k) \nabla_{zz} G_j(z^k) d^k - u_j^{\Phi,k}(d^k)^T \nabla_H \Phi_j^\epsilon(z^k) \nabla_{zz} H_j(z^k) d^k \\
&\quad + \frac{2(G_j(z^k) + p_j^k)(H_j(z^k) + p_j^k)}{(G_j(z^k) + H_j(z^k) + 2p_j^k)^3} u_j^{\Phi,k} (\kappa_G - \kappa_H)^2 \\
&= - u_j^{\Phi,k}(d^k)^T \nabla_G \Phi_j^\epsilon(z^k) \nabla_{zz} G_j(z^k) d^k - u_j^{\Phi,k}(d^k)^T \nabla_H \Phi_j^\epsilon(z^k) \nabla_{zz} H_j(z^k) d^k \\
&\quad + \frac{2(G_j(z^k) + p_j^k)(H_j(z^k) + p_j^k)}{(G_j(z^k) + H_j(z^k) + 2p_j^k)^3} u_j^{\Phi,k} \\
&= - u_j^{\Phi,k}(d^k)^T \nabla_G \Phi_j^\epsilon(z^k) \nabla_{zz} G_j(z^k) d^k - u_j^{\Phi,k}(d^k)^T \nabla_H \Phi_j^\epsilon(z^k) \nabla_{zz} H_j(z^k) d^k \\
&\quad + \frac{2}{G_j(z^k) + H_j(z^k) + 2p_j^k} \nabla_G \Phi_j^\epsilon(z^k) \nabla_H \Phi_j^\epsilon(z^k) u_j^{\Phi,k}.
\end{aligned}
$$

The first two terms are bounded. As for the last term, $\nabla_G \Phi_j^\epsilon(z^k), \nabla_H \Phi_j^\epsilon(z^k)$, and $u_j^{\Phi,k}$ tend to $\theta_j > 0, 1 - \theta_j > 0$, and $\bar{u}_j^\Phi < 0$, respectively, while $G_j(z^k), H_j(z^k)$, and p_j^k tend to zero. Therefore,

$$
-u_j^{\Phi,k}(d^k)^T \nabla_{zz} \Phi_j^\epsilon(z^k) d^k \to -\infty. \tag{35}
$$

Since all other terms in the reduced Hessian $(d^k)^T \nabla_{zz} \mathcal{L}(z^k, u^k) d^k$ are bounded, the relation (35) contradicts (A2). Hence, the assumption must be false and $\bar{z}$ is M-stationary. □

4.6 Inequality relaxation of BA

To further understand and explore the convergence properties of the Bounding Algorithm, it is beneficial to take advantage of an inequality variant of the problem BA(ϵ), which is given by

$$
\begin{array}{llll}
\text{MLF}(\epsilon): & \min & f(z) & \text{multipliers} \\
& \text{s.t.} & g(z) \le 0, & u^g \\
& & h(z) = 0, & u^h \\
& & -\epsilon/2 \le \Phi_i^\epsilon(z) \le 0, \quad i = 1 \ldots m. & u_{L,i}^\Phi, u_{U,i}^\Phi
\end{array}
\tag{36}
$$

For a sequence of positive scalars $\epsilon^k \to 0$, solving problems MLF(ϵ^k) generates a sequence $\{z^k\} \to \bar{z}$, where every z^k is a KKT point of MLF(ϵ^k). At every point z^k we have multipliers

$u^k = (u^{g,k}, u^{h,k}, u_L^{\Phi,k}, u_U^{\Phi,k})$ with $u^{g,k} \geq 0$, such that

$$0 = \nabla f(z^k) + \sum_{i \in I_g(z^k)} u_i^{g,k} \nabla g_i(z^k) + \sum_{i=1}^{n_h} u_i^{h,k} \nabla h_i(z^k) - \sum_{i=1}^{m} (u_{L,i}^{\Phi,k} - u_{U,i}^{\Phi,k}) \nabla \Phi_i^{\epsilon}(z^k), \tag{37}$$

where $0 \leq u_{L,i}^{\Phi,k} \perp u_{U,i}^{\Phi,k} \geq 0, i = 1, \ldots, m$. Comparing the program formulations (24) and (36), and the KKT conditions (29) and (37), gives the relations between BA(ϵ) and MLF(ϵ):

$$\begin{aligned} p_i = \epsilon/2 &\Leftrightarrow \text{lower bound of } \Phi_i(\epsilon) \text{ is active, and } u_{L,i}^{\Phi} \geq 0, \\ p_i = 0 &\Leftrightarrow \text{upper bound of } \Phi_i(\epsilon) \text{ is active, and } u_{U,i}^{\Phi} \geq 0, \\ u^{\Phi} &= u_L^{\Phi} - u_U^{\Phi}. \end{aligned} \tag{38}$$

Substituting the last relation into (31) gives the MPCC multipliers at a limit point $\bar{z}$:

$$\begin{aligned} \bar{\lambda}^g &= \bar{u}^g = \lim_{k \to \infty} u^{g,k}, \\ \bar{\lambda}^h &= \bar{u}^h = \lim_{k \to \infty} u^{h,k}, \\ \bar{\lambda}_i^G &= \begin{cases} \bar{u}_{L,i}^{\Phi} - \bar{u}_{U,i}^{\Phi} = \lim_{k \to \infty} (u_{L,i}^{\Phi,k} - u_{U,i}^{\Phi,k}), & i \in \alpha(\bar{z}) \\ (\bar{u}_{L,i}^{\Phi} - \bar{u}_{U,i}^{\Phi})\theta_i = \lim_{k \to \infty} (u_{L,i}^{\Phi,k} - u_{U,i}^{\Phi,k}) \left[\dfrac{H_i(z^k)}{G_i(z^k) + H_i(z^k)} \right], & i \in \beta(\bar{z}), \end{cases} \\ \bar{\lambda}_i^H &= \begin{cases} \bar{u}_{L,i}^{\Phi} - \bar{u}_{U,i}^{\Phi} = \lim_{k \to \infty} (u_{L,i}^{\Phi,k} - u_{U,i}^{\Phi,k}), & i \in \gamma(\bar{z}) \\ (\bar{u}_{L,i}^{\Phi} - \bar{u}_{U,i}^{\Phi})(1 - \theta_i) = \lim_{k \to \infty} (u_{L,i}^{\Phi,k} - u_{U,i}^{\Phi,k}) \left[\dfrac{G_i(z^k)}{G_i(z^k) + H_i(z^k)} \right], & i \in \beta(\bar{z}). \end{cases} \end{aligned} \tag{39}$$

Obviously, if $(\bar{u}_{L,i}^{\Phi} - \bar{u}_{U,i}^{\Phi}) \geq 0$ for all $i \in \beta(\bar{z})$, then $\bar{\lambda}_i^G, \bar{\lambda}_i^H \geq 0$ for all $i \in \beta(\bar{z})$, and $\bar{z}$ is S-stationary; otherwise, if there exist indices $i \in \beta(\bar{z})$ such that $(\bar{u}_{L,i}^{\Phi} - \bar{u}_{U,i}^{\Phi}) < 0$, then for these indices i, $\bar{\lambda}_i^G, \bar{\lambda}_i^H \leq 0$. The C- and M- stationarity of the limit point established in the previous subsections for BA can be extended to MLF.

In particular, numerical experience reveals the feature that a sequence $\{z^k\}$ converges from the upper bounds of MLF(ϵ^k) to $\bar{z}$, when $\bar{z}$ is not S-stationary, paralleling the observation that the Bounding Algorithm sets the parameters p_i^k which correspond to non-strongly stationary complementarity constraints $i \in \beta(\bar{z})$ to be zero and thus $u_i^{\Phi,k} < 0$ as $\epsilon^k \to 0$. This feature also has a theoretical reason. At a feasible point z of MLF(ϵ^k), define the index sets

$$\begin{aligned} I_L^{\Phi}(z) &= \{i \,|\, \Phi_i^{\epsilon}(z) = -\epsilon^k/2\}, \\ I_U^{\Phi}(z) &= \{i \,|\, \Phi_i^{\epsilon}(z) = 0\}. \end{aligned}$$

Definition (25) of the NCP function implies the following at the boundaries of $\Phi_i^{\epsilon}(z)$:

$$\begin{aligned} i \in I_L^{\Phi}(z) &\implies G_i(z) + \tfrac{\epsilon^k}{2} > 0,\ H_i(z) + \tfrac{\epsilon^k}{2} > 0,\ (G_i(z) + \tfrac{\epsilon^k}{2})(H_i(z) + \tfrac{\epsilon^k}{2}) = (\epsilon^k)^2/4, \\ i \in I_U^{\Phi}(z) &\implies G_i(z) > 0,\ H_i(z) > 0,\ G_i(z)H_i(z) = (\epsilon^k)^2/4. \end{aligned}$$

The constraints $-\epsilon^k/2 \leq \Phi_i^{\epsilon}(z) \leq 0$ require that

$$\begin{aligned} (G_i(z) + \tfrac{\epsilon^k}{2})(H_i(z) + \tfrac{\epsilon^k}{2}) &\geq (\epsilon^k)^2/4, \\ G_i(z)H_i(z) &\leq (\epsilon^k)^2/4. \end{aligned}$$

Therefore, the feasible region of MLF(ϵ^k) includes the feasible region of MPCC (1), while it restricts the feasible region of RNLP (5) from above by enforcing $\Phi_i^\epsilon(z) \leq 0$. Thus, for every $\epsilon^k > 0$ suitably small, a local minimizer z^k of MLF(ϵ^k) is also a local minimizer of the RNLP constrained additionally by $\Phi_i^\epsilon(z) \leq 0$. Suppose that there exists a subset $\Omega \subseteq \{1 \dots m\}$, such that RNLP is minimized at $G_\Omega(z) > 0$ and $H_\Omega(z) > 0$. In such circumstance, MLF(ϵ^k) achieves the minimal cost on the boundaries of $\Phi_\Omega^\epsilon(z) \leq 0$ for every $\epsilon^k > 0$ suitably small. This gives rise to the phenomenon that the upper bounds of the constraints $-\epsilon^k/2 \leq \Phi_\Omega^\epsilon(z) \leq 0$ are active at every z^k as $\epsilon^k \to 0$; moreover, $\Omega \subseteq \beta(\bar{z})$. Since the solutions of RNLP locate outside of the feasible region of the MPCC, no local minimizer of the MPCC can be S-stationary.

Now we reconsider a limit point $\bar{z}$ of BA(ϵ) or MLF(ϵ), at which there exists a subset $\Omega \subseteq \beta(\bar{z})$ such that $\bar{u}_\Omega^\Phi < 0$ (BA), or $\bar{u}_{L,\Omega}^\Phi - \bar{u}_{U,\Omega}^\Phi < 0$ (MLF). This indicates that for k sufficiently large, at their KKT points z^k, the BA multipliers $u_\Omega^{\Phi,k} < 0$, while the MLF multipliers $u_{L,\Omega}^{\Phi,k} - u_{U,\Omega}^{\Phi,k} < 0$. According to the relations (38), it can be observed at z^k for k sufficiently large that the BA parameters $p_\Omega^k = 0$, while the upper bounds of the MLF constraints Φ_Ω^ϵ are active. We aim to verify whether such $\bar{z}$ is a B-stationary point of the MPCC.

Theorem 2.2 states that a local minimizer of MPCC (1) at which MPCC-GCQ holds, is "piecewise M-stationarity". Since the above discussion has shown that such a limit point $\bar{z}$ usually indicates the absence of S-stationary solutions, then, if $\bar{z}$ is a local minimizer of the MPCC, the MPCC multipliers at $\bar{z}$ must satisfy that $\bar{\lambda}_i^G \bar{\lambda}_i^H = 0$ for all $i \in \beta(\bar{z})$. Therefore, for every $(\beta_1, \beta_2) \in \mathcal{P}(\beta(\bar{z}))$, checking whether $d = 0$ solves LP$_{(\beta_1,\beta_2)}$, or equivalently, checking whether LP$_{(\beta_1,\beta_2)}$ is bounded below (see Theorem 3.2), can be simplified to checking the following problem:

$$\begin{aligned} \min \quad & obj(d) = \nabla f(\bar{z})^T d \\ \text{s.t.} \quad & \nabla g_I(\bar{z})^T d \leq 0, \\ & \nabla h(\bar{z})^T d = 0, \\ & \nabla G_\alpha(\bar{z})^T d = 0, \\ & \nabla H_\gamma(\bar{z})^T d = 0, \\ & \nabla G_{\beta_1}(\bar{z})^T d = 0, \\ & \nabla H_{\beta_2}(\bar{z})^T d = 0, \end{aligned} \tag{40}$$

where the inequalities $\nabla H_{\beta_1}(\bar{z})^T d \geq 0$ and $\nabla G_{\beta_2}(\bar{z})^T d \geq 0$ can be removed because the multipliers $\lambda_{\beta_1}^H = \lambda_{\beta_2}^G = 0$ indicate that the associated constraints are locally inactive. For every $(\beta_1, \beta_2) \in \mathcal{P}(\beta(\bar{z}))$, if the problem (40) is bounded, then its dual problem

$$\begin{aligned}
\max \quad & obj^{dual}(\eta) = \eta^T \cdot 0 \\
\text{s.t.} \quad & \eta_I^g \geq 0, \\
& \eta^h \, free, \\
& \eta_\alpha^G \, free, \\
& \eta_\gamma^H \, free, \\
& \eta_{\beta_1}^G \, free, \\
& \eta_{\beta_2}^H \, free, \\
& 0 = \nabla f(\bar{z}) + \eta_I^g \nabla g_I(\bar{z}) + \eta^h \nabla h(\bar{z}) - \eta_\alpha^G \nabla G_\alpha(\bar{z}) - \eta_\gamma^H \nabla H_\gamma(\bar{z}) \\
& \quad - \eta_{\beta_1}^G \nabla G_{\beta_1}(\bar{z}) - \eta_{\beta_2}^H \nabla H_{\beta_2}(\bar{z})
\end{aligned} \tag{41}$$

is feasible; thus $d = 0$ is optimal for each partition because $obj(d) = obj^{dual}(\eta) = 0$. It follows that $\bar{z}$ solves LPEC (17) and is B-stationary, and MPCC-GCQ holds at $\bar{z}$ because of Lemma 2.1. The equality constrained problems (40) offer convenience in verifying B-stationarity at $\bar{z}$, in contrast to problems (18) composing the LPEC, which have both equality and inequality constraints.

5 Practical issues

This section discusses the behaviors of NCP-based algorithms, when converging to a limit point which is not S-stationary.

5.1 MPCC multipliers by NCP-based methods

The M-stationarity pertaining to a local minimizer of MPCC (1) is essentially a piecewise property. When approaching a local minimizer $\bar{z}$ which is not S-stationary, an algorithm should not converge to the multipliers satisfying the M-stationarity conditions for a particular partition in $\mathcal{P}(\beta(\bar{z}))$. And, unless $\bar{z}$ is S-stationary, there does not exist a set of MPCC multipliers which is suitable for all the partitions in $\mathcal{P}(\beta(\bar{z}))$.

Example: ***ex9.2.2***

This example shows that the solutions of the NCP-based methods (BA and MLF) provide MPCC multipliers satisfying C-stationarity at a non-strongly stationary local minimizer, and the multipliers satisfying M-stationarity can be easily derived.

The problem *ex9.2.2* from the MacMPEC collection [14] is given by

$$
\begin{array}{lll}
\min & x^2 + (y-10)^2 & \text{multipliers} \\
\text{s.t.} & x \le 15, & \text{(inactive)} \\
& -x + y \le 0, & \lambda_1 \\
& -x \le 0, & \text{(inactive)} \\
& x + y + s_1 = 20, & \lambda_2 \\
& -y + s_2 = 0, & \lambda_3 \\
& y + s_3 = 20, & \lambda_4 \\
& 2x + 4y + l_1 - l_2 + l_3 = 60, & \lambda_5 \\
& 0 \le s_i \perp l_i \ge 0, \; i = 1 \ldots 3. & \sigma^{si}, \sigma^{li}
\end{array}
$$

The NCP-based methods converge to the solution $\bar{z} = (\bar{x}, \bar{y}, \bar{s}, \bar{l})$:

$$
\begin{aligned}
\bar{x} &= 10, \\
\bar{y} &= 10, \\
\bar{s} &= (0, 10, 10), \\
\bar{l} &= (0, 0, 0).
\end{aligned}
$$

Since the problem has linear constraints, MPCC-ACQ holds at every feasible point of the problem, and Theorem 3.1 indicates that $\bar{z}$ is B-stationary. The weak stationarity conditions (3) at $\bar{z}$ require that

$$
\begin{aligned}
2\bar{x} - \lambda_1 + \lambda_2 + 2\lambda_5 &= 0, \\
2(\bar{y} - 10) + \lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 + 4\lambda_5 &= 0, \\
\lambda_2 - \sigma^{s1} &= 0, \\
\lambda_3 &= 0, \\
\lambda_4 &= 0, \\
\lambda_5 - \sigma^{l1} &= 0, \\
-\lambda_5 - \sigma^{l2} &= 0, \\
\lambda_5 - \sigma^{l3} &= 0,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\sigma^{s1} &= -3\lambda_5 - 10, \\
\sigma^{l1} &= \lambda_5.
\end{aligned}
$$

Thus the MPCC multipliers σ^{s1}, σ^{l1} for the biactive complementary components s_1, l_1 cannot be both nonnegative, and the local minimizer $\bar{z}$ cannot be S-stationary.

Now we check the multipliers at $\bar{z}$. For the set $\beta(\bar{z}) = \{1\}$, solutions of the NCP-based methods give the corresponding NLP multipliers (at $\epsilon = 10^{-6}$)

$$
\begin{aligned}
\text{BA}: &\quad u^{\Phi} = -5.74, \\
\text{MLF}: &\quad u_L^{\Phi} = 0, \; u_U^{\Phi} = 5.74.
\end{aligned}
$$

According to (31) and (39),

$$\bar{\lambda}_i^G + \bar{\lambda}_i^H = \lim_{k\to\infty} u_i^{\Phi,k} = \lim_{k\to\infty} (u_{L,i}^{\Phi,k} - u_{U,i}^{\Phi,k}), \quad \forall i \in \beta(\bar{z}). \tag{42}$$

By enforcing $\sigma^{s1}+\sigma^{l1} = -5.74$, we obtain the MPCC multipliers $(\sigma^{s1}, \sigma^{l1}) = (-3.61, -2.13)$, which satisfies C-stationarity. This shows that the solutions of the NCP-based methods provide a set of MPCC multipliers that satisfy C-stationarity at $\bar{z}$, which can be interpreted as a balance/compromise between the partitions. As indicated by Theorem 2.2, a local minimizer is piecewise M-stationary provided that MPCC-GCQ holds. To find the multipliers that satisfy M-stationarity, we let $\sigma^{s1} = 0$ or $\sigma^{l1} = 0$, then obtain the MPCC multipliers $(\sigma^{s1}, \sigma^{l1}) = (0, -10/3)$ or $(\sigma^{s1}, \sigma^{l1}) = (-10, 0)$. These two sets of multipliers reflect optimality of $\bar{z}$ for their respective partitions.

5.2 Unbounded NLP multipliers and inaccurate solutions

NLP subproblems of MPCC algorithms may encounter unbounded multipliers when approaching a limit point which is not S-stationary for the MPCC. However, in view of numerical experience to date NCP-based reformulations BA(ϵ) and MLF(ϵ) avoid these unbounded NLP multipliers. The following confirms this observation, by comparing them with the typical regularization scheme proposed in [19]:

$$\begin{array}{llll} \text{REG}(\epsilon): & \min & f(z) & \text{multipliers} \\ & \text{s.t.} & g(z) \le 0, & v^g \\ & & h(z) = 0, & v^h \\ & & G(z) \ge 0, & v^G \\ & & H(z) \ge 0, & v^H \\ & & G_i(z)H_i(z) \le \epsilon, \quad i = 1\ldots m. & v_i^{REG} \end{array}$$

Solving a sequence of programs REG(ϵ^k) with the positive scalar $\epsilon^k \to 0$, generates a sequence $\{z^k\} \to \bar{z}$. Based on stationarity of z^k for REG(ϵ^k), namely,

$$\begin{aligned} 0 =& \nabla f(z^k) + \sum_{i\in I_g(z^k)} v_i^{g,k} \nabla g_i(z^k) + \sum_{i=1}^{n_h} v_i^{h,k} \nabla h_i(z^k) \\ & - \sum_{i=1}^m v_i^{G,k} \nabla G_i(z^k) - \sum_{i=1}^m v_i^{H,k} \nabla H_i(z^k) + \sum_{i=1}^m v_i^{REG,k} \left[H_i(z^k) \nabla G_i(z^k) + G_i(z^k) \nabla H_i(z^k) \right], \end{aligned}$$

the relations between the multipliers $v^k = (v^{g,k}, v^{h,k}, v^{G,k}, v^{H,k}, v^{REG,k})$ and the MPCC multipliers at $\bar{z}$ can be expressed by (see also [19, Theorem 3.1]):

$$\begin{aligned} \bar{\lambda}^g &= \bar{v}^g = \lim_{k\to\infty} v^{g,k}, \\ \bar{\lambda}^h &= \bar{v}^h = \lim_{k\to\infty} v^{h,k}, \\ \bar{\lambda}_i^G &= \lim_{k\to\infty} \left[v_i^{G,k} - v_i^{REG,k} H_i(z^k) \right], \quad i = 1, \ldots, m, \\ \bar{\lambda}_i^H &= \lim_{k\to\infty} \left[v_i^{H,k} - v_i^{REG,k} G_i(z^k) \right], \quad i = 1, \ldots, m. \end{aligned} \tag{43}$$

It has been proved that $\bar{z}$ is a strongly stationary point of MPCC (1) if and only if it is a stationary point of REG(0) [7, Proposition 4.1].

Consider the case where $\bar{z}$ is not S-stationary. Then $\bar{z}$ is not a stationary point of REG(0). When the algorithm converges to MPCC multipliers satisfying C-stationarity, then there exist indices $i \in \beta(\bar{z})$ such that $\bar{\lambda}_i^G < 0, \bar{\lambda}_i^H < 0$. According to (43), the NLP multipliers $v_i^{G,k} < 0, v_i^{H,k} < 0$ for k sufficiently large, which are not allowed in REG(ϵ). Since

$$v_i^{G,k} = \bar{\lambda}_i^G + v_i^{REG,k} H_i(z^k),$$
$$v_i^{H,k} = \bar{\lambda}_i^H + v_i^{REG,k} G_i(z^k),$$

the multipliers $v_i^{REG,k}$ become very large to enforce $v_i^{G,k}$ and $v_i^{H,k}$ nonnegative. At the same time, $G_i(z^k)$ and $H_i(z^k)$ are prevented from being very close to zero, otherwise $v_i^{REG,k} G_i(z^k)$ and $v_i^{REG,k} H_i(z^k)$ would be ineffective. As a consequence, it can be observed for k sufficiently large that $v_i^{G,k} = 0, v_i^{H,k} = 0, v_i^{REG,k} \to \infty$, and $G_i(z^k), H_i(z^k)$ cannot converge accurately to zero.

On the other hand, the multipliers for the programs BA(ϵ^k) and MLF(ϵ^k) do not have this difficulty. As indicated by the relations (31) and (39), there is no contradiction between the signs of the MPCC multipliers $\bar{\lambda}_i^G, \bar{\lambda}_i^H$ and of the NLP multipliers $u_i^{\Phi,k}$ and $u_{L,i}^{\Phi,k} - u_{U,i}^{\Phi,k}$. In addition, the underlying relation (42) indicates that the NLP multipliers exist whenever the MPCC multipliers do. Therefore, whether $\bar{z}$ is S-stationary or not has little influence on the performance of BA and MLF methods, which is an important difference from the behavior of the REG method.

Examples: Multiplier Comparisons

We review the examples in sections 3.1 and 5.1 to illustrate the difference in behavior between the NCP-based methods and REG regularization method.

As we discussed in the previous sections, the examples *scholtes4* and *ex9.2.2* have non-strongly stationary local minimizers. Numerical results of these two examples are presented in Tables 1 and 2. The results indicate that REG method gives rise to large NLP multipliers for the constraints corresponding to the biactive complementary components, and the multipliers get even larger when the regularization parameter ϵ becomes smaller; at the same time, the convergence is slow and inaccurate, compared to the magnitude of ϵ.

On the other hand, the multipliers of the NCP-based methods (BA and MLF) are well behaved. According to (42), their multipliers can be derived from the MPCC multipliers at a limit point and vice versa. In addition, the accuracy of their solutions is about O(ϵ).

ϵ	scholtes4	BA		MLF		REG		
10^{-6}		p	u^{Φ}	u_L^{Φ}	u_U^{Φ}	v^{z_1}	v^{z_2}	v^{REG}
	multipliers	0	-2	0	2	0	0	6.32E+2
	z_1	1.25E-6		1.25E-6		0.001581		
	z_2	1.25E-6		1.25E-6		0.001581		
	z_3	5E-6		5E-6		0.006324		
10^{-9}		p	u^{Φ}	u_L^{Φ}	u_U^{Φ}	v^{z_1}	v^{z_2}	v^{REG}
	multipliers	0	-2	0	2	0	0	1.87E+4
	z_1	1.25E-9		1.25E-9		0.000054		
	z_2	1.25E-9		1.25E-9		0.000054		
	z_3	5E-9		5E-9		0.000214		

Table 1: Results of scholtes4.

ϵ	ex9.2.2	BA		MLF		REG		
10^{-6}		p	u^{Φ}	u_L^{Φ}	u_U^{Φ}	v^{s_1}	v^{l_1}	v^{REG}
	multipliers	0	-5.74	0	5.74	0	0	1.83E+3
	s_1	9.6E-7		9.6E-7		0.000913		
	l_1	1.63E-6		1.63E-6		0.002739		
10^{-9}		p	u^{Φ}	u_L^{Φ}	u_U^{Φ}	v^{s_1}	v^{l_1}	v^{REG}
	multipliers	0	-5.63	0	5.63	0	0	5.71E+4
	s_1	9.12E-10		9.13E-10		0.000029		
	l_1	1.48E-9		1.49E-9		0.000088		

Table 2: Results of ex9.2.2.

6 Conclusions

This study explores characteristics of local minimizers of MPCCs and their influence on convergence behavior for NLP-based MPCC algorithms. First, we develop conditions for M-stationarity of a local minimizer of an MPCC, under the assumption of MPCC-GCQ (Theorem 2.2). A key point is that the M-stationarity is a piecewise property. For a local minimizer $\bar{z}$ at which MPCC-LICQ fails, there exist multiple sets of MPCC multipliers, each corresponding to one partition of $\beta(\bar{z})$ and satisfying M-stationarity on that partition.

Second, we aim to capture conditions that guarantee B-stationarity convergence. By applying the main results (D1), (D2), and (D3) of duality theory to the LPEC at a weakly stationary point of an MPCC, we prove that every weakly stationary point of an MPCC is B-stationary, provided that MPCC-ACQ holds at the point (Theorem 3.1). However, under a weaker assumption MPCC-GCQ, combinatorial checking with respect to the LPEC is generally required (Theorem 3.2).

To investigate convergence properties of the Bounding Algorithm we proposed in [22], in the absence of MPCC-LICQ, we consider stationarity of a limit point of this method based on local optimality of a sequence of NLP solutions approaching to it. We have established C-stationarity of a limit point by using attributes of the NCP function involved (Theorem 4.1), and M-stationarity by introducing additional assumptions on rank of the active gradients and on curvature of the reduced Hessian (Theorem 4.2). Further investigation from the perspective of an inequality variant of this algorithm provides a reason to simplify the LPEC when verifying B-stationarity of a limit point.

Finally, we discuss a few practical issues related to local minimizers of MPCCs which are not S-stationary. It has been illustrated that the solutions of the NCP-based methods (BA and MLF) usually provide MPCC multipliers that satisfy C-stationarity conditions at a non-strongly stationary solution (Section 5.1). On the other hand, the REG method, which is a typical regularization method, usually encounters unbounded NLP multipliers and inaccurate convergence when approaching a non-strongly stationary solution (Section 5.2). This analysis shows some advantages of NCP-based methods to converge to MPCC solutions that are not S-stationary.

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