

Exact Matrix Completion via High-Rank Matrices in Sum-of-Squares Relaxations

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November 25, 2023

Abstract

We study exact matrix completion from partially available data with hidden connectivity patterns. Exact matrix completion was shown to be possible recently by Cosse and Demanet in 2021 with Lasserre’s relaxation using the trace of the variable matrix as the objective function with given data structured in a chain format. In this study, we introduce a structure for the objective function so that the resulting sum-of-squares (SOS) relaxation, the dual of Lasserre’s SDP relaxation, produces a rank- $(N-1)$ solution, where N denotes the size of variable matrix in the SOS relaxation. Specifically, the arrowhead structure is employed for the coefficient matrix of the objective function. We show that a matrix can be exactly completed through the SOS relaxation when the connectivity of given data is not explicitly displayed or follows a chain format. The theoretical exactness is proved using the rank of the Gram matrix for the SOS relaxation. We also present numerical algorithms designed to find the coefficient matrix in the SOS relaxation. Numerical experiments illustrate the validity of the proposed algorithm.

Key words. Matrix completion, exact sum of squares of relaxations, arrowhead pattern matrix, linear combinations of given data, algorithms for matrix completion.

AMS Classification. 90C22, 90C25, 90C26.

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1 Introduction

Matrix completion is to find or estimate the entries of a matrix from partially given entries [7, 9, 15]. It has been a very widely studied subject in recent years for its wide-range applications such as compressed sensing [5, 8, 13], multi-class learning [2], dimension reduction [27]. A basic assumption in the study of matrix completion is that the matrix is low-rank or approximately low-rank [6, 7, 10, 16, 20, 26].

Recently, it was shown in [12] that the rank-1 matrix can be recovered by the Lasserre's semidefinite (SDP) relaxation under some assumptions on the given data. The problem of finding the rank-1 matrix $X = \mathbf{x}\mathbf{y}^T \in \mathbb{R}^{n \times m}$ with $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$ for noiseless case is written as

$$\begin{aligned} \text{find } & X \in \mathbb{R}^{n \times m} \\ \text{s.t. } & \text{rank}(X) = 1 \\ & X_{ij} = (X_0)_{ij} \quad (i, j) \in \Lambda, \end{aligned} \tag{1}$$

where Λ is a subset of $\{1, \dots, n\} \times \{1, \dots, m\}$ and $(X_0)_{ij}$ of $X_0 \in \mathbb{R}^{n \times m}$ for $(i, j) \in \Lambda$ are given entries.

The condition for the exact recovery in their work [12] was presented with the connected bipartite graph \mathcal{G} associated with X in (1) (see Lemma 1 in [12]). By letting $X = \mathbf{x}\mathbf{y}^T$, they assumed that at least one connected path in Λ for each variable in \mathbf{x} and \mathbf{y} from x_1 exist. Here, $X_{ij} = x_i y_j$ for $(i, j) \in \Lambda$ means that $x_i \rightarrow y_j$ or $y_j \rightarrow x_i$. For instance, for $i_L \in \{2, \dots, n\}$, a chain for x_{i_L} can be expressed as

$$x_1 \rightarrow y_{j_1} \rightarrow x_{i_1} \rightarrow y_{j_2} \rightarrow \dots \rightarrow y_{j_{L-1}} \rightarrow x_{i_{L-1}} \rightarrow y_{j_L} \rightarrow x_{i_L}$$

with $\{(1, j_1), (i_1, j_1), (i_1, j_2), \dots, (i_{L-1}, j_{L-1}), (i_{L-1}, j_L), (i_L, j_L)\} \subset \Lambda$. For each variable $y_{j_L} \in \{1, \dots, m\}$, we also have a chain that passes through or arrives at y_{j_L} . By expressing the variables \mathbf{x} and \mathbf{y} as one vector $\mathbf{z} := [x_2, x_3, \dots, x_n, y_1, y_2, \dots, y_m]$, the chain can be written with \mathbf{z} as

$$z_0 \rightarrow z_{i_1} \rightarrow z_{i_2} \rightarrow z_{i_3} \rightarrow \dots \rightarrow z_{i_{L-1}} \rightarrow z_{i_L}, \tag{2}$$

where $z_0 := x_1 = 1$. The vector \mathbf{z} includes the variables from $x_2 (= z_1)$, thus, the length of \mathbf{z} (the number of variables in (1)) is $s := n + m - 1$.

In their work [12], the rank of the Gram matrix for the sum of squares (SOS) relaxation [25], the dual of the Lasserre's SDP relaxation [22], was shown to be $N - 1$ using the condition (2), where N is the size of the matrix. As a result, the rank-1 solution was obtained by the Lasserre's SDP relaxation. We should mention that their SDP relaxation for (1) minimized the trace of the variable matrix, and each constraint $X_{ij} = (X_0)_{ij}$ $(i, j) \in \Lambda$ of (1) involves only one element of X . For instance, we consider the constraints with $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{y} \in \mathbb{R}^3$ and $X_{ij} = x_i y_j$:

$$\begin{aligned} h_1(\mathbf{z}) = y_1 - (X_0)_{11} &= 0, \quad h_2(\mathbf{z}) = x_2 y_1 - (X_0)_{21} = 0, \quad h_3(\mathbf{z}) = x_2 y_2 - (X_0)_{22} = 0, \\ h_4(\mathbf{z}) = x_3 y_1 - (X_0)_{31} &= 0, \quad h_5(\mathbf{z}) = x_3 y_3 - (X_0)_{33} = 0. \end{aligned} \tag{3}$$

The chain structure corresponding to (3) can be represented as

$$\begin{array}{ccccccc} x_1 & \longrightarrow & y_1 & \longrightarrow & x_2 & \longrightarrow & y_2 \\ & & & & & \searrow & \\ & & & & & & x_3 & \longrightarrow & y_3. \end{array}$$

If data is given as a linear combination of $h_k(\mathbf{z}) = 0$ ($k = 1, \dots, 5$), for example,

$$\begin{aligned}\bar{h}_1(\mathbf{z}) &= 6y_1 - 8x_2y_1 + 8x_2y_2 + 9x_3y_1 - 10x_3y_3 - \bar{b}_1 = 0, \\ \bar{h}_2(\mathbf{z}) &= 5x_2y_1 + 9x_2y_2 - \bar{b}_2 = 0,\end{aligned}$$

for some $\bar{b}_1, \bar{b}_2 \in \mathbb{R}$, then we need to find a chain that runs through \mathbf{z} from the given data to see if the problem can be exactly solved. The underlying connectivity is called the hidden connectivity in this paper. Data with the hidden connectivity may contain more or less information for a chain. For the above instance, $\bar{h}_1(\mathbf{z})$ and $\bar{h}_2(\mathbf{z})$ may contain some or all monomials $x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_2y_3, x_3y_1, x_3y_2, x_3y_3$ with $x_1 = 1$. In this case, a chain can be constructed with $x_1y_1, x_2y_1, x_2y_2, x_3y_1, x_3y_3$. The method in [12] for constructing the SOS relaxation separately dealt with each monomial depending on their degree since each constraint in (1) involves only one variable X_{ij} . Their method cannot deal with data expressed as a linear combination of $h_k(\mathbf{z}) = 0$.

In this paper, we first prove exact matrix completion for the given data with a chain structure (2) by an SOS relaxation, the dual of the Lasserre's SDP relaxation. Then, we show that the matrix completion problem with the constraints expressed as linear combinations of $X_{ij} - (X_0)_{ij}$ ($i, j \in \Lambda$) in (1) can be exactly solved by the SOS relaxation. Our approach is to solve a minimization problem by employing the objective function with the arrowhead sparsity pattern [24] and converting the equality constraints to inequality constraints. We demonstrate that the objective function for the minimization problem can be constructed such that it includes the given data and leads to the exact optimal solution. We also present an algorithm for constructing the objective function with the given data.

1.1 Main contributions

Our main contributions are:

- While the problem (1) has been studied as an optimization problem with various formulations, there has been limited investigation into the sparsity pattern on the objective function, which may be crucial for the exact recovery of the matrix. The arrowhead sparsity structure on the objective function is employed to achieve the exact recovery of the matrix.
- For data with hidden connectivity, we show that our proposed method can successfully complete the matrix. This is attained through converting equality constraints into polynomial inequality constraints of degree up to 4, which effectively reduces the degree of the dual variables in the SOS relaxation.
- We also introduce a two-stage algorithm. The initial stage involves the determination of the objective function, implicitly incorporating the data provided by the constraints, utilizing the arrowhead structured elementary matrix. Subsequently, in the second stage, we solve the SOS relaxation with the objective function obtained in the first stage and obtain an exact solution.

1.2 Outline of the paper

Our paper is organized as follows: In Section 2, we introduce some notation and symbols used in this paper and describe the matrix completion problem with an example. The problem formulation in [12] is also described with its SOS relaxation. In Section 3, we present our problem formulation using the method in [18, 21]. In Section 4, we show theoretical results on the exact matrix recovery for data with a chain structure and hidden connectivity. In particular, we construct the objective function with the arrowhead sparsity structure in Section 4.1. The framework of the proposed algorithm is described in Section 5.1 and numerical results are presented in Section 5.2. We conclude in Section 6.

2 Preliminaries

2.1 Notation and symbols

Let \mathbb{N}^s and \mathbb{R}^s be the sets of the s -dimensional nonnegative integers and real numbers, respectively. For $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{N}^s$, a monomial of \mathbf{z} is denoted as $\mathbf{z}^\alpha := \prod_{i=1}^s z_i^{\alpha_i}$ and its degree $\sum_{i=1}^s \alpha_i$. A vector of all monomials \mathbf{z}^α of degree d or less is described as

$$\mathbf{u}_d(\mathbf{z}) := [\mathbf{z}^\alpha]_{\sum_{i=1}^s \alpha_i \leq d} = [1 \quad z_1 \quad \dots \quad z_s \quad z_1^2 \quad z_1 z_2 \quad \dots \quad z_s^d] \in \mathbb{R}^{s_d},$$

where $s_d = \binom{s+d}{d}$. We frequently write $\mathbf{u}_d(\mathbf{z})$ as \mathbf{u}_d , in particular,

$$\begin{aligned} \mathbf{u}_1 &= [1, \mathbf{z}] := [1, x_2, x_3, \dots, x_n, y_1, y_2, \dots, y_m] \in \mathbb{R}^{s+1}, \\ \mathbf{u}_2 &= [1, x_2, \dots, x_n, y_1, \dots, y_m, x_2 x_3, x_2 x_4, \dots, y_{m-1} y_m, y_m^2] \in \mathbb{R}^N, \end{aligned} \quad (4)$$

where $N = \binom{n+m+1}{2}$.

The set of all polynomials in variable vector \mathbf{z} is denoted by $\mathbb{R}[\mathbf{z}]$, and the set of all polynomial of degree at most d by $\mathbb{R}_d[\mathbf{z}]$, the set of all *nonnegative* polynomials in $\mathbb{R}[\mathbf{z}]$ by $\mathcal{N}[\mathbf{z}]$. A polynomial $p(\mathbf{z}) \in \mathcal{N}[\mathbf{z}]$ is an SOS polynomial if and only if there exist r polynomials $q_1(\mathbf{z}), \dots, q_r(\mathbf{z}) \in \mathbb{R}[\mathbf{z}]$ such that $p(\mathbf{z}) = \sum_{j=1}^r q_j(\mathbf{z})^2$. The set of SOS polynomials of degree $2d$ in $\mathcal{N}[\mathbf{z}]$ is denoted by $\Sigma_{2d}[\mathbf{z}]$ and its precise definition is given by

$$\Sigma_{2d}[\mathbf{z}] := \left\{ \sum_{j=1}^r q_j(\mathbf{z})^2 : r \geq 1, q_j(\mathbf{z}) \in \mathbb{R}_d[\mathbf{z}] \right\}.$$

Obviously, $\Sigma_{2d}[\mathbf{z}] \subseteq \mathcal{N}[\mathbf{z}] \subseteq \mathbb{R}[\mathbf{z}]$ holds. An SOS relaxation by SOS polynomials in $\Sigma_{2d}[\mathbf{z}]$ is denoted by SOS_{2d} . We also use $\Sigma[\mathbf{z}]$ defined by

$$\Sigma[\mathbf{z}] := \left\{ \sum_{j=1}^r q_j(\mathbf{z})^2 : r \geq 1, q_j(\mathbf{z}) \in \mathbb{R}[\mathbf{z}] \right\}.$$

Lemma 2.1. [11] *A polynomial $p(\mathbf{z}) \in \mathbb{R}_{2d}[\mathbf{z}]$ is an SOS polynomial in $\Sigma_{2d}[\mathbf{z}]$ if and only if there exists $W \in \mathbb{S}_+^{s_d}$ such that*

$$p(\mathbf{z}) = \mathbf{u}_d(\mathbf{z})^T W \mathbf{u}_d(\mathbf{z}),$$

where \mathbb{S}^{s_d} and $\mathbb{S}_+^{s_d}$ denote the sets of symmetric and positive semidefinite matrices of size s_d , respectively. Such a matrix W is called the Gram matrix of $p(\mathbf{z})$.

A positive semidefinite matrix R is also denoted by $R \succeq O$ or $R \in \mathbb{S}_+$. A polynomial matrix is a matrix whose elements are polynomials in $\mathbb{R}[\mathbf{z}]$. Let $\mathbb{R}^{n \times m}[\mathbf{z}]$ denote the set of all $n \times m$ polynomial matrices. For $M \in \mathbb{S}^{s_d}$, we use M_I to denote the submatrix of M constructed by the rows and columns of M indexed by $I \subset \{1, \dots, s_d\}$.

The graph $\mathcal{G} = (V, E)$ denotes an undirected graph with the vertex set V and the edge set E . In particular, a bipartite graph is a graph $\mathcal{G} = (V, E)$ in which V can be divided to two subsets $L, R \subseteq V$ such that $L \cap R = \emptyset$ and every edge connects a vertex in L and one in R . It is also be expressed by $\mathcal{G} = (L, R, E)$.

2.2 The rank-1 matrix completion problem

To describe the rank-1 matrix completion problem with partial data, an illustrative example is presented:

Example 2.2.

$$X_{\text{part}} = \begin{bmatrix} 7 & 3 & ? \\ -35 & ? & 10 \\ ? & 9 & ? \end{bmatrix} \quad (5)$$

The goal of finding the rank-1 matrix completion here is to find $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$ from (5), where $X = \mathbf{x}\mathbf{y}^T \in \mathbb{R}^{n \times m}$ with $x_1 = 1$.

If given data has a unique solution, any elements in a partial matrix X_{part} can be computed. In the above example, we notice that the second row of X_{part} is obtained by multiplying the first row by -5 , thus, we see that the (2,2)nd element of X_{part} must be -15 .

We can formulate the matrix completion of (5) as:

$$\begin{aligned} \text{find } & \mathbf{z} := [x_2, \dots, x_n, y_1, \dots, y_m]^T \\ \text{s.t. } & y_1 = 7, y_2 = 3, x_2y_1 = -35, x_2y_3 = 10, x_3y_2 = 9. \end{aligned}$$

Let $\mathbf{x}_0 \in \mathbb{R}^n$ and $\mathbf{y}_0 \in \mathbb{R}^m$ denote unknown true values of the rank-1 decomposition $X_0 = \mathbf{x}_0\mathbf{y}_0^T$ of X , and $\mathbf{z}_0 := [(x_0)_2, \dots, (x_0)_n, (y_0)_1, \dots, (y_0)_m]^T \in \mathbb{R}^s$. Obviously, if \mathbf{z}_0 contains a zero element, then either one of the rows or columns consists entirely of zeros, and some variables in z_i may not be connected to x_1 through a chain in Λ . When chains are disconnected, we can decompose the problem into subproblems corresponding to the disconnected components. We can then apply the discussion in this paper to each of these components. Thus, the following is assumed without loss of generality.

Assumption 2.3. *There are no zeros in the true value vectors \mathbf{x}_0 and \mathbf{y}_0 , i.e., \mathbf{z}_0 .*

To express the matrix completion problem in general, we define a function $h_k(\mathbf{z}) := x_i y_j - (\mathbf{x}_0)_i (\mathbf{y}_0)_j$ where $i \in \{2, \dots, n\}, j \in \{1, \dots, m\}, k = 1, \dots, K$ and K is the number of partially given data, i.e., the cardinality of Λ . Then, the matrix completion problem can be written as

$$\begin{aligned} \text{find } & \mathbf{z} \\ \text{s.t. } & h_k(\mathbf{z}) = 0 \quad (k = 1, \dots, K). \end{aligned} \quad (MC)$$

The problem can also be described as a minimization problem with some continuous objective function $h_0(\mathbf{z})$, for instance, $h_0(\mathbf{z}) = \sum_{i=2}^n x_i^2 + \sum_{j=1}^m y_j^2$:

$$\begin{aligned} \min \quad & h_0(\mathbf{z}) \\ \text{s.t.} \quad & h_k(\mathbf{z}) = 0 \quad (k = 1, \dots, K). \end{aligned} \tag{MC^1}$$

Let F be the feasible region of (MC^1) :

$$F = \{\mathbf{z} \in \mathbb{R}^s : h_k(\mathbf{z}) = 0 \ (k = 1, \dots, K)\}.$$

We assume that F is nonempty and bounded. Then, (MC^1) has a finite optimal value ζ^* at an optimal solution $\mathbf{z}^* \in F$.

It was shown in [12] that problem (MC) can be solved exactly by the Lasserre's SDP relaxation. They proved in [12, Proposition 3] that for the unique recovery of X_0 , it is sufficient, in addition to the injectivity of the constraints, to find an SOS polynomial $\sum_{j=1}^r q_j(\mathbf{z})^2 \in \Sigma_4[\mathbf{z}]$ for some $r \geq 1$ with the rank of the associated Gram matrix $N - 1$, polynomials $\lambda_k(\mathbf{z}) \in \mathbb{R}_{2d-d_k}[\mathbf{z}]$ with $d = 2$, and $\rho \in \mathbb{R}$ such that

$$q(\mathbf{z}) = \sum_{\alpha \in \mathbb{N}_2^N} \mathbf{z}^{2\alpha} - \rho + \sum_{k=1}^K \lambda_k(\mathbf{z}) h_k(\mathbf{z}) \quad \text{with } q(\mathbf{z}_0) = 0,$$

where $\mathbb{N}_2^N := \{\alpha \in \mathbb{N}^N \mid 1 \leq \sum_{i=1}^N \alpha_i \leq 2\}$. Since $\sum_{\alpha \in \mathbb{N}_d^N} \mathbf{z}^{2\alpha} = (\mathbf{u}_d)^T \mathbf{u}_d$, the SOS_{2d} relaxation in [12], with $d = 2$, can be written as

$$\begin{aligned} \max_{\rho, \lambda} \quad & \rho \\ \text{s.t.} \quad & q(\mathbf{z}) := (\mathbf{u}_d)^T \mathbf{u}_d - \rho + \sum_{k=1}^K h_k(\mathbf{z}) \lambda_k(\mathbf{z}) \in \Sigma_{2d}[\mathbf{z}] \\ & \lambda_k(\mathbf{z}) \in \mathbb{R}_{2d-d_k}[\mathbf{z}] \quad (k = 1, \dots, K). \end{aligned} \tag{6}$$

3 Problem formulation

We describe our problem formulation for (MC) in this section. The primary goal of our formulation is to handle the given data with hidden connectivity for the exact matrix recovery and to efficiently compute $\lambda(\mathbf{z})$. We employ the objective function $Q \in \mathbb{S}^N$ with the arrowhead structured matrix [24] so that the rank of the Gram matrix for $q(\mathbf{z})$ becomes $N - 1$. Moreover, we replace $h_k(\mathbf{z}) = 0$ ($k = 1, \dots, K$) in (MC) with $h_k(\mathbf{z})^2 \leq 0$.

3.1 Matrix completion problem with connectivity

Since $h_k(\mathbf{z}) = 0$ is equivalent to an inequality $h_k(\mathbf{z})^2 \leq 0$ for any $\mathbf{z} \in \mathbb{R}^s$ and $k = 1, \dots, K$, the matrix completion (MC^1) can be equivalently expressed as

$$\begin{aligned} \min \quad & h_0(\mathbf{z}) \\ \text{s.t.} \quad & -h_k(\mathbf{z})^2 \geq 0 \quad (k = 1, \dots, K). \end{aligned} \tag{MC^2}$$

We define the Lagrangian function $\mathcal{L}(\mathbf{z}, \boldsymbol{\lambda}) : \mathbb{R}^s \times (\Sigma[\mathbf{z}])^K \rightarrow \mathbb{R}$ for (MC^2)

$$\mathcal{L}(\mathbf{z}, \boldsymbol{\lambda}) = h_0(\mathbf{z}) + \sum_{k=1}^K h_k(\mathbf{z})^2 \lambda_k(\mathbf{z}). \quad (7)$$

Since F corresponds to the feasible region in (MC^2) , it holds that for any $\boldsymbol{\lambda} \in (\Sigma[\mathbf{z}])^K$,

$$\mathcal{L}(\mathbf{z}, \boldsymbol{\lambda}) \leq h_0(\mathbf{z}) \quad \text{if } \mathbf{z} \in F.$$

We define a Lagrangian dual of (MC^2) :

$$\max \mathcal{L}^*(\boldsymbol{\lambda}) \quad \text{s.t. } \boldsymbol{\lambda} \in (\Sigma[\mathbf{z}])^K, \quad (8)$$

where

$$\mathcal{L}^*(\boldsymbol{\lambda}) = \inf \{ \mathcal{L}(\mathbf{z}, \boldsymbol{\lambda}) : \mathbf{z} \in \mathbb{R}^s \}.$$

Let η be the maximum of the degree of $h_k(\mathbf{z})$, $k = 0, 1, \dots, K$ in (MC^2) , and we take $d \in \mathbb{N}$ such that $d \geq \lceil \eta/2 \rceil$. We also let $2d_k$ be the degree of $h_k(\mathbf{z})^2$. Then, we consider a subproblem of the Lagrangian dual (8):

$$\max \mathcal{L}^*(\boldsymbol{\lambda}) \quad \text{s.t. } \lambda_k(\mathbf{z}) \in \Sigma_{2d-2d_k} \quad (k = 1, \dots, K). \quad (9)$$

We can rewrite the problem (9) as

$$\begin{aligned} \max_{\rho, \boldsymbol{\lambda}} \quad & \rho \\ \text{s.t.} \quad & \mathcal{L}(\mathbf{z}, \boldsymbol{\lambda}) - \rho \geq 0 \quad \text{for every } \mathbf{z} \in \mathbb{R}^s, \\ & \lambda_k(\mathbf{z}) \in \Sigma_{2d-2d_k} \quad (k = 1, \dots, K). \end{aligned}$$

By replacing the inequality constraints $\mathcal{L}(\mathbf{z}, \boldsymbol{\lambda}) - \rho \geq 0$ ($\forall \mathbf{z} \in \mathbb{R}^s$) with an SOS condition $\mathcal{L}(\mathbf{z}, \boldsymbol{\lambda}) - \rho \in \Sigma_{2d}$, we obtain an SOS relaxation of (MC^2) :

$$\begin{aligned} \max_{\rho, \boldsymbol{\lambda}} \quad & \rho \\ \text{s.t.} \quad & \mathcal{L}(\mathbf{z}, \boldsymbol{\lambda}) - \rho \in \Sigma_{2d} \quad \text{for every } \mathbf{z} \in \mathbb{R}^s, \\ & \lambda_k(\mathbf{z}) \in \Sigma_{2d-2d_k} \quad (k = 1, \dots, K). \end{aligned}$$

For $Q \in \mathbb{S}^N$, we let $h_0(\mathbf{z}) = (\mathbf{u}_2)^T Q \mathbf{u}_2$ where \mathbf{u}_2 is given by (4). Then the resulting SOS relaxation of (MC^2) with $d = 2$ is

$$\begin{aligned} \max_{\rho, \boldsymbol{\lambda}} \quad & \rho \\ \text{s.t.} \quad & q(\mathbf{z}) := (\mathbf{u}_2)^T Q \mathbf{u}_2 - \rho + \sum_{k=1}^K h_k(\mathbf{z})^2 \lambda_k(\mathbf{z}) \in \Sigma_4[\mathbf{z}], \\ & \lambda_k(\mathbf{z}) \in \Sigma_{4-2d_k}[\mathbf{z}] \quad (k = 1, \dots, K). \end{aligned} \quad (10)$$

Reducing (10) to a semidefinite program is based on Lemma 2.1. By applying Lemma 2.1 to $q(\mathbf{z})$ and $h_k(\mathbf{z})^2 \lambda_k(\mathbf{z})$, we obtain a semidefinite program for (10). See [18] for details on the reduction.

We derive Lasserre's SDP relaxation [22] of (MC^2) based on the interpretation in [21]. Let $\nu_k = d - d_k$ ($k = 1, \dots, K$). The following polynomial optimization problem on positive semidefinite cones (a nonlinear semidefinite program) is equivalent to (MC^2) :

$$\begin{aligned} \min \quad & \mathbf{u}_d(\mathbf{z})^T Q \mathbf{u}_d(\mathbf{z}) \\ \text{s.t.} \quad & -h_k(\mathbf{z})^2 \mathbf{u}_{\nu_k}(\mathbf{z}) \mathbf{u}_{\nu_k}(\mathbf{z})^T \succeq O \quad (k = 1, \dots, K), \\ & \mathbf{u}_d(\mathbf{z}) \mathbf{u}_d(\mathbf{z})^T \succeq O. \end{aligned} \quad (11)$$

By applying the linearization in [21] to (11), we obtain a linear semidefinite program, which is the dual of the semidefinite program obtained from (10). If d is taken to be 2, then (11) corresponds to the Lasserre's SDP relaxation of (MC^2) . We call the linear SDP problems obtained through linearizing (11) and (10) with $d = 2$ *the primal SDP relaxation* and *the dual SDP relaxation* of (MC^2) , respectively.

In our formulation (10), the objective function $\mathbf{u}_d(\mathbf{z})^T Q \mathbf{u}_d(\mathbf{z})$ is employed instead of $\mathbf{u}_d(\mathbf{z})^T \mathbf{u}_d(\mathbf{z}) = \mathbf{u}_d(\mathbf{z})^T I \mathbf{u}_d(\mathbf{z})$ in [12], where I denotes the identity matrix. The problem (10) can be considered as a generalization of (6) in the sense that the coefficient matrix Q of the objective function can be chosen to be any symmetric matrix. We select the structure of Q in a manner that ensures the existence of an exact solution. The construction of Q with the arrowhead sparsity pattern is discussed in Lemma 4.1 and Theorem 4.2.

3.2 Matrix completion problem with hidden connectivity

Let $\mathbf{h}(\mathbf{z}) = [h_1(\mathbf{z}), h_2(\mathbf{z}), \dots, h_K(\mathbf{z})]^T$ be a column vector of the constraints $x_i y_j - (\mathbf{x}_0)_i (\mathbf{y}_0)_j = 0$. We consider a case where $h_k(\mathbf{z})$ includes several terms such as $x_i y_j - (\mathbf{x}_0)_i (\mathbf{y}_0)_j$, $j = 1, \dots, \bar{j}$ for some $\bar{j} > 1$, a case more general than the previous subsection. In such a case, the given data may not display the chain format, instead, they appear as a linearly combination of data $x_i y_j - (\mathbf{x}_0)_i (\mathbf{y}_0)_j$, $j = 1, \dots, \bar{j}$, as mentioned in Section 1.

A linear combination of $\mathbf{h}(\mathbf{z})$ can be expressed as $\bar{\mathbf{h}}(\mathbf{z}) = C \mathbf{h}(\mathbf{z})$ with some matrix $C \in \mathbb{R}^{\bar{K} \times K}$. We assume that $C^T C$ is invertible for the connectivity of the data. Then the given data of the problem with hidden connectivity can be represented by $\bar{h}_{\bar{k}}(\mathbf{z}) = 0$ ($\bar{k} = 1, \dots, \bar{K}$).

As in (MC^2) , we consider $(\bar{h}_{\bar{k}}(\mathbf{z}))^2 \leq 0$ for $\bar{h}_{\bar{k}}(\mathbf{z}) = 0$. The resulting problem is

$$\begin{aligned} \min \quad & h_0(\mathbf{z}) \\ \text{s.t.} \quad & -(\bar{h}_{\bar{k}}(\mathbf{z}))^2 \geq 0 \quad (\bar{k} = 1, \dots, \bar{K}), \end{aligned} \quad (MC^3)$$

and an SOS relaxation of (MC^3) is

$$\begin{aligned} \max_{\rho, \lambda} \quad & \rho \\ \text{s.t.} \quad & (\mathbf{u}_2)^T Q \mathbf{u}_2 - \rho + \sum_{\bar{k}=1}^{\bar{K}} (\bar{h}_{\bar{k}}(\mathbf{z}))^2 \bar{\lambda}_{\bar{k}}(\mathbf{z}) \in \Sigma_4[\mathbf{z}], \\ & \bar{\lambda}_{\bar{k}}(\mathbf{z}) \in \Sigma_{4-2d_{\bar{k}}}[\mathbf{z}] \quad (\bar{k} = 1, \dots, \bar{K}). \end{aligned} \quad (12)$$

4 Theoretical results

In this section, we construct Q in (10) in Lemma 4.1 and Theorem 4.2 and provide a theoretical proof for the exact recovery with the SOS relaxation (10). In Section 4.2, we show that Q constructed in Section 4.1 implicitly includes the given data with $\mathbf{h}(\mathbf{z})$. The problem with hidden connectivity is discussed in Section 4.3.

4.1 Exact recovery with SOS relaxation (10)

We define the constraint of the SOS relaxation (10) as

$$q(\mathbf{z}; Q; \rho, \boldsymbol{\lambda}) := (\mathbf{u}_2)^\top Q \mathbf{u}_2 - \rho + \sum_{k=1}^K h_k(\mathbf{z})^2 \lambda_k(\mathbf{z}) \in \Sigma_4[\mathbf{z}]. \quad (13)$$

Since $q(\mathbf{z}; Q; \rho, \boldsymbol{\lambda}) \in \Sigma_4[\mathbf{z}]$, there exists a positive semidefinite matrix $\Gamma \in \mathbb{S}_+^N$ such that $q(\mathbf{z}; Q; \rho, \boldsymbol{\lambda}) = (\mathbf{u}_2)^\top \Gamma (\mathbf{u}_2)$. If the rank of Γ is $N - 1$, the positive semidefiniteness of Γ guarantees that there exists a unique vector \mathbf{u}_2 such that $(\mathbf{u}_2)^\top \Gamma (\mathbf{u}_2) = 0$ and $(\mathbf{u}_2)_1 = 1$. This implies that there exists a unique point $\bar{\mathbf{z}} \in \mathbb{R}^s$ such that $q(\bar{\mathbf{z}}; Q; \rho, \boldsymbol{\lambda}) = 0$. When searching for a point $\bar{\mathbf{z}}$ that leads $q(\bar{\mathbf{z}}; Q; \rho, \boldsymbol{\lambda}) = 0$, a natural candidate is a vector consisting of the true data values, denoted by \mathbf{z}_0 . Let $(\mathbf{u}_2)_0 := \mathbf{u}_2(\mathbf{z}_0)$. Since $\mathbf{h}(\mathbf{z}_0) = 0$, $\sum_{k=1}^K h_k(\mathbf{z}_0)^2 \lambda_k(\mathbf{z}_0) = 0$ in (13). Thus,

$$0 = q(\mathbf{z}_0; Q; \rho, \boldsymbol{\lambda}) = (\mathbf{u}_2)_0^\top Q (\mathbf{u}_2)_0 - \rho = (\mathbf{u}_2)_0^\top \Gamma (\mathbf{u}_2)_0,$$

which implies

$$(\mathbf{u}_2)_0^\top Q (\mathbf{u}_2)_0 - \rho = 0. \quad (14)$$

Therefore, we can take $\bar{\mathbf{z}} = \mathbf{z}_0$.

For the uniqueness of $\bar{\mathbf{z}}$, the rank of Γ needs to be $N - 1$. Thus, Q must be chosen such that the rank of Q is $N - 1$. Recall that $Q \in \mathbb{S}^N$ can be any symmetric matrix in (10). Specifically, we utilize the arrowhead sparsity pattern for the rank of Q to be $N - 1$.

To construct Q whose rank is $N - 1$ using (14), we first consider $(\mathbf{u}_2)^\top Q \mathbf{u}_2 - (\mathbf{u}_2)_0^\top Q (\mathbf{u}_2)_0$ in (13). Let $E^r \in \mathbb{S}^N$ be a diagonal matrix whose (r, r) th element is one and all others are zeros.

Lemma 4.1. *For any $r \in \{2, \dots, N\}$, there exists $Q^r \in \mathbb{S}_+^N$ such that*

$$(\mathbf{u}_2)^\top Q^r \mathbf{u}_2 - (\mathbf{u}_2)_0^\top Q^r (\mathbf{u}_2)_0 = (\mathbf{u}_2 - (\mathbf{u}_2)_0)^\top E^r (\mathbf{u}_2 - (\mathbf{u}_2)_0). \quad (15)$$

Proof. We define two monomials $z_{i(r)}$ and $z_{j(r)}$ of degree 0 or 1 so that the r th component of \mathbf{u}_2 corresponds to the product of $z_{i(r)}$ and $z_{j(r)}$, i.e., $(\mathbf{u}_2)_r = z_{i(r)} z_{j(r)}$. Let $Q^r \in \mathbb{S}_+^N$ be a matrix with only four nonzero elements in

$$[Q^r]_{11} = ((\mathbf{z}_0)_{i(r)})^2 ((\mathbf{z}_0)_{j(r)})^2, \quad [Q^r]_{1r} = [Q^r]_{r1} = -(\mathbf{z}_0)_{i(r)} (\mathbf{z}_0)_{j(r)}, \quad [Q^r]_{rr} = 1. \quad (16)$$

Then, we have

$$\begin{aligned} (\mathbf{u}_2)_0^\top Q^r (\mathbf{u}_2)_0 &= ((\mathbf{z}_0)_{i(r)})^2 ((\mathbf{z}_0)_{j(r)})^2 - 2(\mathbf{z}_0)_{i(r)} (\mathbf{z}_0)_{j(r)} (\mathbf{z}_0)_{i(r)} (\mathbf{z}_0)_{j(r)} + ((\mathbf{z}_0)_{i(r)})^2 ((\mathbf{z}_0)_{j(r)})^2 \\ &= 0. \end{aligned} \quad (17)$$

By (16) and (17),

$$\begin{aligned} (\mathbf{u}_2)^\top Q^r \mathbf{u}_2 - (\mathbf{u}_2)_0^\top Q^r (\mathbf{u}_2)_0 &= z_{i(r)}^2 z_{j(r)}^2 - 2(\mathbf{z}_0)_{i(r)} (\mathbf{z}_0)_{j(r)} z_{i(r)} z_{j(r)} + ((\mathbf{z}_0)_{i(r)})^2 ((\mathbf{z}_0)_{j(r)})^2 \\ &= (\mathbf{u}_2 - (\mathbf{u}_2)_0)^\top E^r (\mathbf{u}_2 - (\mathbf{u}_2)_0). \end{aligned}$$

□

Next, we show that for any $\boldsymbol{\lambda}$, there exists a matrix Q such that the rank of the Gram matrix for $q(\mathbf{z}; Q; \rho, \boldsymbol{\lambda})$ is of $N - 1$.

Theorem 4.2. *Suppose that Assumption 2.3 holds. Then, there exist $Q \in \mathbb{S}_+^N, \bar{E} \in \mathbb{S}_+^N, \Gamma \in \mathbb{S}_+^N$ such that*

$$(\mathbf{u}_2)^\top Q \mathbf{u}_2 - (\mathbf{u}_2)_0^\top Q (\mathbf{u}_2)_0 = (\mathbf{u}_2 - (\mathbf{u}_2)_0)^\top \bar{E} (\mathbf{u}_2 - (\mathbf{u}_2)_0) = (\mathbf{u}_2)^\top \Gamma \mathbf{u}_2, \quad (18)$$

and the ranks of Q, \bar{E} and Γ are all $N - 1$. As a result, $(\mathbf{u}_2)^\top Q \mathbf{u}_2 - (\mathbf{u}_2)_0^\top Q (\mathbf{u}_2)_0$ is an SOS polynomial. In particular, Q can be constructed with the arrowhead sparsity pattern.

Proof. By Lemma 4.1, there exists $Q^r \in \mathbb{S}_+^n$ satisfying (15) for $r \in \{2, \dots, N\}$. We can obtain the first equality of (18) by letting $Q = \sum_{r=2}^N Q^r$ and $\bar{E} = \sum_{r=2}^N E^r$. Obviously, $\text{rank } \bar{E} = N - 1$. Next, we check the rank of the matrix Q . Since each Q^r in Lemma 4.1 is a rank-1 matrix, the subadditivity of the rank implies $\text{rank } Q \leq N - 1$. By (15), the submatrix $Q_{\{2, \dots, N\}}$ is a diagonal matrix with ones on the diagonal. Hence, it follows $\text{rank } Q = N - 1$. Furthermore, we can take $\Gamma = Q$ by (17), and this implies that $\text{rank } \Gamma = N - 1$. □

It was shown in [3] that the exact solution of quadratic programs with quadratic constraints (QCQPs) can be obtained when the aggregated sparsity pattern of the QCQP forms a tree. We mention that the arrowhead sparsity pattern forms a tree.

The following remark is a consequence of Theorem 4.2 for $\rho = (\mathbf{u}_2)_0^\top Q (\mathbf{u}_2)_0$ and $\boldsymbol{\lambda} = \mathbf{0}$.

Remark 4.3. *Assume that $\sum_{k=1}^K h_k(\mathbf{z})^2 \lambda_k(\mathbf{z})$ with $\boldsymbol{\lambda} = \mathbf{0}$ are added to the left hand side of (18). Then, we have*

$$(\mathbf{u}_2)^\top Q \mathbf{u}_2 - (\mathbf{u}_2)_0^\top Q (\mathbf{u}_2)_0 + \sum_{k=1}^K h_k(\mathbf{z})^2 \lambda_k(\mathbf{z}) = (\mathbf{u}_2 - (\mathbf{u}_2)_0)^\top \bar{E} (\mathbf{u}_2 - (\mathbf{u}_2)_0). \quad (19)$$

Then, it can be viewed that the results in Theorem 4.2 holds with the choice of $\boldsymbol{\lambda} = \mathbf{0}$ in (19). In other words, when considering (MC^2) for any rank-1 matrix completion problem, there always exists $Q \in \mathbb{S}_+^N$ of rank exactly $N - 1$ guaranteeing the existence of $\rho \in \mathbb{R}$ and $\lambda_k(\mathbf{z}) \in \Sigma_{4-2d_k}[\mathbf{z}]$ such that $q(\mathbf{z}; Q; \rho, \boldsymbol{\lambda}) \in \Sigma_4[\mathbf{z}]$. Moreover, $q(\mathbf{z}; Q; \rho, \boldsymbol{\lambda})$ can be represented with a rank- $(N - 1)$ Gram matrix.

For nonzero $\lambda_k(\mathbf{z})$, we show that $h_k(\mathbf{z})^2\lambda_k(\mathbf{z})$ does not decrease the rank of the Gram matrix of (13).

Theorem 4.4. *Let Q^r be an $N \times N$ symmetric positive semidefinite matrix defined by (16), and $Q := \sum_{r=2}^N Q^r$. For any $\rho \in \mathbb{R}$ and $\lambda_k(\mathbf{z}) \in \Sigma_{4-2d_k}$ ($k = 1, \dots, K$) such that $q(\mathbf{z}; Q; \rho, \boldsymbol{\lambda})$ is an SOS polynomial, the rank of the Gram matrix $\Gamma \in \mathbb{S}_+^N$ for $q(\mathbf{z}; Q; \rho, \boldsymbol{\lambda})$, i.e., $(\mathbf{u}_2)_2^\top \Gamma \mathbf{u}_2 = q(\mathbf{z}; Q; \rho, \boldsymbol{\lambda})$, is greater than or equal to $(N - 1)$.*

Proof. In the submatrix $Q_{\{2, \dots, N\}}$ that includes the second to the N th rows and columns of Q , the diagonal elements are positive numbers and all the other elements are zeros by (16). Hence, the submatrix $Q_{\{2, \dots, N\}}$ is a positive definite matrix. For any $k = 1, \dots, K$, the polynomial $h_k^2(\mathbf{z})\lambda_k(\mathbf{z}) \in \Sigma(\mathbf{z})$ from $\lambda_k(\mathbf{z}) \in \Sigma_{4-2d_k}(\mathbf{z})$ and $h_k^2(\mathbf{z}) \in \Sigma_{2d_k}(\mathbf{z})$, and there exists an $N \times N$ symmetric matrix $\Delta \in \mathbb{S}_+^N$ such that

$$(\mathbf{u}_2)^\top \Delta \mathbf{u}_2 = \sum_{k=1}^K h_k^2(\mathbf{z})\lambda_k(\mathbf{z}).$$

Obviously, $\Delta_{\{2, \dots, N\}} \succeq O$. For $p \in \mathbb{N}$, let $A \in \mathbb{S}^p$ be a positive definite matrix, and let B be a positive semidefinite matrix. It is easy to show that $A + B$ is also a positive definite matrix, i.e., the rank of $A + B$ is p . Thus, $Q_{\{2, \dots, N\}} + \Delta_{\{2, \dots, N\}}$ is a positive definite matrix. The rank of $Q - \rho E^1 + \Delta$ is greater than or equal to $N - 1$ and $Q - \rho E^1 + \Delta$ is a Gram matrix for $q(\mathbf{z}; Q; \rho, \boldsymbol{\lambda})$. \square

Remark 4.5. *The rank- $(N - 1)$ Gram matrix is essential for the exactness of the primal SDP relaxation which can be obtained by linearizing (11).*

Since the feasible region F of (MC^2) is nonempty and bounded as assumed in Section 2.2, its primal SDP relaxation is feasible. In fact, $\mathbf{z} \in F$ satisfies $h_k(\mathbf{z})^2 = 0$, thus, $-h_k(\mathbf{z})^2 \mathbf{u}_{\nu_k}(\mathbf{z}) \mathbf{u}_{\nu_k}(\mathbf{z})^\top$ is the zero matrix.

On the other hand, in (10), we can take $\lambda_k(\mathbf{z}) = 1$ for each $k = 1, \dots, K$ and a negative ρ with very large magnitude. Then, we can find a positive definite matrix \bar{W} satisfying $(\mathbf{u}_2)^\top Q \mathbf{u}_2 - \rho + \sum_{k=1}^K h_k(\mathbf{z})^2 \lambda_k(\mathbf{z}) = \mathbf{u}_d(\mathbf{z})^\top \bar{W} \mathbf{u}_d(\mathbf{z})$. Thus, the dual SDP relaxation has an interior feasible point, and strong duality holds for the primal and dual SDP relaxation. Therefore, by the discussion in Section 3 of [3], when the Gram matrix is of rank- $(N - 1)$, the primal SDP relaxation provides an exact solution to (MC) . More precisely, let X^ and $S(\mathbf{z}^*)$ be the solutions of the primal and dual SDP relaxation. Then, $X^* S(\mathbf{z}^*) = O$ by strong duality. From the Sylvester's rank inequality [1], $\text{rank } X^* + \text{rank } S(\mathbf{z}^*) \leq N + \text{rank } X^* S(\mathbf{z}^*)$ holds for X^* and $S(\mathbf{z}^*)$. Therefore, $\text{rank } X^* \leq N + \text{rank } O - \text{rank } S(\mathbf{z}^*) \leq N + 0 - (N - 1) = 1$. Consequently, this rank-1 condition ensures the exactness of SDP relaxation [3].*

4.2 Given partial data and the SOS expression

While the SOS relaxation (13) includes $\sum_{k=1}^K h_k(\mathbf{z})^2 \lambda_k(\mathbf{z})$, (18) holds without explicit dependence on $\mathbf{h}(\mathbf{z})$. We show that $\mathbf{h}(\mathbf{z})$ is implicitly included in $(\mathbf{u}_2 - (\mathbf{u}_2)_0)^\top \bar{E} (\mathbf{u}_2 - (\mathbf{u}_2)_0)$ in (18) in this subsection.

Suppose that we have an arbitrary chain from the chains expressed as (2). Then, the indices for the elements of $\mathbf{h}(\mathbf{z})$ may not coincide with the data indices, for instance, $h_2(\mathbf{z}) = z_7 z_3 - (\mathbf{z}_0)_7 (\mathbf{z}_0)_3$. In this case, $\mathbf{h}(\mathbf{z})$ can be renumbered according to a chain structure as follows:

$$h_{i_1}(\mathbf{z}) = z_{i_1} - (\mathbf{z}_0)_{i_1}, \quad h_{i_\ell}(\mathbf{z}) = z_{i_{\ell-1}} z_{i_\ell} - (\mathbf{z}_0)_{i_{\ell-1}} (\mathbf{z}_0)_{i_\ell}, \quad \ell = 2, \dots, L. \quad (20)$$

By the definition of \mathbf{u}_2 in (4), we see that $(\mathbf{u}_2 - (\mathbf{u}_2)_0)^\top \bar{E} (\mathbf{u}_2 - (\mathbf{u}_2)_0)$ includes the terms of form $(z_i - (\mathbf{z}_0)_i)^2$ ($i = 1, \dots, s$) or $(z_i z_j - (\mathbf{z}_0)_i (\mathbf{z}_0)_j)^2$ ($1 \leq i \leq j \leq s$).

First, let us focus on the chain in (20) and express $(z_{i_\ell} - (\mathbf{z}_0)_{i_\ell})^2$ ($\ell = 1, \dots, L$) as $(\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)^\top U_{i_\ell} (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)$ where \otimes is the Kronecker product and $U_{i_\ell} \in \mathbb{S}_+^{KN}$. We see that $h_k(\mathbf{z}) z_i$ is included in the vectors $\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1$ and $\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2$, and $h_k(\mathbf{z}) z_i z_j$ in $\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2$. To simplify the discussion with U_{i_ℓ} , we use the notation $I^1(h_k(\mathbf{z}) z_i)$ and $I^2(h_k(\mathbf{z}) z_i z_j)$ to denote the indices that correspond to $h_k(\mathbf{z}) z_i$ in $\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1$ and $h_k(\mathbf{z}) z_i z_j$ in $\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2$, i.e., $(\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1)_{I^1(h_k(\mathbf{z}) z_i)} = h_k(\mathbf{z}) z_i$ and $(\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)_{I^2(h_k(\mathbf{z}) z_i z_j)} = h_k(\mathbf{z}) z_i z_j$. We also use the same notation for $z_i = 1$ or $z_j = 1$. For example, the first element of \mathbf{u}_2 is 1, thus, $I^2(h_k(\mathbf{z})) = (k-1)N + 1$ and $(\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)_{I^2(h_k(\mathbf{z}))} = h_k(\mathbf{z})$. In addition, we also use \mathbf{e}_i to denote a vector whose i th component is 1 and all other components are 0.

Next, we examine each term corresponding to a constraint which forms the chain in (20). The term $(z_{i_1} - (\mathbf{z}_0)_{i_1})^2$ in $(\mathbf{u}_2 - (\mathbf{u}_2)_0)^\top \bar{E} (\mathbf{u}_2 - (\mathbf{u}_2)_0)$ corresponds to the first constraint in the chain and it can be written as

$$z_{i_1} - (\mathbf{z}_0)_{i_1} = h_{i_1}(\mathbf{z}) = (\mathbf{e}_{I^1(h_{i_1}(\mathbf{z}))})^\top (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1), \quad (21)$$

from (20). Therefore,

$$(z_{i_1} - (\mathbf{z}_0)_{i_1})^2 = (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1)^\top U_{i_1}^1 (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1)$$

holds with $U_{i_1}^1 := E^{I^1(h_{i_1}(\mathbf{z}))} \in \mathbb{S}_+^{s+1}$.

For the second term $(z_{i_2} - (\mathbf{z}_0)_{i_2})^2$, we use a relation:

$$\begin{aligned} z_{i_2} - (\mathbf{z}_0)_{i_2} &= \frac{-1}{(\mathbf{z}_0)_{i_1}} (z_{i_1} - (\mathbf{z}_0)_{i_1}) z_{i_2} + \frac{1}{(\mathbf{z}_0)_{i_1}} (z_{i_1} z_{i_2} - (\mathbf{z}_0)_{i_1} (\mathbf{z}_0)_{i_2}) \\ &= \frac{-1}{(\mathbf{z}_0)_{i_1}} h_{i_1}(\mathbf{z}) z_{i_2} + \frac{1}{(\mathbf{z}_0)_{i_1}} h_{i_2}(\mathbf{z}) \\ &= \left(\frac{-1}{(\mathbf{z}_0)_{i_1}} \mathbf{e}_{I^1(h_{i_1}(\mathbf{z}) z_{i_2})} + \frac{1}{(\mathbf{z}_0)_{i_1}} \mathbf{e}_{I^1(h_{i_2}(\mathbf{z}))} \right)^\top (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1). \end{aligned} \quad (22)$$

Therefore,

$$(z_{i_2} - (\mathbf{z}_0)_{i_2})^2 = (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1)^\top U_{i_2}^1 (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1)$$

can be obtained with

$$U_{i_2}^1 := \left(\frac{-1}{(\mathbf{z}_0)_{i_1}} \mathbf{e}_{I^1(h_{i_1}(\mathbf{z}) z_{i_2})} + \frac{1}{(\mathbf{z}_0)_{i_1}} \mathbf{e}_{I^1(h_{i_2}(\mathbf{z}))} \right) \left(\frac{-1}{(\mathbf{z}_0)_{i_1}} \mathbf{e}_{I^1(h_{i_1}(\mathbf{z}) z_{i_2})} + \frac{1}{(\mathbf{z}_0)_{i_1}} \mathbf{e}_{I^1(h_{i_2}(\mathbf{z}))} \right)^\top.$$

We apply the induction step on $\ell = 3, \dots, L$, i.e., $(z_{i_\ell} - (\mathbf{z}_0)_{i_\ell})^2$ is examined. Assume that there exists a vector $\mathbf{a}_{i_{\bar{\ell}}} \in \mathbb{R}^{K(s+1)}$ for each $\bar{\ell} = 1, \dots, \ell - 1$ such that $z_{i_{\bar{\ell}}} - (z_{i_{\bar{\ell}}}) =$

$\mathbf{a}_{i_\ell}^\top(\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1)$. In fact, (21) and (22) indicate that we can take $\mathbf{a}_{i_1} = \mathbf{e}_{I^1(h_{i_1}(\mathbf{z}))}$ and $\mathbf{a}_{i_2} = \frac{-1}{(\mathbf{z}_0)_{i_1}} \mathbf{e}_{I^1(h_{i_1}(\mathbf{z})z_{i_2})} + \frac{1}{(\mathbf{z}_0)_{i_1}} \mathbf{e}_{I^1(h_{i_2}(\mathbf{z}))}$, respectively. As a result, we have

$$\begin{aligned} z_{i_\ell} - (\mathbf{z}_0)_{i_\ell} &= \frac{1}{(\mathbf{z}_0)_{i_{\ell-2}}(\mathbf{z}_0)_{i_{\ell-1}}} (z_{i_{\ell-1}}z_{i_\ell} - (\mathbf{z}_0)_{i_{\ell-1}}(\mathbf{z}_0)_{i_\ell})z_{i_{\ell-2}} \\ &\quad - \frac{1}{(\mathbf{z}_0)_{i_{\ell-2}}(\mathbf{z}_0)_{i_{\ell-1}}} (z_{i_{\ell-2}}z_{i_{\ell-1}} - (\mathbf{z}_0)_{i_{\ell-2}}(\mathbf{z}_0)_{i_{\ell-1}})z_{i_\ell} \\ &\quad + \frac{(\mathbf{z}_0)_{i_\ell}(\mathbf{z}_0)_{i_{\ell-1}}}{(\mathbf{z}_0)_{i_{\ell-2}}(\mathbf{z}_0)_{i_{\ell-1}}} (z_{i_{\ell-2}} - (\mathbf{z}_0)_{i_{\ell-2}}) \\ &= \mathbf{a}_{i_\ell}^\top(\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1), \end{aligned}$$

where \mathbf{a}_{i_ℓ} is defined by

$$\mathbf{a}_{i_\ell} = \frac{1}{(\mathbf{z}_0)_{i_{\ell-2}}(\mathbf{z}_0)_{i_{\ell-1}}} \mathbf{e}_{I^1(h_{i_\ell}(\mathbf{z})z_{i_{\ell-2}})} - \frac{1}{(\mathbf{z}_0)_{i_{\ell-2}}(\mathbf{z}_0)_{i_{\ell-1}}} \mathbf{e}_{I^1(h_{i_{\ell-1}}(\mathbf{z})z_{i_\ell})} + \frac{(\mathbf{z}_0)_{i_\ell}(\mathbf{z}_0)_{i_{\ell-1}}}{(\mathbf{z}_0)_{i_{\ell-2}}(\mathbf{z}_0)_{i_{\ell-1}}} \mathbf{a}_{i_{\ell-2}}. \quad (23)$$

By letting $U_{i_\ell}^1 = \mathbf{a}_{i_\ell} \mathbf{a}_{i_\ell}^\top \in \mathbb{S}_+^{K(s+1)}$, we have that

$$(z_{i_\ell} - (\mathbf{z}_0)_{i_\ell})^2 = (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1)^\top U_{i_\ell}^1 (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1) \quad (24)$$

for $\ell = 3, \dots, L$. Since the elements of \mathbf{u}_2 include those of \mathbf{u}_1 , $U_{i_\ell}^1 \in \mathbb{S}_+^{K(s+1)}$ can be expanded into a matrix $U_{i_\ell}^2 \in \mathbb{S}_+^{KN}$ such that $(\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1)^\top U_{i_\ell}^1 (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1) = (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)^\top U_{i_\ell}^2 (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)$ by inserting zeros in $U_{i_\ell}^2$ appropriately.

By applying the discussion above to each chain starting from $z_0 = x_1$, we can find $U_i^2 \in \mathbb{S}_+^{KN}$ for each $i = 1, \dots, s$ such that

$$(z_i - (\mathbf{z}_0)_i)^2 = (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)^\top U_i^2 (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2).$$

Now, we consider $(z_i z_j - (\mathbf{z}_0)_i (\mathbf{z}_0)_j)^2$ ($1 \leq i < j \leq s$) since $(\mathbf{u}_2 - (\mathbf{u}_2)_0)^\top \bar{E} (\mathbf{u}_2 - (\mathbf{u}_2)_0)$ includes not only $(z_i - (\mathbf{z}_0)_i)^2$ but also $(z_i z_j - (\mathbf{z}_0)_i (\mathbf{z}_0)_j)^2$ ($1 \leq i < j \leq s$). By the relation (23), we can find $\hat{\mathbf{a}}_i$ and $\hat{\mathbf{a}}_j$ such that $(z_i - (\mathbf{z}_0)_i) = (\hat{\mathbf{a}}_i)^\top (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1)$ and $(z_j - (\mathbf{z}_0)_j) = (\hat{\mathbf{a}}_j)^\top (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1)$. We also notice that \mathbf{u}_1 includes z_j . As a result, we write as follows:

$$\begin{aligned} z_i z_j - (\mathbf{z}_0)_i (\mathbf{z}_0)_j &= (z_i - (\mathbf{z}_0)_i)z_j + (\mathbf{z}_0)_i (z_j - (\mathbf{z}_0)_j) \\ &= (\hat{\mathbf{a}}_i)^\top (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1) ((\mathbf{e}_{I^1(z_j)})^\top \mathbf{u}_1) + (\mathbf{z}_0)_i (\hat{\mathbf{a}}_j)^\top (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1). \end{aligned} \quad (25)$$

We see that the Kronecker product $\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2$, i.e., the componentwise product of $\mathbf{h}(\mathbf{z})$ and \mathbf{u}_2 , includes $\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_1$. Thus, there exists $U_{ij}^2 \in \mathbb{S}_+^{KN}$ such that $(z_i z_j - (\mathbf{z}_0)_i (\mathbf{z}_0)_j)^2 = (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)^\top U_{ij}^2 (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)$.

Consequently, it holds that

$$\begin{aligned}
& (\mathbf{u}_2 - (\mathbf{u}_2)_0)^\top \bar{E} (\mathbf{u}_2 - (\mathbf{u}_2)_0) \\
&= \sum_{i=1}^s (z_i - (\mathbf{z}_0)_i)^2 + \sum_{i=1}^s \sum_{j=i}^s (z_i z_j - (\mathbf{z}_0)_i (\mathbf{z}_0)_j)^2 \\
&= \sum_{i=1}^s (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)^\top U_i^2 (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2) + \sum_{i=1}^s \sum_{j=i}^s (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)^\top U_{ij}^2 (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2) \\
&= (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)^\top U (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2),
\end{aligned}$$

where

$$U := \sum_{i=1}^s U_i^2 + \sum_{i=1}^s \sum_{j=i}^s U_{ij}^2. \quad (26)$$

It should be mentioned that different ways of expressing (25) exist, for example,

$$z_i z_j - (\mathbf{z}_0)_i (\mathbf{z}_0)_j = (z_j - (\mathbf{z}_0)_j) z_i + (\mathbf{z}_0)_j (z_i - (\mathbf{z}_0)_i),$$

therefore, the matrix U is not always unique.

In view of Remark 4.3, we can find $Q \in \mathbb{S}_+^N$, $\lambda_k(\mathbf{z}) \in \Sigma_{4-2d_k}[\mathbf{z}]$ ($k = 1, \dots, K$) and $U \in \mathbb{S}_+^{KN}$ such that

$$(\mathbf{u}_2)^\top Q \mathbf{u}_2 - (\mathbf{u}_2)_0^\top Q (\mathbf{u}_2)_0 + \sum_{k=1}^K h_k(\mathbf{z})^2 \lambda_k(\mathbf{z}) = (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)^\top U (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2). \quad (27)$$

Consequently, we have shown that $\mathbf{h}(\mathbf{z})$ is included in $(\mathbf{u}_2 - (\mathbf{u}_2)_0)^\top \bar{E} (\mathbf{u}_2 - (\mathbf{u}_2)_0)$ in (18).

4.3 Exact recovery of the problem with hidden connectivity

In this subsection, we discuss the exact recovery of problems with hidden connectivity in Section 3.2.

Using $\bar{\mathbf{h}}(\mathbf{z}) = C\mathbf{h}(\mathbf{z})$ in (12), the equation (27) turns out to be

$$(\mathbf{u}_2)^\top Q \mathbf{u}_2 - (\mathbf{u}_2)_0^\top Q (\mathbf{u}_2)_0 + \sum_{\bar{k}=1}^{\bar{K}} (\bar{h}_{\bar{k}}(\mathbf{z}))^2 \bar{\lambda}_{\bar{k}}(\mathbf{z}) = (\bar{\mathbf{h}}(\mathbf{z}) \otimes \mathbf{u}_2)^\top \bar{U} ((\bar{\mathbf{h}}(\mathbf{z})) \otimes \mathbf{u}_2), \quad (28)$$

with $\bar{\lambda}_{\bar{k}}(\mathbf{z}) \in \Sigma_{4-2d_{\bar{k}}}(\mathbf{z})$ and some $\bar{U} \in \mathbb{S}_+^{\bar{K}N}$.

From the assumption that $C^\top C$ is invertible in Section 3.2, $\bar{\mathbf{h}}(\mathbf{z}) = 0$ if and only if $\mathbf{h}(\mathbf{z}) = 0$. Since $C \in \mathbb{R}^{\bar{K} \times K}$ is a matrix, there exists $\bar{C} \in \mathbb{R}^{(\bar{K}N) \times (KN)}$ such that $(C\mathbf{h}(\mathbf{z})) \otimes \mathbf{u}_2 = \bar{C}(\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)$. By letting $U = \bar{C}^\top \bar{U} \bar{C} \in \mathbb{S}_+^{KN}$, (28) can be written as

$$(\mathbf{u}_2)^\top Q \mathbf{u}_2 - (\mathbf{u}_2)_0^\top Q (\mathbf{u}_2)_0 + \sum_{\bar{k}=1}^{\bar{K}} (\bar{h}_{\bar{k}}(\mathbf{z}))^2 \bar{\lambda}_{\bar{k}}(\mathbf{z}) = (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)^\top U (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2).$$

In addition, $\sum_{\bar{k}=1}^{\bar{K}} (\bar{h}_{\bar{k}}(\mathbf{z}))^2 \bar{\lambda}_{\bar{k}}(\mathbf{z})$ can be expressed as $(\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)^\top V (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)$ for some $V \in \mathbb{S}_+^{KN}$. Therefore, the discussion in Section 4.3 can be reduced to that of Section 4.2, and we can obtain the exact recovery of the problem with $\bar{\mathbf{h}}(\mathbf{z}) = 0$ by solving (28).

5 Algorithms and numerical results

We propose a two-stage algorithm for solving the SOS relaxation (10). In the first stage, Q is constructed as described in Lemma 4.1 and Theorem 4.2 with the given data $\mathbf{h}(\mathbf{z}) = 0$. In the second stage, we solve (10).

We also present numerical results in Section 5.2 by implementing the proposed algorithm on Example 2.2 and test problems with data generated based on bipartite graphs.

5.1 Algorithms

The first stage of our proposed algorithm constructs Q in Theorem 4.2 by finding Q^r ($r = 2, \dots, N$) and then letting $Q = \sum_{r=2}^N Q^r$. Each Q^r has nonzero elements only in $[Q^r]_{11}$, $[Q^r]_{rr}$, $[Q^r]_{1r}$, and $[Q^r]_{r1}$. The resulting Q is an arrowhead matrix. The second stage is to solve the SOS relaxation to determine Γ in Theorem 4.4.

Specifically, in the first stage, for any $r \in \{2, \dots, N\}$, Q^r is determined as follows:

$$\begin{aligned} & \text{find } Q^r \in \mathbb{S}_+^N, U^r \in \mathbb{S}_+^{KN} \\ & \text{s.t. } \begin{bmatrix} 1 \\ [\mathbf{u}_2]_r \end{bmatrix}^T Q_{\{1,r\}}^r \begin{bmatrix} 1 \\ [\mathbf{u}_2]_r \end{bmatrix} = (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)^T U^r (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2), \quad (\mathcal{S}_1(r)) \\ & \quad [Q^r]_{rr} = 1, \quad [Q^r]_{ij} = 0, \quad (i, j) \in (N \times N), \quad i \notin \{1, r\} \vee j \notin \{1, r\}, \end{aligned}$$

where $Q_{\{1,r\}}^r$ denotes a 2×2 submatrix that includes the first and r th columns and rows of Q^r . We note that $(\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)^T R^r (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)$ can be expressed as $\mathbf{h}(\mathbf{z})^T \widehat{U}^r \mathbf{h}(\mathbf{z})$ if \widehat{U}^r is a $K \times K$ symmetric polynomial matrix. For the feasibility of $(\mathcal{S}_1(r))$, we present the following proposition:

Proposition 5.1. *For $r \in \{2, \dots, N\}$, the problem $(\mathcal{S}_1(r))$ has a feasible solution.*

Proof. It is easy to see that $(\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)^T U^r (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)$ in $(\mathcal{S}_1(r))$ is a general form of $\sum_{k=1}^K h_k(\mathbf{z})^2 p_k(\mathbf{z})$, where $p_k(\mathbf{z}) \in \Sigma_{4-2d_k}[\mathbf{z}]$. As discussed in Section 4.2, there exists U^r such that

$$\begin{bmatrix} 1 \\ [\mathbf{u}_2]_r \end{bmatrix}^T Q_{\{1,r\}}^r \begin{bmatrix} 1 \\ [\mathbf{u}_2]_r \end{bmatrix} = (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)^T U^r (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2).$$

□

By letting $Q = \sum_{r=2}^N Q^r$, we obtain the following problem from $(\mathcal{S}_1(r))$:

$$\begin{aligned} & \text{find } Q \in \mathbb{S}_+^N, U \in \mathbb{S}_+^{KN} \\ & \text{s.t. } (\mathbf{u}_2)^T Q \mathbf{u}_2 = (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)^T U (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2), \quad (\mathcal{S}_{1,\text{sum}}) \\ & \quad Q_{ii} = 1, \quad i \in \{2, \dots, N\}, \\ & \quad Q_{ij} = 0, \quad (i, j) \in \{2, \dots, N\} \times \{2, \dots, N\}, \quad i \neq j, \end{aligned}$$

which has a feasible solution by Proposition 5.1. For $\sum_{r=2}^N Q^r$, $(\mathcal{S}_1(r))$ needs to be solved $(N - 1)$ times for each r in $\{2, \dots, N\}$, while $(\mathcal{S}_{1,\text{sum}})$ needs to be solved once. Hence, solving $(\mathcal{S}_{1,\text{sum}})$ is more efficient than $(\mathcal{S}_1(r))$ and is employed in our algorithm.

Let $f_{\text{sum}}(\mathbf{z}) = (\mathbf{u}_2)^\top Q \mathbf{u}_2$. In the second stage, we solve the following SDP for a solution that can recover the matrix.

$$\begin{aligned}
& \max_{\rho, \boldsymbol{\lambda}, \Delta_1, \dots, \Delta_K} \quad \rho \\
& \text{s.t.} \quad f_{\text{sum}}(\mathbf{z}) - \rho + \sum_{k=1}^K h_k(\mathbf{z})^2 \lambda_k(\mathbf{z}) \in \Sigma_4[\mathbf{z}] \\
& \quad \lambda_k(\mathbf{z}) = \mathbf{u}_1^\top \Delta_k \mathbf{u}_1 \quad (k = 1, \dots, K) \\
& \quad \mu I - \Delta_k \succeq O \quad (k = 1, \dots, K) \\
& \quad \rho \in \mathbb{R}, \lambda_k(\mathbf{z}) \in \Sigma_{4-2d_k}[\mathbf{z}], \Delta_k \in \mathbb{S}_+^{m+n} \quad (k = 1, \dots, K).
\end{aligned} \tag{\mathcal{S}_2(f_{\text{sum}})}$$

where $\mu \in \mathbb{R}$ is a sufficiently large constant. Compared with (10), the problem $(\mathcal{S}_2(f_{\text{sum}}))$ includes the constraints $\lambda_k(\mathbf{z}) = \mathbf{u}_2^\top \Delta_k \mathbf{u}_2$ and $\mu I - \Delta_k \succeq O$ for all $k = 2, \dots, N$ for the boundedness of the dual variables $\boldsymbol{\lambda}$. By Slater's condition, strong duality holds between $(\mathcal{S}_2(f_{\text{sum}}))$ and its dual. In the numerical experiments, we use $\mu = 10^6$.

The proposed algorithm is described in Algorithm 1.

Algorithm 1 for solving (MC^2)

Input: $\mathbf{h}(\mathbf{z})$

Output: A solution \mathbf{z}^* to (MC^2)

- 1: Solve the problem $(\mathcal{S}_{1,\text{sum}})$ and obtain a solution (Q^*, U^*) .
- 2: $f_{\text{sum}} \leftarrow (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)^\top U^* (\mathbf{h}(\mathbf{z}) \otimes \mathbf{u}_2)$.
- 3: Solve the problem $(\mathcal{S}_2(f_{\text{sum}}))$ using f_{sum} and obtain a solution $(\rho^*, \boldsymbol{\lambda}^*)$.
- 4: Find the Gram matrix $\Gamma \in \mathbb{S}^N$ such that

$$(\mathbf{u}_2)^\top \Gamma \mathbf{u}_2 = f_{\text{sum}}(\mathbf{z}) - \rho^* + \sum_{k=1}^K h_k(\mathbf{z})^2 \lambda_k^*(\mathbf{z}).$$

- 5: Find a vector $\mathbf{u}_2^* \in \mathbb{R}^N$ in the null space of Γ .
 - 6: $\mathbf{z}^* \leftarrow \frac{1}{(\mathbf{u}_2^*)_1} [(\mathbf{u}_2^*)_2, (\mathbf{u}_2^*)_3, \dots, (\mathbf{u}_2^*)_{s+1}]^\top$ and return \mathbf{z}^* .
-

If the rank of Γ at Step 4 is $(N - 1)$, the dimension of the null space of Γ must be 1 and \mathbf{u}_2^* is uniquely determined up to a scalar. At Step 6, the multiplier $\frac{1}{(\mathbf{u}_2^*)_1}$ is used to have $(\mathbf{u}_2^*)_1 = 1$ by the definition of \mathbf{u}_2 ; more precisely, the first $s + 1$ elements of $\mathbf{u}_2^*/(\mathbf{u}_2^*)_1$ correspond to $[1; \mathbf{z}^*]$.

5.2 Numerical results

We present numerical results by Algorithm 1 on Example 2.2 in Section 5.2.1 and randomly generated problems with bipartite graphs in Section 5.2.2. The bipartite graph \mathcal{G} is employed as $x_1 = 1$, x_i ($i = 2, \dots, n$), and y_j ($j = 1, \dots, m$) form two groups and the partially given data for $x_i y_j$ can be regarded as an edge between the two groups.

The purpose of the experiments is to verify that Algorithm 1 can recover the exact solution of (MC^2) under the assumption of Theorem 4.2 and the connectivity of the bipartite

graph \mathcal{G} . In the discussion of numerical results by Algorithm 1, the error is computed by

$$\text{Er}_{\mathbf{z}^*} := \frac{\|\mathbf{z}^* - \mathbf{z}_0\|_2}{\|\mathbf{z}_0\|_2}. \quad (29)$$

All computations were performed on Intel Core i9-12900K (3.2GHz) with Julia 1.9.2 and Mosek 10.0.2.

5.2.1 Illustrative example

For Example 2.2, the first stage of the algorithm solves $(\mathcal{S}_{1,\text{sum}})$ with $\mathbf{h}(\mathbf{z}) = [y_1 - 7; y_2 - 3; x_2y_1 + 35; x_2y_3 - 10; x_3y_2 - 9]$. The resulting polynomial f_{sum} is used in the second-stage problem $(\mathcal{S}_2(f_{\text{sum}}))$:

$$\begin{aligned} & \max_{\rho, \lambda, \Delta_1, \Delta_2} \quad \rho \\ & \text{s.t.} \quad f_{\text{sum}} - \rho + (y_1 - 7)^2\lambda_1 + (y_2 - 3)^2\lambda_2 \\ & \quad \quad \quad + (x_2y_1 + 35)^2\lambda_3 + (x_2y_3 - 10)^2\lambda_4 + (x_3y_2 - 9)^2\lambda_5 \in \Sigma_4[\mathbf{z}] \\ & \quad \quad \quad \lambda_k(\mathbf{z}) = \mathbf{u}_1^T \Delta_k \mathbf{u}_1 \quad (k = 1, 2) \\ & \quad \quad \quad \mu I - \Delta_k \succeq O \quad (k = 1, 2) \\ & \quad \quad \quad \rho \in \mathbb{R}, \Delta_k \in \mathbb{S}_+^{n+m}(k = 1, 2), \lambda_3, \lambda_4, \lambda_5 \in [0, \mu]. \end{aligned}$$

The matrices Q and U in (26) can be obtained after $(\mathcal{S}_{1,\text{sum}})$ is solved. Their size is so large that the matrices cannot be displayed here, instead, shown in GitHub (<https://github.com/godazm/exact-matrix-completion>).

Table 1 displays the total time for the two stages, the optimal value of $(\mathcal{S}_2(f_{\text{sum}}))$, the recovered solution \mathbf{z}^* of Algorithm 1, and its error $\text{Er}_{\mathbf{z}^*}$. We can see that the total time for the first stage is longer than that of the second stage, although $(\mathcal{S}_{1,\text{sum}})$ was implemented instead of $(\mathcal{S}_1(r))$. We observe that $(\mathcal{S}_2(f_{\text{sum}}))$ can recover an exact solution of the rank-1 matrix for Example 2.2.

Table 1: Average total time in seconds and relative errors.

Total time		Optimal value	Recovered solution \mathbf{z}^*	$\text{Er}_{\mathbf{z}^*}$
$(\mathcal{S}_{1,\text{sum}})$	$(\mathcal{S}_2(f_{\text{sum}}))$	$(\mathcal{S}_2(f_{\text{sum}}))$		
7.62	1.03	1.59×10^{-6}	$[1.00; -5.00; 3.00; 7.00; 3.00; -2.00]$	9.91×10^{-7}

5.2.2 Tests on random data

We performed numerical experiments on two types of random data: data in a chain format and data with hidden connectivity. Each experiment involved 100 randomly generated instances. The instances were generated by Algorithm 2 for connected bipartite graphs. With $\text{rand}(\cdot)$ in Algorithm 2, each element of (\cdot) is generated with the same probability. Ten samples of \mathbf{z}_0 were generated simultaneously where each element is chosen by $(\mathbf{z}_0)_i \in \text{rand}([-5, 5])$ for $i \in \{1, \dots, s\}$.

Algorithm 2 Generating random bipartite graphs

Input: The number of vertices v

Output: A bipartite graph $(\{1, \dots, n\}, \{1, \dots, m\}, \Gamma)$

- 1: $n \leftarrow \text{rand}(\{1, \dots, v-1\})$, $m \leftarrow v - n$
 - 2: $L_{\text{remain}} \leftarrow \{1, \dots, n\}$, $R_{\text{remain}} \leftarrow \{1, \dots, m\}$
 - 3: $l_{\text{first}} \leftarrow \text{rand}(L_{\text{remain}})$ and then $L_{\text{remain}} \leftarrow L_{\text{remain}} \setminus \{l_{\text{first}}\}$
 - 4: $r_{\text{first}} \leftarrow \text{rand}(R_{\text{remain}})$ and then $R_{\text{remain}} \leftarrow R_{\text{remain}} \setminus \{r_{\text{first}}\}$
 - 5: $\Gamma \leftarrow \{(l_{\text{first}}, r_{\text{first}})\}$
 - 6: **while** $|L_{\text{remain}}| + |R_{\text{remain}}| \geq 1$ **do**
 - 7: Select *Left* or *Right* randomly
 - 8: **if** *Left* and $L_{\text{remain}} \neq \emptyset$ **then**
 - 9: $r \leftarrow \text{rand}(\{1, \dots, m\} \setminus R_{\text{remain}})$
 - 10: $l \leftarrow \text{rand}(L_{\text{remain}})$ and then $L_{\text{remain}} \leftarrow L_{\text{remain}} \setminus \{l\}$
 - 11: $\Omega \leftarrow \Omega \cup \{(l, r)\}$
 - 12: **else if** *Right* and $R_{\text{remain}} \neq \emptyset$ **then**
 - 13: $l \leftarrow \text{rand}(\{1, \dots, n\} \setminus L_{\text{remain}})$
 - 14: $r \leftarrow \text{rand}(R_{\text{remain}})$ and then $R_{\text{remain}} \leftarrow R_{\text{remain}} \setminus \{r\}$
 - 15: $\Omega \leftarrow \Omega \cup \{(l, r)\}$
 - 16: **end if**
 - 17: **end while**
-

Table 2: Bipartite graphs.

	\mathcal{G}_1	\mathcal{G}_2	\mathcal{G}_3	\mathcal{G}_4	\mathcal{G}_5	\mathcal{G}_6	\mathcal{G}_7	\mathcal{G}_8	\mathcal{G}_9	\mathcal{G}_{10}
n	9	7	6	2	6	2	1	8	1	6
m	1	3	4	8	4	8	9	2	9	4
max. degree	9	5	3	5	4	5	9	6	9	4

We generated 10 bipartite graphs $\mathcal{G}_1, \dots, \mathcal{G}_{10}$ associated with \mathbf{X} in (MC^2) with 10 vertices fixed. For the connectivity, each graph \mathcal{G}_p always included 9 edges for $p = 1, \dots, 10$. The graph property of \mathcal{G}_p is summarized on Table 2. The lengths of \mathbf{x} and \mathbf{y} are shown in the rows of n and m , respectively, and the maximum degree among all vertices in \mathcal{G}_p is also shown in the last row. For each \mathbf{z}_0 and each bipartite graph \mathcal{G}_p , we generated 100 instances of (MC^2) as follows: (i) The initial $n - 1$ elements of $\mathbf{z}_0 \in \mathbb{R}^{n+m-1}$ are regarded as the true values $(\mathbf{x}_0)_2, \dots, (\mathbf{x}_0)_n$, while the last m elements are considered as \mathbf{y}_0 . (ii) For every edge $(i, j) \in \mathbb{N}^2$ of \mathcal{G}_p , an equality constraint h_k is constructed to determine $(X_0)_{ij}$. For example, when $n = 7$ and $m = 3$, the edge $(3, 2)$ corresponds to a constraint of form $h_k(\mathbf{z}) = x_3 y_2 - (\mathbf{x}_0)_3 (\mathbf{y}_0)_2 = 0$, where $\mathbf{x}_0 = [1, (\mathbf{z}_0)_1, \dots, (\mathbf{z}_0)_6]^\top$ and $\mathbf{y}_0 = [(\mathbf{z}_0)_7, (\mathbf{z}_0)_8, (\mathbf{z}_0)_9]^\top$.

Table 3 shows the results on random data in a chain structure. Each row displays the mean value of times in seconds for the first and second stages, and the mean of errors (29) from 10 samples. We observe that the first stage spent much computational time as $(\mathcal{S}_{1, \text{sum}})$ involves a large matrix of size 495×495 . Here 495 is computed from 9×55 with the size of

Table 3: Average total time in seconds and relative errors.

	Total time		Mean errors
	$(\mathcal{S}_{1,\text{sum}})$	$(\mathcal{S}_2(f_{\text{sum}}))$	
\mathcal{G}_1	68.9	0.325	6.21×10^{-8}
\mathcal{G}_2	119	0.338	2.81×10^{-7}
\mathcal{G}_3	138	0.338	1.93×10^{-6}
\mathcal{G}_4	87.7	0.349	3.58×10^{-7}
\mathcal{G}_5	148	0.340	7.51×10^{-6}
\mathcal{G}_6	88.2	0.354	1.40×10^{-6}
\mathcal{G}_7	42.7	0.354	4.65×10^{-8}
\mathcal{G}_8	105	0.343	7.02×10^{-7}
\mathcal{G}_9	45.1	0.365	3.46×10^{-8}
\mathcal{G}_{10}	138	0.348	3.28×10^{-7}

$\mathbf{h}(\mathbf{z})$ and \mathbf{u}_2 , which is $K = 9$ and $N = \binom{11}{2} = 55$, respectively. We also see that the total time is shorter for the instances with larger differences between n and m than the instances with small differences. The errors shown in the last column of Table 3 are small, thus we see that the exact solution of (MC) can be recovered for all instances.

To test the second type of data for (MC^3) , test instances were generated with a matrix C as discussed in Section 3.2. The i th row of C scales each component of $\mathbf{h}(\mathbf{z})$, i.e., if the first row of C is $[1 \ 0 \ -3 \ 0 \ 0 \ \dots \ 0]$, the first constraint of (MC^3) must be $(h_1(\mathbf{z}) - 3h_3(\mathbf{z}))^2 \leq 0$. In our experiments, we generated 10 invertible matrices $C_1, \dots, C_{10} \in \mathbb{R}^{9 \times 9}$ for C . Each element of them was randomly chosen from $[-5, 5]$, and then each row of C_k was normalized using the Euclidean norm for $k = 1, \dots, 10$. For each matrix C_k , we tested the algorithm for all the combination of graphs $\mathcal{G}_1, \dots, \mathcal{G}_{10}$ and 10 samples of \mathbf{z}_0 , and thus 100 instances were used.

Figures 1 and 2 show the performance of the algorithm for each matrix C_k . Note that (MC^2) can be regarded as a special case of (MC^3) , in other words, (MC^3) with $C = I$. From the error $\text{Er}_{\mathbf{z}^*}$ from 100 instances in Figure 1, the proposed algorithm can extract the solution with accuracy even for the problems with the hidden connectivity. We also notice that the mean and maximum errors for the instances with $C = C_1, C_2, \dots, C_{10}$ are larger than that for the instances with $C = I$. The computing time for the first and second stages is shown in Figure 2. The time unit in this figure is second. We see that the total time for the instances with $C = C_1, C_2, \dots, C_{10}$ is much longer than that for the instances with $C = I$, since $\bar{\mathbf{h}}(\mathbf{z}) = C\mathbf{h}(\mathbf{z})$ involves more nonzero coefficients than $\mathbf{h}(\mathbf{z})$.

6 Concluding remarks

We have studied a minimization problem arising from the matrix completion under the assumption that the known values of the matrix elements form a chain format or possess the hidden connectivity. For the minimization problem, we have proved that the exact solution to the problem can be obtained by the SOS relaxation or the SDP relaxation

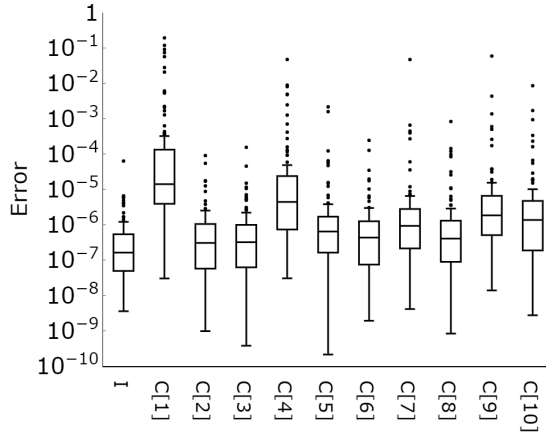


Figure 1: Relative errors for the instances with hidden connectivity, I, C_1, \dots, C_{10} . Errors are shown in the log-scale.

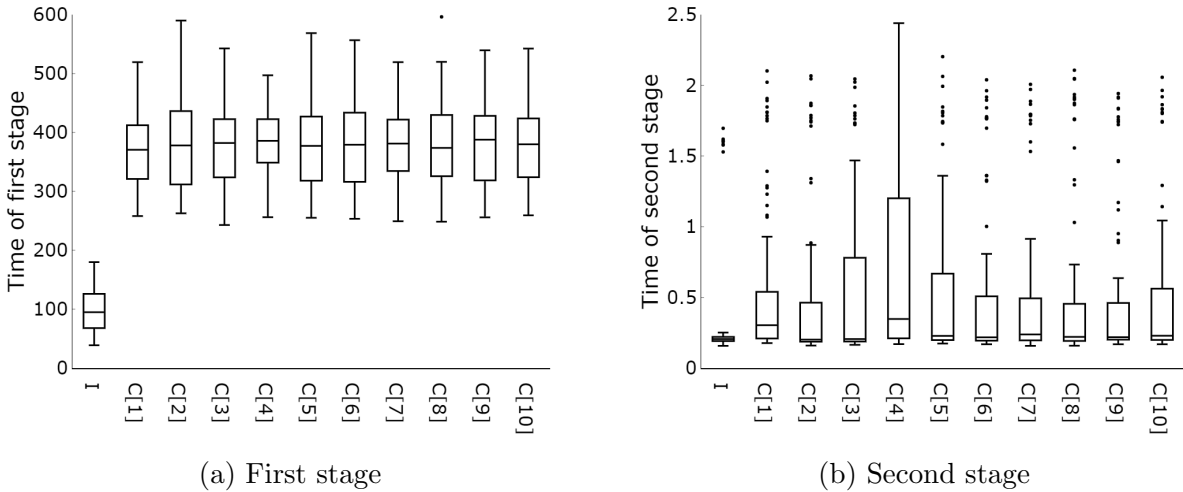


Figure 2: Total time in seconds for the instances with hidden connectivity, I, C_1, \dots, C_{10} .

by formulating the problem with the constraints as polynomials of degree up to 4 and incorporating an objective function with the arrowhead sparsity pattern.

Exploiting the sparsity of a given problem has been an essential subject for improving the computational efficiency of SDP relaxations [14, 17, 19, 23]. It has been commonly employed to reduce the size of the variable matrix so that the resulting problem can be solved fast. The bipartite structure, on the other hand, has been used to show the exactness of the SDP relaxation for some class of quadratically constrained quadratic programs [3, 4].

The sparsity discussed in this paper falls into two categories: one is for the objective function and the other for the constraints. In the case of the proposed problem (MC^2), the sparsity in the objective function is not obtained from the given problem. Rather, it is enforced onto the objective function through the matrix Q in (10) with the arrowhead sparsity pattern and the information from the given data. The constraint sparsity, characterized by the chain format, additionally preserves the rank of Γ in Theorem 4.4 as $N - 1$. In

this context, both forms of sparsity have been utilized to show the exactness of the solution obtained from the SOS relaxation. It will be interesting to explore the applicability of our approach to other problems where the objective function can be chosen.

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