Asset liability management under sequential stochastic dominance constraints

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November 20, 2023

Abstract

We formulate a long-term multi-period institutional Asset-Liability Management (ALM) problem, in which the decision maker seeks the minimization of the initial capital invested in a dedicated immunized portfolio and the risk stemming from two sources: the investment losses and the shortfall with respect to an exogenous capital requirement. The asset portfolio is required to stochastically dominate a liability portfolio under a rich set of exogenous uncertainties. The problem is formulated as a sequential decision problem with second-order stochastic-dominance constraints that are enforced in a time-consistent manner. The risk associated with investment losses and regulatory capital is managed by optimizing a time-consistent dynamic measure of risk in the objectives, while the risk associated with the liability coverage is controlled by the sequential stochastic dominance constraints resulting in a robust optimal risk-averse policy. We devise an efficient decomposition method for solving the risk-averse multi-stage problem and discuss its convergence. The proposed methodology is validated computationally on a case study developed on a property and casualty ALM problem.

Keywords: *Asset-liability management, stochastic dominance, multistage stochastic programming, time consistency, decomposition method.*

Introduction

We consider a multi-period *asset-liability management* (ALM) problem with assets and liabilities exposed to several risk sources and formulated as a *multistage stochastic programming* (MSP) problem. From a modeling and financial perspective, the ALM problem is characterised by a solvency requirement under which the decision maker seeks the minimization of the shortfall with respect to an exogenous risk capital estimate. Inspired by a corporate case study, the ALM problem features a large insurance and financial intermediary, whose business structure is summarized by a technical division, an investment and a risk management division, that are responsible for *liability* policies, *asset* management and *risk* assessment, respectively. The decision maker, based on the risk assessment and the current asset-liability portfolios, intends to minimize the dedicated capital and preserve over time a sufficient funding to cover the liabilities and comply with regulatory policies.

Various mathematical models have been proposed in the literature to devise a policy that manages the risk of liability coverage and investment losses with a minimal initial capital. The framework of risk-averse multi-stage stochastic programming problems is particularly suitable to this end. In such a framework, the decision maker determines an optimal portfolio immunization strategy based on the evolution of the assets and the liabilities over the time-horizon. In this problem formulation, we identify several sources of risk and we propose to control the risk in multiple ways. We impose a risk-averse constraint in the form of a stochastic ordering relation that ensures statistically the ability to cover the liabilities over the time horizon. The initial capital dedicated to the liability portfolio is minimized in the objective of the first stage decision, while the risk of falling below the required regulatory capital and the potential investment losses are handle by a time-consistent dynamic measure of risk. The immunization of the dedicated investment portfolio includes matching of its duration to that of the liability portfolio.

Optimization problems with stochastic dominance constraints were first introduced in Dentcheva and Ruszczyński [2003] and further developed in Dentcheva and Ruszczyński [2004]. The consistency of measures of risk with the stochastic dominance relations was first investigated by Ogryczak and Ruszczyński [1999] with a focus on the class of semideviation risk measures and in Ogryczak and Ruszczyński [2002] with a dual characterization of stochastic dominance conditions. A general consistency result of law-invariant coherent measures of risk and second order stochastic dominance is stated in Leitner [2005]. A solid foundation for the adoption of SD as a decision paradigm in portfolio theory is provided by Levy [2006]. Optimization with SD constraints relate to many other risk-averse models such as optimization using coherent measures of risk, see Dentcheva and Ruszczyński [2008a], utility functions, see Dentcheva and Ruszczyński [2003], distortion, see Dentcheva and Ruszczyński [2006a], chance constraints, or Average (Conditional) Value-at-Risk constraints, see Dentcheva and Ruszczyński [2006b]. Numerical methods for static optimization problems involving stochastic dominance relations as constraints or as multivariate objectives are proposed in Dentcheva and Ruszczyński [2003, 2010]; Fábián et al. [2011]; Luedtke [2008]; Noyan and Ruszczyński [2008]; Roman et al. [2006]; Rudolf and Ruszczyński [2008].

While a lot of literature analyses stochastic order relations for scalar random variables and their implications for decision making under uncertainty, much fewer works are devoted to the comparison of sequences, although dynamical systems are prevalent in practice. We refer to Müller and Stoyan [2002] for an overview of stochastic orderings for sequences and processes. The challenge in such comparisons arise from time-consistency considerations which are crucial in the context of sequential decision making.

The majority of the sequential comparisons result in a time-inconsistent decision problems. Stochastic-ordering constraints in dynamic stochastic optimization are discussed in Dentcheva and Ruszczyński [2008b]; Dentcheva et al. [2022]; Escudero et al. [2016, 2017]; Haskell and Jain [2013]. The proposal in Haskell and Jain [2013] addresses the limiting distribution in an infinite-time horizon average-cost Markov decision problem. In Escudero et al. [2016, 2017], time-consistent constraints are imposed in a manner akin to Average (Conditional) Value-at-Risk constraints for a multi-stage stochastic optimization problem. Average Value-at-Risk can be represented as the expected value of a nonlinear function, which facilitates the application of the theory and methods for risk-neutral optimization. The dynamic (two-stage version) of the Conditional Value-at-Risk is consistent with the SD comparison, see Pflug and Pichler [2014]; Pflug and Ruszczyński [2005]. In this paper, we use a stochastic comparison within the framework presented in Dentcheva et al. [2022], which develops a multistage stochastic program constrained by time-consistent sequential comparisons. Our approach differs from the one presented in Dentcheva et al. [2022] because we do not benchmark the recourse function sequences and our stochastic comparison is limited to one step look-ahead. Further details are provided in Section 1.

Very popular and powerful tools for controlling risk in sequential decision problems are provided by the theory and methods of time-consistent coherent measures of risk. We refer to Föllmer and Schied [2008]; Shapiro et al. [2021] for an overview. We use this framework in our paper to address the risk associated with the overall capital involved in the operation. The time-consistency definition, which we adopt in this paper, is based on Ruszczyński [2010]. In particular, a scenario decomposition method for multi-stage stochastic programming problems with coherent measures of risk is proposed in Collado et al. [2012]. In Gülten and Ruszczyński [2015] an extended two-stage problem formulation with coherent measures of risk is formulated and a numerical method for solving the problem is proposed. In our paper, we use the techniques form Gülten and Ruszczyński [2015] extending them to the multi-stage setting and integrating with a suitable numerical treatment of the sequential stochastic dominance constraints.

When considering the current state-of-the-art in ALM modelling approaches, our starting point is the comprehensive ALM problem formulation, developed from Consigli et al. [2011a, 2018]. The introduction of market based asset and liability valuation and the enforcement of funding and solvency feasibility conditions have been noted as essential contributions in recent years. To mention just a few: Urban et al. [2004] analyses the risk capital allocation problem for an insurance portfolio, Gatzert and Schmeiser [2008] combines fair pricing and capital requirements for non-life insurance companies, Alessandri and Drehmann [2010] develops the analysis on risk capital requirements for a banking intermediary, Dhaene et al. [2012] establishes optimal capital allocation principles in presence of several risk sources. A risk management approach to capital allocation by financial intermediaries is proposed in Maume-Deschamps et al. [2015]. In Consigli et al. [2018] the implications of capital constraints on the optimization of risk-adjusted returns in a dynamic model are analysed. In Lauria et al. [2022] a dynamic stochastic control approach based on an open-loop linear feedback policy has been applied to a defined-benefit pension fund manager problem combining a stochastic control approach, with a chance constraint on pension fund funding ratio. A recent review on dynamic risk measures in financial optimization can be found in Chen et al. [2017], which discusses the distinction between terminal, additive, and recursive risk measures.

Any such application requires the definition of a set of stochastic models for scenario generation and, specifically for this application, the approximation of the liability probability distributions. Following the adopted risk capital-based problem formulation, we capture a rich set of risk sources through the definition of a 2-level statistical model with the yield curve and inflation as core risk processes, see Christensen et al. [2009]; Nelson and Siegel [1987]. The adopted scenario generation method follows classical approaches, see for instance Dupačová et al. [2000]; Heitsch and Römisch [2009]; Maggioni and Pflug [2016, 2019]; Narum et al. [2023].

The adoption of stochastic dominance principles in an ALM context is not new. To date however, the models are either static or use a time-inconsistent comparisons in a multi-stage setting. Additionally, the computational burden has been a major deterrent for the use of sequential stochastic comparisons in a multistage stochastic programming when either a direct linear programming method or the one based on the approach proposed in Luedtke [2008] is applied. On the other hand, the ALM problems are a natural candidate for introducing a sequential SD requirement with positive managerial and decision making implications. An early application of SD criteria in a multistage ALM problem was due to Yang et al. [2010] with a focus on risk control at specific stages. More recently, yet enforcing SD constraints at individual stages, under an independence assumption, Consigli et al. [2020] solved an individual ALM problem over a long term horizon. A similar approach was previously adopted by Kopa et al. [2018] to solve an optimal pension allocation problem based on a multi-criteria optimization problem formulation with SD constraints at an intermediate and at the final stage. We also refer to an ALM problem over several stages and a relatively long planning horizon subject discussed in Moriggia et al. [2019] and Consigli et al. [2020]. In that work the SD relation is used to compare the performance of the constructed portfolio to a benchmark portfolio stage-wise at selected stages. The numerical solution is based on the approach proposed in Luedtke [2008]. In Mei et al. [2022], the authors discuss a portfolio selection problem in a multi-stage setting using the comparison proposed in Dentcheva and Ruszczyński [2008b]. These references show that the application of stochastic dominance as a risk constraint leads to relevant computational implications.

The contribution of our paper can be summarized as follows.

- We introduce a novel multi-stage ALM model using time-consistent stochastic-ordering relations and dynamic measures of risk for constructing immunized portfolios; the model includes a comprehensive risk control enabling the AL manager to satisfy capital requirements, preserve the company funding status, and fulfil liability obligations;

- We develop an efficient decomposition method to solve the associated risk-averse multi-stage optimization problem and discuss its convergence;

- We provide an extended set of financial-based validation evidence and sensitivity results analysing the impact of the stochastic dominance conditions.

The paper is organized as follows. We introduce the necessary notions and information related to stochastic dominance in Section 1. The ALM problem is then formulated in

Section 2. In Section 3, we present the stochastic models adopted to derive the full set of coefficients of the ALM model. The numerical approach developed to solve the ALM problem is presented in Section 4. Section 5 discusses the computational evidences collected to validate the methodology and to support the decision-making process. Finally, conclusions are drawn in Section 6. We leave to the appendices the detailed description of the stochastic models supporting scenario generation and the algorithm developed for this project.

1 Sequential stochastic dominance

First, we introduce the notions of stochastic dominance of first and higher order. The right-continuous cumulative distribution function (CDF) $F_Z(\eta)$ of Z is defined as $F_Z(\eta) = P(Z \leq \eta)$ and the survival function of Z is given by $\bar{F}_Z(\eta) = P(Z > \eta)$. The integrated distribution function $F_Z^{(2)}(\eta)$ is defined as follows:

$$F_Z^{(2)}(\eta) = \int_{-\infty}^{\eta} F_Z(t)\, dt \ \text{ for } \eta \in \mathbb{R}.$$

Clearly, the function $F_Z^{(2)}(\cdot)$ is finite everywhere whenever Z is integrable and it is convex as an integral of a non-decreasing function. For a random variables Z with a finite k-th moment, $k \geq 2$, we define recursively the functions

$$F_Z^{(k+1)}(\eta) = \int_{-\infty}^{\eta} F_Z^{(k)}(\alpha)\ d\alpha \quad \text{for } \eta \in \mathbb{R}. \tag{1}$$

Definition 1. *(i) A random variable V is stochastically larger than a random variable Z with respect to the first order stochastic dominance (denoted $V \succeq_{(1)} Z$) if $F_V(\eta) \leq F_Z(\eta)$ for all $\eta \in \mathbb{R}$.*

(ii) For two random variables V and Z, it is said that the variable V is stochastically larger than Z with respect to the k-th order stochastic dominance (denoted $V \succeq_{(k)} Z$) if $F_V^{(k)}(\eta) \leq F_Z^{(k)}(\eta)$ for all $\eta \in \mathbb{R}$.

Notice that the relation $V \succeq_{(1)} Z$ is also equivalent to $\bar{F}_Z(\eta) \leq \bar{F}_V(\eta)$ for all $\eta \in \mathbb{R}$, meaning that V takes larger values more frequently but comparisons of integrated survival functions lead to different relations than the k-order dominance.

We use the shorthand notation $a_+ = \max(0, a)$ for any $a \in \mathbb{R}$. Changing the order of integration in the definition of the function $F_Z^{(2)}(\cdot)$, we obtain the following equivalent representation of the second-order relation:

$$V \succeq_{(2)} Z \quad \Leftrightarrow \quad \mathbb{E}[\eta - V]_+ \leq \mathbb{E}[\eta - Z]_+, \quad \text{for all } \eta \in \mathbb{R}. \tag{2}$$

The second-order stochastic dominance relation can also be characterized by the respective quantile functions, which turned out to be very useful. Let $F_Z^{-1}(\cdot)$ be the left continuous inverse of the cumulative distribution function $F_Z(\cdot)$ defined by

$$F_Z^{-1}(p) = \inf\{\eta : F_Z(\eta) \geq p\}, \ \text{for } 0 < p < 1.$$

The absolute Lorenz function $L_Z : [0,1] \to \mathbb{R}$, introduced in the seminal work of Lorenz, see Lorenz [1905], is defined as the cumulative quantile function:

$$L_Z(p) = \int_0^p F_Z^{-1}(t)dt \text{ for } 0 < p \le 1.$$

The definition of the function beyond the interval $(0,1]$ is extended by setting $L_Z(0) = 0$ and $L_Z(p) = \infty$ for $p \notin [0,1]$. The Lorenz function is widely used in economics for comparison of income streams.

Interestingly the integrated distribution function and the Lorenz function are related via conjugate duality. It is shown in Ogryczak and Ruszczyński [2002] that $L_Z(\cdot)$ and $F_Z^{(2)}(\cdot)$ are Fenchel conjugate functions. This result implies that relating the Lorenz functions of two integrable random variables provides an equivalent characterizations of the stochastic ordering relations, i.e.,

$$V \succeq_{(2)} Z \Leftrightarrow L_V(p) \ge L_Z(p) \quad \text{for all } p \in [0,1]. \tag{3}$$

It is clear that $Z \succeq_{(1)} V$ if and only if $F_Z^{(-1)}(\eta) \ge F_V^{(-1)}(\eta)$ for all $\eta \in \mathbb{R}$ but a quantile characterization for the relations of order $k > 2$ is not available.

Let us turn to comparison of sequences. Given probability space $(\Omega, \mathcal{F}, P)$, a filtration $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$, denote $\mathcal{Z} = \mathcal{L}_1(\Omega, \mathcal{F}_1, P) \times \cdots \times \mathcal{L}_1(\Omega, \mathcal{F}_T, P)$. We assume that the filtration is generated by the random data process $\{\xi_t\}_{t=1}^T$ and denote the history of the data process until time t by $\xi_{[t]}$.

We wish to compare two sequences $\mathbf{X} = (X_1, X_2, \ldots, X_T)$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_T)$ in $\mathcal{Z}$ at any time $t = 1, \ldots T$ in a consistent way in the context of a sequential decision problem whose description is based on the data process $\{\xi_t\}_{t=1}^T$. To this end, we introduce the cumulative sum of $\mathbf{X}$ until time t as a function of the history path $\xi_{[t]}$, i.e.,

$$\mathbf{x}_t(\xi_{[t]}) = (X_1 + X_2 + \cdots + X_t)(\xi_{[t]}), \quad t = 1, \ldots T.$$

We shall denote the projected future value for the sequence $\mathbf{X}$ at time t when $\xi_{[t]}$ is fixed defined as follows:

$$\mathbf{X}_{t+1}|\xi_{[t]} = X_{t+1}|\xi_{[t]} + \mathbb{E}_{t+1}\Big[X_{t+2}|\xi_{[t]} + \ \mathbb{E}_{t+2}\Big[X_{t+3}|\xi_{[t]} + \cdots + \mathbb{E}_{T-1}[X_T|\xi_{[t]}]\Big]\Big].$$

A time-consistent comparison, proposed in Dentcheva et al. [2022], is obtained in the following way. We call the sequence $\mathbf{X} \in \mathcal{Z}$ stochastically larger than the sequence $\mathbf{Y} \in \mathcal{Z}$ if at any time t and history $\xi_{[t]}$ the following holds

$$\mathbf{x}_t(\xi_{[t]}) + \mathbf{X}_{t+1}\,|\xi_{[t]} \succeq_{\xi_{[t]}} \ \mathbf{y}_t(\xi_{[t]}) + \mathbf{Y}_{t+1}\,|\xi_{[t]},$$

where the comparison $\succeq_{\xi_{[t]}}$ is an appropriately chosen stochastic ordering for scalar-valued random variables. The choice may depend on the current state of the data process, or simply on the time of comparison. In particular, we may postulate

$$\mathbf{x}_t(\xi_{[t]}) + \mathbf{X}_{t+1}\,|\xi_{[t]} \succeq_{(2)} \ \mathbf{y}_t(\xi_{[t]}) + \mathbf{Y}_{t+1}\,|\xi_{[t]}.$$

If we denote the accumulated difference between $\mathbf{X}$ and $\mathbf{Y}$ until time t along the path (scenario) $\xi_{[t]}$ by $\sigma_t(\xi_{[t]})$, then at time t we have the following

$$\sigma_t(\xi_{[t]}) + \mathbf{X}_{t+1}\,|\xi_{[t]} \succeq_{(2)} \ \mathbf{Y}_{t+1}\,|\xi_{[t]}. \tag{4}$$

A simpler comparison is to look only one step ahead and requires

$$\sigma_t(\xi_{[t]}) + X_{t+1}|\xi_{[t]} \ \succeq_{(2)} \ Y_{t+1}|\xi_{[t]}. \tag{5}$$

Notice that comparison (4) is better suited for situations when the decision maker is interested in comparing the total reward (profit) associated with $\mathbf{X}$ to that of $\mathbf{Y}$. Comparison (5) is more suitable for situation when dominating at every state of the process is essential. For this reason, we utilize the latter comparison in this paper.

The sequences of random variables that we consider are modeled as scenario tree processes. We apply the SSD principles via comparison (5) specifically to the asset portfolio distribution (as $\mathbf{X}$) relative to the probability distribution of the liabilities (as $\mathbf{Y}$): in financial economics both quantities define stock variables, the former representing the current value of all assets of an intermediary and the latter the obligations still pending at current time. Modern accounting principles, see European Parliament [2009], require those quantities to be *marked-to-market* and recent solvency regulations, to limit widespread financial instability in the markets, ask for *sufficient* capital to hedge future negative scenarios and the stability over time of the funding conditions. These latter require the value of the portfolio at every point in time to be sufficient to cover the current liabilities. These considerations motivate an ALM problem formulation capturing both aspects.

The model formulation in Section 2 relies on an exogenous specification of the liability sequence to be stochastically dominated. As control process we consider the portfolio allocations that *generate* the portfolio distribution in every stage of the problem. This is then naturally defined as a multistage problem with a long but finite planning horizon. By enforcing an SSD ordering between the portfolio and the liability dynamics, the *ALM manager* may allow for a few scenarios under which the solvency condition may actually worsen, but overall she/he will preserve an effective liability hedge. Furthermore she/he will hold over time a sufficient capital to account for future losses. Any ALM manager is however well aware of the cost of capital that she/he will typically try to minimize. The recent 2020-2021 global crisis provides a very good motivation for the adopted SSD approach and helps clarifying the model rationale. Indeed, consider a generic insurance intermediary issuing policies for property and casualty as well as health and life insurance policies (see Consigli et al. [2012]): this intermediary was confronted in 2020 with increasing liabilities and cash outflows, heavy negative asset returns and very low if not negative interest rates that did further deteriorate the liability condition. In this project we then impose SSD conditions to preserve a good asset-liability ratio and determine a sufficient, yet minimal, capital to hedge those type of negative scenarios. A stressed scenario analysis is also conducted in the case study.

2 Asset-liability management: problem description and formulation

We consider the decision problem of a financial intermediary managing assets and liabilities exposed to several risk sources over a finite time horizon T. The financial intermediary needs to define in which assets to invest at any of the discrete times $t \in \mathcal{T}' := \{0, 1, \ldots, T-1\}$. We consider a set $\mathcal{I}$ of liquid assets, divided into fixed income assets $\mathcal{I}_1$ and equity assets $\mathcal{I}_2$. The asset universe $\mathcal{I}'$ includes also a cash account labelled as 0, such as $\mathcal{I}' := \mathcal{I} \cup \{0\}$. At stage $t = 0$, we assume that the financial intermediary holds an initial amount $\hat{x}_{i,0}$ for liquid asset $i \in \mathcal{I}$ and that the proportion of each liquid asset $i \in \mathcal{I}$ in the portfolio must respect specific lower and upper bounds θ_i^{m} and θ_i^{M} relative to an evolving scenario-dependent portfolio value. In addition, fixed income assets $i \in \mathcal{I}_1$ are characterized by a deterministic duration parameter $\delta_{i,t}^x$ at each stage t. Assets price returns and gain-loss coefficients are instead considered as random parameters evolving as discrete-time stochastic processes.

Uncertainty in assets price returns and gain-loss coefficients is represented by means of a non recombining scenario tree, with $\mathcal{N}$ indicating its set of nodes. For each stage $t \in \mathcal{T} := \{\mathcal{T}' \cup T\}$, there is a discrete set of nodes $\mathcal{N}_t$. The final set $\mathcal{N}_T$ is the set of nodes called leaves, while the set $\mathcal{N}_0$ is composed of a unique node, i.e., the root. Each node at stage t, except the root, is connected to a unique node at stage $t-1$, which is called ancestor node $a(n)$, and to nodes at stage $t+1$, called successors. For each node n except the leaves (i.e., $n \in \mathcal{N}_t,\ t < T$) there exists a non-empty set of children nodes $\mathcal{C}(n) \in \mathcal{N}_{t+1}$. A scenario is a path through nodes from the root node to a leaf node. We represent with p_n the probability of node n and we have $\sum_{n \in \mathcal{N}_t} p_n = 1,\ t \in \mathcal{T}$. For each liquid asset $i \in \mathcal{I}$, we represent by $r_{i,n}$ and $g_{i,n}$ the realization at node $n \in \mathcal{N}_t, t \in \mathcal{T}$ of price returns and gain-loss coefficients respectively. Details on their evolution will be provided later in Section 3.3.

The other source of uncertainty is given by due payments $L_{j,n}$ generated by liability class $j \in \mathcal{J}$, reflected in random values in both the liability durations $\delta_{j,n}^{\lambda}$ and the nominal value of liability contracts $\lambda_{j,n}$. At each stage t, the total value of liabilities incurred by the financial intermediary is therefore a stochastic parameter, represented by $\Lambda_n := \sum_{j \in \mathcal{J}} \lambda_{j,n},\ n \in \mathcal{N}_t$. In terms of durations, a small mismatch $\overline{\Delta}^{(x,\lambda)}$ between asset durations and liability durations is allowed. Finally, the core business of the financial intermediary generates over the time horizon uncertain revenues $c_n,\ n \in \mathcal{N}_t$. Thus, the difference $c_n - \sum_{j \in \mathcal{J}} L_{j,n},\ n \in \mathcal{N}_t$ represents the stochastic core technical profit at stage t.

In such a framework, the financial intermediary wants to determine an optimal portfolio immunization strategy, by defining the evolution over the time horizon of the assets portfolio to cover liabilities. Specifically, to represent the amount of each asset $i \in \mathcal{I}'$ held at stage $t \in \mathcal{T}$, the continuous variables $x_{i,n}, n \in \mathcal{N}_t$ are introduced. The assets portfolio composition may change over the time due to buying and selling decisions that can occur in any stage except the final one. We thus introduce the two continuous variables $x_{i,n}^+$ and $x_{i,n}^-, n \in \mathcal{N}_t$ to represent the quantity of each liquid asset $i \in \mathcal{I}$ purchased and sold at stage $t \in \mathcal{T}'$. It is worth mentioning that assets can also be purchased in the root node. This is done by drawing the quantity $\hat{x}_{0,0}$ on the initial deposit in the cash account. Moreover, transaction costs associated with buying and selling decisions are represented by parameters ϕ^+ and ϕ^- respectively. The asset portfolio rebalancing occurring at stage $t \geq 1$ may determine either gains or losses $z_n,\ n \in \mathcal{N}_t$ for the financial intermediary. Finally, we assume the existence at each stage $t > 1$ of a regulatory capital K_t representing a minimum capital requirement needed to hedge current liabilities and any potential loss over the next stages. Thus, the financial intermediary also needs to control the evolution of the risk capital $k_n,\ n \in \mathcal{N}_t$ over the set of stages.

We define the following notation:

Sets:

- $\mathcal{T} = \{t \ : \ t = 0, \dots, T\}$: set of stages;
- $\mathcal{T}' = \{t \ : \ t = 0, \dots, T-1\}$: set of stages (last stage excluded);
- $\mathcal{N} = \{n \ : \ n = 0, \dots, N\}$: set of nodes of the scenario tree;
- $\mathcal{N}_t \subset \mathcal{N}$: set of the scenario tree nodes at stage $t \in \mathcal{T}$;
- $a(n)$: ancestor of node $n \in \mathcal{N}_t,\ t \in \mathcal{T} \setminus \{0\}$;
- $\mathcal{C}(n)$: set of children of node $n \in \mathcal{N}_t,\ t \in \mathcal{T}'$;
- $\mathcal{I}' = \{i \ : \ i = 0, \dots, I\}$: set of assets (cash included $i = 0$);
- $\mathcal{I} = \{i \ : \ i = 1, \dots, I\} = \mathcal{I}_1 \cup \mathcal{I}_2$: set of liquid assets;
- $\mathcal{I}_1 = \{i \ : \ i = 1, \dots, I_1\} \subset \mathcal{I}$: subset of fixed income assets;
- $\mathcal{I}_2 = \{i \ : \ i = 1, \dots, I_2\} \subset \mathcal{I}$: subset of equity assets;
- $\mathcal{J} = \{j \ : \ j = 1, \dots, J\}$: set of liabilities.

Deterministic Parameters:

- $\hat{x}_{i,0}$: initial amount of liquid asset $i \in \mathcal{I}$ held;
- θ_i^{m} minimum proportion of liquid asset $i \in \mathcal{I}$ in the portfolio;
- θ_i^{M}: maximum proportion of liquid asset $i \in \mathcal{I}$ in the portfolio;
- $\delta_{i,t}^{x}$: duration of fixed income asset $i \in \mathcal{I}_1$ in stage $t \in \mathcal{T}$;
- $\overline{\Delta}^{(x,\lambda)}$: maximum duration mismatching;
- ϕ^{+}: investment unit transaction cost coefficient;
- ϕ^{-}: selling unit transaction cost coefficients;
- K_t: regulatory capital in stage $t \in \mathcal{T}$;
- $\alpha \in [0,1]$: parameter of the convex combination of the shortfall and the opposite of the investment profits;
- β: weight in the objective function of the initial cash account deposit.

Stochastic Parameters:

- $r_{i,n}$: price return of asset $i \in \mathcal{I}'$ in node $n \in \mathcal{N}_t,\ t \in \mathcal{T} \setminus \{0\}$;
- $g_{i,n}$: gain-loss coefficient of asset $i \in \mathcal{I}$ in node $n \in \mathcal{N}_t,\ t \in \mathcal{T} \setminus \{0\}$;

- $L_{j,n}$: cash outflows associated with liability $j \in \mathcal{J}$ in node $n \in \mathcal{N}_t,\ t \in \mathcal{T} \setminus \{0\}$;
- $\delta^{\lambda}_{j,n}$: duration of liability $j \in \mathcal{J}$ in node $n \in \mathcal{N}$;
- $\lambda_{j,n}$: value of liability $j \in \mathcal{J}$ in node $n \in \mathcal{N}$;
- Λ_n: total liability value $\sum_{j\in\mathcal{J}} \lambda_{j,n}$ in node $n \in \mathcal{N}$;
- c_n: core business revenues in node $n \in \mathcal{N}_t,\ t \in \mathcal{T} \setminus \{0\}$.

Decision Variables:

- $x^+_{i,n} \in \mathbb{R}^+$: amount of asset $i \in \mathcal{I}$ purchased in node $n \in \mathcal{N}$;
- $x^-_{i,n} \in \mathbb{R}^+$: amount of asset $i \in \mathcal{I}$ sold in node $n \in \mathcal{N}$;
- $x_{i,n} \in \mathbb{R}^+$: amount of asset $i \in \mathcal{I}'$ held in node $n \in \mathcal{N}$;
- $\hat{x}_{0,0} \in \mathbb{R}^+$: initial cash account deposit;
- $z_n \in \mathbb{R}$: cumulative investment profit from portfolio rebalancing in node $n \in \mathcal{N}_t,\ t \in \mathcal{T} \setminus \{0\}$. $z_0 = 0$ at the root;
- $k_n \in \mathbb{R}^+$: capital value in node $n \in \mathcal{N}$.

The corresponding optimization model is formulated as follows:

$$\min \rho_0 \circ \cdots \circ \rho_{T-1} \left[\sum_{t\in\mathcal{T}\setminus\{0\}} \sum_{n\in\mathcal{N}_t} \left[\alpha (K_t - k_n)_+ - (1-\alpha) \sum_{i\in\mathcal{I}} g_{i,n} x^-_{i,n} \right] \right] + \beta \hat{x}_{0,0} \tag{6a}$$

$$\text{s.t.}\quad x_{i,0} = \hat{x}_{i,0} + x^+_{i,0} - x^-_{i,0}, \qquad i \in \mathcal{I}, \tag{6b}$$

$$x_{0,0} = \hat{x}_{0,0} + \sum_{i\in\mathcal{I}} x^-_{i,0}(1-\phi^-) - \sum_{i\in\mathcal{I}} x^+_{i,0}(1+\phi^+), \tag{6c}$$

$$x_{i,n} = x_{i,a(n)}(1+r_{i,n}) + x^+_{i,n} - x^-_{i,n}, \qquad i \in \mathcal{I},\quad n \in \mathcal{N}_t,\quad t \in \mathcal{T} \setminus \{0\}, \tag{6d}$$

$$z_n = \sum_{i\in\mathcal{I}} g_{i,n} x^-_{i,n} + z_{a(n)}, \quad n \in \mathcal{N}_t,\quad t \in \mathcal{T} \setminus \{0\}, \tag{6e}$$

$$\begin{aligned} x_{0,n} &= x_{0,a(n)}(1+r_{0,a(n)}) + \sum_{i\in\mathcal{I}} x^-_{i,n}(1-\phi^-) + \\ &- \sum_{i\in\mathcal{I}} x^+_{i,n}(1+\phi^+) + c_n - \sum_{j\in\mathcal{J}} L_{j,n}, \quad n \in \mathcal{N}_t,\quad t \in \mathcal{T} \setminus \{0\}, \end{aligned} \tag{6f}$$

$$- \sum_{j\in\mathcal{J}} \lambda_{j,n} \overline{\Delta}^{(x,\lambda)} \leq \sum_{i\in\mathcal{I}_1} x_{i,n} \delta^x_{i,t} - \sum_{j\in\mathcal{J}} \lambda_{j,n} \delta^{\lambda}_{j,n} \leq \sum_{j\in\mathcal{J}} \lambda_{j,n} \overline{\Delta}^{(x,\lambda)}, \quad n \in \mathcal{N}_t,\ t \in \mathcal{T}', \tag{6g}$$

$$k_n = \sum_{i\in\mathcal{I}} x_{i,n} - \sum_{j\in\mathcal{J}} \lambda_{j,n} + z_n, \quad n \in \mathcal{N}_t,\quad t \in \mathcal{T}, \tag{6h}$$

$$\sum_{i\in\mathcal{I}} x_{i,n}(1+r_{i,\mathcal{C}(n)}) \succeq_{(k)} \sum_{j\in\mathcal{J}} \lambda_{j,\mathcal{C}(n)}, \quad n \in \mathcal{N}_t,\quad t \in \mathcal{T}', \tag{6i}$$

$$\theta^{\mathrm{m}}_i \sum_{i\in\mathcal{I}} x_{i,n} \leq x_{i,n} \leq \theta^{\mathrm{M}}_i \sum_{i\in\mathcal{I}} x_{i,n}, \quad i \in \mathcal{I},\quad n \in \mathcal{N}_t,\quad t \in \mathcal{T}', \tag{6j}$$

$$x^+_{i,n} = x^-_{i,n} = 0, \quad n \in \mathcal{N}_T. \tag{6k}$$

Denoting with ρ_t a one-period conditional risk measure, the objective function (6a) minimizes the sum of the weighted initial amount of cash invested $\beta\hat{x}_{0,0}$ and the risk at each stage t of a convex combination of the shortfall $\sum_{n\in\mathcal{N}_t}(K_t - k_n)_+$ and the opposite of the investment profits $-\sum_{n\in\mathcal{N}_t}\sum_{i\in\mathcal{I}} g_{i,n}x^-_{i,n}$. The coefficients β and α are non-negative fixed inputs, the first to calibrate the problem with a possible default value $\beta = 1$ and the latter to span alternative risk profiles through different convex combinations. Constraint (6b) provides the quantities to purchase and sell at time $t = 0$, given the initial portfolio allocation $\hat{x}_{i,0}$ for each liquid asset $i \in \mathcal{I}$. Equation (6c) computes the cash flows associated with selling and buying decisions at stage $t = 0$. Equation (6d) represents the rebalancing constraints for asset $i \in \mathcal{I}$. Equation (6e) computes the cumulative gain and loss process in node n by focusing on the investment portfolio and helps distinguishing the realized gains, here considered, from the potential, though unrealized gains associated with the portfolio value evolution. The cash flow balance equation (6f) traces the evolution of cash surpluses at each stage taking into account interest accrual from previous stage cash balance, selling $x^-_{i,n}$ and buying $x^+_{i,n}$ decisions on the asset portfolio and revenues c_n and costs $\sum_{j\in\mathcal{J}} L_{j,n}$ associated with the intermediary core business. Constraint (6g) models the assets-liabilities duration matching, which allows a small duration mismatch $\overline{\Delta}^{(x,\lambda)}$ between fixed income assets and liabilities. Equation (6h) computes the agent capital endowment, reflecting the market-to-market difference $\sum_{i\in\mathcal{I}} x_{i,n} - \sum_{j\in\mathcal{J}} \lambda_{j,n}$ between the asset portfolio and the liability, plus the cumulative investment profit z_n. Every year such endowment increases if profits accumulate and decreases if new liabilities are issued or losses are accounted for. We do not account here for core technical profit because those will be captured by a positive cash balance $x_{0,n}$ which is already part of the capital definition. The agent solvency under this assumption is consistently determined by the ratio between the asset minus the liability and the capital growth induced by cumulative profits. Constraint (6i) enforces along the scenario tree the stochastic dominance of order k of the the asset portfolio values over the liability portfolio on the children nodes, thus conditionally along the tree, consistently with the decomposition method that will be employed. Finally, constraint (6j) enforces the portfolio diversification, by imposing minimum and maximum shares for each asset in the portfolio and (6k) rules out possible selling or buying decisions at the end of the planning horizon.

3 Uncertainty model

The ALM model implementation requires the specification of a rich set of random coefficients, assumed, in this setting, to follow a discrete non recombining tree process. We present in the following subsections the stochastic models adopted to derive the full set of coefficients of problem (6). The following economic and financial risk sources are accounted for and motivate the model specification based on investment horizon T and a liability evaluation horizon T_λ:

- The fluctuations of the term structure of interest rates and inflation have a joint impact on the asset portfolio and the liability of the intermediary through their pricing and duration mismatching.
- Credit risk fluctuations in the economy have an impact on corporate returns and

exogenous liability costs.

- Every asset class has specific risk factors driving their future behaviour.
- All the above jointly determine the risk exposure of the intermediary and the associated capital requirements.

3.1 Yield curve, inflation and credit spread models

Asset returns and liability costs depend on the evolution of inflation and the term structure of interest rates. For the latter we have implemented the popular *Nelson-Siegel-Svensson* model in the dynamic arbitrage-free version proposed by Christensen et al. [2009]. Let $y_{t,\tau}$ denote the yield quoted at time t over the term τ. We have:

$$y_{t,\tau} = b^y_{1,t} + b^y_{2,t} e^{-\frac{\tau}{\lambda_t}} + b^y_{3,t} \frac{\tau}{\lambda_t} e^{-\frac{\tau}{\lambda_t}}. \tag{7}$$

Eq. (7) relies on a 3-factor model with factors reflecting level, slope and convexity of the curve, as functions of $b^y_{1,t}$, $b^y_{2,t}$ and $b^y_{3,t}$, from which a long-term yield $b^y_{1,t}$ and an instantaneous yield $b^y_{1,t} + b^y_{2,t}$ for the instantaneous short rate dynamics $y_{t,0} = y_t$ can be derived. The parameter λ_t represents a decay factor, here expressed and estimated as a linear function of the coefficients $b^y_{j,t}$, $j = 1, 2, 3$:

$$\lambda_t = a^y_0 + a^y_1 b^y_{1,t} + a^y_2 b^y_{2,t} + a^y_3 b^y_{3,t} + \epsilon^y_t, \tag{8}$$

where $a^y_j, j = 0, 1, 2, 3$, are the coefficient processes and ϵ^y_t are the residuals, here supposed to be normally distributed and correlated. We have adopted multivariate Ordinary Least Squares (OLS) estimation and calibrated the model to enforce arbitrage free conditions, following Christensen et al. [2009].

We consider as reference for the inflation process π_t the annual Consumer Price Index (CPI) dynamics, specified as a simple mean-reverting model with the long term mean set at the European Central Bank (ECB) 2% target. Given an initial state π_0, for $t \in \mathcal{T}$ with monthly increments Δt, we assume a dynamic:

$$\pi_t = \pi_{t-\Delta t} + a^\pi (0.02 - \pi_{t-\Delta t}) \Delta t + \sigma^\pi \sqrt{\pi_{t-\Delta t}} \sqrt{\Delta t} \epsilon^\pi_t, \tag{9}$$

where ϵ^π_t are the residuals that we assume normally distributed and correlated, while parameters a^π and σ^π are the coefficient processes to be estimated on the data history.

Jointly with the yield curve, the spread model for Investment Grade (IG) borrowers in the euro area defines an explanatory variable for corporate returns as specified next. Credit spread s^{IG}_t random dynamics are described by the following autoregressive model:

$$s^{IG}_t = c_0 + c_1 s^{IG}_{t-\Delta t} + c_2 y_t + \epsilon^s_t, \tag{10}$$

where $c_j, j = 0, 1, 2$, are the coefficient processes to be estimated on data, ϵ^s_t are the residuals, supposed to be normally distributed and correlated and y_t is the instantaneous short rate.

We refer to Appendix A for the estimation of the coefficients associated with the statistical models introduced in this paragraph.

3.2 Liability model

We consider a liability process of a representative insurance intermediary with an *investment grade* credit rating that over time funds its activity relying on incoming premiums generated by claims and life contracts. We rely in this respect on the modeling framework in Consigli et al. [2011a]. Based on current market conditions, the intermediary is classified as an investment-grade BBB borrower in terms of credit rating. Our primary interest in this application is twofold: (i) analyse the impact on the company solvency of potentially disruptive liability scenarios, as those emerged recently due to the pandemics, and (ii) verify the effectiveness of multistage stochastic dominance feasibility conditions. To our knowledge this has not being attempted before neither in an ALM context nor more generally in dynamic stochastic problems.

We consider an insurance intermediary whose revenues and costs evolve according to a simple linear stochastic process. Specifically, see Consigli et al. [2011b], given initial long term estimates c_0 and $L_{j,0}$, cash inflows and revenues c_t at time t and insurance costs and compensations to underwriters $L_{j,t}$ for liability class j at time t are described by the following equations:

$$
\begin{aligned}
c_{t+1} &= c_t(1+\rho_{t+1}) && (11)\\
\rho_t &= \mu_\rho + \sigma_\rho \epsilon_t^\rho && (12)\\
L_{j,t+1} &= L_{j,t}(1+\xi_{j,t+1}) && (13)\\
\xi_{j,t} &= \mu_{j,\xi} + \sigma_{j,\xi}\epsilon_{j,t}^\xi, && (14)
\end{aligned}
$$

where μ_ρ ($\mu_{j,\xi}$) is the average increase for cash inflows (outflows), σ_ρ ($\sigma_{j,\xi}$) is the volatility of cash inflows (outflows), and ϵ_t^ρ ($\epsilon_{j,t}^\xi$) are the normally distributed residuals. By subjectively modifying the premiums and claims distributions, we can generate *stressed* liability scenarios and verify the impact on the optimal solvency conditions and investment policy.

Given the liability cash flows $L_{j,t}$, the current liability obligation $\lambda_{j,t}$ is defined as the discounted value of the expected payments as follows:

$$
\lambda_{j,t} = \mathbb{E}_t\left[\sum_{h\in\mathcal{T}_{\lambda,-t}} e^{-y_{t,h}(h-t)} L_{j,h}\right]. \qquad (15)
$$

We estimate the nodal liability values as scenario dependent discounted cash flows in descending nodes over a stage-dependent horizon $\mathcal{T}_\lambda$: at the root node this estimation includes all cash flows projected over T_λ years, then for every t we update the estimation to $T_\lambda - t = T_{\lambda,-t}$. Thus at the horizon T, future liability cash flows are accounted for $T_\lambda - T$ as future random cash-flows. As a result, as time evolves, the pressure on the asset portfolio decreases due to the shortening of the liability evaluation horizon and the associated duration $\delta_{j,t}^\lambda$, which is defined as follows:

$$
\delta_{j,t}^\lambda = \mathbb{E}_t\left[\frac{\sum_{h\in\mathcal{T}_{\lambda,-t}}(h-t)e^{-y_{t,h}(h-t)}L_{j,h}}{\lambda_{j,t}}\right]. \qquad (16)
$$

Discounting is then attained by backward recursion along the tree relying on the nodal realizations of the yield curve from the model described above.

3.3 Asset returns

We consider a partition of the asset universe $\mathcal{I}$ into two classes: for fixed income ETFs $\mathcal{I}_1 := \{i = 1, \dots, I_1\}$ and for equity ETFs $\mathcal{I}_2 := \{i = I_1 + 1, \dots, I_2\}$. The set $\mathcal{I}_1$ is further partitioned into treasury fixed income assets $i = 1, \dots, I_3,\ I_3 < I_1$, and corporate fixed income assets $i = I_3 + 1, \dots, I_1$.

Asset price returns are computed as $r_{i,t} = \frac{v_{i,t}}{v_{i,t-\Delta t}} - 1$, with $v_{i,t}$ to denote the ETF value at time t. We have:

$$r_{i,t} = b_{i,0} + b_{i,1} r_{i,t-1} + b_{i,2} y_{t-1,\delta^x_{i,t}} + b_{i,3}\pi_t + \epsilon_{i,t}, \quad i = 1, \dots, I_3, \tag{17}$$

where $y_{t-1,\delta^x_{i,t}}$ is the yield to maturity at time $t-1$ of asset i with underlying duration $\delta^x_{i,t}$, π_t is the EU inflation rate at time t given in (9) and $\epsilon_{i,t}$ are the residuals. Ordinary least squares estimation is employed to determine the regression coefficients $b_i = (b_{i,0}, b_{i,1}, b_{i,2}, b_{i,3})^\top,\ i = 1, \dots, I_3$.

Furthermore, price returns $r_{i,t}$ for corporate assets are described by the following process:

$$r_{i,t} = b_{i,0} + b_{i,1} r_{i,t-1} + b_{i,2} y_{t-1,\delta^x_{i,t}} + b_{i,3} s^{IG}_t + b_{i,4} r_{I_1+1,t} + \epsilon_{i,t}, \quad i = I_3 + 1, \dots, I_1, \tag{18}$$

which assumes a dependence on the credit spread variable s^{IG}_t and on small cap returns $r_{I_1+1,t}$ of equity $I_1 + 1 \in \mathcal{I}_2$. The vector of regression coefficients to be estimated is given by $b_i = (b_{i,0}, b_{i,1}, b_{i,2}, b_{i,3}, b_{i,4})^\top$.

Equity asset returns for $i \in \mathcal{I}_2$ are considered for large and small caps in the equity market plus emerging markets. These assets become of primary importance in the long term when trying to stochastically dominate the liability costs that may increase significantly upon increasing uncertainty. The correspondent autoregressive model is:

$$r_{i,t} = b_{i,0} + b_{i,1} r_{i,t-1} + b_{i,2} y_{t,1} + b_{i,3}\pi_t + b_{i,4} f_t + \epsilon_{i,t}, \quad i = I_1 + 1, \dots, I_2. \tag{19}$$

Model (19) postulates dependence of equity returns on previous returns, on the 1-year interest rate $y_{t,1}$ and on the *term spread* f_t between the 10- and the 1-year interest rates, whose relevance is related to its ability to capture long-term economic expectations. The corresponding vector of regression coefficients is $b_i = (b_{i,0}, b_{i,1}, b_{i,2}, b_{i,3}, b_{i,4})^\top,\ i = I_1 + 1, \dots, I_2$. We refer to Appendix A for the results of the estimation of the regression coefficients $b_i = (b_{i,0}, b_{i,1}, b_{i,2}, b_{i,3}, b_{i,4})^\top,\ i \in \mathcal{I}$.

Once defined the asset returns, we can compute the gain and loss coefficients $g_{i,t}$, which are associated with selling decisions of any type of asset. For $h \in \mathcal{T}, i \in \mathcal{I}$, let $\rho_{i,h} := \Pi^h_{s=1}(1 + r_{i,s}) - 1$. We define the average gain and loss coefficient per unit selling at time t as:

$$g_{i,t} := \frac{1}{t}\sum_{h=1}^{t} \rho_{i,h}. \tag{20}$$

We refer to Appendix B for the scenario generation algorithm associated wit the corresponding uncertain parameters.

4 Numerical solution

The idea is to use the dynamic programming formulation of the multi-stage problem and to solve it recursively by using a version of the multi-cut method (see Ruszczyński and Shapiro [2003]), in which additional event cuts approximate the stochastic order constraints and further cuts approximating the risk measures in the objective functions. The objective is in form (6a), which represents the time-consistent dynamic risk measures.

We introduce the extra variables $s_n \in \mathbb{R}^+$, representing the shortfall of capital below the regulatory capital in node $n \in \mathcal{N}_t,\ t = 1, \dots, T$. The problem can be solved recursively as follows. At the last stage, we calculate for every leaf node $n \in \mathcal{N}_T$ the shortfall below the regulatory capital:

$$
\begin{aligned}
Q_{n,T} = \min\ & \alpha s_n \\
\text{s.t.}\ & x_{i,n} = x_{i,a(n)}(1 + r_{i,n}), \\
& x_{0,n} = x_{0,a(n)}(1 + r_{0,a(n)}) + c_n - \sum_{j \in \mathcal{J}} L_{j,n}, \\
& z_n = z_{a(n)}, \\
& k_n = \sum_{i \in \mathcal{I}} x_{i,n} - \sum_{j \in \mathcal{J}} \lambda_{j,n} + z_n, \\
& s_n \geq K_T - k_n, \quad s_n \geq 0.
\end{aligned}
$$

For the nodes $n \in \mathcal{N}_t$, $t = 1 \dots, T-1$, we calculate:

$$
\begin{aligned}
Q_{n,t} = \min\ & \alpha s_n - (1-\alpha) \sum_{i \in \mathcal{I}} g_{i,n} x^-_{i,n} + \varrho_n[\mathcal{Q}_{t+1}|n] \\
\text{s.t.}\ & x_{i,n} = x_{i,a(n)}(1 + r_{i,n}) + x^+_{i,n} - x^-_{i,n}, \quad i \in \mathcal{I}, && (21) \\
& x_{0,n} = x_{0,a(n)}(1 + r_{0,a(n)}) + \sum_{i \in \mathcal{I}} x^-_{i,n}(1 - \phi^-) + \\
& - \sum_{i \in \mathcal{I}} x^+_{i,n}(1 + \phi^+) + c_n - \sum_{j \in \mathcal{J}} L_{j,n}, && (22) \\
& \theta^{\mathrm{m}}_i \sum_{i \in \mathcal{I}} x_{i,n} \leq x_{i,n} \leq \theta^{\mathrm{M}}_i \sum_{i \in \mathcal{I}} x_{i,n}, \quad i \in \mathcal{I}, && (23) \\
& - \sum_{j \in \mathcal{J}} \lambda_{j,n} \overline{\Delta}^{(x,\lambda)} \leq \sum_{i \in \mathcal{I}_1} x_{i,n} \delta^x_{i,t} - \sum_{j \in \mathcal{J}} \lambda_{j,n} \delta^\lambda_{j,n} \leq \sum_{j \in \mathcal{J}} \lambda_{j,n} \overline{\Delta}^{(x,\lambda)}, && (24) \\
& \sum_{i \in \mathcal{I}} x_{i,n}(1 + r_{i,\mathcal{C}(n)}) \succeq_{(k)} \sum_{j \in \mathcal{J}} \lambda_{j,\mathcal{C}(n)}, && (25) \\
& z_n = \sum_{i \in \mathcal{I}} g_{i,n} x^-_{i,n} + z_{a(n)}, && (26) \\
& k_n = \sum_{i \in \mathcal{I}} x_{i,n} - \sum_{j \in \mathcal{J}} \lambda_{j,n} + z_n, && (27) \\
& s_n \geq K_t - k_n, \quad s_n \geq 0.
\end{aligned}
$$

For the root node:

$$\begin{aligned} \min\ & \beta \hat{x}_{0,0} + \varrho_0[\mathcal{Q}_1] \\ \text{s.t.}\ & x_{i,0} = \hat{x}_{i,0} + x_{i,0}^+ - x_{i,0}^-, \quad i \in \mathcal{I}, \\ & x_{0,0} = \hat{x}_{0,0} + \sum_{i\in\mathcal{I}} x_{i,0}^-(1-\phi^-) - \sum_{i\in\mathcal{I}} x_{i,0}^+(1+\phi^+), \\ & (21) - (25), (27). \end{aligned}$$

We start with solving the problems at the leaf nodes with some initial guess for the variables at their ancestor nodes. Assume that we carry out iteration ℓ. For each problem at node $n \in \mathcal{N}_t$, we obtain its optimal value, denoted $\bar{v}_n^\ell$, the optimal solution, denoted $(x_n^\ell, (x_n^+)^\ell, (x_n^-)^\ell, z_n^\ell, k_n^\ell)$, and the optimal Lagrange multipliers $d_n^\ell \in \mathbb{R}^{|\mathcal{I}|+2}$ associated with constraints about re-balancing of the assets, cash, and the cumulative profit. This information provides an objective cut at the ancestor node of n of form:

$$\begin{aligned} v_{a(n)} &\geq \bar{v}_n^\ell + \langle -T_n^\top d_n^\ell, (x_{a(n)}, z_{a(n)}) - (x_n^\ell, z_n^\ell)\rangle \\ &= -\langle T_n^\top d_n^\ell, (x_{a(n)}, z_{a(n)})\rangle + \alpha_n^\ell, \quad \text{with } \alpha_n^\ell = \bar{v}_n^\ell + \langle T_n^\top d_n^\ell, (x_{a(n)}^\ell, z_{a(n)}^\ell)\rangle. \end{aligned}$$

Here T_n is the matrix containing the coefficients associated with the ancestor variables of node n; it is a diagonal matrix with elements $d_{ii} = 1 + r_{i,n}$ for $i \in \mathcal{I}$, $d_{00} = 1 + r_{0,a(n)}$, and $d_{ii} = 1$ for $i = |\mathcal{I}| + 2$. We shall gather the objective cuts for the objective function of node n constructed until iteration ℓ in the set $J_o^\ell(n)$.

Furthermore, at node n, having solved the problems for all successor nodes $m \in \mathcal{C}(n)$, we solve an auxiliary problem

$$\max_{\mu \in \mathcal{A}_\varrho} \sum_{m \in \mathcal{C}(n)} p_{n,m} \mu_m \bar{v}_m^\ell, \tag{28}$$

where $\mathcal{A}_\varrho$ is the convex subdifferential $\varrho[0]$ in the dual representation of the risk measure ϱ. Let μ_n^ℓ be the solution of that problem. As it is a subgradient of the risk measure, it provides a cut in the approximating problem for node n of the following form:

$$w_n \geq \langle \mu_n^\ell, v\rangle.$$

These cuts approximating the risk function at node n that are constructed until iteration ℓ are gathered in the set $J_r^\ell(n)$. We also need a parameter $\underline{w}$ to impose a lower bound on the value of the risk measure ϱ_n.

The ordering constraint is approximated according to the quantile method presented in Dentcheva and Martinez [2012], see also Dentcheva and Ruszczyński [2010]. This means that at node $n \in \mathcal{N}_t$, we compare the random variable Λ_n with realizations $\Lambda_{n,m} = \sum_{j\in\mathcal{J}} \lambda_{j,m}$, $m \in \mathcal{C}(n)$, and the random variable X_n with realizations $X_{n,m} = \sum_{i\in\mathcal{I}} x_{i,n}(1 + r_{i,m})$ where $r_{i,m}$ is associated with node $m \in \mathcal{C}(n)$.

In order to impose the stochastic dominance constraints (6i) assuming the order $k = 2$, we use the following method. We denote $S^1 = \{1, \ldots, |\mathcal{C}(n)|\}$.

Algorithm to impose stochastic dominance

Step 0: Set $\iota = 1$, $J_e^\iota(n) = \{S^1\}$, and $X_{n,m}^1 = \sum_{i\in\mathcal{I}} x_{i,n}^\ell(1 + r_{i,m})$ for all $m \in \mathcal{C}(n)$.

Step 1: Solve the problem:

$$\begin{aligned}
\min\ & \alpha s_n - (1-\alpha)\sum_{i\in\mathcal{I}} g_{i,n}x^-_{i,n} + w_n + \beta\hat{x}_{0,0} \\
\text{s.t.}\ & w_n \geq \langle \mu^j_n, v\rangle \quad j\in J^\ell_r(n),\ v\in\mathbb{R}^{|\mathcal{C}(n)|} \\
& v_m \geq -\langle T^\top_m d^j_m, (x_n,z_n)\rangle + \alpha^j_m \quad j\in J^\ell_o(m),\ m\in\mathcal{C}(n), \\
& x_{i,n} = x^{\ell-1}_{i,a(n)}(1+r_{i,n}) + x^+_{i,n} - x^-_{i,n}, \quad i\in\mathcal{I}, \\
& x_{0,n} = x^{\ell-1}_{0,a(n)}(1+r_{0,a(n)}) + \sum_i x^-_{i,n}(1-\phi^-)+ \\
& \quad -\sum_{i\in\mathcal{I}} x^+_{i,n}(1+\phi^+) + c_n - \sum_{j\in\mathcal{J}} L_{j,n}, \\
& \theta^{\mathrm{m}}_i \sum_{i\in\mathcal{I}} x_{i,n} \leq x_{i,n} \leq \theta^{\mathrm{M}}_i \sum_{i\in\mathcal{I}} x_{i,n}, \quad i\in\mathcal{I}, \\
& -\sum_{j\in\mathcal{J}} \lambda_{j,n}\overline{\Delta}^{(x,\lambda)} \leq \sum_{i\in\mathcal{I}_1} x_{i,n}\delta^x_{i,t} - \sum_{j\in\mathcal{J}} \lambda_{j,n}\delta^\lambda_{j,n} \leq \sum_{j\in\mathcal{J}} \lambda_{j,n}\overline{\Delta}^{(x,\lambda)} \\
& z_n = \sum_{i\in\mathcal{I}} x^-_{i,n} g_{i,n} + z^{\ell-1}_{a(n)}, \\
& k_n = \sum_{i\in\mathcal{I}} x_{i,n} + z_n - \sum_{j\in\mathcal{J}} \lambda_{j,n} \\
& \frac{1}{P(S^j)}\sum_{m\in S^j} p_{n,m}X^\iota_{n,m} \geq \frac{1}{P(S^j)}F^{(-2)}(\Lambda_n; P(S^j)), \quad S^j\in J^\iota_e(n), \\
& s_n \geq K_t - k_n, \\
& s_n \geq 0 \ \ w_n \geq \underline{w}.
\end{aligned} \tag{29}$$

Let X^ι_n be the new random variable associated with the solution of problem (29).

Step 2: Consider the sets $A^\iota_\eta = \{X^\iota_n \leq \eta\}$ and let

$$\delta_\iota = \sup_\eta \left\{ \frac{1}{P(A^\iota_\eta)}F^{(-2)}(\Lambda_n; P(A^\iota_\eta)) - \frac{1}{P(A^\iota_\eta)}\sum_{m\in A^\iota_\eta} p_{n,m}X^\iota_{n,m} : P(A^\iota_\eta) > 0 \right\}. \tag{30}$$

If $\delta_\iota \leq 0$, then index the solutions of problem (29) by ℓ and stop. Otherwise, continue.

Step 3: Find η^ι such that $P(X^\iota_n \leq \eta^\iota) > 0$ as well as

$$\frac{1}{P(A^\iota_{\eta^\iota})}\sum_{m\in A^\iota_{\eta^\iota}} p_{n,m}X^\iota_{n,m} - \frac{1}{P(A^\iota_{\eta^\iota})}F^{(-2)}(\Lambda_n; P(A^\iota_{\eta^\iota})) \leq -\frac{\delta_\iota}{2} \tag{31}$$

are satisfied.

Step 4: Set $S^\iota = A^\iota_{\eta^\iota}$, $J^{\iota+1}_e(n) = J^\iota_e(n)\cup\{S^\iota\}$, increase ι by one, and go to Step 1.

The solution of the problem v^ℓ_n provides a lower bound for the recourse function $Q_n(x_{a(n)}, z_{a(n)})$, while w^ℓ_n is a lower bound for the risk measure associated with node n.

If the problem is infeasible, we can construct a feasibility cut,

$$\gamma^\ell_n + \langle \tilde{d}^\ell_n, (x_{a(n)}, z_{a(n)})\rangle \leq 0. \tag{32}$$

The feasibility cuts remain valid for the true cost-to-go function.

We refer to the approximate problem (29) at each node of the scenario tree as $\mathcal{P}(n)$. Each of the problems $\mathcal{P}(n)$ maintains and updates the following data: its current solution $(x_n, (x_n^+), (x_n^-), z_n, k_n)$, convex polyhedral models of the cost-to-go functions $\underline{Q}^{(j)}(\cdot)$ of its successors $m \in \mathcal{C}(n)$ (if any), and the current approximation v_n and w_n of the optimal value of its own cost-to-go function and the risk measure at n. The operation of each subproblem is as follows:

Step 1. If n is not the root node, retrieve from the ancestor problem $\mathcal{P}(a(n))$ its current approximate solution $(x_{a(n)}, z_{a(n)})$.

Step 2. If n is not a leaf node, retrieve from each successor problem $\mathcal{P}(m)$, $m \in \mathcal{C}(n)$, all new objective and feasibility cuts and update the approximations of their cost-to-go functions $\underline{Q}_n(\cdot)$. Update the approximation of its risk measure by solving problem (28).

Step 3. Solve the problem (29).

(a) If it is solvable, update its solution and its optimal value. If n is not the root node and v_n increased, construct a new objective cut.

(b) If the problem is infeasible, and n is not the root node, construct a new feasibility cut. If n is the root node, then stop, because the entire problem is infeasible.

Step 4. Wait for the command to activate again, and then go to Step 1.

It remains to describe the way in which these subproblems are initiated, activated in the course of the solution procedure, and terminated. We assume that we know a sufficiently large number M such that each cost-to-go function can be bounded from below by $-M$. Our initial approximations of the successors' functions are just

$$\underline{Q}^{(j)}(\cdot) = -M.$$

At the beginning, no ancestor solutions are available, but we can initiate each subproblem with some arbitrary point $(x_{a(n)}, z_{a(n)})$.

There is much freedom in determining the order in which the subproblems are solved. Three rules have to be observed.

1. There is no sense to activate a subproblem $\mathcal{P}(n)$ whose ancestor's solution did not change, and whose successor problems $\mathcal{P}(m)$, $m \in \mathcal{C}(n)$, did not generate any new cuts since this problem was activated last.

2. If a subproblem $\mathcal{P}(n)$ has a new solution, each of its successors $\mathcal{P}(m)$, $m \in \mathcal{C}(n)$ has to be activated some time after this solution has been obtained.

3. If a subproblem $\mathcal{P}(n)$ generates a new cut, i.e., if it is infeasible or has a new optimal value v_n, its ancestor $\mathcal{P}(a(n))$ has to be activated some time after this cut has been generated.

We shall terminate the method if Rule 1 applies to all subproblems, in which case we claim that the current solutions constitute the optimal solution of the entire problem. The other stopping test is the infeasibility test at Step 3(a) for the root node. It is obvious, because we operate with relaxations here, and if a relaxation is infeasible, so is the true problem.

Now we argue that the method discovers infeasibility of the problem or converges to a solution of it. If the method stops because the Rule 1 applies to all subproblems, i.e., no subproblem needs to be activated, then, we claim that the current solutions constitute the optimal policy of the entire problem. If the method stops because of infeasibility at the root node, then the whole problem is infeasible, because we operate with relaxations. Hence, if a relaxation is infeasible, so is the true problem. We also observe that for each set of decisions, the algorithm employed to impose the SD constraint terminates in finitely many steps discovering infeasibility or identifying an optimal solution. This is due to the fact that the Lorenz functions of random variables with finitely many realizations are piece-wise linear; cf. also Dentcheva and Martinez [2012], Theorem 4. The approximation of the risk measure for a fixed random variable converges to its true value due to the convergence of the cutting plane method because the subdifferential set $\mathcal{A}_\varrho$ is a closed and bounded convex set. In the case of the mean-semideviation of first order, or the average value at risk combined with the expected value, we shall obtain an exact calculation after finitely many steps because the dual set is polyhedral, otherwise, we need to terminate the approximation when a prescribed numerical accuracy is reached. In such a case, only finitely many cuts are used to approximate the measure of risk up to the prescribed accuracy. Finally, the multicut method approximates the optimal value of the recourse function by objective cuts and its domain by feasibility cuts. It is convergent due to the convergence of the cutting plane method. For polyhedral functions, the method converges in finitely many iterations (see, Ruszczyński and Shapiro [2003], Chapter 3).

5 Computational evidence

In this section, an extended set of computational results is presented with the aim of validating the proposed methodology and discussing the most relevant financial evidence. Following the ALM problem in (6), we consider an ALM manager seeking a minimal initial capital injection, sufficient however to fund an investment strategy, with periodic revision, able to cover all liabilities and minimize the shortfall with respect to an exogenously defined regulatory capital over the following 10 years. We present through the section the results collected assuming either a *base liability scenario*, we may also refer to as *ongoing ALM* scenario, or a *stressed liability scenario*, as the one recently experienced in insurance markets. In particular, the main features of the data set used to generate the assets' and liability scenario trees are first summarized with their statistical properties in Section 5.1. We then present in Section 5.2 the evidence collected on the decomposition method developed to solve the optimization problem. Section 5.3 focuses on the results on risk capital allocation and interest rates exposures induced by the optimal solutions. The impact of stochastic dominance constraints is analyzed specifically in Section 5.4 with final results on the Intermediary solving conditions over a 10 year planning horizon.

5.1 Data inputs and experimental design

We take the perspective of a generic European insurance intermediary with a 10-year planning horizon for strategic asset allocation and liability hedging, see Consigli et al. [2012]. The asset universe includes the following *Exchange Traded Funds* (ETF), or benchmarks (in round brackets the ID ticker for *Yahoo! Finance*, see Yahoo!Finance [2023]):

- Money market index: *UCITS ETF C-EUR* (SMART.MI);
- 1-3 year bond index: *iShares Govt Bond 1-3yr UCITS ETF* (IBGS.L);
- 5-7 year bond index: *Xtrackers II Eurozone Govt Bond 5-7 UCITS ETF* (DBXR.DE);
- 10 year bond index: SPDR Bloomberg 10+ Year Euro Govt Bond UCITS ETF (SYBV.DE);
- IG corporate bond index: *iShares iBoxx Investment Grade Corporate Bond ETF* (LQD);
- Inflation linked bond index: *iShares Eur Inflation Linked Govt Bond UCITS ETF* (IBCI.AS);
- Large cap equity index: *iShares Core MSCI Europe UCITS ETF EUR* (IMEU.AS);
- Small cap equity index: *iShares Russell 2000 ETF* (IWM);
- Emerging markets equity index: *iShares MSCI Emerging Markets ETF* (EEM).

The statistical models are calibrated with a data history of monthly observations from December 2018 to December 2022. The subset $\mathcal{I}_1$ of the assets includes five fixed income treasury ETFs (SMART.MI, IBGS.L, DBXR.DE, SYBV.DE, IBCI.AS), and one IG corporate bond ETF (LQD). The subset $\mathcal{I}_2$ includes a global equity ETF (IMEU.AS), the small cap equity ETF (IWM) and an ETF for emerging markets (EEM). The decision space is thus $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$. Following the details in Table 1, the ETFs in class $\mathcal{I}_1$ are classified as *constant-to-maturity* (CTM) fixed income benchmarks carrying by construction a relatively stable duration coefficient. The asset-liability duration matching will rely on such simplifying assumptions. Insurance liabilities do instead carry a time dependent declining duration, as explained below.

Index $i \in \mathcal{I}_1$	ETF code	Duration δ_i^x
Money market index	Smart.MI	0.25 year
1-3 year bond index	IBGS.L	1.6 years
5-7 year bond index	DBXR.DE	5.6 years
10 year bond index	SYBV.DE	14.5 years
IG corporate bond index	LQD	8.4 years
Inflation linked bond index	IBCI.AS	8.1 years

Table 1: Durations $\delta_{i,t}^x$ of fixed-income assets $i \in \mathcal{I}_1$.

The following settings are assumed in the case study:

- ALM planning horizon of 10 years with stages $\mathcal{T} := \{0, \frac{1}{2}, 1, 3, 5, 7, 10\}$ years.
- A scenario tree with branching $[12-10-4-2-2-2]$ resulting into $\mathcal{N}_6 = 3840$ scenarios and $N = 7333$ nodes. Stochastic values for asset returns and liability costs are determined by applying the models described in Section 3 and the scenario generation procedure summarized in Appendix B.
- Different risk-reward trade-offs in the objective function can be considered by varying the coefficients α and β in eq. (6a). Parameter α determines the trade-off between the risk capital shortfall and the cumulative investment profit. By contrast, β acts uniquely on the initial portfolio estimate and will determine the optimal initial risk capital to be allocated in the form of a given root node portfolio allocation. In our numerical experiments, we assume $\alpha = \{0.25, 0.5, 0.75\}$ and $\beta = \{1, 2\}$.
- Liability evaluation horizon $T_\lambda = 20$ years.
- Investment lower and upper bounds $\theta_i^{\mathrm{m}} = 0.05$ and $\theta_i^{\mathrm{M}} = 0.4$ respectively on all assets $i \in \mathcal{I}$ to facilitate portfolio diversification. Transaction costs are set to $\phi^+ = \phi^- = 0.001$.
- Maximum duration mismatching between assets and liabilities $\overline{\Delta}^{(x,\lambda)} = \{1, 0.75\}$ years. These values help analysing the effectiveness of the portfolio immunization strategies. The AL manager will allocate the investment portfolio to hedge against liability interest rate exposure.
- Regulatory capital is determined relying on a very simple model, surely not intended to be on it's own compliant with the ongoing regulatory reforms in this context. Still we believe that the assumptions are sufficient to validate the overall ALM problem formulation and derive interesting insights. We assume in particular that the regulatory capital K_t increases over the 10 years at a constant 1-year interest rate, determined from the yield curve estimated at $t = 0$, with continuous compounding. At the initial stage, K_0 is determined based on two alternative assumptions: (i) based on a rule of thumb very approximate estimate, or (ii) based on potential asset and liability losses. See Section 5.3 for further details.

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Asset statistics and scenario tree process Table 2 compares the average monthly returns and standard deviation from historical data to those associated at the horizon with the tree process for each asset.

Liability scenarios The liability estimates follow the case study developed in Consigli et al. [2011a] for a large P&C company with an estimated first year cash inflows c_0 due to collected premiums of 4.2 Mln €, with mean $\mu_\rho = 0.5\%$ and volatility $\sigma_\rho = 1\%$, and cash outflows due to casualties associated with a liability $L_{1,n}$ of 2.2 Mln € and estimated 1% average annual increase $\mu_{1,\xi}$ and a 3% volatility $\sigma_{1,\xi}$. These forecasts correspond to the base scenario case. We consider a stressed scenario by assuming an annual average increase of $\mu_{1,\xi} = 5\%$. Based on these estimates we derive two possible evolutions of the

	Historical moments		Moments at $t = 10$	
	Mean	Std	Mean	Std
Smart.MI	−0.0003	0.0002	−0.0003	0.0001
IBGS.L	0.0023	0.0208	0.0022	0.0069
DBXR.DE	0.0013	0.0073	0.0013	0.0024
SYBV.DE	0.0013	0.0237	0.0012	0.0079
LQD	0.0019	0.0187	0.0020	0.0063
IBCI.AS	0.0025	0.0130	0.0025	0.0044
EEM	0.0038	0.0496	0.0038	0.0165
IMEU.AS	0.0038	0.0452	0.0038	0.0151
IWM	0.0097	0.0641	0.0096	0.0214

Table 2: Mean and standard deviation (Std) of returns $r_{i,t}$ of assets $i \in \mathcal{I}$: historical versus simulation evidence at the end of year 10.

insurance liability reserves Λ_n, shown in Fig. 1: in particular under the base scenario the current (time 0) liability estimate is $\Lambda_0 = 496.34$ Mln €, while under the stressed scenario, *ceteris paribus*, this amount increases to $\Lambda_0 = 519.85$ Mln €.

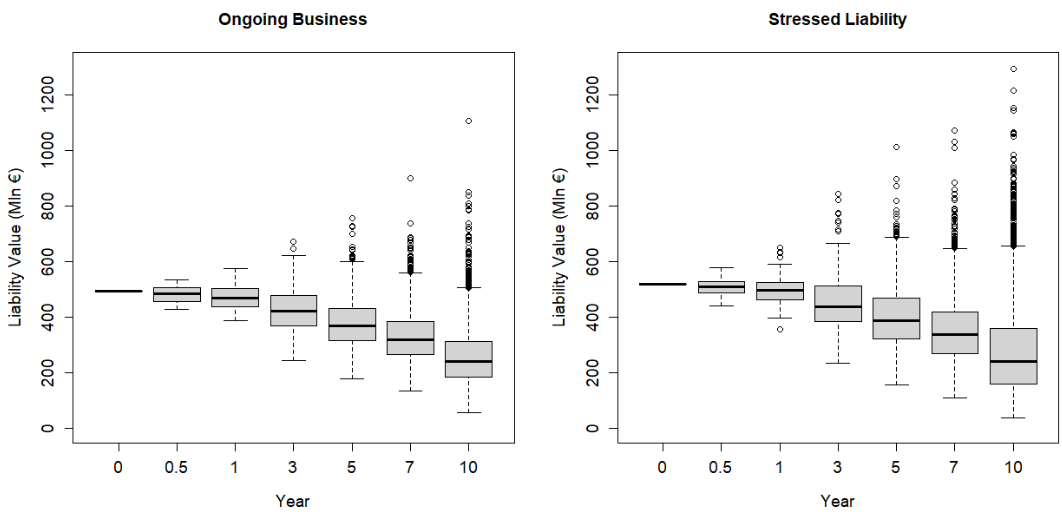

Fig. 1: Liability estimates $\Lambda_n, n \in \mathcal{N}_t, t \in \mathcal{T}$ in the *ongoing business* scenario (left) and in the *stressed liability* scenario (right).

The purpose of introducing a stressed liability scenario is twofold: first, between 2020 and 2021 most insurance companies have faced an even more negative scenario due to unprecedented increase of health-policies insurance costs, further impacted by persisting very low interest rates. Second, to further validate the adopted methodology and problem formulation, we consider a greater penalty on the initial capital allocation and reduce the duration maximum mismatching between asset and liabilities.

Stage $t \in \mathcal{T}$	0	1	2	3	4	5	6
N. of problems $\mathcal{P}(n),\ n \in \mathcal{N}_t$	1	12	120	480	960	1920	3840
N. of $\mathcal{P}(n)$ continuos variables	46	43	37	35	35	35	32
N. of $\mathcal{P}(n)$ constraints	85	78	54	46	46	46	14
CPU time spent on stage t	0.02%	0.19%	1.82%	6.62%	13.20%	26.41%	51.74%

Table 3: Number and size of problems $\mathcal{P}_n, n \in \mathcal{N}_t$ and CPU time allocation over stages $t \in \mathcal{T}$.

5.2 Problem decomposition: numerical results

On the available data set we applied the solution method described in Section 4. We chose the mean-semideviation of order 1 as a risk measure ρ_t (see Ruszczyński and Shapiro [2003]), which allowed as to formulate problems (29) as linear programs. Specifically, we set the weighting parameter between mean and semideviations in the risk-measure specification to 0.1. As a lower bound on the value of the risk measure we considered $\underline{w} = -10^6$. Parameters α and β in the objective function (6a) have been set to 0.5 and 1 respectively. Solutions for different values of these parameters are investigated in the following paragraph. All computational experiments were run on an ASUS laptop with a 3 GHz Intel Core i7-5500U Processor and 4 GB of RAM using solver Gurobi under GAMS 24.7.4 environment. We start with solving the problems at the leaf nodes by considering the solution of the worst-case liability scenario as initial guess for the variables at their ancestor nodes $(x^0_{a(n)}, z^0_{a(n)})$. This choice allows us to start the numerical procedure with a feasible solution in the leaf nodes. One iteration of the algorithm consists in solving all $N = 7333$ problems $\mathcal{P}(n)$ in the scenario tree. At each iteration, problems $\mathcal{P}(n),\ n \in \mathcal{N}$ are solved following a backward approach, from the leaf nodes to the root. In terms of order in which the subproblems are solved, to take into account the three rules described in Section 4, we control the set of problems to be solved by means of a binary parameter $ON(n)$, which is set to 1 if problem $\mathcal{P}(n)$ needs to be solved and to 0 otherwise. Once problem $\mathcal{P}(n)$ is solved, we set $ON(n) = 0$ and we compare the solution provided at the current iteration with the previous one: if problem $\mathcal{P}(n)$ has a new solution, we send an activation signal to the ancestor and to the children nodes of node n (i.e., $ON_{a(n)} = 1$ and $ON_m = 1,\ m \in \mathcal{C}(n)$). At each iteration, we only solve problems with $ON_n = 1$, since $ON_n = 0$ implies that the solution of the problem $\mathcal{P}(n)$ does not differ from the one determined before. The number of iterations required by the algorithm described above in the base case instance is 4. Thanks to the specific initialization procedure, problems $\mathcal{P}(n),\ n \in \mathcal{N}$ are always feasible in all iterations: this prevents from the generation of feasibility cuts. The total CPU time needed to solve the problem is 18817 seconds, corresponding to 5 hours 13 minutes and 37 seconds. With respect to the stressed liability case, this stressed situation does not affect the performance of the solution algorithm, keeping the number of iterations the same as in the base case, not requiring further feasibility cuts. We present in Table 3 additional evidences on the solution times and the dimension of the subproblems at the last iteration of the solution algorithm. As can be noticed, the size of the problems $\mathcal{P}(n)$ decreases over the stages due to the decreasing branching degree assumed in our instance. Despite the smaller size of problems in the leaf nodes, due to their large cardinality, most of the CPU time is spent solving the problem at the last stage. Results on stochastic dominance constraints will be described later in Section 5.4.

5.3 Capital allocation and risk management

The ALM manager first decision is related to the optimal amount of capital to be allocated in $t = 0$: this amount will evolve in the following stages, following eq. (6h) as a result of *ongoing* portfolio evolution relative to liability reserves and investment profit or losses. Alternatively, we may consider the stressed scenario case. From an economic viewpoint the risk capital is understood as that amount of financial resources the intermediary will allocate to hedge against negative asset-liability scenarios over the next stage. This amount will then be negatively affected by decreasing asset portfolio values and increasing liability costs. We assume a regulatory capital $K_t = K_0 e^{y_{0,1} t}$ where $y_{0,1}$ is the 1 year risk-free interest rate estimated at time 0 and K_0 is determined by considering two alternative approaches: a standard capital model and an internal capital model. More precisely, in the standard capital model we simply assume an exogenuous capital requirement K_0 of 100 Mln €. Instead, in the internal capital approach, the joint risks of the asset portfolio devaluation and liability increase are taken into account by computing the initial regulatory capital as $K_0 = [qX_0 + \Lambda_1^{0.99} - \Lambda_0]$, where X_0 is the optimal portfolio value at stage 0 in the standard capital model approach, $q \in [0, 1]$ expresses the possible asset portfolio loss and $\Lambda_1^{0.99}$ is the 99% quantile of the liabilities distribution at the end of stage 1 (i.e., at 6 months). We first consider a $q = 10\%$ loss for the asset portfolio, leading to a capital requirement $K_0 = 178$ Mln €. We then further increase the possible loss to $q = 15\%$, obtaining an initial regulatory capital of 212.65 Mln €. The evidence is summarized in Table 4.

The first two lines refer to the ongoing and stressed scenario results under the standard capital model: we show the average evolution of the risk capital k_n in each stage and its standard deviation. Lines 3 and 4 of the table refer to the stressed liability scenario under the internal capital model for the regulatory capital K_t, respectively with $q = 10\%$ and $q = 15\%$. As can be noticed, the four instances show a similar evolution of the risk capital k_n, which increases in the first five years and dramatically falls at the seventh year, when part of the portfolio is liquidated to attain high investment profits. In addition, the stressed condition for liabilities, together with the higher capital requirements, results into a significant increase of the capital to be allocated over the 10 years: due to the high cost of capital, such a scenario would be highly undesirable. Table 5 provides for each stage the percentage of nodes in the different instances with a shortfall, which is the positive part of the difference between regulatory capital and risk capital $s_n = (K_t - k_n)_+$. As can be seen, since the risk capital is usually above the regulatory capital, the shortfall occurs only in a limited number of cases. However, with respect to the base case scenario (line 1), the stressed values for liabilities and the higher capital requirements make regulatory capital harder to meet, inducing an increase of the shortfall frequency at the horizon (see lines 2, 3 and 4).

$\mu_{1,\xi}$	α	β		Year 0	1/2	1	3	5	7	10
1%	0.5	1	K_t	100	100.50	101.05	103.05	105.13	107.25	110.52
			k_n	135.83	152.97	170.41	277.97	360.89	142.66	221.54
				(0)	(39.97)	(53.34)	(94.94)	(127.86)	(54.12)	(97.59)
5%	0.5	1	K_t	100	100.50	101.05	103.05	105.13	107.25	110.52
			k_n	161.84	177.36	189.31	281.55	364.35	171.77	254.71
				(0)	(48.99)	(64.29)	(110.61)	(133.66)	(77.46)	(165.57)
5%	0.5	1	K_t	178	178.90	179.80	183.46	187.20	191.01	196.87
			k_n	241.95	264.02	275.58	381.23	464.61	253.05	345.44
				(0)	(49.49)	(65.87)	(113.82)	(138.84)	(86.98)	(174.99)
5%	0.5	1	K_t	212.65	213.72	214.80	219.18	223.64	228.19	235.20
			k_n	270.09	292.36	309.76	419.83	503.42	287.29	385.27
				(0)	(49.27)	(65.52)	(114.91)	(140.45)	(88.39)	(178.78)
5%	0.5	2	K_t	212.65	213.72	214.80	219.18	223.64	228.19	235.20
			k_n	181.44	220.41	258.23	419.92	503.41	287.29	385.26
				(0)	(48.15)	(66.13)	(114.90)	(140.44)	(88.38)	(178.78)
1%	0.25	1	K_t	100	100.50	101.05	103.05	105.13	107.25	110.52
			k_n	111.80	142.95	148.54	277.05	131.35	129.27	218.17
				(0)	(39.52)	(54.20)	(109.65)	(83.85)	(74.96)	(95.69)
1%	0.75	1	K_t	100	100.50	101.05	103.05	105.13	107.25	110.52
			k_n	192.27	185.33	184.85	278.31	406.63	142.87	225.04
				(0)	(40.05)	(55.27)	(96.28)	(119.62)	(50.23)	(88.79)

Table 4: Evolution of the regulatory capital K_t and mean (with in parenthesis standard deviation) of allocated risk capital k_n, $n \in \mathcal{N}_t$ in each stage $t \in \mathcal{T}$ for different values of parameters $\mu_{1,\xi}$, α, β and K_0.

$\mu_{1,\xi}$	α	β	K_0	Year 0	1/2	1	3	5	7	10
1%	0.5	1	100	0%	8.33%	8.33%	3.33%	0.83%	0.16%	5.70%
5%	0.5	1	100	0%	8.33%	9.17%	7.29%	1.46%	0.16%	13.80%
5%	0.5	1	178	0%	8.33%	7.50%	5.83%	0.94%	0.10%	13.93%
5%	0.5	1	212.65	0%	8.33%	7.50%	4.79%	0.83%	0.16%	14.14%
5%	0.5	2	212.65	100%	41.67%	25.83%	4.79%	0.83%	0.16%	14.14%
1%	0.25	1	100	0%	25.00%	22.50%	2.71%	1.04%	0.21%	7.16%
1%	0.75	1	100	0%	0%	5.83%	3.13%	0.63%	0.05%	4.60%

Table 5: Percentage at each stage $t \in \mathcal{T}$ of nodes with positive shortfall $s_n = (K_t - k_n)_+$, $n \in \mathcal{N}_t$ for different values of parameters $\mu_{1,\xi}$, α, β and K_0.

5.3.1 Risk preferences and initial capital allocation

The coefficient $\alpha \in [0, 1]$ defines a convex combination between the risk capital expected semideviation from the exogenous regulatory capital and the expected investment profit over all stages but the first. The coefficient β represents instead a penalty coefficient on the initial investment. We can then associate different risk profiles of the ALM manager to each pair. A $\{0.5, 1\}$-type of AL manager would assign same relevance to the shortfall minimization and cumulative investment profits, while being sufficiently safe with the current capital endowment. A $\{0.75, 1\}$-type of AL manager would instead be more concerned with capital requirement, while a $\{0.5, 2\}$-type of AL manager would seek an initial capital minimization. Lines 6 and 7 of Tables 4 and 5 are associated with the solution of the base case scenario with different relevance assigned to the shortfall minimization and cumulative investment profits. Table 4 shows how the different weights assigned to α have a limited impact on $\hat{x}_{0,0}$, and thus on the initial risk capital value k_0. Over the following stages, the three instances associated with the base case scenario show a similar evolution of the risk capital, with, however, an exception for $\alpha = 0.25$ in year 5: here the sharp decrease of the risk capital in year 5 is caused by the liquidation of part of the asset portfolio to attain a higher investment profit. From a financial perspective, this is consistent with the higher relevance of investment profits in the objective function of the problem. From Table 5 it can be noticed how higher values of α reduce the occurrence of the shortfall, especially in the early stages. From a financial perspective this is consistent with the extra-weight assigned to the risk capital shortfall minimization in the objective function of the problem. With regard to parameter β, line 5 of Tables 4 and 5 assesses the impact of a higher weight $\beta = 2$ for the initial cash account deposit under the highest capital requirement K_0 scenario. With respect to the case $\beta = 1$ (line 4), results show a significant reduced risk capital k_0, which leads to a higher occurrence of shortfall, especially in the early stages.

Table 6 displays the optimal asset allocation in the root node for selected risk profiles, by providing the portfolio initial value $\sum_{i\in\mathcal{I}} x_{i,0}$ and its division into fixed income ETFs, equity ETFs and cash.

$\mu_{1,\xi}$	α	β	K_0	$\sum_{i\in\mathcal{I}} x_{i,0}$ (Mln €)	Fixed Income ETFs	Equity ETFs	Cash
1%	0.5	1	100	632.17	64%	36%	0%
5%	0.5	1	100	681.69	63%	37%	0%
5%	0.5	1	178	761.98	57%	43%	0%
5%	0.5	1	212.65	789.95	57%	43%	0%
5%	0.5	2	212.65	701.30	61%	39%	0%
1%	0.25	1	100	608.14	64%	36%	0%
1%	0.75	1	100	688.61	66%	34%	0%

Table 6: Optimal root node portfolio value $\sum_{i\in\mathcal{I}} x_{i,0}$ and equity-bond portfolio allocation for different values of parameters $\mu_{1,\xi}$, α, β and K_0.

5.3.2 Interest rate risk

The ALM model (6) includes a specific set of constraints associated with the exposure to yield curve fluctuations. The AL manager seeks an optimal strategy while imposing a relatively strict constraint on duration mismatching between assets and liabilities. We are then considering only first-order impact of yield curve movements along the tree jointly on the asset and the liability portfolios. In presence of an excess asset portfolio duration over liabilities, then increasing interest rates will affect negatively the exposure to risk. On the contrary, in presence of an excess duration of liabilities over assets, decreasing interest rates will be detrimental.

We assess the interest rate risk exposure through the duration-matching constraint (6g). For duration mismatch we distinguish the following four cases, under each of the above bounds:

- $-\overline{\Delta}^{(x,\lambda)}$: constraint (6g) is active and liabilities duration exceed assets duration.
- $(-\overline{\Delta}^{(x,\lambda)}; 0]$: constraint (6g) is not active with liabilities durations exceeding assets durations.
- $(0; \overline{\Delta}^{(x,\lambda)})$: constraint (6g) is not active with assets durations exceeding liabilities duration.
- $\overline{\Delta}^{(x,\lambda)}$: constraint (6g) is active and assets durations exceed liabilities durations.

Fig. 2 shows for the ongoing business scenario the percentage of nodes with duration mismatching across the four groups in stages $t \in \mathcal{T}'$ for different weights assigned to parameter α and for $\overline{\Delta}^{(x,\lambda)} = 1$. The evidence is of a prevalent exposure to decreasing interest rates over the investment horizon. For $\beta = 1$, when reducing the weight on the risk capital shortfall in favour of trading profit ($\alpha = 0.25$), we see in Fig. 2 that a slightly more balanced A-L duration matching takes place from the early stages.

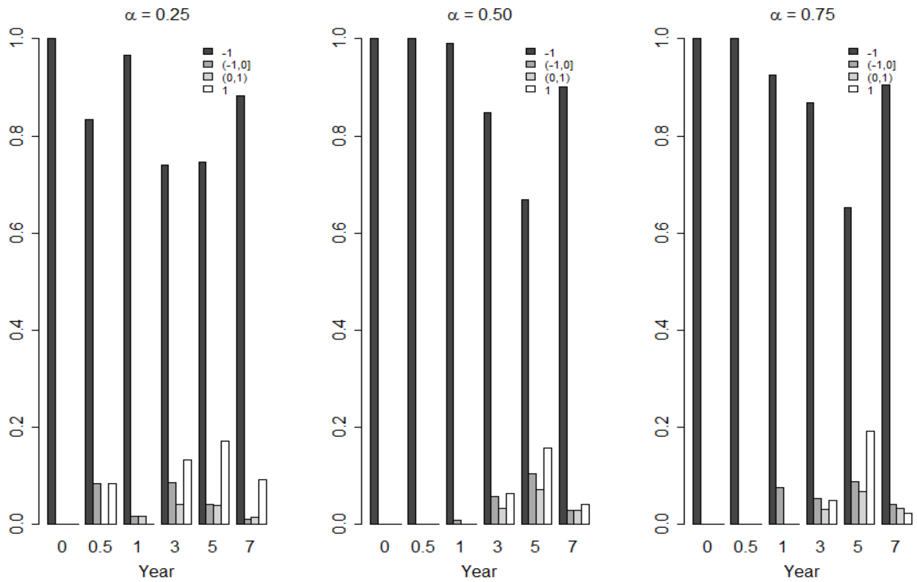

Fig. 2: Percentage of nodes with specific assets-liabilities duration mismatches at different stages for $\beta = 1$ and $\alpha = 0.25$ (left), $\alpha = 0.50$ (center) and $\alpha = 0.75$ (right) in the ongoing business scenario.

We also consider a tightening of the duration matching constraint to $[-0.75; 0.75]$ (9 months). This was motivated by observing that in the base case most of the nodes showed a duration mismatch in the extreme values -1 and 1. As expected, by reducing to 0.75 the maximum duration mismatch, most of the nodes still lie at the boundaries of the domain. In addition, when tightening the duration constraint and at the same time considering the stressed liability scenario, the incentive to invest in fixed income assets increases but, overall, the search of superior portfolio performance leads to a relevant equity investment in the problem instance $\{\mu_{1,\xi}, \alpha, \beta\} = \{5\%, 0.5, 1\}$ (see Table 6).

5.4 Impact of multi-period stochastic dominance constraints

In our numerical experiments, we test the quantile function decomposition method using the event cuts as described in Section 4. Second order stochastic dominance constraints significantly affect the optimal investment policy, especially in the last stages of the planning horizon. Indeed, in the optimal solution with coefficients $\alpha = 0.5$ and $\beta = 1$, stochastic dominance constraints are active in 752 nodes (i.e., 21.5% of the total number of nodes on which these constraints are imposed): 746 are at stage 5 (i.e., year 7), and the remaining 6 nodes are at stage 4 (i.e., year 5). The activation of the constraints in the last stages of the planning horizon is consistent with the numerical results presented in the previous paragraph. In fact, as previously shown in Tables 4 and 5, in order to obtain large investment profits, a relevant fraction of assets is sold in year 7, causing a dramatic fall of the asset portfolio value and therefore increasing the shortfall at the end of the planning horizon. However, the imposition of second order stochastic dominance constraints guarantees a stable solvency condition for the financial intermediary also at the horizon. With respect to the stressed liability scenario, by increasing the annual liability growth rate $\mu_{1,\xi}$ from 1% to 5%, to cover the higher liabilities, stochastic dominance constraints become active in a larger number of nodes, namely 30% instead of 21.5% of the base case. To further analyze the impact of stochastic dominance constraints, the base case solution has been compared with the solution of the model (6a)−(6h), (6j), (6k), not including stochastic dominance constraints (6i) of order $k = 2$. Fig. 3 shows second order (left panel) and first order (right panel) CDFs for assets and liabilities in the children nodes of node $3345 \in \mathcal{N}_5$, that we choose for the sake of presentation. Similar results are obtained for all other nodes where constraints (6i) are active. In particular, three distributions are compared in Fig. 3: the distribution of liabilities (solid line), of the assets portfolio with (dashed line) and without (dotted line) stochastic dominance constraints. As can be noticed, second order stochastic dominance constraints shift the CDFs for the assets portfolio to the right so that they dominate the distribution of liabilities. The model increases the initial deposit of the cash account $\hat{x}_{0,0}$ in order to purchase more assets at $t = 0$, thus rising the portfolio value over all stages. Indeed, when no stochastic constraints are considered, the model allocates an initial cash account deposit $\hat{x}_{0,0}$ of 611.53 Mln €, which is 21.27 Mln € lower than the solution with stochastic dominance constraints, not allowing to cover liabilities.

6 Conclusions

In this paper, we provide a novel formulation for a long-term ALM problem under interest rate, inflation and credit risk exposure, with solvency and funding protection. The proposed model represents a significant extension of a practically and operationally relevant ALM model for a large insurer. For the first time in the literature, a sufficient funding condition is enforced in the model through multistage second-order stochastic dominance

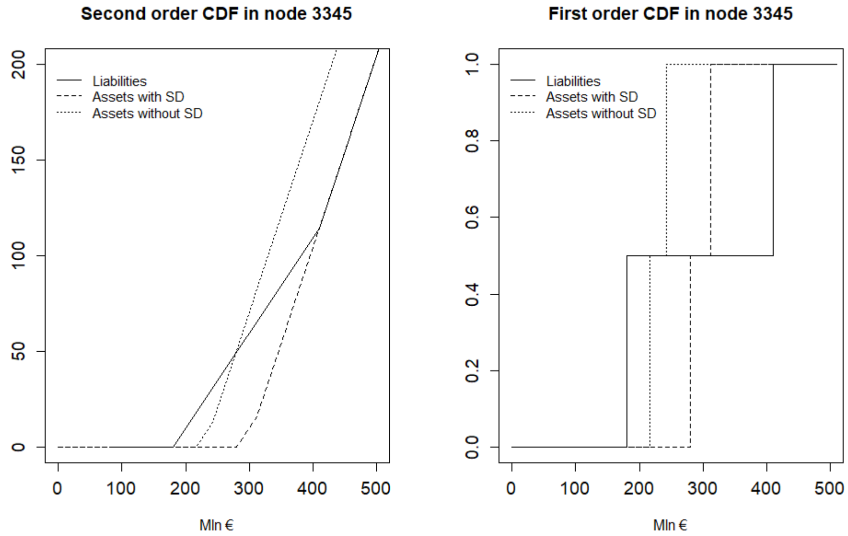

Fig. 3: Second order (left) and first order (right) CDFs in node 3345 for liabilities and assets portfolio with and without stochastic dominance constraints (6i) of order $k = 2$.

of the asset portfolio distribution with respect to the liability distribution over time. Although the adoption of stochastic dominance principles in an ALM context is not new, for the first time in the literature stochastic-ordering relations and dynamic risk measures for constructing immunized portfolios are included in a multistage framework. The model here formulated jointly manages the initial capital injection, the dynamic control of external regulatory requirements and the internal profit generation. Thus, the solution to such a model enables the AL managers to satisfy capital requirements, preserve the company funding status and fulfil liability obligations, while spanning different risk profiles.

To tackle the problem, we develop an efficient decomposition scheme and discuss its convergence. Specifically, by using the dynamic programming formulation of the problem, we propose a recursive solution approach based on a version of the multi-cut method in which additional cuts approximate the stochastic order constraints and the risk measures in the objective function.

The proposed methodology is validated on a case study inspired by an European insurance intermediary over a 10-year planning horizon, with portfolio rebalancing occurring in seven stages, assuming either a base or a stressed liability scenario. Computational results show the effectiveness of the proposed method, which converges to the optimal solution in 4 iterations with a scenario tree with 7333 nodes and 3840 scenarios. Moreover, the stressed scenario does not affect the computational performances of the proposed method. From a financial perspective, we notice how the stressed condition for liabilities implies a significant increase of the dedicated capital, which would make such a scenario highly undesirable due to the high cost of capital. However, the base and the stressed liability scenarios show a similar evolution for the assets portfolio, which is managed so as to limit the occurrence of the shortfall, while pursuing investment profits.

A post-optimality analysis based on a sensitivity of the weights of shortfall, profits and initial invested capital shows that, when more relevance is assigned to the investment profits in the objective function, the occurrence of the shortfall is increased, especially in the early stages, and the allocated risk capital is modified to attain higher investment profits. On the other hand, higher penalties for the initial investment determine a significant reduction of the initial risk capital, making the shortfall more frequent. We further assess the interest rate risk exposure through the duration-matching constraint. Numerical experiments show a prevalent exposure of the financial intermediary to decreasing interest rates over the investment horizon, having liabilities duration exceeding assets duration. When further tightening the duration constraint by reducing the maximum duration mismatch, we observe that the incentive to invest in fixed income assets increases but, overall, the search of superior portfolio performance leads to a relevant equity investment.

Finally, we assess the impact of SD constraints on the optimal solution. Results show that second order SD constraints significantly affect the optimal investment policy, especially under the stressed liability scenario, raising from 21.5% to 30% of the nodes in which they are active. Furthermore SD constraints imply an increased initial investment value in order to purchase more assets at the beginning of the investment horizon, thus rising the portfolio value over all stages to cover the liabilities.

Acknowledgements: Giorgio Consigli acknowledges the support from Khalifa University of Science and Technology, Grant FSU2022-010 award 000634-00001, project no. 8474000393.
Francesca Maggioni acknowledges the support from the PRIN2020 project "ULTRA OPTYMAL - Urban Logistics and sustainable TRAnsportation: OPtimization under uncertainTY and MAchine Learning", funded by the Italian University and Research Ministry (Grant 20207C8T9M, https://ultraoptymal.unibg.it).

Declarations: The authors have no competing interests to declare that are relevant to the content of this article. The data that support the findings of this study are openly available at http://finance.yahoo.com, and upon request to the authors.

References

Alessandri, P. and Drehmann, M. (2010). An economic capital model integrating credit and interest rate risk in the banking book. *Journal of Banking & Finance*, 34(4):730–742.

Chen, Z., Consigli, G., Liu, J., Li, G., Fu, T., and Hu, Q. (2017). Multi-period risk measures and optimal investment policies. *In: Optimal Financial Decision Making under Uncertainty. Springer,*, pages 1–34.

Christensen, J. H. E., Diebold, F. X., and Rudebusch, G. D. (2009). An arbitrage-free generalized nelsonâsiegel term structure model. *The Econometrics Journal*, 12(3):C33–C64.

Collado, R. A., Papp, D., and Ruszczyński, A. (2012). Scenario decomposition of risk-averse multistage stochastic programming problems. *Annals of Operations Research*, 200:147–170.

Consigli, G., Di Tria, M., Gaffo, M., Iaquinta, G., Moriggia, V., and Uristani, A. (2011a). Dynamic portfolio management for property and casualty insurance. In M. Bertocchi, G. C. and Dempster, M., editors, *Handbook on Stochastic Optimization Methods in Finance and Energy*, Fred Hillier International Series in Operations Research and Management Science, chapter 5, pages 99–124. Springer USA.

Consigli, G., Iaquinta, G., Moriggia, V., and Di Tria, M. (2011b). Optimal management of life insurance household portfolios in the long run. *Pacific Journal of Optimization*, 7:197–219.

Consigli, G., Iaquinta, G., Moriggia, V., Di Tria, M., and Musitelli, D. (2012). Retirement planning in individual asset–liability management. *IMA Journal of Management Mathematics*, 23(4):365–396.

Consigli, G., Moriggia, V., and Vitali, S. (2020). Long-term individual financial planning under stochastic dominance constraints. *Annals of Operations Research*, 292(2):973–1000.

Consigli, G., Moriggia, V., Vitali, S., and Mercuri, L. (2018). Optimal insurance portfolios risk-adjusted performance through dynamic stochastic programming. *Computational Management Science*, 15(3-4):599–632.

Dentcheva, D. and Martinez, G. (2012). Two-stage stochastic optimization problems with stochastic ordering constraints on the recourse. *European Journal of Operational Research*, 219:1–8.

Dentcheva, D. and Ruszczyński, A. (2003). Optimization with stochastic dominance constraints. *SIAM Journal on Optimization*, 14(2):548–566.

Dentcheva, D. and Ruszczyński, A. (2004). Semi-infinite probabilistic optimization: first-order stochastic dominance constraints. *Optimization*, 53(5-6):583–601.

Dentcheva, D. and Ruszczyński, A. (2006a). Inverse stochastic dominance constraints and rank dependent expected utility theory. *Mathematical Programming*, 108(2-3):297–311.

Dentcheva, D. and Ruszczyński, A. (2006b). Portfolio optimization with stochastic dominance constraints. *Journal of Banking & Finance*, 30(2):433–451.

Dentcheva, D. and Ruszczyński, A. (2008a). Duality between coherent risk measures and stochastic dominance constraints in risk-averse optimization. *Pacific Journal of Optimization*, 4(3):433–446.

Dentcheva, D. and Ruszczyński, A. (2008b). Stochastic dynamic optimization with discounted stochastic dominance constraints. *SIAM journal on control and optimization*, 47(5):2540–2556.

Dentcheva, D. and Ruszczyński, A. (2010). Inverse cutting plane methods for optimization problems with second-order stochastic dominance constraints. *Optimization*, 59(3):323–338.

Dentcheva, D., Ye, M., and Yi, Y. (2022). Risk-averse sequential decision problems with time-consistent stochastic dominance constraints. In *2022 IEEE 61st Conference on Decision and Control (CDC)*, pages 3605–3610. IEEE.

Dhaene, J., Tsanakas, A., Valdez, E. A., and Vanduffel, S. (2012). Optimal capital allocation principles. *Journal of Risk and Insurance*, 79(1):1–28.

Dupačová, J., Consigli, G., and Wallace, S. W. (2000). Scenarios for multistage stochastic programs. *Annals of Operations Research*, 100(1-4):25–53.

Escudero, L. F., Garín, M. A., Merino, M., and Pérez, G. (2016). On time-consistent stochastic dominance induced by mixed integer-linear recourse in multistage stochastic programs. *European Journal of Operational Research*, 249(1):164–176.

Escudero, L. F., Monge, J. F., and Morales, D. R. (2017). On the time-consistent stochastic dominance risk averse measure for tactical supply chain planning under uncertainty. *Computers & Operations Research*.

European Parliament (2009). European Parliament, Solvency II Directive.

Fábián, C. I., Mitra, G., and Roman, D. (2011). Processing second-order stochastic dominance models using cutting-plane representations. *Mathematical Programming*, 130(1):33–57.

Föllmer, H. and Schied, A. (2008). *Stochastic Finance An Introduction in Discrete Time. Second Edition*. De Gruyter.

Gatzert, N. and Schmeiser, H. (2008). Combining fair pricing and capital requirements for non-life insurance companies. *Journal of Banking & Finance*, 32(12):2589–2596.

Gülten, S. and Ruszczyński, A. (2015). Two-stage portfolio optimization with higher-order conditional measures of risk. *Annals of Operations Research*, 229:409–427.

Haskell, W. B. and Jain, R. (2013). Stochastic dominance-constrained markov decision processes. *SIAM Journal on Control and Optimization*, 51(1):273–303.

Heitsch, H. and Römisch, W. (2009). Scenario tree modeling for multistage stochastic programs. *Mathematical Programming*, 118:371–406.

Kopa, M., Moriggia, V., and Vitali, S. (2018). Individual optimal pension allocation under stochastic dominance constraints. *Annals of Operations Research*, 260:255–291.

Lauria, D., Consigli, G., and Maggioni, F. (2022). Optimal chance-constrained pension fund management through dynamic stochastic control. *OR Spectrum*, 44(3):967–1007.

Leitner, J. (2005). A short note on second-order stochastic dominance preserving coherent risk measures. *Mathematical Finance*, 15:649–651.

Levy, H. (2006). *Stochastic dominance: Investment decision making under uncertainty*, volume 12. Springer Science & Business Media.

Lorenz, M. (1905). Methods of measuring concentration of wealth. *Journal of the American Statistical Association*, 9:209–219.

Luedtke, J. (2008). New formulations for optimization under stochastic dominance constraints. *SIAM Journal on Optimization*, 19(3):1433–1450.

Maggioni, F. and Pflug, G. C. (2016). Bounds and approximations for multistage stochastic programs. *SIAM Journal on Optimization*, 26(1):831–855.

Maggioni, F. and Pflug, G. C. (2019). Guaranteed bounds for general nondiscrete multistage risk-averse stochastic optimization programs. *SIAM Journal on Optimization*, 29(1):454–483.

Maume-Deschamps, V., Rullière, D., and Said, K. (2015). A risk management approach to capital allocation. *arXiv preprint arXiv:1506.04125*.

Mei, Y., Chen, Z., Liu, J., and Ji, B. (2022). Multi-stage portfolio selection problem with dynamic stochastic dominance constraints. *Journal of Global Optimization*, 83(3):585–613.

Moriggia, V., Kopa, M., and Vitali, S. (2019). Pension fund management with hedging derivatives, stochastic dominance and nodal contamination. *Omega*, 87:127–141.

Müller, A. and Stoyan, D. (2002). *Comparison Methods for Stochastic Models and Risks*. John Wiley & Sons, Chichester.

Narum, B. S., Maggioni, F., and Wallace, S. W. (2023). On the safe side of stochastic programming: bounds and approximations. *International Transactions in Operational Research*.

Nelson, C. R. and Siegel, A. F. (1987). Parsimonious modeling of yield curves. *Journal of business*, pages 473–489.

Noyan, N. and Ruszczyński, A. (2008). Valid inequalities and restrictions for stochastic programming problems with first order stochastic dominance constraints. *Mathematical programming*, 114(2):249–275.

Ogryczak, W. and Ruszczyński, A. (1999). From stochastic dominance to mean-risk models: semideviations as risk measures. *European Journal of Operational Research*, 116:33–50.

Ogryczak, W. and Ruszczyński, A. (2002). Dual stochastic dominance and related mean-risk models. *SIAM Journal on Optimization*, 13(1):60–78.

Pflug, G. and Pichler, A. (2014). *Multistage Stochastic Optimization*. Springer Series, New York.

Pflug, G. and Ruszczyński, A. (2005). Measuring risk for income streams. *Computational Optimization and Applications*, 32:161–178.

Roman, D., Darby-Dowman, K., and Mitra, G. (2006). Portfolio construction based on stochastic dominance and target return distributions. *Mathematical Programming*, 108:541–569.

Rudolf, G. and Ruszczyński, A. (2008). Optimization problems with second order stochastic dominance constraints: duality, compact formulations, and cut generation methods. *SIAM Journal on Optimization*, 19(3):1326–1343.

Ruszczyński, A. (2010). Risk-averse dynamic programming for markov decision processes. *Mathematical Programming 125*, 125:235–261.

Ruszczyński, A. and Shapiro, A. (2003). Stochastic programming models. *Handbooks in operations research and management science*, 10:1–64.

Shapiro, A., Dentcheva, D., and Ruszczyński, A. (2021). *Lectures on stochastic programming: modeling and theory*. SIAM.

Urban, M., Dittrich, J., Klüppelberg, C., and Stölting, R. (2004). Allocation of risk capital to insurance portfolios. *Blätter der DGVFM*, 26(3):389–406.

Yahoo!Finance (accessed: 01.05.2023). https://finance.yahoo.com.

Yang, X., Gondzio, J., and Grothey, A. (2010). Asset liability management modelling with risk control by stochastic dominance. *Journal of Asset Management*, 11(2-3):73–93.

A Statistical models parameters estimation

In this appendix we present results for the estimation by OLS method of the coefficients associated with the statistical models introduced in Section 3. Input data are monthly observations from December 2018 to December 2022. The statistical evidences for yield curve parameters are presented in Tables A.1 and A.2. Specifically, Table A.1 provides statistics for the estimated coefficients $b^y_{j,t}$, $j = 1, 2, 3$ and λ_t of the *Nelson-Siegel-Svensson* model (7), while Table A.2 shows the symmetric variance and covariance matrix of coefficients $b^y_{j,t}$, $j = 1, 2, 3$, which is needed in order to implement the arbitrage free calibration method described in Christensen et al. [2009]. Table A.3 provides estimates and standard errors for coefficients a^y_j, $j = 0, 1, 2, 3$ of the decay factor model (8), α^π and σ^π of the inflation model (9), and c_j, $j = 0, 1, 2$, of the credit spread model (10). With

	$b^y_{1,t}$	$b^y_{2,t}$	$b^y_{3,t}$	λ_t
Mean	0.0247	−0.0187	−0.0179	4.9924
Std	0.0160	0.0096	0.01199	2.7213
q(0.25)	0.0111	−0.0254	−0.0269	3.6704
q(0.5)	0.0263	−0.0185	−0.0184	4.5455
q(0.75)	0.0403	−0.0114	−0.0114	5.6551

Table A.1: Statistics for coefficients $b^y_{j,t}$, $j = 1, 2, 3$ and λ_t of the yield curve model (7).

	$b^y_{1,t}$	$b^y_{2,t}$	$b^y_{3,t}$
$b^y_{1,t}$	0.000257	−	−
$b^y_{2,t}$	−0.000063	0.000093	−
$b^y_{3,t}$	0.000012	−0.000062	0.000144

Table A.2: Variance and covariance symmetric matrix for coefficients $b^y_{j,t}$, $j = 1, 2, 3$.

regard to asset price returns, Table A.4 provides estimates of the regression coefficients $b_i = (b_{i,0}, b_{i,1}, b_{i,2}, b_{i,3}, b_{i,4})^\top$, $i \in \mathcal{I}$, in models (17), (18), and (19) and the corresponding coefficients of determination R^2.

	Decay factor (8)				Inflation (9)		Credit Spread (10)		
	a^y_0	a^y_1	a^y_2	a^y_3	α^π	σ^π	c_0	c_1	c_2
Mean	7.0549	47.7621	121.3425	50.8006	−0.0053	0.8569	0.0614	0.9479	4.1689
Std	0.3831	11.0799	21.8113	16.4272	0.0002	0.0280	0.0351	0.0189	1.1905

Table A.3: Estimates and standard errors for coefficients a^y_j, $j = 0, 1, 2, 3$ of the decay factor model (8), α^π and σ^π of the inflation model (9), and c_j, $j = 0, 1, 2$ of the credit spread model (10).

Asset $i \in \mathcal{I}$	$b_{i,0}$	$b_{i,1}$	$b_{i,2}$	$b_{i,3}$	$b_{i,4}$	R^2
Smart.MI	0.0001	0.2997	0.0398	−0.0032		0.82
IBGS.L	0.0324	−0.1358	4.4568	−0.4895		0.79
DBXR.DE	0.0029	−0.0354	0.4042	−0.2334		0.74
SYBV.DE	−0.0007	0.0048	0.3902	0.3065		0.68
LQD	0.0019	−0.0156	−0.1251	−0.0757		0.59
IBCLAS	0.0148	−0.1042	0.0876	−0.9850	0.0675	0.71
EEM	−0.0252	0.0154	−5.7660	−1.2073	0.8823	0.63
IMEU.AS	−0.0380	−0.1206	−7.0718	0.9165	−0.8562	0.66
IWM	−0.0089	−0.0422	−5.4131	−1.1704	−2.0308	0.64

Table A.4: Estimation of the regression coefficients $b_i = (b_{i,0}, b_{i,1}, b_{i,2}, b_{i,3}, b_{i,4})^\top$, $i \in \mathcal{I}$.

B Scenario generation algorithm

Let θ be the vector including all statistical coefficients estimated by OLS method according to the models from (7) to (20) and all the parameters of the yield curve model, the inflation and the credit spread process specified at $t = 0$. Let $\xi_n := \{r_{i,n}, g_{i,n}, \lambda_{j,n}, \Lambda_n, L_{j,n}, c_n, \delta^\lambda_{j,n}\}$ be a coefficient tree process on the node $n \in \mathcal{N}$ of the scenario tree. This vector will then include all the random parameters specified in the ALM model. Since returns $r_{i,n}$ of assets $i \in \mathcal{I}$ in node n and values $\lambda_{j,n}$ of liability $j \in \mathcal{J}$ in node n depend on the yield curve $y_{t,\tau}$, the inflation process π_t and the credit spread process s^{IG}_t, values for the vector ξ_n are generated by applying a two-step procedure, with the first step being the generation of a random vector process $\omega_n := (y_{n,\tau}, \pi_n, s^{IG}_n)$ for yield curve rates, inflation and credit spread, and the second step being the generation of the stochastic ALM model coefficients. Values for the random vector process ω_n, referred to as the *core economic model*, are determined by applying Algorithm 1. Specifically, the input to the algorithm is represented by the vector θ. According to the planning horizon T, to the stage composition $\mathcal{T}$ and to the branching degree vector, the *Nodal Partition Matrix* (NPM) is generated. Such a matrix has N_T rows and $T + 1$ columns. Each row of the matrix is a scenario for the core economic model, determined by applying models (7), (9) and (10) with monthly increments from $t = 0$ to T.

The scenarios of the core economic model are input to the asset returns and liability costs models, which determine the scenarios of the coefficient process ξ_n, as detailed in Algorithm 2. The initial conditions of the scenario generation are defined at the root node by $\xi_0 := \{r_{i,0}, g_{i,0}, \lambda_{j,0}, L_{j,0}, c_0\}$. We distinguish here between the investment horizon T and the liability valuation horizon T_λ: this term reflects the number of years in the future in which liabilities are accounted for to determine the evolution of Λ_t, from which the capital requirements can be derived. The algorithm to determine values for ξ_n consists of a forward pass and a backward pass. Similarly to the previous algorithm, in the forward pass we generate scenarios for parameters $r_{i,n}$, $g_{i,n}$, $L_{j,n}$, and c_n by applying the statistical models introduced in Sections 3.2 and 3.3 with monthly increments from $t = 0$ to T_λ. We then determine with a backward recursion the stochastic values of parameters $\lambda_{j,n}$ and $\delta^\lambda_{j,n}$ by discounting the expected payments according to equations (15) and (16). Finally, the total liability value in each node $\Lambda_n = \sum_{j \in \mathcal{J}} \lambda_{j,n}$ is computed.

Algorithm 1 Scenario generation - core economic model

Input Vector θ:

(a) Nelson-Siegel-Svensson parameters $b^y_{1,0}, b^y_{2,0}, b^y_{3,0}, \lambda_0$ plus stochastic coefficients for $b^y_{j,t}$, $j = 1, 2, 3$ and for $\lambda_t = \lambda(b^y_{j,t})$.

(b) inflation process coefficients α^π, σ^π and π_0.

(c) credit spread process coefficients c_0, c_1, c_2 and initial condition s^{IG}_0.

(d) errors distributions for each model.

1. Specify planning horizon T and stage composition $\mathcal{T}$.
2. Generate the *Nodal Partition Matrix* (NPM) of $\mathcal{N}_T$ rows and $T + 1$ columns.
3. **For** $t = 1 : T$

    For $n \in \mathcal{N}_t$

    For $h = (t_{a(n)}, t_{a(n)} + \Delta t, ..., t_n - \Delta t, t_n)$, Δt *monthly increments between nodes*

    * generate yield curve inter-stage increments from (7): $y_{h,\tau} = y(b^y_{1,h}, b^y_{2,h}, b^y_{3,h}, \lambda_h)$,
    * generate inflation increments (9) $\pi_h = \pi(\alpha^\pi, \sigma^\pi)$,
    * generate credit spread increments from (10) $s^{IG}_h = s(c_0, c_1, c_2)$.

    End For

    End For

    End For

Output $y_{n,\tau}$, π_n, s^{IG}_n scenario paths.

Algorithm 2 Scenario generation - coefficient process

Input Initial conditions for $\xi_0 := \{r_{i,0} = g_{i,0} = 0, \lambda_{j,0}, L_{j,0}, c_0\}$.
Input ALM horizon T, stage composition $\mathcal{T}$ and liability evaluation horizon T_λ.
Input Initial term structure of interest rates $y_{0,\tau}$.

1. Generate NPM for liability evaluation: $\mathcal{N}_T$ rows and $T_\lambda + 1$ columns.
2. **For** $t = 1 : T_\lambda$ *forward pass*

   For $n \in \mathcal{N}_t$

   For $h = (t_{a(n)}, t_{a(n)} + \Delta t, ..., t_n - \Delta t, t_n)$
   Compute $\{r_{i,h}, g_{i,h}, L_{j,h}, c_h\}$ from (11), (12), (13), (14), (17), (18), (19), (20)

   End For

   End For

   End For
3. **For** $t = T_\lambda : 0$ *backward recursion*

   For $n \in \mathcal{N}_t$

   – **For** $j = 1, 2, .., J$
   Compute $\lambda_{j,n}$ and $\delta^\lambda_{j,n}$ from (15) and (16).
   End For

   – Compute $\Lambda_n = \sum_{j \in \mathcal{J}} \lambda_{j,n}$.

   End For

   End For

Output $r_{i,n}, g_{i,n}, L_{j,n}, c_n, \lambda_{j,n}, \delta^\lambda_{j,n}, \Lambda_n$

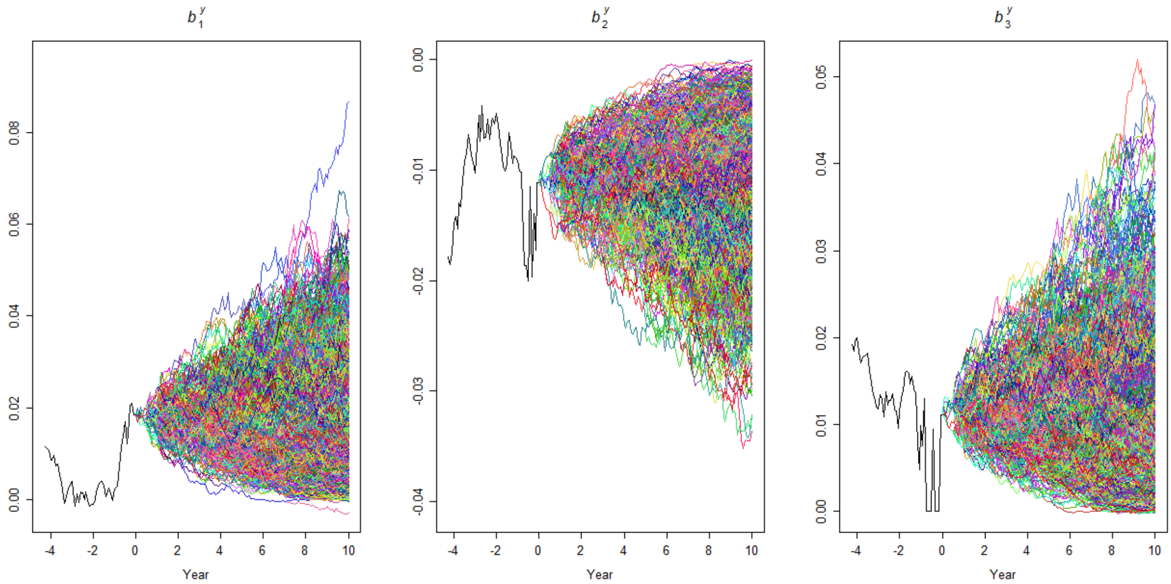

Fig. B.1: 10-year simulation outputs for *Nelson-Siegel-Svensson* parameters $b^y_{j,t},\ j = 1, 2, 3$.

We report in Fig. from B.1 to B.3 the simulation outputs of the scenario generation algorithms. Specifically, Fig. B.1 and B.2 show the 4-year data history and the 10-year simulation outputs for the parameters $b^y_{j,t},\ j = 1, 2, 3$ of the *Nelson-Siegel-Svensson* model (7), the inflation process π_t (9) and the credit spread s_t^{IG} (10). Scenarios for cash outflows and infows are illustrated in Fig. B.3. Notice that here we assume a valuation horizon of $T_\lambda = 20$ years to properly determine the liability value.

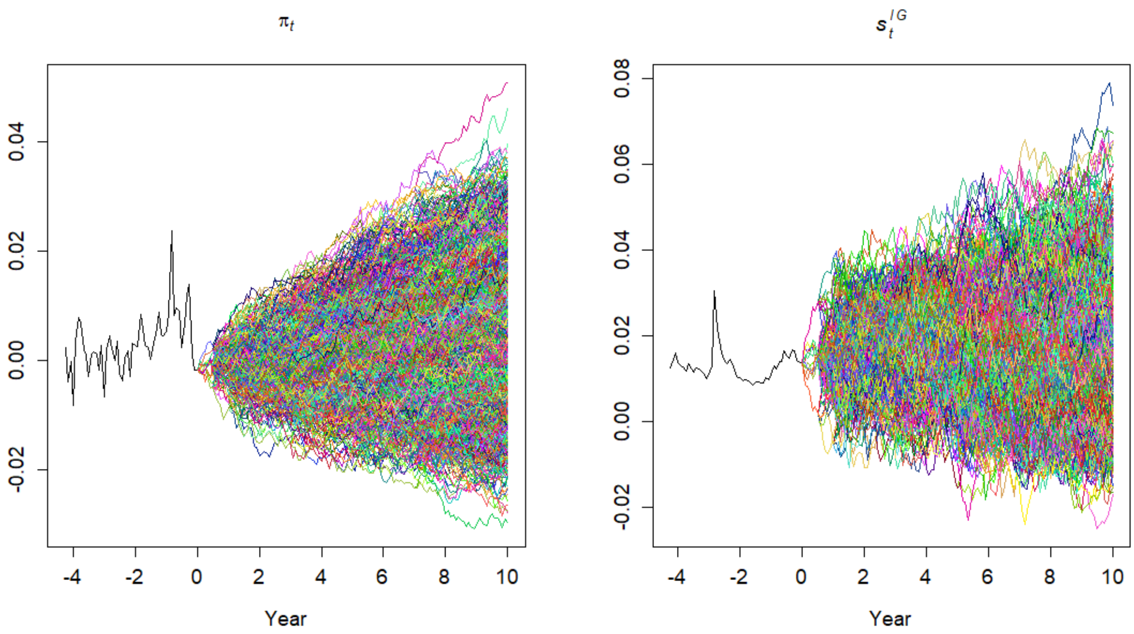

Fig. B.2: 10-year simulation outputs for inflation process π_t and credit spread s_t^{IG}.

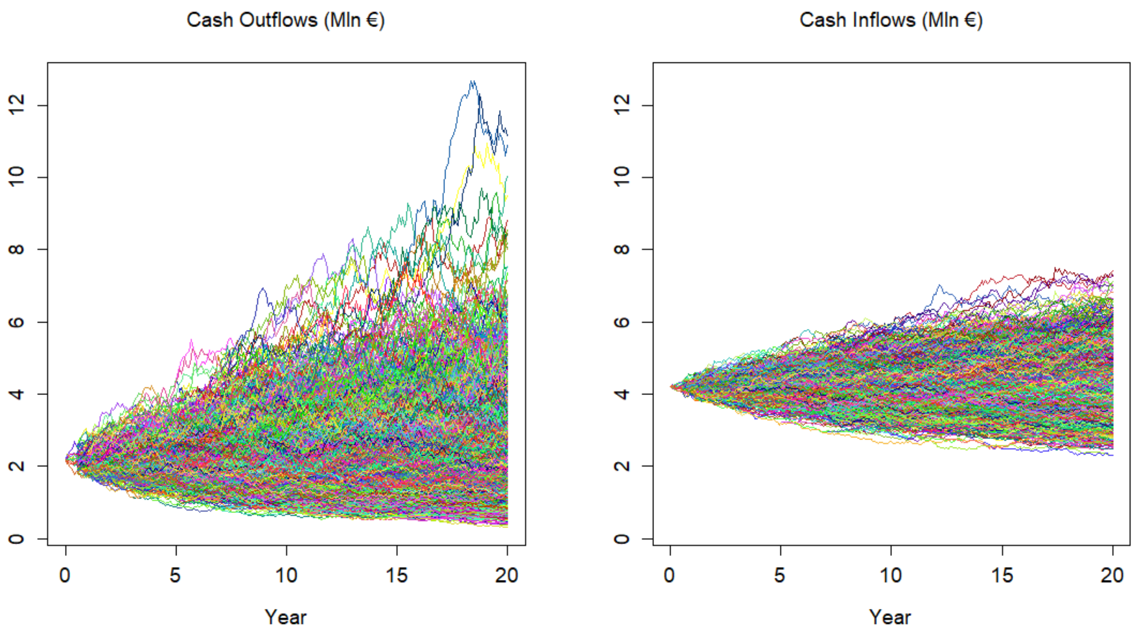

Fig. B.3: 20-year simulation outputs for cash outflows $L_{1,t}$ and cash inflows c_t.