# Facets of the knapsack polytope from non-minimal covers

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**Abstract**

We propose two new classes of valid inequalities (VIs) for the binary knapsack polytope, based on non-minimal covers. We also show that these VIs can be obtained through neither sequential nor simultaneous lifting of well-known cover inequalities. We further provide conditions under which they are facet-defining. The usefulness of these VIs is demonstrated using computational experiments on fixed charge transportation problems, a well-known class of NP-hard problems, which help improve their lower bounds by more than 9% on average. This helps save CPU time by around 77% to 94% when used in the absence of CPLEX-generated cuts, depending on the problem parameters. This also reduces the CPU time by around 28% to 16% when used in conjunction with CPLEX-generated cuts.

**Keywords:** `Knapsack polytope, Valid inequalities, Facets, Non-minimal cover, Integer programming, Fixed charge transportation problem`

# 1 Introduction

Consider a binary knapsack set $Y = P \cap \mathbb{B}^n$, where $P = \{y \in \mathbb{R}^n : \sum_{i \in N} a_i y_i \leq b\}$, $a_i > 0 \; \forall i \in N, b > 0$. Let $KP$ denote the knapsack polytope, i.e., $KP = Conv(Y)$. We assume that $KP$ is full-dimensional, i.e., $dim(KP) = |N|$, which happens only when $a_i <= b \; \forall i \in N$. A set $C \subseteq N$ is a cover of $Y$ if $\sum_{i \in C} a_i > b$, and its surplus $\lambda$ is defined as the extra weight included in $C$ beyond the knapsack capacity, $b$, i.e., $\lambda = \sum_{i \in C} a_i - b$. A cover is minimal if $\lambda \leq \min\limits_{i \in C} \{a_i\}$, else non-minimal, i.e., when $\lambda > \min\limits_{i \in C} \{a_i\}$. Given a cover $C$, it is well-known that the following is valid inequality (VI) for KP:

$$\sum_{i \in C} y_i \leq |C| - 1 \tag{1}$$

(1) is popularly referred to in the literature as Cover Inequality (CI). Let $KP^C = \{y \in KP : y_i = 0 \; \forall i \in N \backslash C\}$. Then, (1) defines a facet to $KP^C$ if and only if $C$ is minimal (Padberg, 1975; Balas, 1975; Wolsey, 1975; Hammer et al., 1975). Hence, (1) defined by a non-minimal cover is dominated by one defined by a minimal cover, and is, therefore, generally not used in the literature. Minimal CIs, which define facets of $KP^C$, are generally not facet-defining for $KP$. Lifting is a popular technique used in the literature to strengthen minimal CIs to make them facet-defining for $KP$ (Balas and Zemel, 1978; Gu et al., 1998, 1999). For a detailed literature review on knapsack polytopes, we suggest the reader refer to (Hojny et al., 2020)

In this paper, we make use of non-minimal covers to propose two new classes of VIs and derive the conditions under which they define facets of $KP$. Further, we show that the facets of $KP$ obtained from one of our proposed classes of VIs can never be obtained through sequential lifting of minimal CIs. Our computational experiments on the fixed charge transportation problem, a well-known class of NP-hard problem, highlight the usefulness of the facets from our proposed VIs, which help improve the lower bounds by more than 9% on average. This helps save the CPU time by around 77% to 94% when used in the absence of CPLEX-generated cuts. This also reduces the CPU time by around 28% to 16% when used in conjunction with CPLEX-generated cuts.

The rest of the paper is organized as follows. We introduce two types of valid inequalities and derive their facet defining condition in section 2. Computational results are presented in section 3.

Finally, we conclude in section 4.

## 2 Valid Inequalities based on Non-Minimal Covers

In this Section, we propose two new classes of VIs based on the idea of partitioning non-minimal covers. In Section 2.1, we partition a non-minimal cover $C$ based on its surplus, whereas its partition in Section 2.2 is based on the idea of an exclusion set, which is defined as a subset of items whose exclusion from $C$ makes it no longer a cover. For each of these classes of VIs, we further derive the conditions under which they define facets of $KP^C$, as well as $KP$. For the rest of the paper, we use the following notation.

| | | |
|---|---|---|
| $C'$ | : | $N \setminus C$ |
| $max(C)$ | : | the highest weight among all items in set $C$ |
| $max_j(C)$ | : | the $j^{th}$ highest weight among all items in set $C$ |
| $min(C)$ | : | the lowest weight among all items in set $C$ |
| $min_j(C)$ | : | the $j^{th}$ lowest weight among all items in set $C$ |
| $c_j$ | : | cardinality of set $C_j$ |
| $\lambda$ | : | surplus of cover C, i.e., $\lambda = \sum_{i \in C} a_i - b$ |

Additionally, for $C = \emptyset$, $max(C) = min(C) = max_j(C) = min_j(C) = 0$.

### 2.1 Surplus-based Partition

**Proposition 1.** *Given a cover $C$ and its partition $C_1 = \{i \in C : a_i < \lambda\}$ and $C_2 = \{i \in C : a_i \geq \lambda\}$,*

*(a) the following is a VI for KP:*

$$\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i \leq c_1 + 2c_2 - 2 \tag{2}$$

*(b) (2) cuts off fractional extreme points $\bar{y} \in P$ characterized as:*

*(i)* $\bar{y}_i = 1 \; \forall i \in (C_2 \cup C_1 \backslash \{j, k\} : a_j + a_k > \lambda)\,, \bar{y}_j = 0, \bar{y}_k = \frac{a_j + a_k - \lambda}{a_k}$

*(ii)* $\bar{y}_i = 1 \; \forall i \in ((C_1 \backslash \{j\}) \cup (C_2 \backslash \{k\}) : a_j + 0.5a_k > \lambda), \bar{y}_j = 0, \bar{y}_k = \frac{a_j + a_k - \lambda}{a_k}$

*(iii)* $\bar{y}_i = 1\ \forall i \in (C_1 \cup (C_2 \backslash \{j\}) : a_j > \lambda), \bar{y}_j = \frac{a_j - \lambda}{a_j}$.

*Proof.* (a) In general, the following are (trivial) VIs for KP:

$$\sum_{i \in C_1} y_i \leq c_1 \tag{3}$$

$$\sum_{i \in C_2} y_i \leq c_2 \tag{4}$$

Multiplying (4) by 2 and adding it to (3), we get the following valid inequality:

$$\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i \leq c_1 + 2c_2$$

Next, we show that any solution $y \in Y$ can be one of the following three mutually exclusive and exhaustive types, and in each case, it satisfies (2):

- $y_i = 1\ \forall i \in C_1$: Since $a_i \geq \lambda\ \forall i \in C_2$, for any feasible solution to KP, $\exists$ at least one $i \in C_2 : y_i = 0$. Therefore, $\sum_{i \in C_2} y_i \leq c_2 - 1$, which implies $\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i \leq c_1 + 2c_2 - 2$.
- $y_i = 1\ \forall i \in C_2$: Since $a_i < \lambda\ \forall i \in C_1$, for any feasible solution to KP, $\exists$ one pair $i, j \in C_1 : j! = i, y_i = y_j = 0$. Therefore, $\sum_{i \in C_1} y_i \leq c_1 - 2$, which implies $\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i \leq c_1 + 2c_2 - 2$.
- $\exists$ at least one $i \in C_1 : y_i = 0$ and $\exists$ at least one $i \in C_2 : y_i = 0$: Since $a_i \geq \lambda\ \forall i \in C_2$, such a solution is always feasible to KP. Further, $\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i \leq c_1 - 1 + 2(c_2 - 1) < c_1 + 2c_2 - 2$.

(b) (i) Substituting the fractional point $\bar{y}$ in the knapsack inequality makes it binding, which immediately shows that it is an extreme point of $P$. Further, substituting it in (2) gives the LHS $= c_1 = 2c_2 - 2 + \frac{a_j + a_k - \lambda}{a_k} > c_1 = 2c_2 - 2 =$ RHS, and hence, violates it.

Proofs for (ii) and (iii) are omitted as the steps involved are exactly the same as for (i).

□

We refer to (2) as a *2-partition cover inequality* (2PCI).

**Remark 1.** *(a) When* $C_1 = \emptyset$*, i.e.,* $\lambda \leq min(C)$*,* (2) *reduces to the following minimal CI:* $\sum_{i \in C_2} y_i \leq c_2 - 1$*.*

*(b) When* $C_2 = \emptyset$*, i.e.,* $\lambda > max(C)$*,* (2) *reduces to the following Extended Cover Inequality (ECI):* $\sum_{i \in C_1} y_i \leq c_1 - 2$*.*

*(c) As a result of the above two remarks, we only consider 2PCIs with* $C_1 \neq \emptyset$ *and* $C_2 \neq \emptyset$*, i.e.,* $min(C) < \lambda \leq max(C)$*.*

We use the following as an illustrative example.

**Example 1.** *Consider* $Y = \{y \in \mathbb{B}^n : \sum_{i \in N} a_i y_i <= b\}$
*with* $n = 15$*,* $a = \{19, 17, 14, 14, 14, 13, 13, 12, 11, 10, 10, 9, 9, 7, 5\}$*, and* $b = 158$*.*
*Clearly,* $C = N$ *is a non-minimal cover since* $\sum_{i \in C} a_i = 177, \lambda = 177 - 158 = 19 > min_{i \in C}\{a_i\} = 5$*.* $C_2 = \{1\}$ *since* $a_1 = 19 \geq \lambda$*,* $C_1 = N \backslash \{1\}$*, and* $c_1 = 14, c_2 = 1$*. Hence, the corresponding 2PCI is* $2y_1 + y_2 + \cdots + y_{15} \leq 14 + 2 \times 1 - 2 = 14$*. Consider the following fractional point:* $\bar{y}_i = 1 \ \forall i \in C_2 \cup C_1 \backslash \{10, 11\}, \bar{y}_{10} = 0, \bar{y}_{11} = 1/10$*. Clearly,* $\sum_{i \in C} a_i \bar{y}_i = 158$*. Hence, this* $\bar{y}$ *is an extreme point of* $P$*. Here,* $a_{10} + a_{11} = 10 + 10 = 20 > 19 = \lambda$*, and* $\bar{y}_{11} = \frac{a_{10} + a_{11} - \lambda}{a_{11}} = \frac{10 + 10 - 19}{10}$*. Hence, this extreme point is of the type as characterized by Proposition 1.b.i. Further,* $\sum_{i \in C_1} \bar{y}_i + 2 \sum_{i \in C_2} \bar{y}_i = 14.1 > 14$*; hence,* $\bar{y}$ *is cut off by the above 2PCI.*

**Theorem 1.** (2) *defines a facet of* $KP^C$ *if and only if the following two conditions are satisfied:*

*(a)* $c_1 \geq 3$

*(b) (i).* $\lambda \leq max(C_1) + min(C_1)$*; (ii).* $\lambda \leq max_2(C_1) + max_3(C_1)$

*Proof.* First, we will prove that condition (a) is necessary. Condition (a) can be violated only in the following two cases: (i) $c_1 = 1$, (ii) $c_1 = 2$. Next, we will show that in neither of the two cases can (2) define a facet of $KP^C$.

(i) $c_1 = 1$: Say $c_2 = k \geq 1$. Then, (2) can be written as: $y_1 + 2(y_2 + \cdots + y_{k+1}) \leq 1 + 2k - 2 = 2k - 1$. This inequality can define a facet of $KP^C$ only if there exist $k+1$ affinely independent points in $KP^C$ for which it is binding. Clearly, the following $k$ points are the only affinely independent points in $KP^C$ for which this inequality is binding. Hence, with $c_1 = 1$, (2) cannot define a facet of $KP^C$.

$$
\begin{array}{c}
\begin{array}{cccccc} y_1 & y_2 & y_3 \cdots\cdots y_k & y_{k+1} & \end{array} \\
\left(\begin{array}{c|cccc}
0 & 0 & 1 \cdots\cdots 1 & 1 \\
0 & 1 & 0 \cdots\cdots 1 & 1 \\
\vdots & & \ddots & \vdots \\
0 & 1 & 1 \quad 1 \quad 0 & 1 \\
0 & 1 & 1 \cdots\cdots 1 & 0
\end{array}\right)
\begin{array}{l} y^1 \\ y^2 \\ \vdots \\ y^{k-1} \\ y^k \end{array}
\end{array}
$$

(ii) $c_1 = 2$: Say $c_2 = k \geq 1$. Then, (2) can be written as: $y_1+y_2+2(y_3+\cdots+y_{k+2}) \leq 2+2k-2 = 2k$. This inequality can define a facet of $KP^C$ only if there exist $k+2$ affinely independent points in $KP^C$ for which it is binding. Clearly, the following $k+1$ points are the only affinely independent points in $KP^C$ for which this inequality is binding. Hence, with $c_1 = 2$, (2) cannot define a facet of $KP^C$.

$$
\begin{array}{c}
\begin{array}{ccccc} y_1 & y_2 & y_3 \cdots\cdots y_k & y_{k+1} & y_{k+2} \end{array} \\
\left(\begin{array}{cc|ccccc}
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 \\
\vdots & \vdots & & & \ddots & & \vdots \\
1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}\right)
\begin{array}{l} y^1 \\ y^2 \\ y^3 \\ \vdots \\ y^k \\ y^{k+1} \end{array}
\end{array}
$$

This proves that condition (a) is necessary, i.e., $c_1 \geq 3$.

Next, we will prove that condition (b) is necessary. For this, we need to show that any set of $c$ affinely independent points in $KP^C$, for which (2) is binding, must satisfy condition (b). For this, let $y^k \in KP^C$ be a point defined as $y_i^k = 1 \forall i \in C_1 \cup (C_2 \backslash \{k\})$ and $y_i^k = 0$ for $i = k$. Clearly, $\{y^k : k \in C_2\}$ is a set of $c_2$ affinely independent points as shown below using matrix $M_1$. Further, for each $y^k$ defined above, (2) is binding since $\sum_{i \in C_1} y_i + 2\sum_{i \in C_2} y_i = c_1 + 2(c_2 - 1) = c_1 + 2c_2 - 2$.

$$
M_1 = \left(P \,\middle|\, Q\right) =
\begin{array}{cccc|ccccc|l}
y_1 & y_2 & \cdots & y_{c_1} & y_{c_1+1} & y_{c_1+2} & \cdots & y_{c-1} & y_c & \\
\hline
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & y^1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & y^2 \\
\vdots & \vdots & & \vdots & & & \ddots & & \vdots & \vdots \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & y^{c_2-1} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & y^{c_2}
\end{array}
$$

Clearly, $Q$ is a $c_2 \times c_2$ matrix with $c_2$ affinely independent rows.

Now, we need to generate the remaining $c_1$ affinely independent points (since $c = c_1 + c_2$). For this, we first show that if $\lambda \leq max(C_1) + min(C_1)$ is violated, then it is not possible to generate $c_1$ affinely independent points in $KP^C$ for which (2) is binding. To that end, let us assume $max(C_1) + min(C_1) < \lambda \leq max(C_1) + min_2(C_1)$. In this case, clearly, the top $c_1 - 2$ points shown below are affinely independent points. Beyond this set, clearly, the remaining two at the bottom are the only ones such that the set of $c_1$ points are affinely independent points. Of these two, $y^{c_1-1} \notin Y$ since $\lambda > max(C_1) + min(C_1)$. This proves $\lambda \leq max(C_1) + min(C_1)$ is a necessary condition.

Now, we show that condition (b) is sufficient. Clearly, when $\lambda \leq max(C_1) + min(C_1)$, the first $c_1 - 1$ points are affinely independent points in $KP^C$ as shown below using matrix $M_2$. In addition, $y^{c_1} \in KP^C$ only if $\lambda \leq max_2(C_1) + max_3(C_1)$.

$$
M_2 = \left(R \,\middle|\, S\right) =
\begin{array}{cccccc|ccc|l}
y_1 & y_2 & y_3 & \cdots & y_{c_1-1} & y_{c_1} & y_{c_1+1} & \cdots & y_c & \\
\hline
0 & 0 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & y^{c_2+1} \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & y^{c_2+2} \\
\vdots & & & \ddots & & & & & \vdots & \vdots \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & \cdots & 1 & y^{c-2} \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & \cdots & 1 & y^{c-1} \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & y^{c} \\
\multicolumn{6}{c|}{\underbrace{\hspace{10em}}_{C_1}} & \multicolumn{3}{c|}{\underbrace{\hspace{5em}}_{C_2}} &
\end{array}
$$

The set of $c_2$ affinely independent points and the set of $c_1$ affinely independent points, as generated above, can be together represented using the following $c \times c$ matrix:

$$
\mathrm{M} = \begin{pmatrix} M_1 \\ \hline M_2 \end{pmatrix} = \left(\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right)
$$

Since the $c_2$ rows of $M_1$ are affinely independent and the $c_1$ rows of $M_2$ are affinely independent, all $c_1 + c_2$ rows of $M$ are affinely independent. This proves that conditions (a) and (b) are necessary and sufficient for (2) to define a facet of $KP^C$.

$\square$

**Example 1** (Continued)**.** $2y_1 + y_2 + \cdots + y_{15} \leq 14$ *defines a facet of* $KP^C$ *since it satisfies both the conditions of Theorem 1 as shown below:*

*(a)* $c_1 = 14 > 3$

*(b)* $\lambda = 19, max(C_1) = 17, min(C_1) = 5, max_2(C_1) = max_3(C_1) = 14$. *Hence,* $\lambda < max(C_1) + min(C_1)$ *and* $\lambda < max_2(C_1) + max_3(C_1)$

**Theorem 2.** (2) *defines a facet of* $KP$ *if and only if it defines a facet of* $KP^C$ *and any of the following conditions is satisfied:*

*(a)* $C = N$

*(b)* $\lambda \leq max(C_2) - max(C')$

*(c)* $\lambda \leq max(C_1) + max_2(C_1) - max(C')$

*Proof.* To prove the theorem, we show that each of the conditions (a), (b), and (c) is necessary in the absence of the remaining two.

(a) It is easy to see that if $C = N$, then a facet of $KP^C$ will also be facet of $KP$ (since $KP = KP^C$).

(b) Suppose that condition (b) is not true. Accordingly, suppose $max(C_2) - max(C') < \lambda \leq max(C_2) - max_2(C')$. In order to prove that (2) defines a facet of $KP$, we need to generate additional $c'$ affinely independent points. However, if $\lambda \leq max(C_2) - max_2(C')$, then we can only generate $c' - 1$ additional affinely independent points as shown below.

Consider $n \times n$ matrix

$$M = \left( P \;\middle|\; Q \;\middle|\; R \right)$$

Where $P$ is a matrix of dimension $n \times c_1$ with all entries equal to 1, $Q$ is a matrix of order $n \times c_2$ with entries equal to $[0, 1, \dots, 1]$, and $R$ is an identity matrix of order $n \times c'$ as shown below.

$$R = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & & 0 \\ & & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \end{array} \begin{array}{l} y^1 \\ y^2 \\ \vdots \\ y^{n-1} \\ y^n \end{array}$$

Now we prove the sufficiency of condition (b). If $\lambda \leq max(C_2) - max(C')$, then we can generate $c'$ affinely independent points. This proves the necessity of condition (b) in the absence of conditions (a) and (c).

(c) We omit the proof of this part as it can be done using similar steps as for part (b).

□

**Example 1** (Continued)**.**

- $2y_1 + y_2 + \cdots + y_{15} \leq 14$ *defines a facet of* $KP$ *since* $C = N$ *(using condition 2.(a).*

- *Let us consider another cover* $C = N \backslash \{15\}$*. For this cover,* $\lambda = 14, C_2 = \{1, \dots, 5\}, C_1 = \{6, \dots, 14\}, C' = \{15\}$*. Hence,* $c_1 = 9, c_2 = 5$ *and the corresponding 2PCI is* $2(y_1 + \cdots + y_5) + y_6 + \cdots + y_{14} \leq 2 \times 5 + 9 - 2 = 17$*. It defines a facet of* $KP^C$ *since it satisfies both the conditions of Theorem 1: (a)* $c_1 = 9 > 3$*; (b)* $\lambda = 14, max(C_1) = 13, min(C_1) = 7, max_2(C_1) = 13, max_3(C_1) = 12$*; hence,* $\lambda < max(C_1) + min(C_1)$ *and* $\lambda < max_2(C_1) + max_3(C_1)$*. Further,* $2(y_1 + \cdots + y_5) + y_6 + \cdots + y_{14} \leq 17$ *also defines a facet of* $KP$ *since it satisfies conditions (b) and (c) of Theorem 2 as follows: (b)* $max(C_2) = 19, max(C') = 5$*; hence,* $\lambda = 14 \leq max(C_2) - max(C') = 14$*; (c)* $max(C_1) = 13, max_2(C_1) = 13$*; hence,* $\lambda = 14 < max(C_1) + max_2(C_1) - max(C') = 21$*.*

**Proposition 2.** (2) *can be obtained through neither sequential nor simultaneous lifting of any minimal CI.*

*Proof.* We prove this for sequential and simultaneous lifting in parts (a) and (b), respectively. For this, Consider (2) corresponding to a 2-partition $C_1, C_2$ of a non-minimal cover $C$. Further, consider a minimal cover $C_0$ obtained by removing a subset of items from $C$.

(a) Clearly, $c_0 \leq c_0^{max} = c_1 + c_2 - 1$. The CI corresponding to a minimal cover with cardinality $c_0^{max}$ has its RHS $= c_1 + c_2 - 2 < c_1 + 2c_2 - 2\ \forall c_1 > 0, c_2 > 0$. We also know that sequential lifting of a minimal CI does not alter its RHS. Hence, (2) can be never obtained through sequential lifting of such a minimal CI. Furthermore, a CI corresponding to any other cover with cardinality strictly less than $c_0^{max}$ will have its RHS $c_1 + c_2 - 2$. Hence, sequential lifting of such a CI can never produce (2).

(b) By definition, $C_0 \ni i\ \forall i \in C_2$. To prove (2) cannot be obtained through simultaneous lifting of any minimal CI, we consider the following two mutually exclusive (and exhaustive) cases:

(i) $C_0 \ni$ at least one $i \in C_1$: Let $C_0^1$ be the set of items from $C_1$ contained in $C_0$. CI corresponding to $C_0$ is $\sum_{i \in C_0} y_i \leq c_0 - 1$. Any inequality obtained through simulatenous lifting of this CI will be of the form:

$$\begin{aligned}
&\sum_{i \in C_0} y_i + \sum_{i \in C_1 \setminus C_0^1} \frac{1}{\alpha} y_i \\
&= \sum_{i \in C_0^1} y_i + \sum_{i \in C_0 \setminus C_0^1} y_i + \sum_{i \in C_1 \setminus C_0^1} \frac{1}{\alpha} y_i \\
&= \sum_{i \in C_0^1} y_i + \sum_{i \in C_2} y_i + \sum_{i \in C_1 \setminus C_0^1} \frac{1}{\alpha} y_i \leq c_0 - 1
\end{aligned}$$

The coefficients of the variables in the set $C_1 \setminus C_0^1$ have 1 in the numerator since the corresponding terms in a 2PCI appear with a coefficient of 1. The above inequality can be rewritten as: $\sum_{i \in C_0^1} \alpha y_i + \sum_{i \in C_2} \alpha y_i + \sum_{i \in C_1 \setminus C_0^1} y_i \leq \alpha(c_0 - 1)$. For this inequality to be a 2PCI, the coefficients of the variables in the set $C_2$ must be 2, i.e., $\alpha = 2$. However, this makes the coefficients of the variables in $C_0^1 \subset C_1$ also equal to $\alpha = 2$, which prevents it from being a 2PCI.

(ii) $C_0 \ni$ no $i \in C_1$: In this case, $C_0 = C_2 \implies c_2 \geq 2$. Also, $c_1 \geq 1$ (since $C$ is a non-minimal cover). So, the CI corresponding to $C_0 = C_2$ is $\sum_{i \in C_2} y_i \leq c_2 - 1$. Any inequality obtained through simulatenous lifting of this CI will be of the form:

$$\sum_{i \in C_2} y_i + \sum_{i \in C_1} \frac{1}{\alpha} y_i \leq c_2 - 1$$
$$\implies \sum_{i \in C_2} \alpha y_i + \sum_{i \in C_1} y_i \leq \alpha(c_2 - 1)$$

For this inequality to be a 2PCI, the coefficients of the variables in the set $C_2$ must be 2, i.e., $\alpha = 2$. Also, the RHS of the inequality must be equal to $c_1 + 2c_2 - 2$. This implies that $2(c_2 - 1) = c_1 + 2c_2 - 2 \implies c_1 = 0$, which contradicts the initial requirement that $c_1 \geq 1$. Hence, a 2PCI can never be obtained from simultaneous lifting of this CI.

□

**Example 1** (Continued)**.** *The facet-defining 2PCI* $2y_1 + y_2 + \cdots + y_{15} \leq 14$ *cannot be obtained from the sequential lifting of any minimal CI: This can be easily seen as follows. First, any CI with RHS = 14 is defined only for* $C = N$*. However, for* $C = N$*, the cover is non-minimal, as discussed earlier. Hence, this facet cannot be obtained from the sequential lifting of a minimal CI.*

*Similarly, it can be shown that the other facet-defining 2PCI* $2(y_1 + \cdots + y_5) + y_6 + \cdots + y_{14} \leq 17$ *can not be obtained through the sequential lifting of any minimal CI.*

Proposition 2 highlights that the facets given by (6) will, in general, complement the facets obtained through sequential liftings of minimal CIs in characterizing $Conv(KP)$.

**Proposition 3.** *Given a cover* $C$*, and its partition* $C_1 = \{\text{any one } i : a_i < \lambda\}$ *or* $\{i, j \in C : j \neq i, a_i + a_j < \lambda\}$*,* $C_2 = \{i, \in C : a_i < \lambda \leq a_i + max(C_1)\}$*, and* $C_3 = \{i \in C : a_i \geq \lambda\}$*,*

*(a) the following is a VI for KP:*

$$\sum_{i \in C_1} y_i + 2\sum_{i \in C_2} y_i + 3\sum_{i \in C_3} y_i \leq c_1 + 2c_2 + 3c_3 - 3 \tag{5}$$

*(b)* (2) *cuts off fractional extreme points* $\bar{y} \in P$ *characterized as:*

*(i)* $\bar{y}_i = 1\ \forall i \in (C_3 \cup C_2 \cup C_1 \backslash \{j, k, l\}\text{: } a_j + a_k + a_l > \lambda), \bar{y}_k = \bar{y}_l = 0, \bar{y}_j = \frac{a_j + a_k + a_l - \lambda}{a_j}$

*(ii)* $\bar{y}_i = 1\ \forall i \in ((C_1 \backslash \{j\}) \cup (C_2 \backslash \{k\}) \cup C_3 : a_j + a_k > \lambda), \left(\bar{y}_j = 0, \bar{y}_k = \frac{a_j + a_k - \lambda}{a_k}\right)$*, or* $\left(\bar{y}_k = 0, \bar{y}_j = \frac{a_j + a_k - \lambda}{a_j}\right)$

*(iii)* $\bar{y}_i = 1 \; \forall i \in (C_1 \cup C_2 \cup (C_3 \backslash \{j\}) : a_j > \lambda), \bar{y}_j = \frac{a_j - \lambda}{a_j}$.

We refer to (5) as a 3-partition cover inequality (3PCI).

*Proof.* (a) Following the steps of the proof for Proposition 1(a), we get the following valid inequality:

$$\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i + 3 \sum_{i \in C_3} y_i \leq c_1 + 2c_2 + 3c_3$$

Next, we show that any solution $y \in Y$ can be one of the following four mutually exclusive and exhaustive types, and in each case, it satisfies (5).

- $y_i = 1 \; \forall i \in C_1 \cup C_2$: Since $a_i \geq \lambda \; \forall i \in C_3$, for any feasible solution to KP, $\exists$ at least one $i \in C_3 : y_i = 0$. Therefore, $\sum_{i \in C_3} y_i \leq c_3 - 1$, which implies $\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i + 3 \sum_{i \in C_3} y_i \leq c_1 + 2c_2 + 3c_3 - 3$.
- $y_i = 1 \; \forall i \in C_2 \cup C_3$: Since $a_i + a_j < \lambda \; \forall i, j : i! = j \in C_1$, for any feasible solution to KP, $\exists$ at least one triplet $i, j, k \in C_1 : i! = j! = k, y_i = y_j = y_k = 0$. Therefore, $\sum_{i \in C_1} y_i \leq c_1 - 3$, which implies $\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i + 3 \sum_{i \in C_3} y_i \leq c_1 + 2c_2 + 3c_3 - 3$.
- $y_i = 1 \; \forall i \in C_1 \cup C_3$: Since $a_i < \lambda \; \forall i \in C_2$, for any feasible solution to KP, $\exists$ at least one pair $i, j \in C_1 : i! = j, y_i = y_j = 0$. Therefore, $\sum_{i \in C_2} y_i \leq c_2 - 2$, which implies $\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i + 3 \sum_{i \in C_3} y_i \leq c_1 + 2c_2 + 3c_3 - 4 < c_1 + 2c_2 + 3c_3 - 3$.
- $y_i = 1 \; \forall i \in C_3$: Since $a_i < \lambda \; \forall i \in C_2$ and $a_i + a_j < \lambda \; \forall i, j : i! = j \in C_1$, for any feasible solution to KP, $\exists$ at least one pair $i, j : i \in C_1, \; j \in C_2, y_i = y_j = 0$. Therefore, $\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i \leq (c_1 - 1) + 2(c_2 - 1)$, which implies $\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i + 3 \sum_{i \in C_3} y_i \leq c_1 + 2c_2 + 3c_3 - 3$.

(b) Proof for part (b) can be done using similar steps as for Proposition 1(b).

□

We refer to (5) as a *3-partition cover inequality* (3PCI).

**Remark 2.** *Given a cover $C$:*

*(a) by definition, $C_1 = \emptyset$ and $C_2 \neq \emptyset$ is not possible.*

(b) *from (a),* $C_1 = C_3 = \emptyset$ *is not possible.*

(c) *when* $C_1 = C_2 = \emptyset$, (5) *reduces to the following minimal CI:* $\sum_{i\in C_3} y_i \le c_3 - 1$.

(d) *when* $C_2 = C_3 = \emptyset$, (5) *reduces to the following ECI:* $\sum_{i\in C_1} y_i \le c_1 - 3$.

(e) *from (b), (c), (d), we only consider only either one of* $C_1$, $C_2$, *and* $C_3$ *is* $\emptyset$ *or none of them is empty.*

(f) *when only* $C_1 = \emptyset$, *then from (a) and (c),* (5) *reduces to CI.*

(g) *As a result of (e) and (f), we only consider 3PCIs with either: (i) only* $C_2 = \emptyset$, *(ii) only* $C_3 = \emptyset$, *or (iii) none is empty.*

**Example 1** (Continued)**.** *Again, we consider a non-minimal cover* $C = N$. *For this cover,* $\lambda = 19$, $C_3 = \{1\}$, $C_2 = \{2,3,\dots,11\}, C_1 = \{12,13,14,15\}$, $c_1 = 4, c_2 = 10, c_3 = 1$. *Hence, the corresponding 3PCI is* $3y_1+2(y_2+\dots+y_{11})+y_{12}+y_{13}+y_{14}+y_{15} \le 4+2\times 10+3\times 1-3 = 24$. *Consider the following fractional point:* $\bar{y}_i = 1\ \forall i \in C_3 \cup C_2 \cup C_1\backslash\{13,14,15\}, \bar{y}_{14} = \bar{y}_{15} = 0, \bar{y}_{13} = 2/9$. *Clearly,* $\sum_{i\in C} a_i\bar{y}_i = 158$. *Hence, this* $\bar{y}$ *is an extreme point of* $P$. *Here,* $a_{13}+a_{14}+a_{15} = 9+7+5 = 21 > 19 = \lambda$, *and* $\bar{y}_{13} = \frac{a_{13}+a_{14}+a_{15}-\lambda}{a_{13}} = \frac{9+7+5-19}{9} = \frac{2}{9}$. *Hence, this extreme point is of the type as characterized by Proposition 3.b.i. Further,* $\sum_{i\in C_1}\bar{y}_i + 2\sum_{i\in C_2}\bar{y}_i + 3\sum_{i\in C_3}\bar{y}_i = 24.22 > 24$*; hence,* $\bar{y}$ *is cut off by the above 3PCI.*

*Similarly, we can verify that the above-described 3PCI also cuts off the following two additional fractional extreme points of* $P$ *as characterized by* $(3).b.ii$: $\bar{y}_i = 1\ \forall i \in C_3 \cup C_2\backslash 2 \cup C_1\backslash\{15\}, \bar{y}_{15} = 0, \bar{y}_2 = \frac{a_{15}+a_2-\lambda}{a_2} = \frac{5+17-19}{19} = \frac{3}{19}$. *or* $\bar{y}_2 = 0, \bar{y}_15 = \frac{a_{15}+a_2-\lambda}{a_{15}} = \frac{5+17-19}{5} = \frac{3}{5}$. *For this example, there exists no extreme point of type* $(3).b.iii$ *because* $a_1 \ge \lambda$.

**Theorem 3.A.** *If* $C_2 = \emptyset$, *then* (5) *defines a facet of* $KP^C$ *if the following conditions are satisfied:*

(a) $c_1 \ge 4$

(b) $\lambda \le max(C_1) + min(C_1) + min_2(C_1)$

(c) $\lambda \le max_2(C_1) + max_3(C_1) + max_4(C_1)$

*Proof.* We omit the proof for the part (a), as it is similar to the proof stated in theorem 1 (a).

To prove that (5) defines a facet we need to show that $c = c_1 + c_3$ affinely independent points exists that are binding. We observe that when $C_2 = \emptyset$, then (5) reduces to $\sum_{i \in C_1} y_i + 3\sum_{i \in C_3} y_i \leq c_1 + 3c_3 - 3$. By definition of set $C_3$, it is easy to see that we can generate $c_3$ affinely independent points. Next, we will show that to generate the remaining $c_1$ affinely independent points, we need conditions (b) and (c). Using condition (b), we can generate $c_1 - 1$ affinely independent points. condition (c) allows the generation of the additional one affinely independent point. □

**Theorem 3.B.** *If $(C_3 = \emptyset)$ or $(C_1 \neq \emptyset, C_2 \neq \emptyset, C_3 \neq \emptyset)$, then* (5) *defines a facet of $KP^C$ if the following conditions are satisfied:*

*(a)* $c_1 \geq 3$

*(b)* $\lambda \leq max(C_2) + min(C_1)$*: This condition allows us to generate $c_1$ affinely independent points.*

*(c)* $\lambda \leq max(C_1) + max_2(C_1) + max_3(C_1)$

*Proof.* Again we omit the proof for the part (a), as it is similar to the proof stated in theorem 1 (a).

First, we prove the case when $C_3 = \emptyset$.

To prove that (5) defines a facet we need to show that $c = c_1 + c_2$ affinely independent points exists that are binding. We observe that when $C_3 = \emptyset$, then (5) reduces to $\sum_{i \in C_1} y_i + 2\sum_{i \in C_2} y_i \leq c_1 + 2c_2 - 3$. This can only be binding in one of the following ways: (i) $y_i = 0 \ i \in C_1$, and $y_j = 0 \ j \in C_2$ (ii) $y_i = y_j = y_k = 0 \ i, j, k \in C_1$

If $\lambda \leq max(c_2) + min(C_1)$ then we can generate $c_1$ affinely independent points as shown in the below matrix.

$$\begin{array}{cc}
\begin{array}{ccccc|cccc} y_1 & y_2 & \cdots & \cdots & y_{c_1} & y_{c_1+1} & y_{c_1+2} & \cdots & y_c \end{array} & \\
\left(\begin{array}{ccccc|cccc}
1 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & \cdots & 1 \\
 & & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 1 & & 1 & 0 & 1 & \cdots & 1 \\
0 & 1 & 1 & & 1 & 0 & 1 & \cdots & 1
\end{array}\right) &
\begin{array}{c} y^1 \\ y^2 \\ \vdots \\ y^{c_1-1} \\ y^{c_1} \end{array}
\end{array}$$

By definition of set $C_1$ and $C_2$, we can see that the following additional $c_2 - 1$ affinely independent points points we can generate as shown in the below matrix.

$$
\begin{array}{cc}
\begin{array}{cccccccccccc}
y_1 & y_2 & y_3 & y_4 & \cdots & \cdots & y_{c_1} & y_{c_1+1} & y_{c_1+2} & y_{c_1+3} & \cdots & y_c
\end{array} & \\
\left(\begin{array}{cccccc|cccccc}
0 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 1 & 1 & \cdots & 1 \\
0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 1 & \cdots & 1 \\
 & & & & & & & & & \ddots & & 1 \\
0 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 1 & \cdots & 0 & 1 \\
0 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 1 & \cdots & 1 & 0 \\
0 & 0 & 0 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1
\end{array}\right) &
\begin{array}{l}
y^{c_1+1} \\
y^{c_2+2} \\
\vdots \\
y^{c-2} \\
y^{c-1} \\
y^{c}
\end{array}
\end{array}
$$

The last point $y^c$ as shown in the above matrix can be generated only if condition $\lambda \leq max(C_1) + max_2(C_1) + max_3(C_1)$ is satisfied.

In the other case, when $(C_1 \neq \emptyset, C_2 \neq \emptyset, C_3 \neq \emptyset)$, we will need additional $C_3$ affinely independent points that are binding to (5). It is easy to see that we can generate additional $C_3$ affinely independent points by definition of set $C_3$.

$\square$

We state the following results without their proof.

**Theorem 4.A.** *If $C_2 = \emptyset$, then* (5) *defines a facet of $KP$ if it defines a facet of $KP^C$ and any of the following conditions is satisfied:*

*(a)* $C = N$

*(b)* $\lambda \leq max(C_3) - max(C')$

*(c)* $\lambda \leq max(C_1) + max_2(C_1) + max_3(c_1) - max(C')$

**Theorem 4.B.** *If $C_3 = \emptyset$, then* (5) *defines a facet of $KP$ if it defines a facet of $KP^C$ and any of the following conditions is satisfied:*

*(a)* $C = N$

*(b)* $\lambda \leq max(C_1) + max(C_2) - max(C')$

*(c)* $\lambda \leq max(C_1) + max_2(C_1) + max_3(C_1) - max(C')$

**Theorem 4.C.** *If $C_1, C_2$, and $C_3$ all are non-empty, then* (5) *defines a facet of $KP$ if it defines a facet of $KP^C$ and any of the following conditions is satisfied:*

(a) $C = N$

(b) $\lambda \leq max(C_3) - max(C')$

(c) $\lambda \leq max(C_1) + max(C_2) - max(C')$

(d) $\lambda \leq max(C_1) + max_2(C_1) + max_3(C_1) - max(C')$

**Proposition 4.** *Given a non-minimal cover* $C$ *and its n-partition* $C_1 = \{i \in C :$ *sum of any n-1 elements* $< \lambda \leq$ *sum of any n elements*$\}$, $C_2 = \{i \in C \backslash C_1 : max(C_1) + \cdots + max_{n-3}(C_1) + a_i < \lambda \leq max(C_1) + \cdots + max_{n-2}(C_1) + a_i\}$, $\ldots$, $C_j = \{i \in C \backslash (C_1 \cup C_2 \cup \ldots C_{i-1}) : max(C_1) + \cdots + max_{n-j-1}(C_1) + a_i < \lambda \leq max(C_1) + \cdots + max_{n-j}(C_1) + a_i\}$, $\ldots$, $C_p = \{i \in C : a_i \geq \lambda\}$ *such that* $\cup_{j=1}^{p} C_j = C$, *the following is a VI for KP:*

$$\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i + \cdots + n \sum_{i \in C_n} y_i \leq c_1 + 2c_2 + \cdots + nc_n - n \tag{6}$$

We refer to (6) as a *n-partition cover inequality* (nPCI).

**Remark 3.** *(a) When* $C_1 = C_2 = \cdots = C_{n-1} = \emptyset$, (6) *reduces to the following minimal CI:* $\sum_{i \in C_n} y_i \leq c_n - 1$.

*(b) When* $C_2 = C_3 = \cdots = C_n = \emptyset$, (6) *reduces to the following ECI:* $\sum_{i \in C_1} y_i \leq |C_1| - n$.

*(c) As a result of the above two remarks, we only consider nPCIs that satisfies the conditions neither in (a) nor in (b).*

**Example 2.** *Consider* $Y = \{y \in \mathbb{B}^n : \sum_{i \in N} a_i y_i <= b\}$ *with* $n = 7$, $a = \{8, 7, 6, 4, 6, 6, 6\}$, *and* $b = 22$.

*Clearly,* $C = N$ *is a non-minimal cover since* $\sum_{i \in C} a_i = 43, \lambda = 43 - 22 = 21 > min_{i \in C}\{a_i\} = 4$. *Clearly, the following is a 4-partition of C:* $C_1 = N \backslash \{1\}$, $C_2 = \{1\}$, $C_3 == c_4 = \emptyset$. *Hence, the corresponding 4PCI is* $2y_1 + y_2 + \cdots + y_7 \leq 2 \times 1 + 6 - 4 = 4$.

## 2.2 Minimal Exclusion-based Partition

We now define our second class of VIs of $KP$ obtained from a non-minimal cover. For this, we first use the following definitions.

**Definition 1.** *Exclusion set: Given a cover $C$, an exclusion set is a subset of $C$ whose exclusion makes $C$ no longer a cover.*

**Example 2** (Continued)**.** *Consider $Y = \{y \in \mathbb{B}^n : \sum_{i \in N} a_i y_i <= b\}$ with $n = 15$, $a = \{19, 17, 14, 14, 14, 13, 13, 12, 11, 10, 10, 9, 9, 7, 5\}$, and $b = 158$. Consider a cover $C = N \setminus \{14, 15\}$. Here, $\sum_{i \in C} a_i = 165, \lambda = 7$. Clearly, any non-empty subset of $C$ is an exclusion set.*

**Definition 2.** *Minimal exclusion set: Given a cover $C$, a minimal exclusion set $C^e$ is an exclusion set such that the inclusion of any $i \in C^e$ back to $C \setminus C^e$ makes $C$ again a cover.*

**Example 2** (Continued)**.** *We discussed above that any non-empty subset of $C = N \setminus \{14, 15\}$ is its exclusion set. Among them, let us consider the following exclusion set: $C^e = \{1, 2\}$. $\sum_{i \in C \setminus C^e} a_i = 165 - 19 - 17 = 129$. Clearly, it is a non-minimal exclusion set since including $i = 1$ back to $C \setminus C^e$ gives $\sum_{i \in C \setminus C^e} a_i = 129 + 19 = 148 < b$; similarly, including $i = 2$ back to $C \setminus C^e$ gives $\sum_{i \in C \setminus C^e} a_i = 129 + 17 = 146 < b$. However, any subset of $C$ containing only one element is a minimal exclusion set of $C$.*

**Definition 3.** *Maximum minimal exclusion set: Given a cover $C$, the maximum minimal exclusion set is that minimal exclusion set that contains the maximum number of elements among all minimal exclusion sets. We use $p$ to denote the cardinality of the maximum minimal exclusion set of a cover.*

**Example 2** (Continued)**.** *In the above-discussed example, all minimal exclusion sets contain only one element. Hence, $p = 1$. Clearly, this is true for any minimal cover. Let us now consider a non-minimal cover $C = N$. Here, $\sum_{i \in C} a_i = 177, \lambda = 19$. For this cover, we have multiple minimal exclusion sets possible, e.g., $C_1^e = \{1\}$, $C_2^e = \{2, 15\}$, $C_3^e = \{13, 14, 15\}$. Clearly, out of these, $C_3^e$ is the maximum minimal exclusion set and $p = 3$. However, $C$ has several other maximum minimal exclusion sets, e.g., $C_4^e = \{11, 14, 15\}$, $C_5^e = \{9, 14, 15\}$, $C_6^e = \{10, 14, 15\}$, all with $p = 3$.*

The problem of finding $p$ can be stated as an optimization problem. For this let, $z_i = 1$ if element $i \in C$ belongs to the exclusion set $C^e$, 0 otherwise.

$$\begin{aligned}
\max \; & \sum_{i \in C} z_i \\
s.t. & \sum_{i \in C} a_i(1 - z_i) \leq b \\
& \sum_{i \in C} a_i(1 - z_i) + a_j \geq (b + \epsilon) z_j \qquad\qquad \forall j \in C
\end{aligned}$$

Clearly, $p = 1$ for a minimal cover. However, for a non-minimal cover, $p > 1$. In Section 2.1, we defined minimal and non-minimal covers based on the surplus of a cover. Now, definition 3 provides alternate the following definitions of minimal and non-minimal covers. A minimal cover is a cover with $p = 1$, while a non-minimal cover has $p > 1$.

**Proposition 5.** *Given a cover $C$, the cardinality $p$ of its maximum minimal exclusion set, and a $p$-partition of $C$ as follows:*

- $C_1 = C^e$
- $C_2 = \{i \in C \backslash C_1 : max(C_1) + max_2(C_1) + \cdots + max_{p-2}(C_1) + a_i \geq \lambda \text{ and } max(C_1) + max_2(C_1) + \cdots + max_{p-3}(C_1) + a_i < \lambda\}$
- $C_j = \{i \in C \backslash (C_1 \cup C_2 \ldots C_{j-1}) : max(C_1) + max_2(C_1) + \cdots + max_{p-j}(C_1) + a_i \geq \lambda \text{ and } max(C_1) + max_2(C_1) + \cdots + max_{p-j-1}(C_1) + a_i < \lambda\} \; \forall j < p$
- $C_p = \{i \in C : a_i \geq \lambda\}$,

*(a) the following is a VI for KP:*

$$\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i + \cdots + p \sum_{i \in C_p} y_i \leq c_1 + 2c_2 + \cdots + pc_p - p \tag{7}$$

*(b) (7) cuts off fractional extreme points $\bar{y} \in P$ characterized as:*

*(i)* $\bar{y}_i = 1 \; \forall i \in (C \setminus C_1 : \sum_{i \in C_1} a_i > \lambda), \bar{y}_i = 0 \; \forall i \in (C_1 \setminus \{j\}), \bar{y}_j = \frac{\sum_{i \in C_1} a_i - \lambda}{a_j}$

*(ii)* $\bar{y}_i = 1 \; \forall i \in (C_1 \cup C_2 \cup \ldots (C_p \backslash \{j\}) : a_j > \lambda), \bar{y}_j = \frac{a_j - \lambda}{a_j}$.

*Proof.* The proof follows using similar steps as used in proving Propositions 1 and 3. □

We refer to (7) as a *maximum minimal exclusion p-partition cover inequality* (MMEpPCI).

**Remark 4.** *(a)* $C_1 \neq \emptyset$ *since* $C_1 = C^e$

*(b) When* $C_2 = C_3 = \cdots = C_p = \emptyset$, *then* (7) *reduces to* $y_i = 0\ \forall i \in C_1$

*(c) If set* $C_1$ *of nPCI and set* $C_1$ *of MMEpPCI are identical, then inequality* (6) *and* (7) *are also identical.*

Further, we state the following additional results without their proof.

**Theorem 5.** *For* $p = 3$, (7) *defines a facet of* $KP^C$ *if and only if the following two conditions are satisfied:*

*(a)* $\lambda \leq max(C_1) + min(C_2)$

*(b)* $\lambda \leq max(C_2) + min(C_1)$

**Theorem 6.** *For* $p = 3$, (7) *defines a facet of* $KP$ *if it defines a facet of* $KP^C$ *and any of the following conditions is satisfied:*

*(a)* $C = N$

*(b)* $\lambda \leq max(C_3) - max(C')$

*(c)* $\lambda \leq max(C_2) + max(C_1) - max(C')$

*(d)* $\lambda \leq max(C_1) + max_2(C_1) + max_3(C_1) - max(C')$

**Proposition 6.** *Given a cover* $C$, *the cardinality* $p$ *of its maximum minimal exclusion set, and a* $(p+1)$*-partition of* $C$ *as follows:*

- $C_1 = \{$ *select any* $p-1$ *element from* $C$ *such that their sum is less than* $\lambda\}$.
- $C_2 = \{i \in C \backslash C_1 : max(C_1) + max_2(C_1) + \cdots + max_{p-2}(C_1) + a_i < \lambda \leq max(C_1) + max_2(C_1) + \cdots + max_{p-1}(C_1) + a_i\}$.
- $C_j = \{i \in C \backslash C_1 \cup C_2 : max(C_1) + max_2(C_1) + \cdots + max_{p-j-1}(C_1) + a_i < \lambda \leq max(C_1) + max_2(C_1) + \cdots + max_{p-j}(C_1) + a_i\}$.

- $C_{p+1} = \{i \in C : a_i \geq \lambda\}$

*the following is a valid inequality for KP:*

$$\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i + \cdots + (p+1) \sum_{i \in C_{p+1}} y_i \leq c_1 + 2c_2 + \cdots + (p+1)c_{p+1} - (p+1) \tag{8}$$

We refer to (8) as a *maximum minimal exclusion p+1-partition cover inequality* (MMEp+1PCI).

# 3 Computational Experiments

We first propose the separation problem for $n$PCI in Section 3.1. We then test the effectiveness of our proposed VIs in efficiently solving the fixed charge transportation problem (FCTP). Our choice of FCTP is motivated by its following two properties: (i) it is NP-hard; (ii) 2PCI and 3PCI occur frequently as VIs for the knapsack substructures derived from FCTP, as shown in Section 3.2. The datasets used in our computational experiments are described in Section 3.3, and the results of our experiments are presented and discussed in Section 3.4.

## 3.1 Separation Problem

Here, we describe the separation problem for 2PCI. For this, let $z_{i1} = 1$ if item $i$ belongs to $C_1$ of a 2-partition cover $C$, 0 otherwise. Similarly, let $z_{i2} = 1$ if item $i$ belongs $C_2$, 0 otherwise. Then, the separation of the most violated 2PCI can be mathematically stated as follows.

$$\text{Maximize} \quad \sum_{i \in C} (\bar{y}_i - 1) z_{i1} + 2 \sum_{i \in C} (\bar{y}_i - 1) z_{i2} + 2 \tag{9}$$

Subject to

$$\sum_{i \in N} a_i (z_{i1} + z_{i2}) \geq b + 1 \tag{10}$$

$$a_i z_{i1} \leq \sum_{i \in N} a_i (z_{i1} + z_{i2}) - b - 1 \tag{11}$$

$$a_i z_{i2} \geq \sum_{i \in N} a_i (z_{i1} + z_{i2}) - b - M(1 - z_{i2}) \tag{12}$$

$$z_{i1} + z_{i2} \leq 1 \quad j = 1 \ldots n \tag{13}$$

$$\sum_{i \in N} z_{i1} \geq 1 \tag{14}$$

$$\sum_{i \in N} z_{i2} \geq 1 \tag{15}$$

$$z_{i1}, z_{i2} \in \{0, 1\} \tag{16}$$

(9) maximizes the violation of a 2PCI. (10) ensures that $C_1 \cup C_2$ is a cover. (11) ensures that item $i$ belongs to $C_1$ only if $a_i < \lambda = \sum_{i \in N} a_i (z_{i1} + z_{i2}) - b$. (12) ensures that item $i$ belongs to $C_2$ only if $a_i \geq \lambda$. (13) forces each item to either belong to $C_1$ or $C_2$ of cover $C$ or lies outside $C$. (14) and (15) ensure that neither $C_1$ not $C_2$ is empty, i.e., $C$ is strictly a non-minimal cover (refer to Remark 1). The separation problem for a general nPCI can be similarly stated.

We show the separation of a 2PCI using the following binary knapsack problem as an illustrative example:

**Example 3.**

$$\textit{Maximize} \quad 2y_1 + 3y_2 + 4y_3 + 5y_4$$

*Subject to*

$$5y_1 + 4y_2 + 3y_3 + 2y_4 <= 10$$

$$y_1, y_2, y_3, y_4 \in \{0, 1\}$$

The optimal solution to its LP relaxation is: $y_1 = 0.2, y_2 = y_3 = y_4 = 1$. The most violated CI by this solution is: $y_1 + y_2 + y_4 <= 2$, while the most violated 2PCI, as obtained by solving (9)-(16), is: $2y_1 + 2y_2 + y_3 + y_4 <= 4$. While it can be easily verified that the violated CI defines a facet to the convex hull of the above knapsack polytope, the violated 2PCI only defines a face. Nonetheless, addition of either VI to the above knapsack problem gives the optimal integer solution as: $y_1 = 0, y_2 = y_3 = y_4 = 1$.

### 3.2 Knapsack Set as a Substructure in FCTP

FCTP is a generalization of the well-known transportation problem that includes a fixed cost of transportation between any source and destination, in addition to the variable cost per unit of transportation. It has a wide range of applications, primarily in distribution, transportation, scheduling, and location (Adlakha and Kowalski, 2003; Mingozzi and Roberti, 2018). Furthermore, FCTP has also been used to solve problems such as process selection (Hirsch and Dantzig, 1968), teacher assignment (Hultberg and Cardoso, 1997), and industrial waste management (Maniezzo et al., 1998).

FCTP is formally defined in the literature as follows. Consider a set of sources (origins) $S = \{1, 2, \ldots s\}$, each with a supply capacity $a_i > 0$, and a set of sinks (destinations) $T = \{1, 2, \ldots t\}$, each with demand $b_j > 0$. We assume that the problem is balanced, i.e. $\sum_{i \in S} a_i = \sum_{j \in T} b_j$. There is a unit shipping cost $c_{ij}$ plus a fixed cost $f_{ij}$ for every $i$-$j$ pair. Let $m_{ij} = \min\{a_i, b_j\}$. If $x_{ij}$ represents the quantity shipped from source $i$ to sink $j$, and $y_{ij} = 1$ if the link from $i$-$j$ is used, 0 otherwise, then FCTP can be mathematically stated as:

$$\min \sum_{i=1}^{s}\sum_{j=1}^{t}(C_{ij}x_{ij} + F_{ij}y_{ij}) \tag{17}$$

subject to

$$\sum_{j=1}^{t} x_{ij} = a_i \qquad i \in S \tag{18}$$

$$\sum_{i=1}^{s} x_{ij} = b_j \qquad j \in T \tag{19}$$

$$x_{ij} \leq m_{ij}y_{ij} \qquad \forall i \in S, j \in T \tag{20}$$

$$x_{ij} \geq 0, y_{ij} \in \{0,1\} \qquad \forall i \in S, j \in T \tag{21}$$

Clearly, it is the presence of the fixed costs in the problem that results in a mixed-integer linear program (MILP) based model for FCTP, as opposed to a pure linear program for the transportation problem. Hence, while the transportation problem is polynomially solvable, FCTP is known to be $\mathcal{NP}$-hard. There have been a few studies on solving FCTP (Agarwal and Aneja, 2012; Roberti et al., 2015) more efficiently using the current MILP solvers, but even the state-of-the-art method struggles to solve general instances of even medium-size. In this paper, we propose a new class of valid inequalities for FCTP that help improve its lower bound, thereby aiding the MILP solver to solve the problem faster.

Clearly, (22) and (23) given below are VIs to (17)-(21) since each term in the LHS of (22) and (23) are upper bounds on the corresponding terms in (18) and (19).

$$\sum_{j=1}^{t} m_{ij}y_{ij} >= a_i \qquad i \in S \tag{22}$$

$$\sum_{i=1}^{s} m_{ij}y_{ij} >= b_j \qquad j \in T \tag{23}$$

(22) and (23) represent knapsack inequalities of the form $\sum_{i\in N} a_i x_i \geq d$. An FCTP with $s = |S|$ supply nodes and $t = |T|$ demand nodes has $s = t$ such knapsack polytopes. To be consistent with the knapsack literature, (22) and (23) can be restated as $\sum_{i\in N} a_i y_i <= b$ using the standard trick of replacing variables by their complements. However, the above VIs are not useful since they are

always satisfied by any fractional solution to (17)-(21). Nonetheless, there exist well-known classes of useful VIs (e.g., CIs) for knapsack polytopes, which can be added to (17)-(21) to strengthen its lower bound. Furthermore, our two classes of VIs based on non-minimal covers can also be used.

The next proposition guarantees the existence of 2PCIs and 3PCIs for any FCTP.

**Proposition 7.** *Let $M_1 = \{i \in S : a_i \leq b_j \ \forall j \in T\}$ and $M_2 = \{j \in T : b_j \leq a_i \ \forall i \in S\}$. Then an FCTP with $s$ supply nodes and $t$ demand nodes will have at least $s + t - m$ 2PCIs at least an equal number of 3PCIs, where $m = |M_1| + |M_2|$.*

*Proof.* The knapsack inequality (22) can be rewritten as:

$$\sum_{j=1}^{t} m_{ij} z_{ij} \leq \sum_{j}^{t} m_{ij} - a_i \qquad i \in S \tag{24}$$

where $z_{ij} = 1 - y_{ij}$. For a given $i \in S$, consider a cover $C$ for the knapsack set given by (24) such that $C = T$, in which case its surplus $\lambda = a_i$. Then, the following three mutually exclusive and exhaustive conditions arise.

(a) $a_i \leq b_j \ \forall j \in T$: In this case, $\lambda \leq min(C)$, and hence, $C$ is a minimal cover. Therefore, no 2PCI or 3PCI exists. Let $M^s = \{i \in S : a_i \leq b_j \ \forall j \in T\}$.

(b) $a_i < b_j \ \forall j \in T' \subset T$: In this case, $min(C) < \lambda \leq max(C)$, and hence, at least one 2PCI exists (using Remark 1(c) and at least one 3PCI exists (using Remark 2(g)).

(c) $a_i > b_j \ \forall j \in T$: In this case, $C_3 = \emptyset$ in a 3-partition of $C$. Further, $C_1 \neq \emptyset, C_2 \neq \emptyset$, and hence, at least one 3PCI exists (using Remark 2(g)). However, for a 2-partition of $C$, $C_2 = \emptyset$. Hence, no 2PCI exists for $C = T$ (using Remark 1(b)). Nonetheless, $\exists C \subset T : min(C) < \lambda \leq max(C)$, and hence, at least one 2PCI will exist for such a cover $C$ (using Remark 1(c)).

A similar argument holds true for knapsack inequality (23). Hence, $m = |M_1| + |M_2|$ represents the number of knapsack inequalities defined by (22) and (23) for which neither 2PCI nor 3PCI exists, and the remaining $s + t - m$ knapsack inequalities are guaranteed to have at least one 2PCI and one 3PCI.

□

**Example 3.** *The binary knapsack set considered in Example 1 is a knapsack substructure in FCTP with* $s = t = 15$ *from Dataset 1 described in Section3.3. The complete convex hull of this knapsack set (excluding the trivial facets* $y_i \geq 0$ *and* $y_i \leq 1$*), as obtained using PANDA Lörwald and Reinelt (2015), is shown in Table 5 in the Appendix. Of the 29 non-trivial facets, the first 19 are CIs. The next 6 (20-25) are nPCIs, while the last 4 (26-29) are minimal exclusion set-based CIs. This highlights the significance of our proposed VIs, besides the well-known CIs, in completely characterizing a knapsack polytope, which appears as a local substructure within FCTP.*

### 3.3 Datasets

For our computational experiments, we consider two benchmark datasets. Dataset 1 is introduced by Agarwal and Aneja (2012). It consists of instances with 15 origins and 15 destinations, while $a_i$ and $b_j$ are randomly generated using the uniform distribution $U \sim [1, 20]$. Fixed and variable costs are generated using $U \sim (200, 800)$. $\theta$ is the ratio between the total variable and fixed costs. Instances with $\theta = 0.0$ represent a pure fixed charge transportation problem (PFCTP). Dataset 2 is introduced by Roberti et al. (2015). It consists of instances similar to Dataset 1, except for the larger number of origins and destinations. In our experiments, we considered instances with 30 origins and 30 destinations[1].

### 3.4 Results

All the runs were conducted on a single core of an Intel(R) Xeon(R) Gold 6240 CPU @ 2.60GHz server with 16GB of RAM. The model of FCTP (17)-(21) is implemented in C++ and solved using CPLEX 22.1. All our VIs are generated only at the root node of the branch-and-bound tree using complete enumeration, which are added if violated by the LP relaxation of the problem and satisfy the facet-defining condition. For a given knapsack inequality, 2PCI with $c = n$ is unique, while there are only $n - 1$ such 2PCIs with $c = n - 1$. So, these VIs can be easily enumerated. However, there can be multiple 3PCIs with $c = n$, and we have arbitrarily generated one of them.

In all the experiments, a CPU time limit of 3,600 seconds is used for instances from Dataset 1. Instances from Dataset 1 are relatively easier to solve; hence, we solve them without the cplex-

[1]In our experiments, instances for Dataset 1 are used as received from the authors, while those for Dataset 2 are randomly generated using the scheme described in the paper

generated cuts to better assess the efficacy of our facets. The results from our computational experiments corresponding to Dataset 1 are shown in Table 1 for $\theta = 0.2$ and Table 2 for $\theta = 0$. As clear from Table 1, the facets from our nPCIs help improve the lower bound of FCTP by 9% (from 81.9% to 89.2% of the IP optimal objective function value), which helps cut down the CPU time by around 77% (from 1,418.6 seconds to 330.1 seconds). The corresponding savings for $\theta = 0$, as evident from Table 2, are around 13.5% and 94% in the lower bound and CPU time, respectively.

Instances from Dataset 2 are much more difficult; hence, we use a higher CPU time limit of 7,200 seconds, and solve them with the cplex-generated cuts. However, we turn off the cover inequalities generted by cplex, again to better assess the efficacy of our facets. The results from our computational experiments corresponding to Dataset 2 are shown in Table 3 for $\theta = 0.2$ and Table 4 for $\theta = 0$. As clear from Table 3, the facets from our nPCIs help improve the lower bound of FCTP by 9% (from 84.2% to 91.8%), which helps cut down the CPU time by around 16.5% (from 3,392.6 seconds to 2,847.1 seconds). The corresponding savings for $\theta = 0$, as evident from Table 4, are around 9% and 28% in the lower bound and CPU time, respectively.

Table 1: Dataset 1, $s = t = 15$, $\theta = 0.2$, Without CPLEX-generated cuts

| | | Without nPCI | | | With nPCI | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Ins. | IP Obj. | LP% | CPU/ (Gap%) | BBTS | 2PCI $c = n$ | 2PCI $c = n - 1$ | 3PCI $c = n$ | LP% | CPU/ (Gap%) | BBTS |
| 1 | 10017 | 81.8 | 1037.9 | 11.31 | 13 | 4 | 2 | 89.0 | 47.3 | 0.19 |
| 2 | 10075 | 82.1 | 687.0 | 15.04 | 14 | 4 | 3 | 88.9 | 42.5 | 0.19 |
| 3 | 9327 | 83.5 | 141.2 | 0.89 | 16 | 9 | 5 | 91.2 | 3.3 | 0.01 |
| 4 | 11093 | 78.7 | 7.83% | 20.83 | 17 | 3 | 3 | 86.6 | 2573.7 | 10.28 |
| 5 | 10312 | 80.3 | 1132.0 | 28.12 | 10 | 4 | 3 | 87.5 | 39.9 | 0.19 |
| 6 | 10086 | 86.6 | 38.0 | 1.02 | 9 | 17 | 3 | 91.9 | 6.0 | 0.02 |
| 7 | 9913 | 82.1 | 323.5 | 1.90 | 9 | 6 | 2 | 88.7 | 21.1 | 0.09 |
| 8 | 10495 | 80.3 | 5.84% | 22.42 | 14 | 5 | 4 | 87.5 | 532.6 | 2.41 |
| 9 | 10137 | 83.5 | 26.3 | 1.10 | 12 | 9 | 1 | 91.3 | 3.1 | 0.01 |
| 10 | 9939 | 80.2 | 0.59% | 25.22 | 13 | 6 | 2 | 89.6 | 31.4 | 0.16 |
| Avg | 10139 | 81.9 | 1418.6(1.43%) | 12.8 | 13 | 7 | 3 | 89.2 | 330.1(0%) | 1.40 |

Ins.: Instance; IP Obj.: IP optimal; LP%: LP relaxation as a % of IP Obj.; CPU/Gap%: Computation time (time limit = 3600 seconds) / Optimality Gap in %; BBTS: Branch-and-bound tree size in millions of nodes

Table 2: Dataset 1, $s = t = 15$, $\theta = 0$, Without CPLEX-generated cuts

| Ins. | IP Obj. | Without nPCI LP% | Without nPCI CPU/ (Gap%) | Without nPCI BBTS | With nPCI 2PCI $c = n$ | With nPCI 2PCI $c = n - 1$ | With nPCI 3PCI $c = n$ | With nPCI LP% | With nPCI CPU/ (Gap%) | With nPCI BBTS |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 6683 | 74.5 | 1184.6 | 11.31 | 19 | 7 | 5 | 86.4 | 38.0 | 0.15 |
| 2 | 6903 | 73.2 | 1542.4 | 15.04 | 17 | 5 | 5 | 80.5 | 88.1 | 0.41 |
| 3 | 6210 | 77.7 | 100.6 | 0.89 | 15 | 16 | 4 | 89.6 | 3.9 | 0.01 |
| 4 | 7753 | 71.1 | 9.8% | 20.83 | 21 | 3 | 3 | 82.1 | 366.0 | 1.50 |
| 5 | 7360 | 69.1 | 2.7% | 28.12 | 12 | 1 | 2 | 80.3 | 88.3 | 0.43 |
| 6 | 6911 | 80.2 | 109.1 | 1.02 | 13 | 14 | 2 | 88.6 | 7.9 | 0.03 |
| 7 | 6434 | 77.7 | 194.2 | 1.90 | 10 | 6 | 1 | 86.6 | 17.6 | 0.08 |
| 8 | 7254 | 73.6 | 7.4% | 22.42 | 14 | 1 | 2 | 83.2 | 285.9 | 1.46 |
| 9 | 7119 | 80.0 | 119.6 | 1.10 | 10 | 8 | 1 | 88.8 | 9.3 | 0.04 |
| 10 | 6843 | 72.1 | 5.4% | 25.22 | 14 | 9 | 3 | 85.4 | 143.5 | 0.65 |
| Avg | 6947 | 74.9 | 1765.1(1.43%) | 12.79 | 15 | 7 | 3 | 85.1 | 104.9(0%) | 0.48 |

Ins.: Instance; IP Obj.: IP optimal; LP%: LP relaxation as a % of IP Obj.; CPU/Gap%: Computation time (time limit = 3600 seconds) / Optimality Gap in %; BBTS: Branch-and-bound tree size in millions of nodes

Table 3: Dataset 2, $s = t = 30$, $\theta = 0.2$, With CPLEX-generated cuts

| Ins. | IP Obj. | Without nPCI LP% | Without nPCI CPU/ (Gap%) | Without nPCI BBTS | With nPCI 2PCI $c = n$ | With nPCI 2PCI $c = n - 1$ | With nPCI 3PCI $c = n$ | With nPCI LP% | With nPCI CPU/ (Gap%) | With nPCI BBTS |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 13367 | 82.8 | 4695.8 | 0.41 | 23 | 62 | 10 | 90.7 | 4082.1 | 0.37 |
| 2 | 13447 | 82.3 | 2.19% | 0.58 | 23 | 45 | 7 | 88.9 | 1.40% | 0.44 |
| 3 | 13953 | 83.0 | 5785.2 | 0.53 | 18 | 53 | 2 | 91.6 | 3978.0 | 0.44 |
| 4 | 13477 | 84.9 | 266.0 | 0.04 | 28 | 47 | 5 | 93.4 | 219.7 | 0.04 |
| 5 | 13387 | 86.0 | 1485.8 | 0.16 | 29 | 71 | 15 | 93.3 | 863.4 | 0.10 |
| 6 | 13704 | 86.1 | 126.2 | 0.03 | 22 | 27 | 5 | 93.0 | 70.2 | 0.01 |
| 7 | 13412 | 84.9 | 2349.2 | 0.25 | 28 | 105 | 11 | 92.5 | 1284.7 | 0.14 |
| 8 | 13600 | 80.2 | 3.00% | 0.62 | 28 | 44 | 11 | 89.8 | 2.59% | 0.60 |
| 9 | 13683 | 86.2 | 1600.6 | 0.16 | 24 | 46 | 10 | 93.0 | 265.3 | 0.04 |
| 10 | 13605 | 85.8 | 3217.4 | 0.32 | 22 | 106 | 8 | 92.2 | 3307.9 | 0.30 |
| Avg | 13564 | 84.2 | 3392.6(0.52%) | 0.31 | 25 | 61 | 8 | 91.8 | 2847.1(0.40%) | 0.25 |

Ins.: Instance; IP Obj.: IP optimal; LP%: LP relaxation as a % of IP Obj.; CPU/Gap%: Computation time (time limit = 7200 seconds) / Optimality Gap in %; BBTS: Branch-and-bound tree size in millions of nodes

Table 4: Dataset 2, $s = t = 30, \theta = 0$, With CPLEX-generated cuts

| | | Without nPCI | | | With nPCI | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Ins. | IP Obj. | LP% | CPU/ (Gap%) | BBTS | 2PCI $c = n$ | 2PCI $c = n - 1$ | 3PCI $c = n$ | LP% | CPU/ (Gap%) | BBTS |
| 1 | 11109 | 80.9 | 2467.2 | 0.24 | 25 | 55 | 3 | 89.1 | 750.2 | 0.09 |
| 2 | 10622 | 83.0 | 2514.6 | 0.30 | 33 | 40 | 8 | 91.4 | 1504.0 | 0.20 |
| 3 | 10886 | 80.0 | 0.31% | 0.74 | 27 | 34 | 10 | 87.9 | 3775.2 | 0.42 |
| 4 | 11316 | 79.5 | 4754.8 | 0.47 | 23 | 84 | 13 | 90.7 | 4381.8 | 0.40 |
| 5 | 10717 | 82.9 | 1711.6 | 0.17 | 34 | 54 | 13 | 91.8 | 684.4 | 0.08 |
| 6 | 10391 | 83.0 | 1458.3 | 0.16 | 34 | 58 | 7 | 92.6 | 1575.1 | 0.16 |
| 7 | 10919 | 81.1 | 2520.1 | 0.20 | 30 | 67 | 14 | 91.5 | 2207.9 | 0.19 |
| 8 | 10908 | 82.8 | 601.3 | 0.10 | 23 | 83 | 3 | 92.4 | 238.3 | 0.05 |
| 9 | 10744 | 77.9 | 4.31% | 0.56 | 33 | 40 | 2 | 86.8 | 3.33% | 0.53 |
| 10 | 10903 | 81.9 | 741.9 | 0.07 | 32 | 63 | 12 | 91.9 | 180.8 | 0.03 |
| Avg | 10852 | 81.3 | 3117.0(0.46%) | 0.30 | 29 | 58 | 9 | 90.6 | 2249.8(0.33%) | 0.22 |

Ins.: Instance; IP Obj.: IP optimal; LP%: LP relaxation as a % of IP Obj.; CPU/Gap%: Computation time (time limit = 7200 seconds) / Optimality Gap in %; BBTS: Branch-and-bound tree size in millions of nodes

# 4 Conclusions and Future Work

In this paper, we studied the polyhedral structure of the binary knapsack polytope. For this, we exploited non-minimal covers of a knapsack, as opposed to minimal covers, popularly used in the literature. Using non-minimal covers, we proposed two new classes of VIs and derived the conditions under which they define facets of $KP$. Further, we proved that the facets of $KP$ obtained from one of our proposed classes of VIs can never be obtained through sequential lifting of minimal cover inequalities. Our computational experiments on a well-known class of NP-hard problems highlighted the usefulness of the facets from our proposed VIs.

In our computational experiments, we used 2PCIs only for $c = n$ and $c = n - 1$ and 3PCIs for $c = n$. For $c = n$, the corresponding cover is unique and so is its 2-partition and a finite number of 3-partitions possible. For $c = n-1$, there are only $n$ different covers, each again resulting in a unique 2-partition. So, for these cases, the resulting 2PCIs and 3PCIs, which are finite in number, could be easily enumerated. However, for $c <= n - 2$, the number of covers becomes large, and so does the number of resulting 2PCIs and 3PCIs, which may be computationally inefficient to enumerate. In such cases, the most violated 2PCI and 3PCI can be separated by solving an optimization problem, as show in Section 3.1 for 2PCI. Since the separation problem for nPCI is $\mathcal{NP}$-hard, we plan to propose heuristics to separate them. We anticipate even greater computational efficiency with the addition of such separated 2PCIs and 3PCIs. We also aim to test the effectiveness of MMEpPCI and MMEp+1PCI.

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# A Facets of Example 1

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a_i$ | 19 | 17 | 14 | 14 | 14 | 13 | 13 | 12 | 11 | 10 | 10 | 9 | 9 | 7 | 5 | b = 158 | |
| S. No. | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $\alpha_5$ | $\alpha_6$ | $\alpha_7$ | $\alpha_8$ | $\alpha_9$ | $\alpha_{10}$ | $\alpha_{11}$ | $\alpha_{12}$ | $\alpha_{13}$ | $\alpha_{14}$ | $\alpha_{15}$ | RHS | Type |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 12 | CI, $C = N \setminus \{11, 15\}$, $\lambda = 4$ |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 12 | CI, $C = N \setminus \{12, 15\}$, $\lambda = 5$ |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 12 | CI, $C = N \setminus \{13, 15\}$, $\lambda = 5$ |
| 4 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 12 | CI, $C = N \setminus \{14, 15\}$, $\lambda = 7$ |
| 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 12 | CI, $C = N \setminus \{10, 14\}$, $\lambda = 2$ |
| 6 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 12 | CI, $C = N \setminus \{12, 13\}$, $\lambda = 1$ |
| 7 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 12 | CI, $C = N \setminus \{13, 14\}$, $\lambda = 3$ |
| 8 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 12 | CI, $C = N \setminus \{11, 14\}$, $\lambda = 2$ |
| 9 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 12 | CI, $C = N \setminus \{12, 14\}$, $\lambda = 3$ |
| 10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 12 | CI, $C = N \setminus \{10, 15\}$, $\lambda = 4$ |
| 11 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 12 | CI, $C = N \setminus \{7, 15\}$, $\lambda = 1$ |
| 12 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 12 | CI, $C = N \setminus \{8, 15\}$, $\lambda = 2$ |
| 13 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 12 | CI, $C = N \setminus \{6, 15\}$, $\lambda = 1$ |
| 14 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 12 | CI, $C = N \setminus \{9, 15\}$, $\lambda = 3$ |
| 15 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 12 | CI, $C = N \setminus \{9, 14\}$, $\lambda = 3$ |
| 16 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 13 | CI, $C = N \setminus \{2\}$, $\lambda = 2$ |
| 17 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 13 | CI, $C = N \setminus \{3\}$, $\lambda = 5$ |
| 18 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 13 | CI, $C = N \setminus \{4\}$, $\lambda = 5$ |
| 19 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 13 | CI, $C = N \setminus \{5\}$, $\lambda = 5$ |
| 20 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 14 | 2PCI, $C = N, \lambda = 19$ |
| 21 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 17 | 2PCI, $C = N \setminus \{15\}, \lambda = 14$ |
| 22 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 20 | 2PCI, $C = N \setminus \{14\}, \lambda = 12$ |
| 23 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 0 | 1 | 1 | 23 | 2PCI, $C = N \setminus \{13\}, \lambda = 10$ |
| 24 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 0 | 1 | 1 | 1 | 23 | 2PCI, $C = N \setminus \{12\}, \lambda = 10$ |
| 25 | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 24 | 3PCI, $C = N, \lambda = 19$ |
| 26 | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 2 | 2 | 2 | 2 | 1 | 1 | 25 | MMEpPCI, $C = N$, $C^e = \{9, 14, 15\}$, $p = 3$ |
| 27 | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 2 | 2 | 2 | 1 | 1 | 25 | MMEpPCI, $C = N$, $C^e = \{10, 14, 15\}$, $p = 3$ |
| 28 | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 2 | 2 | 1 | 1 | 25 | MMEpPCI, $C = N$, $C^e = \{11, 14, 15\}$, $p = 3$ |
| 29 | 4 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 33 | MMEp+1PCI, $C = N$, $C^e = \{9, 14, 15\}$, $p = 3$ |

Table 5: Facets of the form $\sum_{i \in N} \alpha_i y_i \leq RHS$ generated using PANDA Lörwald and Reinelt (2015)