Facets of the knapsack polytope from non-minimal covers

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Abstract

In this paper, we study the polyhedral structure of the binary knapsack polytope and propose a new class of valid inequalities derived from non-minimal covers. Our proposed inequalities generalize several well-known inequalities, including cover, extended cover, $(1, k)$-configuration, and generalized $(1, k)$-configuration inequalities. Moreover, we show that the proposed inequalities cannot generally be obtained through either sequential or simultaneous lifting of any minimal cover inequality. Further, we provide sufficient conditions under which the proposed inequalities are facet-defining for the knapsack polytope. The usefulness of these facets is demonstrated using computational experiments on fixed charge transportation problems, a well-known class of NP-hard problems, which help improve their lower bounds by more than 9% on average. This reduces CPU time by around 95% (respectively, 28% to 31%) when used in the absence (respectively, in the presence) of CPLEX-generated cuts, depending on the problem parameters.

Keywords: `Knapsack polytope, Valid inequalities, Facets, Non-minimal cover, Integer programming, Fixed charge transportation problem`

1 Introduction

Given a set of items with their corresponding weights and a knapsack of a given capacity, a binary knapsack set is the set of all subsets of items that can fit in the knapsack. Binary knapsack sets appear as a substructure in a wide range of mixed-integer programs (MIPs), including cutting stock, facility location, fixed-charge problems, and various scheduling problems (Kellerer et al., 2004; Wolsey, 2021). Hence, any valid inequality for the knapsack polytope (KP) – the convex hull of the knapsack set – is also valid for the MIPs in which they appear.

Crowder et al. (1983) used cover inequalities, which are valid inequalities for KP, in a branch-and-cut framework to efficiently solve large-scale binary programs. Van Roy and Wolsey (1987) used cover inequalities and extended cover inequalities, besides several other inequalities, to efficiently solve a variety of medium-to-large size MIPs. In a more recent work, Achterberg

and Wunderling (2013) demonstrate that knapsack cuts—particularly cover inequalities—are among the most impactful families of cutting planes in MIPs. Turning them off in CPLEX 12.5 leads to notable performance degradation, with an average increase of 14% in CPU time across benchmark instances and up to 35% on affected problems.

The popularity of KP has resulted in significant research interest in its polyhedral study and the discovery of several classes of valid inequalities, besides cover inequalities. These include extended cover inequalities (ECIs) (Balas, 1975), lifted cover inequalities (LCIs) (Balas and Zemel, 1978; Gu et al., 1998), $(1, k)$-configuration inequalities (Padberg, 1980), weight inequalities (Weismantel, 1997), and lifted pack inequalities (Atamtürk, 2005). Fenchel cutting planes (Boyd, 1994), which are not specific to any particular class of inequalities (such as cover inequalities or weight inequalities), have also been studied for KP.

The majority of the VIs in the literature are based on minimal covers since cover inequalities based on them are stronger than those based on non-minimal covers. Moreover, at least one of the sequential lifted cover inequalities (LCIs) obtained from a minimal cover is facet-defining for KP (Balas, 1975; Wolsey, 1975). Therefore, it is not surprising that there are only a few works that propose the generation of VIs from non-minimal covers. For example, Appleget (1997) introduced non-minimal cover cuts and a deficit lifting procedure to strengthen them. More recently, Letchford and Souli (2019) introduced a modified lifting procedure that can also be applied if the underlying cover C is not minimal and can yield facet-defining inequalities. Similarly, Del Pia et al. (2024) extended $(1, k)$-configuration inequalities by relaxing the condition of minimal cover. To the best of our knowledge, these are among the few works that use non-minimal covers for KP.

In this paper, we make use of non-minimal covers to propose a new class of VIs and derive the conditions under which they define facets of KP. Further, we establish conditions under which the proposed inequalities cannot be obtained through sequential or simultaneous lifting of minimal CIs. Our computational experiments on the fixed-charge transportation problem, a well-known class of NP-hard problems, highlight the usefulness of the facets from our proposed VIs, which help improve the lower bounds by more than 9% on average. This helps reduce the CPU time by around 95% when used in the absence of CPLEX-generated cuts. When used in conjunction with CPLEX-generated cuts, these inequalities also reduce the CPU time by around 28% to 31%.

The rest of the paper is organized as follows. We introduce a new class of valid inequalities and derive their facet-defining conditions in Section 2. Computational results are presented in Section 3. Finally, we conclude in Section 4.

2 Valid Inequalities based on Non-Minimal Covers

Consider a binary knapsack set $Y = P \cap \mathbb{B}^{|N|}$, where

$$P = \left\{ y \in [0,1]^{|N|} : \sum_{i \in N} a_i y_i <= b, a_i > 0 \; \forall i \in N, b > 0 \right\}.$$

Let KP denote the knapsack polytope, i.e., $KP = Conv(Y)$. We assume that KP is full-dimensional, i.e., $dim(KP) = |N|$, which happens only when $a_i \leq b \; \forall i \in N$. A set $C \subseteq N$ is a cover of Y if $\sum_{i \in C} a_i > b$, and its surplus λ is defined as the extra weight included in C beyond the knapsack capacity, b, i.e., $\lambda = \sum_{i \in C} a_i - b$. A cover is minimal if $\lambda \leq \min_{i \in C} \{a_i\}$, else non-minimal, i.e., when $\lambda > \min_{i \in C} \{a_i\}$. Let $KP^C = \{y \in KP : y_i = 0 \; \forall i \in N \backslash C\}$.

In this section, we propose a new class of VIs based on the idea of partitioning non-minimal covers. For some of these VIs, we further derive the conditions under which they define facets of KP. For the rest of the paper, we use the following notation.

C'	:	$N \setminus C$
$max(C)$	:	the highest weight among all items in set C
$max_j(C)$	:	the j^{th} highest weight among all items in set C
$min(C)$	:	the lowest weight among all items in set C
$min_j(C)$	:	the j^{th} lowest weight among all items in set C
c	:	cardinality of set C
λ	:	surplus of cover C, i.e., $\lambda = \sum_{i \in C} a_i - b$

Additionally, for $C = \emptyset$, $max(C) = min(C) = max_j(C) = min_j(C) = 0$.

Proposition 1. *Given a non-minimal cover C and its partition $C_1 = \{i \in C : a_i < \lambda\}$ and $C_2 = \{i \in C : a_i \geq \lambda\}$,*

(a) the following is a VI for KP, which we refer to as a two-partition cover inequality (2PCI):

$$\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i \leq c_1 + 2c_2 - 2 \tag{2PCI}$$

(b) (2PCI) *cuts off fractional extreme points $\bar{y} \in P$ characterized as:*

(i) $\bar{y}_i = 1 \; \forall i \in C_2 \cup (C_1 \backslash \{j,k\})$, *where* $j,k \in C_1$, $j \neq k$, *and* $a_j + a_k > \lambda$; $\bar{y}_j = 0$, $\bar{y}_k = \frac{a_j + a_k - \lambda}{a_k}$, *and* $\bar{y}_i = 0 \; \forall i \in C'$

(ii) $\bar{y}_i = 1 \; \forall i \in (C_1 \backslash \{j\}) \cup (C_2 \backslash \{k\})$, *where* $j \in C_1$, $k \in C_2$, *and* $a_j + 0.5a_k > \lambda$; $\bar{y}_j = 0$, $\bar{y}_k = \frac{a_j + a_k - \lambda}{a_k}$, *and* $\bar{y}_i = 0 \; \forall i \in C'$

(iii) $\bar{y}_i = 1 \; \forall i \in C_1 \cup (C_2 \backslash \{j\})$, *where* $j \in C_2$ *and* $a_j > \lambda$; $\bar{y}_j = \frac{a_j - \lambda}{a_j}$, *and* $\bar{y}_i = 0 \; \forall i \in C'$.

Proof. (a) In general, the following are (trivial) VIs for KP:

$$\sum_{i \in C_1} y_i \leq c_1 \tag{1}$$

$$\sum_{i \in C_2} y_i \leq c_2 \tag{2}$$

Multiplying (2) by 2 and adding it to (1), we get the following valid inequality:

$$\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i \leq c_1 + 2c_2$$

Consider any solution $y \in Y : y_i = 0 \ \forall i \in C'$. The remaining part of such a solution $(y_i : i \in C)$ can be categorized as one of the following three mutually exclusive and exhaustive types and for each type, we show the solution satisfies (2PCI):

- $y_i = 1 \ \forall i \in C_1$: Since $a_i \geq \lambda \ \forall i \in C_2$, for any feasible solution to KP, $\exists$ at least one $i \in C_2 : y_i = 0$. Therefore, $\sum_{i \in C_2} y_i \leq c_2 - 1$, which implies $\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i \leq c_1 + 2c_2 - 2$.
- $y_i = 1 \ \forall i \in C_2$: Since $a_i < \lambda \ \forall i \in C_1$, for any feasible solution to KP, $\exists$ one pair $i, j \in C_1 : j \neq i, y_i = y_j = 0$. Therefore, $\sum_{i \in C_1} y_i \leq c_1 - 2$, which implies $\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i \leq c_1 + 2c_2 - 2$.
- $\exists$ at least one $i \in C_1 : y_i = 0$ and $\exists$ at least one $i \in C_2 : y_i = 0$: Since $a_i \geq \lambda \ \forall i \in C_2$, such a solution is always feasible to KP. Further, $\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i \leq c_1 - 1 + 2(c_2 - 1) < c_1 + 2c_2 - 2$.

(b) (i) Substituting the fractional point $\bar{y}$ in the knapsack inequality makes it binding. Further, since all variables except $\bar{y}_k$ are at their bounds, there are $n - 1$ linearly independent active bound constraints. Together with the binding knapsack constraint, these give n linearly independent active constraints, which shows that $\bar{y}$ is an extreme point of P. Further, substituting it in (2PCI) gives the LHS $= c_1 + 2c_2 - 2 + \frac{a_j + a_k - \lambda}{a_k} > c_1 + 2c_2 - 2 =$ RHS, and hence, violates it.

Proofs for (ii) and (iii) are omitted as the steps involved are exactly the same as for (i). □

Remark 1. *(a) When $C_1 = \emptyset$, i.e., $\lambda \leq min(C)$,* (2PCI) *reduces to the following minimal CI: $\sum_{i \in C_2} y_i \leq c_2 - 1$.*

(b) When $C_2 = \emptyset$, i.e., $\lambda > max(C)$, (2PCI) *reduces to the following Extended Cover Inequality (ECI): $\sum_{i \in C_1} y_i \leq c_1 - 2$.*

(c) As a result of the above two remarks, we only consider 2PCIs with $C_1 \neq \emptyset$ and $C_2 \neq \emptyset$, i.e., $min(C) < \lambda \leq max(C)$.

Example 1. *Consider $Y = \{y \in \mathbb{B}^n : \sum_{i \in N} a_i y_i \leq b\}$ with $n = 8$, $a = \{12, 11, 10, 8, 7, 4, 3, 2\}$, and $b = 45$.*

Clearly, $C = N$ is a non-minimal cover since $\sum_{i\in C} a_i = 57$, $\lambda = 57 - 45 = 12 > \min_{i\in C}\{a_i\} = 2$. $C_2 = \{1\}$ since $a_1 = 12 \geq \lambda$, $C_1 = N\backslash\{1\}$, and $c_1 = 7, c_2 = 1$. Hence, the corresponding 2PCI is $2y_1 + y_2 + \cdots + y_8 \leq 2\times 1 + 1\times 7 - 2 = 7$. Consider the following fractional point: $\bar{y}_i = 1\ \forall i \in C_2 \cup C_1\backslash\{4,5\}$, $\bar{y}_4 = 0$, $\bar{y}_5 = \frac{3}{7}$. Clearly, $\sum_{i\in C} a_i\bar{y}_i = 45$. Hence, this $\bar{y}$ is an extreme point of P. Here, $a_4 + a_5 = 8+7 = 15 > 12 = \lambda$, and $\bar{y}_5 = \frac{a_4+a_5-\lambda}{a_5} = \frac{8+7-12}{7} = \frac{3}{7}$. Hence, this extreme point is of the type as characterized by Proposition 1.b.i. Further, $\sum_{i\in C_1}\bar{y}_i + 2\sum_{i\in C_2}\bar{y}_i = \frac{52}{7} > 7$; hence, $\bar{y}$ is cut off by the above 2PCI.

Theorem 1. (2PCI) *defines a facet of KP^C if and only if the following three conditions are satisfied:*

(a) $c_1 \geq 3$

(b) $\lambda \leq max(C_1) + min(C_1)$

(c) $\lambda \leq max_2(C_1) + max_3(C_1)$

Proof. First, we will prove that condition (a) is necessary. Condition (a) can be violated only in the following two cases: (i) $c_1 = 1$, (ii) $c_1 = 2$. Next, we will show that in neither of the two cases can (2PCI) define a facet of KP^C.

(i) $c_1 = 1$: Say $c_2 = k \geq 1$. Then, (2PCI) can be written as: $y_1 + 2(y_2 + \cdots + y_{k+1}) \leq 1 + 2k - 2 = 2k - 1$. This inequality can define a facet of KP^C only if there exist $k+1$ affinely independent points in KP^C for which it is binding. However, only those vectors with $y_1 = 1$ and exactly one $y_i = 0$ from C_2 can be binding, and there exist only k such vectors (as shown below), not $k+1$. Therefore, with $c_1 = 1$, (2PCI) cannot define a facet of KP^C.

$$\begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & \cdots & y_k & y_{k+1} \end{array} \\ \left(\begin{array}{c|ccccc} 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & & & \ddots & & \vdots \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{array}\right) \begin{array}{c} y^1 \\ y^2 \\ \vdots \\ y^{k-1} \\ y^k \end{array} \end{array}$$

(ii) $c_1 = 2$: Say $c_2 = k \geq 1$. Then, (2PCI) can be written as: $y_1 + y_2 + 2(y_3 + \cdots + y_{k+2}) \leq 2 + 2k - 2 = 2k$. This inequality can define a facet of KP^C only if there exist $k+2$ affinely independent points in KP^C for which it is binding. Clearly, there are at most $k+1$ points in KP^C for which this inequality is binding, as shown below. Hence, with $c_1 = 2$, (2PCI) cannot define a facet of KP^C.

$$\begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & \cdots & y_k & y_{k+1} & y_{k+2} \end{array} \\ \left(\begin{array}{cc|ccccc} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ \vdots & \vdots & & & \ddots & & \vdots \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right) \begin{array}{c} y^1 \\ y^2 \\ y^3 \\ \vdots \\ y^k \\ y^{k+1} \end{array} \end{array}$$

This proves that condition (a) is necessary, i.e., $c_1 \geq 3$.

Next, we will prove that condition (b) is necessary. For this, we need to show that any set of c affinely independent points in KP^C, for which (2PCI) is binding, must satisfy condition (b). For this, let $y^k \in KP^C$ be a point defined as $y_i^k = 1 \forall i \in C_1 \cup (C_2 \backslash \{k\})$ and $y_i^k = 0$ for $i = k$. Clearly, $\{y^k : k \in C_2\}$ is a set of c_2 affinely independent points as shown below using matrix M_1. Further, for each y^k defined above, (2PCI) is binding since $\sum_{i \in C_1} y_i + 2\sum_{i \in C_2} y_i = c_1 + 2(c_2 - 1) = c_1 + 2c_2 - 2$.

$$M_1 = \Big(P \;\Big|\; Q\Big) = \left(\begin{array}{cccc|cccc|c} y_1 & y_2 & \cdots & y_{c_1} & y_{c_1+1} & y_{c_1+2} & \cdots & y_{c-1} & y_c & \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & y^1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & y^2 \\ \vdots & \vdots & & \vdots & & & \ddots & & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & y^{c_2-1} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & y^{c_2} \end{array}\right)$$

Clearly, Q is a $c_2 \times c_2$ matrix with c_2 affinely independent rows.

Now, we need to generate the remaining c_1 affinely independent points (since $c = c_1 + c_2$). For this, we first show that if $\lambda \leq max(C_1) + min(C_1)$ is violated, then it is not possible to generate c_1 affinely independent points in KP^C for which (2PCI) is binding. To that end, let us assume $max(C_1) + min(C_1) < \lambda$. Let $j \in C_1$ be such that $a_j = min(C_1)$. Then, for any $i \in C_1 \backslash \{j\}$, $a_i + a_j \leq max(C_1) + min(C_1) < \lambda$. Hence, no feasible point in KP^C, for which (2PCI) is binding, can have $y_i = y_j = 0$. Further, any binding point having exactly one zero in C_2 has $y_j = 1$. Therefore, $y_j = 1$ for every point in KP^C for which (2PCI) is binding, and hence it is not possible to generate c affinely independent points. This proves $\lambda \leq max(C_1) + min(C_1)$ is a necessary condition.

Now, we prove that condition (c) is necessary. For this, we need to show that any set of c affinely independent points in KP^C, for which (2PCI) is binding, must satisfy condition (c). It is easy to see that we can generate the required c_2 affinely independent points corresponding to C_2, exactly as constructed earlier in matrix M_1.

$$M_2 = \Big(R \;\Big|\; S\Big) = \left(\begin{array}{ccccccc|ccc} y_1 & y_2 & y_3 & \cdots & y_{c_1-1} & y_{c_1} & y_{c_1+1} & \cdots & y_c & \\ 0 & 0 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & y^{c_2+1} \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & y^{c_2+2} \\ \vdots & & & \ddots & & & & & \vdots & \vdots \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & \cdots & 1 & y^{c-2} \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & \cdots & 1 & y^{c-1} \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & y^c \end{array}\right)$$

Next, we need to generate the remaining c_1 affinely independent points (since $c = c_1 + c_2$). For this, we show that if $max_2(C_1) + max_3(C_1)$ is smaller than λ, then it is not possible to generate these remaining c_1 affinely independent points for which (2PCI) is binding. To see this, assume $max_2(C_1) + max_3(C_1) < \lambda$. In this case, the last candidate point y^c in matrix M_2 would require excluding items with weights $max_2(C_1)$ and $max_3(C_1)$ together, but this combination would exceed the capacity λ. Hence, $y^c \notin Y$. Further, among all pairs of items in C_1 excluding the

item with weight $max(C_1)$, the largest possible total weight is $max_2(C_1) + max_3(C_1)$. Hence, if $max_2(C_1) + max_3(C_1) < \lambda$, no other candidate point can be feasible and affinely independent of the previously constructed points. This shows that it is not possible to construct all c affinely independent points satisfying (2PCI) unless $max_2(C_1) + max_3(C_1) \geq \lambda$. Therefore, $\lambda \leq max_2(C_1) + max_3(C_1)$ is a necessary condition.

Finally, we show that conditions (a), (b), and (c) are sufficient. Since condition (a) requires $c_1 \geq 3$, the indices $max(C_1)$, $max_2(C_1)$, and $max_3(C_1)$ are well-defined. Clearly, when $\lambda \leq max(C_1) + min(C_1)$, the first $c_1 - 1$ points are affinely independent points in KP^C, as shown below using matrix M_2. In addition, $y^c \in KP^C$ only if $\lambda \leq max_2(C_1) + max_3(C_1)$.

The set of c_2 affinely independent points and the set of c_1 affinely independent points generated above can be jointly represented by the following $c \times c$ matrix:

$$M = \left(\frac{M_1}{M_2}\right) = \left(\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right)$$

From the block structure of M, it is easy to verify that all $c_1 + c_2$ rows of M are affinely independent.

This proves that conditions (a), (b), and (c) are necessary and sufficient for (2PCI) to define a facet of KP^C.

□

Example 1 (Continued)**.** $2y_1 + y_2 + \cdots + y_8 \leq 7$ *defines a facet of* KP^C *since it satisfies all the conditions of Theorem 1 as shown below:*

(a) $c_1 = 7 \geq 3$.

(b) $\lambda = 12,\ max(C_1) = 11,\ min(C_1) = 2$. *Hence,* $\lambda \leq max(C_1) + min(C_1)$.

(c) $max_2(C_1) = 10,\ max_3(C_1) = 8$. *Hence,* $\lambda \leq max_2(C_1) + max_3(C_1)$.

Theorem 2. (2PCI) *defines a facet of* KP *if and only if it defines a facet of* KP^C *and any of the following conditions is satisfied:*

(a) $C = N$

(b) $\lambda \leq max(C_2) - max(C')$

(c) $\lambda \leq max(C_1) + max_2(C_1) - max(C')$

Proof. To prove the theorem, we show that each of the conditions (a), (b), and (c) is necessary in the absence of the remaining two.

(a) It is easy to see that if $C = N$, then a facet of KP^C will also be facet of KP (since $KP = KP^C$).

(b) Suppose that condition (b) does not hold. That means $max(C_2) - max(C') < \lambda$. To show that (2PCI) defines a facet of KP, we must have enough affinely independent feasible points. We already have the points for KP^C. To reach the full dimension, we need one extra feasible point for each variable in C'.

These extra points are constructed by setting one variable in C' equal to one. To keep the knapsack feasible, we must create enough space for this extra weight. This is done by setting one variable in the cover to zero. We choose the largest item in C_2 for this because it creates the maximum possible space.

If the surplus is small enough, we can repeat this process for each variable in C'. However, if the surplus is too large, the largest item in C_2 does not create enough space for the largest item in C'. In that case, turning on that variable would make the total weight exceed the knapsack capacity, so that point is infeasible. Therefore, we lose one point.

Further, since condition (c) does not hold, $max(C_1) + max_2(C_1) - max(C') < \lambda$. Hence, even by setting the two largest variables in C_1 to zero, we cannot generate a binding feasible point with the variable corresponding to $max(C')$ equal to one.

As a result, we can only get $c' - 1$ extra feasible points instead of c'. This is not enough to reach the full dimension. Therefore, condition (b) is necessary in the absence of conditions (a) and (c).

Now we prove the sufficiency of condition (b). If $\lambda \leq max(C_2) - max(C')$, then we can generate c' affinely independent points as shown below.

Consider $c' \times n$ matrix

$$M = \Big(P \;\Big|\; Q \;\Big|\; R \Big)$$

where P is a matrix of dimension $c' \times c_1$ with all entries equal to 1, Q is a matrix of order $c' \times c_2$ with entries equal to $[0, 1, \ldots, 1]$, where the zero corresponds to an item with weight $max(C_2)$, and R is an identity matrix of order $c' \times c'$ as shown below.

$$R = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{matrix} y^{c+1} \\ y^{c+2} \\ \vdots \\ y^{n-1} \\ y^{n} \end{matrix}$$

This proves that condition (b) is necessary and sufficient in the absence of conditions (a) and (c).

(c) We omit the proof of this part, as it follows from arguments analogous to those used in part (b), by setting two variables in C_1 to zero instead of one variable in C_2.

□

Example 1 (Continued)**.**

- $2y_1 + y_2 + \cdots + y_8 \leq 7$ *defines a facet of KP since* $C = N$ *(using condition (a) of Theorem 2).*

- *Let us consider another cover* $C = N \backslash \{8\}$. *For this cover,* $\sum_{i \in C} a_i = 55$, *hence* $\lambda = 55 - 45 = 10$, $C_2 = \{1, 2, 3\}, C_1 = \{4, 5, 6, 7\}, C' = \{8\}$. *Hence,* $c_1 = 4, c_2 = 3$ *and the corresponding 2PCI is* $2(y_1 + y_2 + y_3) + y_4 + y_5 + y_6 + y_7 \leq 2 \times 3 + 4 - 2 = 8$. *It defines a facet of* KP^C *since it satisfies the conditions of Theorem 1: (a)* $c_1 = 4 \geq 3$*; (b)* $\lambda = 10$, $max(C_1) = 8$, $min(C_1) = 3$ *so* $\lambda \leq max(C_1) + min(C_1) = 11$*; (c)* $max_2(C_1) = 7$, $max_3(C_1) = 4$ *so* $\lambda \leq max_2(C_1) + max_3(C_1) = 11$. *Further,* $2(y_1 + y_2 + y_3) + y_4 + y_5 + y_6 + y_7 \leq 8$ *also defines a facet of KP since it satisfies condition (b) of Theorem 2:* $max(C_2) = 12$, $max(C') = 2$, *hence* $\lambda = 10 \leq max(C_2) - max(C') = 10$ *(and condition (c)* $\lambda \leq max(C_1) + max_2(C_1) - max(C')$ *also holds:* $10 \leq 8 + 7 - 2 = 13$*).*

Proposition 2. (2PCI) *can be obtained through neither sequential nor simultaneous lifting of any minimal CI.*

Proof. We prove this for sequential and simultaneous lifting in parts (a) and (b), respectively. For this, consider (2PCI) corresponding to a 2-partition C_1, C_2 of a non-minimal cover C. Further, consider a minimal cover C_0 obtained by removing a subset of items from C. It is sufficient to consider $C_0 \subseteq C$, since if C_0 contains an item from C', the coefficient of the corresponding variable in the CI remains nonzero after lifting, whereas its coefficient in (2PCI) is zero.

(a) Clearly, $C_0 \ni i \; \forall i \in C_2$, since for any $i \in C_2$, $\sum_{j \in C \backslash \{i\}} a_j = b + \lambda - a_i \leq b$. We consider the following two cases. If $C_0 \ni$ at least one $i \in C_1$, then the variables corresponding to this item and to all items in C_2 have coefficient 1 in the CI corresponding to C_0. Sequential lifting does not alter the coefficients of the variables already contained in C_0. Hence, even under a positive scaling, these variables will have the same coefficient. However, in (2PCI), the variables in C_1 and C_2 have coefficients 1 and 2, respectively. Hence, (2PCI) cannot be obtained through sequential lifting in this case.

If $C_0 \ni$ no $i \in C_1$, then $C_0 = C_2$. The CI corresponding to $C_0 = C_2$ is $\sum_{i \in C_2} y_i \leq c_2 - 1$. Sequential lifting does not alter its RHS. For an inequality obtained through sequential lifting of this CI to be proportional to (2PCI), the coefficients of the variables in C_2 must be scaled from 1 to 2. Hence, its RHS becomes $2(c_2 - 1)$. However, the RHS of (2PCI) is $c_1 + 2c_2 - 2$. This implies that $2(c_2 - 1) = c_1 + 2c_2 - 2 \implies c_1 = 0$, which contradicts $c_1 \geq 1$. Hence, (2PCI) cannot be obtained through sequential lifting in this case either.

(b) Clearly, $C_0 \ni i \; \forall i \in C_2$, as shown in part (a). To prove (2PCI) cannot be obtained through simultaneous lifting of any minimal CI, we consider the following two mutually exclusive (and exhaustive) cases:

(i) $C_0 \ni$ at least one $i \in C_1$: Let C_0^1 be the set of items from C_1 contained in C_0. CI corresponding to C_0 is $\sum_{i \in C_0} y_i \leq c_0 - 1$. Any inequality obtained through

simultaneous lifting of this CI will be of the form:

$$\begin{aligned}
&\sum_{i\in C_0} y_i + \sum_{i\in C_1\setminus C_0^1} \frac{1}{\alpha_i} y_i \\
&= \sum_{i\in C_0^1} y_i + \sum_{i\in C_0\setminus C_0^1} y_i + \sum_{i\in C_1\setminus C_0^1} \frac{1}{\alpha_i} y_i \\
&= \sum_{i\in C_0^1} y_i + \sum_{i\in C_2} y_i + \sum_{i\in C_1\setminus C_0^1} \frac{1}{\alpha_i} y_i \le c_0 - 1.
\end{aligned}$$

For this inequality to be a 2PCI, the coefficients of the variables in the set C_2 must be 2, and hence the inequality must be multiplied by 2. However, this also makes the coefficients of the variables in $C_0^1 \subset C_1$ equal to 2, whereas their coefficients in (2PCI) are 1. Hence, a 2PCI cannot be obtained through simultaneous lifting of this CI.

(ii) $C_0 \ni$ no $i \in C_1$: In this case, $C_0 = C_2 \implies c_2 \ge 2$. Also, $c_1 \ge 1$ (since C is a non-minimal cover). So, the CI corresponding to $C_0 = C_2$ is $\sum_{i\in C_2} y_i \le c_2 - 1$. Any inequality obtained through simultaneous lifting of this CI will be of the form:

$$\begin{aligned}
&\sum_{i\in C_2} y_i + \sum_{i\in C_1} \frac{1}{\alpha_i} y_i \le c_2 - 1 \\
\implies &2\sum_{i\in C_2} y_i + \sum_{i\in C_1} \frac{2}{\alpha_i} y_i \le 2(c_2 - 1).
\end{aligned}$$

For this inequality to be a 2PCI, the coefficients of the variables in the set C_1 must be 1, which requires $\alpha_i = 2\ \forall i \in C_1$. Also, the RHS of the inequality must be equal to $c_1 + 2c_2 - 2$. This implies that $2(c_2 - 1) = c_1 + 2c_2 - 2 \implies c_1 = 0$, which contradicts the initial requirement that $c_1 \ge 1$. Hence, a 2PCI can never be obtained from the simultaneous lifting of this CI.

□

Example 1 (Continued)**.** *The facet-defining 2PCI* $2y_1 + y_2 + \cdots + y_8 \le 7$ *cannot be obtained through the sequential lifting of any minimal CI. This can be easily seen as follows. First, any CI with RHS = 7 is defined only for* $C = N$*. However, for* $C = N$*, the cover is non-minimal, as discussed earlier. Hence, this facet cannot be obtained from the sequential lifting of a minimal CI.*

Similarly, it can be shown that the other facet-defining 2PCI $2(y_1+y_2+y_3)+y_4+y_5+y_6+y_7 \le 8$ *cannot be obtained through the sequential lifting of any minimal CI.*

Proposition 2 highlights that the facets given by (2PCI) will, in general, complement the facets obtained through sequential liftings of minimal CIs in characterizing KP.

Before discussing our next VI, we first clarify the general procedure for constructing the sets $C_1, C_2, \ldots, C_q$.

Remark 2 (Sequential Partition Construction)**.** *The sets C_k $(k = 1, \ldots, q)$ are constructed sequentially in the order of increasing k. For all $k \geq 2$, the weight of every item in C_k is less than or equal to that of any item in subsequent sets, i.e.,*

$$a_i \leq a_j \quad \forall i \in C_k,\ \forall j \in C_{k'},\ k' > k.$$

The set C_1 is an exception, where some items may have weights greater than those in later sets.

Proposition 3. *Given a cover C and its partition as follows:*

$$C_1 = \Big\{ i, j \in C : \begin{cases} a_i + a_j < \lambda, & \text{if } i \neq j, \\ a_i + a_j < 2\lambda, & \text{if } i = j \end{cases} \Big\}.$$
$$C_2 = \big\{ i \in C \setminus C_1 : a_i < \lambda \big\},$$
$$C_3 = \big\{ i \in C \setminus (C_1 \cup C_2) : a_i \geq \lambda \big\}.$$

(a) the following is a VI for KP, which we refer to as a three-partition cover inequality (3PCI):

$$\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i + 3 \sum_{i \in C_3} y_i \leq c_1 + 2c_2 + 3c_3 - 3 \tag{3PCI}$$

(b) (3PCI) *cuts off fractional extreme points $\bar{y} \in P$ characterized as:*

(i) $\bar{y}_i = 1\ \forall i \in (C_3 \cup C_2 \cup C_1 \backslash \{j,k,l\} \colon a_j + a_k + a_l > \lambda), \bar{y}_k = \bar{y}_l = 0, \bar{y}_j = \frac{a_j + a_k + a_l - \lambda}{a_j}$

(ii) $\bar{y}_i = 1\ \forall i \in ((C_1 \backslash \{j\}) \cup (C_2 \backslash \{k\}) \cup C_3 : a_j + a_k > \lambda), \Big(\bar{y}_j = 0, \bar{y}_k = \frac{a_j + a_k - \lambda}{a_k}\Big)$, *or* $\Big(\bar{y}_k = 0, \bar{y}_j = \frac{a_j + a_k - \lambda}{a_j}\Big)$

(iii) $\bar{y}_i = 1\ \forall i \in (C_1 \cup C_2 \cup (C_3 \backslash \{j\}) : a_j > \lambda), \bar{y}_j = \frac{a_j - \lambda}{a_j}$.

Remark: Note that C_1 is not unique, as multiple subsets of C may satisfy the above given condition. However, for any such choice of C_1, the sets C_1, C_2, and C_3 form a partition of C: they are pairwise disjoint, and their union is C.

Proof. (a) Following the steps of the proof for Proposition 1(a), we get the following valid inequality:

$$\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i + 3 \sum_{i \in C_3} y_i \leq c_1 + 2c_2 + 3c_3$$

Since C is a cover, for any $y \in Y$, we have $\sum_{i \in C} a_i(1 - y_i) \geq \lambda$. Next, we show that any solution $y \in Y$ can be one of the following four mutually exclusive and exhaustive types, and in each case, it satisfies (3PCI).

- $\exists$ at least one $i \in C_3 : y_i = 0$: Since at least one item from C_3 is excluded, $\sum_{i \in C_3} y_i \leq c_3 - 1$, which implies $\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i + 3 \sum_{i \in C_3} y_i \leq c_1 + 2c_2 + 3c_3 - 3$.
- $y_i = 1\ \forall i \in C_2 \cup C_3$: Since $a_i < \lambda\ \forall i \in C_1$ and $a_i + a_j < \lambda\ \forall i, j \in C_1, i \neq j$, for any feasible solution to KP, at least three distinct items from C_1 must be excluded.

Therefore, $\sum_{i\in C_1} y_i \leq c_1 - 3$, which implies $\sum_{i\in C_1} y_i + 2\sum_{i\in C_2} y_i + 3\sum_{i\in C_3} y_i \leq c_1 + 2c_2 + 3c_3 - 3$.

- $y_i = 1\ \forall i \in C_3$ and at least two items from C_2 are excluded: Therefore, $\sum_{i\in C_2} y_i \leq c_2 - 2$, which implies $\sum_{i\in C_1} y_i + 2\sum_{i\in C_2} y_i + 3\sum_{i\in C_3} y_i \leq c_1 + 2c_2 + 3c_3 - 4 < c_1 + 2c_2 + 3c_3 - 3$.
- $y_i = 1\ \forall i \in C_3$ and exactly one item $j \in C_2$ is excluded: Since $a_j < \lambda$, for any feasible solution to KP, at least one item from C_1 must also be excluded. Therefore, $\sum_{i\in C_1} y_i \leq c_1 - 1$ and $\sum_{i\in C_2} y_i = c_2 - 1$, which implies $\sum_{i\in C_1} y_i + 2\sum_{i\in C_2} y_i + 3\sum_{i\in C_3} y_i \leq c_1 + 2c_2 + 3c_3 - 3$.

(b) Proof for part (b) can be done using similar steps as for Proposition 1(b).

□

Remark 3. *Given a cover C:*

(a) by definition, $C_1 = \emptyset$ and $C_2 \neq \emptyset$ is not possible.

(b) from (a), $C_1 = C_3 = \emptyset$ is not possible.

(c) when $C_1 = C_2 = \emptyset$, (3PCI) *reduces to the following minimal CI: $\sum_{i\in C_3} y_i \leq c_3 - 1$.*

(d) when $C_2 = C_3 = \emptyset$, (3PCI) *reduces to the following ECI: $\sum_{i\in C_1} y_i \leq c_1 - 3$.*

(e) from (b), (c), (d), we therefore consider only the cases in which exactly one of C_1, C_2, and C_3 is $\emptyset$ or none of them is empty.

(f) when only $C_1 = \emptyset$, then from (a) and (c), (3PCI) *reduces to CI.*

(g) As a result of (e) and (f), we only consider 3PCIs with either: (i) only $C_2 = \emptyset$, (ii) only $C_3 = \emptyset$, or (iii) none is empty.

Example 1 (Continued)**.** *Again, we consider a non-minimal cover $C = N$. For this cover, $\lambda = 12$, $C_3 = \{1\}$, $C_2 = \{2,3,4\}$, $C_1 = \{5,6,7,8\}$, $c_1 = 4, c_2 = 3, c_3 = 1$. Hence, the corresponding 3PCI is $3y_1 + 2(y_2 + y_3 + y_4) + (y_5 + y_6 + y_7 + y_8) \leq 4 + 2 \times 3 + 3 \times 1 - 3 = 10$. Consider the following fractional point: $\bar{y}_i = 1\ \forall i \in C_3 \cup C_2 \cup C_1 \backslash \{5,6,7\}$, $\bar{y}_5 = \frac{2}{7}$, $\bar{y}_6 = 0$, $\bar{y}_7 = 0$. Clearly, $\sum_{i\in C} a_i \bar{y}_i = 45$. Here, $a_5 + a_6 + a_7 = 7 + 4 + 3 = 14 > 12 = \lambda$, and $\bar{y}_5 = \frac{a_5 + a_6 + a_7 - \lambda}{a_5} = \frac{2}{7}$. Hence, this extreme point is of the type characterized by Proposition 3.b.i. Further, $\sum_{i\in C_1} \bar{y}_i + 2\sum_{i\in C_2} \bar{y}_i + 3\sum_{i\in C_3} \bar{y}_i = \frac{72}{7} > 10$; hence, $\bar{y}$ is cut off by the above 3PCI.*

The next set of theorems establishes the conditions under which (3PCI) defines a facet of KP^C.

Theorem 3.A. *If $C_2 = \emptyset$, then* (3PCI) *defines a facet of KP^C if the following conditions are satisfied:*

(a) $c_1 \geq 4$

(b) $\lambda \leq max(C_1) + min(C_1) + min_2(C_1)$

(c) $\lambda \leq max_2(C_1) + max_3(C_1) + max_4(C_1)$

Proof. We omit the proof of part (a), as it follows the same argument as part (a) of Theorem 1.

To prove that (3PCI) defines a facet, we need to show that $c = c_1 + c_3$ affinely independent points exist that are binding. We observe that when $C_2 = \emptyset$, then (3PCI) reduces to $\sum_{i \in C_1} y_i + 3\sum_{i \in C_3} y_i \leq c_1 + 3c_3 - 3$. By definition of set C_3, it is easy to see that we can generate c_3 affinely independent points. Next, we show that conditions (b) and (c) allow us to generate the remaining c_1 affinely independent points. Using condition (b), we obtain $c_1 - 1$ affinely independent points, while condition (c) yields the remaining affinely independent point.

□

Theorem 3.B. *If* $(C_3 = \emptyset)$ *or* $(C_1 \neq \emptyset, C_2 \neq \emptyset, C_3 \neq \emptyset)$*, then* (3PCI) *defines a facet of* KP^C *if the following conditions are satisfied:*

(a) $c_1 \geq 3$

(b) $\lambda \leq max(C_2) + min(C_1)$

(c) $\lambda \leq min(C_2) + max(C_1)$

(d) $\lambda \leq max(C_1) + max_2(C_1) + max_3(C_1)$

Proof. Again, we omit the proof of part (a), as it follows the same argument as part (a) of Theorem 1.

First, we prove the case when $C_3 = \emptyset$.

To prove that (3PCI) defines a facet, we need to show that $c = c_1 + c_2$ affinely independent points exist that are binding. We observe that when $C_3 = \emptyset$, then (3PCI) reduces to $\sum_{i \in C_1} y_i + 2\sum_{i \in C_2} y_i \leq c_1 + 2c_2 - 3$. This can only be binding in one of the following ways: (i) $y_i = 0$ $i \in C_1$, and $y_j = 0$ $j \in C_2$ (ii) $y_i = y_j = y_k = 0$ $i, j, k \in C_1$

If $\lambda \leq max(C_2) + min(C_1)$ then we can generate c_1 affinely independent points as shown in the matrix below.

$$\begin{array}{c} \begin{array}{ccccc|ccccc} y_1 & y_2 & \cdots & y_{c_1-1} & y_{c_1} & y_{c_1+1} & y_{c_1+2} & \cdots & y_c & \end{array} \\ \left(\begin{array}{ccccc|ccccc} 1 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & 1 & \end{array}\right. y^1 \\ \left(\begin{array}{ccccc|ccccc} 1 & 1 & 1 & 0 & 1 & 0 & 1 & \cdots & 1 & \end{array}\right. y^2 \\ \left(\begin{array}{ccccc|ccccc} & & \iddots & & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}\right. \vdots \\ \left(\begin{array}{ccccc|ccccc} 1 & 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & 1 & \end{array}\right. y^{c_1-1} \\ \left(\begin{array}{ccccc|ccccc} 0 & 1 & 1 & \cdots & 1 & 0 & 1 & \cdots & 1 & \end{array}\right. y^{c_1} \end{array}$$

If $\lambda \leq min(C_2) + max(C_1)$, we can generate the following additional $c_2 - 1$ affinely independent points as shown in the matrix below.

$$\begin{array}{c} \begin{array}{ccccccc|ccccc} y_1 & y_2 & y_3 & y_4 & \cdots & y_{c_1-1} & y_{c_1} & y_{c_1+1} & y_{c_1+2} & y_{c_1+3} & \cdots & y_c \end{array} \\ \left(\begin{array}{ccccccc|ccccc} 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 1 & \cdots & 1 \\ & & & & & & & & & \ddots & & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 0 & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{array}\right) \begin{array}{l} y^{c_1+1} \\ y^{c_2+2} \\ \vdots \\ y^{c-2} \\ y^{c-1} \\ y^{c} \end{array} \end{array}$$

The last point y^c as shown in the above matrix can be generated only if the condition $\lambda \leq max(C_1) + max_2(C_1) + max_3(C_1)$ is satisfied.

In the other case, when $C_1 \neq \emptyset$, $C_2 \neq \emptyset$, and $C_3 \neq \emptyset$, we need an additional c_3 affinely independent binding points. By the definition of C_3, it is easy to see that these c_3 additional affinely independent points can be generated.

□

The next set of theorems establishes the conditions under which (3PCI) defines a facet of KP.

Theorem 4.A. *If $C_2 = \emptyset$, then* (3PCI) *defines a facet of KP if it defines a facet of KP^C and any of the following conditions is satisfied:*

(a) $C = N$

(b) $\lambda \leq max(C_3) - max(C')$

(c) $\lambda \leq max(C_1) + max_2(C_1) + max_3(C_1) - max(C')$

Proof. To prove the theorem, we show that each of the conditions (a), (b), and (c) is sufficient in the absence of the remaining two.

(a) It is easy to see that if $C = N$, then a facet of KP^C will also be a facet of KP, since in this case $KP = KP^C$.

(b) To prove that (3PCI) defines a facet of KP, we need to show that an additional $(n - c)$ affinely independent points exist in KP that are binding to (3PCI). One of the ways in which (3PCI) can be binding is when $y_i = 1 \ \forall i \in C_1$ and $y_i = 0 \ i \in C_3$. If $\lambda \leq max(C_3) - max(C')$, then we can generate $(n - c)$ affinely independent points as shown in the matrix below:

$$M = \Big(P \ \Big| \ Q \ \Big| \ R \Big)$$

where P is a matrix of dimension $(n - c) \times c_1$ with all entries equal to 1, Q is a matrix of order $(n - c) \times c_3$ with entries equal to $[0, 1, \ldots, 1]$, and R is an identity matrix of order $(n - c) \times (n - c)$.

(c) Another way in which (3PCI) can be binding is when $y_i = y_j = y_k = 0 \ i, j, k \in C_1$ and $y_i = 1 \ \forall i \in C_3$. In this case, if $\lambda \leq max(C_1) + max_2(C_1) + max_3(C_1) - max(C')$, then we can generate $(n - c)$ affinely independent points as shown in the matrix below:

$$M = \Big(P \;\Big|\; Q \;\Big|\; R \Big)$$

where P is a matrix of dimension $(n-c) \times c_1$ with entries equal to $[0, 0, 0, 1, \ldots, 1]$, Q is a matrix of order $(n-c) \times c_3$ with all entries equal to 1, and R is an identity matrix of order $(n-c) \times (n-c)$.

This completes the proof. □

Theorem 4.B. *If $C_3 = \emptyset$, then* (3PCI) *defines a facet of KP if it defines a facet of KP^C and any of the following conditions is satisfied:*

(a) $C = N$

(b) $\lambda \leq max(C_1) + max(C_2) - max(C')$

(c) $\lambda \leq max(C_1) + max_2(C_1) + max_3(C_1) - max(C')$

Proof. To prove the theorem, we show that each of the conditions (a), (b), and (c) is sufficient in the absence of the remaining two. We observe that when $C_3 = \emptyset$, then (3PCI) reduces to $\sum_{i \in C_1} y_i + 2 \sum_{i \in C_2} y_i \leq c_1 + 2c_2 - 3$.

(a) It is easy to see that if $C = N$, then a facet of KP^C will also be a facet of KP, since in this case $KP = KP^C$.

(b) To prove that (3PCI) defines a facet of KP, we need to show that an additional $(n-c)$ affinely independent points exist in KP that are binding to (3PCI). One of the ways in which (3PCI) can be binding is when $y_i = 0$ for exactly one $i \in C_1$ and $y_i = 0$ for exactly one $i \in C_2$. If $\lambda \leq max(C_1) + max(C_2) - max(C')$, then we can generate $(n-c)$ affinely independent points as shown in the matrix below:

$$M = \Big(P \;\Big|\; Q \;\Big|\; R \Big)$$

where P is a matrix of dimension $(n-c) \times c_1$ with entries equal to $[0, 1, \ldots, 1]$, Q is a matrix of order $(n-c) \times c_2$ with entries equal to $[0, 1, \ldots, 1]$, and R is an identity matrix of order $(n-c) \times (n-c)$.

(c) Another way in which (3PCI) can be binding is when $y_i = y_j = y_k = 0$ $i, j, k \in C_1$ and $y_i = 1$ $\forall i \in C_2$.

In this case, if $\lambda \leq max(C_1) + max_2(C_1) + max_3(C_1) - max(C')$, then we can generate $(n-c)$ affinely independent points as shown in the matrix below:

$$M = \Big(P \;\Big|\; Q \;\Big|\; R \Big)$$

where P is a matrix of dimension $(n-c) \times c_1$ with entries equal to $[0, 0, 0, 1, \ldots, 1]$, Q is a matrix of order $(n-c) \times c_2$ with all entries equal to 1, and R is an identity matrix of order $(n-c) \times (n-c)$.

This completes the proof. □

Theorem 4.C. *If C_1, C_2, and C_3 are all non-empty, then* (3PCI) *defines a facet of KP if it defines a facet of KP^C and any of the following conditions is satisfied:*

(a) $C = N$

(b) $\lambda \leq max(C_3) - max(C')$

(c) $\lambda \leq max(C_1) + max(C_2) - max(C')$

(d) $\lambda \leq max(C_1) + max_2(C_1) + max_3(C_1) - max(C')$

Proof. To prove the theorem, we show that each of the conditions (a), (b), (c), and (d) is sufficient in the absence of the remaining three.

(a) It is easy to see that if $C = N$, then a facet of KP^C will also be a facet of KP, since in this case $KP = KP^C$.

(b) To prove that (3PCI) defines a facet of KP, we need to show that an additional $(n - c)$ affinely independent points exist in KP that are binding to (3PCI). One of the ways in which (3PCI) can be binding is when $y_i = 1\ \forall i \in C_1$, $y_i = 1\ \forall i \in C_2$, and $y_i = 0$ for exactly one $i \in C_3$. If $\lambda \leq max(C_3) - max(C')$, then we can generate $(n - c)$ affinely independent points as shown in the matrix below:

$$M = \Big(P \;\Big|\; Q \;\Big|\; R \;\Big|\; S\Big)$$

where P is a matrix of dimension $(n - c) \times c_1$ with all entries equal to 1, Q is a matrix of dimension $(n - c) \times c_2$ with all entries equal to 1, R is a matrix of order $(n - c) \times c_3$ with entries equal to $[0, 1, \ldots, 1]$, and S is an identity matrix of order $(n - c) \times (n - c)$.

(c) Another way in which (3PCI) can be binding is when $y_i = 0$ for exactly one $i \in C_1$, $y_i = 0$ for exactly one $i \in C_2$, and $y_i = 1\ \forall i \in C_3$.

In this case, if $\lambda \leq max(C_1) + max(C_2) - max(C')$, then we can generate $(n - c)$ affinely independent points as shown in the matrix below:

$$M = \Big(P \;\Big|\; Q \;\Big|\; R \;\Big|\; S\Big)$$

where P is a matrix of dimension $(n - c) \times c_1$ with entries equal to $[0, 1, \ldots, 1]$, Q is a matrix of order $(n - c) \times c_2$ with entries equal to $[0, 1, \ldots, 1]$, R is a matrix of order $(n-c) \times c_3$ with all entries equal to 1, and S is an identity matrix of order $(n-c) \times (n-c)$.

(d) One more way in which (3PCI) can be binding is when $y_i = y_j = y_k = 0\ i, j, k \in C_1$, $y_i = 1\ \forall i \in C_2$, and $y_i = 1\ \forall i \in C_3$.

In this case, if $\lambda \leq max(C_1) + max_2(C_1) + max_3(C_1) - max(C')$, then we can generate $(n - c)$ affinely independent points as shown in the matrix below:

$$M = \Big(P \;\Big|\; Q \;\Big|\; R \;\Big|\; S\Big)$$

where P is a matrix of dimension $(n-c)\times c_1$ with entries equal to $[0,0,0,1,\ldots,1]$, Q is a matrix of order $(n-c)\times c_2$ with all entries equal to 1, R is a matrix of order $(n-c)\times c_3$ with all entries equal to 1, and S is an identity matrix of order $(n-c)\times(n-c)$.

This completes the proof. □

Proposition 4. *Given a cover C and its q-partition, where $4 \le q \le 6$, and:*

C_1 *is a subset of* C *such that* $\sum_{j\in S} a_j < \lambda,\ \forall\, S \subseteq C_1,\ |S| \le q-1,$

C_j *is a subset of* $C \setminus \bigcup_{k=1}^{j-1} C_k$ *such that for each of its elements* $i, a_i + \sum_{k=1}^{q-j-1} max_k(C_1) < \lambda,$

$a_i + \sum_{k=1}^{\left\lceil \frac{q-j}{j-1}\right\rceil -1} max_k(C_{j-1}) < \lambda,$ *and* $\sum_{s\in S} a_s < \lambda,\ \forall\, S \subseteq C_j,\ |S| \le \left\lceil \frac{q}{j}\right\rceil - 1,$ *for* $2 \le j \le q-2,$

C_{q-1} *is a subset of* $C \setminus \bigcup_{k=1}^{q-2} C_k$ *such that for each of its elements* $i, a_i < \lambda,$

C_q *is a subset of* $C \setminus \bigcup_{k=1}^{q-1} C_k$ *such that for each of its elements* $i, a_i \ge \lambda,$

and when $q=6,\ a_i + a_r + a_s < \lambda,\ \forall\, i \in C_1,\ \forall\, r,s \in C_2,\ r \ne s,$

the following is a VI for KP, which we refer to as a q-partition cover inequality (qPCI):

$$\sum_{i\in C_1} y_i + 2\sum_{i\in C_2} y_i + \cdots + q\sum_{i\in C_q} y_i \le c_1 + 2c_2 + \cdots + qc_q - q \qquad \text{(qPCI)}$$

Proof. We begin by noting that for any feasible solution $y \in Y$, the following trivial inequalities hold for each partition set:

$$\sum_{i\in C_j} y_i \le c_j, \quad \text{for all } j = 1,\ldots,q.$$

Multiplying each inequality by its respective group index j and summing over all partitions yields:

$$\sum_{j=1}^{q} j \sum_{i\in C_j} y_i \le \sum_{j=1}^{q} jc_j.$$

Our objective is to prove that every feasible solution $y \in Y$ also satisfies the q-partition cover inequality stated in the proposition. Since C is a cover with surplus λ, any feasible solution must exclude items from C whose total weight is at least λ. To do so, we classify feasible solutions based on the sets from which items are excluded, and consider three exhaustive cases:

Case 1: Items excluded only from a single set C_j

Consider a feasible solution $y \in Y$ such that $y_i = 0$ for one or more items from a single set C_j (for some $1 \le j \le q$), and $y_i = 1$ for all $i \in C \setminus C_j$. We now show that any such feasible solution satisfies the proposed qPCI.

- For C_1: By construction, any subset of fewer than q elements in C_1 has total weight less than the surplus λ. Therefore, at least q items must be excluded from C_1 to ensure feasibility. This yields a reduction of at least q in the left-hand side of the qPCI.
- For $2 \leq j \leq q-2$: The definition of C_j ensures that any subset of fewer than $t_j := \left\lceil \frac{q}{j} \right\rceil$ items has total weight less than λ. Hence, at least t_j items must be excluded to achieve feasibility. The resulting reduction in the left-hand side is at least $j \cdot t_j \geq q$.
- For C_{q-1}: All items satisfy $a_i < \lambda$, so no single item can account for the full surplus. Thus, at least two items must be excluded to obtain a feasible solution. This leads to a reduction of $2(q-1) = 2q-2$, which is strictly greater than q for all $q \geq 4$.
- For C_q: Each item satisfies $a_i \geq \lambda$. Since each such exclusion contributes q units to the reduction in the left-hand side, the inequality is satisfied.

In all cases, the total reduction in the left-hand side of (qPCI) due to the excluded items is at least q, i.e.,

$$\sum_{j=1}^{q} j \sum_{i \in C_j} y_i \leq \sum_{j=1}^{q} j c_j - q,$$

and thus the inequality is valid for such feasible solutions.

Case 2: Items excluded from exactly two sets

We consider two subcases: (a) items excluded from C_1 and another set C_j; and (b) items excluded from any two sets other than C_1.

(a) Items excluded from C_1 and any other set C_j:

For $j \geq q-1$, excluding at least one item from each of C_1 and C_j yields a reduction of at least $1+j \geq q$. Thus, we only need to consider $2 \leq j \leq q-2$.

From the definition of C_j, we have

$$a_i + \sum_{k=1}^{q-j-1} \max_k (C_1) < \lambda,$$

that is, the sum of weights of any $q-j-1$ items from C_1 together with the weight of any single item from C_j is strictly less than λ.

Hence, if exactly one item is excluded from C_j, at least $q-j$ items must be excluded from C_1 to obtain a feasible solution. The resulting reduction in the left-hand side of (qPCI) is at least

$$j + (q-j) = q.$$

If two or more items are excluded from C_j, where $j \geq 3$, the reduction is at least

$$1 + 2j \geq 7 > q.$$

For $j = 2$, excluding at least two items from C_2 and one item from C_1 yields a reduction of at least $1 + 2(2) = 5$, which is sufficient for $q \leq 5$.

For $q = 6$, excluding at least two items from C_1 and two items from C_2, or at least one item from C_1 and three items from C_2, yields a reduction of at least 6. The only remaining possibility with a reduction less than 6 is the exclusion of one item from C_1 and two items from C_2. However, the additional condition for $q = 6$ ensures that

$$a_i + a_r + a_s < \lambda, \quad \forall\, i \in C_1,\ \forall\, r, s \in C_2,\ r \neq s.$$

Thus, such an exclusion cannot yield a feasible solution.

In all cases, the total reduction in the left-hand side of (qPCI) due to excluded items is at least q. Hence, the inequality remains valid.

(b) **Items excluded from any two sets other than C_1.**

For $4 \leq q \leq 5$, excluding items from any two sets other than C_1 yields a reduction of at least $2 + 3 = 5 \geq q$.

For $q = 6$, the only possibility with a reduction less than 6 is the exclusion of exactly one item from C_2 and one item from C_3, yielding a reduction of $2 + 3 = 5$.

From the definition of C_j, for $j = 3$ and $q = 6$, we have

$$a_i + \sum_{k=1}^{\left\lceil \frac{q-j}{j-1} \right\rceil - 1} \max_k\, (C_{j-1}) < \lambda,$$

which reduces to

$$a_i + \max_1\, (C_2) < \lambda, \quad \forall\, i \in C_3.$$

Thus, the sum of weights of any one item from C_2 and one item from C_3 is strictly less than λ. Hence, such an exclusion cannot yield a feasible solution.

Therefore, the total reduction in the left-hand side of (qPCI) is at least q for every feasible solution in this subcase, ensuring that the inequality remains valid.

Case 3: Items excluded from more than two sets

For $4 \leq q \leq 6$, the smallest possible reduction in the left-hand side of (qPCI) occurs when one item is excluded from each of C_1, C_2, and C_3, giving

$$1 + 2 + 3 = 6.$$

Any other combination of three sets, or exclusion from more than three sets, yields a larger reduction since the sum of their indices exceeds $1 + 2 + 3 = 6$. Thus, the total reduction is always at least $6 \geq q$, and the inequality (qPCI) remains valid.

□

Remark 4. *(a) When $C_1 = C_2 = \cdots = C_{q-1} = \emptyset$,* (qPCI) *reduces to the following minimal CI: $\sum_{i \in C_q} y_i \leq c_q - 1$.*

(b) When $C_2 = C_3 = \cdots = C_q = \emptyset$, (qPCI) *reduces to the following ECI: $\sum_{i \in C_1} y_i \leq |C_1| - q$.*

(c) *As a result of the above two remarks, we only consider qPCIs that satisfy neither condition (a) nor (b).*

(d) *Consider the following generalized* $(1,k)$*-configuration inequality for a given* $S \subseteq N \setminus \{t\}$ *and* $k < |S|$ *(Del Pia et al., 2024):*

$$(s' - k + 1)y_t + \sum_{i \in S} y_i \leq s',$$

where s' *is the smallest integer such that the inequality* $\sum_{i \in S} y_i \leq s'$ *is valid for* KP*. Let* $q = |S| - k + 1$ *and* $r = s' - k + 1$*. It can be obtained from a qPCI with (i)* $C_1 = S$*, (ii)* $C_r = \{t\}$*, and (iii)* $C_i = \emptyset \; \forall i \in \{2, \cdots, q\} \setminus \{r\}$*. The corresponding qPCI is*

$$\begin{aligned} \sum_{i \in S} y_i + r y_t &\leq |S| + r - q \\ &= |S| + (s' - k + 1) - (|S| - k + 1) \\ &= s', \end{aligned}$$

which is the same as the generalized $(1,k)$*-configuration inequality. We provide examples showing the relationship of* $(1,k)$*-configuration and generalized* $(1,k)$*-configuration inequalities with qPCI in Appendix B.*

Example 2. *Consider* $Y = \{y \in \mathbb{B}^n : \sum_{i \in N} a_i y_i \leq b\}$ *with* $n = 7$*,* $a = \{8, 7, 6, 4, 6, 6, 6\}$*, and* $b = 22$*.*
Clearly, $C = N$ *is a non-minimal cover since* $\sum_{i \in C} a_i = 43, \lambda = 43 - 22 = 21 > min_{i \in C}\{a_i\} = 4$*. Clearly, the following is a 4-partition of* C*:* $C_1 = N \setminus \{1\}$*,* $C_2 = \{1\}$*,* $C_3 = C_4 = \emptyset$*. Hence, the corresponding 4PCI is* $2y_1 + y_2 + \cdots + y_7 \leq 2 \times 1 + 6 - 4 = 4$*.*

Proposition 5. (qPCI) *cannot be obtained through:*

(a) *Either sequential or simultaneous lifting of any minimal CI if* $C_q \neq \emptyset$*.*

(b) *Sequential lifting of any minimal CI if* C_1 *is a minimal cover satisfying either of the following conditions: (i)* $|C_{q-1}| \geq 2$*; (ii)* $|C_{q-1}| = 1$ *and* $\sum_{i=2}^{q-2} |C_i| \geq 1$*.*

(c) *Sequential lifting of the CI corresponding to a minimal cover* C_0 *if* C_1 *is a non-minimal cover with* $|C_1| > |C_0| + (q - 3)$*.*

Proof. We show that (qPCI) cannot be obtained through sequential lifting of a minimal CI under any of the conditions stated in the proposition. Further, under condition (a), we show that (qPCI) cannot be obtained through simultaneous lifting of any minimal CI. For this, let C_0 be a minimal cover and $C_1, C_2, \ldots, C_q$ be the q partitions of a non-minimal cover C. Since sequential lifting does not change the coefficients of the variables in C_0, we must have $C_0 \subseteq C_1$, and hence $c_1 \geq c_0$. Further, sequential lifting of a minimal CI does not alter its right-hand side, which remains $c_0 - 1$.

(a) $C_q \neq \emptyset$: Since C_q is nonempty, there is at least one element in C_q. The minimum possible value of the right-hand side of the qPCI is obtained when $|C_q| = 1$ and $C_2, \ldots, C_{q-1}$ are

empty. In this case, the right-hand side of the qPCI becomes $c_1 + q - q = c_1 \geq c_0$. Moving any element from C_1 to any other set $C_2, \ldots, C_{q-1}$, or having more than one element in C_q, only increases the value of the right-hand side. However, the minimal cover inequality corresponding to C_0 has right-hand side $c_0 - 1$. Since sequential lifting of this inequality leaves its right-hand side unchanged, the qPCI cannot, in this case, be obtained through sequential lifting of any minimal CI.

We now consider simultaneous lifting for the case $C_q \neq \emptyset$. Since simultaneous lifting does not change the coefficients of the variables in the original minimal CI, any minimal cover C_0 whose CI can be lifted to obtain (qPCI) must satisfy $C_0 \subseteq C$.

For any $i \in C_q$, we have $a_i \geq \lambda$. Therefore,

$$\sum_{j \in C \setminus \{i\}} a_j = b + \lambda - a_i \leq b.$$

Hence, any cover $C_0 \subseteq C$ must contain all the elements of C_q, i.e., $C_q \subseteq C_0$. Since the coefficients of the variables in C_0 are not changed through simultaneous lifting, C_0 cannot contain any element from C_j, $j < q$, as the coefficients of such variables in (qPCI) are different from those in C_q. Hence, we must have $C_0 = C_q$.

The CI corresponding to $C_0 = C_q$ is $\sum_{i \in C_q} y_i \leq c_q - 1$. For this inequality to have coefficient q for the variables in C_q, it must be multiplied by q, and hence its right-hand side becomes $q(c_q - 1)$. However, the right-hand side of the qPCI is

$$\sum_{j=1}^{q} jc_j - q = q(c_q - 1) + \sum_{j=1}^{q-1} jc_j.$$

Since C is a non-minimal cover and C_q contains only elements with $a_i \geq \lambda$, at least one of the sets $C_1, \ldots, C_{q-1}$ must be nonempty. Therefore, $\sum_{j=1}^{q-1} jc_j > 0$, and hence the right-hand side of the qPCI is strictly greater than $q(c_q - 1)$. Thus, (qPCI) cannot be obtained through simultaneous lifting of any minimal CI when $C_q \neq \emptyset$.

(b) C_1 is a minimal cover satisfying either of the following two conditions: (i) $|C_{q-1}| \geq 2$; (ii) $|C_{q-1}| = 1$ and $\sum_{i=2}^{q-2} |C_i| \geq 1$: Since $C_0 \subseteq C_1$ and both C_0 and C_1 are minimal covers, we have $C_0 = C_1$, and hence $c_0 = c_1$. Further, from part (a), we only need to consider the case $C_q = \emptyset$.

If $|C_{q-1}| \geq 2$, then the minimum possible value of the right-hand side of the qPCI is obtained when $|C_{q-1}| = 2$ and $C_i = \emptyset \; \forall i \in \{2, \ldots, q-2\}$. In this case, the right-hand side becomes $c_1 + 2(q-1) - q = c_1 + q - 2 > c_1 - 1 = c_0 - 1$.

Similarly, if $|C_{q-1}| = 1$ and $\sum_{i=2}^{q-2} |C_i| \geq 1$, then the minimum possible value of the right-hand side is obtained when $|C_2| = 1$ and $C_i = \emptyset \; \forall i \in \{3, \ldots, q-2\}$. In this case, the right-hand side becomes $c_1 + 2 + (q-1) - q = c_1 + 1 > c_1 - 1 = c_0 - 1$. Hence, in either case, the qPCI cannot be obtained through sequential lifting of any minimal CI.

(c) C_1 is a non-minimal cover with $|C_1| > |C_0| + (q-3)$, where C_0 is the minimal cover being lifted: Since the cardinalities are integer, we have $c_1 \geq c_0 + (q-2)$. From Remark 4(c),

we know that at least one of the sets $C_2, \ldots, C_q$ must be nonempty. Let us assume that $|C_2| = 1$, which gives the minimum possible value of the right-hand side of the qPCI. In this case, the right-hand side becomes $c_1 + 2 \cdot 1 - q \geq c_0 + (q-2) + 2 - q = c_0$. If any higher-order set (i.e., some C_j with $j \geq 3$) is made nonempty, then the right-hand side of the qPCI increases further. However, the minimal CI corresponding to C_0 has right-hand side $c_0 - 1$, and sequential lifting of this inequality does not change its right-hand side. Therefore, the qPCI, whose right-hand side is at least c_0, cannot be obtained through sequential lifting of the CI corresponding to C_0 in this case.

□

Example 1 (Continued). *The facet-defining 3PCI:* $3y_1 + 2y_2 + 2y_3 + 2y_4 + y_5 + y_6 + y_7 + y_8 \leq 3\times1+2\times3+1\times4-3 = 10$ *cannot be obtained through either sequential or simultaneous lifting of any minimal CI. This can be easily seen as follows. First, any CI with RHS* $= 7$ *is defined only for* $C = N$*. However, for* $C = N$*, the cover is non-minimal, as discussed earlier. Hence, any minimal CI has RHS* ≤ 6*, whereas the RHS of the 3PCI is 10. Since sequential lifting does not change the RHS, this facet cannot be obtained through sequential lifting of any minimal CI. Further, since* $C_3 = \{1\} \neq \emptyset$*, it follows from Proposition 5(a) that this facet cannot be obtained through simultaneous lifting of any minimal CI.*

3 Computational Experiments

We first propose the separation problem for qPCI in Section 3.1. We then test the effectiveness of our proposed VIs in efficiently solving the fixed-charge transportation problem (FCTP). Our choice of FCTP is motivated by its following two properties: (i) it is NP-hard; (ii) 2PCI and 3PCI occur frequently as VIs for the knapsack substructures derived from FCTP, as shown in Section 3.2. The datasets used in our computational experiments are described in Section 3.3, and the results of our experiments are presented and discussed in Section 3.4.

3.1 Separation Problem

Here, we describe the separation problem for 2PCI. For the formulation of the separation problem shown below, we assume that the item weights a_i and the knapsack capacity b are integral, as is the case in our computational application, i.e., FCTP. Let $z_{i1} = 1$ if item i belongs to C_1 of a two-partition cover C, 0 otherwise. Similarly, let $z_{i2} = 1$ if item i belongs to C_2, 0 otherwise.

Then, the separation of the most violated 2PCI can be mathematically stated as follows.

$$\text{Maximize} \quad \sum_{i \in N} (\bar{y}_i - 1) z_{i1} + 2 \sum_{i \in N} (\bar{y}_i - 1) z_{i2} + 2 \tag{3}$$

Subject to

$$\sum_{i \in N} a_i (z_{i1} + z_{i2}) \geq b + 1 \tag{4}$$

$$a_i z_{i1} \leq \sum_{j \in N} a_j (z_{j1} + z_{j2}) - b - 1 \qquad \forall i \in N \tag{5}$$

$$a_i z_{i2} \geq \sum_{j \in N} a_j (z_{j1} + z_{j2}) - b - M(1 - z_{i2}) \qquad \forall i \in N \tag{6}$$

$$z_{i1} + z_{i2} \leq 1 \qquad \forall i \in N \tag{7}$$

$$\sum_{i \in N} z_{i1} \geq 1 \tag{8}$$

$$\sum_{i \in N} z_{i2} \geq 1 \tag{9}$$

$$z_{i1}, z_{i2} \in \{0, 1\} \qquad \forall i \in N \tag{10}$$

(3) maximizes the violation of a 2PCI. (4) ensures that $C_1 \cup C_2$ is a cover. (5) ensures that item i belongs to C_1 only if $a_i < \lambda = \sum_{i \in N} a_i (z_{i1} + z_{i2}) - b$. (6) ensures that item i belongs to C_2 only if $a_i \geq \lambda$. (7) forces each item either to belong to C_1 or C_2 of cover C or to lie outside C. (8) and (9) ensure that neither C_1 nor C_2 is empty, i.e., C is strictly a non-minimal cover (refer to Remark 1). The separation problem for a general qPCI can be similarly stated.

3.2 Knapsack Set as a Substructure in FCTP

For the reader's convenience, we reproduce the FCTP formulation. Consider a set of sources (origins) $S = \{1, 2, \ldots, i, \ldots, s\}$, each with an integral supply capacity $a_i > 0$ and a set of sinks (destinations) $T = \{1, 2, \ldots, j, \ldots, t\}$, each with an integral demand $b_j > 0$. We assume that the problem is balanced, i.e. $\sum_{i \in S} a_i = \sum_{j \in T} b_j$. There is a unit shipping cost c_{ij} plus a fixed cost f_{ij} for every i-j pair. Let $m_{ij} = \min \{a_i, b_j\}$. If x_{ij} represents the quantity shipped from source i to sink j, and $y_{ij} = 1$ if there is a positive quantity shipped from i to j, 0 otherwise, then FCTP can be mathematically stated as:

$$\min \sum_{i=1}^{s}\sum_{j=1}^{t}(c_{ij}x_{ij} + f_{ij}y_{ij}) \tag{11}$$

subject to

$$\sum_{j=1}^{t} x_{ij} = a_i \qquad i \in S \tag{12}$$

$$\sum_{i=1}^{s} x_{ij} = b_j \qquad j \in T \tag{13}$$

$$x_{ij} \leq m_{ij}y_{ij} \qquad \forall i \in S, j \in T \tag{14}$$

$$x_{ij} \geq 0, y_{ij} \in \{0,1\} \qquad \forall i \in S, j \in T \tag{15}$$

Clearly, (16) and (17) given below are VIs to (11)-(15) since each term in the LHS of (16) and (17) is an upper bound on the corresponding terms in (12) and (13).

$$\sum_{j=1}^{t} m_{ij}y_{ij} \geq a_i \qquad i \in S \tag{16}$$

$$\sum_{i=1}^{s} m_{ij}y_{ij} \geq b_j \qquad j \in T \tag{17}$$

(16) and (17) represent knapsack inequalities of the form $\sum_{i\in N} a_i x_i \geq d$. An FCTP with $s = |S|$ supply nodes and $t = |T|$ demand nodes has $s + t$ such knapsack polytopes. To be consistent with the knapsack literature, (16) and (17) can be restated as $\sum_{i\in N} a_i y_i \leq b$ using the standard trick of replacing variables by their complements. However, the above VIs are not useful since they are always satisfied by any fractional solution to (11)-(15). Nonetheless, there exist well-known classes of useful VIs (e.g., CIs) for knapsack polytopes, which can be added to (11)-(15) to strengthen its lower bound. Furthermore, our two classes of VIs based on non-minimal covers can also be used.

The next proposition establishes sufficient conditions for the existence of 2PCIs and guarantees the existence of 3PCIs for any FCTP.

Proposition 6. *Let $M_1 = \{i \in S : a_i \leq b_j \ \forall j \in T\}$ and $M_2 = \{j \in T : b_j \leq a_i \ \forall i \in S\}$. Then an FCTP with s supply nodes and t demand nodes, satisfying $\max_{i\in S} a_i = \max_{j\in T} b_j$, will have at least $s+t-m$ 2PCIs and at least an equal number of 3PCIs, where $m = |M_1| + |M_2|$. Moreover, the lower bound of $s+t-m$ for 3PCIs remains valid even when this condition does not hold.*

Proof. The knapsack inequality (16) can be rewritten as:

$$\sum_{j=1}^{t} m_{ij}z_{ij} \leq \sum_{j=1}^{t} m_{ij} - a_i \qquad i \in S \tag{18}$$

where $z_{ij} = 1 - y_{ij}$. For a given $i \in S$, consider a cover C for the knapsack set given by (18) such that $C = T$, in which case its surplus $\lambda = a_i$. Then, the following three mutually exclusive and exhaustive conditions arise.

(a) $a_i \leq b_j \ \forall j \in T$: In this case, $\lambda = min(C)$, and hence, C is a minimal cover. Further, all the coefficients $m_{ij} = a_i$, so no proper subset of T is a cover. Therefore, no 2PCI or 3PCI exists.

(b) $min_{j \in T} b_j < a_i \leq max_{j \in T} b_j$: In this case, $min(C) < \lambda \leq max(C)$, and hence, at least one 2PCI exists (using Remark 1(c)) and at least one 3PCI exists (using Remark 3(g)).

(c) $a_i > b_j \ \forall j \in T$: In this case, $C_3 = \emptyset$ in a 3-partition of C. By choosing any $k \in T$ and setting $C_1 = \{k\}$ and $C_2 = T \setminus \{k\}$, we have $C_1 \neq \emptyset$ and $C_2 \neq \emptyset$. Hence, at least one 3PCI exists (using Remark 3(g)). However, for a 2-partition of C, $C_2 = \emptyset$. Hence, no 2PCI exists for $C = T$ (using Remark 1(b)), and its existence for a proper subset of T is not guaranteed.

A similar argument holds true for knapsack inequality (17). Under the condition $\max_{i \in S} a_i = \max_{j \in T} b_j$, case (c) cannot occur for either supply or demand nodes. Hence, $m = |M_1| + |M_2|$ represents the number of knapsack inequalities defined by (16) and (17) for which neither 2PCI nor 3PCI exists, and the remaining $s + t - m$ knapsack inequalities are guaranteed to have at least one 2PCI and one 3PCI. Moreover, since case (c) also guarantees at least one 3PCI, the lower bound of $s + t - m$ for 3PCIs remains valid even without the additional condition. □

Example 2. *Consider* $Y = \{y \in \mathbb{B}^n : \sum_{i \in N} a_i y_i \leq b\}$
with $n = 15$, $a = \{19, 17, 14, 14, 14, 13, 13, 12, 11, 10, 10, 9, 9, 7, 5\}$, *and* $b = 158$.
This knapsack instance corresponds to a substructure of FCTP with $s = t = 15$ *from Dataset 1 described in Section 3.3. The complete convex hull of this knapsack set (excluding the trivial facets* $y_i \geq 0$ *and* $y_i \leq 1$*), as obtained using PANDA (Lörwald and Reinelt, 2015), is shown in Table 5 in the Appendix A. Of the 29 nontrivial facets, the first 19 are CIs. The next five (20–24) are 2PCIs, the next four (25–28) are 3PCIs, and the last one (29) is a 4PCI. This highlights the significance of our proposed VIs, in addition to the well-known CIs, in characterizing the knapsack polytope that appears as a local substructure within FCTP.*

3.3 Datasets

For our computational experiments, we considered two benchmark datasets. Dataset 1 was introduced by Agarwal and Aneja (2012). It consists of instances with 15 origins and 15 destinations, while a_i and b_j are randomly generated using the uniform distribution $U \sim [1, 20]$. Fixed and variable costs are generated using $U \sim (200, 800)$. θ is the ratio between the total variable and fixed costs. Instances with $\theta = 0.0$ represent a pure fixed-charge transportation problem (PFCTP). Dataset 2 was introduced by Roberti et al. (2015). It consists of instances similar to Dataset 1, except for the larger number of origins and destinations. In our experiments, we considered instances with 30 origins and 30 destinations.

3.4 Results

All the runs were conducted on a single core of an Intel(R) Xeon(R) Gold 6240 CPU @ 2.60GHz server with 16GB of RAM. The model of FCTP (11)-(15) is implemented in C++ and solved

using CPLEX 22.1. All proposed VIs are generated through complete enumeration at the root node of the branch-and-bound tree and are added only if they violate the current LP relaxation solution and satisfy the corresponding facet-defining conditions. For a given knapsack inequality, a 2PCI with $c = n$ is unique, while there are only $n - 1$ such 2PCIs with $c = n - 1$. So, these VIs can be easily enumerated. However, there can be multiple 3PCIs with $c = n$, and we have arbitrarily generated one of them.

The computational results are shown in Table 1 - Table 4. Each Table reports the following information for each of the instances:

- IP Obj: Optimal objective value or best objective value obtained within the specified CPU time limit.
- LP%: Linear programming (LP) relaxation value of FCTP model (11)-(15) as a % of IP Obj.
- CPU/Gap(%): It reports the CPU time (in seconds). For the instances that could not be solved within their respective CPU time limits (3,600 seconds for Dataset 1 and 7,200 seconds for Dataset 2), the table reports the corresponding optimality gap at termination.
- BBTS: Branch-and-bound tree size in millions of nodes.

Further, the row "Average" gives the average CPU time across those instances that are solved "With VIs" and "Without VIs" within the specified CPU time limit. It also provides (within parentheses) the average gap for the remaining instances. The row "#Solved" shows the number of instances solved within the CPU time limit, and the row "#Best" shows the number of instances for which a given method ("With VIs" or "Without VIs") is the best.

In all the experiments, a CPU time limit of 3,600 seconds is used for instances from Dataset 1. Instances from Dataset 1 are relatively easier to solve; hence, we solve them without the CPLEX-generated cuts to better assess the efficacy of our facets. The results from our computational experiments corresponding to Dataset 1 are shown in Table 1 for $\theta = 0.2$ and Table 2 for $\theta = 0$. As is clear from Table 1, the facets from our qPCIs help improve the lower bound of FCTP by 9% (from 81.9% to 89.2% of the IP optimal objective function value), which helps cut down the CPU time by around 95% (from 483.70 seconds to 23.32 seconds). The corresponding savings for $\theta = 0$, as evident from Table 2, are around 13.5% and 95% in the lower bound and CPU time, respectively.

Instances from Dataset 2 are much more difficult; hence, we use a higher CPU time limit of 7,200 seconds and solve them with the CPLEX-generated cuts. However, we turn off the cover inequalities generated by CPLEX, again, to better assess the efficacy of our facets. The results from our computational experiments corresponding to Dataset 2 are shown in Table 3 for $\theta = 0.2$ and Table 4 for $\theta = 0$. As is clear from Table 3, the facets from our qPCIs help improve the lower bound of FCTP by 9% (from 84.2% to 91.8%), which helps cut down the CPU time by around 28% (from 2,440.78 seconds to 1,758.88 seconds). The corresponding savings for $\theta = 0$, as evident from Table 4, are around 9% and 31.3% in the lower bound and CPU time, respectively.

Table 1: Dataset 1, $s = t = 15$, $\theta = 0.2$, Without CPLEX-generated cuts

Ins.	Obj.	Without qPCI			With qPCI					
		LP%	CPU	BBTS	#2PCI		#3PCI	LP%	CPU	BBTS
			(Gap%)		$c = n$	$c = n - 1$	$c = n$		(Gap%)	
1	10017	81.8	1037.86	11.31	13	4	2	89.0	**47.28**	0.19
2	10075	82.1	687.03	15.04	14	4	3	88.9	**42.49**	0.19
3	9327	83.5	141.22	0.89	16	9	5	91.2	**3.32**	0.01
4	11093	78.7	(7.83%)	20.83	17	3	3	86.6	**2573.70**	10.28
5	10312	80.3	1131.99	28.12	10	4	3	87.5	**39.92**	0.19
6	10086	86.6	38.03	1.02	9	17	3	91.9	**6.00**	0.02
7	9913	82.1	323.52	1.90	9	6	2	88.7	**21.12**	0.09
8	10495	80.3	(5.84%)	22.42	14	5	4	87.5	**532.61**	2.41
9	10137	83.5	26.25	1.10	12	9	1	91.3	**3.13**	0.01
10	9939	80.2	(0.59%)	25.22	13	6	2	89.6	**31.43**	0.16
Average	10139	81.9	483.70(4.75%)	12.79	13	7	3	89.2	**23.32(0%)**	1.36
#Solved			7						**10**	
#Best			0						**10**	

Ins.: Instance; IP Obj.: Optimal IP objective function value; LP%: LP relaxation as a % of Obj.; CPU/Gap%: Computation time in seconds/ Optimality Gap in %; BBTS: Branch-and-bound tree size in millions of nodes #2PCI: Number of 2PCI cuts; #3PCI: Number of 3PCI cuts

Table 2: Dataset 1, $s = t = 15$, $\theta = 0$, Without CPLEX-generated cuts

Ins.	Obj.	Without qPCI			With qPCI					
		LP%	CPU	BBTS	#2PCI		#3PCI	LP%	CPU	BBTS
			(Gap%)		$c = n$	$c = n - 1$	$c = n$		(Gap%)	
1	6683	74.54	1184.61	11.31	19	7	5	86.44	**37.99**	0.15
2	6903	73.23	1542.38	15.04	17	5	5	80.45	**88.14**	0.41
3	6210	77.66	100.57	0.89	15	16	4	89.57	**3.89**	0.01
4	7753	71.06	(9.84%)	20.83	21	3	3	82.12	**365.97**	1.50
5	7360	69.14	(2.65%)	28.12	12	1	2	80.26	**88.33**	0.43
6	6911	80.19	109.12	1.02	13	14	2	88.61	**7.88**	0.03
7	6434	77.68	194.22	1.90	10	6	1	86.63	**17.60**	0.08
8	7254	73.55	(7.39%)	22.42	14	1	2	83.24	**285.92**	1.46
9	7119	79.98	119.61	1.10	10	8	1	88.79	**9.34**	0.04
10	6843	72.14	(5.43%)	25.22	14	9	3	85.39	**143.52**	0.65
Average	6947	74.92	541.75(6.33%)	12.79	15	7	3	85.1	**27.47(0%)**	0.48
#Solved			6						**10**	
#Best			0						**10**	

Ins.: Instance; IP Obj.: Optimal IP objective function value; LP%: LP relaxation as a % of Obj.; CPU/Gap%: Computation time in seconds/ Optimality Gap in %; BBTS: Branch-and-bound tree size in millions of nodes #2PCI: Number of 2PCI cuts; #3PCI: Number of 3PCI cuts

Table 3: Dataset 2, $s = t = 30$, $\theta = 0.2$, With CPLEX-generated cuts

Ins.	Obj.	Without qPCI			With qPCI					
		LP%	CPU	BBTS	#2PCI		#3PCI	LP%	CPU	BBTS
			(Gap%)		$c = n$	$c = n - 1$	$c = n$		(Gap%)	
1	13367	82.81	4695.83	0.41	23	62	10	90.67	**4082.09**	0.37
2	13447	82.27	(2.19%)	0.58	23	45	7	88.87	**(1.40%)**	0.44
3	13953	83.01	5785.2	0.53	18	53	2	91.56	**3978.00**	0.44
4	13477	84.90	266.0	0.04	28	47	5	93.35	**219.65**	0.04
5	13387	86.02	1485.8	0.16	29	71	15	93.26	**863.35**	0.10
6	13704	86.06	126.2	0.03	22	27	5	93.02	**70.15**	0.01
7	13412	84.86	2349.2	0.25	28	105	11	92.46	**1284.65**	0.14
8	13600	80.21	(3.00%)	0.62	28	44	11	89.79	**(2.59%)**	0.60
9	13683	86.18	1600.6	0.16	24	46	10	92.98	**265.27**	0.04
10	13605	85.83	**3217.4**	0.32	22	106	8	92.16	3307.90	0.30
Average	13564	84.21	2440.78(2.60%)	0.31	25	61	8	91.81	**1758.88(2.00%)**	0.25
#Solved			8						8	
#Best			1						**9**	

Ins.: Instance; IP Obj.: Optimal IP objective function value; LP%: LP relaxation as a % of Obj.; CPU/Gap%: Computation time in seconds/ Optimality Gap in %; BBTS: Branch-and-bound tree size in millions of nodes #2PCI: Number of 2PCI cuts; #3PCI: Number of 3PCI cuts

Table 4: Dataset 2, $s = t = 30$, $\theta = 0$, With CPLEX-generated cuts

Ins.	Obj.	Without qPCI			With qPCI					
		LP%	CPU	BBTS	#2PCI		#3PCI	LP%	CPU	BBTS
			(Gap%)		c=n	c=n-1	c=n		(Gap%)	
1	11109	80.85	2467.21	0.24	25	55	3	89.12	**750.22**	0.09
2	10622	82.98	2514.58	0.30	33	40	8	91.38	**1504.03**	0.20
3	10886	80.03	(0.31%)	0.74	27	34	10	87.91	**3775.22**	0.42
4	11316	79.46	4754.79	0.47	23	84	13	90.67	**4381.82**	0.40
5	10717	82.94	1711.58	0.17	34	54	13	91.81	**684.43**	0.08
6	10391	83.04	**1458.28**	0.16	34	58	7	92.64	1575.11	0.16
7	10919	81.08	2520.10	0.20	30	67	14	91.50	**2207.92**	0.19
8	10908	82.82	601.32	0.10	23	83	3	92.38	**238.31**	0.05
9	10744	77.89	(4.31%)	0.56	33	40	2	86.81	**(3.33%)**	0.53
10	10903	81.93	741.87	0.07	32	63	12	91.90	**180.78**	0.03
Average	10852	81.30	2096.22(2.31%)	0.30	29	58	9	90.61	**1440.33(1.67%)**	0.22
#Solved			8						**9**	
#Best			1						**9**	

Ins.: Instance; IP Obj.: Optimal IP objective function value; LP%: LP relaxation as a % of Obj.; CPU/Gap%: Computation time in seconds/ Optimality Gap in %; BBTS: Branch-and-bound tree size in millions of nodes #2PCI: Number of 2PCI cuts; #3PCI: Number of 3PCI cuts

4 Conclusions

In this paper, we studied the polyhedral structure of the binary knapsack polytope. For this, we exploited non-minimal covers of a knapsack, as opposed to minimal covers, popularly used in the literature. Using non-minimal covers, we proposed a new class of VIs and derived the conditions under which they define facets of KP. Further, we established conditions under

which the proposed inequalities cannot be obtained through sequential or simultaneous lifting of minimal cover inequalities. Our computational experiments on a well-known class of NP-hard problems highlighted the usefulness of the facets from our proposed VIs.

In our computational experiments, we used 2PCIs for covers containing either all items or all but one item, and 3PCIs for covers containing all items. In these cases, only a limited number of covers and associated partitions need to be considered, so the resulting 2PCIs and 3PCIs can be easily enumerated. However, when covers exclude two or more items, the number of possible covers increases substantially, as does the number of resulting 2PCIs and 3PCIs, making complete enumeration computationally expensive. In such cases, the most violated 2PCI and 3PCI can be separated by solving an optimization problem, as shown in Section 3.1 for 2PCIs. A natural extension is to develop heuristic separation procedures for 2PCIs and 3PCIs, which we anticipate will improve overall computational efficiency.

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A Facets of Example 2

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15		
a_i	19	17	14	14	14	13	13	12	11	10	10	9	9	7	5	b = 158	
S. No.	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	RHS	Type
1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	0	12	CI, $C = N \setminus \{11, 15\}$, $\lambda = 4$
2	1	1	1	1	1	1	1	1	1	1	1	0	1	1	0	12	CI, $C = N \setminus \{12, 15\}$, $\lambda = 5$
3	1	1	1	1	1	1	1	1	1	1	1	1	0	1	0	12	CI, $C = N \setminus \{13, 15\}$, $\lambda = 5$
4	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	12	CI, $C = N \setminus \{14, 15\}$, $\lambda = 7$
5	1	1	1	1	1	1	1	1	1	0	1	1	1	0	1	12	CI, $C = N \setminus \{10, 14\}$, $\lambda = 2$
6	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1	12	CI, $C = N \setminus \{12, 13\}$, $\lambda = 1$
7	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	12	CI, $C = N \setminus \{13, 14\}$, $\lambda = 3$
8	1	1	1	1	1	1	1	1	1	1	0	1	1	0	1	12	CI, $C = N \setminus \{11, 14\}$, $\lambda = 2$
9	1	1	1	1	1	1	1	1	1	1	1	0	1	0	1	12	CI, $C = N \setminus \{12, 14\}$, $\lambda = 3$
10	1	1	1	1	1	1	1	1	1	0	1	1	1	1	0	12	CI, $C = N \setminus \{10, 15\}$, $\lambda = 4$
11	1	1	1	1	1	1	0	1	1	1	1	1	1	1	0	12	CI, $C = N \setminus \{7, 15\}$, $\lambda = 1$
12	1	1	1	1	1	1	1	0	1	1	1	1	1	1	0	12	CI, $C = N \setminus \{8, 15\}$, $\lambda = 2$
13	1	1	1	1	1	0	1	1	1	1	1	1	1	1	0	12	CI, $C = N \setminus \{6, 15\}$, $\lambda = 1$
14	1	1	1	1	1	1	1	1	0	1	1	1	1	1	0	12	CI, $C = N \setminus \{9, 15\}$, $\lambda = 3$
15	1	1	1	1	1	1	1	1	0	1	1	1	1	0	1	12	CI, $C = N \setminus \{9, 14\}$, $\lambda = 3$
16	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	13	CI, $C = N \setminus \{2\}$, $\lambda = 2$
17	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	13	CI, $C = N \setminus \{3\}$, $\lambda = 5$
18	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	13	CI, $C = N \setminus \{4\}$, $\lambda = 5$
19	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	13	CI, $C = N \setminus \{5\}$, $\lambda = 5$
20	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14	2PCI, $C = N, \lambda = 19$
21	2	2	2	2	2	1	1	1	1	1	1	1	1	1	0	17	2PCI, $C = N \setminus \{15\}, \lambda = 14$
22	2	2	2	2	2	2	2	2	1	1	1	1	1	0	1	20	2PCI, $C = N \setminus \{14\}, \lambda = 12$
23	2	2	2	2	2	2	2	2	2	2	2	1	0	1	1	23	2PCI, $C = N \setminus \{13\}, \lambda = 10$
24	2	2	2	2	2	2	2	2	2	2	2	0	1	1	1	23	2PCI, $C = N \setminus \{12\}, \lambda = 10$
25	3	2	2	2	2	2	2	2	2	2	2	1	1	1	1	24	3PCI, $C = N, \lambda = 19, C_1 = \{12, 13, 14, 15\}$
26	3	2	2	2	2	2	2	2	1	2	2	2	2	1	1	25	3PCI, $C = N, \lambda = 19, C_1 = \{9, 14, 15\}$
27	3	2	2	2	2	2	2	2	2	1	2	2	2	1	1	25	3PCI, $C = N, \lambda = 19, C_1 = \{10, 14, 15\}$
28	3	2	2	2	2	2	2	2	2	2	1	2	2	1	1	25	3PCI, $C = N, \lambda = 19, C_1 = \{11, 14, 15\}$
29	4	3	3	3	3	3	3	3	2	2	2	2	2	1	1	33	4PCI, $C = N, \lambda = 19, C_1 = \{14, 15\}$

Table 5: Facets of the form $\sum_{i \in N} \alpha_i y_i \leq RHS$ generated using PANDA (Lörwald and Reinelt, 2015)

B $(1, k)$-configurations, generalized $(1, k)$-configuration, and qPCI

Using illustrative examples, we show the relationship of our proposed qPCI with $(1, k)$-configuration and generalized $(1, k)$-configuration inequalities. For this, we first consider an example from Padberg (1980) to show the relationship between qPCI and a $(1, k)$-configuration inequality.

Example 3. *Consider* $Y = \{y \in \mathbb{B}^n : \sum_{i \in N} a_i y_i \leq b\}$ *with* $n = 6$, $a = \{12, 2, 3, 3, 3, 4\}$, *and* $b = 16$. *Since* $2 + 3 + 3 + 3 + 4 = 15 < 16$, $12 + 4 \leq 16$, *and* $12 + 2 + 3 = 17 > 16$, *we have* $k = 2$, $N' = \{2, 3, 4, 5, 6\}$, *and* $t = 1$ *(corresponding to a* $(1, k)$*-configuration).*

Let $S = \{4, 5, 6\}$*; the corresponding* $(1, k)$*-configuration is* $2y_1 + y_4 + y_5 + y_6 \leq 3$. *Corresponding to this* S, *consider the cover* $C = S \cup \{t\} = \{1, 4, 5, 6\}$, *and the corresponding* $\lambda = 12 + 3 + 3 + 4 - 16 = 6$. $C_1 = \{a_i < \lambda\} = \{4, 5, 6\}$, *and* $C_2 = \{a_i \geq \lambda\} = \{1\}$. *The corresponding 2PCI is* $2y_1 + y_4 + y_5 + y_6 \leq 3 + 2 - 2 = 3$, *which is the same as the above* $(1, k)$*-configuration inequality. Hence, this shows that for the special case when* $C_1 = C \setminus \{t\}$, *2PCI reduces to a* $(1, k)$*-configuration inequality.*

Next, we consider an example from Del Pia et al. (2024) to show the relationship between qPCI and a generalized $(1, k)$-configuration inequality.

Example 4. *Given* $Y = \{y \in \mathbb{B}^n : \sum_{i \in N} a_i y_i \leq b\}$ *with* $n = 6$, $a = \{2, 4, 7, 10, 10, 20\}$, *and* $b = 30$.

Consider $N' = \{2, 3, 4, 5\}$, $t = 6$, *and* $k = 2$. $\sum_{i \in N'} y_i \leq s' = 3$ *is valid for* KP. *The corresponding generalized* $(1, k)$*-configuration inequality is* $y_2 + y_3 + y_4 + y_5 + 2y_6 \leq 3$. *Note that one assumption of the* $(1, k)$*-configuration is violated here because* $\sum_{i \in N'} a_i = 4 + 7 + 10 + 10 = 31 > b$.

Corresponding to this N', *consider the cover* $C = N' \cup \{t\} = \{2, 3, 4, 5, 6\}$, *and the corresponding* $\lambda = 4 + 7 + 10 + 10 + 20 - 30 = 21$. *Here,* $q = |N'| - k + 1 = 3$ *and* $r = s' - k + 1 = 2$. *Since the sum of any two elements in* N' *is less than* λ, *we have* $C_1 = N' = \{2, 3, 4, 5\}$, $C_2 = \{t\} = \{6\}$, *and* $C_3 = \phi$. *The corresponding 3PCI is* $y_2 + y_3 + y_4 + y_5 + 2y_6 \leq 4 + 2 - 3 = 3$, *which is the same as the generalized* $(1, k)$*-configuration inequality. Hence, this shows that for the special case when* $C_1 = C \setminus \{t\}$ *and* $C_2 = \{t\}$, *3PCI reduces to a generalized* $(1, k)$*-configuration inequality.*