Budget-constrained Maximization of "Cobb-Douglas with Linear Components" Utility Function

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Abstract

In what follows, we provide the demand analysis associated with budget-constrained linear utility maximization for each of several categories of goods, with the marginal rate of consumption expenditure-as a share of wealth- being a positive constant less than or equal to one. The marginal rate of consumption expenditure is endogenously determined, by a budget-constrained "Cobb-Douglas with Linear Components" utility function maximization problem, where the utility function includes the possibility of savings as a variable, and which reduces to the category specific budget-constrained linear utility maximization problems we are concerned with here. We also show that the budget-constrained Cobb-Douglas with Linear Components utility function maximization problem of a single consumer can be reduced to a finite number of budget-constrained linear utility maximizations problems all having the same number of variables, where the number of such budget-constrained linear utility maximizations problems is equal to the number of "categories" of the non-monetary goods that are consumed.

1. Introduction: In classical demand analysis for individual consumers (see Chapter 3 of Mas-Colell, Whinston and Green (1995)), a very important role is played by budget-constrained Cobb-Douglas utility function maximization. However, the problem with all consumers having a demand function arising out of such a maximization, is that the aggregate Marshallian market demand curve will then be totally incompatible with imperfect competition, particularly when it arises out of strictly increasing total variable costs of production.

An alternative possibility is to consider budget-constrained linear utility function maximization. There is substantial work on the linear exchange model, i.e., pure exchange model with consumer preferences representable by linear utility functions, beginning with the works of Gale (1957, 1976), Eisenberg and Gale (1959) and thereafter pursued by Eaves (1976), Mantel (1976), Trockel (1989, 1992) among many others. A more recent contribution to the topic is Cornet (2004). The problem with applying budget-constrained linear objective function maximization at the aggregate level or even for the entire consumer choice behaviour at the individual level, is that it leads to an extremely high degree of specialization in the consumption basket of the consumer, which is contrary to what is observed in reality. It is hardly the case that a consumer would consume just plain rice and nothing else, although for many Asians rice is a staple food and a meal would be incomplete without it. Keeping this in mind we propose in this paper a utility function which structurally combines component-wise linear utility with the components interacting with each other as in a Cobb-Douglas utility function.

We consider a model where there are several categories of non-monetary goods with each category consisting of possibly more than one type of good. We provide the demand analysis associated with budget-constrained linear utility function maximization for each category of goods, with the marginal rate of consumption expenditure-as a share of wealth- for each category being a positive constant and the aggregate marginal rate of consumption expenditure being less than or equal to one. The aggregate marginal rate of consumption expenditure is often and alternatively referred to as "marginal propensity to consume". The marginal rate of consumption expenditure for each category of goods as well as the marginal propensity to consume, is endogenously determined, by a budget-constrained "Cobb-Douglas with Linear Components" utility function maximization problem, where the utility function includes the possibility of savings as a variable, and which reduces to a budget-constrained linear utility function maximization problem for each category of goods. We obtain the Marshallian demand curves and the "Willingness To Pay" functions for all non-monetary goods. Marshallian demand curves and the "Willingness To Pay" functions, which are the workhorse of the related normative economics, are introduced, discussed and rigorously presented in Lahiri (2022a, 2022b), the genesis of which is available in Lahiri (2020).

In reality, a consumer may be simply allocating its aggregate wealth between savings, the different categories of goods that it consumes and then solve a budget-constrained linear utility function maximization problem for each category to obtain the quantities of goods consumed in each and every category. This could be conceived as a two-level optimization problem, where at the first level the "wealth allocation" problem is solved using a Cobb-Douglas utility function defined on savings and the different categories of consumption expenditure all measured in money units and hence each having a price of one per unit. The parameters of the Cobb-Douglas utility function for over-all budget allocation, determine the marginal rates of consumption expenditure and the marginal propensity to save. At the second stage, the expenditure allocated to each category of goods is used for buying goods in that category based on budget-constrained linear utility maximization. Alternatively, it could be viewed as a budget-constrained "Cobb-Douglas with Linear Components" utility function optimization problem, which determines the marginal propensity to save, the marginal rates of consumption expenditure for each category of goods as well as the quantity consumed of each good in each category as we have done here. The set of arrays of quantities of the goods in a particular category that are consumed by the consumer according to this composite maximization problem, coincides with the set of optimal solutions for the budget-constrained linear utility maximization problem for that category used in the two-level procedure. We also show how the budget-constrained Cobb-Douglas with Linear Components utility function maximization problem of a single consumer can be reduced to a finite number of budget-constrained linear utility maximizations problems all having the same number of variables. The number of such budget-constrained linear utility maximizations problems is equal to the number of "categories" of the non-monetary goods that are consumed. This helps to reduce the "equilibrium existence problem" in an exchange economy with preferences of consumers represented by Cobb-Douglas with Linear Components utility function to the "equilibrium existence problem" in a linear exchange model, whose solution can be computed using a finite algorithm due to Eaves (1976).

An important issue that is often not explicitly discussed in consumer demand theory, including the model we discuss here is the one concerning complementarities. In fact, the seminal and path-breaking contributions of Arrow and Hurwicz (1958) and Arrow, Block and

Hurwicz (1959) on the stability of competitive equilibrium assume, that the commodities under consideration are gross substitutes. While such a restriction may appear "irksome" to many, deeper thought may provide a justification for such an apparent lack of concern for complementarities. Take coffee, milk, and sugar for instance. There is a considerable amount of complementarity among the three goods. However, if we consider black coffee, black coffee with sugar, sugar-free coffee with milk and coffee with both milk and sugar added to it as four separate commodities in the category of hot beverages, then we have four goods that substitute one another while simultaneously incorporating the complementarity among their ingredients within our model. Consumers "purchase" complements from the grocery store, but "consume" a combination of the complements, that is possibly a substitute of a different combination of the complements. Hence, the goods and prices that the consumers are concerned with are not complements and their prices but various combinations of the complements and the prices of the various combinations. These combinations are indeed substitutes of one another. Hence, complementarities are "implicitly" taken care of in our model of consumer demand which "explicitly" concerns itself only with substitutes.

2. Demand analysis for budget-constrained linear utility maximization: Consider a consumer who consumes $L \geq 1$ non-monetary goods indexed by $j \in \{1,\ldots,L\}$ and money or monetary savings which is good L+1.

The price of the j^{th} good is denoted by p_j and is assumed to be strictly positive for all $j \in \{1,\ldots,L\}$. Let $w > 0$ be the "monetary wealth" of the consumer which is assumed to be variable.

Given $p_j > 0$ for $j \in \{1,\ldots,L\}$and $w > 0$, let $(<p_j| j \in \{1,\ldots,L\}>, w)$ denote the "price array-monetary wealth" pair.

For $j \in \{1,\ldots,L\}$ the initial endowment of the j^{th} non-monetary good is $\omega_j \geq 0$. We will assume that the initial endowments of the non-monetary goods are constant.

The "monetary value of the wealth" of the consumer $M(<p_j| j \in \{1,\ldots,L\}>, w) = \sum_{j=1}^{L} p_j\ \omega_j + w$. It is strictly positive, strictly increasing in "monetary wealth" and weakly increasing (non-decreasing) in the prices of the non-monetary goods. Clearly, $w \leq M(<p_j| j \in \{1,\ldots,L\}>, w)$.

When there is no scope for confusion we will write M instead of $M(<p_j| j \in \{1,\ldots,L\}>, w)$. When it is necessary to emphasize the dependence of M on the price of good j, we write $M(p_j)$ instead of M. When it is necessary to emphasise the dependence of M on w, we will write $M(w)$.

Let $S \geq 0$ denote the monetary savings of the consumer. It may be used for future consumption or for expenditure on non-monetary goods other than the L monetary goods we are concerned with here.

In this section we will consider savings to be a fixed non-negative share of the monetary value of the wealth owned by the consumer. Let $c \in (0,1]$ be the marginal propensity to consume so that the marginal propensity to save is the non-negative constant 1-c which is strictly less than one.

The utility function of the consumer for the L goods is given by the function $u:\mathbb{R}^L_+ \to \mathbb{R}$, such that $u(\xi) = \sum_{j=1}^{L} u_j\, \xi_j$ for all $\xi \in \mathbb{R}^L_+$, where for each $j \in \{1,\ldots,L\}$, u_j is a strictly positive real number.

The bundle of non-monetary goods chosen by the consumer ξ^* is an optimal solution to the linear programming maximization problem:

Maximize $\sum_{j=1}^{L} u_j\, \xi_j$

Subject to $\sum_{j=1}^{L} p_j\, \xi_j \leq cM$,

$\xi \in \mathbb{R}^L_+$.

It is easy to see that $\sum_{j=1}^{L} u_j\, \xi_j^*$ must be strictly positive and the budget constraint is satisfied with equality at the optimal solution.

Thus, $\xi_k^* > 0$ implies $k \in \underset{j \in \{1,\ldots,L\}}{\operatorname{argmax}} \frac{u_j}{p_j}$ and since we know that $\sum_{j=1}^{L} u_j\, \xi_j^* > 0$, it must be that $\underset{j \in \{1,\ldots,L\}}{\operatorname{argmax}} \frac{u_j}{p_j} \neq \phi$.

Given $p_j > 0$ for $j \in \{1,\ldots,L\}$and $w > 0$, let ($<p_j|\, j \in \{1,\ldots,L\}>$, w) let denote the "price array-monetary wealth" pair.

For L = 1, the entire consumption expenditure will be devoted to the single non-monetary consumption and hence its quantity demanded by the consumer will be $\frac{cM}{p_1}$.

Hence suppose L > 1.

Given $k \in \{1,\ldots,L\}$ and $p_j > 0$ for $j \in \{1,\ldots,L\}\backslash\{k\}$, let $a_k(<p_j|\, j \in \{1,\ldots,L\}\backslash\{k\}>) = \max\{\frac{u_j}{p_j}|\, j \in \{1,\ldots,L\}\backslash\{k\}\}$. When there is no scope for confusion we will let a_k denote $a_k(<p_j|\, j \in \{1,\ldots,L\}\backslash\{k\}>)$. $\frac{u_k}{a_k}$ is called the "**choke price**" of good k.

Given the "price array-monetary wealth" pair ($<p_j|\, j \in \{1,\ldots,L\}>$, w) consider the linear programming maximization problem formulated above.

Any $\xi \in \mathbb{R}^L_+$ such that for all $k \in \{1,\ldots,L\}$ (a) $\xi_k = 0$ if $p_k > \frac{u_k}{a_k}$, (b) $\xi_k = \frac{cM(p_k)}{p_k}$ if $p_k < \frac{u_k}{a_k}$, (c) $\xi_k \in [0, \frac{cM(\frac{u_k}{a_k})}{\frac{u_k}{a_k}}]$ if $p_k = \frac{u_k}{a_k}$

<u>satisfying</u>

$\sum_{\{k|p_k=\frac{u_k}{a_k}\}} p_k\, \xi_k = cM(p_k)$,

is a solution to the above linear programming maximization problem.

A "second level" of preference of the consumer, which acts as a tie-breaking rule, may be represented by an L-tuple of strictly positive real numbers α satisfying $\sum_{j=1}^{L} \alpha_j = 1$, such that at prices given by the L-tuple $p \in \mathbb{R}^L_{++}$, for $k \in \underset{j \in \{1,\ldots,L\}}{\operatorname{argmax}} \frac{u_j}{p_j}$, the consumer consumes

$(\frac{\alpha_k}{\sum_{i \in \underset{j \in \{1,\ldots,L\}}{\operatorname{argmax}} \frac{u_j}{p_j}} \alpha_i})\frac{\mathrm{cM}(p_k)}{p_k}$ of good k, and for $k \in \{1,\ldots,L\} \setminus \underset{j \in \{1,\ldots,L\}}{\operatorname{argmax}} \frac{u_j}{p_j}$, the consumer consumes nothing of the corresponding good.

Note that if $\frac{u_k}{p_k} > a_k$, then $\underset{j \in \{1,\ldots,L\}}{\operatorname{argmax}} \frac{u_j}{p_j} = \{k\}$.

For $j \in \{1,\ldots,L\}$, $\frac{u_j}{p_j}$ is called the "**bang per buck** for good j".

For $k \in \{1,\ldots,L\}$, the array $<X_k(<p_j|\, j \in \{1,\ldots,L\}>, w)|k \in \{1,\ldots,L\}>$ defined by (a) $X_k(<p_j|\, j \in \{1,\ldots,L\}>, w) = 0$ if $p_k > \frac{u_k}{a_k}$, (b) $X_k(<p_j|\, j \in \{1,\ldots,L\}>, w) = \frac{\mathrm{cM}(p_k)}{p_k}$ if $p_k < \frac{u_k}{a_k}$ and (c) $X_k(<p_j|\, j \in \{1,\ldots,L\}>, w) = (\frac{\alpha_k}{\sum_{i \in \underset{j \in \{1,\ldots,L\}}{\operatorname{argmax}} \frac{u_j}{p_j}} \alpha_i})\frac{cM(\frac{u_k}{a_k})}{(\frac{u_k}{a_k})}$ if $p_k = \frac{u_k}{a_k}$, is clearly a solution to the linear programming maximization problem formulated above.

For $k \in \{1,\ldots,L\}$, let X_k be the function on the set of all "price array-monetary wealth" pairs defined above. X_k is the **demand function** for good k or simply **demand** for good k.

However, what turns out to be more useful for demand analysis are the following concepts.

The **Marshallian demand curve** for good k, given $(<p_j|\, j \in \{1,\ldots,L\} \setminus \{k\}>, w)$ is the function $\mu_k(.|<p_j|\, j \in \{1,\ldots,L\} \setminus \{k\}>, w): \mathbb{R}_{++} \rightarrow \mathbb{R}_{+}$ such that:

$\mu_k(\xi_k|<p_j|\, j \in \{1,\ldots,L\} \setminus \{k\}>, w) = a_k$ for all $\xi_k \in (0, \frac{cM(\frac{u_k}{a_k})}{(\frac{u_k}{a_k})}]$

$\mu_k(\xi_k|<p_j|\, j \in \{1,\ldots,L\} \setminus \{k\}>, w) = \frac{c(\sum_{j \neq k} p_j \omega_j + w)}{\xi_k - c\omega_k}$ for all $\xi_k > \frac{cM(\frac{u_k}{a_k})}{(\frac{u_k}{a_k})}$,

Clearly $\mu_k(.|<p_j|\, j \in \{1,\ldots,L\} \setminus \{k\}>, w)$ is continuous on $\mathbb{R}_{++}$.

Note that as w increases, so does M(w) and hence $X_k(<p_j|\, j \in \{1,\ldots,L\} \setminus \{k\}>, w)$ for all values of $p_k < \frac{u_k}{a_k}$. Further $(\frac{\alpha_k}{\sum_{i \in \underset{j \in \{1,\ldots,L\}}{\operatorname{argmax}} \frac{u_j}{p_j}} \alpha_i})\frac{cM(\frac{u_k}{a_k})}{(\frac{u_k}{a_k})}$ increases as 'w' increases.

Thus, the Marshallian demand curve for good k, moves outwards (upwards) as w increases, although a_k and hence $\frac{u_k}{a_k}$ (the "choke price" for good k) remains invariant and hence the good is a normal good.

Further if $j,k \in \{1,\ldots,L\}$ with $j \neq k$, then if p_j increases, $\mu_k(\xi_k|<p_j|\, j \in \{1,\ldots,L\} \setminus \{k\}>, w)$ does not decrease for any value of ξ_k. In fact, if $\omega_j > 0$, then if p_j increases so does $M(p_j)$ and hence $\mu_k(\xi_k|<p_j|\, j \in \{1,\ldots,L\} \setminus \{k\}>, w)$ increases for all values of ξ_k in the region where the curve was strictly decreasing. If good j is the unique maximiser of the "bang per buck", then $\frac{u_k}{a_k}$ increases. Overall, the Marshallian demand curve for good k, does not decrease anywhere and may increase in the region where it was strictly decreasing if $\omega_j > 0$. If good j is the unique maximiser of the "bang per buck", then the value of its choke price rises. Thus, goods j and k are substitutes.

Hence all goods are normal and substitutes of one another.

In the case of L = 1, the **Marshallian demand curve** for good 1, given w is the function $\mu_k(.|w):\mathbb{R}_{++}\rightarrow \mathbb{R}_{+}$ such that:

$\mu_k(\xi_k|w) = \frac{cM}{\xi_1}$ for all $\xi_1 > 0$.

Thus, the Marshallian demand curve for good k, moves outwards (upwards) as w increases, and hence the good is a normal good.

The **Marshallian "market" demand curve** for a good, arising out of horizontal summation of individual Marshallian demand curves for the same good, will look like a step function, with each step connected to the subsequent one by a downward sloping curved (and not straight) line, with successive curved lines being "flatter".

Since in the case of L = 1, it is easy to see that the Marshallian market demand curve is a "rectangular hyperbola" we assume for the rest of this section that L > 1.

Let us focus on a market for a single non-monetary good among the L available to the consumers.

Suppose that some positive integer K, there are K + 1 different "choke prices" at which consumers enter the market. Let $<\theta-\kappa\Delta| \kappa = 0, 1, \ldots, K>$ for some θ, Δ satisfying $\theta - K\Delta > 0$ be the K different choke prices and suppose that for $\kappa = 0, 1, \ldots, K$, the maximum total quantity the consumers who enter the market at choke price $\theta-\kappa\Delta$ would want to buy of this non-monetary good is $\lambda^{(\kappa+1)} > 0$.

Letting $\lambda > 0$ be the aggregate quantity of good demanded in the market $\eta:\mathbb{R}_{++}\rightarrow \mathbb{R}_{+}$ be the function such that,

$$\eta(\lambda) = \theta \text{ for } \lambda\in(0, \lambda^{(1)}]$$

$$= \frac{\lambda^{(1)}\theta}{\lambda} \text{ for } \lambda\in(\lambda^{(1)}, \frac{\lambda^{(1)}\theta}{\theta-\Delta})$$

and for $\kappa = 1, \ldots, K-1$,

$$\eta(\lambda) = \theta-\kappa\Delta \text{ for } \lambda\in[\frac{1}{\theta-\kappa\Delta}\sum_{\chi=0}^{\kappa-1}\lambda^{(\chi+1)}(\theta-\chi\Delta), \frac{1}{\theta-\kappa\Delta}\sum_{\chi=0}^{\kappa-1}\lambda^{(\chi+1)}(\theta-\chi\Delta) + \lambda^{(\kappa+1)}]$$

$$= \frac{\sum_{\chi=0}^{\kappa}\lambda^{(\chi+1)}(\theta-\chi\Delta)}{\lambda} \text{ for } \lambda\in(\frac{1}{\theta-\kappa\Delta}\sum_{\chi=0}^{\kappa-1}\lambda^{(\chi+1)}(\theta-\chi\Delta) + \lambda^{(\kappa+1)}, \frac{\sum_{\chi=0}^{\kappa}\lambda^{(\chi+1)}(\theta-\chi\Delta)}{\theta-(\kappa+1)\Delta})$$

$$\eta(\lambda) = \theta-K\Delta \text{ for } \lambda\in[\frac{1}{\theta-K\Delta}\sum_{\chi=0}^{K-1}\lambda^{(\chi+1)}(\theta-\chi\Delta), \frac{1}{\theta-K\Delta}\sum_{\chi=0}^{K-1}\lambda^{(\chi+1)}(\theta-\chi\Delta) + \lambda^{(K+1)}]$$

$$= \frac{\sum_{\chi=0}^{K}\lambda^{(\chi+1)}(\theta-\chi\Delta)}{\lambda} \text{ for } \lambda > \frac{1}{\theta-K\Delta}\sum_{\chi=0}^{K-1}\lambda^{(\chi+1)}(\theta-\chi\Delta) + \lambda^{(K+1)}.$$

It is easy to show that a uniform price monopolist facing constant unit cost of production, will produce and sell (if it does so at all) <u>either</u> $\lambda^{(1)}$ at price θ per unit <u>or</u> $\frac{1}{\theta-\kappa\Delta}\sum_{\chi=0}^{\kappa-1}\lambda^{(\chi+1)}(\theta-\chi\Delta) + \lambda^{(\kappa+1)}$ at price $\theta-\kappa\Delta$ for some $\kappa\in\{1, \ldots, K\}$.

3. The "Willingness To Pay" function of an individual consumer: The concept discussed here is introduced, discussed and rigorously presented in Lahiri (2022a, 2022b), the genesis of which is available in Lahiri (2020).

Suppose $L > 1$.

The **"Willingness To Pay" function** for good k, given ($<p_j|\ j\in\{1,\ldots,L\}\backslash\{k\}>$, w) is the function $W_k(.|<p_j|\ j\in\{1,\ldots,L\}\backslash\{k\}>, w)$: $\mathbb{R}_{++}\rightarrow\mathbb{R}$ such that:

$W_k(\xi_k|<p_j|\ j\in\{1,\ldots,L\}\backslash\{k\}>, w) = a_k\xi_k$ for all $\xi_k\in(0, \frac{cM(\frac{u_k}{a_k})}{(\frac{u_k}{a_k})}]$

$W_k(\xi_k|<p_j|\ j\in\{1,\ldots,L\}\backslash\{k\}>, w) = a_k \frac{cM(\frac{u_k}{a_k})}{(\frac{u_k}{a_k})} + c(\sum_{j\neq k} p_j\omega_j + w)[\log(\xi_k-\omega_k) - \log\frac{cM(\frac{u_k}{a_k})}{(\frac{u_k}{a_k})}]$.

$W_k(\xi_k|<p_j|\ j\in\{1,\ldots,L\}\backslash\{k\}>, w)$ is the area under the Marshallian demand curve for good k, up to ξ_k.

If $L = 1$, then we consider an arbitrarily small positive real number $\varepsilon > 0$, the **"Willingness To Pay" function** for good 1, given w is the function $W_1(.|w)$: $\mathbb{R}_{++}\rightarrow\mathbb{R}$ such that:

$W(\xi_1|w) = cM[\log\xi_1 - \log\varepsilon]$ for all $\xi_1 > 0$.

4. Model of demand analysis for multiple categories of goods: We will now extend the preceding analysis to the situation where there are multiple categories of non-monetary goods consumed by the consumer, e.g., non-vegetarian foods, cereals, fruits, clothes, fuel etc. Thus, suppose there are 'm' categories of non-monetary goods for some positive integer 'm' greater than or equal to 2, and for each category $j\in\{1,\ldots,m\}$ there are $L(j)\geq 1$ non-monetary goods indexed by $i\in\{1,\ldots,L(j)\}$. In addition, there is money or monetary savings which is good L+1, where $L = \sum_{j=1}^{m} L(j)$.

The initial endowment of the i^{th} non-monetary good in the j^{th} category is $\omega_j(i)\geq 0$, the initial endowment of money is $w > 0$. In what follows, we will consider $\omega_j(i)$ for $(i,j)\in\bigcup_{k=1}^{m}\{1,\ldots,L(k)\}\times\{k\}$ to be fixed whereas the value of w is variable.

The price of the i^{th} good in the j^{th} category is denoted by $p_j(i)$ and is assumed to be strictly positive for all $(i,j)\in\bigcup_{k=1}^{m}\{1,\ldots,L(k)\}\times\{k\}$. For prices given by the L-tuple $p = (p_1,\ldots, p_m)\in\mathbb{R}_{++}^{L}$, where for each $j\in\{1,\ldots,m\}$, $p_j\in\mathbb{R}_{++}^{L(j)}$ with its i^{th} coordinate being $p_j(i)$ the monetary value of the wealth of the consumer is $\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\ \omega_j(i)]$ +w.

Let $S\geq 0$ denote the monetary savings of the consumer. It may be used for future consumption or for expenditure on non-monetary goods other than the L monetary goods we are concerned with here.

The utility function of the consumer for the L+1 goods is given by the function $U:\mathbb{R}_{+}^{L+1}\rightarrow\mathbb{R}$, such that $U(\xi,S) = \prod_{j=1}^{m}[(\sum_{i=1}^{L(j)} u_j(i)\ \xi_j(i))^{\beta_j}]S^{1-\beta}$ for all $(\xi,S)\in\mathbb{R}_{+}^{L+1}$, where $\xi = (\xi_1,\ldots,\xi_m)\in\prod_{j=1}^{m}\mathbb{R}_{+}^{L(j)}$ with the i^{th} coordinate of ξ_j being $\xi_j(i)$, and for each $j\in\{1,\ldots,L\}$, (a) $u_j(i)$ is a strictly positive real number for all $i\in\{1,\ldots,L(j)\}$, (b) β_j is a strictly positive real number with $\beta = \sum_{j=1}^{m}\beta_j\leq 1$.

Such a utility function may be referred to as a **Cobb-Douglas with Linear Components** utility function.

In case β = 1, the utility function reduces to the form $U(\xi,S) = \prod_{j=1}^{m}[(\sum_{i=1}^{L(j)} u_j(i)\ \xi_j(i))^{\beta_j}]$ for all $(\xi,S)\in\mathbb{R}_+^{L+1}$.

The consumption bundle chosen by the consumer (ξ^*,S^*) is an optimal solution to the maximization problem

Maximize U(ξ,S)

Subject to $\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\ \xi_j(i)] + S \le \sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\ \omega_j(i)] + w$

$(\xi,S)\in\mathbb{R}_+^{L+1}$.

It is easy to see that for each j∈{1,…,m}, $\sum_{i=1}^{L(j)} u_j(i)\ \xi_j^*(i)$ must all be strictly positive and the budget constraint is satisfied with equality at the optimal solution. S^* is strictly positive <u>if and only if</u> β < 1,

First let us suppose β < 1, so that $S^* > 0$.

Then, $\frac{\beta_k u_k(i)}{\sum_{i=1}^{L(k)} u_j(i)\xi_j^*(i)} - \frac{(1-\beta)p_k(i)}{\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\omega_j(i)] + w - \sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\xi_j^*(i)]} \le 0$ and $[\frac{\beta_k u_k(i)}{\sum_{i=1}^{L(k)} u_j(i)\xi_j^*(i)} - \frac{(1-\beta)p_k(i)}{\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\omega_j(i)] + w - \sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\xi_j^*(i)]}]\xi_k^*(i) = 0$, for all (i,k)∈ $\bigcup_{j=1}^{m}\{1,\dots,L(j)\}\times\{j\}$, where $S^* = \sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\ \omega_j(i)]\ +\ w\ -\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\ \xi_j^*(i)] > 0$.

If β < 1, then for all (i,k)∈ $\bigcup_{j=1}^{m}\{1,\dots,L(j)\}\times\{j\}$, $\frac{\beta_k u_k(i)\xi_k^*(i)}{\sum_{i=1}^{L(k)} u_j(i)\xi_j^*(i)} = \frac{(1-\beta)p_k(i)\xi_k^*(i)}{\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\omega_j(i)] + w - \sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\xi_j^*(i)]}$, which when summed over 'i' yields $\beta_k = \frac{(1-\beta)\sum_{i=1}^{L(k)} p_k(i)\xi_k^*(i)}{S^*}$, i.e., $\beta_k S^* = (1-\beta)\sum_{i=1}^{L(k)} p_k(i)\ \xi_k^*(i)$ for all k∈{1,…,m}.

Summing over k, we get $\beta S^* = (1-\beta)\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\ \xi_j^*(i)] = (1-\beta)[[\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\ \omega_j(i)] + w - S^*]$.

Thus, $S^* = (1-\beta)[\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\ \omega_j(i)] + w]$.

1-β is the "marginal propensity to save".

Thus, $\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\ \xi_j^*(i)] = \beta[\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\ \omega_j(i)] + w]$, whence β is the "marginal propensity to consume".

Further, $\beta_k S^* = (1-\beta)\sum_{i=1}^{L(k)} p_k(i)\ \xi_k^*(i)$ for all k∈{1,…,m} implies $\beta_k(1-\beta)[\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\ \omega_j(i)] + w] = (1-\beta)\sum_{i=1}^{L(k)} p_k(i)\ \xi_k^*(i)$ for all k∈{1,…,m}.

Thus, $\sum_{i=1}^{L(k)} p_k(i)\ \xi_k^*(i) = \beta_k\ [\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\ \omega_j(i)] + w]$ for all k∈{1,…,m}.

We may refer to β_k as the "marginal rate of consumption expenditure for the kth category of goods.

$\frac{\beta_k u_k(i)}{\sum_{i=1}^{L(k)} u_j(i)\xi_j^*(i)}$ - $\frac{(1-\beta)p_k}{S^*}$ ≤ 0 and $[\frac{\beta_k u_k(i)}{\sum_{i=1}^{L(k)} u_j(i)\xi_j^*(i)}$ - $\frac{(1-\beta)p_k}{S^*}]\xi_k^*(i)=0$, for all (i,k)∈ $\bigcup_{j=1}^{m}\{1,\ldots,L(j)\}\times\{j\}$ implies $\frac{u_k(i)}{p_k}$ - $\frac{(1-\beta)\sum_{i=1}^{L(k)} u_j(i)\xi_j^*(i)}{\beta_k S^*}$ ≤ 0, for all for all (i,k)∈ $\bigcup_{j=1}^{m}\{1,\ldots,L(j)\}\times\{j\}$ and $\frac{u_k(i)}{p_k}$ - $\frac{(1-\beta)\sum_{i=1}^{L(k)} u_j(i)\xi_j^*(i)}{\beta_k S^*}$ = 0 whenever $\xi_k^*(i) > 0$.

Thus for all k∈{1,…,m}, $\xi_k^*(i) > 0$ implies i∈ $\underset{h\in\{1,\ldots,L(k)\}}{\operatorname{argmax}} \frac{u_k(h)}{p_k(h)}$.

Now suppose β = 1. Then there exists η> 0 such that $\frac{\beta_k u_k(i)}{\sum_{i=1}^{L(k)} u_j(i)\xi_j^*(i)}$ - ηp_k(i) ≤ 0 and $[\frac{\beta_k u_k(i)}{\sum_{i=1}^{L(k)} u_j(i)\xi_j^*(i)}$ - ηp_k(i)]$\xi_k^*(i)=0$, for all (i,k)∈ $\bigcup_{j=1}^{m}\{1,\ldots,L(j)\}\times\{j\}$.

Thus, for all (i,k)∈ $\bigcup_{j=1}^{m}\{1,\ldots,L(j)\}\times\{j\}$, $\frac{\beta_k u_k(i)\xi_k^*(i)}{\sum_{i=1}^{L(k)} u_j(i)\xi_j^*(i)}$ = η$p_k(i)p_k\xi_k^*(i)$, which when summed over 'i' yields β_k = η$\sum_{i=1}^{L(k)} p_k(i)\,\xi_k^*(i)$.

$\sum_{k=1}^{m}\beta_k$= β = 1 and $\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\,\xi_j^*(i)]$ = $[\sum_{k=1}^{m}[\sum_{i=1}^{L(k)} p_k(i)\,\omega_k(i)] + \mathrm{w}]$ implies η = $\frac{1}{\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\omega_j(i)] + \mathrm{w}}$.

Thus, $\frac{\beta_k u_k(i)}{\sum_{i=1}^{L(k)} u_j(i)\xi_j^*(i)}$ - $\frac{p_k(i)}{\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\omega_j(i)] + \mathrm{w}}$ ≤ 0 and $[\frac{\beta_k u_k(i)}{\sum_{i=1}^{L(k)} u_j(i)\xi_j^*(i)}$ - $\frac{p_k(i)}{\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\omega_j(i)] + \mathrm{w}}]\xi_k^*(i)=0$, for all (i,k)∈ $\bigcup_{j=1}^{m}\{1,\ldots,L(j)\}\times\{j\}$.

Hence, $\frac{u_k(i)}{p_k(i)}$ - $\frac{\sum_{i=1}^{L(k)} u_j(i)\xi_j^*(i)}{\beta_k(\sum_{j=1}^{m}\left[\sum_{i=1}^{L(j)} p_j(i)\omega_j(i)\right] + \mathrm{w})}$ ≤ 0 and $[\frac{u_k(i)}{p_k(i)}$ - $\frac{\sum_{i=1}^{L(k)} u_j(i)\xi_j^*(i)}{\beta_k(\sum_{j=1}^{m}\left[\sum_{i=1}^{L(j)} p_j(i)\omega_j(i)\right] + \mathrm{w})}]\ \xi_k^*(i)=0$, for all (i,k)∈ $\bigcup_{j=1}^{m}\{1,\ldots,L(j)\}\times\{j\}$.

Thus, $\frac{u_k(i)}{p_k(i)}$ - $\frac{\sum_{i=1}^{L(k)} u_j(i)\xi_j^*(i)}{\beta_k(\sum_{j=1}^{m}\left[\sum_{i=1}^{L(j)} p_j(i)\omega_j(i)\right] + \mathrm{w})}$ ≤ 0 and $\frac{u_k(i)}{p_k(i)}$ - $\frac{\sum_{i=1}^{L(k)} u_j(i)\xi_j^*(i)}{\beta_k(\sum_{j=1}^{m}\left[\sum_{i=1}^{L(j)} p_j(i)\omega_j(i)\right] + \mathrm{w})}$ = 0 whenever $\xi_k^*(i) > 0$, whence once again we get that for all k∈{1,…,m}, $\xi_k^*(i) > 0$ implies i∈ $\underset{h\in\{1,\ldots,L(k)\}}{\operatorname{argmax}} \frac{u_k(h)}{p_k(h)}$.

$\frac{u_k(i)}{p_k}$ is the **"bang per buck"** for good i in the jth category.

Thus, for all k∈{1,…,m},ξ_k^* is an optimal solution for the linear programming maximization problem:

Maximize $\sum_{i=1}^{L(k)} u_k(i)\,\xi_k(i)$

Subject to $\sum_{i=1}^{L(k)} p_k(i)\xi_k(i)$ ≤ β_k $[\sum_{j=1}^{m}[\sum_{i=1}^{L(j)} p_j(i)\,\omega_j(i)] + \mathrm{w}]$

$\xi_k \in \mathbb{R}_+^{L(k)}$.

From here on, the analysis for each category of goods follows exactly as in sections 2 and 3.

If, as in the case of linear exchange models studied by Gale (1957, 1976) and Eisenberg and Gale (1959), we allow for linear utility functions with some- though not all- coefficients of the utility function to be zero, then each of the 'm' linear programming problems can be represented as the budget-constrained linear utility function maximization problem of a distinct consumer with $L = \sum_{j=1}^{m} L(j)$ non-monetary goods. Thus for $k \in \{1,\ldots,m\}$, the linear programming maximization problem faced by consumer k is:

Maximize $\sum_{h=1}^{m} \sum_{i=1}^{L(h)} u_h^k(i)\, \xi_h^k(i)$

Subject to $\sum_{h=1}^{m} \sum_{i=1}^{L(h)} p_h(i) \xi_h^k(i) \ \leq \beta_k \left[\sum_{j=1}^{m}\left[\sum_{i=1}^{L(j)} p_j(i)\, \omega_j(i)\right] + w\right]$

$\xi_h^k(i) \geq 0$ for all $i \in \{1,\ldots,L(h)\}$, $h \in \{1,\ldots,m\}$,

where for each $h \in \{1,\ldots,m\}$ and $i \in \{1,\ldots,L(h)\}$: $u_h^k(i) = u_h(i)$ if $k = h$ and $u_h^k(i) = 0$ otherwise.

It is easy to see that any optimal solution of the above problem must satisfy the property that for all $h \in \{1,\ldots,m\}\backslash\{k\}$ and $i \in \{1,\ldots,L(h)\}$, the quantity of the i^{th} good in category h <u>must be</u> zero.

This would reduce the "equilibrium existence problem" in an economy with preferences of consumers represented by Cobb-Douglas with Linear Components utility function to the "equilibrium existence problem" in a linear exchange model.

5. Conclusion: In the economy with a category of goods consisting of more than one good, it is natural to expect that more than one good in the category may get consumed although our discussion above points towards the possibility of a marked degree of specialization in individual consumption. For instance, with $L(j) = 2$, and with $p_j(1)$, $p_j(2)$ both strictly positive, the preferences of the consumers may be sufficiently heterogeneous. For some, the "bang per buck for good 1 in category j" is greater than the "bang per buck for good 2 in category j", whereas for others the "bang per buck for good 2 in category j" is greater than the "bang per buck for good 1 category j". The first group of consumers consume good 1 in category j and not good 2, whereas the second group of consumers consume 2 in category j and not good 1. A good example is one that has "fish" and "chicken" as the two goods in the category of non-vegetarian food items. For some the "bang per buck for fish" is greater than the "bang per buck for chicken", whereas for some others the "bang per buck for chicken" is greater than the "bang per buck for fish". The first group will consume fish, whereas the second group will consume chicken. Those whose "bang per buck for fish" is equal to "bang per buck for chicken", may wish to consume both fish and chicken and split their total expenditure among the two in a way that is determined by a "second level" of their preferences.

Note: In my gastronomic world, a "vegetarian meal" simply does not exist. Imposing "vegetarianism" on me is nothing short of "unwanted penetration" into my private life.

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