Price of Anarchy in Paving Matroid Congestion Games

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Abstract

Congestion games allow to model competitive resource sharing in various distributed systems. Pure Nash equilibria, that are stable outcomes of a game, could be far from being socially optimal. Our goal is to identify combinatorial structures that limit the inefficiency of equilibria. This question has been mainly investigated for congestion games defined over networks. Instead, we focus on symmetric matroid congestion games, where the strategies of every player are the bases of a given matroid. We derive new upper bounds on the Price of Anarchy (PoA) of congestion games defined over k-uniform matroids and paving matroids with delay functions in class $\mathcal{D}$. For both affine and polynomial delay functions, our bounds indicate that the inefficiency of pure Nash equilibria is limited by these combinatorial structures.

1 Introduction

Congestion games are a class of strategic games that provide an appealing paradigm to model resource sharing among selfish players. In a congestion game, a set of resources is given, and each player selects a feasible subset of the resources in order to minimize their cost function. The cost of a player's strategy is the sum of the delays of the resources selected by the player, and the delay of each resource is a function of the total number of players using it. The game is called *symmetric* if all players have the same strategy set. An example are *network congestion games*, where the resources are the arcs of a given digraph and the strategies of each player are paths in the network. Congestion games are practically relevant for various problems related to resource sharing in distributed systems, e.g., routing, network design and scheduling.

A *pure Nash equilibrium* (PNE) is a configuration where no player can decrease their cost by unilaterally deviating to another strategy, and it represents a stable outcome of the game. However, since the players act selfishly and independently in a non-cooperative fashion, a PNE might be far from minimizing the social cost, which is commonly defined as the sum of all players' costs. Two classic metrics for quantifying the inefficiency of equilibria are the *Price of Anarchy* (PoA) [19] and the *Price of Stability* (PoS) [3].

Congestion games always admit a PNE [27]. However, the complexity of computing a PNE in a congestion can be significantly affected by its combinatorial structure. While symmetric congestion games and asymmetric network congestion games are PLS-complete [13], Fabrikant et al. gave a strongly polynomial-time algorithm to find a PNE in symmetric network congestion games [13], which was later extended to symmetric totally unimodular congestion games by Del Pia et al. [12].

Our main goal is to better understand how the combinatorial structure of a congestion game might affect the inefficiency of equilibria. For *nonatomic* congestion games, where each single player has a negligible impact on congestion, structure has no impact on the PoA. In fact, Roughgarden proved that the worst-case PoA is equal to $\rho(\mathcal{D})$, a function that only depends on the class $\mathcal{D}$ of delay functions [29][1]. On the other hand, in *atomic* games, where each single player can affect the other players' decisions, there are structures that might reduce the inefficiency of equilibria. In the absence of structure, Awerbuch et al. [4, 5] and Christodoulou and Koutsoupias [10] independently provided an upper bound of 5/2 on the PoA for general atomic congestion games with affine delays. This bound can be improved to $(5N-2)/(2N+1)$ if the game is symmetric [10], where N is the number of players. For atomic congestion games with polynomial delays of highest degree p, Aland et al. [2] obtained exact values for the worst-case PoA, see also [10, 4, 5]. These exact values admit a lower bound of $\lfloor \phi_p \rfloor^{p+1}$ and an upper bound of ϕ_p^{p+1}, where $\phi_p \in \Theta(p/\ln p)$ is the unique nonnegative real solution to $(x+1)^p = x^{p+1}$. In the general case, Bhawalkar et al. [6] proved that the worst-case PoA can be achieved in symmetric games.

However, in the symmetric case the PoA can significantly decrease if the players' strategy sets have a special structure. Most of the existing literature has focused on *graph* structures in network congestion games. Lücking et al. [21, 22] studied symmetric congestion games on parallel links and proved that the PoA is 4/3 for linear delay functions. Fotakis later extended this result to network congestion games defined over extension-parallel networks and proved that for these networks the worst-case PoA is equal to $\rho(\mathcal{D})$, if the delays belong to class $\mathcal{D}$ [15]. Recently, Hao and Michini explored a further extension to the larger family of series-parallel networks. For affine delays, they proved that the worst-case PoA is in $[27/19, 2]$ [16]; for polynomial delays of highest degree p they showed that the worst-case PoA is at most $2^{p+1}-1$, which is significantly smaller than the worst-case PoA in general network congestion games [17].

In this paper we focus on another combinatorial structure, namely matroids. *Matroid congestion games* are congestion games where each player's strategy set is the set of bases of a given matroid. For this class of games, a PNE equilibrium can be efficiently computed, both in the symmetric and in the asymmetric case [1, 12]. Concerning the inefficiency of equilibria, Kleer and Schäfer [18] showed that the PoS in general matroids is upper bounded by $\rho(\mathcal{D})$ when the delay functions belong to class $\mathcal{D}$. However, the PoA of matroid congestion games is not well understood. For affine delays, the worst-case PoA of general congestion games, that is equal to 5/2, can be asymptotically achieved in asymmetric instances of singleton congestion games —that coincide with 1-uniform matroid congestion games— when the number of players goes to infinity [9]. In the symmetric case, the PoA of general matroid congestion games is still not completely understood. For graphic matroids and $N = 2, 3, 4$ or infinity the PoA can be as large as the worst-case PoA of symmetric congestion games, which is equal to $\frac{5N-2}{2N+1}$ [14]. However, for arbitrary N or different delay functions we don't know whether the the worst-case PoA of symmetric congestion games can be achieved by symmetric matroid congestion games. Interestingly, the worst-case PoA of k-uniform matroid congestion games with affine delays cannot exceed 1.4131 and it is equal 1.35188 when the number of players goes to infinity [11]. Moreover, for symmetric k-uniform matroid congestion games with polynomial delays of highest degree p the worst-case PoA is in $O(2^{p(p+1)})$ and in $\Omega(2^p)$ [20]. This indicates that the combinatorial structure of k-uniform matroids significantly limits the inefficiency of equilibria. However, k-uniform matroids are very special matroids, since every subset of the ground set of size at most k is independent. *Are there weaker matroid structures that affect the inefficiency of equilibria?* In this paper we focus on *paving matroids*, i.e., matroids whose circuits have cardinality greater than or equal to the matroid rank. Unlike k-uniform matroids, paving matroids exhibit a notable predominance within the enumeration of matroids. It has been conjectured that, in an asymptotic sense, the majority of matroids are paving matroids [23]. This conjecture holds if the ground set has size at most 9 [7, 24]. Pendavingh and van der Pol [26] more recently proved that, as the size of the ground set goes to infinity, the ratio of logarithms between the total number of

[1]The formal definition of $\rho(\mathcal{D})$ is recalled later in equation (2)

matroids and the number of sparse paving matroids, a subclass of paving matroids, converges to 1.

Our contributions. First, we provide a lower bound of 13/9 on the worst-case PoA for symmetric paving matroid congestion games with affine delays. This ratio is worse than the previously known best upper bound ≈ 1.41 on the PoA of symmetric congestion games with affine delay functions over k-uniform matroids, which are a subclass of paving matroids. Thus, relaxing the structure of players' strategy sets from uniform matroids to paving matroids can increase the inefficiency of pure Nash equilibria.

Theorem 1 *The worst-case PoA of symmetric paving matroid congestion games with affine delay functions is at least* 13/9.

We next turn to the question of finding upper bounds on the PoA of symmetric paving matroid congestion games. Given the class of delay functions $\mathcal{D}$, we define the parameter $z(\mathcal{D})$ as

$$z(\mathcal{D}) = \sup_{d \in \mathcal{D},\ x \in \mathbb{N}^+} \frac{d(x+1)}{d(x)}.$$

Since the delay functions $d(x)$ are non-negative and non-decreasing, we have $z(\mathcal{D}) \geq 1$. Our first main result is an upper bound on the worst-case PoA in symmetric paving matroid congestion games with delay functions in class $\mathcal{D}$.

Theorem 2 *The PoA of symmetric paving matroid congestion games with delay functions in class $\mathcal{D}$ is at most $z(\mathcal{D})^2\rho(\mathcal{D})$.*

When $\mathcal{D}$ is the class of polynomial functions of maximum degree p, we have $z(\mathcal{D}) = 2^p$ and $\rho(\mathcal{D}) \in \Theta(p/\ln p)$. Thus, the worst-case PoA is in $O(4^p p/\ln p)$. For $p \geq 6$ our bound is smaller than the worst-case PoA that can be achieved in general symmetric congestion games, that is in $\Theta(p/\ln p)^{p+1}$ [2]. Thus, the worst-case PoA of symmetric congestion games *cannot* be achieved in paving matroids.

We also prove —with a substantially different approach— that this is the case for $p = 1$, i.e., when the delay functions are affine. In this case, the worst-case PoA for general symmetric congestion games is 5/2.

Theorem 3 *The PoA of symmetric paving matroid congestion games with affine delay functions is at most* 17/7.

Finally, the approach used to prove Theorem 2 also provides a new upper bound on the worst-case PoA in symmetric k-uniform matroid congestion games with delay functions in class $\mathcal{D}$.

Theorem 4 *The PoA of symmetric k-uniform matroid congestion games with delay functions in class $\mathcal{D}$ is at most $z(\mathcal{D})\rho(\mathcal{D})$.*

When $\mathcal{D}$ is the class of polynomial functions of maximum degree p, we obtain that the worst-case PoA is in $O(2^p p/\ln p)$. This significantly improves on the previously known upper bound of $O(2^{p(p+1)})$ [20] and partially closes the gap with the lower bound of $\Omega(2^p)$ [20].

Our approach. Our approach is based on representing the "difference" between a PNE f and a social optimum o of a matroid congestion game as a flow on a complete directed graph, whose nodes correspond to the resources. Each unit of flow on arc (r, r') corresponds to a player replacing r with r' in their strategy. The *overloaded* resources (those with more players in f than in the o) act as supply nodes and the *underloaded* resources (those with more players in the o than in the f) act as demand nodes. If every path from supply u to demand v is such that the costs of

u and v in the PNE are related through a constant α, then we can establish that the PoA is at most $\alpha\rho(\mathcal{D})$ (Theorem 7). When the delay functions are in class $\mathcal{D}$, we can determine values of α for the case where the matroid is k-uniform (Lemma 8) or paving (Lemma 10). These results allow us to establish Theorems 2 and 4. Note that our definition of flows generalizes the idea of the "augmenting paths" used by de Jong et al. [11], extending it from k-uniform matroids with affine delay functions to general matroids with delay functions in class $\mathcal{D}$.

For a paving matroid congestion game with affine delays we require a different approach in order to prove Theorem 3. Given f, o and the associated flow, we construct another congestion game with two states s and q such that $\frac{\mathrm{cost}(f)}{\mathrm{cost}(o)} \leq \frac{\mathrm{cost}(s)}{\mathrm{cost}(q)}$. We show that s and q and their associated flow satisfy some special properties, which are used to establish that $\mathrm{cost}(s)/\mathrm{cost}(q) \leq 17/7$ (Theorem 12).

2 Preliminaries

In this section, we first recall some basics of matroid theory and then we introduce some fundamental notions of congestion games.

Matroids. A *matroid* is a pair $(R, \mathcal{I})$ where the *ground set* R consists of a finite set of elements and $\mathcal{I}$ is a nonempty collection of subsets of R such that: (i) if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$; and (ii) if $I, J \in \mathcal{I}$ and $|I| < |J|$, then $I \cup \{z\} \in \mathcal{I}$ for some $z \in J \setminus I$. Given a matroid $M = (R, \mathcal{I})$, a subset I of R is called *independent* if I belongs to $\mathcal{I}$, and dependent otherwise. A subset $B \subseteq R$ is called a *basis* if B is an inclusion-wise maximal independent subset. That is, $B \in \mathcal{I}$ and there is no $Z \in \mathcal{I}$ with $B \subset Z \subseteq R$. The common size of all bases is called the *rank* of the matroid, denoted by $r(M)$. A *circuit* of a matroid is an inclusion-wise minimal dependent set. For every basis B and every element in $R \setminus B$, there is a unique circuit contained in $B \cup \{x\}$, that is called a *fundamental circuit*. Next, we introduce the *bijective basis-exchange property*:

Theorem 5 ([8]) *Let $\mathcal{B}$ be the collection of bases of a matroid. For any $B, B' \in \mathcal{B}$, there is a bijection $\pi : B \to B'$ from B to B', such that for every $x \in B \setminus B'$, $B \setminus \{x\} \cup \{\pi(x)\}$ is a basis.*

A matroid is called k*-uniform* matroid if its independent sets are all the subsets of R of cardinality at most k, i.e. every $k+1$-element subset of R is a circuit. A matroid is called *paving* matroid if every circuit of M has cardinality $r(M)$ or $r(M) + 1$. The following proposition characterizes paving matroids in terms of their circuits.

Proposition 6 ([25]) *Let $\mathcal{C}$ be a collection of non-empty subsets of a set R such that each each member of $\mathcal{C}$ has size either t or $t+1$. Let $\mathcal{C}' \subseteq \mathcal{C}$ consist only of the t-element members of $\mathcal{C}$. Then $\mathcal{C}$ is the set of circuits of a paving matroid on R of rank t if and only if*

1. *if two distinct members C_1 and C_2 of $\mathcal{C}'$ have $t-1$ common elements, then every t-element subset of $C_1 \cup C_2$ is in $\mathcal{C}'$; and*

2. *$\mathcal{C} \setminus \mathcal{C}'$ consists of all the $(t+1)$-element subsets of R that contains no member of $\mathcal{C}'$.*

Congestion games. We consider a congestion game with N players and resources set R. For $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, \ldots, n\}$. The set $X^i \subseteq R$ is the strategy set of player i. We call the game *symmetric* if all the players have the same strategy set, i.e. $X^i = X^j$ for all $i, j \in [N]$. A *state* of the game is a strategy profile $s = (s^1, \ldots, s^N)$ where $s^i \in X^i$ is the strategy chosen by player i, for $i \in [N]$. The set of states of the game is denoted by $X = X^1 \times \cdots \times X^N$.

For each $r \in R$ we have a nondecreasing delay function $d_r : [N] \to \mathbb{R}_{\geq 0}$. Given a state s we denote the number of players using resource r by s_r. Each player using r incurs a cost equal to $d_r(s_r)$, i.e., the cost of r depends on the total number of players that use r in s. Since d_r is a nondecreasing function, $d_r(j+1) \geq d_r(j)$ for $j \in [N-1]$, which models the effect of congestion. We denote the cost of a resource r with respect to state s by $\text{cost}_s(r) = d_r(s_r)$. We also define $\text{cost}_s^+(r) = d_r(s_r+1)$. Finally, the *social cost* of state s is denoted by $\text{cost}(s) = \sum_{r\in R} s_r d_r(s_r) = \sum_{r\in R} s_r \text{cost}_s(r)$.

Matroid congestion games. A *matroid congestion game* is a congestion game where the strategy set of each player i is the set of bases $\mathcal{B}_i$ of a given matroid $M_i = (R_i \subseteq R, \mathcal{I}_i)$. For an arbitrary state s of the matroid congestion game, we denote by B_s^i the strategy of player i in s. A *paving* matroid congestion game is a matroid congestion game where M_i is a paving matroid for all $i \in [N]$. A *k-uniform* matroid congestion game is a congestion game where M_i is a k-uniform matroid for all $i \in [N]$ and $k \in [\min_i |R_i|]$.

Pure Nash Equilibria and social optima. A *pure Nash equilibrium* (PNE) is a state $s = (s^1, \ldots, s^i, \ldots, s^N)$ such that, for each $i \in [N]$ we have

$$\text{cost}_s(s^i) \leq \text{cost}_{\tilde{s}}(\tilde{s}^i) \qquad \forall \tilde{s} = (s^1, \ldots, \tilde{s}^i, \ldots, s^N) \in X.$$

A PNE represents a stable outcome of the game, since no player $i \in [N]$ can improve their cost if they select a different strategy $\tilde{s}^i$.

We are also interested in a *social optimum* (SO), which is a state that minimizes $\text{cost}(s)$ over all the states $s \in X$. The *Price of Anarchy* (PoA) is the maximum ratio $\frac{\text{cost}(f)}{\text{cost}(o)}$ such that o is a SO and f is a PNE. In other words, to compute the PoA we consider the "worst" PNE, i.e., a PNE whose social cost is as large as possible.

3 Upper bounds on the PoA for delays in class $\mathcal{D}$

In this section, our goal is to prove Theorems 2 and 4. For a matroid congestion game over resource set R, we let $G = (R, E)$ be a complete directed graph, where the nodes correspond to the resources in R. Let s and q be two states of the congestion game. We define the following two sets:

$$R^-(s,q) = \{r \in R : s_r > q_r\} \qquad\qquad R^+(s,q) = \{r \in R : s_r < q_r\},$$

and we let $l = \sum_{r\in R^-(s,q)}(s_r - q_r) = \sum_{r\in R^+(s,q)}(q_r - s_r)$. In G, every node $r \in R^-$ has supply $s_r - q_r$, and every node $r \in R^+$ has demand $q_r - s_r$. A (single-commodity) flow $F \in \mathbb{Z}^{R\times R}$ in G is a non-negative vector such that for every node $r \in R$

$$F(\delta^-(r)) - F(\delta^+(r)) = q_r - s_r, \tag{1}$$

where $\delta^-(r)$ contains all the arcs whose head is r and $\delta^+(r)$ contains all the arcs whose tail is r. We call F a (s,q)-*difference flow.* Note that the above definitions can be applied to a generic congestion game. For a matroid congestion game, we can construct a special (s,q)-difference flow F, that we call (s,q)-*exchange flow*, as follows. According to Theorem 5, for each pair (B_s^i, B_q^i), there is a bijection $\pi^i(x) : B_s^i \to B_q^i$ such that for every $r \in B_s^i \setminus B_q^i$ there is an unique $\pi^i(r) \in B_q^i \setminus B_s^i$ and $B_s^i \setminus \{r\} \cup \{\pi^i(r)\} \in \mathcal{B}$. Starting from the zero vector, for every $i \in [N]$, $r \in B_s^i \setminus B_q^i$, we add one unit of flow to the arc $(r, \pi^i(r))$ to G in order to obtain F. We observe that F can be decomposed into l paths, each one starting from a node in $R^-(s,q)$ and ending at a node in $R^+(s,q)$, and carrying one unit of flow. Each path in the exchange flow can be interpreted as a sequence of resource exchanges such that each arc (r, r') in the path corresponds to some player replacing resource r with resource r' in their strategy.

In the next theorem, we consider an (f,o)-exchange flow. For any (u,v)-path from $R^-(f,o)$ to

$R^+(f,o)$, if $\text{cost}_f(u)$ is equal to at least a fraction α of $\text{cost}^+_f(v)$, then we can upper bound the ratio between the social costs of f and o by $\alpha\rho(\mathcal{D})$. We recall that the function $\rho(\mathcal{D})$, initially introduced by Roughgarden [28], is defined as $\rho(\mathcal{D}) := \sup_{d\in\mathcal{D}} \rho(d)$, where

$$\rho(d) = \sup_{x\geq y\geq 0} \frac{xd(x)}{yd(y) + (x-y)d(x)}. \tag{2}$$

Theorem 7 *Let F be an (f,o)-exchange flow. Let $R^- = R^-(f,o)$ and $R^+ = R^+(f,o)$. For all paths p contained in F from $u \in R^-$ to $v \in R^+$, if $\alpha \text{cost}^+_f(v) \geq \text{cost}_f(u)$ for some $\alpha \geq 1$ then we have $\text{cost}(f) \leq \alpha\rho(\mathcal{D})\text{cost}(o)$.*

Proof. For every resource $r \in R^-$, inequality (2) and $\alpha \geq 1$ imply

$$\begin{aligned} f_r \text{cost}_f(r) = f_r d_r(f_r) &\leq \rho(\mathcal{D})(o_r d(o_r) + (f_r - o_r)d(f_r)) \\ &\leq \rho(\mathcal{D})(\alpha o_r d(o_r) + (f_r - o_r)d(f_r)) \end{aligned} \tag{3}$$

Let $\{p_1, \ldots, p_l\}$ be an arbitrary decomposition of the flow F, where each p_k is from r_k^- to r_k^+ such that $r_k^- \in R^-$ and $r_k^+ \in R^+$. We have

$$\begin{aligned} \sum_{r\in R^+} (o_r - f_r)\text{cost}_o(r) &= \sum_{k=1}^{l} \text{cost}_o(r_k^+) \geq \sum_{k=1}^{l} \text{cost}^+_f(r_k^+) \\ &\geq \sum_{k=1}^{l} \frac{1}{\alpha}\text{cost}_f(r_k^-) = \frac{1}{\alpha} \sum_{r\in R^-} (f_r - o_r)\text{cost}_f(r), \end{aligned} \tag{4}$$

where the equalities hold by the definition of F and equality (1), the first inequality holds because of the definition of R^+, and the second inequality holds because by assumption. Let $\bar{R} = \{r \in R : f_r = o_r\} = R \setminus (R^- \cup R^+)$.

$$\begin{aligned} \text{cost}(f) &= \sum_{r\in R^-} f_r\text{cost}_f(r) + \sum_{r\in R^+} f_r\text{cost}_f(r) + \sum_{r\in \bar{R}} f_r\text{cost}_f(r) \\ &\leq \sum_{r\in R^-} f_r\text{cost}_f(r) + \sum_{r\in R^+} f_r\text{cost}_o(r) + \sum_{r\in \bar{R}} o_r\text{cost}_o(r) \\ &\leq \rho(\mathcal{D}) \sum_{r\in R^-} \alpha o_r\text{cost}_o(r) + \rho(\mathcal{D}) \sum_{r\in R^-} (f_r - o_r)\text{cost}_f(r) \\ &\quad + \alpha\rho(\mathcal{D}) \sum_{r\in R^+} f_r\text{cost}_o(r) + \sum_{r\in \bar{R}} o_r\text{cost}_o(r) \\ &\leq \rho(\mathcal{D}) \sum_{r\in R^-} \alpha o_r\text{cost}_o(r) + \rho(\mathcal{D}) \sum_{r\in R^+} (o_r - f_r)\text{cost}_o(r) \\ &\quad + \alpha\rho(\mathcal{D}) \sum_{r\in R^+} f_r\text{cost}_o(r) + \sum_{r\in \bar{R}} o_r\text{cost}_o(r) \\ &= \alpha\rho(\mathcal{D}) \sum_{r\in R^-\cup R^+} o_r\text{cost}_o(r) + \sum_{r\in \bar{R}} o_r\text{cost}_o(r) \leq \alpha\rho(\mathcal{D})\text{cost}(o). \end{aligned}$$

The first inequality holds because of the definition of R^+ and $\bar{R}$; the second inequality holds because of inequality (3) and $\alpha \geq 1$, $\rho(\mathcal{D}) \geq 1$; the third inequality follows by applying (4); the last inequality follows because $\alpha \geq 1$, $\rho(\mathcal{D}) \geq 1$. $\square$

We emphasize that the bound on the PoA provided by Theorem 7 is not restricted to the class of paving matroids. In fact, the assumption of the theorem involves an exchange flow, which is defined for any matroid, and a parameter α. Thus, for any matroid, if we are able to find such α, we are able to bound the PoA.

The next lemma implies that for k-uniform matroids $\alpha = z(\mathcal{D})$ satisfies the assumption of Theorem 7. This lemma is an extension of Lemma 5 in [11] from affine delay functions to general delay functions. Moreover, it can be verified that for polynomial delay functions the bound established in Lemma 8 is tight.

Lemma 8 *Suppose M is a k-uniform matroid. Let q be an arbitrary state of the game. For every $u \in R^-(f,q)$ and $v \in R^+(f,q)$ we have $z(\mathcal{D})\mathit{cost}_f^+(v) \geq \mathit{cost}_f(u)$.*

Proof. Let u^* be the most expensive resource in $R^-(f,q)$, i.e., $\text{cost}_f(r) \leq \text{cost}_f(u^*)$ for every resource $r \in R^-(f,q)$. To prove the lemma, we will show that for every $v \in R^+(f,q)$ we have $z(\mathcal{D})\text{cost}_f^+(v) \geq \text{cost}_f(u^*)$. By contradiction, suppose there exists a resource $v \in R^+(f,q)$ such that

$$z(\mathcal{D})\text{cost}_f^+(v) < \text{cost}_f(u^*). \tag{5}$$

Since $q_v > f_v$, we have $f_v < N$, thus there exists at least one player i who does not use v in f, i.e., $v \notin B_f^i$. We claim that, for all $r \in B_f^i$, we have

$$\text{cost}_f(r) \leq \text{cost}_f^+(v). \tag{6}$$

This follows from the fact that, since M is a k-uniform matroid $B_f^i \setminus \{r\} \cup \{v\}$ is a basis of M for all $r \in B_f^i$. Thus, if (6) did not hold, player i could deviate from $r \in B_f^i$ to v to decrease their cost. As a consequence, $z(\mathcal{D})\text{cost}_f^+(v) < \text{cost}_f(u^*)$ implies that $v \notin B_f^i$. Moreover, recalling that $z(\mathcal{D}) \geq 1$, we have $\text{cost}_f^+(r) \leq z(\mathcal{D})\text{cost}_f(r)$ for all $r \in R$. Combining this with (5) and (6), we obtain that, for all $r \in B_f^i$

$$\text{cost}_f^+(r) < \text{cost}_f(u^*). \tag{7}$$

Note that (7) implies $u^* \notin B_f^i$. Since $u^* \in R^-(f,q)$, $f_{u^*} > o_{u^*} \geq 0$, thus there is at least one player j using u^* in f, i.e., $u^* \in B_f^j$. Since M is a k-uniform matroid, $B_f^j \setminus \{u^*\} \cup \{r\}$ is a basis of M for all $r \in B_f^i$. Moreover, since $u^* \notin B_f^i$ and $|B_f^i| = |B_f^j| = k$, we can conclude that $|B_f^i \setminus B_f^j| \geq 1$. I.e. there exists at least one resource $r^* \in B_f^i$ such that $r^* \notin B_f^j$. Thus, by (7), player j could deviate from u^* to r^* to decrease their cost. This contradicts the fact that f is a PNE. $\square$

Applying Theorem 7 and Lemma 8, we can immediately derive Theorem 4.

Next, we show that for paving matroids $\alpha = z(\mathcal{D})^2$ satisfies the assumption of Theorem 7. To this purpose, we first introduce an auxiliary result.

Lemma 9 *Consider a symmetric matroid congestion game with delays in class $\mathcal{D}$. Let f be a PNE, and o a SO. Let v be a resource that is not used by player i in f and let C_v^i be the unique circuit in $B_f^i \cup \{v\}$. Then, for all $r \in C_v^i$ we have $\mathit{cost}_f^+(r) \leq z(\mathcal{D})\mathit{cost}_f(r) \leq z(\mathcal{D})\mathit{cost}_f^+(v)$.*

Proof. Assume that there exists a resource $r \in C_v^i$ such that $\text{cost}_f(r) > \text{cost}_f^+(v)$. Since C_v^i is the unique circuit that satisfies $C_v^i \setminus \{v\} \subseteq B_f^i$, we have that $B_f^i \setminus \{r\} \cup \{v\} \in \mathcal{B}$, i.e., exchanging

r and v defines a feasible strategy for player i. By performing this exchange player i is able to lower their cost, thus contradicting the fact that f is a PNE. Thus, we can conclude that for each $r \in C_v^i$ we have $\text{cost}_f(r) \le \text{cost}_f^+(v)$. This implies that $z(\mathcal{D})\text{cost}_f(r) \le z(\mathcal{D})\text{cost}_f^+(v)$. Finally, by the definition of $z(\mathcal{D})$, thus we have $\text{cost}_f^+(r) \le z(\mathcal{D})\text{cost}_f(r)$. □

For an arbitrary state q, consider an (f,q)-exchange flow F and any path contained in it starting from a node $u \in R^-(f,q)$ and ending at a node $v \in R^+(f,q)$. If the matroid is paving, the next lemma implies that $\text{cost}_f^+(v)$ cannot be smaller than a fraction of $\text{cost}_f(u)$.

Lemma 10 *Suppose M is a paving matroid with $r(M) = t \ge 1$. Let q be an arbitrary state of the game and let F be an (f,q)-exchange flow. Let $R^- = R^-(f,q)$ and $R^+ = R^+(f,q)$. For all paths p contained in F from $u \in R^-$ to $v \in R^+$, and for every resource r in p we have $cost_f(r) \le z(\mathcal{D})^2 cost_f^+(v)$.*

Proof. Let r^* be the most expensive resource of path p in f, i.e., $\text{cost}_f(r) \le \text{cost}_f(r^*)$ for every resource r in p. Since $t \ge 1$ we know that r^* is used by at least one player in f. We will prove $\text{cost}_f(r^*) \le z(\mathcal{D})^2\text{cost}_f^+(v)$. By contradiction, suppose

$$\text{cost}_f(r^*) > z(\mathcal{D})^2\text{cost}_f^+(v). \tag{8}$$

Define

$$S = \{r \in R : \text{cost}_f^+(r) < \text{cost}_f(r^*)\}, \quad \bar{S} = \{r \in R : z(\mathcal{D})\text{cost}_f^+(r) < \text{cost}_f(r^*)\}.$$

Since $z(\mathcal{D}) \ge 1$, we have $\bar{S} \subseteq S$. Moreover, we have the following property.

Claim 1. $|\bar{S}| \ge t$.

Proof of claim. Since v is the last node in p, there exists a player j such that $v \notin B_f^j$. Let C_v^j be the fundamental circuit in $B_f^j \cup \{v\}$. By Lemma 9, for all $r \in C_v^j$ we have $\text{cost}_f^+(r) \le z(\mathcal{D})\text{cost}_f^+(v)$. Thus:

$$z(\mathcal{D})\text{cost}_f^+(r) \le z(\mathcal{D})^2\text{cost}_f^+(v) < \text{cost}_f(r^*),$$

where the last inequality comes from (8). This implies that $C_v^j \subseteq \bar{S}$. Since in a paving matroid of rank t every circuit has size at least t we obtain $|\bar{S}| \ge t$. ◇

Note that $v \in S$, since $z(\mathcal{D}) \ge 1$, and $r^* \notin S$. Since p traverses both r^* and v, there is an arc (a,b) in p such that $a \notin S$ and $b \in S$. Since (a,b) is contained in F there exists a player i such that $a \in B_f^i$, $b \notin B_f^i$ and $B_f^i \setminus \{a\} \cup \{b\} \in \mathcal{B}$.

First, $a \in B_f^i \setminus S = B_f^i \setminus (B_f^i \cap S)$. Thus $1 \le t - |B_f^i \cap S|$. We have

$$|\bar{S} \setminus B_f^i| = |\bar{S}| - |\bar{S} \cap B_f^i| \ge t - |\bar{S} \cap B_f^i| \ge t - |S \cap B_f^i| \ge t + (1-t) = 1,$$

where the first inequality follows from Claim 1. Thus $\bar{S} \setminus B_f^i \neq \emptyset$. Let $w \in \bar{S} \setminus B_f^i$. Let C_w^i be the fundamental circuit in $B_f^i \cup \{w\}$. By Lemma 9 for all $r \in C_w^i$ we have

$$\text{cost}_f^+(r) \le z(\mathcal{D})\text{cost}_f^+(w) < \text{cost}_f(r^*),$$

where the last inequality holds because $w \in \bar{S}$.

This implies $C_w^i \subseteq S$. Recall that $C_w^i \setminus \{w\} \subseteq B_f^i$. Since the matroid is paving, $|C_w^i \setminus \{w\}| \geq t-1$. Finally, as $a \in B_f^i \setminus S$ we can conclude that $B_f^i \setminus \{a\} = C_w^i \setminus \{w\} \subseteq S$. Since $b \in S$, we have $B_f^i \setminus \{a\} \cup \{b\} \subseteq S$. We now prove that every t-element subset of S is a circuit. This immediately contradicts the fact that $B_f^i \setminus \{a\} \cup \{b\}$ is a basis.

Claim 2. *Every t-element subset of S is a circuit of the paving matroid M.*

Proof of claim. Let h be a player such that $r^* \in B_f^h$ and let r be an arbitrary resource in $S \setminus B_f^h$. We show that $B_f^h \setminus \{r^*\} \cup \{r\}$ is a circuit. Consider the fundamental circuit C_r^h in $B_f^h \cup \{r\}$. We argue that r^* is not in C_r^h. If that was the case, we would have $\text{cost}_f^+(r) \geq \text{cost}_f(r^*)$ by Lemma 9, which contradicts $r \in S$. Since we have a paving matroid $C_r^h \geq t$, thus $C_r^h = \{r\} \cup B_f^h \setminus \{r^*\}$. This proves that $B_f^h \setminus \{r^*\}$ forms a circuit with every resource $r \in S \setminus B_f^h$. By applying the first statement in Proposition 6 we can conclude that every t-element subset of $S \cup B_f^h \setminus \{r^*\}$ is a circuit. By the definition of S we have $r^* \notin S$, so $S \subseteq S \cup B_f^h \setminus \{r^*\}$ and every t-element subset of S is a circuit. $\diamond$

$\square$

Lemma 10 implies that for paving matroids $\alpha = z(D)^2$ satisfies the assumption of Theorem 7. Thus, Theorem 2 directly follows.

Remark 11 *It can be verified that the bound of Lemma 10 is tight for polynomial delay functions, however we conjecture that the bound of Theorem 2 is* not *tight for the same class of delays. In fact, instances where the bound of Lemma 10 is tight can have PoA smaller than the upper bound of Theorem 2. An intuitive explanation is the following: when the bound in Lemma 10 is tight, in the PNE there is an "expensive" resource used by many players and a "cheap" resource used by few players. For this state to be a PNE, the circuits of the matroid must prevent single player deviations where the expensive resource is replaced by the cheap one. The existence of these circuits requires the existence of other resources with comparable costs both in the PNE and in the SO (this is implied by Lemma 9). As a result, the PoA in these instances will be lower than the upper bound of Theorem 2.*

4 Lower bound on the PoA of paving matroid congestion games with affine delays

In this section, we consider symmetric paving matroid congestion game with affine delays, i.e., we assume that the delay function of each resource $r \in R$ is of the form $d_r(x) = a_r x + b_r$ with $a_r \geq 0$ and $b_r \geq 0$. Our goal is to prove Theorem 1, stating that the worst-case PoA is at least 13/9. This lower bound is higher than the previously best known lower bound of ≈ 1.35, which is achieved in the symmetric k-uniform matroid congestion games [11]. Moreover, this lower bound indicates that the upper bound of ≈ 1.41 for symmetric k-uniform matroid congestion games does not hold for paving matroids.

Proof. [Proof of Theorem 1] We prove the theorem by constructing an instance of a symmetric paving matroid congestion game with affine delays that achieves the PoA of 13/9. Let $R = \{r_1\} \cup R_2 \cup R_3$, where $R_2 = \{r_2, r_3, r_4, r_5\}$ and $R_3 = \{r_6, \ldots, r_{13}\}$. Let

$$C_1 = \{\{r_1, r_{6+2i}, r_{6+2i+1}\} : \forall i \in \{0, 1, 2, 3\}\},$$
$$C_2 = \{S \subset R : |S| = 4 \text{ and } S' \not\subset S, \forall S' \in C_1\}.$$

Let $C = C_1 \cup C_2$. Using Proposition 6 with $\mathcal{C}' = C_1$ and $\mathcal{C} = C$ we can easily check that C is the set of circuits for a paving matroid of rank 3 defined over R.

Next we define a symmetric congestion game over $\mathcal{M}$. Let the delay function of r_1 be $d_{r_1}(x) = 1$, and for $i \in \{2, 3, \ldots, 13\}$ let $d_{r_i}(x) = x$. Let the number of players be $N = 6$. The strategy set of each player is the set of bases of the paving matroid. In a PNE, players 1 and 2 select resources $\{r_1, r_2, r_3\}$ and for $i \in \{3, 4, 5, 6\}$, player i selects resources $\{r_4, r_6, r_7\}$, $\{r_4, r_8, r_9\}$, $\{r_5, r_{10}, r_{11}\}$, $\{r_5, r_{12}, r_{13}\}$, respectively. Note that players will not deviate from r_4 or r_5 to r_1, since this would form a circuit in C_1. The social cost of this PNE state is 26. In the SO, each player $i \in [N]$ selects resources $\{r_1, r_{1+i}, r_{7+i}\}$. It can be easily checked that those strategies contain no circuit and the social cost is 18. Thus, the PoA of this instance is at least $26/18 = 13/9$. $\square$

5 Upper bound on the PoA of paving matroid congestion games with affine delays

In this section, we prove Theorem 3. Consider a symmetric matroid congestion game with N players over resource set R, and suppose that every delay function is affine. Let s and q be two arbitrary states of the game such that $\text{cost}(s) \geq \text{cost}(q)$, and let $R^- = R^-(s, q)$, $R^+ = R^+(s, q)$. We consider the graph G defined in Section 3, where each node $r \in R^-$ has supply $s_r - q_r$ and each node $r \in R^+$ has demand $q_r - s_r$, and we let Φ be a (s, q)-difference flow in G. The following theorem identifies some special properties of Φ that can be used to upper bound $\frac{\text{cost}(s)}{\text{cost}(q)}$. The proof of the theorem is deferred to the end of the section.

Theorem 12 *Suppose that Φ is an acyclic (s, q)-exchange flow satisfying the following properties:*

1. *For every arc (u, v) with positive flow in Φ, $cost_s(u) \leq cost_s^+(v)$.*
2. *For every path p from $u \in R^-$ to $v \in R^+$, $cost_s^+(v) \geq \frac{1}{4} cost_s(u)$.*
3. *Let (v, w) be an arc with positive flow in Φ. If for every path to v starting at a node $u \in R^-$ we have $cost_s(v) \geq \frac{1}{2} cost_s(u)$, then $w \notin R^+$.*
4. *For all $r \in R^+$, $s_r = 0$ and $\Phi(\delta^+(r)) = 0$.*
5. *For all $r \notin R^+$, the delay function of r is linear.*

Then $cost(s)/cost(q) \leq 17/7$.

Now consider a symmetric paving matroid congestion game with N players over resource set R, and suppose that the delay functions $d = (d_r)_{r \in R}$ are affine. Let f and o be a PNE and a SO, respectively, that achieve the PoA. We consider an (f, o)-exchange flow F. We then apply five steps, to map $\mathcal{S} = (R, d, f, o, F)$ to a tuple $\mathcal{S}' = (R', d', s, q, \Phi)$ that defines a symmetric 1-uniform matroid congestion game over R' with affine delays $d' = (d'_r)_{r \in R}$, where s and q are two states of the game, and Φ is a (s, q)-exchange flow satisfying the assumptions in Theorem 12, and such that

$$\frac{\text{cost}(f)}{\text{cost}(o)} \leq \frac{\text{cost}(s)}{\text{cost}(q)}.$$

Then using Theorem 12 we can conclude that the worst-case PoA of symmetric paving matroid congestion games is at most 17/7.

Let $\mathcal{S}^0 = (R, d, f, o, F)$. F is an (f, o)-exchange flow of a matroid congestion game, thus for every arc (u, v) with positive flow in F there exists a player i who could replace resource u with resource

v in their strategy. Since f is a PNE, player i is not able to decrease their cost by exchanging u and v, implying that F satisfies property 1. Moreover, since for affine delays $z(\mathcal{D}) = 2$, Lemma 10 implies that also property 2 is satisfied. We apply the following four steps, that preserve properties 1 and 2. Moreover, the construction guarantees $\sum_{r\in R} s_r = \sum_{r\in R} q_r$ in every step. This implies that in every step we can construct an instance of a symmetric 1-uniform matroid congestion game on resource set R where s and q are two states that are obtained by assigning players to resources so that for each $r \in R$ we have s_r players using r in s and q_r players using r in q. The corresponding (s, q)-exchange flow is redefined accordingly. Note that s and q are not necessarily a PNE and a SO of the game.

Step 1. First, we let $s = f$, $q = o$ and $\Phi = F$. We redefine (R, d, s, q, Φ) as follows. For every resource $v \in R^+(f, o)$ such that $f_v > 0$, we add a new resource v' with constant delay equal to $\text{cost}_o(v)$. We set $s_v = q_v = f_v$, $s_{v'} = 0$ and $q_{v'} = o_v - f_v > 0$. Note that $q_{v'} > s_{v'}$, i.e., $v' \in R^+(s, q)$, while $q_v = s_v$, i.e., $v \notin R^+(s, q)$. Moreover we define the flow Φ on arc (v, v') to be $o_v - f_v$. At the end, Φ is a (s, q)-exchange flow that satisfies property 4. Finally we show that $\frac{\text{cost}(f)}{\text{cost}(o)} \le \frac{\text{cost}(s)}{\text{cost}(q)}$ after Step 1. Denote the set of nodes we added in this step by V'. According to the construction in Step 1, we have

$$\text{cost}(s) = \sum_{r\in R} s_r \text{cost}_s(r) + \sum_{r\in V'} s_r \text{cost}_s(r) = \sum_{r\in R} f_r \text{cost}_f(r) + 0 = \text{cost}(f),$$

and

$$\begin{aligned}
\text{cost}(q) &= \sum_{r\in R\setminus V} q_r \text{cost}_q(r) + \sum_{r\in V} q_r \text{cost}_q(r) + \sum_{r\in V'} q_r \text{cost}_q(r) \\
&= \sum_{r\in R\setminus V} o_r \text{cost}_o(r) + \sum_{r\in V} f_r \text{cost}_q(r) + \sum_{r\in V} (o_r - f_r) \text{cost}_o(r) \\
&< \sum_{r\in R\setminus V} o_r \text{cost}_o(r) + \sum_{r\in V} f_r \text{cost}_o(r) + \sum_{r\in V} (o_r - f_r) \text{cost}_o(r) = \text{cost}(o),
\end{aligned}$$

where the inequality holds because $\text{cost}_q(r) = d_r(f_r) < d_r(o_r) = \text{cost}_o(r)$. By combining the above inequalities we obtain $\frac{\text{cost}(f)}{\text{cost}(o)} \le \frac{\text{cost}(s)}{\text{cost}(q)}$.

Step 2. For each resource $v \in R^+(s, q)$ receiving $t_1, \dots, t_h$ units of flow from $h \ge 2$ resources $u_1, \dots, u_h$ through arcs $(u_1, v), \dots, (u_h, v)$ in Φ, we redefine (R, d) by replacing v with h new nodes $v_1, \dots, v_h$, each having delay function d_v. We redefine (s, q) by setting $s_{v_i} = 0$ and $q_{v_i} = t_i$ for all $i \in [h]$. Next, we redefine Φ by replacing arc (u_i, v) with (u_i, v_i) having flow value t_i, for all $i \in [h]$. After this step, for each $v \in R^+(s, q)$ there is only one resource sending flow to v. Let $(s, q), (s', q')$ denote the input and output states of Step 2, respectively. We show that $\frac{\text{cost}(s)}{\text{cost}(q)} \le \frac{\text{cost}(s')}{\text{cost}(q')}$ holds after Step 2. For each $v \in R^+(s, q)$ that we selected in Step 2, we replaced it with $v_1, \dots, v_h$. By the construction we have:

$$s_v \text{cost}_s(v) = \sum_{i=1}^{h} s'_{v_i} \text{cost}_{s'}(v_i) = 0,$$

and

$$q_v \text{cost}_q(v) = \sum_{i=1}^{h} q'_{v_i} \text{cost}_q(v) = \sum_{i=1}^{h} q'_{v_i} d_v(q_v) \ge \sum_{i=1}^{h} q'_{v_i} d_v(q'_v) = \sum_{i=1}^{h} q'_{v_i} \text{cost}_{q'}(v_i).$$

Thus, the social cost of s stays the same and the social cost of q decreases after Step 2, so we have

$\frac{\text{cost}(s)}{\text{cost}(q)} \le \frac{\text{cost}(s')}{\text{cost}(q')}$.

Step 3. For each resource $v \in R^+(s,q)$, let r^* be the most expensive resource in $R^-(s,q)$ that is connected to v along a path carrying at least one unit of flow in Φ. Let u be the only resource sending flow to v in Φ, and let h be the flow of Φ on arc (u,v). If $\text{cost}_s^+(v) > \frac{1}{2}\text{cost}_s(r^*)$, we redefine (R,d) by replacing v with h new nodes $v_1,\dots,v_h$ having delay function $\frac{1}{2}\text{cost}_s^+(v)x$ for $i \in [h]$. Moreover, we add h new resource $w_1,\dots,w_h$ with constant delay function $\frac{1}{2}\text{cost}_s^+(v)$ for $i \in [h]$. We redefine (s,q) by setting $s_{v_i} = 1$, $s_{w_i} = 0$ and $q_{v_i} = q_{w_i} = 1$ for $i \in [h]$. Thus, property 4 is preserved. Finally, we redefine Φ by setting to one the flow of arcs (u,v_i) and (v_i,w) for $i \in [h]$. We repeat this step until for all $v \in R^+(s,q)$ we have $\text{cost}_s^+(v) \le \frac{1}{2}\text{cost}_s(r^*)$, thus achieving property 3. As in Step 2, let $(s,q),(s',q')$ denote the input and output states of each iteration in Step 3, respectively. We show that $\frac{\text{cost}(s)}{\text{cost}(q)} \le \frac{\text{cost}(s')}{\text{cost}(q')}$ holds after each iteration of Step 3. Note that for each $v \in R^+(s,q)$ that we selected in an iteration of Step 3, v is replaced by $v_1,\dots,v_h$ and $w_1,\dots,w_h$. By our construction we have:

$$s_v\text{cost}_s(v) = 0 < \sum_{i=1}^{h}\left(v_i\text{cost}_{s'}(v_i) + w_i\text{cost}_{s'}(w_i)\right) = \sum_{i=1}^{h}\frac{1}{2}\text{cost}_s^+(v),$$

and

$$\begin{aligned} q_v\text{cost}_q(v) = h\text{cost}_q(v) \ge h\text{cost}_s^+(v) &= \sum_{i=1}^{h}\frac{1}{2}\text{cost}_s^+(v) + \sum_{i=1}^{h}\frac{1}{2}\text{cost}_s^+(v) \\ &= \sum_{i=1}^{h} q'_{v_i}\text{cost}_{q'}(v_i) + \sum_{i=1}^{h} q'_{w_i}\text{cost}_{q'}(w_i). \end{aligned}$$

The above inequalities imply that after each iteration of Step 3 the social cost of s increases and the social cost of q decreases, so we have $\frac{\text{cost}(s)}{\text{cost}(q)} \le \frac{\text{cost}(s')}{\text{cost}(q')}$.

Step 4. For every resource $r \notin R^+(s,q)$, suppose $d_r(x) = ax + b$ where $a, b \ge 0$. We redefine the delay function of r as $\frac{\text{cost}_s(r)}{s_r}x = \frac{as_r+b}{s_r}x$. Next we show that $\frac{\text{cost}(s)}{\text{cost}(q)} \le \frac{\text{cost}(s')}{\text{cost}(q')}$, where $(s,q),(s',q')$ are the input and output states of Step 4, respectively. According to the definition of the new delay functions, it is easy to conclude that $\text{cost}(s) = \text{cost}(s')$. For every resource $r \in R \setminus R^+(s',q')$, since we have $s_r = s'_r \ge q'_r = q_r$, then $\text{cost}_{q'}(r) \le \text{cost}_q(r)$. For every resource $r \in R^+(s',q')$, since we did not change the associated delay function, we have $\text{cost}_{q'}(r) = \text{cost}_q(r)$. Thus, we can conclude that $\text{cost}(q') \le \text{cost}(q)$, implying $\frac{\text{cost}(s)}{\text{cost}(q)} \le \frac{\text{cost}(s')}{\text{cost}(q')}$.

Step 5. We delete all the cycles in Φ to make the flow acyclic. At the end, we set $\mathcal{S}' = \{R,d,s,q,\Phi\}$ and $\mathcal{S} = \mathcal{S}^0$. Thus, we achieve property 5.

Based on our discussion we obtain the following lemma.

Lemma 13 *$\mathcal{S}'$ satisfies the six assumptions in Theorem 12 and $\frac{cost(f)}{cost(o)} \le \frac{cost(s)}{cost(q)}$.*

Remark 14 *The construction that we use in the proof of Theorem 3 relies on Lemma 10 to satisfy property 2 in Theorem 12. As discussed in Remark 11, although there exist instances where the bound in Lemma 10 is tight, these instances might still have PoA smaller than the upper bound of Theorem 3. Thus, we conjecture that the upper bound of Theorem 3 is not tight.*

We are now left with proving Theorem 12.

Proof. [Proof of Theorem 12]By property 5, every node $r \notin R^+$ has a linear delay function. Thus, for all $r \notin R^+$ we have

$$\mathrm{cost}_s(r) = \frac{s_r}{s_r+1}\mathrm{cost}_s^+(r), \tag{9}$$

$$\mathrm{cost}_q(r) = \frac{q_r}{s_r+1}\mathrm{cost}_s^+(r). \tag{10}$$

Then we can write

$$1 \le \frac{\mathrm{cost}(s)}{\mathrm{cost}(q)} = \frac{\sum_{r\in R\setminus R^+} s_r \mathrm{cost}_s(r)}{\sum_{r\in R} q_r \mathrm{cost}_q(r)} \tag{11}$$

$$= \frac{\sum_{r\in R\setminus R^+} \frac{s_r^2}{s_r+1}\mathrm{cost}_s^+(r)}{\sum_{r\in R\setminus R^+} \frac{q_r^2}{s_r+1}\mathrm{cost}_s^+(r) + \sum_{r\in R^+} q_r d_r(q_r)}, \tag{12}$$

$$\le \frac{\sum_{r\in R\setminus R^+} \frac{s_r^2}{s_r+1}\mathrm{cost}_s^+(r)}{\sum_{r\in R\setminus R^+} \frac{q_r^2}{s_r+1}\mathrm{cost}_s^+(r) + \sum_{r\in R^+} q_r \mathrm{cost}_s^+(r)}. \tag{13}$$

Note that equality (11) holds because we have $s_r = 0$ for all $r \in R^+$ according to property 4, while equality (12) is implied by (9) and (10) and inequality (13) follows from the fact that $d_r(q_r) \ge \mathrm{cost}_s^+(r)$ for all $r \in R^+$.

Let $r \in R$. We define $\lambda(r) = \mathrm{cost}_s(r) - \frac{1}{2}\mathrm{cost}_s^+(r)$. Let $p = r_0, \dots, r_k$ be a path in Φ carrying one unit of flow, where $r_0 \in R^-$ and $r_k \in R^+$. For each $h \in [k-1]$ and $i \in [h-1]$ we define:

$$\Psi^0(p,h) := \left(\frac{1}{2}\right)^h \mathrm{cost}_s(r_0), \qquad \Psi^i(p,h) := \left(\frac{1}{2}\right)^i \lambda(r_{h-i}),$$

$$\Omega(p,0) := \sum_{j=1}^{k-1} \Psi^0(p,j), \qquad \Omega(p,i) := \sum_{j=i+1}^{k-1} \Psi^{j-i}(p,j) = \sum_{j=i+1}^{k-1} \left(\frac{1}{2}\right)^{j-i} \lambda(r_i).$$

Moreover, we set $\Omega(p,k-1) = 0$, $\Psi(p,0) = 0$, and for $h \in [k-1]$ we let $\Psi(p,h) := \sum_{i=0}^{h-1} \Psi^i(p,h)$. It can be checked that

$$\sum_{j=0}^{k-1} \Psi(p,j) = \sum_{j=0}^{k-1} \Omega(p,j). \tag{14}$$

Claim 3. *We have that $\Psi(p,h) \le \frac{1}{2}\mathit{cost}_s^+(r_h)$.*

Proof of claim. We prove the claim by induction on h. Let $h = 1$. Since (r_0, r_1) is an arc in the path p, by property 1 we have

$$\Psi(p,1) = \frac{1}{2}\mathrm{cost}_s(r_0) \le \frac{1}{2}\mathrm{cost}_s^+(r_1).$$

Now assume that for r_h with $h < k-1$, the claim holds. Then for r_{h+1}, we have

$$
\begin{aligned}
\frac{1}{2}\mathrm{cost}_s^+(r_{h+1}) &\geq \frac{1}{2}\mathrm{cost}_s(r_h) && (15)\\
&= \frac{1}{2}\lambda(r_h) + \frac{1}{4}\mathrm{cost}_s^+(r_h) && (16)\\
&\geq \frac{1}{2}\lambda(r_h) + \frac{1}{2}\Psi(p,h) && (17)\\
&= \frac{1}{2}\lambda(r_h) + \frac{1}{2}\left(\left(\frac{1}{2}\right)^h \mathrm{cost}_s(r_0) + \sum_{i=1}^{h-1}\left(\frac{1}{2}\right)^i \lambda(r_{h-i})\right)\\
&= \left(\frac{1}{2}\right)^{h+1} \mathrm{cost}_s(r_0) + \sum_{i=1}^{h}\left(\frac{1}{2}\right)^i \lambda(r_{h+1-i}) = \Psi(p,h+1),
\end{aligned}
$$

where (15) follows from applying property 1 to the arc (r_h, r_{h+1}) in the path p, equality (16) follows from the definition of $\lambda(r_h)$, and inequality (17) holds because of our inductive hypothesis. $\diamond$

Now let $P = \{p_1, \ldots, p_l\}$ be an arbitrary decomposition of the flow Φ where each path starts at a node in R^- and ends at a node in R^+ and carries one unit of flow. By property 4, $\Phi(\delta^+(r)) = 0$ for each $r \in R^+$, thus in every path $p \in P$ the only node in R^+ is the sink of the path, denoted by $t(p)$. Moreover, for each resource $r \in R$ we denote by $P(r)$ the paths in P that contain r and by $P^0(r)$ the paths in P starting at r. Finally, for each resource $r \in R$ and path $p \in P(r)$ we use the notation $p(r)$ to identify the position of r in p, precisely $p(r) = 0$ if r is the start node of p, and $p(r) = i$ if r is the i-th node appearing after the start node of p. After summing up inequalities (14) for every path $p \in P$ we have

$$
\sum_{r \in R \setminus R^+} \sum_{p \in P(r)} \Psi(p, p(r)) = \sum_{r \in R \setminus R^+} \sum_{p \in P(r)} \Omega(p, p(r)). \tag{18}
$$

For each resource $r \in R$, we define $a_r := \Phi(\delta^-(r))$, $b_r := \Phi(\delta^+(r)) - \Phi(\delta^-(r))$. Note that we have $a_r = |P(r) \setminus P^0(r)|$ and $b_r = |P^0(r)|$ because Φ is acyclic and thus each path $p \in P$ is simple. For each $r \in R \setminus R^+$ and each path in $P(r) \setminus P^0(r)$ we apply Claim 3. Summing up we obtain

$$
\Theta_r := \frac{a_r}{2}\mathrm{cost}_s^+(r) - \sum_{p \in P(r) \setminus P^0(r)} \Psi(p, p(r)) \geq 0.
$$

Since the fraction in (13) is at least 1, by subtracting the non-negative constant $\sum_{r \in R \setminus R^+} \Theta_r$ to both the numerator and the denominator we obtain an upper bound. By using (18) and the fact that $\Psi(p, 0) = 0$ for every $p \in P$ we obtain

$$
\frac{\mathrm{cost}(s)}{\mathrm{cost}(q)} \leq \frac{\sum_{r \in R \setminus R^+} A_r}{\sum_{r \in R \setminus R^+} B_r} \leq \max_{r \in R \setminus R^+} \frac{A_r}{B_r}, \tag{19}
$$

where

$$A_r = \frac{(s_r)^2}{s_r+1}\mathrm{cost}_s^+(r) + \sum_{p\in P(r)} \Omega(p,p(r)) - \frac{a_r}{2}\mathrm{cost}_s^+(r), \tag{20}$$

$$B_r = \frac{(q_r)^2}{s_r+1}\mathrm{cost}_s^+(r) + \sum_{p\in P(r)} \Omega(p,p(r)) + \sum_{p\in P^0(r)} \mathrm{cost}_s^+(t(p)) - \frac{a_r}{2}\mathrm{cost}_s^+(r). \tag{21}$$

In the remaining part of the proof we will show that $\frac{A_r}{B_r} \le \frac{17}{7}$ for all $r \in R \setminus R^+$.

Let $z_r := s_r - a_r - b_r = q_r - a_r$. Thus we have $s_r = z_r + a_r + b_r$ and $q_r = z_r + a_r$. First, we remark that $z_r \ge 0$ for all $r \in R$. This is because Φ is a (s,q)-exchange flow, thus for every node $r \in R$ we have that $\Phi(\delta^+(r))$ is exactly the number of players using r in s and not in q. This number is clearly upper bounded by s_r, the number of players using r in s, thus $s_r \ge \Phi(\delta^+(r)) = a_r + b_r$, implying $z_r \ge 0$. For $r \in R\setminus R^+$ we have $s_r \ge q_r$. If $q_r = s_r$ then we have $B_r - A_r = \sum_{p\in P^0(r)} \mathrm{cost}_s^+(t(p)) \ge 0$, which implies that $\frac{A_r}{B_r} \le 1$. Thus, to upper bound $\frac{A_r}{B_r}$ we now assume that $s_r \ge q_r + 1$, i.e., $r \in R^-$. Since $s_r = z_r + a_r + b_r$ and $q_r = z_r + a_r$, this implies that we have $b_r \ge 1$. Moreover, since $s_r \ge 1$ we have that $\lambda(r) = \mathrm{cost}_s(r) - \frac{1}{2}\mathrm{cost}_s^+(r) = (\frac{s_r}{s_r+1} - \frac{1}{2})\mathrm{cost}_s^+(r) \ge 0$. This implies:

$$\Omega(p,p(r)) \ge 0 \qquad \forall\, p \in P(r) \setminus P^0(r) \tag{22}$$

$$\Omega(p,p(r)) \ge \frac{1}{2}\lambda(r) \qquad \forall\, p \in P(r) \setminus P^0(r) : t(p) > p(r)+1. \tag{23}$$

Let

$$\begin{aligned} P_1^0(r) &:= \{p \in P^0(r) : \mathrm{cost}_s^+(t(p)) \ge \mathrm{cost}_s(r)\} \\ P_2^0(r) &:= \{p \in P^0(r) : \frac{1}{2}\mathrm{cost}_s(r) \le \mathrm{cost}_s^+(t(p)) < \mathrm{cost}_s(r)\} \\ P_3^0(r) &:= \{p \in P^0(r) : \frac{1}{4}\mathrm{cost}_s(r) \le \mathrm{cost}_s^+(t(p)) < \frac{1}{2}\mathrm{cost}_s(r)\}. \end{aligned}$$

Note that we have $P^0 = P_1^0 \cup P_2^0 \cup P_3^0$. In fact, Lemma 10 and $z(\mathcal{D}) = 2$ for the class of affine delay functions imply $\mathrm{cost}_s^+(r) \le 4\mathrm{cost}_s^+(t(p))$ for every path $p \in P^0$.

First, recalling the definition of $\Omega(p,0)$ and the fact that delay functions are nonnegative, we obtain that for every path $p \in P^0(r)$

$$\Omega(p,0) \ge 0. \tag{24}$$

Secondly, we prove that for every path $p \in P_2^0(r)$ we have

$$\Omega(p,0) - \frac{1}{2}\mathrm{cost}_s(r) \ge 0. \tag{25}$$

In fact, for every path $p \in P_2^0(r)$, there must exists at least one resource between r and $t(p)$. Otherwise $(r, t(p))$ would be an arc in p and $\mathrm{cost}_s^+(t(p)) < \mathrm{cost}_s(r)$, which contradicts property 1. Thus, from the definition of $\Omega(p,0)$ and the fact that p has at least three nodes, we have $\Omega(p,0) \ge \frac{1}{2}\mathrm{cost}_s(r)$, which implies (25).

Finally, for every path $p \in P_3^0(r)$ there must exists at least two resources between r and $t(p)$. Otherwise, if there is no resource between them, then $(r, t(p))$ would be an arc in p and $\mathrm{cost}_s^+(t(p)) < \mathrm{cost}_s(r)$, which contradicts property 1. If there is one resource r' between r and $t(p)$, then we have two edges (r, r') and $(r', t(p))$. By property 1 and the definition of $z(\mathcal{D})$ we have $\mathrm{cost}_s(r) \le \mathrm{cost}_s^+(r') \le z(\mathcal{D})\mathrm{cost}_s(r')$. Since for affine delays $z(\mathcal{D}) = 2$, so $\frac{1}{2}\mathrm{cost}_s(r) \le \mathrm{cost}_s(r')$.

Because $p \in P_3^0(r)$, we also have $\mathrm{cost}_s^+(t(p)) < \frac{1}{2}\mathrm{cost}_s(r) \leq \mathrm{cost}_s(r')$, which contradicts property 1 on the edge $(r', t(p))$. Thus, we have $\Omega(p, 0) \geq (\frac{1}{2} + \frac{1}{4})\mathrm{cost}_s(r)$ which implies

$$\Omega(p, 0) - \frac{3}{4}\mathrm{cost}_s(r) \geq 0. \tag{26}$$

From now on we denote by r^* the most expensive resource with respect to state s among all the resources $u \in R^-$ such that there exists a path from u to r in the directed graph induced by Φ. We denote by p^* a path from r^* to r in this graph. Next we need to analyze the following two cases.

Case(i): $\mathbf{cost}_s(r) \leq \frac{1}{2}\mathbf{cost}_s(r^*)$

In this case, we show that if $\frac{A_r}{B_r} \geq 1$, then $\frac{A_r}{B_r} \leq \frac{7}{3}$. We first argue that for every path $p \in P^0(r)$, we have

$$\mathrm{cost}_s^+(t(p)) \geq \frac{1}{2}\mathrm{cost}_s(r). \tag{27}$$

By contradiction, suppose there exists a path $p \in P^0(r)$ such that $\mathrm{cost}_s^+(t(p)) < \frac{1}{2}\mathrm{cost}_s(r)$. Then, since we are assuming $\mathrm{cost}_s(r) \leq \frac{1}{2}\mathrm{cost}_s(r^*)$, we have that $\mathrm{cost}_s^+(t(p)) < \frac{1}{4}\mathrm{cost}_s(r^*)$. Then we can combine the path p^* from r^* to r and the path p from r to $t(p)$ to obtain a path from $r^* \in R^-$ to $t(p) \in R^+$ that carries at least one unit of flow in Φ. This contradicts to the property 2.

By (22) we have $\sum_{p \in P(r) \setminus P^0(r)} \Omega(p, p(r)) \geq 0$. Since $A_r/B_r \geq 1$ we can subtract from both the numerator and the denominator this nonnegative constant and obtain an upper bound. Thus, by using (20) and (21) we get

$$\frac{A_r}{B_r} \leq \frac{\frac{(s_r)^2}{s_r+1}\mathrm{cost}_s^+(r) + \sum_{p \in P^0(r)} \Omega(p, 0) - \frac{a_r}{2}\mathrm{cost}_s^+(r)}{\frac{(q_r)^2}{s_r+1}\mathrm{cost}_s^+(r) + \sum_{p \in P^0(r)} (\Omega(p, 0) + \mathrm{cost}_s^+(t(p))) - \frac{a_r}{2}\mathrm{cost}_s^+(r)}. \tag{28}$$

Because of inequality (27), we have $P^0(r) = P_1^0(r) \cup P_2^0(r)$. Thus, from (28) we obtain

$$\frac{A_r}{B_r} \leq \frac{\frac{(s_r)^2}{s_r+1}\mathrm{cost}_s^+(r) + \sum_{i=1}^2 \sum_{p \in P_i^0(r)} \Omega(p, 0) - \frac{a_r}{2}\mathrm{cost}_s^+(r)}{\frac{(q_r)^2}{s_r+1}\mathrm{cost}_s^+(r) + \sum_{i=1}^2 \sum_{p \in P_i^0(r)} \left(\Omega(p, 0) + \frac{1}{2^{i-1}}\mathrm{cost}_s(r)\right) - \frac{a_r}{2}\mathrm{cost}_s^+(r)}. \tag{29}$$

To upper bound the right-hand-side of (29) we do the following. First, for every $p \in P_1^0(r)$, we subtract $\Omega(p, 0) - \frac{1}{2}\mathrm{cost}_s(r)$ from the numerator and subtract $\Omega(p, 0)$ from the denominator. Because $\Omega(p, 0) \geq 0$ for all $p \in P_1^0(r)$ and $\frac{A_r}{B_r} \geq 1$, this will increase the right-hand-side of inequality (29). Secondly, for every $p \in P_2^0(r)$, we subtract $\Omega(p, 0) - \frac{1}{2}\mathrm{cost}_s(r)$ from both the numerator and the denominator. This will also increase the right-hand-side of (29) because $\Omega(p, 0) - \frac{1}{2}\mathrm{cost}_s(r) \geq 0$ and $\frac{A_r}{B_r} \geq 1$. We obtain

$$\frac{A_r}{B_r} \leq \frac{\frac{(s_r)^2}{s_r+1}\mathrm{cost}_s^+(r) + \sum_{p \in P^0(r)} \frac{1}{2}\mathrm{cost}_s(r) - \frac{a_r}{2}\mathrm{cost}_s^+(r)}{\frac{(q_r)^2}{s_r+1}\mathrm{cost}_s^+(r) + \sum_{p \in P^0(r)} \mathrm{cost}_s(r) - \frac{a_r}{2}\mathrm{cost}_s^+(r)} \tag{30}$$

$$= \frac{\frac{(s_r)^2}{s_r+1}\mathrm{cost}_s^+(r) + \sum_{p \in P^0(r)} \frac{1}{2}\frac{(s_r)}{s_r+1}\mathrm{cost}_s^+(r) - \frac{a_r}{2}\mathrm{cost}_s^+(r)}{\frac{(q_r)^2}{s_r+1}\mathrm{cost}_s^+(r) + \sum_{p \in P^0(r)} \frac{(s_r)}{s_r+1}\mathrm{cost}_s^+(r) - \frac{a_r}{2}\mathrm{cost}_s^+(r)}, \tag{31}$$

where equation (31) follows from (9) and $r \notin R^+$. Finally, we rewrite the right-hand-side of (31) by factoring out $\mathrm{cost}_s^+(r)$ and exploiting $|P^0(r)| = b_r$, $s_r = a_r + b_r + z_r$, $q_r = a_r + z_r$. Next, we derive an upper bound by considering the maximum over all possible values of a_r, b_r and z_r. We obtain

$$\frac{A_r}{B_r} \le \frac{\frac{(a_r+b_r+z_r)^2}{a_r+b_r+z_r+1} + b_r\frac{1}{2}\frac{(a_r+b_r+z_r)}{a_r+b_r+z_r+1} - \frac{a_r}{2}}{\frac{(a_r+z_r)^2}{a_r+b_r+z_r+1} + b_r\frac{(a_r+b_r+z_r)}{a_r+b_r+z_r+1} - \frac{a_r}{2}} \le \max_{a,z\ge 0, b\ge 1} \frac{\frac{(a+b+z)^2}{a+b+z+1} + \frac{1}{2}b\frac{(a+b+z)}{a+b+z+1} - \frac{a}{2}}{\frac{(a+z)^2}{a+b+z+1} + b\frac{(a+b+z)}{a+b+z+1} - \frac{a}{2}} \le \frac{7}{3}.$$

Case(ii): $\mathbf{cost}_s(r) > \frac{1}{2}\mathbf{cost}_s(r^*)$ In this case, we show that if $\frac{A_r}{B_r} \ge 1$, then $\frac{A_r}{B_r} \le \frac{17}{7}$. Let p be a path in $P(r) \setminus P^0(r)$. We first argue that there exists at least one resource between r and $t(p)$ in p. By assumption we have $\text{cost}_s(r) > \frac{1}{2}\text{cost}_s(r^*)$, where by definition r^* is such that $\text{cost}_s(r^*) \ge \text{cost}_s(u)$ for every resource $u \in R^-$ such that there is a path from u to r in the directed graph induced by Ψ. Thus, property 3 implies that each arc of the form (r, w) with positive flow in Φ has $w \notin R^+$. We can the conclude that $w \ne t(p) \in R^+$. This implies $p(t(p)) > p(r) + 1$. By (23) we then have $\Omega(p, p(r)) \ge \frac{1}{2}\lambda(r)$ for all $p \in P(r) \setminus P^0(r)$. We have

$$\begin{aligned}
\frac{A_r}{B_r} &= \frac{\frac{(s_r)^2}{s_r+1}\text{cost}_s^+(r) + \sum_{p\in P(r)} \Omega(p, p(r)) - \frac{a_r}{2}\text{cost}_s^+(r)}{\frac{(q_r)^2}{s_r+1}\text{cost}_s^+(r) + \sum_{p\in P(r)} \Omega(p, p(r)) + \sum_{p\in P^0(r)} \text{cost}_s^+(t(p)) - \frac{a_r}{2}\text{cost}_s^+(r)} \\
&\le \frac{\frac{(s_r)^2}{s_r+1}\text{cost}_s^+(r) + \frac{1}{2}\lambda(r) + \sum_{p\in P^0(r)} \Omega(p, 0) - \frac{a_r}{2}\text{cost}_s^+(r)}{\frac{(q_r)^2}{s_r+1}\text{cost}_s^+(r) + \frac{1}{2}\lambda(r) + \sum_{p\in P^0(r)} (\Omega(p, 0) + \text{cost}_s^+(t(p))) - \frac{a_r}{2}\text{cost}_s^+(r)} && (32) \\
&\le \frac{\frac{(s_r)^2}{s_r+1}\text{cost}_s^+(r) + \frac{1}{2}\lambda(r) + \sum_{i=1}^3 \sum_{p\in P_i^0(r)} \Omega(p, 0) - \frac{a_r}{2}\text{cost}_s^+(r)}{\frac{(q_r)^2}{s_r+1}\text{cost}_s^+(r) + \frac{1}{2}\lambda(r) + \sum_{i=1}^3 \sum_{p\in P_i^0(r)} \left(\Omega(p, 0) + \frac{1}{2^{i-1}}\text{cost}_s(r)\right) - \frac{a_r}{2}\text{cost}_s^+(r)} && (33)
\end{aligned}$$

Inequality (32) holds because $\Omega(p, p(r)) \ge \frac{1}{2}\lambda(r)$ for all $p \in P(r) \setminus P^0(r)$ and $\frac{A_r}{B_r} \ge 1$. Inequality (33) holds because we have $P^0 = P_1^0 \cup P_2^0 \cup P_3^0$.

To derive an upper bound on the right-hand-side of (33), we do the following. First, for every $p \in P_1^0(r)$, we subtract $\Omega(p, 0) - \frac{3}{4}\text{cost}_s(r)$ from the numerator and subtract $\Omega(p, 0)$ from the denominator. Because $\Omega(p, 0) \ge 0$ for all $p \in P_1^0(r)$ by inequality (24) and $\frac{A_r}{B_r} \ge 1$, this yields an upper bound. Secondly, for every $p \in P_2^0(r)$, we subtract $\Omega(p, 0) - \frac{3}{4}\text{cost}_s(r)$ from the numerator and we subtract $\Omega(p, 0) - \frac{1}{2}\text{cost}_s(r)$ from the denominator. This will also produce an upper bound, because $\Omega(p, 0) - \frac{1}{2}\text{cost}_s(r) \ge 0$ by inequality (25) and $\frac{A_r}{B_r} \ge 1$. Finally, for every $p \in P_3^0(r)$, we subtract $\Omega(p, 0) - \frac{3}{4}\text{cost}_s(r)$ from both the numerator and the denominator. This will yield an upper bound because $\Omega(p, 0) - \frac{3}{4}\text{cost}_s(r) \ge 0$ by inequality (26) and $\frac{A_r}{B_r} \ge 1$. Then from (33) we have

$$\begin{aligned}
\frac{A_r}{B_r} &\le \frac{\frac{(s_r)^2}{s_r+1}\text{cost}_s^+(r) + \frac{1}{2}\lambda(r) + \sum_{p\in P^0(r)} \frac{3}{4}\text{cost}_s(r) - \frac{a_r}{2}\text{cost}_s^+(r)}{\frac{(q_r)^2}{s_r+1}\text{cost}_s^+(r) + \frac{1}{2}\lambda(r) + \sum_{p\in P^0(r)} \text{cost}_s(r) - \frac{a_r}{2}\text{cost}_s^+(r)} && (34) \\
&= \frac{\frac{(s_r)^2}{s_r+1}\text{cost}_s^+(r) + \frac{a_r}{2}(\frac{s_r}{s_r+1} - \frac{1}{2})\text{cost}_s^+(r) + \sum_{p\in P^0(r)} \frac{3}{4}\frac{(s_r)}{s_r+1}\text{cost}_s^+(r) - \frac{a_r}{2}\text{cost}_s^+(r)}{\frac{(q_r)^2}{s_r+1}\text{cost}_s^+(r) + \frac{a_r}{2}(\frac{s_r}{s_r+1} - \frac{1}{2})\text{cost}_s^+(r) + \sum_{p\in P^0(r)} \frac{(s_r)}{s_r+1}\text{cost}_s^+(r) - \frac{a_r}{2}\text{cost}_s^+(r)}, && (35)
\end{aligned}$$

where the equation (35) follows from (9) and $\lambda(r) = \text{cost}_s(r) - \frac{1}{2}\text{cost}_s^+(r)$. Finally, we rewrite the right-hand-side of (35) by factoring out $\text{cost}_s^+(r)$ and exploiting $|P^0(r)| = b_r$, $s_r = a_r + b_r + z_r$, and $q_r = a_r + z_r$. Next, we derive an upper bound by considering the maximum over all possible values of a_r, b_r and z_r. We obtain

$$\frac{A_r}{B_r} \leq \frac{\frac{(a_r+b_r+z_r)^2}{a_r+b_r+z_r+1} + \frac{a_r}{2}\left(\frac{a_r+b_r+z_r}{a_r+b_r+z_r+1} - \frac{1}{2}\right) + b_r \frac{3}{4}\frac{(a_r+b_r+z_r)}{a_r+b_r+z_r+1} - \frac{a_r}{2}}{\frac{(a_r+z_r)^2}{a_r+b_r+z_r+1} + \frac{a_r}{2}\left(\frac{a_r+b_r+z_r}{a_r+b_r+z_r+1} - \frac{1}{2}\right) + b_r \frac{(a_r+b_r+z_r)}{a_r+b_r+z_r+1} - \frac{a_r}{2}} \tag{36}$$

$$\leq \max_{a,z\geq 0, b\geq 1} \frac{\frac{(a+b+z)^2}{a+b+z+1} + \frac{a}{2}\left(\frac{a+b+z}{a+b+z+1} - \frac{1}{2}\right) + \frac{3}{4} b\frac{(a+b+z)}{a+b+z+1} - \frac{a}{2}}{\frac{(a+z)^2}{a+b+z+1} + \frac{a}{2}\left(\frac{a+b+z}{a+b+z+1} - \frac{1}{2}\right) + b\frac{(a+b+z)}{a+b+z+1} - \frac{a}{2}} \leq \frac{17}{7}. \tag{37}$$

This completes the proof of the theorem. □

6 Conclusion

We have investigated the impact of matroid structures on the PoA of symmetric congestion games. In the symmetric case, the PoA of general matroid congestion games is still not completely understood. For graphic matroids and $N = 2, 3, 4$ or infinity with affine delay functions, the PoA can be as large as the worst-case PoA of symmetric congestion games, which is equal to $\frac{5N-2}{2N+1}$ [14]. However, for arbitrary N or different delay functions we don't know whether the the worst-case PoA of symmetric congestion games can be achieved by symmetric matroid congestion games. Our results indicate that if we restrict to paving matroid, the worst-case PoA is significantly smaller than that of symmetric congestion games. A similar result had been previously established by De Jong et al. [11] for k-uniform matroids and affine delays. However, k-uniform matroids are only a mild generalization of singleton congestion games. Paving matroid, on the other hand, are a substantial generalization of k-uniform matroids, since they are conjectured to represent the vast majority of matroids. Since paving matroid are quite more complex that k-uniform matroids, it is not as easy to characterize the worst-case PoA. There is still a gap between our upper and lower bounds, and we conjecture that the our upper bounds are not tight (see Remarks 11 and 14) .

Our approach to bound the PoA relies on a constant α that we have quantified for both k-uniform matroids and paving matroids (Theorem 7). In particular, we can set $\alpha = z(\mathcal{D})$ for k-uniform matroids and $\alpha = z(\mathcal{D})^2$ for paving matroids. Since paving matroids of rank k contain circuits whose size is smaller than the circuit size of k-uniform matroids, this suggests that the difference between the sizes of bases and circuits might impact the PoA. Let δ be a parameter that is equal to the rank of the matroid minus the size of the smallest circuit in the matroid. We conjecture that for $\delta \geq 0$ we can satisfy the assumptions of Theorem 7 with $\alpha = z(\mathcal{D})^{2(\delta+1)}$. Thus, we would get an upper bound on the PoA which is equal to $\rho(\mathcal{D})z(\mathcal{D})^{2(\delta+1)}$. For polynomial delays of highest degree p, this bound is in $O((C^p)(p/lnp))$, where $C = 4^{\delta+1}$. For fixed δ and large p this bound is still better than the PoA of general congestion games, that is in $O((p/lnp)^{p+1})$. To summarize, it is possible that our approach could be extended to upper bound the PoA in arbitrary matroid congestion games where we have an upper bound on δ. On the other hand, our approach might fail to provide meaningful upper bounds for small values of p or when the circuits can be much smaller than the rank. Besides the *size* of the circuits, we suspect that the way in which the circuits overlap can affect the PoA. For example, circuits of k-uniform matroids are highly symmetric. When dealing with paving matroids, we observed that instances with highly symmetric circuits displayed a lower PoA. On the other hand, the paving matroid congestion game example in Section 4, whose PoA is larger than the worst-case PoA of uniform matroid congestion games, has circuits that more often overlap on a single resource. In conclusion, it is open to find lower and upper bounds of symmetric matroid congestion games that depend on the size of the matroid circuits and/or on their degree of symmetry.

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