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Efficient Proximal Subproblem Solvers for a Nonsmooth Trust-Region Method

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Abstract In [R. J. Baraldi and D. P. Kouri, *Mathematical Programming*, (2022), pp. 1-40], we introduced an inexact trust-region algorithm for minimizing the sum of a smooth nonconvex and nonsmooth convex function. The principle expense of this method is in computing a trial iterate that satisfies the so-called fraction of Cauchy decrease condition—a bound that ensures the trial iterate produces sufficient decrease of the subproblem model. In this paper, we expound on various proximal trust-region subproblem solvers that generalize traditional trust-region methods for smooth unconstrained and convex-constrained problems. We introduce a simplified spectral proximal gradient solver, a truncated nonlinear conjugate gradient solver, and a double dogleg method. We compare algorithm performance on examples from data science and PDE-constrained optimization.

Keywords Nonsmooth Optimization · Nonlinear Programming · Trust Regions · Large-Scale Optimization · Proximal Newton's Method

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1 Introduction

In [3], we developed a trust-region method for the nonsmooth optimization problem

$$\min_{x\in X} \ f(x) + \phi(x), \tag{1}$$

where X is a Hilbert space, $f : X \to \mathbb{R}$ is Fréchet differentiable with Lipschitz continuous gradient, and $\phi : X \to [-\infty, +\infty]$ is proper, closed and convex. The method introduced in [3] permits and systematically controls inexactness in the evaluations of f and its gradient ∇f, while guaranteeing convergence. This enables the numerical solution of infinite-dimensional optimization problems, where finite-dimensional approximations are indispensable for evaluating f and ∇f.

Inexactness notwithstanding, typical trust-region methods measure progress using a Cauchy point (CP) or, more generally, a fraction of Cauchy decrease (FCD) condition [13,29,33,43]. For smooth unconstrained problems, the CP is the minimizer of a quadratic model in the steepest descent direction. When simple constraints are present, the CP is any point along the projected gradient path that produces sufficient decrease of the model [33,43]. In [3], we generalized the CP to a point along the proximal gradient path and computed it using a bidirectional proximal search, cf. [3, Alg. 2]. In this paper, we develop various trust-region subproblem solvers that improve upon the CP and are guaranteed to satisfy the FCD condition, thereby ensuring convergence of the trust-region algorithm [3, Alg. 1]. Moreover, our subproblem solvers ensure rapid superlinear, even quadratic, convergence of the trust-region algorithm when the problem data in (1) permits [4].

Since the inception of trust-region methods, numerous subproblem solvers have been proposed, primarily for smooth problems. Early methods were so-called *dogleg* approaches because they employ a piecewise linear interpolation between the CP and unconstrained Newton point to guarantee fraction of Cauchy decrease; cf. Powell [35,36]. Powell's dogleg method was extended in [16] to a double dogleg path by adding an additional piecewise linear segment that biases the Newton point, yielding improved local convergence. Dogleg methods are computationally simple but produce potentially poor trial iterates near the trust-region radius. To overcome this, Moré and Sorensen computed trial iterates by solving the reformulated subproblem first-order optimality conditions with Newton's method [32]. One could similarly solve the subproblem using Gaussian quadrature [20]. Around the same time as [32], Steihaug [41] and Toint [42] introduced the truncated conjugate gradient (CG) method, which approximately solves the subproblem using CG, modified with custom stopping conditions that account for negative curvature and the trust-region constraint. Motivated by truncated CG, the authors in [21], proposed solving the subproblem using a truncated Lanczos method. Truncated CG has also been used for solving various constrained optimization problems [22, 23,29]. More recently, [24,30] employed the spectral projected gradient method [6] to compute a trial iterate for smooth unconstrained and convex-constrained problems.

The trust-region subproblem used in [3, Alg. 1] is

$$\min_{x\in X} \{m_k(x) := f_k(x) + \phi(x)\} \qquad \text{subject to} \qquad \|x - x_k\| \le \Delta_k, \tag{2}$$

where $x_k \in X$ is the current iterate, f_k is a local approximation of f around x_k, and $\Delta_k > 0$ is the current trust-region radius. The presence of nonsmooth ϕ in (2) renders most of the aforementioned methods irrelevant. To rectify this, we introduce extensions of these classical methods that verifiably produce trial iterates satisfying the FCD condition. We establish three main solvers: 1) a spectral proximal gradient (SPG) method; 2) a nonsmooth truncated CG method; and 3) a nonsmooth double dogleg method. Our SPG method simplifies the algorithm proposed in [3] by using a simplified spectral CP and handling the trust-region constraint separately from the proximity operator computation. These modifications typically result in fewer evaluations of the proximity operator. Our truncated CG approach is based on nonlinear CG with modifications that account for the nonsmooth term as well as the trust-region constraint. For our double dogleg framework, we compute the Newton point using damped semismooth Newton, which requires the application of a generalized Jacobian of the proximity operator. Fortunately, the proximity operators for numerous ϕ are semismooth [7]. In the appendix, we include a specialized orthant-based subproblem solver for L^1-regularized problems based on [9].

We organize the paper as follows. Section 2 introduces the notation and problem assumptions. Section 3 reviews the trust-region algorithm from [3] and highlights its basic functionality. Section 4 discusses global and local convergence of the algorithm. Section 5 details the subproblem solvers, and Section 6 compares their performance on five numerical examples arising from data science and optimization problems constrained by partial differential equations (PDE).

2 Notation and Problem Assumptions

Let X be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let $\mathcal{L}(X)$ denote the space of continuous linear operators that map X into itself. Recall that $\mathcal{L}(X)$ is a Banach space endowed with the usual operator norm

$$\|B\| = \sup\{\|Bx\| \mid \|x\| \le 1\} \qquad \forall\, B \in \mathcal{L}(X).$$

To simplify the presentation, we identify the topological dual space X^* with X via Riesz representation. Following standard convex analysis notation, we denote the subdifferential of a proper, closed and convex function $\psi : X \to [-\infty,\infty]$ by

$$\partial\psi(x) := \{\eta \in X \mid \psi(y) \ge \psi(x) + \langle \eta, y - x\rangle \quad \forall\, y \in X\}.$$

and the effective domains of ψ and $\partial\psi$ by

$$\operatorname{dom}\psi := \{x \in X \mid \psi(x) < +\infty\} \qquad \text{and} \qquad \operatorname{dom}\partial\psi := \{x \in X \mid \partial\psi(x) \neq \emptyset\},$$

respectively. Furthermore, the proximity operator of ψ is

$$\operatorname{Prox}_{r\psi}(x) := \operatorname*{arg\,min}_{y\in X} \{\psi(y) + \tfrac{1}{2r}\|y - x\|^2\}, \tag{3}$$

for $r > 0$. When $\psi = \iota_{\mathcal{C}}$ is the indicator function of a nonempty, closed and convex set $\mathcal{C} \subset X$ (i.e., $\iota_{\mathcal{C}}(x) = 0$ if $x \in \mathcal{C}$ and $\iota_{\mathcal{C}}(x) = +\infty$ if $x \notin \mathcal{C}$), $\mathrm{Prox}_{r\psi}(x)$ is the projection of x onto $\mathcal{C}$. We repeatedly utilize the proximity operator's firm nonexpansivity [5, Prop. 12.27]. For other useful properties of the proximity operator, see [3, Sec. 2.2];

The convergence theory in [3] requires the following standard assumptions on the problem data in (1).

Assumption 1 (Problem Data) *The components of the objective function*

$$F(x) \coloneqq f(x) + \phi(x)$$

in (1) *satisfy the following conditions.*

1. *The function $\phi : X \to [-\infty, +\infty]$ is proper, closed and convex.*
2. *The function $f : X \to \mathbb{R}$ is L-smooth on* $\mathrm{dom}\,\phi$*. That is, f is Fréchet differentiable and its gradient ∇f is Lipschitz continuous with modulus $L > 0$ on an open set $U \subseteq X$ containing* $\mathrm{dom}\,\phi$*.*
3. *The objective function F is bounded below, i.e., there exists $\kappa_{\mathrm{lb}} \in \mathbb{R}$ such that $F(x) \geq \kappa_{\mathrm{lb}}$ for all $x \in X$.*

Recall that if $\bar{x} \in X$ is a local minimizer for (1), then it satisfies

$$-\nabla f(\bar{x}) \in \partial\phi(\bar{x}) \qquad \Longleftrightarrow \qquad \bar{x} = \mathrm{Prox}_{r\phi}(\bar{x} - r\nabla f(\bar{x}))$$

for arbitrary, fixed $r > 0$. The second condition above motivates a natural algorithmic stopping condition. Commonly, algorithms for (1) will stop iterating if the current iterate $x \in X$ satisfies

$$\tfrac{1}{r}\left\|x - \mathrm{Prox}_{r\phi}(x - r\nabla f(x))\right\| \leq \tau,$$

for a user-specified tolerance $\tau > 0$ and fixed $r > 0$. For use in later sections, we define the functions $G : X \times X \times [0,\infty) \to X$, $G_f : X \times [0,\infty) \to X$, $H : X \times X \times [0,\infty) \to \mathbb{R}$ and $h : X \times [0,\infty) \to \mathbb{R}$ by

$$\begin{aligned} &G(x,g,r) \coloneqq \tfrac{1}{r}(x - \mathrm{Prox}_{r\phi}(x - rg)), \quad G_f(x,r) \coloneqq G(x, \nabla f(x), r) \\ &H(x,g,r) \coloneqq \|G(x,g,r)\| \quad \text{and} \quad h(x,r) \coloneqq \|G_f(x,r)\|, \end{aligned} \tag{4}$$

respectively. The next proposition catalogues important properties of G and H.

Proposition 1 (Properties of G and H)

a: *For fixed x, $g \in X$, $r \mapsto rH(x,g,r)$ is nondecreasing on $(0,\infty)$. In particular, if $r \geq t > 0$, then $rH(x,g,r) \geq tH(x,g,t)$. Moreover, this inequality is strict if $rG(x,g,r) \neq tG(x,g,t)$.*

b: *For fixed x, $g \in X$, $r \mapsto H(x,g,r)$ is nonincreasing for $r > 0$.*

c: *For fixed x, $g \in X$ and $r > 0$, the following inequality holds*

$$-r\langle g, G(x,g,r)\rangle + \phi(x - rG(x,g,r)) - \phi(x) \leq -rH(x,g,r)^2. \tag{5}$$

d: *The maps $(x,g,r) \mapsto G(x,g,r)$ and $(x,g,r) \mapsto H(x,g,r)$ are continuous on $X \times X \times (0,\infty)$.*

e: *For fixed $r > 0$, $(x, g) \mapsto H(x, g, r)$ satisfies*

$$|H(x,g,r) - H(x',g',r)| \le \frac{1}{r}\|x - x'\| + \|g - g'\| \quad \forall\, x,\, x',\, g,\, g' \in X. \tag{6}$$

In particular, $(x, g) \mapsto H(x, g, r)$ is Lipschitz continuous.

Proof Parts a, b and c are direct consequences of [3, Lem. 2 & 3] with $d = -g$. For part d, we recall that $r \mapsto \mathrm{Prox}_{r\phi}(y)$ is continuous for fixed $y \in X$ [3, Lem. 3] and $y \mapsto \mathrm{Prox}_{r\phi}(y)$ is Lipschitz continuous with unit modulus for fixed $r \in (0, \infty)$. Now, suppose $\{(y_n, r_n)\} \subset X \times (0, \infty)$ with $y_n \to y$ and $r_n \to r > 0$. By Lipschitz continuity, we have that

$$\|\mathrm{Prox}_{r_n\phi}(y_n) - \mathrm{Prox}_{r\phi}(y)\| \le \|y_n - y\| + \|\mathrm{Prox}_{r_n\phi}(y) - \mathrm{Prox}_{r\phi}(y)\|.$$

Consequently, $(y, r) \mapsto \mathrm{Prox}_{r\phi}(y)$ is continuous. Hence, the composition of this map with the continuous map $(x, g, r) \mapsto (x - rg, r)$ is also continuous. Part e follows from the firm nonexpansivity of the proximity operator. □

3 Trust-Region Algorithm

At the k-th iteration of the trust-region algorithm introduced in [3], one computes a trial iterate x_k^+ that approximately solves the trust-region subproblem (2). To facilitate subproblem solver development, we assume that f_k is the quadratic model

$$f_k(x) := \frac{1}{2}\langle B_k(x - x_k), x - x_k\rangle + \langle g_k, x - x_k\rangle, \tag{7}$$

where $B_k \in \mathcal{L}(X)$ is self adjoint and $g_k \in X$ is an approximation of $\nabla f(x_k)$. The operator B_k encapsulates the curvature of f at x_k and is often the Hessian $\nabla^2 f(x_k)$ or a secant approximation thereof.

We require that the trial iterate x_k^+ satisfies the trust-region constraint

$$\left\|x_k^+ - x_k\right\| \le \kappa_{\mathrm{rad}}\Delta_k \tag{8a}$$

and FCD condition

$$m_k(x_k) - m_k(x_k^+) \ge \kappa_{\mathrm{fcd}} h_k \min\left\{\frac{h_k}{1 + \|B_k\|}, \Delta_k\right\}, \tag{8b}$$

where κ_{rad}, $\kappa_{\mathrm{fcd}} > 0$ are independent of k,

$$h_k := H(x_k, g_k, t_k) \tag{9}$$

with H as in (4), and $t_k > 0$. Note that (8b) ensures that $x_k^+ \in \mathrm{dom}\,\phi_k$ since the left-hand side would be $-\infty$ otherwise. Additionally, note that (8b) is a slight generalization of [3, Eq. (12b)], where t_k is a constant independent of k.

Given a trial iterate x_k^+ that satisfies (8), the trust-region algorithm accepts or rejects x_k^+ based on the ratio of actual and predicted reduction

$$\rho_k^* := \frac{\mathrm{ared}_k}{\mathrm{pred}_k},$$

where

$$\text{ared}_k := F(x_k) - F(x_k^+) \qquad \text{and} \qquad \text{pred}_k := m_k(x_k) - m_k(x_k^+).$$

Here, ared_k is the actual reduction of the objective function F achieved by x_k^+ relative to x_k and pred_k is the reduction predicted by the model m_k. In many practical applications, the objective function F cannot be computed accurately [14,19,26,27], necessitating the replacement of ared_k in ρ_k^* with an approximation denoted cred_k—the *computed reduction.* Algorithmically, we decide whether or not to accept x_k^+ based on the ratio of computed and predicted reduction

$$\rho_k := \frac{\text{cred}_k}{\text{pred}_k}. \tag{10}$$

We set $x_{k+1} = x_k^+$ if $\rho_k \geq \eta_1$ and $x_{k+1} = x_k$ otherwise. The trust-region algorithm then increases the radius Δ_k if $\rho_k \geq \eta_2$ and reduces Δ_k if $\rho_k < \eta_1$. The algorithmic parameters $0 < \eta_1 < \eta_2 < 1$ are user-specified with common values $\eta_1 = 10^{-4}$ and $\eta_2 = 0.75$.

To ensure cred_k is a sufficiently accurate approximation of ared_k, we require the following assumption.

Assumption 2 (Inexact Objective Function) *The accuracy of the computed reduction* cred_k *can be refined to satisfy the condition: there exists a constant* $\kappa_{\text{obj}} \geq 0$*, independent of* k*, such that*

$$|\text{ared}_k - \text{cred}_k| \leq \kappa_{\text{obj}} \left[\eta \min\{\text{pred}_k, \theta_k\}\right]^\zeta \quad \forall\, k, \tag{11}$$

where ζ*,* η*, and* θ_k *are (user-specified) positive real numbers that satisfy*

$$\zeta > 1, \qquad 0 < \eta < \min\{\eta_1, (1-\eta_2)\}, \qquad \textit{and} \qquad \lim_{k\to+\infty} \theta_k = 0.$$

Here, ζ *and* η *are independent of* k*.*

Condition (11) was first used in [26], where it was motivated by [45, Sec. 5.3.3]. Assumption 2 allows us to inexactly evaluate the objective function F. Moreover, since $\theta_k \to 0$ as $k \to +\infty$, we have that $\theta_k \leq \kappa_{\text{obj}}^{-1/(\zeta-1)}$ for sufficiently large k and

$$|\text{ared}_k - \text{cred}_k| \leq \kappa_{\text{obj}}[\eta \min\{\text{pred}_k, \theta_k\}]\theta_k^{\zeta-1} \leq \eta \min\{\text{pred}_k, \theta_k\}. \tag{12}$$

The consequence of (12) on the accuracy of ρ_k is summarized in the next lemma.

Lemma 1 (Lemma 6 in [3]) *If Assumption 2 holds, then there exists a positive integer* K_η *satisfying*

$$\rho_k^* = \frac{\text{ared}_k}{\text{pred}_k} \in [\rho_k - \eta, \rho_k + \eta] \qquad \forall\, k \geq K_\eta. \tag{13}$$

Lemma 1 ensures that successful steps produce a fraction of Cauchy decrease as demonstred in the following corollary.

Corollary 1 (Corollary 2 in [3]) *Let Assumption 2 hold and suppose x_k^+ is a trial iterate that satisfies* (8b) *with $\rho_k \geq \eta_1$, then*

$$\mathrm{cred}_k \geq \eta_1 \kappa_{\mathrm{fcd}} h_k \min\left\{ \frac{h_k}{1+\|B_k\|}, \Delta_k \right\}.$$

Moreover, if $k \geq K_\eta$ where K_η is defined in Lemma 1, then

$$\mathrm{ared}_k \geq (\eta_1 - \eta) \kappa_{\mathrm{fcd}} h_k \min\left\{ \frac{h_k}{1+\|B_k\|}, \Delta_k \right\}.$$

As with the computed reduction, the gradient of f often cannot be evaluated exactly in practice [14, 19, 25, 27]. Instead, we require that the approximate gradient satisfies the following assumption.

Assumption 3 (Subproblem Model) *The accuracy of the model gradient g_k can be refined to satisfy the condition: there exists a constant $\kappa_{\mathrm{grad}} \geq 0$, independent of k, such that*

$$\|g_k - \nabla f(x_k)\| \leq \kappa_{\mathrm{grad}} \min\{h_k, \Delta_k\} \quad \forall\, k. \tag{14}$$

With Assumptions 2 and 3, we can now state the trust-region algorithm for solving (1), listed in Algorithm 1.

Algorithm 1 Inexact Nonsmooth Trust-Region Algorithm

Require: Initial guess $x_0 \in \mathrm{dom}\,\phi$, initial radius $\Delta_0 > 0$, $0 < \eta_1 < \eta_2 < 1$, and $0 < \gamma_1 \leq \gamma_2 < 1 \leq \gamma_3$

for $k = 0, 1, 2, \ldots$ **do**
  Gradient Approximation: Compute g_k that satisfies Assumption 3
  Model Selection: Choose self-adjoint $B_k \in \mathcal{L}(X)$ and build m_k using (7)
  Step Computation: Compute $x_k^+ \in X$ that satisfies (8)
  Computed Reduction: Compute cred_k that satisfies Assumption 2
  Step Acceptance and Radius Update: Compute ρ_k as in (10)
  if $\rho_k < \eta_1$ **then**
    $x_{k+1} \leftarrow x_k$
    $\Delta_{k+1} \in [\gamma_1 \Delta_k, \gamma_2 \Delta_k]$
  else
    $x_{k+1} \leftarrow x_k^+$
    if $\rho_k \in [\eta_1, \eta_2)$ **then**
      $\Delta_{k+1} \in [\gamma_2 \Delta_k, \Delta_k]$
    else
      $\Delta_{k+1} \in [\Delta_k, \gamma_3 \Delta_k]$
    end if
  end if
end for

4 Convergence Analysis

The global convergence analysis of Algorithm 1 is essentially the same as in [3], despite the more general h_k definition. As such, we state the basic convergence results without proof unless significant modification is required.

Theorem 1 (Convergence of Algorithm 1) *Let $\{x_k\}$ be the sequence of iterates generated by Algorithm 1. If Assumptions 1, 2 and 3 hold and if*

$$\sum_{k=0}^{\infty}(1+\max_{i=1,\ldots,k}\|B_i\|)^{-1}=+\infty, \tag{15}$$

then

$$\liminf_{k\to\infty} h_k = 0 \qquad \text{and} \qquad \liminf_{k\to\infty} h(x_k,t_k)=0. \tag{16}$$

Under mild additional assumptions, we can improve upon Theorem 1 to show that the limit of $h(x_k, t)$, not just the lower limit, is zero for all $t > 0$.

Theorem 2 *Let the assumptions of Theorem 1 hold. In addition, suppose there exist $\kappa_{\rm curv} > 0$ and $t_{\max} > 0$ such that $\|B_k\| \le \kappa_{\rm curv}$ and $t_k \le t_{\max}$ for all k. Then,*

$$\lim_{k\to\infty} H(x_k,g_k,t)=0 \qquad \text{and} \qquad \lim_{k\to\infty} h(x_k,t)=0 \quad \forall\, t>0.$$

Proof By Theorem 1, the existence of $t_{\max}$ and Proposition 1, we have that

$$\liminf_{k\to\infty} H(x_k,g_k,t_{\max})=0 \qquad \text{and} \qquad \liminf_{k\to\infty} h(x_k,t_{\max})=0.$$

The result then follows from [4, Th. 1]. □

To derive a convergence rate for Algorithm 1, we require the following assumptions on the method used to generate the trial iterate x_k^+.

Assumption 4 (Subproblem Solver) *There exists $\mu \in (0, \frac{1}{2})$, independent of k, such that the trial iterate x_k^+ satisfies the decrease condition*

$$m_k(x_k^+)-m_k(x_k)\le \mu\left(\left\langle g_k, x_k^+-x_k\right\rangle+\phi(x_k^+)-\phi(x_k)\right) \coloneqq \mu\mathcal{Q}_k, \tag{17}$$

the trust-region constraint (8a)*, and either*

$$H(x_k^+,\nabla f_k(x_k^+),t_k)\le \tau_k h_k \quad \text{or} \quad \left\|x_k^+-x_k\right\| = \kappa_{\rm rad}\Delta_k, \tag{18}$$

where $\{\tau_k\} \subset [0,\infty)$ is a bounded sequence of relative tolerances. Moreover, let $x_k^{\rm n} \in X$ be any point that satisfies the first condition in (18)*. If there exists $x_k^{\rm n}$ with $\|x_k^{\rm n} - x_k\| \le \kappa_{\rm rad}\Delta_k$, then x_k^+ also satisfies the first condition in* (18)*.*

The assumption that x_k^+ eventually behaves like an inexact Newton iterate x_k^n is common in the trust-region literature. For instance, similar conditions are used in [34, Th. 4.9] for smooth unconstrained optimization and [29] smooth convex-constrained optimization.

Two of our subproblem solvers are iterative, in which case x_k^+ is selected as the final element in a sequence of iterates, $\{x_{k,0}, x_{k,1}, \ldots, x_{k,n_k}\}$ with $x_{k,0} = x_k$ and $x_{k,n_k} = x_k^+$. For these solvers, we employ the iteration decrease condition

$$\begin{aligned} m_k(x_{k,\ell+1}) - & m_k(x_{k,\ell}) \\ & \le \mu\left(\langle \nabla f_k(x_{k,\ell}), x_{k,\ell+1}-x_{k,\ell}\rangle + \phi(x_{k,\ell+1})-\phi(x_{k,\ell})\right) \end{aligned} \tag{19}$$

for $\ell = 0, 1, \ldots, n_k$, instead of (17). We further assume that the number of iterations is limited to `maxit`, i.e., $n_k \le$ `maxit` for $k = 1, 2, \ldots$. See [4] for the convergence analysis of iterative subproblem solvers. Under stronger assumptions than the preceding two theorems, the next result demonstrates that Algorithm 1 ultimately accepts every x_k^+, which eventually satisfiesi the first condition in (18).

Theorem 3 *Let the assumptions of Theorem 2 and Assumption 4 hold, and suppose there exists an open set* $U_0 \subseteq X$ *containing a stationary point* $\bar{x}$ *of* (1) *on which* f *is twice continuously Fréchet differentiable. Furthermore, suppose that* $x_k \to \bar{x}$, $g_k = \nabla f(x_k)$, *and* B_k *in* (7) *satisfies:*

1. *There exists* $K_0 \in \mathbb{N}$ *such that* B_k *is uniformly strongly monotone and bounded for* $k \geq K_0$, *i.e., there exist* $m > 0$ *and* $\kappa_{\text{curv}} > 0$ *such that*
$$m \|s\|^2 \leq \langle B_k s, s\rangle \quad \text{and} \quad \|B_k\| \leq \kappa_{\text{curv}} \tag{20}$$
for all $s \in X$ *and* $k \geq K_0$*; and*
2. *The Dennis-Moré condition holds, i.e.,*
$$\lim_{k\to\infty} \frac{\|(B_k - \nabla^2 f(\bar{x}))(x_k^+ - x_k)\|}{\|x_k^+ - x_k\|} = 0. \tag{21}$$

Then, there exists a positive integer K_1 *such that* $x_{k+1} = x_k^+$ *and* $\Delta_{k+1} \geq \Delta_k$ *for all* $k \geq K_1$.

Proof The main distinction between this result and [4, Th. 2] is the use of a quadratic model f_k whose Hessian satisfies the Dennis-Moré condition (21), in place of the gradient consistency condition M5 in [4].

We bound $|\rho_k^* - 1| = |(\text{ared}_k - \text{pred}_k)|/\text{pred}_k$ and show that it converges to zero if $s_k := x_k^+ - x_k \to 0$. Suppose $s_k \to 0$. To bound the numerator, we note that the nonsmooth terms cancel. Therefore, Taylor's theorem applied to the twice continuously differentiable function $\sigma \mapsto f(x_k + \sigma s_k)$ (for k sufficiently large) ensures the existence of $\sigma_k \in [0,1]$ for which

$$\begin{aligned}
|\text{ared}_k - \text{pred}_k| &= |f(x_k) - f(x_k^+) - f_k(x_k) + f_k(x_k^+)| \\
&= \tfrac{1}{2}\left|\left\langle (B_k - \nabla^2 f(x_k + \sigma_k s_k))s_k, s_k\right\rangle\right| \\
&\leq \tfrac{1}{2}\left(\left\|(B_k - \nabla^2 f(\bar{x}))s_k\right\| + \left\|(\nabla^2 f(\bar{x}) - \nabla^2 f(x_k + \sigma_k s_k))s_k\right\|\right)\|s_k\|.
\end{aligned}$$

Since $x_k \to \bar{x}$, $\nabla^2 f$ is continuous, $\{\sigma_k\} \subset [0,1]$, and (21) holds, we have that

$$|\text{ared}_k - \text{pred}_k| \leq o(\|s_k\|)\,\|s_k\| \qquad \text{as} \qquad \|s_k\| \to 0.$$

Moreover, the sufficient decrease condition (17) ensures that

$$\begin{aligned}
\mu \mathcal{Q}_k \geq m_k(x_k^+) - m_k(x_k) &= \mathcal{Q}_k + \tfrac{1}{2}\langle B_k s_k, s_k\rangle \\
\iff \qquad -(1-\mu)\mathcal{Q}_k &\geq \tfrac{1}{2}\langle B_k s_k, s_k\rangle.
\end{aligned}$$

Combining this with (20) yields

$$\text{pred}_k \geq -\mu \mathcal{Q}_k = -\frac{\mu}{1-\mu}(1-\mu)\mathcal{Q}_k \geq \frac{\mu}{1-\mu}\frac{m}{2}\|s_k\|^2 =: \kappa_0 \|s_k\|^2.$$

Combining the numerator and denominator bounds, we arrive at

$$|\rho_k^* - 1| \leq o(\|s_k\|)\frac{\|s_k\|}{\kappa_0 \|s_k\|^2} = o(1) \qquad \text{as} \qquad \|s_k\| \to 0.$$

Hence, if $\Delta_k \to 0$, then $\|s_k\| \to 0$ and consequently $|\rho_k^* - 1| \to 0$. Therefore, $\rho_k \geq \eta_2$ and $\Delta_{k+1} \geq \Delta_k$ for all k sufficiently large, which contradicts $\Delta_k \to 0$. The result then follows from [4, Cor. 2]. □

Our final result provides convergence rates for $\{x_k\}$ generated by Algorithm 1, when the trial iterates x_k^+ satisfy Assumption 4.

Theorem 4 *Let the assumptions of Theorem 3 hold.*

1. *If* $\tau_k \to \bar{\tau}$ *with*
$$0 < \bar{\tau} < \frac{m}{r_0 L + 1} \min\left\{ r_0, \frac{2m}{\kappa_{\text{curv}}^2} \right\},$$
then x_k *converges q-linearly to* $\bar{x}$*.*
2. *If* $\tau_k \to 0$*, then* x_k *converges q-superlinearly to* $\bar{x}$*.*
3. *If* $\nabla^2 f(\cdot)$ *is Lipschitz continuous on* U_0 *and* $\tau_k \leq \tau h_k^{1+\alpha}$ *for fixed* $\tau > 0$ *and* $\alpha \geq 0$*, then* x_k *converges q-quadratically to* $\bar{x}$*.*

Proof The result follows from the proof of [4, Th. 3] with [4, Cor. 2] replaced by Theorem 3. □

Before concluding this section, we provide a technical lemma that is useful for verifying that the subproblem solvers described in the subsequent section satisfy the sufficient decrease conditions (17) or (19).

Lemma 2 *Consider* $p : \mathbb{R} \to (-\infty, +\infty]$ *defined by* $p(t) = \frac{1}{2}\kappa t^2 + \psi(t)$*, where* $\kappa > 0$ *and* $\psi : \mathbb{R} \to (-\infty, +\infty]$ *is closed, convex and satisfies* $\psi(0) = 0$*.*

1. *The map* $t \mapsto \psi(t)/t$ *is nondecreasing on* $(0, +\infty)$*.*
2. *If there exists* $t_0 > 0$ *such that* $\psi(t_0) < 0$*, then there exists* $\bar{t} > 0$ *such that* $p(t) \leq \frac{1}{2}\psi(t) < 0$ *for all* $t \in [0, \bar{t}]$*.*
3. *Let* $t_\star \in (0, t_1]$ *denote a minimizer of* p *over* $[0, t_1]$ *for* $t_1 > 0$*. If there exists* $t_0 > 0$ *such that* $\psi(t_0) < 0$*, then* $\psi(t_\star) < 0$ *and* $p(t_\star) \leq \frac{1}{2}\psi(t_\star)$*.*

Proof To prove the first claim, let $0 < s \leq t$ and define $\tau = s/t$. The convexity of ψ and the assumption that $\psi(0) = 0$ ensure that
$$\psi(s)/s = \psi(\tau t)/(\tau t) \leq ((1-\tau)\psi(0) + \tau\psi(t))/(\tau t) = \psi(t)/t,$$
as desired.

For the second claim, suppose there exists $t_0 > 0$ such that $\psi(t_0) < 0$. We notice that
$$p(t) \leq \tfrac{1}{2}\psi(t) \quad\iff\quad t \leq -\psi(t)/(\kappa t).$$
Let $\bar{t} = -\psi(t_0)/(\kappa t_0) > 0$. Then, for any $t \in [0, \bar{t}]$, the first claim ensures that
$$t \leq \bar{t} = -\psi(t_0)/(\kappa t_0) \leq -\psi(t)/(\kappa t) \quad\implies\quad p(t) \leq \tfrac{1}{2}\psi(t) < 0,$$
as desired.

Finally, assume there exists $t_0 > 0$ such that $\psi(t_0) < 0$. The proof of this claim follows, in part, from the optimality of $t_\star$. In particular, we make repeated use of the first-order optimality condition
$$-\kappa t_\star \in \partial(\psi + \iota_{[0,t_1]})(t_\star),$$
which implies
$$\psi(t) \geq \psi(t_\star) + \kappa t_\star(t_\star - t) \qquad \forall\, t \in [0, t_1]. \tag{22}$$

Now, if $t_\star < t_0$, then $\psi(t_\star) < 0$ by the first claim. Otherwise, substituting $t = t_0$ in (22) and noting that $0 < t_0 \le t_\star \le t_1$, we obtain

$$0 > \psi(t_0) \ge \psi(t_\star) + \kappa t_\star (t_\star - t_0) \ge \psi(t_\star).$$

For the second part of this proof, we substitute $t = \frac{1}{2} t_\star$ in (22) to obtain

$$p(t_\star) = \psi(t_\star) + (\kappa t_\star)(t_\star - \tfrac{1}{2} t_\star) \le \psi(\tfrac{1}{2} t_\star) \le \tfrac{1}{2}\psi(t_\star),$$

where the final inequality follows because ψ is convex and satisfies $\psi(0) = 0$. □

Remark 1 (Sufficient Decrease for Iterative Subproblem Solvers) For iterative subproblem solvers, $x_{k,\ell+1}$ typically has the form $x_{k,\ell+1} = x_{k,\ell} + \alpha_{k,\ell} s_{k,\ell}$ for a step $s_{k,\ell} \in X$ and step length $\alpha_{k,\ell} > 0$. In this setting, (19) can be rewritten as

$$p(\alpha_{k,\ell}) \le \mu \psi(\alpha_{k,\ell}),$$

where p is defined in Lemma 2 with κ and ψ given by

$$\kappa = \langle B_k s_{k,\ell}, s_{k,\ell} \rangle \quad \text{and} \quad \psi(t) = t \langle \nabla f_k(x_{k,\ell}), s_{k,\ell} \rangle + \phi(x_{k,\ell} + t s_{k,\ell}) - \phi(x_{k,\ell}).$$

Suppose there exists $t_0 > 0$ such that $\psi(t_0) < 0$, then for all $t \in (0, t_0]$, $\psi(t) < 0$ by the first part of Lemma 2. Therefore, if $\kappa \le 0$, then we have that

$$p(t) \le \psi(t) \le \mu \psi(t) \quad \forall\, t \in (0, t_0]$$

and there exists $\alpha_{k,\ell}$ such that (19) holds. On the other hand, if $\kappa > 0$, then Lemma 2 ensures the existence of $\alpha_{k,\ell}$ for which (19) holds. In fact, $\alpha_{k,\ell}$ can be the minimizer of $p(t)$ over some bounded interval $[0, \bar{\alpha}_{k,\ell}]$ for any $\bar{\alpha}_{k,\ell} > 0$.

5 Trust-Region Subproblem Solvers

To achieve guaranteed global convergence as well as rapid local convergence, we generate trial iterates x_k^+ that improve upon the CP. The CP used in [3] is an extension of that used for smooth convex-constrained optimization [24, 29, 43] and requires Goldstein-type conditions to be satisfied. The trust-region algorithm proposed in [3] employs a bidirectional proximal search to satisfy these conditions and may require multiple evaluations of the proximity operator of ϕ. To avoid this computational expense, we introduce a simplified CP based on the SPG step [6]. This *spectral* CP requires a single evaluation of the proximity operator. Motivated by methods for smooth unconstrained and convex-constrained optimization, we introduce three subproblem solvers that improve upon the CP, producing trial iterates x_k^+ that satisfy the FCD condition (8b). The first is a double dogleg approach [16], the second a simplified version of the algorithm described in [24] that produces trial iterates using the SPG method, and the third generalizes the truncated CG method [41, 42].

5.1 Spectral Cauchy Points

We define the spectral CP at the k-th iteration of Algorithm 1 by

$$x_k^c \coloneqq x_k + \alpha_k(p_k(t_k) - x_k), \tag{23}$$

for $\alpha_k \in [0,1]$ and $t_k \in [t_{\min}, t_{\max}]$, where $p_k(t)$ is the proximal gradient path

$$p_k(t) \coloneqq \mathrm{Prox}_{t\phi}(x_k - tg_k) \tag{24}$$

and $0 < t_{\min} \le t_{\max} < +\infty$ are user-specified parameters. In general, $t_k \in [t_{\min}, t_{\max}]$ can be arbitrary. However, in our numerical examples we set

$$t_k = \min\{t_{\max}, \max\{t_{\min}, t_{k,0}\}\} \quad \text{for} \quad t_{k,0} \coloneqq \begin{cases} \frac{\|g_k\|^2}{\langle B_k g_k, g_k\rangle}, \text{ if } \langle B_k g_k, g_k\rangle > 0 \\ \frac{t_0}{\|g_k\|}, \text{ otherwise,} \end{cases}$$

where $t_0 > 0$ is user specified. To determine α_k, we first define

$$\alpha_{k,\max} \coloneqq \min\left\{1, \frac{\Delta_k}{\|s_k\|}\right\},$$

where $s_k \coloneqq (p_k(t_k) - x_k)$ and then define α_k to be the minimizer of the quadratic upper bound $q_k(\alpha)$, defined by

$$\begin{aligned} m_k(x_k + \alpha s_k) - m_k(x_k) &= f_k(x_k + \alpha s_k) + \phi(x_k + \alpha s_k) - f_k(x_k) - \phi(x_k) \\ &\le \alpha^2 \tfrac{1}{2}\langle B_k s_k, s_k\rangle + \alpha(\langle g_k, s_k\rangle + \phi(x_k + s_k) - \phi(x_k)) =: q_k(\alpha), \end{aligned} \tag{25}$$

over the interval $[0, \alpha_{k,\max}]$. The upper bound (25) follows from the convexity of ϕ. Note that since $\alpha_k \le \alpha_{k,\max}$, we have that (8a) is satisfied. In the following proposition, we prove that x_k^c satisfies the FCD condition (8b).

Proposition 2 *Let x_k^c be defined by (23) with $t_{\min} \le t_k \le t_{\max}$ and $0 \le \alpha_k \le 1$ given as the minimizer of the quadratic optimization problem*

$$\min_{\alpha \in \mathbb{R}} \; q_k(\alpha) \qquad \text{subject to} \qquad 0 \le \alpha \le \alpha_{k,\max}.$$

If $h_k > 0$, then x_k^c satisfies (8) with $\kappa_{\mathrm{fcd}} = \frac{1}{2}\min\{1, t_{\min}\}$ and $\kappa_{\mathrm{rad}} = 1$.

Proof Suppose $h_k > 0$. For simplicity, we define the following quantities

$$\kappa_k \coloneqq \langle B_k s_k, s_k\rangle \qquad \text{and} \qquad d_k \coloneqq \langle g_k, s_k\rangle + \phi_k(x_k + s_k) - \phi_k(x_k),$$

and note that if $\kappa_k > 0$, then the unconstrained minimizer of q_k is $-d_k/\kappa_k$; note $d_k \le -t_k h_k^2$ from (5). In this case, we have $\alpha_k = \min\{-d_k/\kappa_k, \alpha_{k,\max}\}$. When $\kappa_k = 0$, we have that $q_k(\alpha) = d_k\alpha \le -t_k h_k^2 \alpha$. Therefore, $\alpha_k = \alpha_{k,\max} > 0$. Finally, if $\kappa_k < 0$, then q_k is concave and α_k is either 0 or $\alpha_{k,\max}$. Considering the two cases that define $\alpha_{k,\max}$, we see that

$$q_k(\alpha_{k,\max}) \le -h_k \min\{t_k h_k, \Delta_k\} < 0 = q_k(0)$$

and hence $\alpha_k = \alpha_{k,\max}$. This demonstrates that there are three cases we must consider: $\alpha_k = 1$, $\alpha_k = \Delta_k / \|s_k\|$, and $\alpha_k = -d_k/\kappa_k$. The remainder of the proof relies heavily on the bound (25), which in the notation of this proof is

$$m_k(x_k) - m_k(x_k + \alpha s_k) \ge -\tfrac{1}{2}\alpha^2 \kappa_k - \alpha d_k = -q_k(\alpha) \quad \forall\, \alpha \in [0,1].$$

Case $\alpha_k = 1$: If $\kappa_k \le 0$, then

$$m_k(x_k) - m_k(x_k^c) \ge -d_k \ge t_k h_k^2 \ge t_{\min} \frac{h_k^2}{1 + \|B_k\|},$$

where we used the facts that $1 + \|B_k\| \ge 1$ and $t_k \ge t_{\min}$. If $\kappa_k > 0$, then the unconstrained minimizer of q_k, $-d_k/\kappa_k$, is greater than or equal to one and so $-\kappa_k \ge d_k$. Consequently,

$$m_k(x_k) - m_k(x_k^c) \ge -\frac{1}{2}\kappa_k - d_k \ge -\frac{1}{2}d_k \ge \frac{t_{\min}}{2} \frac{h_k^2}{1 + \|B_k\|}.$$

Case $\alpha_k = \Delta_k / \|s_k\|$: If $\kappa_k \le 0$, then $\alpha_k \le 1$ and

$$m_k(x_k) - m_k(x_k^c) \ge -\alpha_k d_k \ge \frac{\Delta_k}{\|s_k\|} t_k h_k^2 = \Delta_k h_k.$$

If $\kappa_k > 0$, then $\alpha_k = \Delta_k / \|s_k\| \le -d_k/\kappa_k$. Consequently,

$$m_k(x_k) - m_k(x_k + \alpha_k s_k) = \alpha_k(-\tfrac{1}{2}\alpha_k \kappa_k - d_k) \ge -\frac{\alpha_k}{2} d_k \ge \frac{1}{2}\Delta_k h_k.$$

Case $\alpha_k = -d_k/\kappa_k$: In this case, $0 < -d_k \le \kappa_k \le \|B_k\| \|s_k\|^2$ and

$$m_k(x_k) - m_k(x_k^c) \ge \frac{1}{2}\frac{d_k^2}{\kappa_k} \ge \frac{1}{2} \frac{t_k^2 h_k^4}{\|B_k\| \|s_k\|^2} \ge \frac{1}{2} \frac{h_k^2}{1 + \|B_k\|}.$$

Combining cases 1, 2 and 3 proves that (8b) holds for x_k^c. □

5.2 Dogleg Subproblem Solver

Dogleg and double dogleg approaches are common trust-region methods that construct piecewise linear paths between the Cauchy and Newton points, and then minimize the quadratic model along these paths. To generalize the dogleg approach to nonsmooth problems of the form (1), we define a Newton point x_k^n to "approximately" solve the trust-region subproblem (2), ignoring the trust-region constraint:

$$\min_{x \in X} \; f_k(x) + \phi(x). \tag{26}$$

A basic approach to computing the Newton point x_k^n is to apply a finite number of iterations of a descent method to (26), starting at x_k^c. However, if the proximal mapping of ϕ is semismooth, we can instead compute x_k^n by applying a semismooth Newton method [38] to solve the first-order optimality condition

$$x - \mathrm{Prox}_{t\phi}(x - t(B_k(x - x_k) + g_k)) = 0 \tag{27}$$

or the normal mapping equation [40]

$$B_k(\mathrm{Prox}_{t\phi}(z) - x_k) + g_k + t^{-1}(z - \mathrm{Prox}_{t\phi}(z)) = 0 \quad \text{with} \quad x = \mathrm{Prox}_{t\phi}(z). \tag{28}$$

Algorithm 2 Dogleg Subproblem Solver

Require: The trust-region radius Δ_k and a relaxation parameter $\theta \in (0,1)$ (e.g., $\theta = 0.7$)

1: Compute a generalized Cauchy point x_k^c
2: **if** $\|s_k^c\| = \Delta_k$ **then**
3: Return $x_k^+ = x_k^c$
4: **else**
5: Compute a point x_k^n that satisfies (29)
6: **if** $\|s_k^n\| \le \Delta_k$ **then**
7: Return $x_k^+ = x_k^n$
8: **else**
9: Compute $\gamma = 1 + \theta(\gamma_0 - 1)$ where $\gamma_0 \in (0,1)$ solves

$$f_k(x_k + \gamma_0 s_k^n) + \gamma_0(\phi(x_k^n) - \phi(x_k)) + \phi(x_k) = m_k(x_k^c)$$

10: **if** $\gamma\|s_k^n\| \le \Delta_k$ **then**
11: Compute the solution $\alpha_k > 0$ to the quadratic optimization problem

$$\min_{\alpha \in [\gamma, \Delta_k/\|s_k^n\|]} \{f_k(x_k + \alpha s_k^n) + \alpha(\phi(x_k^n) - \phi(x_k)) + \phi(x_k)\}$$

12: Return $x_k^+ = x_k + \alpha_k s_k^n$
13: **else**
14: Compute $\alpha_{k,\max} \in (0,1)$ such that

$$\|s_k^c + \alpha_{k,\max}(\gamma s_k^n - s_k^c)\| = \Delta_k$$

15: Compute the solution $\alpha_k \in [0,1]$ to the quadratic optimization problem

$$\min_{\alpha \in [0, \alpha_{k,\max}]} \{f_k(x_k^c + \alpha(\gamma s_k^n - s_k^c)) + \alpha(\phi(x_k + \gamma s_k^n) - \phi(x_k^c)) + \phi(x_k^c)\}$$

16: Return $x_k^+ = x_k^c + \alpha_k(\gamma s_k^n - s_k^c)$
17: **end if**
18: **end if**
19: **end if**

One advantage of (28) over (27) is that $x_k^n \in \operatorname{dom}\phi$ by construction. Independent of the approach for generating the Newton point, we assume that x_k^n satisfies the basic model decrease condition

$$m_k(x_k^n) < m_k(x_k^c) < m_k(x_k). \tag{29}$$

We denote the Cauchy and Newton steps by $s_k^c := x_k^c - x_k$ and $s_k^n := x_k^n - x_k$, respectively. The dogleg algorithm is listed in Algorithm 2.

The root $\gamma_0 \in (0,1)$ in line 9 of Algorithm 2 exists since

$$q(\alpha) = f_k(x_k + \alpha s_k^n) + \alpha(\phi(x_k^n) - \phi(x_k)) + \phi(x_k)$$

is a continuous quadratic polynomial that satisfies

$$q(0) = m_k(x_k) > m_k(x_k^c) > m_k(x_k^n) = q(1).$$

See Figure 1 for an illustration of this fact. Now, if the condition on line 10 holds, then the trial iterate x_k^+ satisfies (8) since

$$m_k(x_k + \alpha s_k^n) \le q(\alpha) \le q(\gamma_0) = m_k(x_k^c) \quad \forall\, \alpha \in [\gamma_0, 1],$$

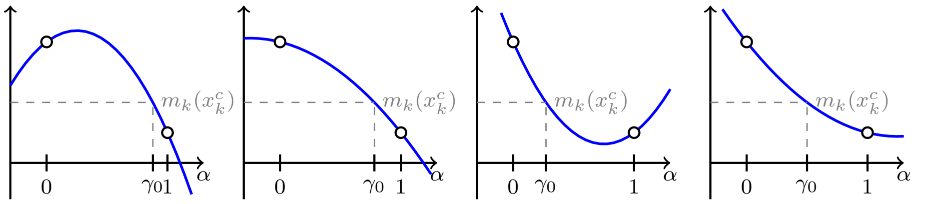

Fig. 1 Possible cases at line 9 of Algorithm 2 when $\langle B_k s_k^n, s_k^n \rangle \neq 0$. The two left images correspond to $\langle B_k s_k^n, s_k^n \rangle < 0$ while the two right images correspond to $\langle B_k s_k^n, s_k^n \rangle > 0$. The blue curve is $q(\alpha)$, which satisfies $q(0) = m_k(x_k)$, $q(\gamma_0) = m_k(x_k^c)$ and $q(1) = m_k(x_k^n)$.

where the first inequality follows from the convexity of ϕ. On the other hand, if the condition on line 10 is violated, then $\alpha_{k,\max}$ in line 14 exists since $\|s_k^c\| < \Delta_k$ and $\gamma \|s_k^n\| > \Delta_k$, and again the convexity of ϕ ensures that

$$m_k(x_k^c + \alpha(\gamma s_k^n - s_k^c)) \leq f_k(x_k^c + \alpha(\gamma s_k^n - s_k^c)) + \alpha(\phi(x_k + \gamma s_k^n) - \phi(x_k^c)) + \phi(x_k^c)$$

for all $\alpha \in [0, 1]$. Consequently, (8) holds and Algorithm 2 produces a viable trial iterate.

In order to achieve rapid convergence as in Theorem 4, we augment the basic model decrease condition (29) with the conditions

$$m_k(x_k^n) - m_k(x_k) \leq \mu(\langle g_k, s_k^n \rangle + \phi(x_k^n) - \phi(x_k)) \tag{30a}$$

$$\|G_k(x_k^n, r_k)\| \leq \tau_k h_k, \tag{30b}$$

where μ and τ_k are as in Assumption 4. The condition (30a) ensures that x_k^n produces sufficient decrease of the model m_k as in (17), while condition (30b) ensures that x_k^n satisfies (18). Consequently, for sufficiently large Δ_k, $x_k^+ = x_k^n$ which satisfies Assumption 4, resulting in superlinear, even quadratic, convergence under the assumptions of Theorem 4.

5.3 Spectral Proximal Gradient Subproblem Solver

Building upon the dogleg approach of Section 5.2, we can improve upon the spectral CP described in Section 5.1 using additional SPG iterations. This approach is closely related to the subproblem solver described in [24] for convex-constrained optimization, which was generalized to our problem class in [3]. In contrast to the subproblem solver described in [3], our solver does not perform a backtracking linesearch to compute the convex combination parameter α, nor does it require the evaluation of the proximity operator of ϕ augmented with the indicator function of the trust-region constraint. Instead, we compute $\alpha \in [0, 1]$ by minimizing a quadratic upper bound for our model, similar to (25). This subproblem solver is listed in Algorithm 3. The algorithm employs stopping conditions similar to those used in truncated CG for unconstrained problems. In particular, if negative curvature is encountered, the algorithm takes the longest possible step in that direction. Similarly, if the computed step violates the trust-region constraint, then the step is truncated. In addition to these stopping conditions, we terminate Algorithm 3 if the iteration limit `maxit` is exceeded or if the stopping criterion

$$h_{k,\ell} \leq \min\{\bar{\tau}, \tau_k h_{k,0}\} \tag{31}$$

is satisfied for $\bar{\tau} > 0$ and $\tau_k > 0$. Here, $x_{k,\ell}$ is the ℓ-th iterate and

$$h_{k,\ell} := H(x_{k,\ell}, \nabla f_k(x_{k,\ell}), \lambda_{k,\ell}),$$

where $\lambda_{k,\ell} \in [t_{\min}, t_{\max}]$ is the safeguarded spectral step length.

Algorithm 3 SPG Trust-Region Subproblem Solver

Require: The initial guess $x_{k,0} = x_k$, $f_{k,0} = f_k(x_k)$, $\phi_{k,0} = \phi(x_k)$, $m_{k,0} = f_{k,0} + \phi_{k,0}$ $d_{k,0} = g_k$, an integer `maxit`, and positive tolerances $\bar{\tau}$ and τ_k, the positive safeguards $t_{\min} \le t_{\max}$, and $\lambda_{k,0} = t_k \in [t_{\min}, t_{\max}]$

1: Set $\ell = 0$
2: **while** $\ell <$ `maxit` **and** $h_{k,\ell} > \min\{\bar{\tau}, \tau_k h_{k,0}\}$ **and** $\|x_{k,\ell} - x_k\| < \Delta_k$ **do**
3:     Set $s \leftarrow \mathrm{Prox}_{\lambda_{k,\ell}\phi}(x_{k,\ell} - \lambda_{k,\ell} d_{k,\ell}) - x_{k,\ell}$
4:     Set $\alpha_{\max} \leftarrow 1$
5:     **if** $\|x_{k,\ell} + s - x_k\| > \Delta_k$ **then**
6:         Set $\alpha_{\max} > 0$ so that $\|x_{k,\ell} + \alpha_{\max} s - x_k\| = \Delta_k$
7:     **end if**
8:     Compute $\hat{\phi}_{k,\ell} \leftarrow \phi(x_{k,\ell} + s)$, $b \leftarrow B_k s$, and $\kappa \leftarrow \langle b, s\rangle$
9:     **if** $\kappa \le 0$ **then**
10:         Set $\alpha \leftarrow \alpha_{\max}$
11:     **else**
12:         Set $\alpha \leftarrow \min\{\alpha_{\max}, -(\langle d_{k,\ell}, s\rangle + \hat{\phi}_{k,\ell} - \phi_{k,\ell})/\kappa\}$
13:     **end if**
14:     Set $x_{k,\ell+1} \leftarrow x_{k,\ell} + \alpha s$, $d_{k,\ell+1} \leftarrow d_{k,\ell} + \alpha b$, and $\phi_{k,\ell+1} \leftarrow \phi(x_{k,\ell+1})$
15:     **if** $\kappa \le 0$ **then**
16:         Set $\bar{\lambda} \leftarrow t_k / \|d_{k,\ell+1}\|$
17:     **else**
18:         Set $\bar{\lambda} \leftarrow \langle s, s\rangle / \kappa$
19:     **end if**
20:     Set $\lambda_{k,\ell+1} \leftarrow \max\{t_{\min}, \min\{t_{\max}, \bar{\lambda}\}\}$
21:     Set $\ell \leftarrow \ell + 1$
22: **end while**
23: Return $x_k^+ \leftarrow x_{k,\ell+1}$ as the approximate solution

The convexity of ϕ and the definition of s in line 3 of Algorithm 3 ensures that

$$\begin{aligned} m_k(x_{k,\ell} + \alpha s) &= f_k(x_{k,\ell} + \alpha s) + \phi(x_{k,\ell} + \alpha s) \\ &\le f_k(x_{k,\ell} + \alpha s) + \alpha(\phi(x_{k,\ell} + s) - \phi(x_{k,\ell})) + \phi(x_{k,\ell}). \end{aligned} \tag{32}$$

The upper bound (32) is quadratic in α since f_k is. Consequently, the α computed in lines 4 through 13 in Algorithm 3 is the minimizer of (32) subject to the constraints that $\alpha \in [0, 1]$ and $\|x_{k,\ell} + \alpha s - x_k\| \le \Delta_k$. One consequence of this is that

$$m_k(x_{k,\ell+1}) \le m_k(x_{k,\ell}) \qquad \forall\, \ell = 1, 2, \ldots.$$

Since the first step $x_{k,1} = x_k^c$, where x_k^c is the Cauchy point defined in (23), we have that (8) is satisfied.

Theorem 4 ensures rapid convergence of Algorithm 3 as long as (19) is satisfied. Similar to the proof of Proposition 2, we can demonstrate that (19) is satisfied by considering three cases: $\alpha = 1$, α solves $\|x_{k,\ell} + \alpha s - x_k\| = \Delta_k$ and $\alpha = -(\langle d_{k,\ell}, s\rangle + \hat{\phi}_{k,\ell} - \phi_{k,\ell})/\kappa = -\psi(1)/\kappa$, where ψ and κ are specified in Remark 1. As described in Remark 1, if $\kappa \le 0$ then (19) holds since the SPG step satisfies

$\psi(1) < 0$. Now suppose $\kappa > 0$. If $\alpha = 1$, then $1 < -\psi(1)/\kappa$ or equivalently $\kappa < -\psi(1)$, which ensures that

$$p(1) = \tfrac{1}{2}\kappa + \psi(1) < \tfrac{1}{2}(-\psi(1)) + \psi(1) = \tfrac{1}{2}\psi(1).$$

In the second case, we have that $\alpha \le \min\{1, -\psi(1)/\kappa\}$ and so $\kappa\alpha \le -\psi(1)$ and $-\psi(\alpha)/\alpha \ge -\psi(1)$ by Lemma 2. These two facts imply

$$p(\alpha) = \tfrac{1}{2}\kappa\alpha^2 + \psi(\alpha) \le \tfrac{1}{2}\alpha(-\psi(1)) + \psi(\alpha) \le \tfrac{1}{2}\alpha(-\psi(\alpha)/\alpha) + \psi(\alpha) = \tfrac{1}{2}\psi(\alpha).$$

Finally, if $\alpha = -\psi(1)/\kappa$, then $-\psi(1)/\kappa \le 1$ and

$$p(\alpha) = \tfrac{1}{2}\psi(1)^2/\kappa + \psi(\alpha) \le \tfrac{1}{2}\alpha(-\psi(1)) + \psi(\alpha) \le \tfrac{1}{2}\alpha(-\psi(\alpha)/\alpha) + \psi(\alpha) = \tfrac{1}{2}\psi(\alpha).$$

Consequently, (19) is satisfied and Theorem 4 ensures rapid convergence.

5.4 Nonlinear Conjugate Gradient Subproblem Solver

Motivated by its efficiency for solving smooth unconstrained [41, 42] and constrained [22, 23, 29] optimization problems, we extend the truncated CG algorithm to solve (2) for potentially nonsmooth nonquadratic term ϕ. There are three locations in the truncated CG algorithm that must be modified: first, we replace the negative gradient computed at each iteration with the SPG step

$$p_{k,\ell} = \frac{1}{\lambda_{k,\ell}}(\mathrm{Prox}_{\lambda_{k,\ell}\phi}(x_{k,\ell} - \lambda_{k,\ell}\nabla f_k(x_{k,\ell})) - x_{k,\ell}), \tag{33}$$

where $x_{k,\ell}$ denotes the ℓ-th CG iteration and $\lambda_{k,\ell} \in [t_{\min}, t_{\max}]$ is the safeguarded spectral step length (see lines 15-19 in Algorithm 3); second, we modify the line search since the model m_k is not necessarily quadratic; and third, we select the the conjugacy parameter β using a nonlinear CG rule such as the nonnegative Dai-Yuan parameter [15]

$$\beta_{k,\ell} = \max\left\{0, \frac{\|p_{k,\ell}\|^2}{\langle p_{k,\ell-1} - p_{k,\ell}, s_{k,\ell-1}\rangle}\right\}.$$

For the line search, we employ an iterative procedure to determine the step length $\alpha > 0$ that approximately minimizes the one-dimensional function

$$q_{k,\ell}(\alpha) \coloneqq m_k(x_{k,\ell} + \alpha s_{k,\ell}).$$

To determine α, we first minimize the quadratic upper bound of $q_{k,\ell}$

$$q_{k,\ell}(t\gamma_{k,\ell}) \le f_k(x_{k,\ell} + t\gamma_{k,\ell}s_{k,\ell}) + t(\phi(x_{k,\ell} + \gamma_{k,\ell}s_{k,\ell}) - \phi(x_{k,\ell})) + \phi(x_k, \ell), \tag{34}$$

for $t \in [0, 1]$, where $\gamma_{k,\ell} \in (0, \bar{\alpha}_{k,\ell}]$ is chosen so that $\phi(x_{k,\ell} + \gamma_{k,\ell}s_{k,\ell}) < +\infty$ and $\bar{\alpha}_{k,\ell} > 0$ is chosen so that

$$\|x_{k,\ell} + \bar{\alpha}_{k,\ell}s_{k,\ell} - x_k\| = \Delta_k. \tag{35}$$

Note that the upper bound in (34) follows from the convexity of ϕ and since it is quadratic, we can compute the exact minimizer, $t_{k,\ell}$. Using this minimizer, we define the initial guess $\alpha^0_{k,\ell} \coloneqq t_{k,\ell}\gamma_{k,\ell}$. We then approximately minimize $q_{k,\ell}$ using finitely many iterations of Brent's method [8], which produces the step length $\alpha_{k,\ell}$. Since Brent's method produces a sequence of decreasing function values, we have that $q_{k,\ell}(\alpha_{k,\ell}) \le q_{k,\ell}(\alpha^0_{k,\ell})$. In addition, we terminate Brent's method when the computed step satisfies (19), i.e., $\alpha_{k,\ell}$ satisfies

$$q_{k,\ell}(\alpha_{k,\ell}) - q_{k,\ell}(0) \le \mu(\alpha_{k,\ell}\langle \nabla f_k(x_{k,\ell}), s_{k,\ell}\rangle + \phi(x_{k,\ell} + \alpha_{k,\ell}s_{k,\ell}) - \phi(x_{k,\ell})). \quad (36)$$

Similar to other nonlinear CG methods, we employ restarts. That is, we set $\beta_{k,\ell} = 0$ (i.e., revert to the SPG step) if the current step $s_{k,\ell}$ does not produce sufficient decrease [37] as given by the inequality

$$\langle \nabla f_k(x_{k,\ell}), s_{k,\ell}\rangle + \phi(x_{k,\ell} + s_{k,\ell}) - \phi(x_{k,\ell}) > -(1-\eta)\, \|p_{k,\ell}\|^2 . \quad (37)$$

With (37) in mind, we set

$$\gamma_{k,\ell} := \begin{cases} \min\{\bar{\alpha}_{k,\ell}, \lambda_{k,\ell}\} & \text{if } \ell = 0 \text{ or } \beta_{k,\ell} = 0, \\ \min\{\bar{\alpha}_{k,\ell}, 1\} & \text{otherwise,} \end{cases} \quad (38)$$

which ensures that $\phi(x_{k,\ell} + \alpha^0_{k,\ell}s_{k,\ell}) < +\infty$. Moreover, since the step length defined in Proposition 2 is feasible with respect to the one-dimensional minimization problem considered here, we have that each step of the truncated CG procedure satisfies (8b). We list the complete routine in Algorithm 4.

By reverting to the SPG step when (37) is satisfied, we ensure that

$$\psi(t) = t\,\langle \nabla f_k(x_{k,\ell}), s_{k,\ell}\rangle + \phi(x_{k,\ell} + ts_{k,\ell}) - \phi(x_{k,\ell}) < 0$$

with $t = \lambda_{k,\ell}$ if (37) holds and with $t = 1$ otherwise. As a consequence, Lemma 2 ensures that there exists an $\alpha_{k,\ell}$ such that (36) is satisfied. Hence, (17) is satisfied by Algorithm 4 and we can expect rapid convergence under the assumptions of Theorem 4.

6 Numerical Results

We demonstrate the performance of each subproblem solver on five numerical examples: the first two arising from data science and the final three from PDE-constrained optimization.

Low-Rank Matrix Completion. Our first example is the rank minimization problem

$$\min_{X \in \mathbb{R}^{M\times N}} \ \frac{1}{2}\, \|\mathcal{A}X - Y\|_F^2 + \|X\|_* , \quad (39)$$

where $\|\cdot\|_F$ is the Frobenius norm, $\|\cdot\|_*$ is the nuclear norm, and $\mathcal{A}$ is a selection matrix that observes 50% of the matrix entries [2,10,39]. In our example, $M = N = 225$ and Y is the observed data, which we corrupt with additive Gaussian noise (mean zero and variance 0.01). The matrix used to generate Y has rank 25. Recall that the nuclear norm has a computable proximity operator that is semismooth [17].

Algorithm 4 Truncated Nonlinear CG Trust-Region Subproblem Solver

Require: The initial guess $x_{k,0} = x_k$, $d_{k,0} = g_k$, an integer `maxit`, and positive tolerances $\bar{\tau}$, τ_k, $t_{\min} \le t_{\max}$, $\lambda_{k,0} = t_k \in [t_{\min}, t_{\max}]$, and $\eta \in (0, 1]$

Set $\ell \leftarrow 0$
Set $p_{k,0} \leftarrow (\mathrm{Prox}_{\lambda_{k,0}\phi}(x_{k,0} - \lambda_{k,0} d_{k,0}) - x_{k,0})/\lambda_{k,0}$ and $s_{k,0} \leftarrow p_{k,0}$
Compute $h_{k,0} \leftarrow \|p_{k,0}\|$
while $\ell <$ `maxit` **and** $h_{k,\ell} > \min\{\bar{\tau}, \tau_k h_{k,0}\}$ **and** $\|x_{k,\ell} - x_k\| < \Delta_k$ **do**
  Set $b_{k,\ell} \leftarrow B_k s_{k,\ell}$ and $\kappa_{k,\ell} \leftarrow \langle b_{k,\ell}, s_{k,\ell}\rangle$
  Compute $\alpha_{k,\ell} \in (0, \bar{\alpha}_{k,\ell}]$ that satisfies (36), where $\bar{\alpha}_{k,\ell}$ is the positive root of (35)
  Set $x_{k,\ell+1} \leftarrow x_{k,\ell} + \alpha_{k,\ell} s_{k,\ell}$ and $d_{k,\ell+1} \leftarrow d_{k,\ell} + \alpha_{k,\ell} b_{k,\ell}$
  if $\kappa_{k,\ell} \le 0$ **then**
    Set $\bar{\lambda} \leftarrow t_k / \|d_{k,\ell+1}\|$
  else
    Set $\bar{\lambda} \leftarrow \|s_{k,\ell}\|^2 / \kappa_{k,\ell}$
  end if
  Set $\ell \leftarrow \ell + 1$
  Set $\lambda_{k,\ell} \leftarrow \max\{t_{\min}, \min\{t_{\max}, \bar{\lambda}\}\}$
  Set $p_{k,\ell} \leftarrow (\mathrm{Prox}_{\lambda_{k,\ell}\phi}(x_{k,\ell} - \lambda_{k,\ell} d_{k,\ell}) - x_{k,\ell})/\lambda_{k,\ell}$
  Set $h_{k,\ell} \leftarrow \|p_{k,\ell}\|$
  Set $\beta_{k,\ell} \leftarrow \max\{0, h_{k,\ell}^2 / \langle p_{k,\ell-1} - p_{k,\ell}, s_{k,\ell-1}\rangle\}$
  Set $s_{k,\ell} \leftarrow p_{k,\ell} + \beta_{k,\ell} s_{k,\ell-1}$
  if $\langle d_{k,\ell}, s_{k,\ell}\rangle + \phi(x_{k,\ell} + s_{k,\ell}) - \phi(x_{k,\ell}) > -(1-\eta)\|p_{k,\ell}\|^2$ **then**
    Set $s_{k,\ell} \leftarrow p_{k,\ell}$ and $\beta_{k,\ell} \leftarrow 0$
  end if
end while
Return $x_k^+ \leftarrow x_{k,\ell}$ as the approximate solution

Support Vector Machine. Our second example is the nonconvex support vector machine problem

$$\min_{x\in\mathbb{R}^n} \frac{1}{m}\sum_{i=1}^{m}\{1 - \tanh(b_i\langle a_i, x\rangle)\} + \lambda\|x\|_1, \tag{40}$$

where $\lambda > 0$, $b_i \in \{-1, 1\}$ are labels, and $a_i \in \mathbb{R}^n$ are data points for $i = 1, \dots, n$. This problem was studied in [12] and is a nonsmooth extension of the problems considered in [31,44]. We use the `phishing` data set [18] from the `LIBSVM` data repository [11]. The number of data points is $m = 11{,}055$ and the number of features is $n = 68$. We set the regularization parameter to $\lambda = 10^{-2}$.

Optimal Control of Burgers' Equation. Our third example is the optimal control of Burgers' equation

$$\min_{z\in L^2(\Omega)} \frac{1}{2}\int_\Omega ([S(z)](x) - w(x))^2 \mathrm{d}x + \frac{\alpha}{2}\int_\Omega z(x)^2\,\mathrm{d}x + \beta\int_\Omega |z(x)|\,\mathrm{d}x \tag{41a}$$

where $\Omega = (0, 1)$ is the physical domain, $\alpha = 10^{-4}$ and $\beta = 10^{-2}$ are penalty parameters, $w(x) = -x^2$ is the target state, and $S(z) = u \in H^1(\Omega)$ solves the weak form of Burgers' equation

$$\begin{aligned} -\nu u'' + u\,u' &= z + f \quad \text{in } \Omega, \\ u(0) = 0, &\quad u(1) = -1, \end{aligned} \tag{42}$$

where $f(x) = 2(\nu + x^3)$ and $\nu = 0.08$. We discretize the state u using continuous piecewise linear finite elements and the control z using piecewise constants on a uniform mesh with $n = 512$ intervals.

Semilinear Optimal Control. Our fourth example is the optimal control of a semilinear elliptic PDE

$$\min_{z\in L^2(\Omega)} \; \frac{1}{2}\int_\Omega ([S(z)](x) - w(x))^2 \mathrm{d}x + \frac{\alpha}{2}\int_\Omega z(x)^2 \,\mathrm{d}x + \beta\int_\Omega |z(x)|\,\mathrm{d}x \tag{43a}$$

$$\text{subject to} \qquad -25 \le z \le 25 \;\text{ a.e.}, \tag{43b}$$

where $\Omega = (0,1)^2$ is the physical domain, $\alpha = 10^{-4}$ and $\beta = 10^{-2}$ are penalty parameters, $w \equiv -1$ is the target state, and $u = S(z) \in H^1(\Omega)$ solves the weak form of the semilinear elliptic PDE

$$-\Delta u + u^3 = z \quad \text{in } \Omega \tag{44a}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{44b}$$

We discretize the state u using continuous piecewise linear finite elements on a uniform triangular mesh with 131,072 elements and the control variable z using piecewise constants on the same mesh, resulting in 131,072 degrees of freedom.

Topology Optimization. Our final example is the compliance minimization problem

$$\min_{\rho\in L^2(\Omega)} \int_{\Gamma_t} T(x)[S(\rho)](x)\,\mathrm{d}x \tag{45a}$$

$$\text{subject to} \quad \int_\Omega \rho(x)\,\mathrm{d}x = v|\Omega|, \quad 0 \le \rho \le 1 \;\text{ a.e.}, \tag{45b}$$

where $\Omega = (0,150)\times(0,50)$ is the physical domain, $v = 0.4$ is the volume fraction, $\Gamma_d = \{0\}\times[0,50]$ is the fixed boundary, $\Gamma_t = \partial\Omega \setminus \Gamma_d$ is the traction boundary and $S(\rho) = u \in H^1(\Omega)^2$ solves the weak form of the linear elasticity equations

$$-\nabla\cdot(K(\rho) : \varepsilon) = 0 \qquad \text{in } \Omega \tag{46a}$$

$$\varepsilon = \tfrac{1}{2}(\nabla u + \nabla u^\top) \qquad \text{in } \Omega \tag{46b}$$

$$K(\rho) : \varepsilon\mathbf{n} = T \qquad \text{in } \Gamma_t \tag{46c}$$

$$u = 0 \qquad \text{in } \Gamma_d. \tag{46d}$$

Here, $\mathbf{n}$ denotes the outward pointing normal vector,

$$K(\rho) \coloneqq [\kappa_{\min} + (1-\kappa_{\min})\mathbb{F}(\rho)^3]K_0,$$

K_0 is the usual isotropic elasticity matrix, $\mathbb{F}$ is the Helmholtz filter [28] with filter radius 0.1, $\kappa_{\min} = 10^{-4}$, and $T \in L^2(\Gamma_t)^2$ is the traction force: $T(x) = (0,0)$ for $x \in \Gamma_t \setminus (\{150\}\times[0,1])$ and $T(x) = (0,-1)$ for $x \in \{150\}\times[0,1]$. For our numerical results, the Young's modulus is 200 and the Poisson ratio is 0.29. We discretize the state u and the filtered density $\mathbb{F}(\rho)$ using continuous piecewise linear finite elements on a 150×50 uniform quadrilateral mesh, and the density ρ using piecewise constants on the same mesh, resulting in 7,500 degress of freedom.

For each example, we employ the quadratic model (7) with $g_k = \nabla f(x_k)$ and $B_k = \nabla^2 f(x_k)$, making Algorithm 1 an inexact proximal Newton method that rigorously handles indefinite Hessians. We test up to six subproblem solvers, depending on the nonsmooth term ϕ:

- `SPG` is the spectral proximal gradient solver described in [3, Alg. 5];
- `SPG2` is the spectral proximal gradient solver Algorithm 3;
- `NCG` is the nonlinear CG solver Algorithm 4;
- `SEMI` is Algorithm 2 with the Newton point computed via (27)[1];
- `NORM` is Algorithm 2 with the Newton point computed via (28)[1];
- `OBM` is the L^1-specific solver described in Appendix A.

We use the following algorithmic parameters for all examples: $\Delta_0 = 50$, $\eta_1 = 0.05$, $\eta_2 = 0.9$, $\gamma_1 = \gamma_2 = 0.25$, $\gamma_3 = 2.5$, $\mu_1 = 10^{-4}$, $\beta_{\text{dec}} = 0.1$, and $\beta_{\text{inc}} = 10$. We employ the bidirectional CP algorithm [3, Alg. 2] for `SEMI`, `NORM` and `OBM`, and allow at most two iterations of increase. We stop Algorithm 1 if $h_k \le 10^{-5}$ and we stop the subproblem solvers using the condition (31) with the absolute tolerance $\bar{\tau} = 10^{-5}$ and tolerance sequence $\tau_k = 10^{-3} h_k$. We set the maximum number of iterations for each subproblem solver to 15. For `NCG`, we set the number of Brent's iterations in (36) to be 10. For `SEMI` and `NORM`, we solve (27) and (28), respectively, using semismooth Newton, globalized with a line search. We compute the semismooth Newton step using GMRES with a maximum of 10 iterations and precondition the solve with a rank 2 perturbation of the identity similar to BFGS. For `OBM`, we set `maxit` $= 1$ and `maxitcg` $= 5$.

We summarize the performance of all subproblem algorithms in Table 1, where we tabulate the number of trust-region iterations (`iter`), the number of f (`fval`) and ∇f (`grad`) evaluations, the number of $\nabla^2 f$ applications (`hess`), the number of ϕ evaluations (`phi`), the number of proximity operator evaluations (`prox`), and the wallclock time in seconds (`time (s)`). Aside from the topology optimization example (`TopOpt`), all subproblem solvers perform comparably in terms of total trust-region iterations. As one might expect, the dogleg methods `SEMI` and `NORM` require the most Hessian applications, which are used to iteratively compute the Newton points. On the other hand, `NCG` requires the most evaluation of ϕ, which is required for the Brent's line search. As a general trend, `SPG`, `SPG2` and `NCG` tend to outperform the other methods with respect to wallclock time because they require fewer applications of the Hessian and proximity operator.

7 Conclusion

We have introduced three new subproblem solvers, each generalizing a smooth counterpart, that improve upon the SPG-based solver described in [3]. Our methods provide guaranteed rapid local convergence under specific assumptions on the problem data. Moreover, we have demonstrated the performance of these subproblem solvers on five numerical examples taken from data science and PDE-constrained optimization. While there is no clearly superior method, we do generally see that the broader class of methods based on SPG tend to excel for most applications.

[1] This method requires the application of a generalized Jacobian of the proximity operator.

Example	AlgType	iter	fval	grad	hess	phi	prox	time (s)
RankMin	SPG	4	5	5	40	99	111	1.80
	SPG2	4	5	5	49	67	52	0.91
	NCG	8	9	7	89	2053	179	3.67
	SEMI	3	4	4	483	25	178	17.25
	NORM	2	3	3	475	17	173	15.77
SVM	SPG	21	22	18	231	603	695	0.64
	SPG2	31	32	28	406	591	425	0.97
	NCG	22	23	16	162	3251	310	0.46
	SEMI	20	21	20	3668	135	1716	7.53
	NORM	12	13	12	2364	79	1160	4.78
	OBM	78	79	74	607	1107	440	1.49
Burgers	SPG	11	12	8	127	331	3940	0.28
	SPG2	13	14	10	154	208	158	0.11
	NCG	15	16	10	115	2033	220	0.13
	SEMI	18	19	15	3828	173	1863	1.24
	NORM	12	13	9	2773	99	881	0.76
	OBM	14	15	11	101	117	81	0.08
Semilinear	SPG	2	3	3	34	91	67	8.67
	SPG2	2	3	3	34	51	37	8.49
	NCG	2	3	3	32	1469	67	9.39
	SEMI	8	9	9	1331	59	333	68.84
	NORM	6	7	7	978	45	299	52.53
TopOpt	SPG	15	16	15	201	463	509	5.74
	SPG2	106	107	105	1735	1860	1835	43.92
	NCG	20	21	19	269	11987	543	7.89
	SEMI	86	87	86	18052	577	20644	401.88
	NORM	109	110	105	34309	739	26213	712.46

Table 1 Algorithmic performance for all examples: `iter` is the number of trust-region iterations, `fval` and `grad` are the numbers of f and ∇f evaluations, respectively, `hess` is the number of $\nabla^2 f$ applications, `phi` is the number of ϕ evaluations, `prox` is the number of proximity operator evaluations, and `time (s)` is the total wallclock time in seconds.

Statements and Declarations

Competing Interests: The authors declare that they have no conflicts of interest.
Data Availability: The support vector machine dataset was taken from `LIBSVM` dataset [18], `http://archive.ics.uci.edu/ml`. The other datasets generated for the numerical results are available from the corresponding author upon reasonable request.

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A L^1-Specific Orthant-Based Method Subproblem Solver

The `OBM` subproblem solver in Section 6 is tailored to L^1-regularization and is adapted from the orthant-based method described in [9]. This solver has close ties to the subproblem solver described in [29] for linearly-constrained optimization. For this method, $X = L^2(D)$ defined on the measurable space $(D, \mathcal{F}, \mu)$ and

$$\phi(x) = \beta \|x\|_1 \coloneqq \beta \int_D |x| \, \mathrm{d}\mu,$$

for $\beta > 0$. Extending the notation in [1], we denote the minimum-norm subgradient of the model m_k at the ℓ-th subproblem iterate $x_{k,\ell}$ by

$$v_{k,\ell}(w) \coloneqq \begin{cases} g_{k,\ell}(w) + \beta & \text{if } x_{k,\ell}(w) > 0 \text{ or } (x_{k,\ell}(w) = 0 \wedge g_{k,\ell}(w) < -\beta) \\ g_{k,\ell}(w) - \beta & \text{if } x_{k,\ell}(w) < 0 \text{ or } (x_{k,\ell}(w) = 0 \wedge g_{k,\ell}(w) > \beta) \\ 0 & \text{if } x_{k,\ell}(w) = 0 \text{ and } g_{k,\ell}(w) \in [-\beta, \beta] \end{cases} \tag{47}$$

for $w \in D$, where $g_{k,\ell} \coloneqq \nabla f_k(x_{k,\ell})$. Note that $-v_{k,\ell}$ is the steepest descent direction for m_k at $x_{k,\ell}$ and the directional derivative $m_k'(x_{k,\ell}; -v_{k,\ell}) < 0$ whenever $-g_{k,\ell} \notin \partial\phi(x_{k,\ell})$, i.e.,

$$m_k'(x_{k,\ell}; -v_{k,\ell}) = \sup_{\eta \in \partial\phi(x_{k,\ell})} \langle g_{k,\ell} + \eta, -v_{k,\ell} \rangle = -\|v_{k,\ell}\|^2 < 0$$

[5, Prop. 17.22]. Using $v_{k,\ell}$, we define the active set $\mathcal{A}_{k,\ell} \coloneqq \{w \in D \,|\, v_{k,\ell}(w) = 0\}$. Roughly speaking, we eliminate the active components from the trust-region subproblem and only solve for the inactive ones $D \setminus \mathcal{A}_{k,\ell}$. Instead of computing a search direction $s_{k,\ell}$ by approximately solving the modified problem

$$\begin{aligned} &\min_{s\in X}\ \frac{1}{2}\langle B_k s, s\rangle + \langle v_{k,\ell}, s\rangle \\ &\text{subject to}\quad \|s\|_2 \le \Delta_k,\ \ s(w) = 0 \quad \text{for a.a. } w \in \mathcal{A}_{k,\ell} \end{aligned} \tag{48}$$

using projected truncated CG [22], we compute $s_{k,\ell}$ by explicitly eliminating the active components. Let $P_{k,\ell} \in \mathcal{L}(X)$ denote the projection onto the inactive set $D \setminus \mathcal{A}_{k,\ell}$, i.e.,

$$[P_{k,\ell}s](w) := s(w)(1 - \mathbb{1}_{\mathcal{A}_{k,\ell}}(w)),$$

where $\mathbb{1}_{\mathcal{A}_{k,\ell}}(w) = 1$ if $w \in \mathcal{A}_{k,\ell}$ and $\mathbb{1}_{\mathcal{A}_{k,\ell}}(w) = 0$ if $w \in D \setminus \mathcal{A}_{k,\ell}$. Then, we can rewrite (48) in reduced form as

$$\min_{s\in X}\ \frac{1}{2}\left\langle (P_{k,\ell}^* B_k P_{k,\ell})s, s\right\rangle + \left\langle P_{k,\ell}^* v_{k,\ell}, s\right\rangle \quad \text{subject to} \quad \|P_{k,\ell}s\|_2 \le \Delta_k, \tag{49}$$

which we approximately solve using truncated CG [41]. Let $\hat{s}_{k,\ell} \in X$ denote an approximate solution to (49), then $s_{k,\ell} = P_{k,\ell}\hat{s}_{k,\ell}$ is an approximate solution to (48). Given $s_{k,\ell}$, we perform a backtracking line search to determine a step length that satisfies the sufficient decrease condition (17). The full routine is described in Algorithm 5.

Algorithm 5 Orthant-based subproblem solver for L^1-regularized problems

Require: The iteration limit `maxit` $\in \mathbb{N}$, decrease factor $\beta_{\text{dec}} \in (0,1)$, descent parameter $\mu \in (0,1)$, CG iteration limit `maxiticg` $\in \mathbb{N}$, and positive tolerances $\bar{\tau}$, τ_k, δ_{abs} and δ_{rel}

Set $\ell \leftarrow 0$ and compute the Cauchy point $x_{k,0} = x_k^c$
Compute $g_{k,0} \leftarrow g_k + B_k(x_{k,0} - x_k)$ and $h_{k,0} \leftarrow H(x_{k,0}, g_{k,0}, t_k)$
while $\ell <$ `maxit` **and** $h_{k,\ell} > \min\{\bar{\tau}, \tau_k h_{k,0}\}$ **and** $\|x_{k,\ell} - x_k\|_2 < \Delta_k$ **do**
    Compute $v_{k,\ell}$ from (47) and the corresponding active set $\mathcal{A}_{k,\ell}$
    Set $r \leftarrow P_{k,\ell}^* v_{k,\ell}$ and $\rho_1 \leftarrow \langle r, r \rangle$
    Set $d \leftarrow -r$ and $s_{k,\ell} \leftarrow 0$
    for $i = 1, \ldots,$ `maxitcg` **do**
        Compute $b \leftarrow (P_{k,\ell}^* B_k P_{k,\ell})d$ and $\kappa \leftarrow \langle b, d \rangle$
        if $\kappa \le 0$ **then**
            Compute $\alpha > 0$ as the solution to $\|x_{k,\ell} + s_{k,\ell} + \alpha d - x_k\| = \Delta_k$
            Set $s_{k,\ell} \leftarrow s_{k,\ell} + \alpha d$
            break
        end if
        Compute $\alpha \leftarrow \rho_i / \kappa$
        if $\|x_{k,\ell} + s_{k,\ell} + \alpha d - x_k\| \ge \Delta_k$ **then**
            Compute $\alpha > 0$ as the solution to $\|x_{k,\ell} + s_{k,\ell} + \alpha d - x_k\| = \Delta_k$
            Set $s_{k,\ell} \leftarrow s_{k,\ell} + \alpha d$
            break
        end if
        Update the step $s_{k,\ell} \leftarrow s_{k,\ell} + \alpha d$
        Update the residual $r \leftarrow r + \alpha b$
        Compute $\rho_{i+1} \leftarrow \langle r, r \rangle$
        if $\sqrt{\rho_{i+1}} \le \min\{\delta_{\text{abs}}, \delta_{\text{rel}}\sqrt{\rho_1}\}$ **then**
            break
        end if
        Compute $\beta \leftarrow \rho_{i+1}/\rho_i$
        Set the trial step $d \leftarrow \beta d - p$
    end for
    Set the step length $\sigma \leftarrow 1$
    Set the trial iterate $x_{k,\ell+1} \leftarrow x_{k,\ell} + \sigma s_{k,\ell}$
    while $m_k(x_{k,\ell+1}) > m_k(x_{k,\ell}) + \mu \min\{0, \langle g_{k,\ell}, x_{k,\ell+1} - x_{k,\ell} \rangle + \phi(x_{k,\ell+1}) - \phi(x_{k,\ell})\}$ **do**
        Set the step length $\sigma \leftarrow \beta_{\text{dec}} \sigma$
        Set the trial iterate $x_{k,\ell+1} \leftarrow x_{k,\ell} + \sigma s_{k,\ell}$
    end while
    Compute $g_{k,\ell+1} \leftarrow g_{k,\ell} + \sigma B_k s_{k,\ell}$ and $h_{k,\ell+1} \leftarrow H(x_{k,\ell+1}, g_{k,\ell+1}, t_k)$
    Update $\ell \leftarrow \ell + 1$
end while
Return $x_k^+ \leftarrow x_{k,\ell}$ as the approximate solution