Upgrading the network in discrete location problems with customers satisfaction

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Abstract

Generally speaking, in a discrete location problem the decision maker chooses a set of facilities among a finite set of possibilities and decides to which facility each customer will be allocated in order to minimize the allocation cost. However, it is natural to consider the more realistic situation in which customers have their own criterion to choose one of the open facilities, based for instance on delivery time or service quality. Giving freedom to the customers results in costs for the decision maker which are greater than those coming from the forced allocation of customers to facilities.

Here we consider several facility location problems on a directed network with two kinds of costs. The so-called *customer cost* is the one that each customer takes into account to select the facility that provides him with the service (like delivery time or a measure of loss of quality). Therefore, once the facilities are located, a customer will choose that of minimum customer cost for him. The so-called *company cost* includes any cost that derives from the allocation of the customers (demand points) to the facilities they have chosen. The aim is to minimize the company cost taking into account the decisions of the customers once the company opens its facilities.

Additionally, the company can reduce its costs by upgrading the network. To this end, a limited amount of money (budget) can be used to reduce (upgrade) the company costs associated to the arcs of the network. Then, the aim is to simultaneously find the location of facilities and the distribution of the budget (or part of it) among the arcs of the graph in order to minimize the total cost, obtained adding up the upgraded company costs and the upgrading of the network.

Different problems arise depending on the criterion used to locate the facilities and the distribution scheme. In this article we will address the upgrading of the p-median location problem, a two-stage facility location problem, a single allocation hub location problem and a tree of hubs location problem.

Keywords: facility location problems; upgrading; p-median; hubs; two-stage; tree of hubs;

1 Introduction

In a classical discrete facility location problem on a network, a graph with a weight (cost) on each arc is considered. The goal is to find locations for some facilities (on the nodes of the graph) which are optimal with respect to an objective function, and to allocate all or part of the nodes to the chosen facilities. Facility location models have wide application in the real world, e.g., in the fields of logistics, economics, emergency response, transport, distribution, just to name a few. Numerous papers and books have contributed from many points of view to this research area. An overview of most of the models, methods and applications can be found in the book edited by Laporte, Nickel and Saldanha-da-Gama ([36]). The situation is usually presented as (i) the selection of a finite set of facilities among a finite set of possibilities and (ii) the subsequent allocation of customers (demand nodes) to facilities that minimizes the resulting cost. Then, customers are forced to use the facility which is cheaper for the decision maker. This can be right when customers represent objects without decision capacity, like depots that will be supplied from the facilities. Nevertheless, there are situations where customers will not follow the decision maker instructions, but will choose the facility on their own. For this reason, or simply because a company (the decision maker) wants to give the best possible service to its customers (once the facilities are open), the facility that each customer will choose can be known in advance and respected, even when the cost of supplying the service is higher due to this greater degree of freedom for the customers.

Then we consider several facility location problems on a network with two kinds of costs. We call *customer cost* the one that each customer takes into account to select the facility that provides him with the service. One can easily imagine that the degree of satisfaction of a customer buying online will be greater if the seller delivers the product sooner (coming from an open facility with less travel time). But perhaps the route followed by the vehicle delivering this product involves a large cost (due to outsourcing, small capacity vehicle or whatever). In any case, the *company costs* will be unknown for the customers but a direct consequence of their decisions. Although the company cannot (or does not want to) change the decisions of the customers, there is something that it can do: reducing the costs of supplying customers by investing a given budget in some parts of the network. And, what is in the heart of the location problems, to change the decisions about the location of facilities, using the knowledge on both, the decisions of the customers and the improvements (and consequent costs reductions) to be carried out on the network.

Several combined problems involving facility location and network decisions have been studied in the literature. The initial model for the facility location/network design problem was introduced by Daskin et al. [12]. Contreras and Fernández [9] presented a unified framework for the general network design problem that addresses classical problems involving combined location and network design decisions. Some recent applications can be found in the papers by Dukkanci et al. [17], Gokbayrak [27] and Laporte et al. [37]. Problems of this type involve two decisions: locating facilities in a set of nodes and selecting a set of links from the demand nodes to their allocated facilities. Several costs are involved in the process, such as design costs (set-up costs of facilities and links), and operating (service) costs to transport the demand through the network. Chrètienne et al. [7] deals with the Location-Dispatching Problem, that consists of determining subsets located at nodes minimizing the sum of two costs: a piecewise linear installation cost and an access cost. Drezner and Wesolowski [16] and Melkote and Daskin [44] proposed models that aimed to determine where to locate facilities and which network links to build in order to minimize the sum of set-up costs and operating costs. Another approach that considers these costs simultaneously is to take into account the operating costs in the objective function while requiring the overall design costs to satisfy a given budget constraint. Melkote and Daskin [44] raised a scenario in which a budget could be used to construct facilities or links on the network and the goal is to minimize the operating cost. In the paper by Contreras et al. [8], the operating costs represent access times of demand nodes to their allocated facilities in the network induced by the selected arcs, and the objective is to minimize the maximum operating cost. The design costs are considered in a budget constraint. Simultaneously, the paths connecting two nodes have to be determined.

Some authors have already studied the upgrading approach in different location problems. Related to median problems, Gassner [24] studied the 1-median problem in a network, and Sepasian and Rahbarnia [51] dealt with the p-median problem on a path. In the continuous case, the Euclidean 1-median problem was analyzed by Plastria [48]. In the work of Alizadeh and Afrashteh [4], the budget-constrained inverse median facility location problem on networks is studied. Concerning the center objective, the 1-center problem was studied by Gassner [26] and Sepasian [49], and the inverse version on trees was considered by Nguyen [45]. Nguyen and Teh [46] consider the uniform cost reverse 1-centdian problem on networks, where edge lengths are reduced within a given budget. In the case of undesirable facilities, obnoxious median location problems on trees have been studied by Afrashteh et al. [2], Alizadeh et al. [3] and Gassner [25]. The upgrading version of the maximal covering location problem with edge length modifications on networks has been recently considered by Baldomero et al. [5], whereas Blanco and Marín [6] have applied upgrading to hub location problems. Finally, some authors have considered upgrading in spanning tree problems (Krumke et al. [15, 32], Sepasian and Monabbati [50]), flow problems (Demgensky et al. [13], Holzhauser et al. [29]), shortest and longest path problems (Fulkerson and Harding [22], Hambrusch and Tu [28]), and the bottleneck constrained network upgrading problem (Krumke et al. [33]).

To the best of our knowledge, none of these papers consider, as we do, a first cost that is used to allocate the customers to the facilities and a second cost which is a kind of operating cost (derived or company cost). Only Espejo and Marín [19] have dealt with a problem where arc upgrading is combined with two kinds of costs associated with the arcs of a network. These authors analyzed the upgrading of arcs in the p-median problem on a bi-network (a graph with two kinds of costs associated to the edges/arcs), and build an objective function that involves only the second cost. Another important difference is related to the budget. As we do here, it is used to reduce only the costs of second type, and does not affect the customers costs. Then, since each demand node will be always assigned to the facility with the lowest customer cost, the path from a demand node to their allocated facility is determined independently of the use of the budget.

Regarding the models, our contribution in this paper is two-fold. We introduce new location/allocation problems on networks dealing simultaneously with two costs (related to and derived from the allocation) on the arcs, and the incorporation of the cost of upgrading the network to the objective function, to determine which part of the budget (if any) should be invested in the cost reduction. On the other hand, we develop mathematical models determining the locations of the facilities, the allocation of the demand nodes to the facilities providing the lowest allocation cost, and the distribution of the budget among the arcs of the network to reduce the derived company costs associated to the allocation, in order to minimize the upgrading investment plus the upgraded company costs. To concrete, we concentrate on modelling versions of the p-median location problem, two-stage facility location problem, the single allocation p-hub location problem and tree of hubs location problem.

Potential applications arise in the context of transportation networks, express shipment and postal delivery and communication networks. In wireless sensor networks, facilities can represent any kind of service provider (like cluster heads, time synchronization hubs, or network sinks) that can be used by client nodes nearby. Specifically, for an energy efficient clustering it is required that clients connect to their closest facilities. Therefore, the allocation costs are given by the distances customer-facility. Additionally, another edge costs are given representing the quality of the link, e.g., based on dissipated energy, latency or interference. This is a kind of communication cost that we are interested in improving. For further details about facility location in wireless sensor networks see the article of Frank [21]. This situation can be modelled as a p-median problem in a network with two costs associated in each arc and upgrading in the communication cost.

Another interesting application is that of solid waste systems. The waste is collected from the districts and transported to landfill areas or to waste processing plants using collection vehicles. The resources used for collection and transport, trucks and labor, can be utilized more effectively if waste is transported to the disposal area via transfer stations. The location of transfer stations becomes especially important, as we can see in the paper by Kirca and Erkip [31]. On the one hand, using transfer stations allows spend less time in transporting and more time in collecting, and the labor force is also more effectively utilized. On the other hand, transporting the waste from transfer stations to landfill is carried out with other vehicles (transfer vehicles) with a lower operating cost than collection vehicles. The aim is to select a set of landfills and transfer stations to open so that each district is assigned to the transfer station and the landfill with the lowest travel time. This allocation provides a operating cost or transportation cost that we want to reduce. The problem can be viewed as a version of a two-stage facility location problem where the districts are the clients, and the waste of the clients has to be collected from each district and sent to the landfill (service facility) through a transfer station (depot).

It is worth insisting on the fact that in contexts where the facilities provide a service and the demand nodes represent users or customers, our model also allows considering the interests of both locator and users. Instead of assigning the users to the facility based only in the interest of the decision maker, the users establish their preferences for the location of the facilities (that can be based on travel time or distances, waiting time, ordering cost, quality of the service, reliability or any other). Once the facilities are located, the users will be allocated to the most preferred of these facilities. The allocations imply a cost from each customer to its allocated facility that the decision maker wants to reduce. This cost could be a transportation cost, a delivery time, delay time or another one. The decision maker, whose budget is limited, must decide where to reduce these derived costs. The reason to reduce these costs is the improvement of the service, saving time or money, influence the purchase decision, among others, but always preserving the customers' preferences for the allocated facility.

The rest of the paper is organized as follows. First, we formally define the facility location problems studied here and introduce some notation to be used in the following sections. The next three sections are devoted to the analysis of the p-median location problem, the two-stage location problem, the single allocation p-hub location problem and the tree of hubs location problem , all on bi-networks with upgrading. Different integer programming formulations are developed for the aforementioned problems. A brief computational study will show the limits of the upgrading to reduce the costs for the company. The paper finishes with a section containing conclusions and some possible future lines of research.

2 General description of the problem

Consider a directed bi-network (V, A, c^1, c^2) given by a directed graph (V, A) and two kinds of costs: c^1 (customer or allocation total cost) and c^2 (company or derived unit cost), associated with the arcs of A. Let $I \subseteq V$ and $K \subseteq V$ be the the sets of customers (demand nodes) and nodes candidates to be selected as facilities, respectively. For each $i \in I$, a demand w_i must be supplied.

A fixed amount of $1 \leq p \leq n-1$ of the nodes in K will be chosen as facilities. Once the subset of facilities $X \subseteq K$ has been determined, $|X| = p$, and depending on the particular model, every demand node i chooses the facility (or facilities) that serves him with less c^1 cost. This cost is produced by the use of some arcs of the network, that we denote $S(i, X)$. The use of the arcs in $S(i, X)$ produces in turn a cost $C^2(i, X)$ to the company. A budget $B > 0$ can be used to reduce the company costs, taking into account that the reduction of the cost in each arc $a \in A$ is limited to

$0 \leq u_a \leq c_a^2$, and that a unit cost $h_a \leq B/u_a$ for reducing c_a^2 is applied.

The *Generalized Induced Facility Location Problem with Upgrading* consists then of simultaneously selecting the subset of facilities X and distributing a budget B or part of it (for reducing the company costs c^2 of the arcs in the sets $S(i,X)$) in order to minimize the sum of the upgraded company costs on the arcs of $S(i,X)$ for all $i \in I$, plus the part of the budget invested in the upgrading. Note that there will be some arcs in the network that will never been used by customers and then, there will be never upgraded. Let then $A^* \subseteq A$ denote the set of arcs susceptible of been upgraded. Denoting by b_a the reduction of the company cost of arc $a \in A$, and by $b(i,X)$ total reduction in the company costs of the set $S(i,X)$, the problem is

$$\min_{\substack{X \subset K\\ |X|=p}} \left\{ \sum_{i \in I} w_i (C^2(i,X) - b(i,X)) + \sum_{a \in A^*} h_a b_a : \sum_{a \in A^*} h_a b_a \leq B;\ 0 \leq b_a \leq u_a,\ \forall a \in A^* \right\}.$$

The closely related Induced p-Median Problem with Upgrading was studied by Espejo and Marín in [19]. They showed that: (i) the location of the medians could change when the arcs are upgraded, and (ii) the demand points were not in a natural way allocated to the facilities with the lowest customer costs. Therefore, this property of the solution must be enforced in any formulation using Closest Assignment Constraints (denoted by CAC, see Espejo et al. [18]).

It is assumed in the rest of the article that $c_a^1, c_a^2 \geq 0\ \forall a \in A$, but no other condition on the costs is required.

3 Generalized Induced p-Median Problem with Upgrading

The classical p-median problem on networks consists in the following. A set of p medians has to be located in the vertices $K \subseteq V$ of a directed network (V, A, c^1). Let $I \subseteq V$ be the customers set. Let $SP^1(s,\ell)$ be the c^1-shortest directed path from $s \in V$ to $\ell \in V$, that is assumed to be unique and has cost $C^1_{s\ell} := \sum_{a \in SP^1(s,\ell)} c_a^1$. Once the set of medians $X \subset K$ has been determined we also assume that for each $i \in I$, there exists only one $k_i \in X$ such that the shortest path from X to i is $SP^1(k_i, i)$. Using the following variables

$$y_k = \begin{cases} 1 & \text{if } k \text{ is chosen as a median} \\ 0 & \text{otherwise} \end{cases} \quad \forall k \in K,$$

$$x_{ki} = \begin{cases} 1 & \text{if } k = k_i \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in I,\ \forall k \in K,$$

the classical formulation of the p-median problem reads

$$
\begin{aligned}
\text{(pM)} \quad \min \quad & \sum_{i\in I} w_i \sum_{k\in K} C^1_{ki} x_{ki} && \\
\text{s.t.} \quad & x_{ki} \le y_k, \quad \forall i \in I,\ \forall k \in K, && (1)\\
& \sum_{k\in K} x_{ki} = 1, \quad \forall i \in I, && (2)\\
& \sum_{k\in K} y_k = p, && (3)\\
& x_{ki} \in \{0,1\}, \quad \forall i \in I,\ \forall k \in K, && (4)\\
& y_k \in \{0,1\}, \quad \forall k \in K. && (5)
\end{aligned}
$$

The p-median problem has been widely studied. We refer the interested reader to the chapter by Marín and Pelegrín in the book [42].

In the generalized induced p-median problem with upgrading, a bi-network (V, A, c^1, c^2) is given. The total company cost associated to $SP^1(s,\ell)$, denoted by $C^2_{s\ell}$, is obtained adding up the costs c^2_a of the arcs $a \in SP^1(s,\ell)$, i.e., $C^2_{s\ell} := \sum_{a\in SP^1(s,\ell)} c^2_a$. In addition to the location of the p medians, a budget can be distributed among the arcs to reduce the company costs associated to the shortest path from the medians to the customers, so that the sum of the company costs after the reduction and the upgrading investment is minimal. Recall that arcs exist where a discount will never be applied, since they do not belong to shortest paths from nodes in K to nodes in I. In particular, the discount will be limited to arcs in the set $A^* = \cup_{i\in I} \cup_{k\in K} SP^1(k,i)$. Using variables $b_a \ge 0$ for the reduction of the cost c^2_a, $a \in A^*$, the *Generalized Induced p-Median Problem with Upgrading* can be formulated as a nonlinear program:

$$
\begin{aligned}
\text{(IpM)} \quad \min \quad & \sum_{i\in I} w_i \sum_{k\in K} (C^2_{ki} - \sum_{a\in SP^1(k,i)} b_a) x_{ki} + \sum_{a\in A^*} h_a b_a && (6)\\
\text{s.t.} \quad & (1)-(5) && \\
& y_k + \sum_{\substack{\ell\in K:\\ C^1_{\ell i} > C^1_{ki}}} x_{\ell i} \le 1, \quad \forall i \in I,\ \forall k \in K, && (7)\\
& \sum_{a\in A^*} h_a b_a \le B, && (8)\\
& b_a \le u_a, \quad \forall a \in A^*, && (9)\\
& b_a \ge 0, \quad \forall a \in A^*. && (10)
\end{aligned}
$$

Constraints (7) guarantee that customers are allocated to their preferred facilities (i.e., those with minimum customer cost) and a budget B or part of it, limited by constraints (8) and (9), is applied in the first addend of (6) to reduce the cost of the arcs, producing a nonlinearity. The amount finally invested in upgrading the arcs is also added in the second term of the objective function as a cost for the company.

A linearization of (6), based on the one developed by Espejo and Marín in [19], consists of introducing variables z_{ia} representing the unit reduction in the path that serves node i when the arc $a \in A$ is upgraded. Let $SP^1(i) \subseteq A$ be the set containing the arcs in any c^1-shortest path to $i \in I$. Then

$$z_{ia} := \sum_{\substack{k\in K:\\ a\in SP^1(k,i)}} b_a x_{ki} = b_a \sum_{\substack{k\in K:\\ a\in SP^1(k,i)}} x_{ki}, \quad \forall i \in I,\ \forall a \in SP^1(i).$$

A linear reformulation of (IpM) is

$$\begin{aligned}
\text{(IpML)} \qquad \min \quad & \sum_{i\in I} w_i \left(\sum_{k\in K} C^2_{ki} x_{ki} - \sum_{a\in SP^1(i)} z_{ia} \right) + \sum_{a\in A^*} h_a b_a \\
\text{s.t.} \quad & (1)-(5), (7)-(10) \\
& z_{ia} \le b_a, \quad \forall i \in I, \forall a \in SP^1(i), \\
& z_{ia} \le u_a \sum_{\substack{k\in K:\\ a\in SP^1(k,i)}} x_{ki}, \quad \forall i \in I, \forall a \in SP^1(i), \\
& z_{ia} \ge 0, \quad \forall i \in I,\ \forall a \in SP^1(i).
\end{aligned}$$

When a customer $i \in I$ is allocated to a median $k \in K$, all other customers $i' \in I$ in the path $SP^1(k,i)$ will be also allocated to k. Using this fact, the following result can be derived:

Proposition 3.1. *The following are valid inequalities for* (IpML)*:*

$$x_{ki} \le x_{k\ell} \qquad \forall i \in I,\ \forall k \in K,\ \forall \ell \in I \cap SP^1(k,i),\ \ell \ne i.$$

4 Generalized Induced Two-Stage Facility Location Problem with Upgrading

A two-stage facility location problem consists of locating facilities and depots (distribution centers) in a network (V, A, c^1) and sending a product from the located facilities to customers through the located depots. Some authors dealing with two-stage facility location and related problems are Aardal et al. [1], De Oliveira et al. [14], Landete and Marín [35], Marín [38] and Marín and Pelegrín [39, 40, 41]. In the version we extend here, q facilities and p depots have to be located in the vertices of the network. Let $J \subseteq V$ and $K \subseteq V$ be the sets of candidates to be selected as facilities and depots, respectively. Let $I \subseteq V$ be the customers set. For each $i \in I$ a demand w_i must be supplied from a facility in J through a depot in K. We assume that the capacity of facilities and depots is unlimited. There are two main formulations for this problem, one using twice indexed variables and another one using variables with three indices. Since they have different properties, we present here both.

The first we introduce, with smaller size but weaker than the other one, is the twice-indexed formulation. Location variable y_k, $k \in K$, takes value 1 if k is chosen as a depot and t_j takes value 1 if $j \in J$ is chosen as a facility. Continuous variable x_{ki}, $i \in I$, $k \in K$, represents the fraction of the demand w_i of customer i met from depot k (note that this variable can also be restricted to take values 0,1), and r_{jk}, $j \in J$, $k \in K$, is the fraction of the total demand $\sum_{i\in I} w_i$ being transported from plant j to depot k. Recall that $C^1_{s\ell}$ is the cost of the shortest path from s to ℓ. We can formulate the problem as

$$
\begin{aligned}
\text{(TS2)} \quad \min \quad & \sum_{i\in I} w_i \sum_{k\in K} C^1_{ki} x_{ki} + \sum_{i\in I} w_i \sum_{j\in J}\sum_{k\in K} C^1_{jk} r_{jk} \\
\text{s.t.} \quad & (1)-(5) \\
& \sum_{i\in I} w_i x_{ki} = \sum_{i\in I} w_i \sum_{j\in J} r_{jk}, \quad \forall k \in K, && (11)\\
& \sum_{j\in J} t_j = q, && (12)\\
& r_{jk} \le t_j, \quad \forall j \in J, \forall k \in K, && (13)\\
& r_{jk} \ge 0, \quad \forall i \in I, \forall j \in J, && (14)\\
& t_j \in \{0,1\}, \quad \forall j \in J. && (15)
\end{aligned}
$$

Note that flow conservation constraints (11) make the product demanded by the customers be served to the corresponding depot from some facility. Note also that constraints $r_{jk} \le y_k$ can be added to the model forall $j \in J$ and $k \in K$.

The second formulation for the classical two-stage location problem we present is the three times indexed formulation. Since each depot is going to be optimally assigned to only one facility, binary variable x_{jki}, $j \in J$, $k \in K$, $i \in I$, takes value 1 if customer i is allocated to depot k and depot k is allocated to facility j.

$$
\begin{aligned}
\text{(TS3)} \quad \min \quad & \sum_{i\in I} w_i \sum_{j\in J}\sum_{k\in K} (C^1_{jk} + C^1_{ki}) x_{jki} \\
\text{s.t.} \quad & (3), (5), (12), (15) \\
& \sum_{j\in J}\sum_{k\in K} x_{jki} = 1, \quad \forall i \in I, && (16)\\
& \sum_{j\in J} x_{jki} \le y_k, \quad \forall i \in I, \forall k \in K, && (17)\\
& \sum_{k\in K} x_{jki} \le t_j, \quad \forall i \in I, \forall j \in J, && (18)\\
& x_{jki} \in \{0,1\}, \quad \forall j \in J,\ \forall k \in K,\ \forall i \in I. && (19)
\end{aligned}
$$

Constraints (16) ensure that all customer demands are satisfied. Constraints (17) and (18) guarantee that all customers are supplied through depots and facilities, respectively.

The *Generalized Induced Two-Stage Facility Location Problem with Upgrading* on a directed bi-network (V, A, c^1, c^2) consists of simultaneously locating q facilities and p depots and distributing at most a budget B to reduce the company costs of sending the product to the customers from the facilities through the depots, so that the total –reduced– company cost plus the upgrading cost is minimum. Once the facilities and depots have been located, every depot will be served from the facility providing the minimum directed c^1-shortest path and every customer will be served from the depot with minimum directed c^1-shortest path as well. Let again $SP^1(i) \subset A$ be the set of arcs that belong to any c^1-shortest path to i. The total customer and company costs when i is served from j through k are given by $C^n_{jk} + C^n_{ki}$, $n = 1, 2$, respectively. We consider again variables b_a to account for the reduction of the company cost of arc $a \in A^*$, where A^* is the set of arcs in the c^1-shortest paths depot–customer or facility–depot.

Based on (TS2), the first non-linear formulation for the problem is

$$\begin{aligned}
\text{(ITS2)} \quad \min \quad & \sum_{i\in I} w_i \sum_{k\in K}(C^2_{ki} - \sum_{a\in SP^1(k,i)} b_a)x_{ki} + \sum_{i\in I} w_i \sum_{j\in J}\sum_{k\in K}(C^2_{jk} - \sum_{a\in SP^1(j,k)} b_a)r_{jk} + \\
& \sum_{a\in A^*} h_a b_a \\
\text{s.t.} \quad & (1)-(5), (7)-(10), (11)-(15), \\
& t_j + \sum_{\substack{\ell\in K:\\ C^1_{\ell k} > C^1_{jk}}} r_{\ell k} \le 1, \quad \forall j \in J,\ \forall k \in K. \qquad (20)
\end{aligned}$$

We are minimizing the total company cost after applying the reduction in arcs in c^1-shortest paths depot–customers (first term in the objective function) and facility–depot (second term), plus the upgrading investment (third term). Constraints (20) guarantee that depots are allocated to the facility with the minimum c^1-shortest path.

A linearization of (ITS2) similar to the one derived for the p-median problem follows. Let z^x_{ia} and

z^r_{ka} be the auxiliary variables. Then

$$
\begin{aligned}
\text{(ITS2L)} \quad \min \quad & \sum_{i\in I} w_i \left(\sum_{k\in K} C^2_{ki} x_{ki} - \sum_{a\in SP^1(i)} z^x_{ia} \right) + \\
& (\sum_{i\in I} w_i) \left(\sum_{k\in K} (\sum_{j\in J} C^2_{kj} r_{jk} - \sum_{a\in SP^1(k)} z^r_{ka}) \right) + \sum_{a\in A^*} h_a b_a \\
\text{s.t.} \quad & (1)-(5), (7)-(10), (11)-(15), (20) \\
& z^x_{ia} \le b_a, \quad \forall i\in I, \forall a\in SP^1(i), \\
& z^x_{ia} \le u_a \sum_{\substack{k\in K:\\ a\in SP^1(k,i)}} x_{ki}, \quad \forall i\in I, \forall a\in SP^1(i), \\
& z^x_{ia} \ge 0, \quad \forall i\in I,\ a\in SP^1(i), \\
& z^r_{ka} \le b_a, \quad \forall k\in K, \forall a\in SP^1(k), \\
& z^r_{ka} \le u_a (\sum_{i\in I} w_i) \sum_{\substack{j\in J:\\ a\in SP^1(k,j)}} r_{jk}, \quad \forall k\in K, \forall a\in SP^1(k), \\
& z^r_{ka} \ge 0, \quad \forall k\in K,\ \forall a\in SP^1(k).
\end{aligned}
$$

Proposition 3.1 still holds for formulation (ITS2L). A similar result can be derived for the first part of the routes:

Proposition 4.1. *The following are valid inequalities for* (ITS2L)*:*

$$r_{jk} \le r_{j\ell} \qquad \forall j\in J,\ \forall k\in K,\ \forall \ell\in K\cap SP^1(j,k),\ \ell\ne k.$$

Through a similar process, and using auxiliary variables z_{ia}, we obtain a linear formulation and valid inequalities for the problem based on (TS3):

$$
\begin{aligned}
\text{(ITS3L)} \quad \min \quad & \sum_{i\in I} w_i \left(\sum_{k\in K} \sum_{j\in J} (C^2_{jk} + C^2_{ki}) x_{jki} - \sum_{a\in SP^1(j,k)} z_{ia} - \sum_{a\in SP^1(k,i)} z_{ia} \right) + \sum_{a\in A^*} h_a b_a \\
\text{s.t.} \quad & (3), (5), (12), (15)-(19) \\
& \sum_{i\in I} \sum_{\substack{\ell\in J:\\ C^1_{\ell k} > C^1_{jk}}} x_{\ell ki} + t_j \le 1,\ \forall k\in K,\ \forall j\in J, \\
& \sum_{j\in J} \sum_{\substack{\ell\in K:\\ C^1_{\ell i} > C^1_{ki}}} x_{j\ell i} + y_k \le 1,\ \forall i\in I,\ \forall k\in K, \\
& z_{ia} \le b_a, \quad \forall i\in I,\ \forall k\in K,\ \forall a\in SP^1(k,i)\cup SP^1(k), \\
& z_{ia} \le u_a \sum_{k\in K} \sum_{\substack{j\in J:\\ a\in SP^1(k,i)\cup SP^1(j,k)}} x_{jki}, \quad \forall i\in I,\ \forall \ell\in K,\ \forall a\in SP^1(\ell,i)\cup SP^1(\ell), \\
& z_{ia} \ge 0, \quad \forall i\in I,\ \forall k\in K,\ \forall a\in SP^1(k,i)\cup SP^1(k).
\end{aligned}
$$

Note that an arc $a \in SP^1(k,i) \cap SP^1(j,k)$ would be discounted twice in the objective function.

Some valid inequalities for (ITS3L) are given now.

Proposition 4.2. *The following are valid inequalities for* (ITS3L)*:*

$$\begin{aligned} x_{jki} &\leq x_{jk\ell} && \forall i \in I,\ \forall j \in J,\ \forall k \in K,\ \forall \ell \neq i:\ \ell \in I \cap SP^1(k,i), \\ x_{jki} + \sum_{\ell \neq j} x_{\ell k s} &\leq y_k && \forall i,s \in I,\ \forall j \in J,\ \forall k \in K. \end{aligned}$$

5 Generalized Induced Single Allocation p-Hub Location Problem with Upgrading

In a classical p-hub location problem, there is an undirected network (V, E, c^1) and a product to be sent between some pairs of nodes. Although a directed network could also be considered, this possibility is almost never approached in the literature of hub location. Since the cost of traversing an edge $e \in E$ can be different depending of the direction followed by the flow, we still define A as the set of arcs obtained from E by splitting every edge in two arcs. Let $I \subseteq V$ represent the set of origins and destinations of this product flow and $w_{ij} \geq 0$ the amount of product sent from origin $i \in I$ to destination $j \in I$. Every flow must be routed via either one or two special nodes called hubs. The hubs act as transshipment nodes by collecting the flows from the origins and redistributing them towards the destinations. Let $K \subseteq V$ be the set of nodes candidates to be selected as hubs. Each unit of product that traverses $a = (s, \ell) \in A$ incurs initially a cost c^1_a, although this cost could be smaller when s and ℓ are both hubs. The goal is to locate p hubs and to route every shipment through the most suitable hub or pair of hubs in such a way that the overall cost is minimized. Note that it is in the essence of the problem that two hubs must be directly connected through only one edge, since the economies of scale associated to the lower cost between hubs use this fact. Initially, for the sake of simplicity, we consider that the subnetwork induced by K is complete, adding arcs with very large c^1-costs when needed. Nevertheless, the flow can follow the shortest paths from hubs to destinations and from origins to hubs. To simplify the notation, we call $O_i := \sum_{j\in I} w_{ij}$ and $D_i = \sum_{j \in I} w_{ji}$. In the forthcoming formulations we do not assume special properties of the costs like satisfaction of the triangle inequality, symmetry or any other.

Special attention deserves the inter-hub discounted flow. To model economies of scale, it is usual in the literature to multiply the c^1-cost between two hubs by a discount factor $0 < \alpha < 1$. Although this way to proceed simplify the formulations of the problem, it is not evident that marking two nodes as hubs make the cost between them be automatically reduced by a factor, regardless the amount of flow sent from one of them to the other. In other words, instead of invest a budget in upgrading arcs one by one, it is generally assumed that the upgrading comes for free due to the (hypothetical) higher amount of flow in the inter-hub network. The kind of model we introduce in this paper fills in part this hole in the literature.

There are two large families of hub location problems. Those that allow a customer to choose different pairs of hubs depending on the destination of the flow (multiple allocation problems) and those that enforce a customer to choose only one hub to send (and receive) all its flow. Since, in the philosophy behind our models, origins are assumed to be customers that choose the facility they are allocated to by comparing c^1-costs, it has sense to focus on the Single Allocation p-Hub Location Problem. This last problem has been formulated in the literature in different ways, see [20], [23], [34], [43]. We present and use here only one formulation never considered before. For $i \in I$, $m \in K$, we define the variables S_{im} that measure the inter-hub transportation cost with origin at i sent to all destinations through any first hub but using $m \in K$ as the second hub. Using variables y and x as before, the classical problem on a network (V, A, c^1) can be formulated as

$$
\begin{aligned}
\text{(Hub)} \qquad \min \quad & \alpha \sum_{i\in I}\sum_{m\in K} S_{im} + \sum_{i\in I}\sum_{k\in K}(O_i C^1_{ik} + D_i C^1_{ki})x_{ki} \\
\text{s.t.} \quad & (1)-(5) \\
& S_{im} \geq \sum_{j\in I} w_{ij} c^1_{km}(x_{ki} + x_{mj} - 1), && \forall i \in I,\ \forall k, m \in K, && (21)\\
& S_{im} \geq 0, && \forall i \in I,\ \forall m \in K. && (22)
\end{aligned}
$$

Constraints (21) only have effect when $x_{ki} = 1$. In such a case, the cost of sending all the flow from origin $i \in I$ between hubs comes from arcs (k, m) for every j such that $x_{mj} = 1$, which is the wished effect of the constraints. Note that the cost c^1_{km} is the cost of a single arc, (k, m), whereas the cost C^1_{ik} in the objective function is obtained from the shortest path $SP^1(i, k)$. If some flow moves from node $s \in I \cap K$ to node $\ell \in K$, the cost to be applied is $c^1_{s\ell}$ when both are hubs (and this cost will be multiplied times α) but it is $C^1_{s\ell}$ when s is not a hub allocated to hub ℓ. It could hold $C^1_{s\ell} < c^1_{s\ell}$. In this second case, the arcs in the shortest path from s to ℓ would not benefit of any discount.

In our extension of the problem, with induced costs and upgrading, we use the bi-network (V, A, c^1, c^2). As in previous sections, in the hub location problem the customer cost c^1 could be based on travel times, distances, preferences or any other, while the derived company cost c^2 could represent the transportation cost (per unit of flow). Each route beginning at origin i, ending at destination j, and traversing hubs k and m in this order, carries a customer cost (possibly to be paid by two customers, i and j) $C^1_{ik} + C^1_{mj}$ and a company cost $C^2_{ik} + c^2_{km} + C^2_{mj}$ where $C^2_{s\ell}$ is again the company cost associated to the c^1-shortest directed path between s and ℓ. A budget $B > 0$ is given and can be used to reduce the company costs of the inter-hub arcs. There is still a limit of $0 \leq u_a \leq c_a$ for the upgrading of arc $a \in A^* := (K \times K)$ and the reduction comes at a penalty of $0 \leq h_a \leq B/u_a$ like in previous models.

The *Generalized Induced Single Allocation p-Hub Location Problem with Upgrading* consists then of locating p hubs, assigning each customer $i \in I$ to a located hub and distributing the budget or part of it to reduce the company costs, in such a way that the total company cost after upgrading plus the upgrading cost is minimized. Now, for each $i \in I$ and $m \in K$, variable S_{im} represents the upgraded inter-hub transportation cost with origin at i sent to all destinations through any first hub but using $m \in K$ as the second hub. Then we can formulate the problem as

$$
\begin{aligned}
\text{(IHub)} \quad \min \quad & \sum_{i\in I}\sum_{m\in K} S_{im} + \sum_{i\in I}\sum_{k\in K}(O_i C^2_{ik} + D_i C^2_{ki}) x_{ki} + \sum_{a\in A^*} h_a b_a, && \\
\text{s.t.} \quad & (1)-(5),(7),(8),(10),(22) && \\
& b_{km} \le u_{km} y_k, && \forall (k,m)\in A^* \quad (23)\\
& b_{km} \le u_{km} y_m, && \forall (k,m)\in A^* \quad (24)\\
& S_{im} \ge \sum_{j\in I} w_{ij}(c^2_{km} - b_{km})(x_{ki} + x_{mj} - 1), && \forall i\in I,\ \forall (k,m)\in A^*. \quad (25)
\end{aligned}
$$

Note that in (IHub) costs $C^1_{s\ell}$ and $C^2_{s\ell}$ can be calculated beforehand since they are the customer and company costs associated to the paths $SP(s,\ell)$ when s and ℓ are not both hubs and therefore, the costs of the arcs of $SP(s,\ell)$ are not upgraded. When the subnetwork $(K, (K\times K)\cap A)$ is sparse, adding all the constraints (25) can be computationally very expensive. An alternative possibility is, when $w_{ij} > 0$ for all $i \ne j \in I$, to keep the arcs as they come, and add inequalities $y_k + y_m \le 1$ for those $k < m \in K$ such that $(k,m) \notin E$. This sparsity can also make solutions with less than p hubs less costly. Then, constraint (3) should be replaced by $\sum_{k\in K} y_k \le p$.

Note also that we have the product of b- and x-variables in the set of constraints (25) instead of having them in the objective function, as in the other problems previously studied. In (25) we find, for fixed $i \in I$ and $k, m \in K$,

$$S_{im} \ge \sum_{j\in I} w_{ij} b_{km}(x_{ki} + x_{mj}) = O_i \sum_{m\in K} b_{km} x_{ki} + \sum_{j\in I} w_{ij} b_{km} x_{mj}.$$

In this case it is not the shortest path but the direct arc which has to be considered. Using variables

z and z', we can obtain the linearization of (IHub):

$$
\begin{aligned}
\text{(IHubL)} \qquad \min \quad & \sum_{i\in I}\sum_{m\in K} S_{im} + \sum_{a\in A^*} h_a b_a + \\
& \sum_{i\in I}\sum_{k\in K}(O_i C_{ik}^2 + D_i C_{ki}^2) x_{ki}, \\
\text{s.t.} \quad & (1)-(5),(7),(8),(10),(22)-(24) \\
& S_{im} \geq \sum_{j\in I} w_{ij} \\
& \left(c_{km}^2(x_{ki}+x_{mj}-1) - z_{ikm} - z'_{jkm} + b_{km}\right), && \forall i\in I,\ \forall k\in K, && (26)\\
& z_{ikm} \leq b_{km}, && \forall i\in I,\ \forall (k,m)\in A^*, && (27)\\
& z_{ikm} \leq u_{km} x_{ki}, && \forall i\in I,\ \forall (k,m)\in A^*, \\
& z_{ikm} \geq 0, && \forall i\in I,\ \forall (k,m)\in A^*, \\
& z'_{jkm} \leq b_{km}, && \forall j\in I,\ \forall (k,m)\in A^*, && (28)\\
& z'_{jkm} \leq u_{km} x_{mj}, && \forall j\in I,\ \forall (k,m)\in A^*, \\
& z'_{jkm} \geq 0, && \forall j\in I,\ \forall (k,m)\in A^*.
\end{aligned}
$$

Some additional constraints to strengthen (IHubL) follow.

Proposition 5.1. *The inequalities in the following family are valid for* (IHubL)

$$S_{im} \geq \sum_{j\in I}\left(\min_{k\in K}\{w_{ij}c_{km}^2 - u_{km}\alpha_{ijkm}\}x_{mj} - \sum_{k\in K}(w_{ij}-\alpha_{ijkm})b_{km}\right), \quad \forall i\in I,\ \forall m\in K,$$

whenever $0 \leq \alpha_{ijkm} \leq w_{ij}\ \forall i,j\in I,\ \forall k,m\in K.$

Proof. Let us consider (x,y,b,S,z) a feasible solution of (IHubL). For $i\in I$ and $m\in K$, since $0\leq \alpha_{ijkm}\leq w_{ij}$ and $u_{km}\leq c_{km}^2$, then $w_{ij}c_{km}^2 - u_{km}\alpha_{ijkm}\geq 0$ and

$$
\begin{aligned}
&\sum_{j\in I}\left(\min_{k\in K}\{w_{ij}c_{km}^2 - u_{km}\alpha_{ijkm}\}x_{mj} - \sum_{k\in K}(w_{ij}-\alpha_{ijkm})b_{km}\right) \leq \\
&\quad \sum_{j\in I}\left(\sum_{k\in K}\{w_{ij}c_{km}^2 - u_{km}\alpha_{ijkm}\}x_{mj} - \sum_{k\in K}(w_{ij}-\alpha_{ijkm})b_{km}\right). \qquad (29)
\end{aligned}
$$

If $y_m=0$, then from (24), (27) and (28) it follows $b_{km}=0$, $z_{ikm}=0$ and $z'_{jkm}=0$, and

$$\sum_{j\in I}\left(\sum_{k\in K}(w_{ij}c_{km}^2 - u_{km}\alpha_{ijkm})x_{mj} - \sum_{k\in K}(w_{ij}-\alpha_{ijkm})b_{km}\right) = \quad \sum_{j\in I}\sum_{k\in K}(w_{ij}c_{km}^2 - u_{km}\alpha_{ijkm})x_{mj},$$

which is lower or equal to S_{im} by (26). Otherwise, if $y_m = 1$, then $b_{km} \leq u_{km}$ (from (23)), and (29) can be rewritten as

$$\begin{aligned}\sum_{j\in I}\left(\sum_{k\in K} w_{ij}\left(c^2_{km}x_{mj} - b_{km}\right) - \sum_{k\in K}\alpha_{ijkm}\left(u_{km} - b_{km}\right)\right) \leq & \quad \sum_{j\in I}\sum_{k\in K} w_{ij}\left(c^2_{km}x_{mj} - b_{km}\right) \\ = & \quad \sum_{j\in I} w_{ij}x_{mj}\sum_{k\in K}\left(c^2_{km} - b_{km}\right),\end{aligned}$$

and this is the inter-hub transportation cost (after reduction) with origin at i sent to all destinations through any first hub but using $m \in K$ as the second hub. □

6 Generalized Induced Tree of Hubs Location Problem with Upgrading

This section deals with a hub location model not requiring direct connection between hubs. Potential applications arise when the costs of the links between hubs are very high and full interconnection between hub nodes is prohibitive (see [30]). The model, developed by Contreras et al. in [10] and [11], is a single-allocation hub location problem on an undirected network (V, E, c^1). Like in the previous chapter, since the cost of traversing an edge $e \in E$ can be different depending of the direction followed by the flow, we still define A as the set of arcs obtained from E by splitting every edge in two arcs. Now exactly p hubs have to be located, with the particularity that it is required that the hubs are connected by means of a (non–directed) tree. Consider two sets $I \subset V$ and $K \subset V$, with $I \cap K = \emptyset$, representing the customers and potential hubs, respectively. For each $i, j \in I$, $w_{ij} \geq 0$ denotes the flow that must be sent from i to j through a path of hubs inside the tree. The so-called Tree of Hubs Location Problem aims to locate $p \geq 2$ hubs, to select $p - 1$ edges of E that connect the hubs, and to assign each customer in I to a located hub in such a way that the overall cost is minimized. The cost (per unit of flow) of the path from i to j is the sum of the cost of (i) the directed path from i to its allocated hub (say k) at minimum c^1-cost, (ii) the directed path between k and the hub allocated to j (say m) through the unique path in the hubs subgraph, and (iii) the directed path from m to j following the c^1-shortest path. Now, C^1_{ki} denotes the directed c^1-shortest path from i to k. Let $0 < \alpha < 1$ be a discount factor for the flow between hubs.

In order to formulate this problem, Contreras et al. [10] considered two new sets of variables to determine the unique path that exists in the tree between each pair origin-destination:

$$y_{km} = \begin{cases} 1 & \text{if edge } (k,m) \text{ belongs to the hubs tree} \\ 0 & \text{otherwise} \end{cases} \qquad \forall (k,m) \in E_K,$$

$$v_{ijkm} = \begin{cases} 1 & \text{if the flow from } i \text{ to } j \text{ traverses the} \\ & \text{two-hubs arc } (k,m) \\ 0 & \text{otherwise} \end{cases} \qquad \forall i,j \in I,\ \forall (k,m) \in A_K.$$

Using variables y and x as before, and preserving the notation of previous sections, the formulation reads

$$
\begin{aligned}
\text{(THLP)}\min \quad & \sum_{i\in I}\sum_{k\in K}(O_iC^1_{ik}+D_iC^1_{ki})x_{ki}+\sum_{i\in I}\sum_{j\in I}\sum_{(k,m)\in A_K}\alpha w_{ij}c^1_{km}v_{ijkm} \\
\text{s.t.} \quad & (1)-(5) \\
& v_{ijkm}+v_{ijmk}\le y_{km}, \quad \forall i,j\in I,\ \forall (k,m)\in E_K, && (30)\\
& \sum_{m\in K:\ (k,m)\in A_K} v_{ijkm}+x_{kj}=\sum_{m\in K:\ (m,k)\in A_K} v_{ijmk}+x_{ki}, \quad \forall i,j\in I, \forall k\in K, && (31)\\
& \sum_{(k,m)\in E_K} y_{km}=p-1, && (32)\\
& y_{km}\le y_k, \quad \forall (k,m)\in A_K, && (33)\\
& y_{km}\le y_m, \quad \forall (k,m)\in A_K, && (34)\\
& v_{ijkm}\in\{0,1\}, \quad \forall i,j\in I,\ \forall (k,m)\in A_K, && (35)\\
& y_{km}\in\{0,1\}, \quad \forall (k,m)\in E_K. && (36)
\end{aligned}
$$

Assuming that y_{km} induces a connected graph in (K,E_K), it will be a tree due to (32). The connection between those hubs that have customers allocated to them is guaranteed by constraints (30). To ensure that all hubs receive allocation, we can include in the formulation $|K|$ dummy customers ℓ_k and fix $x_{k\ell_k}=1$. The set of constraints (31) define paths in the tree between customers. Constraints (33) and (34) ensure that if a customer is allocated to an element of K, the latter has been chosen as a hub.

In our extension, the bi-network (V,A,c^1,c^2) is considered. Note that c^1-costs between elements of K and c^2-costs between elements of I will not be required. Once the hubs in X have been located, every customer will follow the c^1-shortest directed path to X to decide its reference hub. The company costs associated to these c^1-shortest paths can be subject to reduction. Let k and m be the hubs to which customers i and j are allocated, respectively. Then, the total company cost to send the product from i to j is the sum of the company cost associated to the c^1-shortest path from i to k, C^2_{ik}, the c^1-shortest path from m to j, C^2_{mj}, and the sum of all company costs of the arcs c^2_{st} between hubs (s,t) in the unique path in the tree of hubs that goes from k to m.

To formulate the *Generalized Induced Tree of Hubs Location Problem with Upgrading, we define* b_a *as the reduction of the transportation cost in the arc* $a\in A_K$ The formulation we propose is then

$$
\begin{aligned}
\text{(ITHLP)}\min \quad & \sum_{i\in I}\sum_{k\in K}(O_iC^1_{ik}+D_iC^1_{ki})x_{ki}+\sum_{a\in A_K}h_ab_a+\\
& \sum_{i\in I}\sum_{j\in I}\sum_{(k,m)\in A_K}w_{ij}(c^2_{km}-b_{km})v_{ijkm},\\
\text{s.t.} \quad & (1)-(5),(7),(30)-(36)\\
& \sum_{(k,m)\in A_K}h_{km}b_{km}\le B, && (37)\\
& b_{km}\le u_{km}y_{km}, \quad \forall (k,m)\in A_K, && (38)\\
& b_{km}\ge 0, \quad \forall (k,m)\in A_K. && (39)
\end{aligned}
$$

A linearization of the products in the objective function can be done using variables

$$
z_{ijk} := \sum_{m:\ (k,m)\in A_K} b_{km}v_{ijkm}
$$

$\forall i,j\in I$, $\forall k\in K$. Then

$$
\begin{aligned}
\text{(ITHLPL)}\min \quad & \sum_{i\in I}\sum_{k\in K}(O_iC^2_{ik}+D_iC^2_{ki})x_{ki}+\sum_{a\in A_K}h_ab_a+\\
& \sum_{i\in I}\sum_{j\in I}\sum_{(k,m)\in A_K}w_{ij}c^2_{km}v_{ijkm}-\sum_{i\in I}\sum_{j\in I}\sum_{k\in K}w_{ij}z_{ijk}\\
\text{s.t.} \quad & (1)-(5),(7),(30)-(39)\\
& z_{ijk}\le \sum_{m:\ (k,m)\in A_K} b_{km} \quad \forall i,j\in I,\ \forall k\in K,\\
& z_{ijk}\le \sum_{m:\ (k,m)\in A_K} u_{km}v_{ijkm} \quad \forall i,j\in I,\ \forall k\in K,\\
& z_{ijk}\ge 0 \quad \forall i,j\in I,\ \forall k\in K.
\end{aligned}
$$

7 Analysis of budget impact on the solution

Although the main objective of the paper was not to carry out computational studies on the performance of the formulations, it would be interesting to know how the amount of available budget affects the solution and, in particular, the optimal value of the problem and the locations of the medians. We have considered, only for the p-median formulation (IpML), several well-known instances from the literature widely used to test p-median algorithms and available in [47].

We have taken, from the OR-Library, the uncapacitated p-median data files, each of them containing information of number of vertices, number of edges, number of medians and, for each edge, the end vertices and the cost. This last value will be our c^1-cost. We have split each edge in two

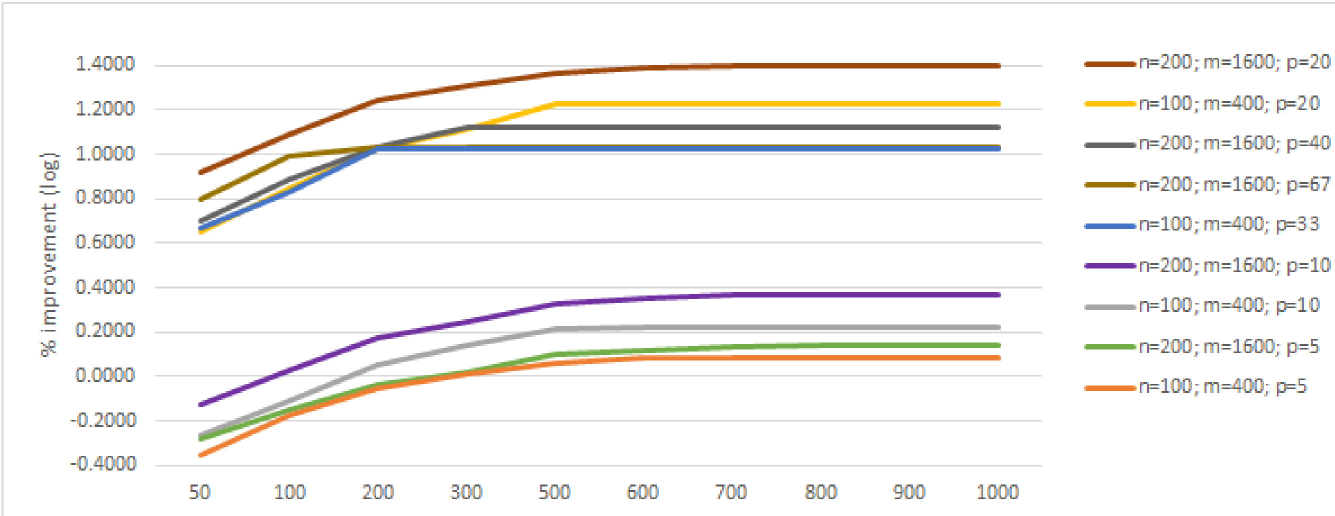

Figure 1: Effect of budget increment on the optimal value

arcs and duplicated the corresponding cost to obtain a directed network. For every cost c_a^1 we have taken $c_a^2 := c_a^1 \cdot (0.8 + 0.4U)$ where U is randomly and uniformly distributed in $(0, 1)$. The maximum of the reduction of the cost, u_a, was fixed to $c_a^2/2$. The cost h_a for reducing c_a^2 was fixed to 1 in each arc, and all the demands w_i were considered equal to 1. The budget was progressively incremented from 0 until no reduction in the optimal value of the problem was possible.

	pmed1	pmed2	pmed3	pmed4	pmed5	pmed6	pmed7	pmed8	pmed9	pmed10
B	$p=5$	$p=10$	$p=10$	$p=20$	$p=33$	$p=5$	$p=10$	$p=20$	$p=40$	$p=67$
50	1/-	0/-	0/-	0/-	1/-	0/-	0/-	0/-	4/-	12/-
100	1/0	1/1	0/-	0/-	4/3	0/-	0/-	0/-	3/2	10/6
200	1/0	1/0	0/-	1/1	4/1	0/-	0/-	0/-	2/1	14/7
300	1/0	1/0	0/-	1/0	2/0	0/-	0/-	0/-	4/4	11/3
500	1/0	1/0	0/-	1/0	3/2	1/1	0/-	0/-	3/0	14/3
600	1/0	1/0	0/-	1/0	1/0	1/0	0/-	0/-	4/2	14/4
700	1/0	1/0	0/-	1/0	2/1	1/0	0/-	0/-	4/2	13/5
800	1/0	1/0	0/-	1/0	1/0	1/0	0/-	0/-	5/2	13/3
900	1/0	1/0	0/-	1/0	2/1	1/0	0/-	0/-	3/3	15/3
1000	1/0	1/0	0/-	1/0	3/2	1/0	0/-	0/-	5/2	16/3

Table 1: Effect of budget increment on the medians location

Figure 1 displays the percentage of improvement (in logarithmic scale) of the optimal value of the objective function (respect to the optimal value when no upgrading is carried out, i.e, with a budget equal to zero) for different instances, depending on the budget. In the figure, m represents the number of arcs of the instance. As can be seen, initially this percentage increases with the budget, but beyond a certain budget value, there is no further improvement. Generally speaking, the larger the value of p, the larger reduction of the cost.

Regarding the medians, the optimal solutions change with the budget, as we can see in Table 1. For each instance and budget value, we compare the optimal medians with i) those of the optimal solution when no upgrading is carried out (on the left of the slash sign), and ii) the optimal solution of the previous row (on the right). Each column shows the number of medians being different in each case. Despite the correlation between c^1- and c^2-costs, for medium and large values of p the optimal set of medians experiment significant changes when a budget is invested in the upgrading of arcs.

8 Concluding remarks

This paper introduces new facility location problems related to upgrading arcs on a network with two kinds of costs. The first cost (customer cost) is used to allocate the customers to the facilities and the second cost (company cost) is a kind of operating cost associated to the allocation. Each customer selects the facility that provides it with the service, once the company opens its facilities, and the aim is to minimize the company cost taking into account the decisions of the customers. Additionally, the company can reduce its costs by upgrading the network.

This problem allows to model more realistic situations where the facilities provide a service and the demand nodes represent users or customers. Instead of assigning the users to the facility based only in the interest of the decision maker, our model allows considering the interests of both locator and users. The customers choose the facility on their own and the decision maker want to reduce the cost of supplying the service, respecting the customers' decisions.

This work can be considered an initial attempt to address facility location problems on networks considering two costs on the arcs, upgrading arcs, and the incorporation of the upgrading cost to the objective function.

Different problems have been considered depending on the criterion used to locate the facilities and the distribution scheme. In this article have addressed the upgrading of the p-median location problem, a two-stage facility location problem, a single allocation hub location problem and a tree of hubs location problem. Different integer programming formulations were developed for the aforementioned problems. A brief computational study has shown the limits of the upgrading to reduce the costs for the company.

The findings of this paper can be the basis of further research concerning facility location problems with upgrading on a network with two costs. From a modelling point of view, the introduction of capacity limits for the facility location problems would be also interesting since, in real life, facilities work with a limited capacity. In future research, the upgrading of arcs in other location problems on a bi-network could be considered.

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References

[1] Aardal K, Labbé M, Leung J, Queyranne M (1996) On the two-level uncapacitated facility location problem. *INFORMS Journal of Computing*, 8(3): 289-301.

[2] Afrashteh E, Alizadeh B, Baroughi F (2020) Optimal approaches for upgrading selective obnoxious p-median location problems on tree networks. *Annals of Operations Research* 289: 153-172.

[3] Alizadeh B, Afrashteh E, Baroughi F (2019) Inverse obnoxious p-median location problems on trees with edge length modifications under different norms. *Theoretical Computer Science*, 772: 73-87.

[4] Alizadeh B, Afrashteh E (2020) Budget-constrained inverse median facility location problem on tree networks. *Applied Mathematics and Computation* 375, 125078.

[5] Baldomero-Naranjo M, Kalcsics K, Marín A, Rodríguez-Chía AM (2022) Upgrading edges in the maximal covering location problem. *European Journal of Operational Research* 303(1): 14–36.

[6] Blanco V, Marín A (2019) Upgrading nodes in tree-shaped hub location. *Computers & Operations Research* 102: 75–90.

[7] Chrètienne P, Fouilhoux P, Gourdin E, Segura J-M (2014) The location-dispatching problem: Polyhedral results and content delivery network design. *Discrete Applied Mathematics* 164(1): 68–85.

[8] Contreras I, Fernández E, Reinelt G (2012) Minimizing the maximum travel time in a combined model of facility location and network design. *Omega*, 40(6): 847-860.

[9] Contreras I, Fernández E (2012) General network design: A unified view of combined location and network design problems. *European Journal of Operational Research*, 219(3): 680-697.

[10] Contreras I, Fernández E, Marín A (2009) Tight bounds from a path based formulation for the tree of hub location problem. *Computers & Operations Research*, 36: 3117-3127.

[11] Contreras I, Fernández E, Marín A (2010) The Tree of Hubs Location Problem. *European Journal of Operational Research*, 202(2): 390-400.

[12] Daskin MS, Hurter AP, VanBuer MG (1993) Toward an integrated model of facility location and transportation network design. Working paper. Northwestern University, Transportation Center.

[13] Demgensky I, Noltemeier H, Wirth HC (2002) On the flow cost lowering problem. *European Journal of Operational Research* 137: 265–271.

[14] De Oliveira PB, Contreras I, De Camargo RS, De Miranda Júnior G (2020) A comparison of separation routines for benders optimality cuts for two-level facility location problems. *Expert Systems with Applications*, 141, 112928.

[15] Drangmeister KU, Krumke SO, Marathe MV, Noltemeier H, Ravi SS (1998) Modifying edges of a network to obtain short subgraphs. *Theoretical Computer Science* 203(1): 91–121.

[16] Drezner Z, Wesolowski GO (2003) Network design: selection and design of links and facility location. *Transportation Research Part A*, 37(3): 241-256.

[17] Dukkanci O, Kara BY, Bektas T (2021) Minimizing energy and cost in range–limited drone deliveries with speed optimization. *Transportation Research Part C: Emerging Technologies*, 125, 102985.

[18] Espejo I, Marín A, Rodríguez-Chía AM (2012) Closest assignment constraints in discrete location problems. *European Journal of Operational Research* 219(1): 49–58.

[19] Espejo I, Marín A (2023) The p-median problem with upgrading of transportation costs and minimum travel time allocation. *Computers and Operations Research* 159, 106354.

[20] Espejo I, Marín A, Muñoz-Ocaña JM, Rodríguez-Chía AM (2023) A new formulation and branch–and–cut method for single–allocation hub location problems. *Computers and Operations Research* 155, 106241.

[21] Frank C (2007) Facility Location. In: Wagner, D., Wattenhofer, R. (eds) *Algorithms for Sensor and Ad Hoc Networks. Lecture Notes in Computer Science*, vol 4621. Springer, Berlin, Heidelberg.

[22] Fulkerson DR, Harding GC (1977) Maximizing the minimum source-sink path subject to a budget constraint. *Mathematical Programming* 13: 116–118.

[23] Ghaffarinasab N, Kara BY (2019) Benders decomposition algorithms for two variants of the single allocation hub location problem. *Network and Spatial Economics* 19, 83–108.

[24] Gassner E (2007) Up- and downgrading the 1-median in a network. *Technical Report* 2007-6, Graz University of Technology.

[25] Gassner E (2008) The inverse 1-maxian problem with edge length modification. *Journal of Combinatorial Optimization*, 16: 50-67.

[26] Gassner E (2009) Up- and downgrading the 1-center in a network. *European Journal of Operational Research* 198: 370-377.

[27] Gokbayrak K (2018) Robust gateway placement in wireless mesh networks. *Computers & Operations Research*, 97: 84-95.

[28] Hambrusch SE, Tu HY (1997). Edge weight reduction problems in directed acyclic graphs. *Journal of Algorithms* 24(1): 66–93.

[29] Holzhauser M, Krumke SO, Thielen C (2016) Budget-constrained minimum cost flows. *Journal of Combinatorial Optimization* 31: 1720–1745.

[30] Kayisoglu B, Akgun (2021) Multiple allocation tree of hubs location problem for non-complete networks. *Computers & Operations Research*, 136, 105478.

[31] Kirca O, Erkip N (1988) Selecting transfer station locations for large solid waste systems. *European Journal of Operational Research*, 38: 339-349.

[32] Krumke SO, Marathe MV, Noltemeier H, Ravi R, Ravi SS (1998) Approximation algorithms for certain network improvement problems. *Journal of Combinatorial Optimization* 2: 257–288.

[33] Krumke SO, Marathe MV, Noltemeier H, Ravi SS, Wirth HC (2001) Upgrading bottleneck constrained forests. *Discrete Applied Mathematics* 108: 129–142.

[34] Labbé M, Yaman H, Gourdin E (2005) A branch and cut algorithm for hub location problems with single assignment. Mathematical Programming 102, 371–405.

[35] Landete M, Marín A. New facets for the two-stage uncapacitated facility location polytope. *Computational Optimization and Applications* 44: 487–519.

[36] Laporte G, Nickel S, Saldanha da Gama, F (2019) Location Science, Second Edition. Springer.

[37] Laporte G, Rancourt M-E, Rodríguez-Pereira J, Silvestri S (2022) Optimizing access to drinking water in remote areas. Application to Nepal, *Computers & Operations Research*, 140, 105669.

[38] Marín A (2007) Lower bounds for the two-stage uncapacitated facility location problem. *European Journal of Operational Research*, 179: 1126-1142.

[39] Marín A, Pelegrín B (1997) A branch-and-bound algorithm for the transportation problem with location of p transshipment points. *Computers & Operations Research* 24: 659-678.

[40] Marín A, Pelegrín B (1998) The return plant location problem: Modelling and resolution. *European Journal of Operational Research*, 104: 375–392.

[41] Marín A, Pelegrín B (1999) Applying Lagrangian relaxation to the resolution of two–stage location problems. *Annals of Operations Research* 86: 179–198.

[42] Marín A, Pelegrín M (2019) p-Median Problems. In Laporte G, Nickel S, Saldanha da Gama, F (eds.) Location Science, Second Edition. Springer.

[43] Meier JF, Clausen U (2017) Solving single allocation hub location problems on Euclidean data. *Transportation Science* 52, 1035–1296.

[44] Melkote S, Daskin MS (2001) An integrated model of facility location and transportation network design. *Transportation Research Part A*, 35(6): 515-538.

[45] Nguyen KT (2016) Inverse 1-median problem on block graphs with variable vertex weights. *Journal of Optimization Theory and Applications* 168: 944-957.

[46] Nguyen KT, Teh WC (2023) The Uniform Cost Reverse 1-Centdian Location Problem on Tree Networks with Edge Length Reduction. *Vietnam Journal of Mathematics* 51: 345-361.

[47] OR-library http://people.brunel.ac.uk/∼mastjjb/jeb/orlib/pmedinfo.html. Retrieved on December 11 2023.

[48] Plastria F (2016) Up- and downgrading the Euclidean 1-median problem and knapsack Voronoi diagrams. *Annals of Operations Research*, 246: 227-251.

[49] Sepasian AR (2018) Upgrading the 1-center problem with edge length variables on a tree. *Discrete Optimization* 29: 1–17.

[50] Sepasian AR, Monabbati E (2017) Upgrading minmax spanning tree problem under various cost functions. *Theoretical Computer Science* 704: 87–91.

[51] Sepasian AR, Rahbarnia F (2015) Upgrading p-median problem on a path. *Journal of Mathematical Modelling and Algorithms* 14: 145–157.