Discrete location models with customers' choice and path improvements

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Abstract

We examine several facility location problems within a directed network involving two distinct cost types. The first, referred to as the "customer cost", represents the expense each customer considers when selecting a facility to obtain service (e.g., delivery time or a measure of quality degradation). Consequently, once facilities are established, each customer chooses the one that minimizes their individual cost. The second type, termed the "company cost", encompasses all expenses incurred by the company due to customer allocation to their chosen facilities. Additionally, the company possesses a budget that can be allocated to reduce company costs associated with the network's arcs (the so-called arcs upgrading).

The company's objective is to simultaneously determine facility locations and distribute the budget (or a portion of it) across the network arcs to minimize the total company cost. This total company cost comprises the post-upgrading company costs and the invested budget, while accounting for customer reactions after facility placement.

Different problem variants emerge based on the facility location criteria and customers' choice strategy. In this paper, we address the following problems: the p-median problem, a two-stage facility location problem, a single-allocation hub location problem, and a tree-of-hubs location problem –all incorporating customers' choice and arc upgrading.

Keywords: facility location problems; upgrading; p-median; hubs; two-stage; tree of hubs

1 Introduction

In classical discrete facility location problems on networks, a weighted graph –where each arc has an associated cost– is used to determine optimal facility locations (placed at nodes) with respect to a given objective function. The problem involves selecting these facilities and allocating some or all nodes to them. Facility location models have broad real-world applications, spanning logistics, economics, emergency response, transportation, and distribution, among others. Extensive research has been conducted in this field, with comprehensive reviews of models, methods, and applications available in the book edited by Laporte, Nickel, and Saldanha-da-Gama ([37]).

Traditionally, the problem is framed in two stages: (i) Facility Selection: Choosing a finite set of facilities from a finite set of candidate locations; (ii) Customer Allocation: Assigning demand nodes (customers) to facilities in a way that minimizes total cost. This conventional approach often assumes that the customer's objective is aligned with the company's objective. However, in many real-world scenarios, customers choose facilities based on their own preferences rather than strictly adhering to a company's cost-minimizing directive. Although there is an extensive literature addressing the cases in which customers decide which facility to use, and their objective is different from that of the company (e.g., competitive facility location problems), our models present a distinct methodology. We utilize a framework focused on a single decision-maker optimizing internal operational costs, where the allocation of customers is pre-determined and induced by the fixed customer cost. This phenomenon is the essence of the "induced allocation" mechanism central to our models. The company must account for this fixed preference when designing its network, as failing to serve customers from their preferred facility would lead to dissatisfaction and non-compliance.

We introduce and analyze several facility location problems within a directed network that involve two distinct cost types on each arc.

(i) *Customer cost*: This represents the expense each customer considers when selecting a service-providing facility. These costs reflect individual customer preferences (e.g., perceived travel time, expected service quality, or convenience). Customers act autonomously, choosing facilities solely to minimize their own customer cost. These customer costs are not altered by the company's strategic decisions, and are considered pre-estimated and fixed inputs for the model at the design stage. The estimation of these costs can be derived in various ways, such as examining past customer choice patterns, making a market research, or using predictive models based on physical attributes of the network or service.

(ii) *Company cost*: This encompasses all expenses incurred by the company due to customer allocation to their chosen facilities. These are the operational costs that the company seeks to manage and reduce.

The company's objective is to simultaneously determine facility locations and strategically distribute a given budget across the network arcs to reduce the company costs. This reduction is usually known as arc upgrading. These upgrades affect only the company's operational expenses and do not influence or alter the customer's preferred assignments. A fundamental characteristic of our approach is that customer-facilities paths are dictated by customers' choices, and the company then optimizes its own costs along these customer-driven paths through targeted arc upgrades.

A critical feature of our models is their ability to incorporate customer autonomy as a critical, unchangeable input to the company's optimization problem. Our framework incorporates customer preferences into the facility selection process. These preferences may depend on various factors, including travel time/distance, waiting times, service quality, ordering costs and service reliability. Once facilities are established, customers are automatically allocated to their most preferred facility based on these parameters. This ensures that from the customer's perspective, their preference or "satisfaction" (as defined by minimizing their customer cost) is paramount and upheld by the model, as any company improvements (reductions in company costs) are assumed not to negatively impact the customer's perception of customer cost value.

1.1 Applications

This dual-cost framework finds broad applicability across various domains. To reinforce the practical motivation of our models, the subsequent examples delineate how customer choice is integrated into the problem structure. Customer choice is defined here as the initial preference or implicit decision for a service point (facility), quantified by the fixed customer cost, which reflects perceived or expected service attributes. The objective of these illustrations is to demonstrate our model's suitability for design scenarios where customer preferences, even if not explicitly declared, are reliably known or can be inferred by the company at the design stage. Critically, these examples underscore that the optimal outcome desired by the customer (e.g., optimal service quality or lowest perceived cost) frequently diverges from the solution that would be most economically advantageous for the company itself to provide.

- Water Management Systems.
  - Context: A Water Management Company (WMC) operates a network of pumping stations and pipelines to deliver water to various consumer nodes. The WMC aims to establish a set of distribution depots (facilities) to efficiently serve its customers, ensuring adequate supply and pressure. However, assigning customers to these stations involves critical trade-offs. While customers implicitly prefer the station that provides them the best perceived water pressure and quality, serving them from their "preferred" station often incurs higher operational costs for the WMC. For instance, that routes might involve older pipes, leading to increased energy consumption, more frequent maintenance, or a higher risk of leaks.
  - Customer Cost: Represents the perceived water pressure and supply quality. Although customers do not actively choose a route, they expect water to flow from their tap with sufficient pressure and to be safe for consumption. Customers therefore implicitly prefer the stations providing the best pressure and quality, which is intrinsically linked to how far the water must travel through pipelines.

- Company Cost: Represents the operational and infrastructure costs for the WMC directly associated with maintaining and operating the pipeline segments that supply customers. This includes energy costs for pumping, the costs of regular inspections, leak repairs, and material and labor costs.
- Upgrading: The WMC can reduce its operational costs on pipeline segments by making strategic investments, such as deploying early fault detection technologies for predictive maintenance on pipes or automating remote monitoring to reduce inspection costs and mitigate leak risks. These upgrades are crucially designed to decrease the company's expenses and improve its internal operational efficiency without altering the perceived water pressure or quality at the tap, which remains the basis for customers' implicit choice of supply route.

- Manufacturing Supply Chain with Hierarchical Logistics.
  - Context: A manufacturing company operates a supply chain from production plants to distribution centers, which then serve various retailers. Retailers (customers) do not directly control the physical routing of goods, but they express a clear preference for service from a specific distribution center based on factors like perceived fastest delivery option or order fulfillment speed. The company then incurs costs for transporting goods between plants and distribution centers, and from distribution centers to retailers to meet these retailer demands.
  - Customer Cost: Perceived delivery time or reliability that a retailer prioritizes when choosing a distribution center. Retailers choose the center that offers them the fastest or most reliable option. This "choice" reflects the retailer's preferred service point, meaning they will demand service from the center that offers them the fastest or most reliable option from their perspective.
  - Company Cost: Operational expenses for the manufacturing company, such as fuel costs, driver wages, vehicle maintenance, and handling associated with using specific routes for inter-plant/distribution center transfers and final deliveries.
  - Upgrading: The company has a budget to invest in upgrading certain routes (e.g., purchasing more fuel-efficient vehicles for specific segments). This investment reduces the company's operational cost for some routes but does not change the retailer's perceived delivery time or reliability.
- Air Cargo Logistics Network.
  - Context: A cargo airline (the company) manages the transport of shipments from origin to destination cities. Each shipment, representing an origin-destination pair, is assigned

to a single consolidation hub (a transfer center) at both the origin and the destination. Shippers (customers) choose their preferred origin hub based on factors like perceived time-in-transit, flight frequency, or initial handling speed (customer cost).

  - Customer Cost: Perceived time-in-transit, flight frequency, or initial handling speed.
  - Company Cost: Fuel consumption, aircraft maintenance, crew wages, fleet depreciation, air navigation fees, and landing/parking fees associated with each flight segment.
  - Upgrading: The airline can invest in upgrading flight legs connecting trunk routes (between hubs), such as acquiring more efficient aircraft for these high-volume flights.

- Specialized Data Back-up and Disaster Recovery Network:
  - Context: A cloud service provider (company) offers data storage and disaster recovery services through a network of data centers. For security or regulatory reasons, data replication might follow a strict tree-like hierarchical structure from client-facing regional centers to a central disaster recovery facility. Corporate clients (customers) choose their primary regional data center based on perceived data latency (customer cost).
  - Customer Cost: Perceived data latency (the delay or lag experienced when accessing or transferring data).
  - Company Cost : Operational costs for the cloud provider, including bandwidth usage for data transfer between data centers along the tree structure, energy consumption for servers, and maintenance of the specialized fiber optic network.
  - Upgrading: The provider has a budget to invest in upgrading specific fiber optic lines within the tree structure (e.g., installing higher-capacity cables, implementing advanced routing protocols) or investing in more energy-efficient cooling systems at certain center branches. These upgrades reduce the company's operational costs but don't alter the client's perceived data latency.

1.2 Main contributions

This paper makes two key contributions to network facility location problems. First, we introduce novel location models that simultaneously address dual-cost allocation (two distinct cost types, customer and company, associated with network arcs) and strategic network investment (incorporation of upgrade costs into the objective function to optimize budget allocation for cost reduction). Second, we develop mathematical formulations that jointly determine optimal facility locations, demand node allocation, and budget distribution across network arcs to minimize total company expenses (post-upgrade company costs plus upgrading investment). To achieve this, we employ a single objective function that explicitly combines both the post-upgrading company operational costs and the investment costs incurred for upgrading arcs. This approach is common and highly relevant in real-world optimization because it allows the decision-maker to directly trade off these two types of costs. By integrating both into a single objective, the model identifies solutions that minimize the total financial outlay, providing a comprehensive view of the most cost-effective strategies. Specifically, we model variants of four classical problems: the p-median problem, the two-stage facility location problem, the single-allocation p-hub location problem and the tree-of-hubs location problem. The applicability of these models across diverse domains is thoroughly demonstrated through the examples provided in Applications Subsection. These include water management systems, which are analyzed as a p-median application reflecting customer-driven quality and infrastructure optimization. We also cover manufacturing supply chains, representing two-stage problems with hierarchical distribution. Additionally, air cargo logistics illustrates single-allocation hub networks for consolidation, and secure data backup systems are tailored for constrained, tree-like network designs.

The rest of the paper is organized as follows. In the next section, we review the related literature. In Section 3, we formally define the facility location problems and introduce some notation to be used in the following sections. The next four sections are devoted to the analysis of the p-median problem, the two-stage location problem, the single allocation p-hub location problem and the tree of hubs location problem, all on networks with two kind of costs and upgrading. Integer programming formulations are developed for the aforementioned problems. The impact of the budget on the solutions is discussed in detail in Section 8. The paper finishes with a section containing conclusions and some possible future lines of research.

2 Literature review

Several problems involving both, facility location and network decisions, have been studied in the literature. The initial model for the facility location/network design problem was introduced by Daskin et al. [12]. Contreras and Fernández [9] presented a unified framework for the general network design problem that addressed classical problems involving combined location and network design decisions. Some recent applications can be found in Dukkanci et al. [17], Gokbayrak [29] and Laporte et al. [38]. Problems of this type involve two decisions: Locating facilities in a set of nodes and selecting a set of links from the demand nodes to their allocated facilities. Several costs are involved in the process, such as design costs (set-up costs of facilities and links) and operating (service) costs to transport the demand through the network. Drezner and Wesolowski [16] and Melkote and Daskin [45] proposed models that aimed to determine where to locate facilities and which network links to build in order to minimize the sum of set-up costs and operating costs. Another approach that considered these costs simultaneously was to incorporate operating costs in the objective function while requiring the overall design costs to satisfy a given budget constraint. Melkote and Daskin [45] raised a scenario in which a budget could be used to construct facilities or links on the network and the goal was to minimize the operating cost. In the paper by Contreras et al. [8], the operating costs represented access times of demand nodes to their allocated facilities in the network induced by the selected arcs, and the objective was to minimize the maximum operating cost. The design costs were considered in a budget constraint. Simultaneously, the paths connecting two nodes had to be determined.

Some authors studied location problems with upgrading. Related to median problems, Gassner [25] studied the 1-median problem in a network, and Sepasian and Rahbarnia [52] dealt with the p-median problem on a path. Antón-Sánchez et al. [5] studied different upgrading strategies for the discrete p-center problem. In the continuous case, the Euclidean 1-median problem was analyzed by Plastria [49]. In the work of Alizadeh and Afrashteh [4], the budget-constrained inverse median facility location problem on networks was studied. Concerning the min-max objective, the 1-center problem was studied by Gassner [27] and Sepasian [50], and the inverse version on trees was considered by Nguyen [46]. Nguyen and Teh [47] considered the uniform cost reverse 1-centdian problem on networks, where edge lengths were reduced within a given budget. In the case of undesirable facilities, obnoxious median location problems on trees were studied by Afrashteh et al. [2], Alizadeh et al. [3] and Gassner [26]. The maximal covering location problem on networks with edge length modifications was considered by Baldomero et al. [6], whereas Blanco and Marín [7] applied upgrading to hub location problems. Finally, some authors considered upgrading in spanning tree problems (Drangmeister et al. [15], Krumke et al. [34], Sepasian and Monabbati [51]), flow problems (Demgensky et al. [13], Holzhauser et al. [31]), and shortest and longest path problems (Fulkerson and Harding [23], Hambrusch and Tu [30]).

To the best of our knowledge, no prior work has simultaneously considered a primary cost governing customer-to-facility allocation, and a secondary operating cost (derived/company cost). The sole exception is Espejo and Marín [20], who examined arc upgrading in networks with two cost types on the arcs in the context of the p-median problem. Unlike prior work where the objective might focus solely on minimizing the post-upgrading transportation cost while treating the budget as a constraint for cost reduction, the present study aims to:

(i) include the upgrading investment in the objective function, to minimize the total company cost, encompassing both the company's costs after improvements and the precise budget invested in those improvements. This is a fundamental shift that enables a more comprehensive optimization by forcing the model to directly balance the benefits of cost reduction against the actual expenditure required for those improvements. This comprehensive formulation allows for a more holistic and economically realistic decision-making process regarding investment.

(ii) apply and adapt this novel dual-cost and upgrading framework to a broader range of classical facility location problems, including the two-stage facility location problem, the single-allocation hub location problem, and the tree-of-hubs location problem. This involves a deep dive into the unique challenges and particularities of extending diverse classical problems to this dual-cost, upgradable bi-network framework. Each of these problems presents distinct structural characteristics and modeling complexities that require specific formulations and analytical approaches. Each problem variant is carefully extended, resulting in distinct integer programming formulations and analyses.

3 General description of the problem

Consider a directed bi-network (V, A, c, c') given by a directed graph (V, A) and two kinds of costs: c (customer or allocation cost) and c' (company or derived cost), associated with the arcs of A. Without loss of generality, we assume that the graph is strongly connected. Let $I \subseteq V$ be the set of customers (demand nodes) and let $K \subseteq V$ be the the set of candidates to be selected as facilities. For each $i \in I$, a demand $w_i \geq 0$ must be supplied.

A number $1 \leq p \leq n-1$ of nodes in K can be chosen as facilities. Once the subset of facilities $X \subseteq K$ has been determined, $|X| \leq p$, and depending on the particular model, every demand node i chooses the facility (or facilities) that serves him with the least total customer cost C. This cost is produced by the use of some arcs of the network, that we denote $S(i, X)$. The use of the arcs in $S(i, X)$ produces in turn a unit cost $C'(i, X)$ to the company.

Moreover, the company can allocate all or a portion of a budget $B > 0$ to reduce its costs. Initially, a given arc $a \in A$ has an associated company cost c'_a (this is the company's operational cost per unit of demand for traversing arc a). However, the company can invest in improving a, thereby reducing its cost from c'_a to a minimum possible cost of $c'_a - u_a$. Here, u_a represents the maximum possible reduction per unit of demand that can be achieved in c'_a for arc a. Each unit of cost reduction of the unit cost c'_a incurs an investment cost of h_a (measured in units of demand). We assume –without loss of generality– that $h_a \leq B/u_a$.

Note that arc upgrading affects only the company costs (c'_a) and does not alter the customer costs (c_a). This means that while c'_a of an arc a can be reduced to $c'_a - u_a$, the c_a of that same arc remains constant. Therefore, the underlying C-values between any pair of nodes do not change. Consequently, the shortest path matrix based on these C-costs, which determines customer assignments, is precomputed and remains unchanged.

The *Generalized Induced Facility Location Problem with Upgrading* (GIFLPU) consists then of simultaneously selecting the subset of p facilities X and distributing a budget B or part of it (for reducing the company costs c' of the arcs in the sets $S(i, X)$) in order to minimize the sum of the upgraded company costs on the arcs of $S(i, X)$ for all $i \in I$, plus the part of the budget invested in the upgrading. Note that there are some arcs in the network that will never be used by customers as part of any shortest customer-cost path and then, they will never be upgraded. Let then $A^* \subseteq A$ denote the set of arcs susceptible of being upgraded (i.e., arcs that belong to at least one shortest path based on customer costs between any origin-destination pair).

A decision variable b_a is used to measure the reduction per unit of demand applied to c'_a for arc $a \in A$. Denoting by $b(i, X)$ the total reduction in the company costs of the set $S(i, X)$, the problem is

$$\min_{\substack{X \subset K \\ |X| \leq p}} \left\{ \sum_{i \in I} w_i(C'(i, X) - b(i, X)) + \sum_{a \in A^*} h_a b_a : \sum_{a \in A^*} h_a b_a \leq B;\ 0 \leq b_a \leq u_a,\ \forall a \in A^* \right\}.$$

Unlike classical facility location problems, where the number of facilities is fixed to exactly p, in the GIFLPU it may be more cost-effective for the company to locate less than p facilities. Computational studies provide further evidence of this claim.

The related *Induced p-Median Problem with Upgrading* has been studied by Espejo and Marín in [20]. They have shown that (i) the location of the medians generally changes when the arcs are upgraded, and (ii) the allocation of the demand points to the facilities with the lowest customer costs requires the explicit addition of closest assignment constraints (CAC) to the formulations, see Espejo et al. [19].

It is assumed in the rest of the article that $c_a, c'_a \geq 0\ \forall a \in A$, but no other condition on the costs is required.

4 Extending the p-Median Problem

The classical p-median problem on networks consists in the following. A set of p medians has to be located in the vertices $K \subseteq V$ of a directed network (V, A, c). Let $I \subseteq V$ be the customers set. Let $SP(s, \ell)$ be the c-shortest directed path from $s \in V$ to $\ell \in V$, that is assumed to be unique and has cost $C_{s\ell} := \sum_{a \in SP(s,\ell)} c_a$. Once the set of medians $X \subset K$ has been determined we also assume that for each $i \in I$, only one $k_i \in X$ exists such that the shortest path from X to i is $SP(k_i, i)$. Using the following variables

$$y_k = \begin{cases} 1 & \text{if } k \text{ is chosen as a median} \\ 0 & \text{otherwise} \end{cases} \quad \forall k \in K,$$

$$x_{ki} = \begin{cases} 1 & \text{if } k = k_i \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in I,\ \forall k \in K,$$

the classical formulation of the p-median problem reads

$$\begin{aligned}
\text{(pM)} \quad \min \quad & \sum_{i\in I} w_i \sum_{k\in K} C_{ki} x_{ki} \\
\text{s.t.} \quad & x_{ki} \le y_k, \quad \forall i \in I,\ \forall k \in K, && (1)\\
& \sum_{k\in K} x_{ki} = 1, \quad \forall i \in I, && (2)\\
& \sum_{k\in K} y_k \le p, && (3)\\
& x_{ki} \in \{0,1\}, \quad \forall i \in I,\ \forall k \in K, && (4)\\
& y_k \in \{0,1\}, \quad \forall k \in K. && (5)
\end{aligned}$$

(pM) can be equivalently formulated with (3) in the form of equality, unlike other problems to be presented afterwards. The p-median problem has been widely studied. We refer the interested reader to the survey by Marín and Pelegrín in [43].

In the *Generalized Induced p-Median Problem with Upgrading* (GIpMPU), a bi-network (V, A, c, c') is given. The total company cost associated to $SP(s,\ell)$, denoted by $C'_{s\ell}$, is obtained adding up the costs c'_a of the arcs $a \in SP(s,\ell)$, i.e., $C'_{s\ell} := \sum_{a\in SP(s,\ell)} c'_a$. In addition to the location of the p medians, a budget can be distributed among the arcs to reduce the company costs associated to the shortest path from the medians to the customers, so that the sum of the company costs after the reduction and the upgrading investment is minimal.

Recall that arcs exist where a discount is never applied, since they do not belong to shortest paths from nodes in K to nodes in I. In particular, the discount is limited to arcs in the set $A^* = \cup_{i\in I} \cup_{k\in K} SP(k,i)$. Using variables $b_a \ge 0$ for the reduction of the cost c'_a, $a \in A^*$, the GIpMPU can be formulated as a nonlinear program:

$$\begin{aligned}
\text{(IpM)} \quad \min \quad & \sum_{i\in I} w_i \sum_{k\in K} (C'_{ki} - \sum_{a\in SP(k,i)} b_a) x_{ki} + \sum_{a\in A^*} h_a b_a && (6)\\
\text{s.t.} \quad & (1) - (5) \\
& y_k + \sum_{\substack{\ell\in K:\\ C_{\ell i} > C_{ki}}} x_{\ell i} \le 1, \quad \forall i \in I,\ \forall k \in K, && (7)\\
& \sum_{a\in A^*} h_a b_a \le B, && (8)\\
& b_a \le u_a, \quad \forall a \in A^*, && (9)\\
& b_a \ge 0, \quad \forall a \in A^*. && (10)
\end{aligned}$$

(7) are CAC that guarantee the allocation of customers to their preferred facilities (i.e., those with minimum customer cost). A budget B or part of it, limited by constraints (8) and (9), is applied in the first part of (6) to reduce the cost of the arcs, producing a nonlinearity. The amount finally

invested in upgrading the arcs is also added in the second term of the objective function as a cost for the company.

A linearization of (6), based on the one developed by Espejo and Marín in [20], consists of introducing variables z_{ia} representing the unit company cost reduction along the path that serves node i when the arc $a \in A$ is upgraded. Let $SP(i) \subseteq A$ be the set containing the arcs in any c-shortest path to $i \in I$. Then

$$z_{ia} := \sum_{\substack{k \in K: \\ a \in SP(k,i)}} b_a x_{ki} = b_a \sum_{\substack{k \in K: \\ a \in SP(k,i)}} x_{ki}, \quad \forall i \in I, \ \forall a \in SP(i).$$

A linear reformulation of (IpM) is

$$\begin{aligned}
\text{(IpML)} \qquad \min \quad & \sum_{i \in I} w_i \left(\sum_{k \in K} C'_{ki} x_{ki} - \sum_{a \in SP(i)} z_{ia} \right) + \sum_{a \in A^*} h_a b_a \\
\text{s.t.} \quad & (1)-(5), (7)-(10) \\
& z_{ia} \le b_a, \quad \forall i \in I, \forall a \in SP(i), \\
& z_{ia} \le u_a \sum_{\substack{k \in K: \\ a \in SP(k,i)}} x_{ki}, \quad \forall i \in I, \forall a \in SP(i), \\
& z_{ia} \ge 0, \quad \forall i \in I, \ \forall a \in SP(i).
\end{aligned}$$

Note that when a customer $i \in I$ is allocated to a median $k \in K$, all other customers $i' \in I$ in the path $SP(k,i)$ are also allocated to k.

5 Extending the Two-Stage Facility Location Problem

The *Two-Stage Facility Location Problem* (TSFLP) consists of locating facilities and depots (also called distribution centers) in a network (V, A, c) and sending a product from the located facilities to customers through the located depots. Some works dealing with two-stage facility location and related problems are Aardal et al. [1], De Oliveira et al. [14], Landete and Marín [36], Marín [39] and Marín and Pelegrín [40, 41, 42]. In the version we extend here, at most q facilities and at most p depots have to be located in the vertices of the network. Let $J \subseteq V$ and $K \subseteq V$ be the sets of candidates to be selected as facilities and depots, respectively. We assume that $I \cap K = \emptyset$ and $J \cap K = \emptyset$. Let $I \subseteq V$ be the customers set. For each $i \in I$ a demand w_i must be supplied from a facility in J through a depot in K. We assume that the capacity of facilities and depots is unlimited.

One of the most studied formulations for the TSFLP is the three-index formulation. Location variable y_k, $k \in K$, takes value 1 if k is chosen as a depot and t_j takes value 1 if $j \in J$ is chosen as a facility. Since each depot is going to be optimally assigned to only one facility, binary variable x_{jki}, $j \in J$, $k \in K$, $i \in I$, takes value 1 if customer i is allocated to depot k and depot k is allocated to facility j. Then, we can formulate TSFLP as

$$
\begin{aligned}
\text{(TS)} \quad \min \quad & \sum_{i\in I} w_i \sum_{j\in J}\sum_{k\in K} C_{jki}x_{jki} \\
\text{s.t.} \quad & (3), (5) \\
& \sum_{j\in J}\sum_{k\in K} x_{jki} = 1, \quad \forall i \in I, && (11)\\
& \sum_{j\in J} x_{jki} \le y_k, \quad \forall i \in I, \forall k \in K, && (12)\\
& \sum_{k\in K} x_{jki} \le t_j, \quad \forall i \in I, \forall j \in J, && (13)\\
& \sum_{j\in J} t_j \le q, && (14)\\
& x_{jki} \in \{0,1\}, \quad \forall j \in J,\ \forall k \in K,\ \forall i \in I, && (15)\\
& t_j \in \{0,1\}, \quad \forall j \in J. && (16)
\end{aligned}
$$

Here C_{jki} is the cost of transporting a unit of product through the shortest path from facility j to client i through depot k, obtained as

$$C_{jki} = C_{jk} + C_{ki}, \qquad \forall j \in J, k \in K, i \in I,$$

where $C_{s\ell}$ is the cost of the shortest path from s to ℓ. Constraints (11) ensure that all customer demands are satisfied. Constraints (12) and (13) guarantee that all customers are supplied through depots and facilities, respectively.

The *Generalized Induced Two-Stage Facility Location Problem with Upgrading* (GITSFLPU) on a directed bi-network (V, A, c, c') consists of simultaneously locating at most q facilities and at most p depots and distributing at most a budget B to reduce the company costs of sending the product to the customers from the facilities through the depots, so that the total –reduced– company cost plus the upgrading investment is minimum. Once the facilities and depots have been located, every depot will be served from the facility providing the minimum directed c-shortest path and every customer will be served from the depot with minimum directed c-shortest path as well. Let $SP(i) \subseteq A$ be the set of arcs that belong to any c-shortest path to $i \in I$. The total company costs when i is served from j through k is given by $C'_{jki} = C'_{jk} + C'_{ki}$. We consider again variables b_a to account for the reduction of the company cost of arc $a \in A^*$. Here $A^* = SP(j,k) \cup SP(k,i)$ is the set of arcs that belong to the c-shortest paths facility–depot or depot–customer. We assume that $SP(j,k) \cap SP(k,i) = \emptyset$ (otherwise, we duplicate the shared arcs).

Based on (TS), a non-linear formulation for GITSFLPU is:

$$
\begin{aligned}
\text{(ITS)} \quad \min \quad & \sum_{i\in I} w_i \sum_{j\in J}\sum_{k\in K}\left(C'_{jki} - \sum_{a\in SP(j,k)\cup SP(k,i)} b_a\right) x_{jki} + \sum_{a\in A^*} h_a b_a \\
\text{s.t.} \quad & (3),(5),(8)-(10),(11)-(16) \\
& \sum_{i\in I}\sum_{\substack{\ell\in J:\\ C_{\ell k}>C_{jk}}} x_{\ell ki} + t_j \le 1, \ \forall k\in K, \ \forall j\in J, && (17)\\
& \sum_{j\in J}\sum_{\substack{\ell\in K:\\ C_{\ell i}>C_{ki}}} x_{j\ell i} + y_k \le 1, \ \forall i\in I, \ \forall k\in K. && (18)
\end{aligned}
$$

We are minimizing the total company cost after applying the reduction in arcs in c-shortest paths facility–depot or and depot–customer, plus the upgrading investment. In order to guarantee that every depot/customer is allocated to the facility/depot with the minimum c-shortest path, CAC (17) and (18) are required. To linearize (ITS) let us denote, for $i\in I$, $SP^K(i) := \{a\in SP(k,i)\cup SP(k) : k\in K\}$, and consider auxiliary variables $z_{ia} := b_a \sum_{\ell\in K}\sum_{\substack{j\in J:\\ a\in SP(\ell,i)\cup SP(j,\ell)}} x_{j\ell i}$, $\forall i\in I$, $\forall a\in SP^K(i)$. Then, the specific linear formulation for GITSFLPU is

$$
\begin{aligned}
\text{(ITSL)} \quad \min \quad & \sum_{i\in I} w_i \left(\sum_{j\in J}\sum_{k\in K} C'_{jki}x_{jki} - \sum_{a\in SP^K(i)} z_{ia}\right) + \sum_{a\in A^*} h_a b_a \\
\text{s.t.} \quad & (3),(5),(8)-(10),(11)-(16),(17),(18) \\
& z_{ia} \le b_a, \quad \forall i\in I, \quad \forall a\in SP^K(i), \\
& z_{ia} \le u_a \sum_{\ell\in K}\sum_{\substack{j\in J:\\ a\in SP(\ell,i)\cup SP(j,\ell)}} x_{j\ell i}, \quad \forall i\in I, \quad \forall a\in SP^K(i), \\
& z_{ia} \ge 0, \quad \forall i\in I, \quad \forall a\in SP^K(i).
\end{aligned}
$$

6 Extending the Single Allocation p-Hub Location Problem

In a classical p-hub location problem, there is an undirected network (V,E,c) and a product to be sent between some pairs of nodes of V. Although a directed network could also be considered, this possibility is almost never considered in the literature of hub location. Since the cost of traversing an edge $e\in E$ can be different depending of the direction followed by the flow, we still define A as the set of arcs obtained from E by splitting every edge in two arcs. Let $I\subseteq V$ represent the set of origins and destinations of this product flow and $w_{ij}\ge 0$ the amount of product sent from origin $i\in I$ to destination $j\in I$. Every flow must be routed via either one or two special nodes called hubs. The hubs act as transshipment nodes by collecting the flows from the origins and redistributing them towards the destinations. Let $K\subseteq V$ be the set of nodes candidates to be selected as hubs. Each unit of product that traverses $a = (s, \ell) \in A$ incurs initially a cost c_a, although this cost could be smaller when s and ℓ are both hubs. The goal is to locate at most p hubs and to route every shipment through the most suitable hub or pair of hubs in such a way that the overall cost is minimized. Note that it is in the essence of the problem that two hubs must be directly connected through only one edge, since the economies of scale associated to the lower cost between hubs use this fact. Initially, for the sake of simplicity, we consider that the subnetwork induced by K is complete, adding arcs with very large c-costs when needed. Nevertheless, the flow can follow the shortest paths from hubs to destinations and from origins to hubs. To simplify the notation, we call $O_i := \sum_{j\in I} w_{ij}$ and $D_i = \sum_{j\in I} w_{ji}$. In the forthcoming formulations we do not assume special properties of the costs like satisfaction of the triangle inequality, symmetry or any other.

Special attention deserves the inter-hub discounted flow. To model economies of scale, it is usual in the literature to multiply the c-cost between two hubs by a discount factor $0 < \alpha < 1$ (usually called transfer factor). Although this way to proceed simplify the formulations of the problem, it is not evident that the mere fact of marking two nodes as hubs make the cost between them be automatically reduced by a factor, regardless the amount of flow sent from one of them to the other. In other words, instead of invest a budget in upgrading arcs one by one, it is generally assumed that the upgrading comes for free due to the (hypothetical) higher amount of flow in the inter-hub network. The kind of model we introduce in this paper fills in part this hole in the literature. The literature also considers the collection (χ) and distribution (δ) factors, applied between origin-hub and hub-destination pairs, respectively.

There are two large families of hub location problems. Those that allow a customer to choose different pairs of hubs depending on the destination of the flow (multiple allocation problems) and those that enforce a customer to choose only one hub to send (and receive) all its flow. Since, in the philosophy behind our models, origins are assumed to be customers that choose the facility they are allocated to by comparing c-costs, it only has sense to extend the Single Allocation p-Hub Location Problem (SApHLP). This problem has been formulated in the literature in different ways, see Espejo et al. [21], Ghaffarinasab and Kara [24], Labbé et al. [35], Meier and Clausen [44]. We present and use here a formulation never considered before. For $i \in I$, $m \in K$, we define the variables S_{im} that measure the inter-hub transportation cost (without the transfer factor α) with origin at i sent to all destinations through any first hub but using $m \in K$ as the second hub. Using variables y and x as before, SApHLP on a network (V, A, c) can be formulated as

$$\text{(Hub)} \quad \min \quad \alpha \sum_{i\in I}\sum_{m\in K} S_{im} + \sum_{i\in I}\sum_{k\in K} (\chi O_i C_{ik} + \delta D_i C_{ki}) x_{ki}$$

$$\text{s.t.} \quad (1) - (5)$$

$$S_{im} \geq \sum_{j\in I} w_{ij} c_{km} (x_{ki} + x_{mj} - 1), \qquad \forall i \in I,\ \forall k, m \in K, \tag{19}$$

$$S_{im} \geq 0, \qquad \forall i \in I,\ \forall m \in K. \tag{20}$$

Constraints (19) only have effect when $x_{ki} = 1$. In such a case, the cost of sending all the flow from origin $i \in I$ between hubs comes from arcs (k, m) for every j such that $x_{mj} = 1$, which is the wished effect of the constraints. Note that the cost c_{km} is the cost of a single arc, (k, m), whereas the cost C_{ik} in the objective function is obtained from the shortest path $SP(i, k)$. If some flow moves from node $s \in I \cap K$ to node $\ell \in K$, the cost to be applied is $c_{s\ell}$ when both are hubs (and this cost will be multiplied times α) but it is $C_{s\ell}$ when s is not a hub allocated to hub ℓ (multiplied times χ or δ). It could hold $C_{s\ell} < c_{s\ell}$. In this second case, the arcs in the shortest path from s to ℓ would not benefit of any discount.

In our extension of the problem, with induced costs and upgrading, we use the bi-network (V, A, c, c'). As in previous sections, in the hub location problem the customer cost c could be based on travel times, distances, preferences or any other, while the derived company cost c' could represent the transportation cost (per unit of flow). Each route beginning at origin i, ending at destination j, and traversing hubs k and m in this order, carries a customer cost (possibly to be paid by two customers, i and j) $C_{ik} + C_{mj}$, and a company cost $\chi C'_{ik} + \alpha c'_{km} + \delta C'_{mj}$, where $C'_{s\ell}$ is again the company cost associated to the c-shortest directed path between s and ℓ. A budget $B > 0$ is given and can be used to reduce the company costs of the inter-hub arcs. There is still a limit of $0 \leq u_a \leq c_a$ for the upgrading of arc $a \in A^* := K \times K$ and the reduction comes at a penalty of $0 \leq h_a \leq B/u_a$ like in previous models.

The *Generalized Induced Single Allocation p-Hub Location Problem with Upgrading* (GISApHLPU) consists then of locating at most p hubs, assigning each customer $i \in I$ to a located hub and distributing the budget or part of it to reduce the company costs, in such a way that the total company cost after upgrading plus the upgrading investment is minimized. Now, for each $i \in I$ and $m \in K$, variable S_{im} represents the upgraded inter-hub transportation cost with origin at i sent to all destinations through any first hub but using $m \in K$ as the second hub. The variable b_{km} represents now the reduction of the company cost of the inter-hub arcs $(k, m) \in A^*$. Then we can formulate GISApHLPU as

$$
\begin{aligned}
\text{(IHub)} \quad \min \quad & \alpha \sum_{i \in I} \sum_{m \in K} S_{im} + \sum_{i \in I} \sum_{k \in K} (\chi O_i C'_{ik} + \delta D_i C'_{ki}) x_{ki} + \sum_{(k,m) \in A^*} h_{km} b_{km}, \\
\text{s.t.} \quad & (1) - (5), (7), (10), (20) \\
& b_{km} \leq u_{km} y_k, \quad \forall (k, m) \in A^*, && (21) \\
& b_{km} \leq u_{km} y_m, \quad \forall (k, m) \in A^*, && (22) \\
& \sum_{(k,m) \in A^*} h_{km} b_{km} \leq B, && (23) \\
& S_{im} \geq \sum_{j \in I} w_{ij} (c'_{km} - b_{km})(x_{ki} + x_{mj} - 1), \quad \forall i \in I, \forall (k, m) \in A^*. && (24)
\end{aligned}
$$

Note that costs $C_{s\ell}$ and $C'_{s\ell}$ can be calculated beforehand since they are the customer and company costs associated to the paths $SP(s, \ell)$ when s and ℓ are not both hubs and therefore, the costs of the arcs of $SP(s,\ell)$ are not upgraded. When the subnetwork $(K,(K\times K)\cap A)$ is sparse, adding all the constraints (24) can be computationally very expensive. An alternative possibility is, when $w_{ij}>0$ for all $i\neq j\in I$, to keep the arcs as they come, and add inequalities $y_k+y_m\leq 1$ for those $k<m\in K$ such that $(k,m)\notin E$.

Note also that we have the product of b- and x-variables in the set of constraints (24) instead of having them in the objective function, as in the other problems previously studied. The nonlinear component in the working out of (24), $i\in I$ and $k,m\in K$, is given by

$$S_{im}\geq\sum_{j\in I}w_{ij}b_{km}(x_{ki}+x_{mj})=O_ib_{km}x_{ki}+\sum_{j\in I}w_{ij}b_{km}x_{mj}.$$

In this case it is not the shortest path but the direct arc what has to be considered. Using variables z and z', we obtain the linear formulation of GISApHLPU:

$$\begin{aligned}
\text{(IHubL)}\quad \min\quad & \alpha\sum_{i\in I}\sum_{m\in K}S_{im}+\sum_{i\in I}\sum_{k\in K}(\chi O_iC'_{ik}+\delta D_iC'_{ki})x_{ki}+\sum_{(k,m)\in A^*}h_{km}b_{km},\\
\text{s.t.}\quad & (1)-(5),(7),(10),(20)-(23)\\
& S_{im}\geq\sum_{j\in I}w_{ij}\left(c'_{km}(x_{ki}+x_{mj}-1)-z_{ikm}-z'_{jkm}+b_{km}\right),\quad\forall i\in I,\ \forall k\in K,\\
& z_{ikm}\leq b_{km},\quad\forall i\in I,\ \forall(k,m)\in A^*,\\
& z_{ikm}\leq u_{km}x_{ki},\quad\forall i\in I,\ \forall(k,m)\in A^*,\\
& z_{ikm}\geq 0,\quad\forall i\in I,\ \forall(k,m)\in A^*,\\
& z'_{jkm}\leq b_{km},\quad\forall j\in I,\ \forall(k,m)\in A^*,\\
& z'_{jkm}\leq u_{km}x_{mj},\quad\forall j\in I,\ \forall(k,m)\in A^*,\\
& z'_{jkm}\geq 0,\quad\forall j\in I,\forall(k,m)\in A^*.
\end{aligned}$$

7 Extending the Tree of Hubs Location Problem

This section deals with a hub location model not requiring direct connection between all pairs of hubs. Potential applications arise when the costs of the links between hubs are very high and full interconnection between hub nodes is prohibitive (see [32]). The model, developed by Contreras et al. in [10] and [11], is a single-allocation hub location problem on an undirected network (V,E,c). Like in the previous chapter, since the cost of traversing an edge $e\in E$ can be different depending on the direction followed by the flow, we still define A as the set of arcs obtained from E by splitting every edge in two arcs. Now at most $p\geq 2$ hubs have to be located, with the particularity that it is required that the hubs are connected by means of a (non–directed) tree. Consider two sets $I\subset V$ and $K\subset V$, with $I\cap K=\emptyset$, representing the customers and potential hubs, respectively. For each $i,j\in I$, $w_{ij}\geq 0$ denotes the flow that must be sent from i to j through a path of hubs inside the tree. The so-called *Tree of Hubs Location Problem* (THLP) aims to locate at most p hubs, to select

$p-1$ edges of E that connect the hubs, and to assign each customer in I to a located hub in such a way that the overall cost is minimized. The cost (per unit of flow) of the path from i to j is the sum of the cost of (i) the directed path from i to its allocated hub (say k) at minimum c-cost, (ii) the directed path between k and the hub allocated to j (say m) through the unique path in the hubs tree, and (iii) the directed path from m to j following the c-shortest path. Now, C_{ki} denotes the directed c-shortest path from i to k. Let $0 < \alpha < 1$ be a discount factor (transfer factor) for the flow between hubs.

In order to formulate THLP, Contreras et al. [10] considered two new sets of variables to determine the unique path that exists in the tree between each pair origin-destination:

$$y_{km} = \begin{cases} 1 & \text{if edge } (k,m) \text{ belongs to the hubs tree} \\ 0 & \text{otherwise} \end{cases} \qquad \forall (k,m) \in E_K,$$

$$v_{ijkm} = \begin{cases} 1 & \text{if the flow from } i \text{ to } j \text{ traverses the two-hubs arc } (k,m) \\ 0 & \text{otherwise} \end{cases} \qquad \forall i,j \in I,\ \forall (k,m) \in A_K.$$

Using variables y and x as before, and preserving the notation of previous sections, the formulation for THLP reads

$$\begin{aligned}
\text{(THub)}\min \quad & \sum_{i\in I}\sum_{k\in K}(O_iC_{ik}+D_iC_{ki})x_{ki} + \alpha\sum_{i\in I}\sum_{j\in I}\sum_{(k,m)\in A_K} w_{ij}c_{km}v_{ijkm} \\
\text{s.t.} \quad & (1)-(5) \\
& v_{ijkm} + v_{ijmk} \le y_{km}, \quad \forall i,j \in I,\ \forall (k,m) \in E_K, && (25)\\
& \sum_{m\in K:\ (k,m)\in A_K} v_{ijkm} + x_{kj} = \sum_{m\in K:\ (m,k)\in A_K} v_{ijmk} + x_{ki}, \quad \forall i,j \in I, \forall k \in K, && (26)\\
& \sum_{(k,m)\in E_K} y_{km} = \sum_{k\in K} y_k - 1, && (27)\\
& y_{km} \le y_k, \quad \forall (k,m) \in A_K, && (28)\\
& y_{km} \le y_m, \quad \forall (k,m) \in A_K, && (29)\\
& v_{ijkm} \in \{0,1\}, \quad \forall i,j \in I,\ \forall (k,m) \in A_K, && (30)\\
& y_{km} \in \{0,1\}, \quad \forall (k,m) \in E_K. && (31)
\end{aligned}$$

Assuming that y_{km} induces a connected graph in (K, E_K), it is a tree due to (27). This constraint ensures the correct number of edges connecting the hubs. The connection between those hubs that have customers allocated to them is guaranteed by constraints (25). To ensure that all hubs receive allocation, we can include in the formulation $|K|$ dummy customers ℓ_k and fix $x_{k\ell_k} = 1$. The set of constraints (26) define paths in the tree between customers. Constraints (28) and (29) ensure that if a customer is allocated to an element of K, the latter has been chosen as a hub.

In our extension of THLP, the bi-network (V, A, c, c') is considered. Note that c-costs between elements of K and c'-costs between elements of I are not required. Once the hubs in X have been located, every customer follows the c-shortest directed path to X to decide its reference hub. The company costs associated to these c-shortest paths can be subject to reduction. Let k and m be the hubs to which customers i and j are allocated, respectively. Then, the total company cost to send the product from i to j is the sum of the company cost associated to the c-shortest path from i to k, C'_{ik}, the c-shortest path from m to j, C'_{mj}, and the sum of all company costs of the arcs between hubs (s,t), c'_{st} times α, in the unique path in the tree of hubs that goes from k to m.

To formulate the so-called *Generalized Induced Tree of Hubs Location Problem with Upgrading* (GITHLPU), we define b_{km} as the reduction of the transportation cost in the arc $(k,m) \in A_K$. The formulation we propose is then

$$
\begin{aligned}
\text{(ITHub)}\min \quad & \sum_{i\in I}\sum_{k\in K}(O_iC'_{ik} + D_iC'_{ki})x_{ki} + \alpha \sum_{i\in I}\sum_{j\in I}\sum_{(k,m)\in A_K} w_{ij}(c'_{km} - b_{km})v_{ijkm} + \sum_{(k,m)\in A_K} h_{km}b_{km}, \\
\text{s.t.} \quad & (1)-(5), (7), (25)-(31) \\
& \sum_{(k,m)\in A_K} h_{km}b_{km} \le B, && (32) \\
& b_{km} \le u_{km}y_{km}, \quad \forall (k,m) \in A_K, && (33) \\
& b_{km} \ge 0, \quad \forall (k,m) \in A_K. && (34)
\end{aligned}
$$

A linearization of the products in the objective function can be done using variables

$$z_{ijk} := \sum_{m:\ (k,m)\in A_K} b_{km}v_{ijkm}$$

$\forall i,j \in I$, $\forall k \in K$. Then, GITHLPU is linearly formulated as

$$
\begin{aligned}
\text{(ITHubL)}\min \quad & \sum_{i\in I}\sum_{k\in K}(O_iC'_{ik} + D_iC'_{ki})x_{ki} + \alpha\sum_{i\in I}\sum_{j\in I}\sum_{(k,m)\in A_K} w_{ij}c'_{km}v_{ijkm} - \alpha\sum_{i\in I}\sum_{j\in I}\sum_{k\in K} w_{ij}z_{ijk} \\
& + \sum_{(k,m)\in A_K} h_{km}b_{km} \\
\text{s.t.} \quad & (1)-(5), (7), (25)-(34) \\
& z_{ijk} \le \sum_{m:\ (k,m)\in A_K} b_{km} \quad \forall i,j \in I,\ \forall k \in K, \\
& z_{ijk} \le \sum_{m:\ (k,m)\in A_K} u_{km}v_{ijkm} \quad \forall i,j \in I,\ \forall k \in K, \\
& z_{ijk} \ge 0 \quad \forall i,j \in I,\ \forall k \in K.
\end{aligned}
$$

8 Sensitivity analysis

In this section we explore the impact of the amount of available budget on the solution, particularly on the optimal value and facility locations. The data considered for the test are taken from the well-known OR-library [48]. We have adapted the data files to our problems as follows:

- GIpMPU and GITSFLPU.

  We adapted the uncapacitated p-median data files. Each file contains information on the number of vertices, number of edges, number of medians and, for each edge, the end vertices and the cost. This last value was our c-cost. We split each edge in two arcs and duplicated the corresponding cost to obtain a directed network. We denoted with m the number of arcs. For every cost c_a we took $c'_a := c_a \cdot (0.8 + 0.4U)$ where U is uniformly distributed in $(0, 1)$. The maximum reduction of the cost, u_a, was fixed to $c'_a/2$. The cost h_a for reducing c'_a was fixed to 1 in each arc, and all the demands w_i were considered equal to 1. The budget was progressively incremented from 0 until no reduction in the optimal value of the problem was possible.

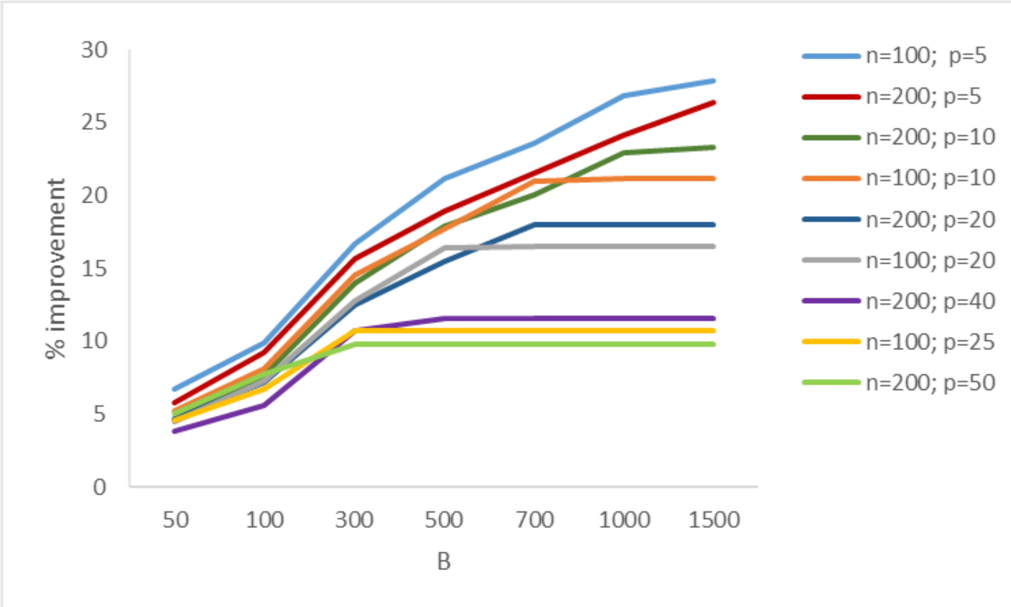
  In the case of GITSFLPU, we also split the set of vertices in three disjoint subsets representing the facilities, depots and customers subsets, and assumed that the number of depots and facilities to locate was the same.

- GISApHLPU and GITHLPU.

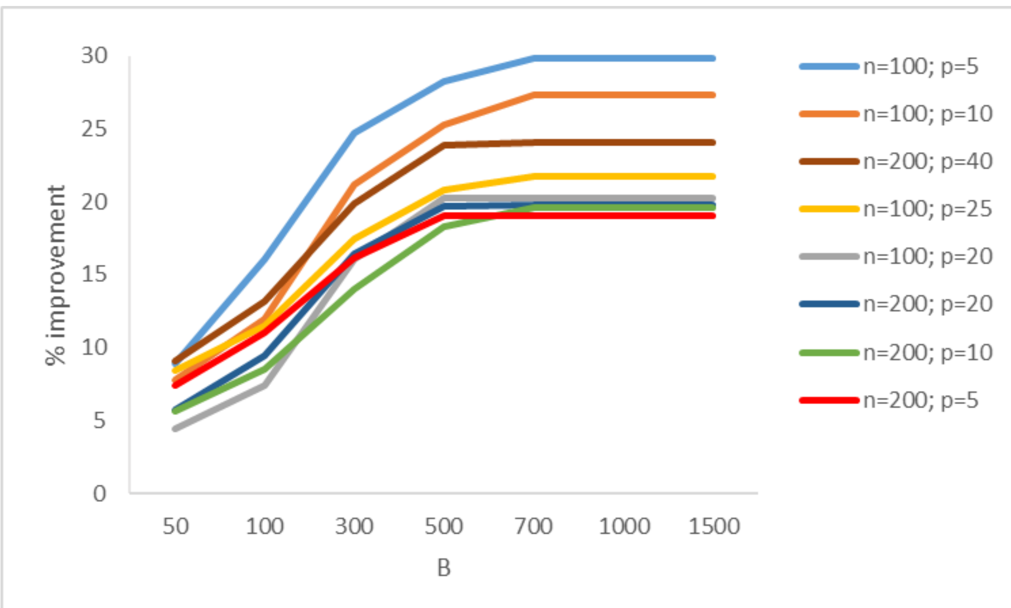
  From the commonly used data set for hub location problems known as AP ([18]) we took the SApHLP data files. To create smaller data sets, the C code available in [28] was used. Each data file contains number of nodes, coordinates of nodes, number of hubs, the flow between nodes and the values of the discount factors α, χ and δ. All these instances considered a complete graph. The c-costs were generated as the Euclidean distances between nodes. The c'-costs were uniformly generated in $[0, 100]$. The maximum reduction of the cost, u_a, was fixed to $c'_a/1.1$. The costs h_a were fixed to 1. The budget was progressively incremented from 0 until no reduction in the optimal value of the problem was possible.

8.1 Impact of budget on the reduction of the company cost

We have analyzed the percentage of improvement of the optimal value of each studied problem (respect to the optimal value when no upgrading is carried out, i.e, with a budget equal to zero) for different instances, depending on the budget (see figures 1 and 2). Thus, for each budget level (parameter B) that we have tested, the percentage of improvement is calculated as follows: $100(v_0 - v_\beta)/v_0$, where v_β and v_0 are the optimal values of the studied problems for the parameter $B = \beta$ and $B = 0$, respectively. Recall that the objective functions always include the part of the budget used to reduce the company costs, that is to say, the improvement shown in the figures is always non negative and the reduction in the company cost is greater than or equal to the cost of upgrading the arcs. As the budget increases initially, we observe a corresponding increase in the percentage of improvement, indicating greater cost reductions. However, beyond a certain budget

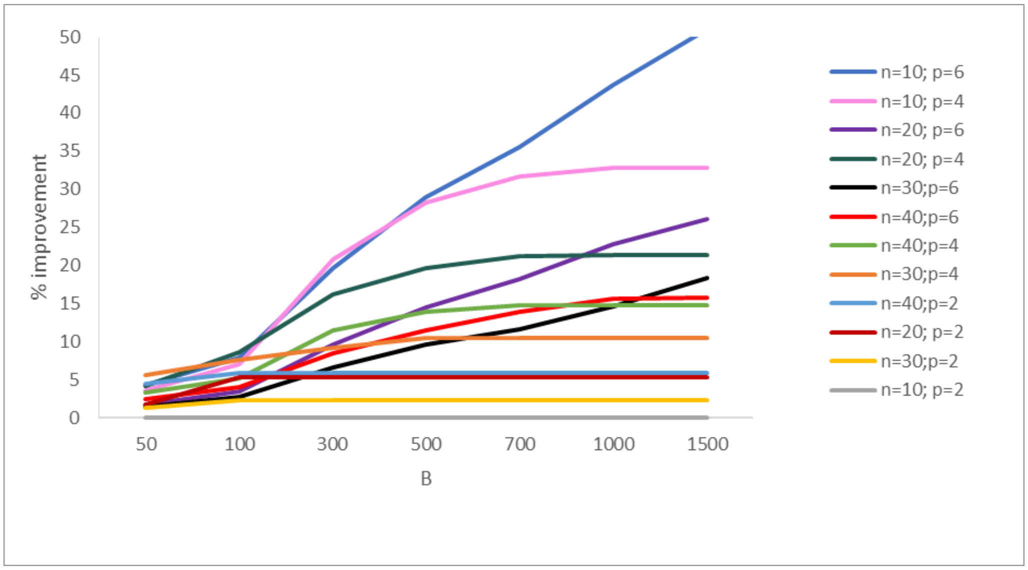


(a) GIpMPU

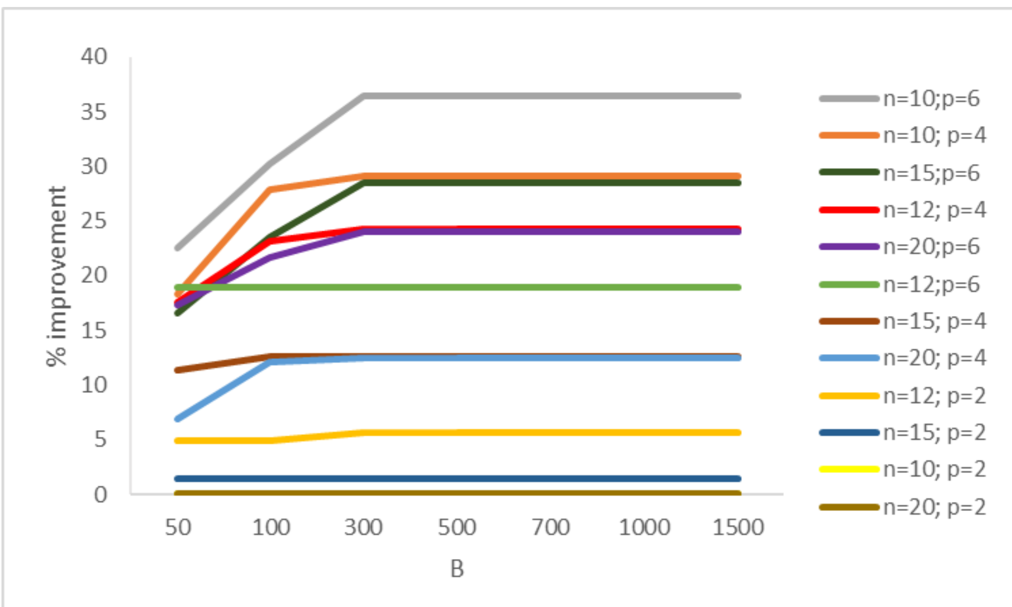


(b) GITSFLPU

Figure 1: Effect of budget on the optimal value for GIpMPU and GITSFLPU



(a) GISApHLPU



(b) GITHLPU

Figure 2: Effect of budget on the optimal value for GISApHLPU and GITHLPU

value, there is no further improvement. This occurs because the potential cost reduction on each individual arc is limited. Once the maximum possible reduction on the most impactful arcs is reached for a given budget level, further increases in the budget do not yield additional improvements in the overall cost. Therefore, the point at which the improvement flatten indicates that the budget has been sufficiently large to exploit the most beneficial cost reduction opportunities within the network's arc capacities. Generally speaking, in the case of GIpMPU (Figure 1, top) the smaller the value of p, the larger reduction of the cost. When fewer medians are open, each one serves more customers. Therefore, reducing costs on the shortest c-paths has a larger overall impact.

However, when GISApHLPU and GITHLPU are considered, the percentage of improvement increases with the value of p (Figure 2). This is because the cost reductions apply only to inter-hub connections. Therefore, the larger the number of hubs the larger the number of these connections and the cost savings.

8.2 Impact of budget on the optimal location of the facilities

GIpMPU									
	$n = 100$				$n = 200$				
B	$p = 5$	$p = 10$	$p = 20$	$p = 25$	$p = 5$	$p = 10$	$p = 20$	$p = 40$	$p = 50$
50	1/-	0/-	0/-	1/-	0/-	0/-	0/-	4/-	8/ -
100	1/0	0/-	0/-	2/1	0/-	0/-	0/-	3/2	8/10
300	1/0	1/1	1/1	3/2	0/-	0/-	0/-	4/4	14/3
500	1/0	1/0	1/0	4/1	0/-	0/-	0/-	4/0	12/6
700	1/0	0/1	1/0	3/1	1/1	0/-	0/-	4/3	15/3
1000	1/0	0/0	1/0	3/0	1/0	0/-	0/-	3/0	14/2

Table 1: Effect of budget increment on the medians location

A comparison of the optimal facilities depending on the budget value was also carried out. Tables 1-3 show, for each instance and budget value, a comparison of the optimal facilities with (i) those of the optimal solution when no upgrading is carried out (on the left of the slash sign), and (ii) the optimal solution of the previous row (on the right). Each column shows the number of facilities being different in each case. The symbol '-' indicates that it is not applicable (there have been no changes). Thus, for the GIpMPU (Table 1), for $n = 200$, $p = 40$ and $B = 100$, the value 3/2 means that (i) the locations of three facilities have changed with respect to the optimal solution of the problem when $B = 0$, and (ii) two facilities have changed their locations from those obtained when the budget was $B = 50$. Although it might be thought that the changes are due to alternative optimal solutions, this is not the case. Varying the budget level leads to the opening of different facilities because other budgets allow a reduction in the cost of new arcs that are used by a greater number of customers.

GITSFLPU												
	$n = 100$						$n = 200$					
B	$p = 5$		$p = 10$		$p = 20$		$p = 10$		$p = 20$		$p = 40$	
	D	F	D	F	D	F	D	F	D	F	D	F
50	0/-	0/-	0/-	0/-	2/-	0/-	0/-	0/-	4/-	2/-	3/-	2/-
100	1/1	1/1	0/-	0/-	4/4	0/-	0/-	0/-	3/3	1/1	3/3	2/2
300	1/1	1/1	4/4	4/4	3/2	0/-	1/1	1/1	4/1	2/1	6/5	2/0
500	1/1	1/0	4/0	4/0	2/3	0/-	1/1	1/1	3/2	1/1	9/3	1/1
700	1/1	0/1	4/2	4/2	3/4	1/1	1/0	1/0	3/3	1/1	7/3	1/0
1000	0/1	0/0	4/0	4/0	2/2	1/0	1/0	1/0	3/0	1/0	7/1	1/0

Table 2: Effect of budget increment on the depots and facilities location

GISApHLPU												
	$n = 10$			$n = 20$			$n = 30$			$n = 40$		
B	$p = 2$	$p = 4$	$p = 6$	$p = 2$	$p = 4$	$p = 6$	$p = 2$	$p = 4$	$p = 6$	$p = 2$	$p = 4$	$p = 6$
50	0/-	0/-	0/-	2/-	4/-	0/-	1/-	0/-	0/-	0/-	0/-	0/-
100	0/-	1/1	0/-	2/0	4/0	0/-	1/0	0/-	0/-	0/-	0/-	0/-
300	0/-	1/0	1/1	2/0	4/0	0/-	1/0	0/-	0/-	0/-	0/-	0/-
500	0/-	1/0	1/0	2/0	4/2	0/-	1/0	0/-	0/-	0/-	0/-	0/-
700	0/-	1/1	1/0	2/0	4/0	0/-	1/0	0/-	0/-	0/-	0/-	0/-
1000	0/-	1/0	1/0	2/0	4/0	0/-	1/0	0/-	3/3	0/-	0/-	0/-

GITHLPU												
	$n = 10$			$n = 12$			$n = 15$			$n = 20$		
B	$p = 2$	$p = 4$	$p = 6$	$p = 2$	$p = 4$	$p = 6$	$p = 2$	$p = 4$	$p = 6$	$p = 2$	$p = 4$	$p = 6$
50	0/-	0/-	3/-	0/-	0/-	0/-	0/-	3/-	0/-	0/-	1/-	0/-
100	0/-	0/-	1/3	0/-	1/1	0/-	0/-	3/0	2/2	0/-	2/1	1/1
300	0/-	0/-	1/0	2/2	1/0	0/-	0/-	3/0	2/0	0/-	2/0	1/1
500	0/-	0/-	1/0	2/0	1/0	0/-	0/-	3/0	2/0	0/-	2/0	1/0
700	0/-	0/-	1/0	2/0	1/0	0/-	0/-	3/0	2/0	0/-	2/0	1/0
1000	0/-	0/-	1/0	2/0	1/0	0/-	0/-	3/0	2/0	0/-	2/0	1/0

Table 3: Effect of budget increment on the hubs location in GISApHLPU and GITHLPU

Note that, despite the correlation between c- and c'-costs, for large values of p the optimal set of medians experiment significant changes when a budget is invested in the upgrading of arcs. The same conclusions are obtained when GITSFLPU is considered (Table 2), where both, depots (D) and facilities (F), experiment changes when a budget is invested. In the case of GISApHLPU and GITHLPU (Table 3), where c- and c'-costs are independent, the location of the hubs also change with the budget. Initially, as the instance size increases, hub locations become more sensitive to budget changes. However, they eventually stabilize and become less responsive.

8.3 Impact of budget on the optimal number of facilities

As previously discussed, it may be more cost-effective for the company to locate less than p facilities. We analyze how budget affect the number of facilities to open.

- GIpMPU and GITSFLPU. In the case of the GIpMPU, the number of facilities to open is always p. This is because c and c' are proportional (minimizing the customer cost, we are minimizing the company cost), and the company can reduce the c'-costs on any arc of the c-shortest path from medians to customers. Therefore, with more medians available, both customers and company have more options to find a path that minimizes their costs.

  When the GITSFLPU is considered, analysis of the solutions reveals that some instances include unused services or depots (since it does not involve any cost). To illustrate the potential savings from opening less than p depots and q facilities, we slightly modified the formulation to open only those that are used. Table 4 shows the number of open depots and facilities when GITSFLPU is solved (denoted by p^* and q^*, respectively). We also compare the optimal values of: (i) solving the GITSFLPU (denoted by v^*), and (ii) solving the same problem when exactly p depots and q facilities must be located (denoted by v^p). We give the reduction in the total cost by means of the parameter $\Delta = 100 \cdot \frac{v^p - v^*}{v^p}\%$. Thus, with an instance size of 100, the company can reduce total costs by about 30% by using fewer than 20 depots and facilities (specifically, 9 and 6). This reduction decreases as the instance size increases, eventually reaching a point where there is no longer a cost benefit to having fewer depots and facilities.

- GISApHLPU and GITHLPU. Now, the optimal number of hubs is often less than p. This is because the company only reduces the c'-costs between hubs, not in arcs origin-hub or hub-destination. On a tight budget, cost savings are achieved by reducing the number of hubs to open. With a larger budget, savings come from reducing the costs between hubs and, consequently, more hubs are opened and customer satisfaction is increased. To provide evidence for this claim, Table 5 shows, for $p = 2$ and different budget values, the number of open facilities in the optimal solution (p^*) when GISApHLPU and GITHLPU are solved, and the reduction in total cost Δ. With a small number of origins/destinations ($n = 10$), opening more than one hub is not cost-effective. By opening only one, no budget is spent on reducing costs (since it can only be invested in connections between hubs). This represents a greater saving than that of opening two hubs and investing in reducing the cost between them. However,

GITSFLPU									
$n = 100$									
	$p = q = 5$			$p = q = 10$			$p = q = 20$		
B	p^*	q^*	Δ	p^*	q^*	Δ	p^*	q^*	Δ
0	5	5	0.0	9	9	0.1	9	6	31.7
50	5	5	0.0	9	9	0.2	9	6	31.7
100	5	5	0.0	9	9	0.2	9	6	30.9
300	4	4	0.3	6	6	3.8	9	6	30.7
500	5	4	0.3	6	6	4.7	9	6	30.8
$n = 200$									
	$p = q = 5$			$p = q = 10$			$p = q = 20$		
B	p^*	q^*	Δ	p^*	q^*	Δ	p^*	q^*	Δ
0	5	5	0.0	10	10	0.0	20	19	2.3
50	5	5	0.0	10	10	0.0	19	17	2.0
100	5	5	0.0	10	10	0.0	20	18	1.8
300	5	5	0.0	10	10	0.0	20	18	1.8
500	5	5	0.0	10	10	0.0	20	18	1.8

Table 4: Effect of budget increment on the number of depots and facilities

with increasing size (more origins/destinations), it becomes worthwhile to open more hubs and invest in reducing costs between hubs.

9 Concluding remarks

This paper introduces new facility location problems related to upgrading arcs on a network with two kinds of costs. The first cost (customer cost) is used to allocate the customers to the facilities and the second cost (company cost) is a kind of operating cost associated to the allocation. Each customer selects the facility that provides it with the service, once the company opens its facilities, and the aim is to minimize the company cost taking into account the decisions of the customers. Additionally, the company can reduce its costs by upgrading the network.

This problem allows to model more realistic situations where the facilities supply a service and the demand nodes represent users or customers. Instead of assigning the users to the facility based only in the interest of the decision maker, our model allows considering the interests of both locator and users. The customers choose the facility on their own and the decision maker wants to reduce the cost of supplying the service, respecting the customers' decisions.

This work represents a significant step in addressing facility location problems on networks by jointly optimizing facility locations and strategic arc upgrading. A key innovation is the integration of

GISApHLPU									
		$n=10$		$n=20$		$n=30$		$n=40$	
	B	p^*	Δ	p^*	Δ	p^*	Δ	p^*	Δ
$p=2$	0	1	22.8	1	3.4	2	-	2	-
	50	1	19.2	2	-	2	-	2	-
	100	1	15.4	2	-	2	-	2	-
	300	1	14.6	2	-	2	-	2	-
	500	1	14.6	2	-	2	-	2	-
GITHLPU									
		$n=10$		$n=12$		$n=15$		$n=20$	
	B	p^*	Δ	p^*	Δ	p^*	Δ	p^*	Δ
$p=2$	0	1	33.2	2	-	2	-	1	10.4
	50	1	28.5	2	-	2	-	1	6.7
	100	1	22.8	2	-	2	-	1	0.7
	300	1	21.5	2	-	2	-	1	0.7
	500	1	21.5	2	-	2	-	1	0.7

Table 5: Effect of budget increment on the number of open hubs

the upgrading investment cost directly into the objective function, enabling a holistic minimization of total company expenses (operational and investment costs). Furthermore, this study extends the dual-cost framework to a variety of classical location problems, each with unique structural characteristics and modeling complexities, providing tailored integer programming formulations.

Different problems have been considered depending on the criterion used to locate the facilities and the distribution scheme. In this article we have addressed the upgrading of the p-median problem, a two-stage facility location problem, a single allocation hub location problem and a tree of hubs location problem. Different integer programming formulations were developed for the aforementioned problems. A brief computational study has shown the limits of the upgrading to reduce the costs for the company.

The findings of this paper can be the basis of further research concerning facility location problems with upgrading on a network with two costs. From a modelling point of view, the introduction of capacity limits for the facility location problems would be also interesting since, in real life, facilities work with a limited capacity. In future research, the upgrading of arcs in other location problems on a bi-network could be considered.

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