

AdaBB: Adaptive Barzilai-Borwein Method for Convex Optimization

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Abstract

In this paper, we propose AdaBB, an adaptive gradient method based on the Barzilai-Borwein stepsize. The algorithm is line-search-free and parameter-free, and essentially provides a convergent variant of the Barzilai-Borwein method for general unconstrained convex optimization. We analyze the ergodic convergence of the objective function value and the convergence of the iterates for solving general unconstrained convex optimization. Compared with existing works along this line of research, our algorithm gives the best lower bounds on the stepsize and the average of the stepsizes. Moreover, we present an extension of the proposed algorithm for solving composite optimization where the objective function is the summation of a smooth function and a nonsmooth function. Our numerical results also demonstrate very promising potential of the proposed algorithms on some representative examples.

Keywords: Adaptive Gradient Descent; Parameter-Free; Line-Search-Free; Automated Gradient Descent; Barzilai-Borwein Stepsize; Locally Lipschitz Gradient.

Mathematical Subject Classifications: 90C25

1 Introduction

In this paper, we propose an adaptive Barzilai-Borwein method (AdaBB) for solving unconstrained convex and smooth optimization problem

$$\min_{x \in \mathbb{R}^n} f(x), \tag{1}$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex with a locally Lipschitz gradient ∇f . Remarkably, AdaBB is an automated gradient descent method – it is line-search-free and parameter-free.

Our AdaBB method is closely related to the pioneering work by Malitsky and Mishchenko [MM20] along the line of automated gradient descent method, as well as its follow-up works [MM23; Lat+23; LTP23]. Recently, it draws great attention on how to design line-search-free and parameter-free gradient descent methods (GD) for solving (1):

$$x^{k+1} = x^k - \alpha_k \nabla f(x^k), \quad k = 0, 1, 2, \dots, \tag{2}$$

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where $\{\alpha_k\}$ is a sequence of stepsizes and $x^0 \in \mathbb{R}^n$ is a starting point. See [GSW23; Gri23; LOZ23; AP23b; AP23a; LL23; DVR23] for a partial list of other works concerning this problem. When f is convex and has globally Lipschitz gradient with Lipschitz constant $L > 0$, (2) can take a fixed stepsize $\alpha_k = \alpha > 0$. It is well known that setting $\alpha < 2/L$ guarantees the convergence and the $O(1/K)$ convergence rate of GD (2). Drori and Teboulle [DT14] first proposed the performance estimation problem (PEP) approach to study the complexity of GD when $\alpha \in (0, 1/L]$. The resulting complexity bound improved the classical one by a factor of two and was shown to be tight. In a recent work, Teboulle and Vaisbourd [TV23] provided an elementary proof to reach this tight bound for $\alpha \in (0, 1/L]$ and designed a dynamic stepsize $\alpha_k \in [1/L, 2/L)$ to improve the tight convergence bound by a constant that converges to two as $k \rightarrow \infty$.

However, these results are not parameter-free, because they require the Lipschitz constant L . Moreover, the Lipschitz constant L can be much larger than the one given by the local curvature information, which will lead to conservative stepsizes and slow convergence in practice.

Since the emergence of applications from large-scale machine learning, it has been an active research area on how to adaptively choose α_k in gradient descent and stochastic gradient descent (SGD) methods. Moreover, the computation of α_k should only be conducted using existing gradient information and should not involve expensive computation. In this sense, GD/SGD with line search are not considered adaptive methods. As explained in a recent paper by Malitsky and Mishchenko [MM23], “(a method is called adaptive), if it automatically adopts a stepsize to (its) local smoothness without additional expensive computation and the method does not deteriorate the rate of the original method in the worst case”, and it was pointed out in [MM23] that AdaGrad [DHS11; MS10] is not adaptive.

In this paper, we focus on GD methods that adaptively compute the stepsizes α_k and are parameter-free. Therefore, this kind of methods are truly automated. The earliest work in this direction is the Barzilai-Borwein (BB) method [BB88], which adaptively computes stepsize using two consecutive gradients. There are two BB stepsizes in the literature:

$$\text{(Long BB)} \quad \beta_k = \frac{\|x^k - x^{k-1}\|^2}{\langle \nabla f(x^k) - \nabla f(x^{k-1}), x^k - x^{k-1} \rangle}, \quad (3)$$

$$\text{(Short BB)} \quad \lambda_k = \frac{\langle \nabla f(x^k) - \nabla f(x^{k-1}), x^k - x^{k-1} \rangle}{\|\nabla f(x^k) - \nabla f(x^{k-1})\|^2}. \quad (4)$$

The BB method, albeit guaranteed to converge only for some special classes of problems, has been very influential in nonlinear optimization. The first time that BB was adopted to SGD was due to Tan et al. [Tan+16], where the authors proposed SGD-BB and SVRG-BB and proved the convergence of the latter one under the assumption that the objective function is strongly convex. We will give a more detailed survey of the BB method in the next section. Recently, Malitsky and Mishchenko have made a significant breakthrough in parameter-free adaptive GD [MM20; MM23]. Specifically, starting with initial point $x^0 \in \mathbb{R}^n$ and initial stepsize $\alpha_0 > 0$, the AdGD algorithm proposed in [MM20; MM23] first computes $x^1 = x^0 - \alpha_0 \nabla f(x^0)$, and then updates the iterates in the k -th iteration as follows for $k = 1, 2, \dots$,

$$\alpha_k = \min \left\{ \alpha_{k-1} \sqrt{1 + \theta_{k-1}}, 1/(\sqrt{2}L_k) \right\}, \quad \text{where } L_k = \frac{\|\nabla f(x^k) - \nabla f(x^{k-1})\|}{\|x^k - x^{k-1}\|},$$

$$x^{k+1} = x^k - \alpha_k \nabla f(x^k), \quad \text{and update } \theta_k \text{ via } \theta_k = \alpha_k / \alpha_{k-1}. \quad (5)$$

We point out that (5) is proposed in [MM23], and the algorithm proposed in [MM20] replaces the factor $\sqrt{2}$ with 2 in the updating formula of α_k . Here we refer (5) as AdGD because the second term

in updating α_k allows larger stepsize. This algorithm is guaranteed to converge for solving (1) for any $\alpha_0 > 0$ and $\theta_0 = 0$. Therefore, it is parameter-free. Note that L_k estimates the local curvature information. Furthermore, in [MM23] a variant of AdGD (we call it AdGD2) is proposed, which allows larger stepsizes. Starting with initial point x^0 and initial stepsize α_0 , the AdGD2 algorithm first computes $x^1 = x^0 - \alpha_0 \nabla f(x^0)$, and then updates the iterates in the k -th iteration as follows for $k = 1, 2, \dots$,

$$\alpha_k = \min \left\{ \sqrt{\frac{2}{3} + \theta_{k-1}} \alpha_{k-1}, \frac{\alpha_{k-1}}{\sqrt{[2\alpha_{k-1}^2 L_k^2 - 1]_+}} \right\}, \text{ where } L_k = \frac{\|\nabla f(x^k) - \nabla f(x^{k-1})\|}{\|x^k - x^{k-1}\|},$$

$$x^{k+1} = x^k - \alpha_k \nabla f(x^k), \text{ and update } \theta_k \text{ via } \theta_k = \alpha_k / \alpha_{k-1}. \quad (6)$$

where $[a]_+ = \max\{a, 0\}$. The AdGDs (5)-(6) are extended to adaptive proximal gradient methods by [Lat+23; MM23; LTP23]. Moreover, Latafat et al. [Lat+23; LTP23] used the BB stepsizes (3) and (4) to estimate the local curvature information. Starting with initial point x^0 , initial stepsize α_0 and $\theta_0 \geq 1$, the basic AdaPGM algorithm [Lat+23, Alg 2.1] first computes $x^1 = x^0 - \alpha_0 \nabla f(x^0)$, and then updates the iterates as follows for $k = 1, 2, \dots$,

$$\alpha_k = \min \left\{ \alpha_{k-1} \sqrt{1 + \theta_{k-1}}, \frac{\alpha_{k-1}}{2\sqrt{[(\alpha_{k-1}/\beta_k)(\alpha_{k-1}/\lambda_k - 1)]_+}} \right\},$$

$$x^{k+1} = x^k - \alpha_k \nabla f(x^k), \text{ and update } \theta_k \text{ via } \theta_k = \alpha_k / \alpha_{k-1}. \quad (7)$$

It is noted that both long BB (3) and short BB (4) are used to estimate the local curvature information in (7). Notably, in a more recent work, Latafat et al. [LTP23] introduced a unified framework, AdaPGM $^{\pi, r}$, that updates α_k through the formula

$$\alpha_k = \min \left\{ \sqrt{\frac{1}{\pi} + \theta_{k-1}}, \sqrt{\frac{1 - (r/\pi)}{[(1-2r) + \alpha_{k-1}^2 L_k^2 + 2\alpha_{k-1}(r-1)/\beta_k]_+}} \right\} \alpha_{k-1}$$

for any $\pi > r \geq \frac{1}{2}$. This modification gives more flexibility to balance the effects of two terms. Specifically, when opting for $r = \frac{1}{2}$ and $\pi = 1$, it aligns with updates of AdaPGM (7) but improves the second term by a factor of $\sqrt{2}$. This variant also guarantees a larger lower bound for the stepsize sequence, as we will discuss later.

Our contributions. The main contributions of our paper lie in several folds.

1. We propose a new adaptive algorithm, AdaBB, for solving (1). There are two prominent features of our AdaBB algorithm: (a) We only use the short BB stepsize (4), and we do not use the long BB stepsize (3) and L_k in (5). Note that the long BB stepsize β_k can be removed from adaPGM (7) but then L_k needs to be used. (b) Our AdaBB has a very simple and intuitive connection with the BB method, and essentially provides a convergent variant of the BB method for general convex optimization.
2. We answer an open question posed by Malitsky and Mishchenko in [MM23] affirmatively. Specifically, Malitsky and Mishchenko posed an open question asking whether there exists an adaptive method in which the sum of the stepsizes $\sum_{i=1}^k \alpha_i$ is close to k/L with readable proof, where L denotes the local Lipschitz constant of ∇f . Note that this indicates the average of the stepsizes is close to $1/L$. We prove that the sum of the stepsizes $\sum_{i=1}^k \alpha_i$ in our AdaBB algorithm is lower bounded by $(k - 2 + \sqrt{2})/L$, $\forall k \geq 1$, which can be further improved to k/L with a suitably chosen initial stepsize.

3. We prove that the stepsize α_k in our AdaBB is always lower bounded by $1/(\sqrt{2}L)$, i.e., $\alpha_k \geq 1/(\sqrt{2}L)$ for any $k \geq 1$. This also improves the existing results in [MM20; MM23; Lat+23; LTP23]. See the detailed comparison in Table 1.

Notation. Throughout the paper, we assume the optimal solution set \mathcal{X}^* of (1) is nonempty and denote its optimal value by f_* . We use x^* to denote one element in \mathcal{X}^* . Let \mathbb{R}_+ be positive real line, and $|\mathcal{W}|$ be the cardinality of the set \mathcal{W} . We use the symbol \times to represent the Cartesian product. We can prove that the sequence $\{x^k\}$ generated by AdaBB is bounded and lies in a ball $B(x^*, R)$ whose center is x^* and radius is R , which will be specified later. We assume that f is L -locally smooth (or ∇f is L -locally Lipschitz) in $B(x^*, R)$, which is defined as

$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle \geq \frac{1}{2L} \|\nabla f(x) - \nabla f(y)\|^2, \forall x, y \in B(x^*, R). \quad (8)$$

According to [Nes04, Theorem 2.1.5], this further implies $\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \forall x, y \in B(x^*, R)$, and

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \frac{1}{L} \|\nabla f(x) - \nabla f(y)\|^2, \forall x, y \in B(x^*, R). \quad (9)$$

We point out that these inequalities are not equivalent in when the points are restricted to a bounded convex set. A counterexample can be found in [Dro18]. However, when f is globally L -smooth in \mathbb{R}^n , they become equivalent (see [Nes04, Theorem 2.1.5]).

Organization. The rest of this paper is organized as follows. In Section 2, we present a simple version of our AdaBB algorithm and discuss its connection with the BB method. In Section 3, we present the full version of our AdaBB algorithm which allows more flexible choices of stepsizes. We also analyze the ergodic convergence of the function value error and the convergence of the iterates. In Section 4, we conduct an in-depth analysis of the stepsizes generated by our AdaBB algorithm. In particular, we provide lower bounds for both α_i and $\sum_{i=1}^k \alpha_i$. In Section 5, we extend our AdaBB algorithm to the case where the objective function is locally strongly convex, and to the composite case where the objective function is the summation of a smooth function and a nonsmooth function. Numerical experimental results are reported in Section 6 to illustrate the effectiveness of the proposed algorithms. Finally, we draw some concluding remarks in Section 7.

2 Our AdaBB Algorithm

The main motivation of the BB method [BB88] is to use a diagonal matrix ($\frac{1}{\eta_k}I$) to approximate the quasi-Newton matrix in the k -th iteration of the quasi-Newton method, where scalar $\eta_k > 0$. In order to satisfy the secant equation, essentially we require $\frac{1}{\eta_k}s^k = y^k$, where $s^k = x^k - x^{k-1}$ and $y^k = \nabla f(x^k) - \nabla f(x^{k-1})$. Since s^k and y^k are both n -dimensional vectors, it is impossible to find a scalar η_k such that this linear equation holds. Therefore, one has to find η_k that minimizes the residual, i.e.,

$$\min_{\eta_k} \|s^k/\eta_k - y^k\|, \text{ or } \min_{\eta_k} \|s^k - \eta_k y^k\|,$$

which leads to the two formulas given in (3) and (4) (with η_k replaced by β_k and λ_k , respectively).

However, the naive BB method

$$x^{k+1} = x^k - \beta_k \nabla f(x^k), \quad \text{or} \quad x^{k+1} = x^k - \lambda_k \nabla f(x^k) \quad (10)$$

does not always converge. In fact, existing convergence results of (10) have been mainly restricted to the special case where f is a strongly convex quadratic function. In the original paper by Barzilai

and Borwein [BB88], it is proved that the BB method (10) converges R -superlinearly, if f is strongly convex quadratic and $n = 2$. When f is strongly convex quadratic with a general dimensionality n , the BB method (10) is proved to converge globally [Ray93] and at an R -linear rate [DL02]. However, when f is not strongly convex quadratic function, i.e., when it is a general convex function, there exist counterexamples showing that the BB method (10) can diverge [BDH19]. To address this limitation, Rayden [Ray97] incorporated the non-monotone line search technique from [GLL86] to the BB method (10) and proved its global convergence when f is a general convex function.

Hence, it has been an open question whether there exists a simple variant of the BB method without line search that globally converges for general convex function. We answer this question affirmatively by proposing our AdaBB algorithm, which is described in Algorithm 1.

Algorithm 1 Adaptive Barzilai-Borwein Algorithm (AdaBB)

Input: $x^0 \in \mathbb{R}^n$, $\alpha_0 > 0$, $\theta_0 \geq 0$

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1:  $x^1 = x^0 - \alpha_0 \nabla f(x^0)$ 
2: for  $k = 1, 2, \dots$ , do
3:    $\lambda_k = \frac{\langle \nabla f(x^k) - \nabla f(x^{k-1}), x^k - x^{k-1} \rangle}{\|\nabla f(x^k) - \nabla f(x^{k-1})\|^2}$ 
4:   if  $\lambda_k \geq \alpha_{k-1}$  then ..... (Case i)
5:      $\alpha_k = \sqrt{1 + \theta_{k-1}} \alpha_{k-1}$ , and  $\theta_k = \frac{\alpha_k}{\alpha_{k-1}}$ 
6:   else if  $\alpha_{k-1}/2 < \lambda_k < \alpha_{k-1}$  then ..... (Case ii)
7:      $\alpha_k = \lambda_k$ , and  $\theta_k = \frac{2\alpha_k}{\alpha_{k-1}} - \frac{\alpha_k}{\lambda_k}$ 
8:   else ..... (Case iii)
9:      $\alpha_k = \frac{\lambda_k}{\sqrt{2}}$ , and  $\theta_k = \frac{\alpha_k}{\alpha_{k-1}}$ 
10:  end if
11:   $x^{k+1} = x^k - \alpha_k \nabla f(x^k)$ 
12: end for

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This is our basic AdaBB algorithm. We will prove its convergence in the next section when we discuss a more general version of AdaBB. We also note that the per-iteration computational cost is almost the same as the BB method (10). We now give some intuitive explanation why AdaBB (Algorithm 1) can overcome the drawbacks of the BB method. Note that BB method may diverge for general convex function because the stepsize is sometimes too aggressive. In AdaBB, we carefully design the stepsize so that if we find that the stepsize is too large in some iteration, then we use a smaller stepsize in the next iteration. This ensures that the risk of taking very large stepsizes is hedged so that the algorithm will not diverge. At the same time, when the stepsize is too small in some iteration, then we use a larger stepsize in the next iteration. This ensures that the stepsize is not always small to ensure a fast convergence. More specifically, the three “if-else” conditions in Algorithm 1 can be interpreted as follows.

- (Case i), when $\lambda_k \geq \alpha_{k-1}$, it means that the stepsize α_{k-1} in the previous iteration is too small. So we set α_k to be larger than α_{k-1} . This ensures that the stepsize is not always too small. The choice of θ_k will be clear from the convergence proof.
- (Case ii), when $\alpha_{k-1}/2 < \lambda_k < \alpha_{k-1}$, it means that the BB stepsize is not too large and not too small. So we just use the BB stepsize λ_k as the stepsize.
- (Case iii), when $\lambda_k \leq \alpha_{k-1}/2$, it means that the stepsize α_{k-1} in the previous iteration is too large. So we shrink the BB stepsize by $\sqrt{2}$ and use it as the stepsize for the current iteration. This ensures that the stepsize in the current iteration is not too large when the previous stepsize is large, to hedge the risk of divergence.

3 The General Version of AdaBB and Convergence Analysis

In this section, we first present the general version of AdaBB in Algorithm 2, which offers more flexibility when choosing stepsize α_k . Our basic AdaBB (Algorithm 1) is a special case of the general version of AdaBB. We then analyze the convergence properties of this algorithm for solving (1). We note that Algorithm 2 offers more choices for α_k in (11) and (12). Choosing (Option II) in

Algorithm 2 The General Version of AdaBB

Input: $x^0 \in \mathbb{R}^n$, $\alpha_0 > 0$, $\theta_0 \geq 0$

1: $x^1 = x^0 - \alpha_0 \nabla f(x^0)$

2: **for** $k = 1, 2, \dots$, **do**

3: $\lambda_k = \frac{\langle \nabla f(x^k) - \nabla f(x^{k-1}), x^k - x^{k-1} \rangle}{\|\nabla f(x^k) - \nabla f(x^{k-1})\|^2}$

4: **if** $\lambda_k \geq \alpha_{k-1}$ **then** (Case i)

5: $\alpha_k = \sqrt{1 + \theta_{k-1}} \alpha_{k-1}$, and $\theta_k = \frac{\alpha_k}{\alpha_{k-1}}$

6: **else if** $\alpha_{k-1}/2 < \lambda_k < \alpha_{k-1}$ **then** (Case ii)

7:

$$\alpha_k = \begin{cases} \min \left\{ \sqrt{\frac{\lambda_k}{2(\alpha_{k-1} - \lambda_k)}}, \sqrt{\frac{(1 + \theta_{k-1})\lambda_k}{2\lambda_k - \alpha_{k-1}}} \right\} \alpha_{k-1}, & \text{(Option I),} \\ \lambda_k, & \text{(Option II),} \end{cases} \quad \text{and } \theta_k = \frac{2\alpha_k}{\alpha_{k-1}} - \frac{\alpha_k}{\lambda_k} \quad (11)$$

8: **else** (Case iii)

9:

$$\alpha_k = \begin{cases} \sqrt{\frac{\alpha_{k-1}}{2(\alpha_{k-1} - \lambda_k)}} \lambda_k, & \text{(Option I),} \\ \frac{\lambda_k}{\sqrt{2}}, & \text{(Option II),} \end{cases} \quad \text{and } \theta_k = \frac{\alpha_k}{\alpha_{k-1}} \quad (12)$$

10: **end if**

11: $x^{k+1} = x^k - \alpha_k \nabla f(x^k)$

12: **end for**

both (11) and (12) recovers the basic AdaBB (Algorithm 1). Therefore, we only need to analyze the convergence of Algorithm 2 and we will devote the rest of this section to it. The following lemma gives some immediate property of α_k .

Lemma 3.1. *In both (11) and (12), the stepsize α_k provided by (Option II) is less than or equal to that provided by (Option I).*

Proof. When $\alpha_k/2 < \lambda_{k+1} < \alpha_k$, we immediately have

$$\sqrt{\frac{(1 + \theta_k)\lambda_{k+1}}{2\lambda_{k+1} - \alpha_k}} \alpha_k \geq \alpha_k \geq \lambda_{k+1}.$$

Moreover, since $v(x) := \sqrt{\frac{\lambda_{k+1}x^2}{2(x - \lambda_{k+1})}}$ is decreasing for $x \in (\lambda_{k+1}, 2\lambda_{k+1}]$, we have

$$v(\alpha_k) = \sqrt{\frac{\lambda_{k+1}}{2(\alpha_k - \lambda_{k+1})}} \alpha_k \geq v(2\lambda_{k+1}) > \lambda_{k+1}.$$

This proves that (Option II) is not larger than (Option I) in (11). The proof when $\lambda_{k+1} \leq \alpha_k/2$ is trivial and thus omitted. \square

We now define some important notation. In particular, we define M_k and P_k for $k \geq 1$ as follows

$$\begin{cases} M_k = 0, & P_k = \frac{\alpha_k^2}{\alpha_{k-1}}, & \text{if } \lambda_k \geq \alpha_{k-1}, \\ M_k = \frac{\alpha_k^2}{\lambda_k \alpha_{k-1}} - \frac{\alpha_k^2}{\alpha_{k-1}^2}, & P_k = \frac{2\alpha_k^2}{\alpha_{k-1}} - \frac{\alpha_k^2}{\lambda_k}, & \text{if } \frac{\alpha_{k-1}}{2} < \lambda_k < \alpha_{k-1}, \\ M_k = \frac{\alpha_k^2}{\lambda_k^2} - \frac{\alpha_k^2}{\alpha_{k-1} \lambda_k}, & P_k = \frac{\alpha_k^2}{\alpha_{k-1}}, & \text{if } 0 < \lambda_k \leq \frac{\alpha_{k-1}}{2}. \end{cases} \quad (13)$$

For convenience, we also define

$$P_0 = P_1 - \alpha_0. \quad (14)$$

The following lemma provides some useful inequalities for M_k and P_k defined in (13) and (14).

Lemma 3.2. *For α_k generated by Algorithm 2, it holds that $P_k = \alpha_k \theta_k$ for all $k \geq 1$, and $2M_{k+1} \leq 1$ and $P_{k+1} \leq P_k + \alpha_k$ for all $k \geq 0$.*

Proof. First, for any $k \geq 1$, we have $P_k = \alpha_k \theta_k$ by the definitions of θ_k in Algorithm 2 and P_k in (13). To show the remaining results, we consider three cases.

- Case (i): when $\lambda_{k+1} \geq \alpha_k$, by recalling $M_{k+1} = 0$, $P_{k+1} = \alpha_{k+1}^2/\alpha_k$, and $\alpha_{k+1} = \sqrt{1 + \theta_k} \alpha_k$, the desired results follow immediately.
- Case (ii): when $\alpha_k/2 < \lambda_{k+1} < \alpha_k$, from Lemma 3.1, we only need to prove the desired results for (Option I) of α_{k+1} in (11). In this case, we have

$$\begin{aligned} M_{k+1} &= \frac{\alpha_{k+1}^2}{\lambda_{k+1} \alpha_k} - \frac{\alpha_{k+1}^2}{\alpha_k^2} \leq \frac{\lambda_{k+1} \alpha_k^2}{2(\alpha_k - \lambda_{k+1})} \left(\frac{1}{\lambda_{k+1} \alpha_k} - \frac{1}{\alpha_k^2} \right) = \frac{1}{2}, \\ P_{k+1} &= \frac{2\alpha_{k+1}^2}{\alpha_k} - \frac{\alpha_{k+1}^2}{\lambda_{k+1}} \leq \frac{(1 + \theta_k) \lambda_{k+1} \alpha_k^2}{2\lambda_{k+1} - \alpha_k} \left(\frac{2}{\alpha_k} - \frac{1}{\lambda_{k+1}} \right) = \alpha_k + P_k. \end{aligned}$$

- Case (iii): when $\lambda_{k+1} \leq \alpha_k/2$, again from Lemma 3.1, we only need to prove the desired results for (Option I) of α_{k+1} in (12). In this case, we have

$$\begin{aligned} M_{k+1} &= \frac{\alpha_{k+1}^2}{\lambda_{k+1}^2} - \frac{\alpha_{k+1}^2}{\alpha_k \lambda_{k+1}} \leq \frac{\lambda_{k+1}^2 \alpha_k}{2(\alpha_k - \lambda_{k+1})} \left(\frac{1}{\lambda_{k+1}^2} - \frac{1}{\alpha_k \lambda_{k+1}} \right) = \frac{1}{2}, \\ P_{k+1} &= \alpha_{k+1}^2/\alpha_k = \frac{\lambda_{k+1}^2}{2(\alpha_k - \lambda_{k+1})} < \alpha_k < P_k + \alpha_k. \end{aligned}$$

Combining these three cases completes the proof. \square

Remark 3.1. *Note that M_1 and P_1 defined in (13) are determined only by λ_1 , α_1 and α_0 . Also, α_1 is determined only by λ_1 , α_0 and θ_0 . Moreover, λ_1 is determined only by x^0 and x^1 , while x^1 is determined only by x^0 and α_0 . Therefore, M_1 and P_1 are both absolute constants determined only by x^0 , θ_0 and α_0 . That is, by slightly abusing the notation, we can denote $M_1 = M_1(x^0, \theta_0, \alpha_0)$ and $P_1 = P_1(x^0, \theta_0, \alpha_0)$.*

The following lemma provides some useful properties about $\|x^{k+1} - x^k\|^2$.

Lemma 3.3. Let $\{x^k\}$ be the sequence generated by Algorithm 2, and M_k and P_k be defined in (13). Then, for any $k \geq 1$, we have

$$\|x^{k+1} - x^k\|^2 = \mathbf{I}_1 := \left(\frac{\alpha_k^2}{\lambda_k} - \frac{\alpha_k^2}{\alpha_{k-1}} \right) \langle \nabla f(x^k) - \nabla f(x^{k-1}), x^k - x^{k-1} \rangle + \frac{\alpha_k^2}{\alpha_{k-1}} \langle \nabla f(x^k), x^{k-1} - x^k \rangle, \quad (15)$$

$$\|x^{k+1} - x^k\|^2 = \mathbf{I}_2 := \alpha_k \langle \nabla f(x^{k+1}) - \nabla f(x^k), x^{k+1} - x^k \rangle + \alpha_k \langle \nabla f(x^{k+1}), x^k - x^{k+1} \rangle, \quad (16)$$

$$\|x^{k+1} - x^k\|^2 \leq \mathbf{E} := M_k \|x^k - x^{k-1}\|^2 + P_k (f(x^{k-1}) - f(x^k)). \quad (17)$$

Proof. Let $k \geq 1$ be arbitrarily fixed. Equation (15) can be proved as follows by using the update $x^{k+1} = x^k - \alpha_k \nabla f(x^k)$ and the identity $\|a\|^2 = \|a - b\|^2 - \|b\|^2 + 2\langle a, b \rangle$:

$$\begin{aligned} & \|x^{k+1} - x^k\|^2 \\ &= \alpha_k^2 \|\nabla f(x^k)\|^2 = \alpha_k^2 \|\nabla f(x^k) - \nabla f(x^{k-1})\|^2 - \alpha_k^2 \|\nabla f(x^{k-1})\|^2 + 2\alpha_k^2 \langle \nabla f(x^k), \nabla f(x^{k-1}) \rangle \\ &= \frac{\alpha_k^2}{\lambda_k} \langle \nabla f(x^k) - \nabla f(x^{k-1}), x^k - x^{k-1} \rangle - \alpha_k^2 \langle \nabla f(x^{k-1}) - \nabla f(x^k), \nabla f(x^{k-1}) \rangle + \alpha_k^2 \langle \nabla f(x^k), \nabla f(x^{k-1}) \rangle \\ &= \frac{\alpha_k^2}{\lambda_k} \langle \nabla f(x^k) - \nabla f(x^{k-1}), x^k - x^{k-1} \rangle - \frac{\alpha_k^2}{\alpha_{k-1}} \langle \nabla f(x^{k-1}) - \nabla f(x^k), x^{k-1} - x^k \rangle + \frac{\alpha_k^2}{\alpha_{k-1}} \langle \nabla f(x^k), x^{k-1} - x^k \rangle, \end{aligned}$$

which proves (15) by noting the definition of λ_k in (4). Equation (16) simply follows from

$$\langle \nabla f(x^{k+1}) - \nabla f(x^k), x^{k+1} - x^k \rangle = \langle \nabla f(x^{k+1}), x^{k+1} - x^k \rangle + \frac{1}{\alpha_k} \|x^{k+1} - x^k\|^2.$$

We now prove (17) by analyzing three cases.

- Case (i): when $\lambda_k \geq \alpha_{k-1}$, we know $1/\lambda_k \leq 1/\alpha_{k-1}$. By the convexity of f and monotonicity of ∇f , we have

$$\|x^{k+1} - x^k\|^2 = \mathbf{I}_1 \leq \frac{\alpha_k^2}{\alpha_{k-1}} \langle \nabla f(x^k), x^{k-1} - x^k \rangle \leq \frac{\alpha_k^2}{\alpha_{k-1}} (f(x^{k-1}) - f(x^k)). \quad (18)$$

- Case (ii): when $\alpha_{k-1}/2 < \lambda_k < \alpha_{k-1}$, we have $2/\alpha_{k-1} - 1/\lambda_k > 0$, and therefore,

$$\begin{aligned} \|x^{k+1} - x^k\|^2 = \mathbf{I}_1 &= \left(\frac{2\alpha_k^2}{\alpha_{k-1}} - \frac{\alpha_k^2}{\lambda_k} \right) \langle \nabla f(x^k), x^{k-1} - x^k \rangle \\ &\quad + \left(\frac{\alpha_k^2}{\lambda_k} - \frac{\alpha_k^2}{\alpha_{k-1}} \right) \left(\langle \nabla f(x^k) - \nabla f(x^{k-1}), x^k - x^{k-1} \rangle + \langle \nabla f(x^k), x^{k-1} - x^k \rangle \right) \\ &\stackrel{(16)}{\leq} \left(\frac{\alpha_k^2}{\lambda_k \alpha_{k-1}} - \frac{\alpha_k^2}{\alpha_{k-1}^2} \right) \|x^k - x^{k-1}\|^2 + \left(\frac{2\alpha_k^2}{\alpha_{k-1}} - \frac{\alpha_k^2}{\lambda_k} \right) (f(x^{k-1}) - f(x^k)), \end{aligned} \quad (19)$$

where the inequality follows from the convexity of f and replacing k by $k-1$ in (16).

- Case (iii): when $\lambda_k \leq \alpha_{k-1}/2$, by using the Young's inequality, we have

$$\begin{aligned} \left(\frac{\alpha_k^2}{\lambda_k} - \frac{\alpha_k^2}{\alpha_{k-1}} \right) \langle \nabla f(x^k) - \nabla f(x^{k-1}), x^k - x^{k-1} \rangle &= \left(\frac{\alpha_k^2}{\lambda_k^2} - \frac{\alpha_k^2}{\alpha_{k-1} \lambda_k} \right) \frac{\langle \nabla f(x^k) - \nabla f(x^{k-1}), x^k - x^{k-1} \rangle^2}{\|\nabla f(x^k) - \nabla f(x^{k-1})\|^2} \\ &\leq \left(\frac{\alpha_k^2}{\lambda_k^2} - \frac{\alpha_k^2}{\alpha_{k-1} \lambda_k} \right) \|x^k - x^{k-1}\|^2, \end{aligned}$$

which directly leads to

$$\|x^{k+1} - x^k\|^2 = \mathbf{I}_1 \leq \left(\frac{\alpha_k^2}{\lambda_k^2} - \frac{\alpha_k^2}{\alpha_{k-1} \lambda_k} \right) \|x^k - x^{k-1}\|^2 + \frac{\alpha_k^2}{\alpha_{k-1}} (f(x^{k-1}) - f(x^k)). \quad (20)$$

Indeed, the inequality (20) holds for all $\lambda_k < \alpha_{k-1}$. Combining these three cases proves (17). \square

Now, we are ready to derive a non-increasing Lyapunov energy. For this purpose, we define

$$w_k := \alpha_k + P_k - P_{k+1}, \quad \forall k \geq 0.$$

Lemma 3.4. *Define Lyapunov function*

$$\Upsilon_k := \|x^k - x^*\|^2 + 2M_k \|x^k - x^{k-1}\|^2 + (2\alpha_{k-1} + 2P_{k-1})(f(x^{k-1}) - f_*). \quad (21)$$

Then for $\{x^k\}$ generated by Algorithm 2, we have

$$\Upsilon_{k+1} \leq \Phi_k := \Upsilon_k - 2w_{k-1}(f(x^{k-1}) - f_*) \leq \Upsilon_k, \quad \forall k \geq 1. \quad (22)$$

Proof. First, from the convexity of f , we have

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|x^k - \alpha_k \nabla f(x^k) - x^*\|^2 \\ &= \|x^k - x^*\|^2 - 2\alpha_k \langle \nabla f(x^k), x^k - x^* \rangle + \alpha_k^2 \|\nabla f(x^k)\|^2 \\ &\leq \|x^k - x^*\|^2 - 2\alpha_k (f(x^k) - f_*) + \alpha_k^2 \|\nabla f(x^k)\|^2. \end{aligned} \quad (23)$$

From (17), we have

$$\|x^{k+1} - x^k\|^2 = \alpha_k^2 \|\nabla f(x^k)\|^2 \leq 2\mathbf{E} - \|x^{k+1} - x^k\|^2 = 2\mathbf{E} - \alpha_k^2 \|\nabla f(x^k)\|^2. \quad (24)$$

Summing (23) and (24) yields

$$\begin{aligned} &\|x^{k+1} - x^*\|^2 + \|x^{k+1} - x^k\|^2 + 2\alpha_k (f(x^k) - f_*) \\ &\leq \|x^k - x^*\|^2 + 2\mathbf{E} \\ &\leq \Upsilon_k - (2\alpha_{k-1} + 2P_{k-1})(f(x^{k-1}) - f_*) + 2P_k (f(x^{k-1}) - f(x^k)), \end{aligned} \quad (25)$$

which further implies

$$\Upsilon_{k+1} \leq \|x^{k+1} - x^*\|^2 + \|x^{k+1} - x^k\|^2 + (2\alpha_k + 2P_k)(f(x^k) - f_*) \leq \Phi_k, \quad (26)$$

where the first inequality is due to $2M_{k+1} \leq 1$ from Lemma 3.2. This proves the first inequality in (22). The second inequality in (22) is trivial because $w_k \geq 0$ from Lemma 3.2. \square

Lemma 3.4 immediately leads to the boundedness of $\{x^k\}$.

Corollary 3.1. *The sequence $\{x^k\}$ generated by Algorithm 2 is bounded. In particular, for all $k \geq 0$, we have $x^k \in B(x^*, R)$, where R is defined as:*

$$R^2 := \|x^0 - x^*\|^2 + \alpha_0^2 (1 + 2M_1) \|\nabla f(x^0)\|^2 + \max\{2P_1 - 2\alpha_0, 0\} (f(x^0) - f_*). \quad (27)$$

Note that from Remark 3.1, $M_1 = M_1(x^0, \theta_0, \alpha_0)$ and $P_1 = P_1(x^0, \theta_0, \alpha_0)$ are both absolute constants.

Proof. From (22), we can obtain $\|x^k - x^*\|^2 \leq \Upsilon_k \leq \Phi_{k-1} \leq \Phi_1$ for all $k \geq 1$. From (14), we have $w_0 = 0$. Therefore,

$$\begin{aligned} \Phi_1 &= \Upsilon_1 = \|x^1 - x^*\|^2 + 2M_1 \|x^1 - x^0\|^2 + (2\alpha_0 + 2P_0)(f(x^0) - f_*) \\ &\stackrel{(23)}{\leq} \|x^0 - x^*\|^2 + \alpha_0^2 (1 + 2M_1) \|\nabla f(x^0)\|^2 + 2(P_1 - \alpha_0)(f(x^0) - f_*) \leq R^2. \end{aligned} \quad (28)$$

Consequently, we have $x^k \in B(x^*, R)$ for all $k \geq 1$. It is trivial to see $x^0 \in B(x^*, R)$. This completes the proof. \square

Remark 3.2. From now on, we assume that both (8) and (9) hold with R defined in (27). From (9), we immediately have the following useful result:

$$\lambda_k \geq 1/L, \quad \forall k \geq 1. \quad (29)$$

The following proposition gives a lower bound on α_k and estimates the order of $\sum_{i=1}^k \alpha_i$.

Proposition 3.1. For $\{\alpha_k\}$ generated by Algorithm 2, we have

(i) If $\alpha_j \geq \frac{1}{\sqrt{2L}}$ for some j , then $\alpha_k \geq \frac{1}{\sqrt{2L}}$ for any $k \geq j$;

(ii) $\alpha_k \geq c := \min\{\alpha_0, \frac{1}{\sqrt{2L}}\} > 0$ for all $k \geq 0$;

(iii) $\sum_{i=1}^k \alpha_i = O(k)$.

Proof. We first prove part (i) by considering the three cases in the $(j+1)$ -th iteration of Algorithm 2. If (Case i) happens, then we have $\alpha_{j+1} \geq \alpha_j \geq \frac{1}{\sqrt{2L}}$. If (Case ii) or (Case iii) happens, then we have $\alpha_{j+1} \geq \frac{\lambda_{j+1}}{\sqrt{2}} \geq \frac{1}{\sqrt{2L}}$, where we used (29) for the second inequality. By induction, this completes the proof of part (i).

We now prove part (ii). Let $r \geq 1$ be the smallest integer that satisfies $\lambda_r < \alpha_{r-1}$. When $r = 1$, we obtain $\alpha_1 \geq \frac{\lambda_1}{\sqrt{2}} \geq \frac{1}{\sqrt{2L}}$. Consequently, applying the result from part (i) yields $\alpha_k \geq \frac{1}{\sqrt{2L}}$ for all $k \geq 1$. When $r > 1$, this implies $\lambda_k \geq \alpha_{k-1}$ for $k = 1, 2, \dots, r-1$, i.e., (Case i) in Algorithm 2 happens for the first $r-1$ iterations. This leads to $\alpha_1 \geq \alpha_0$, and $\alpha_k \geq \sqrt{2}\alpha_{k-1}$ for $k = 2, \dots, r-1$. Therefore, we have $\alpha_k \geq \sqrt{2}^{k-1}\alpha_0 \geq \alpha_0$ for $k = 1, 2, \dots, r-1$. Moreover, $\lambda_r < \alpha_{r-1}$ also implies that either (Case ii) or (Case iii) in Algorithm 2 happens for the r -th iteration. In both cases, we have $\alpha_r \geq \frac{\lambda_r}{\sqrt{2}} \geq \frac{1}{\sqrt{2L}}$, where we used (29) for the second inequality. Now from part (i), we know that $\alpha_k \geq \frac{1}{\sqrt{2L}}$ for any $k \geq r$. This completes the proof of part (ii).

Part (iii) follows from part (ii) immediately. \square

Now we are ready to present the main convergence result of Algorithm 2.

Theorem 3.1 (Ergodic convergence). For sequence $\{x^k\}$ generated by Algorithm 2, define

$$\bar{x}^k = \frac{(\alpha_k + P_k)x^k + \sum_{i=1}^{k-1} w_i x^i}{S_k}, \quad \text{with } S_k = P_1 + \sum_{i=1}^k \alpha_i.$$

Then we have

$$f(\bar{x}^k) - f_* \leq \frac{\Phi_1}{2S_k} = O\left(\frac{1}{k}\right). \quad (30)$$

Proof. We only need to prove the inequality in (30), because $\frac{\Phi_1}{2S_k} = O\left(\frac{1}{k}\right)$ follows directly from (28), Remark 3.1 and Proposition 3.1 part (iii). From (22) we have $\Upsilon_{i+1} \leq \Upsilon_i - 2w_{i-1}(f(x^{i-1}) - f_*)$. Summing this inequality over $i = 1, \dots, k$ yields (note $w_0 = 0$)

$$\Upsilon_{k+1} + 2 \sum_{i=1}^{k-1} w_i (f(x^i) - f_*) \leq \Upsilon_1 = \Phi_1.$$

Using (21), we know that

$$\|x^{k+1} - x^*\|^2 + 2M_{k+1}\|x^{k+1} - x^k\|^2 + (2\alpha_k + 2P_k)(f(x^k) - f_*) + 2 \sum_{i=1}^{k-1} w_i (f(x^i) - f_*) \leq \Phi_1,$$

which further leads to

$$(\alpha_k + P_k)(f(x^k) - f_*) + \sum_{i=1}^{k-1} w_i (f(x^i) - f_*) \leq \frac{\Phi_1}{2}. \quad (31)$$

Since $w_i = \alpha_i + P_i - P_{i+1}$, we have $\sum_{i=1}^{k-1} w_i = \sum_{i=1}^{k-1} \alpha_i + P_1 - P_k$. We thus have

$$(\alpha_k + P_k) + \sum_{i=1}^{k-1} w_i = (\alpha_k + P_k) + \sum_{i=1}^{k-1} \alpha_i + P_1 - P_k = P_1 + \sum_{i=1}^k \alpha_i = S_k.$$

Utilizing the convexity of f , we obtain

$$f(\bar{x}^k) = f\left(\frac{(\alpha_k + P_k)x^k + \sum_{i=1}^{k-1} w_i x^i}{S_k}\right) \leq \frac{\alpha_k + P_k}{S_k} f(x^k) + \sum_{i=1}^{k-1} \frac{w_i}{S_k} f(x^i),$$

which leads to

$$f(\bar{x}^k) - f_* \leq \frac{\alpha_k + P_k}{S_k} (f(x^k) - f_*) + \sum_{i=1}^{k-1} \frac{w_i}{S_k} (f(x^i) - f_*) \leq \frac{\Phi_1}{2S_k},$$

where the last inequality follows from (31). \square

Next, we present a variant of the Opial lemma, which is useful in our convergence analysis.

Lemma 3.5 ([MM20, Lemma 2]). *Let $\{x^k\}$ and $\{a_k\}$ be two sequences in \mathbb{R}^n and \mathbb{R}_+ , respectively. Suppose that $\{x^k\}$ is bounded, its cluster points belong to $\mathcal{X} \subset \mathbb{R}^n$ and it also holds that*

$$\|x^{k+1} - x\|^2 + a_{k+1} \leq \|x^k - x\|^2 + a_k, \quad \forall x \in \mathcal{X},$$

then $\{x^k\}$ converges to some element in \mathcal{X} .

Theorem 3.2 (Pointwise convergence). *The sequence $\{x^k\}$ generated by Algorithm 2 globally converges to an optimal solution of (1). Moreover, we have the following sublinear convergence rate of $\min_{1 \leq i \leq k} \|\nabla f(x^i)\|^2$:*

$$\min_{1 \leq i \leq k} \|\nabla f(x^i)\|^2 = O\left(\frac{1}{k}\right). \quad (32)$$

Proof. By using (8), we have the following improved analysis for (23):

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|x^k - x^*\|^2 - 2\alpha_k \langle \nabla f(x^k), x^k - x^* \rangle + \alpha_k^2 \|\nabla f(x^k)\|^2 \\ &\leq \|x^k - x^*\|^2 - 2\alpha_k \left(f(x^k) - f_* + \frac{1}{2L} \|\nabla f(x^k)\|^2 \right) + \alpha_k^2 \|\nabla f(x^k)\|^2, \end{aligned} \quad (33)$$

and then (26) can be changed to:

$$\Upsilon_{k+1} + \frac{\alpha_k}{L} \|\nabla f(x^k)\|^2 \leq \|x^{k+1} - x^*\|^2 + \|x^{k+1} - x^k\|^2 + (2\alpha_k + 2P_k)(f(x^k) - f_*) + \frac{\alpha_k}{L} \|\nabla f(x^k)\|^2 \leq \Phi_k \leq \Upsilon_k,$$

which yields to $\Phi_{k+1} + \frac{\alpha_k}{L} \|\nabla f(x^k)\|^2 \leq \Phi_k$ for all $k \geq 1$. Change the index k to i , and sum this inequality over $i = 1, \dots, k$, we obtain $\sum_{i=1}^k \alpha_i \|\nabla f(x^i)\|^2 \leq \Phi_1 L$. Using Proposition 3.1 part (ii), we have

$$\sum_{i=1}^k \|\nabla f(x^i)\|^2 \leq \frac{\Phi_1 L}{c}. \quad (34)$$

Therefore, $\nabla f(x^k) \xrightarrow{k \rightarrow \infty} 0$. This proves that all cluster points of $\{x^k\}$ belong to the solution sets of (1). Now using $\Phi_{k+1} \leq \Phi_k$ and applying Lemma 3.5 by letting $\mathcal{X} = \mathcal{X}^*$ and $a_k = 2M_k \|x^k - x^{k-1}\|^2 + 2P_k(f(x^{k-1}) - f_*)$, we obtain that $\{x^k\}$ converges to an optimal solution of (1).

Moreover, (34) immediately leads to the convergence rate (32). \square

4 Improved Lower Bound for $\sum_{i=1}^k \alpha_i$

Existing analysis of gradient method (2) for convex minimization (1) requires stepsize $\alpha_k \leq 1/L$ to achieve the $O(1/k)$ convergence rate $f(x^k) - f_* = O(1/k)$. In this kind of analysis, choosing $\alpha_k \leq 1/L$ guarantees that the function value has a sufficient decrease in each iteration. In practice, $\alpha_k = 1/L$ is usually chosen because it is the largest stepsize in this setting. In the case where adaptive stepsize is used, a natural question to ask is whether we can guarantee that on average the stepsize is approximately equal to $1/L$. This has been posed as an open question recently by Malitsky and Mishchenko [MM23]. More precisely, the open question posed in [MM23] asks whether one can design an adaptive algorithm such that the lower bound for $\sum_{i=1}^k \alpha_i$ is close to k/L . In this section, we answer this question affirmatively: the stepsizes generated by our AdaBB algorithm satisfy

$$\sum_{i=i_0+1}^k \alpha_i \geq (k - i_0)/L, \text{ where } i_0 \in \{0, 1, 2\} \text{ depending on the choice of } \alpha_0. \quad (35)$$

This further implies $\sum_{i=1}^k \alpha_i \geq \frac{k-2+\sqrt{2}}{L}$ for all $k \geq 1$. With a suitably chosen α_0 , this can be further improved to $\sum_{i=1}^k \alpha_i \geq k/L$.

Our result requires the following choice of θ_0 :

$$\theta_0 = \begin{cases} \lambda_1^2/(2\alpha_0^2) - 1, & \text{if } \lambda_1 \geq \sqrt{2}\alpha_0, \\ 0, & \text{otherwise.} \end{cases} \quad (36)$$

Note that λ_1 is fully determined by α_0 and x^0 . So θ_0 can be pre-given to the algorithm. In the rest of this paper, we assume that θ_0 is chosen as in (36). This choice of θ_0 immediately leads to the following lower bound for α_i and $\sum_{i=1}^k \alpha_i$.

Lemma 4.1. *For any given $x^0 \in \mathbb{R}^n$ and $\alpha_0 > 0$, the stepsizes generated by our AdaBB (Algorithm 2) satisfy $\alpha_i \geq \frac{1}{\sqrt{2}L}$ for all $i \geq 1$, and hence $\sum_{i=1}^k \alpha_i \geq \frac{k}{\sqrt{2}L}$ for any $k \geq 1$.*

Proof. We first prove $\alpha_1 \geq \lambda_1/\sqrt{2}$. There are three cases to consider.

- (a). If $\lambda_1 \geq \sqrt{2}\alpha_0$, then (Case i) in Algorithm 2 happens. From the definition of θ_0 in (36), we have $\alpha_1 = \lambda_1/\sqrt{2}$.
- (b). If $\alpha_0 \leq \lambda_1 < \sqrt{2}\alpha_0$, then (Case i) happens. In this case, we have $\theta_0 = 0$, and therefore, $\alpha_1 = \alpha_0 > \lambda_1/\sqrt{2}$.
- (c). If $\lambda_1 < \alpha_0$, then (Case ii) or (Case iii) happens. In this case, recall Lemma 3.1, we have $\alpha_1 \geq \lambda_1/\sqrt{2}$.

Combining these three cases proves $\alpha_1 \geq \lambda_1/\sqrt{2}$, which further implies $\alpha_1 \geq 1/(\sqrt{2}L)$ due to (29). It then follows from Proposition 3.1 (i) that $\alpha_i \geq \frac{1}{\sqrt{2}L}$ for all $i \geq 1$. As a consequence, we obtain $\sum_{i=1}^k \alpha_i \geq \frac{k}{\sqrt{2}L}$ for any $k \geq 1$. \square

For the ease of presentation, we partition the index set $\{1, 2, 3, \dots\}$ into three categories which correspond to the three cases in our AdaBB (Algorithm 2):

$$I_1 := \{k \geq 1 \mid \lambda_k \geq \alpha_{k-1}\}, I_2 := \{k \geq 1 \mid \alpha_{k-1}/2 < \lambda_k < \alpha_{k-1}\} \text{ and } I_3 := \{k \geq 1 \mid 0 < \lambda_k \leq \alpha_{k-1}/2\}. \quad (37)$$

We first establish a useful lemma.

Lemma 4.2. For any given $x^0 \in \mathbb{R}^n$ and $\alpha_0 > 0$, the stepsizes generated by our AdaBB (Algorithm 2) have the following properties.

- (a) If $i \in I_2$, then $\alpha_i \geq \frac{1}{L}$;
- (b) If $(i+1) \in I_3$, then $\alpha_i \geq \frac{2}{L}$ and $\alpha_i + \alpha_{i+1} \geq \frac{4+\sqrt{2}}{2L}$;
- (c) If $i \in I_1 \cup I_2$ and $(i+1) \in I_1$, then $\alpha_{i+1} \geq \frac{1}{L}$;
- (d) If $i \in I_3$, then $\alpha_{i-1} + \alpha_i + \alpha_{i+1} \geq \frac{2+\sqrt{2}}{L} > \frac{3}{L}$.

Proof. (a) Since $i \in I_2$, by Lemma 3.1 and (29) we have $\alpha_i \geq \lambda_i \geq \frac{1}{L}$.

(b) By definition, $(i+1) \in I_3$ implies that $\alpha_i \geq 2\lambda_{i+1} \geq \frac{2}{L}$. This combining with Lemma 4.1 yields $\alpha_i + \alpha_{i+1} \geq \frac{1}{\sqrt{2}L} + \frac{2}{L} = \frac{4+\sqrt{2}}{2L}$.

(c) If $i \in I_1$, we have $\alpha_i = \sqrt{1+\theta_{i-1}}\alpha_{i-1} \geq \alpha_{i-1}$, which gives $\theta_i = \frac{\alpha_i}{\alpha_{i-1}} \geq 1$. Since $(i+1) \in I_1$, we obtain $\alpha_{i+1} = \sqrt{1+\theta_i}\alpha_i \geq \sqrt{2}\alpha_i \geq \frac{1}{L}$. If $i \in I_2$, we have $\alpha_i \geq \lambda_i \geq \frac{1}{L}$. Then, $(i+1) \in I_1$ implies $\alpha_{i+1} = \sqrt{1+\theta_i}\alpha_i \geq \alpha_i \geq \frac{1}{L}$.

(d) Since $i \in I_3$, from part (b) we have $\alpha_{i-1} + \alpha_i \geq \frac{4+\sqrt{2}}{2L}$. The result follows by noting $\alpha_{i+1} \geq \frac{1}{\sqrt{2}L}$. \square

We now define some useful notation. Let $1 \leq p \leq q$ be integers. We define (p, \dots, q) as the ordered sequence of indices from p to q , and $\{p, \dots, q\}$ as the set of indices from p to q without regard to order.

Definition 4.1 (Break index). An index $i \geq 1$ is called a break index if $i \in I_1$ and $(i+1) \notin I_3$. For $j \geq 1$, we let i_j be the j th smallest break index within $\{1, 2, 3, \dots\}$.

To carry out a more elaborate analysis, we define $i_0 \in \{0, 1, 2\}$ as follows:

$$i_0 = \begin{cases} 0, & \text{if } 1 \in I_2, \text{ or } (1, 2) \in I_1 \times I_3, \text{ or } (1, 2) \in I_3 \times I_3, \\ 1, & \text{if } (1, 2) \in I_1 \times I_1, \text{ or } (1, 2) \in I_1 \times I_2, \text{ or } (1, 2) \in I_3 \times I_2, \text{ or } (1, 2, 3) \in I_3 \times I_1 \times I_3, \\ 2, & \text{if } (1, 2, 3) \in I_3 \times I_1 \times I_1, \text{ or } (1, 2, 3) \in I_3 \times I_1 \times I_2. \end{cases} \quad (38)$$

Note that every index belongs to one of the three categories (37). Moreover, the nine conditions in (38) cover all possibilities for the first three indices 1, 2, and 3. Our idea to prove the improved bound (35) is to divide the ordered sequence of indices (i_0+1, i_0+2, \dots, k) into many shorter pieces, and for each piece, say, (p, \dots, q) , we shall show that $\sum_{i=p}^q \alpha_i \geq (q-p+1)/L$. In the rest of this section, we assume $k \geq 3$ is an arbitrarily fixed integer. For fixed $k \geq 3$, we assume that there are $(m-1)$ break indices within $\{1, 2, \dots, k\}$, which satisfy $1 \leq i_1 < i_2 < \dots < i_{m-1} \leq k$. For convenience, we define

$$T_j := (i_{j-1} + 1, i_{j-1} + 2, \dots, i_j) \text{ for } j = 1, 2, \dots, m-1, \text{ and } T_m := (i_{m-1} + 1, i_{m-1} + 2, \dots, k). \quad (39)$$

That is,

$$\overbrace{i_0+1, \dots, i_1}^{T_1}, \overbrace{i_1+1, \dots, i_2}^{T_2}, \dots, \overbrace{i_{m-2}+1, \dots, i_{m-1}}^{T_{m-1}}, \overbrace{i_{m-1}+1, \dots, k}^{T_m}.$$

Note that (p, \dots, q) is an empty set if $p > q$. Therefore, $T_1 = \emptyset$ if $i_0 \geq i_1$, and $T_m = \emptyset$ if $i_{m-1} = k$. Due to the definition of break index, if there is an index $p \in I_1$ such that $i_{j-1} + 1 \leq p \leq i_j - 1$ for some

j , then $(p+1) \in I_3$ must hold. For $i \geq 1$ and $j \geq i+1$, we define the following sets, which contain ordered and continuous indices.

$$\mathcal{A} := \{(i, i+1, \dots, j) \mid \{i+1, \dots, j\} \subseteq I_3\}, \quad (40)$$

$$\mathcal{Q} := \{(i, i+1, \dots, j) \mid i \in I_2, \{i+1, \dots, j\} \subseteq I_2 \cup I_3\}, \quad (41)$$

$$\mathcal{M} := \{(i, i+1, \dots, j) \mid i \notin I_3, j \notin I_1, \text{ and, for any } i \leq \ell \leq j-1, \ell \in I_1 \Rightarrow (\ell+1) \in I_3\}. \quad (42)$$

To establish our improved bound (35), it is sufficient to show that $\sum_{i \in T_j} \alpha_i \geq \frac{|T_j|}{L}$ for $j = 1, 2, \dots, m$. To show this, we first prove the following key lemma.

Lemma 4.3. *Let $i \geq 1$ and $j \geq i+1$. If $(i, i+1, \dots, j) \in \mathcal{A} \cup \mathcal{Q} \cup \mathcal{M}$, where \mathcal{A} , \mathcal{Q} and \mathcal{M} are defined in (40)-(42), then $\sum_{\ell=i}^j \alpha_\ell \geq (j-i+1)/L$.*

Proof. There are three cases to consider.

- (a). $(i, i+1, \dots, j) \in \mathcal{A}$. In this case, we have $\{i+1, \dots, j\} \subseteq I_3$. Hence, it follows from Lemma 4.2 (b) that $\alpha_\ell \geq \frac{2}{L}$ for $i \leq \ell \leq j-1$. Thus, we have $\sum_{\ell=i}^j \alpha_\ell \geq \frac{2(j-i)}{L} + \frac{1}{\sqrt{2}L} \geq \frac{j-i+1}{L}$ as $j-i \geq 1$.
- (b). $(i, i+1, \dots, j) \in \mathcal{Q}$. In this case, we can always divide it into shorter pieces as $(i, i+1, \dots, j) = (p_1, \dots, q_1 \mid p_2, \dots, q_2 \mid \dots \mid p_s, \dots, q_s)$ for some $s \geq 1$, with $p_1 = i$, $q_s = j$, $p_{t+1} = q_t + 1$ for $t = 1, 2, \dots, s-1$, such that for each (p_t, \dots, q_t) , $t = 1, 2, \dots, s$, we have either $p_t = q_t \in I_2$, or $q_t \geq p_t + 1$, $p_t \in I_2$ and $\{p_t + 1, \dots, q_t\} \subseteq I_3$. In the former case, $p_t = q_t \in I_2$, the corresponding piece has length one, and by Lemma 4.2 (a), we have $\alpha_{p_t} \geq 1/L$. In the latter case, we have (p_t, \dots, q_t) belongs to \mathcal{A} , and therefore $\sum_{\ell=p_t}^{q_t} \alpha_\ell \geq (q_t - p_t + 1)/L$.
- (c). $(i, i+1, \dots, j) \in \mathcal{M}$. Similarly, in this case it can always be divided into shorter pieces as $(p_1, \dots, q_1 \mid p_2, \dots, q_2 \mid \dots \mid p_{s'}, \dots, q_{s'})$ for some $s' \geq 1$, with $p_1 = i$, $q_{s'} = j$, $p_{t+1} = q_t + 1$ for $t = 1, 2, \dots, s'-1$, such that for each (p_t, \dots, q_t) , $t = 1, 2, \dots, s'$, we have either $p_t = q_t \in I_2$, or $q_t \geq p_t + 1$, $\{p_t + 1, \dots, q_t\} \subseteq I_3$. In the former case, $p_t = q_t \in I_2$, the corresponding piece has length one, and by Lemma 4.2 (i), we have $\alpha_{p_t} \geq 1/L$. In the latter case, again we have (p_t, \dots, q_t) belongs to \mathcal{A} , and the result $\sum_{\ell=p_t}^{q_t} \alpha_\ell \geq (q_t - p_t + 1)/L$ follows.

For Cases (b) and (c), the result $\sum_{\ell=i}^j \alpha_\ell \geq \frac{j-i+1}{L}$ follows immediately since each piece (p_t, \dots, q_t) satisfies $\sum_{\ell=p_t}^{q_t} \alpha_\ell \geq (q_t - p_t + 1)/L$. \square

Equipped with Lemma 4.3, we next show that $\sum_{i \in T_j} \alpha_i \geq |T_j|/L$ for $1 \leq j \leq m$. We first prove the case $2 \leq j \leq m-1$, and then prove the cases $j = m$ and $j = 1$.

Lemma 4.4. *For $2 \leq j \leq m-1$, there holds $\sum_{i \in T_j} \alpha_i \geq |T_j|/L$.*

Proof. For $2 \leq j \leq m-1$, we have $T_j = (i_{j-1} + 1, \dots, i_j)$. By the definition of i_j , we have $i_{j-1} + 1 \in I_1 \cup I_2$ and $i_j \in I_1$. This also implies $i_j - 1 \notin I_1$. Moreover, if there exists $\ell \in I_1$ satisfying $i_{j-1} + 1 \leq \ell \leq i_j - 2$, then there must hold $(\ell + 1) \in I_3$. We need to prove the result for two cases: (a) $|T_j \cap I_1| = 1$, and (b) $|T_j \cap I_1| \geq 2$.

- (a). In this case, we have $|T_j \cap I_1| = 1$. There are three cases to consider: (a1) $i_{j-1} + 1 \in I_1$; (a2) $i_{j-1} + 1 \in I_2$ and $i_j - 1 \in I_2$; (a3) $i_{j-1} + 1 \in I_2$ and $i_j - 1 \in I_3$.
 - (a1). If $i_{j-1} + 1 \in I_1$, then $i_{j-1} + 1 = i_j$ and $T_j = \{i_j\}$. Since $i_{j-1} \in I_1$ and $i_j \in I_1$, from Lemma 4.2 (c) we have $\alpha_{i_j} \geq \frac{1}{L}$.

- (a2). If $i_{j-1} + 1 \in I_2$ and $i_j - 1 \in I_2$, we partition T_j into two parts: $T_j = (i_{j-1} + 1, \dots, i_j - 1 \mid i_j)$. Since $i_j \in I_1$, there is no index in $(i_{j-1} + 1, \dots, i_j - 1)$ belonging to I_1 . Therefore, $(i_{j-1} + 1, \dots, i_j - 1) \in \mathcal{Q}$. Moreover, $i_j \in I_1$ and $i_j - 1 \in I_2$ imply that $\alpha_{i_j} \geq \frac{1}{L}$ (see Lemma 4.2 (c)). Using Lemma 4.3, we obtain $\sum_{i=i_{j-1}+1}^{i_j} \alpha_i \geq |T_j|/L$.
- (a3). If $i_{j-1} + 1 \in I_2$ and $i_j - 1 \in I_3$, we partition T_j into two parts: $T_j = (i_{j-1} + 1, \dots, i_j - 3 \mid i_j - 2, i_j - 1, i_j)$. Again, since $i_j \in I_1$, there is no index in $(i_{j-1} + 1, \dots, i_j - 3)$ belonging to I_1 . Therefore, $(i_{j-1} + 1, \dots, i_j - 3)$, if nonempty, must belong to \mathcal{Q} . Moreover, it follows from Lemma 4.2 (d) that $\sum_{\ell=i_{j-2}}^{i_j} \alpha_\ell \geq \frac{3}{L}$. Using Lemma 4.3, we obtain $\sum_{i=i_{j-1}+1}^{i_j} \alpha_i \geq |T_j|/L$.
- (b). In this case, we have $|T_j \cap I_1| \geq 2$. There are again three cases to consider: (b1) $i_j - 1 \in I_2$; (b2) $i_j - 1 \in I_3$ and $i_j - 3 \in I_1$; (b3) $i_j - 1 \in I_3$ and $i_j - 3 \notin I_1$.
- (b1). If $i_j - 1 \in I_2$, then we can partition T_j as $T_j = (i_{j-1} + 1, \dots, i_j - 1 \mid i_j)$. It is easy to see that $(i_{j-1} + 1, \dots, i_j - 1) \in \mathcal{M}$. Moreover, according to Lemma 4.2 (c), we have $\alpha_{i_j} \geq \frac{1}{L}$. Using Lemma 4.3, we have $\sum_{i=i_{j-1}+1}^{i_j} \alpha_i \geq |T_j|/L$.
- (b2). If $i_j - 1 \in I_3$ and $i_j - 3 \in I_1$, then we must have $i_j - 2 \in I_3$, because otherwise $i_j - 3$ is a break index. Moreover, $i_j - 4 \notin I_1$, because otherwise, $i_j - 4$ is a break index. We then partition T_j into $T_j = (i_{j-1} + 1, \dots, i_j - 4 \mid i_j - 3 \mid i_j - 2, i_j - 1, i_j)$. It is easy to see that $(i_{j-1} + 1, \dots, i_j - 4)$, if nonempty, must belong to \mathcal{M} . Moreover, since $i_j - 2 \in I_3$, we have by Lemma 4.2 (b) that $\alpha_{i_j-3} \geq \frac{2}{L}$, and since $i_j - 1 \in I_3$, we have $\alpha_{i_j-2} + \alpha_{i_j-1} + \alpha_{i_j} \geq \frac{3}{L}$ by Lemma 4.2 (d). Using Lemma 4.3, we have $\sum_{i=i_{j-1}+1}^{i_j} \alpha_i \geq |T_j|/L$.
- (b3). If $i_j - 1 \in I_3$ and $i_j - 3 \notin I_1$, then we partition T_j into $T_j = (i_{j-1} + 1, \dots, i_j - 3 \mid i_j - 2, i_j - 1, i_j)$. It is easy to see that $(i_{j-1} + 1, \dots, i_j - 3)$, if nonempty, must belong to \mathcal{M} . Since $i_j - 1 \in I_3$, we have $\alpha_{i_j-2} + \alpha_{i_j-1} + \alpha_{i_j} \geq \frac{3}{L}$ by Lemma 4.2 (d). Using Lemma 4.3, we have $\sum_{i=i_{j-1}+1}^{i_j} \alpha_i \geq |T_j|/L$.

This completes the proof. \square

Lemma 4.5. For $j = m$ or $j = 1$, there holds $\sum_{i \in T_j} \alpha_i \geq |T_j|/L$.

Proof. We first consider the case $j = m$ and in this case $T_m = (i_{m-1} + 1, \dots, k)$. If $i_{m-1} = k$, then T_m equals to \emptyset . So we only need to consider the case when $i_{m-1} < k$. If $k \in I_1$, then the proof is exactly the same as for the case $2 \leq j \leq m - 1$ in Lemma 4.4. If $k \notin I_1$, then $T_m \in \mathcal{M}$ as by the construction of T_m we have $i_{m-1} + 1 \notin I_3$ and if $\ell \in I_1$ within $\{i_{m-1} + 1, \dots, k - 1\}$, we have $\ell + 1 \in I_3$. In both cases, we have $\sum_{\ell=i_{m-1}+1}^k \alpha_\ell \geq |T_m|/L$.

We now consider the case $j = 1$ and in this case $T_1 = (i_0 + 1, \dots, i_1)$. There are nine cases to consider according to the definition of i_0 in (38). Among them, in the following four cases, we have $T_1 = \emptyset$ and no proof is needed: $(1, 2, 3) \in I_3 \times I_1 \times I_1$ ($i_0 = i_1 = 2$), or $(1, 2, 3) \in I_3 \times I_1 \times I_2$ ($i_0 = i_1 = 2$), or $(1, 2) \in I_1 \times I_1$ ($i_0 = i_1 = 1$), or $(1, 2) \in I_1 \times I_2$ ($i_0 = i_1 = 1$). It remains to consider the following five scenarios: (a) $1 \in I_2$; (b) $(1, 2) \in I_1 \times I_3$; (c) $(1, 2) \in I_3 \times I_2$; (d) $(1, 2, 3) \in I_3 \times I_1 \times I_3$; and (e) $(1, 2) \in I_3 \times I_3$.

- (a). $1 \in I_2$. In this case, $i_0 = 0$ and $T_1 = (i_0 + 1, \dots, i_1)$ reduces to $(1, \dots, i_1)$ with $i_1 \geq 2$. This case is the same as the one in Lemma 4.4 because $1 \notin I_3$ and $i_1 \in I_1$.
- (b). $(1, 2) \in I_1 \times I_3$. In this case, $i_0 = 0$ and $T_1 = (i_0 + 1, \dots, i_1)$ reduces to $(1, \dots, i_1)$ with $i_1 \geq 3$. This case is again the same as the one in Lemma 4.4 because $1 \notin I_3$ and $i_1 \in I_1$.

- (c). $(1, 2) \in I_3 \times I_2$. In this case, $i_0 = 1$ and $T_1 = (i_0 + 1, \dots, i_1)$ reduces to $(2, \dots, i_1)$ with $i_1 \geq 3$. This case is again the same as the one in Lemma 4.4 because $2 \notin I_3$ and $i_1 \in I_1$.
- (d). $(1, 2, 3) \in I_3 \times I_1 \times I_3$. In this case, $i_0 = 1$ and $T_1 = (i_0 + 1, \dots, i_1)$ reduces to $(2, \dots, i_1)$ with $i_1 \geq 4$. This case is again the same as the one in Lemma 4.4 because $2 \notin I_3$ and $i_1 \in I_1$.
- (e). $(1, 2) \in I_3 \times I_3$. In this case, $i_0 = 0$ and $T_1 = (i_0 + 1, \dots, i_1)$ reduces to $(1, \dots, i_1)$ with $i_1 \geq 3$. In this case, we partition $T_1 = (1, \dots, p \mid p + 1, \dots, i_1)$, where $p \geq 2$, $\{1, \dots, p\} \subseteq I_3$, and $(p + 1) \notin I_3$. If $i_1 > p + 1$, then $(1, \dots, p)$ belongs to \mathcal{A} and $(p + 1, \dots, i_1)$ is the same as the one in Lemma 4.4. If $(p + 1) = i_1 \in I_1$, we then partition $T_1 = (1, \dots, i_1 - 3 \mid i_1 - 2, i_1 - 1, i_1)$ and there are three cases to consider for the first part $(1, \dots, i_1 - 3)$:
- (e1). It is empty.
- (e2). It contains the index 1 only, in which case we have $\alpha_1 \geq \frac{2}{L}$ due to Lemma 4.2 (b) and $2 \in I_3$.
- (e3). It belongs to \mathcal{A} .

Moreover, for the second part $(i_1 - 2, i_1 - 1, i_1)$, it follows from Lemma 4.2 (d) that $\alpha_{i_1-2} + \alpha_{i_1-1} + \alpha_{i_1} \geq \frac{3}{L}$ because $i_1 - 1 \in I_3$. Using Lemma 4.3, we have shown $\sum_{\ell \in T_1} \alpha_\ell \geq |T_1|/L$ for all cases.

This completes the proof. \square

Combining Lemmas 4.4 and 4.5, we obtain the following theorem immediately.

Theorem 4.1. *For any given $x^0 \in \mathbb{R}^n$ and $\alpha_0 > 0$, the stepsizes generated by our AdaBB (Algorithm 2) satisfy $\sum_{i=i_0+1}^k \alpha_i \geq \frac{k-i_0}{L}$ for all $k \geq 3$, where i_0 is defined in (38). This also implies $\sum_{i=1}^k \alpha_i \geq \frac{k-2+\sqrt{2}}{L}$ for all $k \geq 1$.*

Proof. The implication $\sum_{i=1}^k \alpha_i \geq \frac{k-2+\sqrt{2}}{L}$ can be verified as follows:

- (a). If $i_0 = 0$, we have $\sum_{i=1}^k \alpha_i \geq \frac{k}{L} > \frac{k-2+\sqrt{2}}{L}$;
- (b). If $i_0 = 1$, we have $\sum_{i=1}^k \alpha_i = \alpha_1 + \sum_{i=2}^k \alpha_i \geq \frac{1}{\sqrt{2}L} + \frac{k-1}{L} > \frac{k-2+\sqrt{2}}{L}$;
- (c). If $i_0 = 2$, we have $\sum_{i=1}^k \alpha_i = \alpha_1 + \alpha_2 + \sum_{i=3}^k \alpha_i \geq \frac{\sqrt{2}}{L} + \frac{k-2}{L} = \frac{k-2+\sqrt{2}}{L}$,

which completes the proof. \square

Remark 4.1. *Similar to [MM23, Algorithm 2], if one can ensure $\alpha_0 \in (\lambda_1, 2\lambda_1)$ through a line search strategy, then it is guaranteed that $1 \in I_2$ and $i_0 = 0$. In this case, our improved bound (35) becomes $\sum_{i=1}^k \alpha_i \geq \frac{k}{L}$ for any $k \geq 3$. For $k = 1$, we have $\sum_{i=1}^k \alpha_i = \alpha_1 \geq \frac{1}{L}$ since $1 \in I_2$. For $k = 2$, there are three cases to consider.*

- (a). *If $2 \in I_1$, we have $\alpha_2 \geq \frac{1}{L}$ follows from Lemma 4.2 (c), hence we get $\alpha_1 + \alpha_2 \geq \frac{2}{L}$;*
- (b). *If $2 \in I_2$, we have $\alpha_2 \geq \frac{1}{L}$ follows from Lemma 4.2 (a), hence we get $\alpha_1 + \alpha_2 \geq \frac{2}{L}$;*
- (c). *If $2 \in I_3$, we have $\alpha_1 + \alpha_2 \geq \frac{4+\sqrt{2}}{2L} \geq \frac{2}{L}$ follows from Lemma 4.2 (b).*

Hence, we prove that $\sum_{i=1}^k \alpha_i \geq \frac{k}{L}$ holds for all $k \geq 1$.

We now give a detailed comparison of our results on the lower bounds of α_k and $\sum_{i=1}^k \alpha_i$ with the existing results in the literature. The results are summarized in Table 1. From Table 1, we first note that these results are all free with θ_0 which is always pre-defined. The results for AdGD2 require a specially chosen α_0 , but other algorithms do not have restrictions on α_0 . Our AdaBB achieves the best lower bound for α_k , i.e., $\alpha_k \geq \frac{1}{\sqrt{2}L}$, $\forall k \geq 1$. While AdGD [MM23] and AdaPGM $^{\pi,r}$ also achieve the same lower bound, their results only hold for $k \geq r_1, r_3$, respectively. Lastly, our AdaBB clearly achieves the best lower bound for $\sum_{i=1}^k \alpha_i$. Overall, we believe that it is fair to claim that our AdaBB achieves the best results for the lower bounds of α_k and $\sum_{i=1}^k \alpha_i$.

	AdGD		AdGD2	AdaPGM	AdaPGM $^{\pi,r}$	AdaBB
θ_0	$+\infty$	0	$\frac{1}{3}$	≥ 1	1	(36)
α_0 free?	✓	✓	✗	✓	✓	✓
$\alpha_k \geq$	$\frac{1}{2L}$	$\frac{1}{\sqrt{2}L}$ ($k \geq r_1$)	$\frac{1}{\sqrt{3}L}$	$\frac{1}{2L}$ ($k \geq r_2$)	$\frac{1}{\sqrt{2}L}$ ($k \geq r_3$)	$\frac{1}{\sqrt{2}L}$
$\sum_{i=1}^k \alpha_i \geq$	$\frac{k}{2L}$	$\frac{k-r_1+1}{\sqrt{2}L}$	$\frac{k}{\sqrt{2}L}$	$\frac{k-r_2+1}{2L}$	$\frac{k-r_3+1}{\sqrt{2}L}$	$\frac{k-2+\sqrt{2}}{L}$

Table 1: Comparison of the lower bounds of α_k and $\sum_{i=1}^k \alpha_i$ among AdGD [MM20; MM23], AdGD2 [MM23], AdaPGM [Lat+23], AdaPGM $^{\pi,r}$ [LTP23], and AdaBB. We listed two results for AdGD corresponding to different values of θ_0 . The case $\theta_0 = +\infty$ is analyzed in [MM20] and the case $\theta_0 = 0$ is analyzed in [MM23]. Here we note that the results of AdGD2 only hold when a special α_0 satisfying $\alpha_0 L_1 \in [\frac{1}{\sqrt{2}}, 2]$ is chosen. Other algorithms do not have restrictions on α_0 . The lower bounds of α_k for AdGD [MM23], AdaPGM and AdaPGM $^{\pi,r}$ only hold for $k \geq r_1, r_2, r_3$, respectively. Here, r_1 is the smallest integer satisfying $\prod_{i=1}^{r_1} \varpi_i \geq \frac{1}{\sqrt{2}L\alpha_0}$ with $\varpi_{n+1} = \sqrt{1 + \varpi_n}$ and $\varpi_1 = 1$; $r_2 = \lfloor 2 \log_2 \frac{1}{\alpha_0 L} \rfloor_+$; $r_3 = 2 \lfloor \log_2 \frac{1}{\alpha_0 L} \rfloor_+$.

5 Extensions

In this section, we extend AdaBB (Algorithm 1) to locally strongly convex problem and composite convex optimization problems.

5.1 When f is Locally Strongly Convex

In this subsection, we extend our analysis to the case where f is locally strongly convex. Specifically, in addition to the locally L -smoothness condition (8), we also assume that f is locally μ -strongly convex in $B(x^*, R)$, i.e.,

$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle \geq \frac{\mu}{2} \|x - y\|^2, \forall x, y \in B(x^*, R), \quad (43)$$

where R is defined in (27), and we can prove that the sequence $\{x^k\}$ generated by the following algorithm lies in $B(x^*, R)$. Additionally, it is worth noting that the parameters μ, L and R are used solely for the purpose of analysis and are not involved in the algorithm. According to [Nes04, Theorem 2.1.10], (43) implies

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \leq \|\nabla f(x) - \nabla f(y)\|^2 / \mu, \forall x, y \in B(x^*, R). \quad (44)$$

We will present an extension of Algorithm 1 to handle this case and establish a linear convergence result.

Recall that λ_k denotes the Short BB stepsize and is given by (4). The new algorithm follows the same iteration scheme as (2), with the only variation from Algorithm 1 being the update rule for the stepsize α_k . Specifically, for $k \geq 1$ we update α_k as follows

$$\alpha_k = \begin{cases} \min\{\sqrt{1 + \eta\theta_{k-1}}\alpha_{k-1}, \lambda_k\}, & \text{update } \theta_k = \frac{\alpha_k}{\alpha_{k-1}}, & \text{if } \lambda_k \geq \alpha_{k-1}, \\ \lambda_k, & \text{update } \theta_k = \frac{2\alpha_k}{\alpha_{k-1}} - 1, & \text{if } \frac{\delta\alpha_{k-1}}{2} < \lambda_k < \alpha_{k-1}, \\ \frac{\lambda_k}{\sqrt{2}}, & \text{update } \theta_k = \frac{\alpha_k}{\alpha_{k-1}}, & \text{if } 0 < \lambda_k \leq \frac{\delta\alpha_{k-1}}{2}, \end{cases} \quad (45)$$

where $\eta \in [0, 1)$ and $\delta \in (1, 2)$ are parameters. In (45), when $\lambda_k \geq \alpha_{k-1}$, a more cautious stepsize is used compared to Algorithm 1. Additionally, the region $\delta\alpha_{k-1}/2 < \lambda_k < \alpha_{k-1}$ is narrower than (Case ii) in Algorithm 1. As a result, the region $0 < \lambda_k \leq \delta\alpha_{k-1}/2$ in (45) becomes broader than (Case iii) in Algorithm 1. Similarly to (13), for all $k \geq 1$ we define M_k and P_k as follows

$$\begin{cases} M_k = 0, & P_k = \frac{\alpha_k^2}{\alpha_{k-1}}, & \text{if } \lambda_k \geq \alpha_{k-1}, \\ M_k = \frac{\alpha_k}{\alpha_{k-1}} - \frac{\alpha_k^2}{\alpha_{k-1}^2}, & P_k = \frac{2\alpha_k^2}{\alpha_{k-1}} - \alpha_k, & \text{if } \frac{\delta\alpha_{k-1}}{2} < \lambda_k < \alpha_{k-1}, \\ M_k = \frac{1}{2} - \frac{\lambda_k}{2\alpha_{k-1}}, & P_k = \frac{\alpha_k^2}{\alpha_{k-1}}, & \text{if } 0 < \lambda_k \leq \frac{\delta\alpha_{k-1}}{2}. \end{cases} \quad (46)$$

For convenience, we define P_0 as $(P_1 - \alpha_0)/\eta$ for $\eta \in (0, 1)$. Otherwise, $P_0 = 0$. We have the following lemma.

Lemma 5.1. *Let $\{\alpha_k\}$ and $\{\theta_k\}$ be generated by (45) and M_k and P_k be defined in (46). Then, we have $M_k \geq 0$, $P_k \geq 0$ and $P_k = \alpha_k\theta_k$ for all $k \geq 1$, and $2M_{k+1} \leq 1$ and $P_{k+1} \leq \alpha_k + \eta P_k$ for $k \geq 0$.*

Proof. Recall that $\delta \in (1, 2)$. First, the nonnegativity of M_k and P_k and $P_k = \alpha_k\theta_k$ for $k \geq 1$ can be verified straightforwardly by their definitions together with (45). Let $k \geq 0$ be fixed. We then show the remaining claims by considering the following three cases.

- (i) $\lambda_{k+1} \geq \alpha_k$. In this case, we have $M_{k+1} = 0$ and $\alpha_{k+1} \leq \sqrt{1 + \eta\theta_k}\alpha_k$. Hence, it follows that $P_{k+1} = \alpha_{k+1}^2/\alpha_k \leq (1 + \eta\theta_k)\alpha_k = \alpha_k + \eta P_k$.
- (ii) $\delta\alpha_k/2 < \lambda_{k+1} < \alpha_k$. In this case, we have $\alpha_{k+1} = \lambda_{k+1}$, and it is easy to verify that

$$M_{k+1} = \frac{\lambda_{k+1}}{\alpha_k} - \frac{\lambda_{k+1}^2}{\alpha_k^2} \leq \frac{1}{4} \quad \text{and} \quad P_{k+1} = \frac{2\lambda_{k+1}^2}{\alpha_k} - \lambda_{k+1} \leq \alpha_k \leq \alpha_k + \eta P_k.$$

- (iii) $0 < \lambda_{k+1} \leq \delta\alpha_k/2$. In this case, we have $\alpha_{k+1} = \lambda_{k+1}/\sqrt{2}$, and it is elementary to verify that

$$M_{k+1} = \frac{1}{2} - \frac{\lambda_{k+1}}{2\alpha_k} \leq \frac{1}{2} \quad \text{and} \quad P_{k+1} = \frac{\lambda_{k+1}^2}{2\alpha_k} \leq \frac{\delta^2\alpha_k}{8} \leq \alpha_k \leq \alpha_k + \eta P_k.$$

In all three cases, we have shown that $2M_{k+1} \leq 1$ and $P_{k+1} \leq \alpha_k + \eta P_k$. □

Again, we emphasize that α_k , θ_k , M_k and P_k are defined in (45)-(46). With these newly defined parameters, we still define $w_k = \alpha_k + P_k - P_{k+1}$ for $k \geq 0$ as in Lemma 3.4, and \mathbf{E} , Υ_k and Φ_k as in (17), (21) and (22), respectively. Next, we present the pointwise convergence of the gradient method (2) with α_k given by (45) and establish bounds on α_k and P_k/α_k . Note that since f is locally strongly convex, it has a unique optimal solution.

Theorem 5.1 (Pointwise convergence). *For any $x^0 \in \mathbb{R}^n$ and $\alpha_0 > 0$, let $\{x^k\}$ be the sequence generated by (2) with α_k given by (45). Then, $x^k \in B(x^*, R)$ for all $k \geq 0$, where R is defined in (27), and $\{x^k\}$ converges to the unique optimal solution x^* of (1).*

Proof. First, by following the proof of Lemma 3.3, it is elementary to verify that (20), and thus (17), holds as well for $\lambda_k < \alpha_{k-1}$. Then, by following the proof of Lemma 3.4, it is also easy to observe that (24) holds as well, with \mathbf{E} defined in (17). Combining Lemma 5.1, which confirms that $2M_{k+1} \leq 1$ and $P_{k+1} \leq \alpha_k + \eta P_k \leq \alpha_k + P_k$ for all $k \geq 0$, with (24), we obtain (22), with Υ_k and w_k defined in (21) and Lemma 3.4, respectively. Consequently, by following the same lines of proof as in Corollary 3.1 and Theorem 3.2, we can show that $x^k \in B(x^*, R)$ for all $k \geq 0$, where R is defined in (27), and $\{x^k\}$ converges to the unique optimal solution of (1). The details are omitted due to the high similarity. \square

Proposition 5.1 (Bounds on α_k and P_k/α_k). *Let $\{\alpha_k\}$ be generated by (45) with any $\alpha_0 > 0$. Then, for $k \geq 1$, we have (i) $c \leq \alpha_k \leq 1/\mu$, where $c := \min\{\alpha_0, 1/(\sqrt{2}L)\} > 0$, and (ii) $P_k/\alpha_k \geq c_0 := \min\{c\mu, \delta - 1\} > 0$.*

Proof. Let $k \geq 1$ be fixed. (i) First, $\alpha_k \geq c = \min\{\alpha_0, 1/(\sqrt{2}L)\}$ follows from the same analysis as in Proposition 3.1 (ii). Second, the definition of α_k in (45) shows that $\alpha_k \leq \lambda_k$. Further considering (44), we obtain $\alpha_k \leq \lambda_k \leq 1/\mu$. For part (ii), we split the analysis into two cases: (a) $\lambda_k \geq \alpha_{k-1}$ or $\lambda_k \leq \delta\alpha_{k-1}/2$, and (b) $\delta\alpha_{k-1}/2 < \lambda_k < \alpha_{k-1}$. For case (a), we have $P_k/\alpha_k = \alpha_k/\alpha_{k-1} \geq c\mu$, where the inequality follows from $c \leq \alpha_k \leq 1/\mu$. For case (b), we have $P_k = \frac{2\alpha_k^2}{\alpha_{k-1}} - \alpha_k$, $\alpha_k = \lambda_k$, and hence $P_k/\alpha_k \geq \delta - 1$. Combining these two cases completes the proof. \square

Now, we are ready to establish the linear convergence result.

Theorem 5.2 (Linear convergence). *For any $x^0 \in \mathbb{R}^n$ and $\alpha_0 > 0$, the sequence $\{x^k\}$ generated by (2) with α_k given by (45) converges linearly to the unique optimal solution x^* of (1).*

Proof. For convenience, we define for $k \geq 0$ that

$$\Phi_{k+1} := \|x^{k+1} - x^*\|^2 + 2\left(1 + \frac{\mu}{2L}\right)M_{k+1}\|x^{k+1} - x^k\|^2 + \frac{2\alpha_k + 2P_k}{\alpha_k + \eta P_k}P_{k+1}(f(x^k) - f_*). \quad (47)$$

Recall that $f_* = f(x^*)$ and $\nabla f(x^*) = 0$. It follows from (43), (8) and (2) that

$$\begin{aligned} \alpha_k \langle \nabla f(x^k), x^* - x^k \rangle &\stackrel{(43)}{\leq} \alpha_k (f_* - f(x^k)) - \frac{\alpha_k \mu}{2} \|x^k - x^*\|^2, \\ \alpha_k \langle \nabla f(x^k), x^* - x^k \rangle &\stackrel{(8)}{\leq} \alpha_k (f_* - f(x^k)) - \frac{\alpha_k}{2L} \|\nabla f(x^k) - \nabla f(x^*)\|^2 \\ &\stackrel{(2)}{=} \alpha_k (f_* - f(x^k)) - \frac{1}{2\alpha_k L} \|x^{k+1} - x^k\|^2 \\ &\leq \alpha_k (f_* - f(x^k)) - \frac{\mu}{2L} \|x^{k+1} - x^k\|^2, \end{aligned}$$

where the last “ \leq ” is due to $\alpha_k \leq 1/\mu$. Combining the above two inequalities to obtain

$$\alpha_k \langle \nabla f(x^k), x^* - x^k \rangle \leq \alpha_k (f_* - f(x^k)) - \frac{\alpha_k \mu}{4} \|x^k - x^*\|^2 - \frac{\mu}{4L} \|x^{k+1} - x^k\|^2. \quad (48)$$

Plugging (48) into (23), we arrive at

$$\|x^{k+1} - x^*\|^2 \leq \left(1 - \frac{\alpha_k \mu}{2}\right) \|x^k - x^*\|^2 - 2\alpha_k (f(x^k) - f_*) - \frac{\mu}{2L} \|x^{k+1} - x^k\|^2 + \alpha_k^2 \|\nabla f(x^k)\|^2. \quad (49)$$

By summing (49) and (24), considering the definition of \mathbf{E} in (17), and reorganizing terms, we can easily derive

$$\begin{aligned} & (1 - \frac{\alpha_k \mu}{2}) \|x^k - x^*\|^2 + 2M_k \|x^k - x^{k-1}\|^2 + 2P_k (f(x^{k-1}) - f_*) \\ & \geq \|x^{k+1} - x^*\|^2 + (1 + \frac{\mu}{2L}) \|x^{k+1} - x^k\|^2 + (2\alpha_k + 2P_k)(f(x^k) - f_*) \geq \Phi_{k+1}, \end{aligned} \quad (50)$$

where the second “ \geq ” follows from (47) and Lemma 5.1. Define

$$c_{1,k} := \max \left\{ 1 - \frac{\alpha_k \mu}{2}, 1 / \left(1 + \frac{\mu}{2L} \right), \frac{\alpha_{k-1} + \eta P_{k-1}}{\alpha_{k-1} + P_{k-1}} \right\} > 0 \quad \text{and} \quad c_1 := \max \left\{ 1 - \frac{c\mu}{2}, 1 / \left(1 + \frac{\mu}{2L} \right), 1 - \frac{c_0(1-\eta)}{c_0+1} \right\},$$

where $c, c_0 > 0$ are defined in Proposition 5.1. From Proposition 5.1 (i), we have $\alpha_k \geq c$ and thus $1 - \frac{\alpha_k \mu}{2} \leq 1 - \frac{c\mu}{2} < 1$. On the other hand, from Proposition 5.1 (ii) we have $\alpha_{k-1}/P_{k-1} \leq 1/c_0$ for $k \geq 2$, and hence

$$\frac{\alpha_{k-1} + \eta P_{k-1}}{\alpha_{k-1} + P_{k-1}} = 1 - \frac{1-\eta}{(\alpha_{k-1}/P_{k-1})+1} \leq 1 - \frac{c_0(1-\eta)}{c_0+1} < 1.$$

Therefore, we have shown that $c_{1,k} \leq c_1 < 1$ for all $k \geq 2$. It then follows from (47), (50) and the definition of $c_{1,k}$ that $\Phi_{k+1} \leq c_{1,k} \Phi_k \leq c_1 \Phi_k$ for all $k \geq 2$. Again, it follows from the definition of (47) and $c_1 < 1$ that $\{x^k\}$ converges linearly to the unique optimal solution of (1). \square

5.2 Composite Convex Optimization Problems

Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be an extended real-valued closed, proper and convex function, which may be non-smooth. In this subsection, we extend AdaBB (Algorithm 1) to solve the composite convex optimization problem

$$\min_{x \in \mathbb{R}^n} F(x) := f(x) + g(x), \quad (51)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is the same as in (1). In particular, f is a locally L -smooth function convex function satisfying (8) in which the radius $R := T$ and T is defined in (68). We assume that the set of optimal solutions of (51), also denoted by \mathcal{X}^* , is non-empty and denote the optimal value of F by F_* . In this section, we consider the proximal gradient method of the form

$$x^{k+1} = \text{prox}_{\alpha_k g}(x^k - \alpha_k \nabla f(x^k)), \quad k \geq 0, \quad (52)$$

where $\alpha_k > 0$ denotes the stepsize and will be chosen adaptively, and for given $\alpha > 0$, $\text{prox}_{\alpha g}(\cdot)$ is defined by

$$\text{prox}_{\alpha g}(x) = \arg \min_{y \in \mathbb{R}^n} g(y) + \frac{1}{2\alpha} \|y - x\|^2, \quad x \in \mathbb{R}^n.$$

An equivalent implicit form of (52) is given by

$$x^{k+1} = x^k - \alpha_k (\nabla f(x^k) + \xi^{k+1}) \quad \text{for some} \quad \xi^{k+1} \in \partial g(x^{k+1}). \quad (53)$$

Our adaptive proximal BB method (AdaPBB) for solving (51) is presented in Algorithm 3.

Remark 5.1. *It is worth noting that in Algorithm 3, if $\lambda_k < \alpha_{k-1}$ (Cases ii and iii), then $\alpha_k \geq \lambda_k / \sqrt{2} \geq 1 / (\sqrt{2}L)$. Similar discussions in Proposition 3.1 will be formally presented in Proposition 5.2 later.*

Before analyzing the convergence of Algorithm 3, we define some useful notation and recall some important inequalities for the scheme (53). For $k \geq 1$, we define B_k and E_k as follows

$$\begin{cases} B_k := 0, & E_k := \frac{1}{\alpha_{k-1}}, & \text{if } \lambda_k \geq \alpha_{k-1}, \\ B_k := 1, & E_k := 0, & \text{if } \frac{\alpha_{k-1}}{2} < \lambda_k \leq \alpha_{k-1}, \\ B_k := \frac{(\alpha_{k-1} - \lambda_k)^2}{\lambda_k^2}, & E_k := 0, & \text{if } 0 < \lambda_k \leq \frac{\alpha_{k-1}}{2}. \end{cases} \quad (54)$$

Algorithm 3 Adaptive Proximal BB Method (AdaPBB)

Input: $x^0 \in \mathbb{R}^n$, $\alpha_0 > 0$, $\theta_0 \geq 0$

```

1:  $x^1 = \text{prox}_{\alpha_0 g}(x^1 - \alpha_0 \nabla f(x^0))$ 
2: for  $k = 1, 2, \dots$ , do
3:    $\lambda_k = \frac{\langle \nabla f(x^k) - \nabla f(x^{k-1}), x^k - x^{k-1} \rangle}{\|\nabla f(x^k) - \nabla f(x^{k-1})\|^2}$ 
4:   if  $\lambda_k \geq \alpha_{k-1}$  then ..... (Case i)
5:      $\alpha_k = \sqrt{1 + \theta_{k-1}} \alpha_{k-1}$ , and  $\theta_k = \frac{\alpha_k}{\alpha_{k-1}}$ 
6:   else if  $\alpha_{k-1}/2 < \lambda_k < \alpha_{k-1}$  then ..... (Case ii)
7:      $\alpha_k = \frac{\alpha_{k-1}}{\sqrt{2}}$ , and  $\theta_k = 0$ 
8:   else ..... (Case iii)
9:      $\alpha_k = \frac{\lambda_k}{\sqrt{2}}$ , and  $\theta_k = 0$ 
10:  end if
11:   $x^{k+1} = \text{prox}_{\alpha_k g}(x^k - \alpha_k \nabla f(x^k))$ 
12: end for

```

For convenience, we also define $E_0 := (E_1 \alpha_1^2 - \alpha_0) / \alpha_0^2$. The following lemma provides useful inequalities for B_k and E_k defined in (54).

Lemma 5.2. *Let $\{\alpha_k\}$ and $\{\theta_k\}$ be generated by Algorithm 3 and B_k and E_k be defined in (54). Then, $\alpha_k E_k = \theta_k$ for $k \geq 1$, and $2B_{k+1} \leq \alpha_k^2 / \alpha_{k+1}^2$ and $E_{k+1} \alpha_{k+1}^2 \leq E_k \alpha_k^2 + \alpha_k$ for $k \geq 0$.*

Proof. The fact that $\alpha_k E_k = \theta_k$ for $k \geq 1$ is obvious. Let $k \geq 0$ be fixed. To establish the remaining results, we split the analysis into the following three cases. Case (i): $\lambda_{k+1} \geq \alpha_k$. In this case, we have $B_{k+1} = 0$, $E_{k+1} = 1/\alpha_k$, $\alpha_{k+1} = \sqrt{1 + \theta_k} \alpha_k$, and thus $E_{k+1} \alpha_{k+1}^2 = (1 + \theta_k) \alpha_k = E_k \alpha_k^2 + \alpha_k$. Case (ii): $\alpha_k/2 < \lambda_{k+1} < \alpha_k$. In this case, we have $B_{k+1} = 1$, $E_{k+1} = 0$, $\alpha_{k+1} = \alpha_k / \sqrt{2}$, and thus $2B_{k+1} = \alpha_k^2 / \alpha_{k+1}^2$. Case (iii): $\lambda_{k+1} \leq \alpha_k/2$. In this case, we have $B_{k+1} = (\alpha_k - \lambda_{k+1})^2 / \lambda_{k+1}^2$, $E_{k+1} = 0$, $\alpha_{k+1} = \lambda_{k+1} / \sqrt{2}$, and thus $2B_{k+1} = 2(\alpha_k - \lambda_{k+1})^2 / \lambda_{k+1}^2 \leq 2\alpha_k^2 / \lambda_{k+1}^2 = \alpha_k^2 / \alpha_{k+1}^2$. The proof is completed by combining the above three cases. \square

The following results are taken from [MM23]. Note that a refined inequality that improves upon (55) will be derived in the proof of Theorem 5.4.

Lemma 5.3 ([MM23, Eq. (34) and Lemmas 11-12]). *Let $\{x^k\}$ be generated by (53) with arbitrarily positive stepsizes $\{\alpha_k\}$. Then, for $k \geq 0$, we have*

$$\|x^{k+1} - x^*\|^2 + 2\alpha_k (F(x^k) - F_*) \leq \|x^k - x^*\|^2 + \alpha_k^2 \|\nabla f(x^k) + \xi^k\|^2, \quad (55)$$

$$\|\nabla f(x^k) + \xi^{k+1}\|^2 = \langle \nabla f(x^{k+1}) + \xi^{k+1}, \nabla f(x^k) + \xi^{k+1} \rangle + \frac{1}{\alpha_k} \langle \nabla f(x^{k+1}) - \nabla f(x^k), x^{k+1} - x^k \rangle, \quad (56)$$

$$\|\nabla f(x^k) + \xi^{k+1}\| \leq \|\nabla f(x^k) + \xi^k\|. \quad (57)$$

Proof. The result (55) follows from [MM23, Eq. (34)], (56) is taken from the proof of [MM23, Lemma 11], and (57) is given as Lemma 12 in [MM23]. \square

Lemma 5.4 (Analogous to (17)). *Let $\{x^k\}$ be generated by Algorithm 3 and B_k and E_k are defined in (54). Then, for $k \geq 1$, we have*

$$\|\nabla f(x^k) + \xi^k\|^2 \leq B_k \|\nabla f(x^{k-1}) + \xi^{k-1}\|^2 + E_k (F(x^{k-1}) - F(x^k)). \quad (58)$$

Proof. From the equality $\|a\|^2 = \|a - b\|^2 - \|b\|^2 + 2\langle a, b \rangle$, we obtain

$$\begin{aligned} \|\nabla f(x^k) + \xi^k\|^2 &= \|\nabla f(x^k) - \nabla f(x^{k-1})\|^2 - \|\nabla f(x^{k-1}) + \xi^k\|^2 + 2\langle \nabla f(x^k) + \xi^k, \nabla f(x^{k-1}) + \xi^k \rangle \\ &= \left(\frac{1}{\lambda_k} - \frac{1}{\alpha_{k-1}}\right) \langle \nabla f(x^k) - \nabla f(x^{k-1}), x^k - x^{k-1} \rangle + \langle \nabla f(x^k) + \xi^k, \nabla f(x^{k-1}) + \xi^k \rangle, \end{aligned} \quad (59)$$

where the second equality follows from (4) and (53). It follows from (56) and (57) that

$$\begin{aligned} \langle \nabla f(x^k) + \xi^k, \nabla f(x^{k-1}) + \xi^k \rangle &\stackrel{(56)}{=} \|\nabla f(x^{k-1}) + \xi^k\|^2 - \frac{1}{\alpha_{k-1}} \langle \nabla f(x^k) - \nabla f(x^{k-1}), x^k - x^{k-1} \rangle \\ &\stackrel{(57)}{\leq} \|\nabla f(x^{k-1}) + \xi^{k-1}\|^2 - \frac{1}{\alpha_{k-1}} \langle \nabla f(x^k) - \nabla f(x^{k-1}), x^k - x^{k-1} \rangle. \end{aligned} \quad (60)$$

Combining (59) and (60), we obtain

$$\|\nabla f(x^k) + \xi^k\|^2 \leq \left(\frac{1}{\lambda_k} - \frac{2}{\alpha_{k-1}}\right) \langle \nabla f(x^k) - \nabla f(x^{k-1}), x^k - x^{k-1} \rangle + \|\nabla f(x^{k-1}) + \xi^{k-1}\|^2. \quad (61)$$

We then prove the desired result (58) by analyzing the following three cases.

- Case (i): $\lambda_k \geq \alpha_{k-1}$. In this case, we have $1/\lambda_k - 1/\alpha_{k-1} \leq 0$. Since $\nabla f(x^{k-1}) + \xi^k = (x^{k-1} - x^k)/\alpha_{k-1}$ and $\langle \nabla f(x^k) - \nabla f(x^{k-1}), x^k - x^{k-1} \rangle \geq 0$, we obtain from (59) that

$$\|\nabla f(x^k) + \xi^k\|^2 \leq \frac{1}{\alpha_{k-1}} \langle \nabla f(x^k) + \xi^k, x^{k-1} - x^k \rangle \leq \frac{1}{\alpha_{k-1}} (F(x^{k-1}) - F(x^k)), \quad (62)$$

where the second “ \leq ” is due to the convexity of F and $\nabla f(x^k) + \xi^k \in \partial F(x^k)$.

- Case (ii): $\alpha_{k-1}/2 < \lambda_k < \alpha_{k-1}$. In this case, we have $1/\lambda_k - 2/\alpha_k \leq 0$ and (61) implies

$$\|\nabla f(x^k) + \xi^k\|^2 \leq \|\nabla f(x^{k-1}) + \xi^{k-1}\|^2. \quad (63)$$

- Case (iii): $0 < \lambda_k \leq \alpha_{k-1}/2$. Then, (4) and (61) imply

$$\begin{aligned} \|\nabla f(x^k) + \xi^k\|^2 &\leq \frac{1}{\lambda_k} \left(\frac{1}{\lambda_k} - \frac{2}{\alpha_{k-1}}\right) \frac{\langle \nabla f(x^k) - \nabla f(x^{k-1}), x^k - x^{k-1} \rangle^2}{\|\nabla f(x^k) - \nabla f(x^{k-1})\|^2} + \|\nabla f(x^{k-1}) + \xi^{k-1}\|^2 \\ &\leq \frac{1}{\lambda_k} \left(\frac{1}{\lambda_k} - \frac{2}{\alpha_{k-1}}\right) \|x^k - x^{k-1}\|^2 + \|\nabla f(x^{k-1}) + \xi^{k-1}\|^2 \\ &\stackrel{(53)}{=} \frac{\alpha_{k-1}^2}{\lambda_k} \left(\frac{1}{\lambda_k} - \frac{2}{\alpha_{k-1}}\right) \|\nabla f(x^{k-1}) + \xi^k\|^2 + \|\nabla f(x^{k-1}) + \xi^{k-1}\|^2 \\ &\stackrel{(57)}{\leq} \frac{(\alpha_{k-1} - \lambda_k)^2}{\lambda_k^2} \|\nabla f(x^{k-1}) + \xi^{k-1}\|^2. \end{aligned} \quad (64)$$

The desired result (58) follows immediately by combining (62)-(64) with (54). \square

We are now ready to establish a result that is analogous to (22) for problem (1). For this purpose, in the rest of this section, we let $x^* \in \mathcal{X}^*$ be an arbitrarily fixed solution of (51) and define for $k \geq 1$ that

$$\begin{cases} w_k := \alpha_k + E_k \alpha_k^2 - E_{k+1} \alpha_{k+1}^2, \\ V_k := \|x^k - x^*\|^2 + 2B_k \alpha_k^2 \|\nabla f(x^{k-1}) + g^{k-1}\|^2 + 2\alpha_{k-1}(1 + E_{k-1} \alpha_{k-1})(F(x^{k-1}) - F_*), \\ U_k := V_k - 2w_{k-1}(F(x^{k-1}) - F_*). \end{cases} \quad (65)$$

It is obvious from Lemma 5.2 that $w_k \geq 0$ for $k \geq 0$. Furthermore, direct calculations show that

$$U_k = \|x^k - x^*\|^2 + 2B_k \alpha_k^2 \|\nabla f(x^{k-1}) + \xi^{k-1}\|^2 + 2E_k \alpha_k^2 (F(x^{k-1}) - F_*). \quad (66)$$

Lemma 5.5 (Analogous to Lemma 3.4). *Let $\{x^k\}$ be generated by Algorithm 3 and U_k and V_k are defined in (65). Then, for $k \geq 1$, we have $V_{k+1} \leq U_k \leq V_k$.*

Proof. Let $k \geq 1$ be fixed. First, (58) is equivalent to

$$\|\nabla f(x^k) + \xi^k\|^2 \leq 2B_k \|\nabla f(x^{k-1}) + \xi^{k-1}\|^2 + 2E_k(F(x^{k-1}) - F(x^k)) - \|\nabla f(x^k) + \xi^k\|^2. \quad (67)$$

By utilizing equation (67) to expand the term $\|\nabla f(x^k) + \xi^k\|^2$ on the right-hand-side of equation (55), and considering the definitions of U_k , V_k , and w_k provided in equation (65), we can rearrange the terms and perform elementary calculations to obtain the following inequality:

$$U_k \geq \|x^{k+1} - x^*\|^2 + \alpha_k^2 \|\nabla f(x^k) + \xi^k\|^2 + 2\alpha_k(1 + E_k\alpha_k)(F(x^k) - F_*) \geq V_{k+1},$$

where the second “ \geq ” follows from Lemma 5.2 and the definition of V_k in (65). Finally, considering $w_i \geq 0$ for all $i \geq 0$, we obtain $V_{k+1} \leq U_k = V_k - 2w_{k-1}(F(x^{k-1}) - F_*) \leq V_k$. \square

Corollary 5.1 (Analogous to Corollary 3.1). *Let $\{x^k\}$ be generated by Algorithm 3. Then, $\{x^k\}$ is bounded. In particular, $x^k \in B(x^*, T)$ for all $k \geq 0$, where T is defined as:*

$$T^2 := \|x^0 - x^*\|^2 + 2\alpha_0^2 \|\nabla f(x^0) + \xi^0\|^2 + \max\{2(E_1\alpha_1^2 - \alpha_0), 0\}(F(x^0) - F_*). \quad (68)$$

Proof. It follows from (66) and Lemma 5.5 that $\|x^k - x^*\|^2 \leq U_k \leq V_k \leq U_{k-1} \leq \dots \leq U_1$ for all $k \geq 1$. Setting $k = 1$ in (66) and (55), and using $2B_1\alpha_1^2 \leq \alpha_0^2$ from Lemma 5.2, we obtain

$$\begin{aligned} U_1 &= \|x^1 - x^*\|^2 + 2B_1\alpha_1^2 \|\nabla f(x^0) + \xi^0\|^2 + 2E_1\alpha_1^2(F(x^0) - F_*) \\ &\stackrel{(55)}{\leq} \|x^0 - x^*\|^2 + 2\alpha_0^2 \|\nabla f(x^0) + \xi^0\|^2 + 2(E_1\alpha_1^2 - \alpha_0)(F(x^0) - F_*). \end{aligned}$$

Moreover, analogous to Remark 3.1, we can claim that $E_1\alpha_1^2$ is a constant entirely determined by x^0 and α_0 . Furthermore, it is trivial to observe that $\|x^0 - x^*\| \leq T$. Combining the above arguments, we conclude that $\|x^k - x^*\| \leq T$ for all $k \geq 0$. \square

Proposition 5.2 (The same as Proposition 3.1). *For $\{\alpha_k\}$ generated by Algorithm 3, we have (i) if $\alpha_j \geq \frac{1}{\sqrt{2L}}$ for some j , then $\alpha_k \geq \frac{1}{\sqrt{2L}}$ for any $k \geq j$; (ii) $\alpha_k \geq c := \min\{\alpha_0, \frac{1}{\sqrt{2L}}\}$ for all $k \geq 0$; and (iii) $\sum_{i=1}^k \alpha_i = O(k)$.*

Proof. The proof is highly similar to that of Proposition 3.1 and is thus omitted. \square

Now, we are ready to derive the ergodic sublinear convergence result of Algorithm 3.

Theorem 5.3 (Analogous to Theorem 3.1). *Let $\{x^k\}$ be generated by Algorithm 3. Then, we have*

$$F(\bar{x}^k) - F_* \leq \frac{U_1}{2S_k} = O\left(\frac{1}{k}\right),$$

where $\bar{x}^k := (\alpha_k(1 + E_k\alpha_k)x^k + \sum_{i=1}^{k-1} w_i x^i) / S_k$ with $S_k := E_1\alpha_1^2 + \sum_{i=1}^k \alpha_i$.

Proof. The proof is similar to that of Theorem 3.1. By telescoping $V_{i+1} \leq V_i - 2w_{i-1}(F(x^{i-1}) - F_*)$ given in Lemma 5.5 for $i = 2, \dots, k$, we derive $V_{k+1} + 2\sum_{i=1}^{k-1} w_i(F(x^i) - F_*) \leq V_2 \leq U_1$. Taking into account the definition of V_k in (65), we further derive

$$\alpha_k(1 + E_k\alpha_k)(F(x^k) - F_*) + \sum_{i=1}^{k-1} w_i(F(x^i) - F_*) \leq \frac{U_1}{2}. \quad (69)$$

Again, from (65) we have $\alpha_k(1 + E_k\alpha_k) + \sum_{i=1}^{k-1} w_i = E_1\alpha_1^2 + \sum_{i=1}^k \alpha_i = S_k \sim \mathcal{O}(k)$. The desired result follows from (69), the convexity of F , and Jensen’s inequality. \square

Before establishing the pointwise convergence of Algorithm 3, we derive a useful inequality.

Lemma 5.6. *Let $\{x^k\}$ be generated by Algorithm 3. Then, for any $k \geq 1$ we have*

$$\frac{1}{\alpha_k} \|x^{k+1} - x^k\|^2 \leq 4\sqrt{2}\alpha_{k+1} \|\nabla f(x^{k+1}) - \nabla f(x^*)\|^2 + 4\alpha_k \|\nabla f(x^k) - \nabla f(x^*)\|^2 + 2(F(x^k) - F(x^{k+1})). \quad (70)$$

Proof. Let $k \geq 1$ and $x \in \mathbb{R}^n$ be arbitrarily fixed. It follows from (52) that

$$\alpha_k (g(x^{k+1}) - g(x)) \leq \langle x^{k+1} - x^k + \alpha_k \nabla f(x^k), x - x^{k+1} \rangle. \quad (71)$$

It follows from (71), the convexity of f and $F = f + g$ that

$$F(x^{k+1}) - F(x) \leq \frac{1}{\alpha_k} \langle x^{k+1} - x^k, x - x^{k+1} \rangle + \langle \nabla f(x^k) - \nabla f(x^{k+1}), x - x^{k+1} \rangle. \quad (72)$$

Setting $x = x^k$ in (72) and using the definition of λ_{k+1} in (4), we derive

$$\frac{1}{\alpha_k} \|x^{k+1} - x^k\|^2 \leq \lambda_{k+1} \|\nabla f(x^{k+1}) - \nabla f(x^k)\|^2 + F(x^k) - F(x^{k+1}). \quad (73)$$

Furthermore, plugging $\langle \nabla f(x^k) - \nabla f(x^{k+1}), x^k - x^{k+1} \rangle \leq \frac{\alpha_k}{2} \|\nabla f(x^k) - \nabla f(x^{k+1})\|^2 + \frac{1}{2\alpha_k} \|x^k - x^{k+1}\|^2$ into (72) with $x = x^k$ to obtain

$$\frac{1}{\alpha_k} \|x^{k+1} - x^k\|^2 \leq \alpha_k \|\nabla f(x^{k+1}) - \nabla f(x^k)\|^2 + 2(F(x^k) - F(x^{k+1})). \quad (74)$$

We split the discussion into two cases. (i) If $\lambda_{k+1} \geq \alpha_k$, then (74) implies

$$\begin{aligned} \frac{1}{\alpha_k} \|x^{k+1} - x^k\|^2 &\leq 2\alpha_k \|\nabla f(x^{k+1}) - \nabla f(x^*)\|^2 + 2\alpha_k \|\nabla f(x^k) - \nabla f(x^*)\|^2 + 2(F(x^k) - F(x^{k+1})) \\ &\leq 2\alpha_{k+1} \|\nabla f(x^{k+1}) - \nabla f(x^*)\|^2 + 2\alpha_k \|\nabla f(x^k) - \nabla f(x^*)\|^2 + 2(F(x^k) - F(x^{k+1})), \end{aligned} \quad (75)$$

where the first “ \leq ” uses $\|a - b\|^2 \leq 2\|a - c\|^2 + 2\|b - c\|^2$, and the second is due to $\alpha_{k+1} = \sqrt{1 + \theta_k} \alpha_k \geq \alpha_k$.

(ii) If $\lambda_{k+1} < \alpha_k$, then $\alpha_{k+1} \geq \frac{1}{\sqrt{2}} \lambda_{k+1}$ and (73) implies

$$\begin{aligned} \frac{1}{\alpha_k} \|x^{k+1} - x^k\|^2 &\leq 2\lambda_{k+1} \|\nabla f(x^{k+1}) - \nabla f(x^*)\|^2 + 2\lambda_{k+1} \|\nabla f(x^k) - \nabla f(x^*)\|^2 + F(x^k) - F(x^{k+1}) \\ &\leq 2\sqrt{2}\alpha_{k+1} \|\nabla f(x^{k+1}) - \nabla f(x^*)\|^2 + 2\alpha_k \|\nabla f(x^k) - \nabla f(x^*)\|^2 + F(x^k) - F(x^{k+1}). \end{aligned} \quad (76)$$

Apparently the expression on the right-hand side of (76) is nonnegative, allowing us to expand it further by multiplying by a factor of 2. Moreover, considering (75), we can derive (70) in both scenarios. \square

Theorem 5.4 (Analogous to Theorem 3.2). *The sequence $\{x^k\}$ generated by Algorithm 3 converges to an optimal solution of (51).*

Proof. First, we derive a refined inequality of (55). Let $k \geq 1$ be arbitrarily fixed. By setting $x = x^*$ in (71), we obtain $\alpha_k (g(x^{k+1}) - g(x^*)) \leq \langle x^{k+1} - x^k + \alpha_k \nabla f(x^k), x^* - x^{k+1} \rangle$, which can be equivalently reformulated as

$$\|x^{k+1} - x^*\|^2 + 2\alpha_k (g(x^{k+1}) - g(x^*)) \leq \|x^k - x^*\|^2 + 2\alpha_k \langle \nabla f(x^k), x^* - x^{k+1} \rangle - \|x^{k+1} - x^k\|^2. \quad (77)$$

By using the inequality in (8) over $B(x^*, T)$ and the convexity of g , we obtain

$$\begin{aligned} \langle \nabla f(x^k), x^* - x^{k+1} \rangle &= \langle \nabla f(x^k), x^* - x^k \rangle + \langle \nabla f(x^k) + \xi^k, x^k - x^{k+1} \rangle + \langle \xi^k, x^{k+1} - x^k \rangle \\ &\leq f(x^*) - f(x^k) - \frac{1}{2L} \|\nabla f(x^k) - \nabla f(x^*)\|^2 + \langle \nabla f(x^k) + \xi^k, x^k - x^{k+1} \rangle + g(x^{k+1}) - g(x^k). \end{aligned} \quad (78)$$

Combining (78) and (77), using $2\alpha_k \langle \nabla f(x^k) + \xi^k, x^k - x^{k+1} \rangle - \|x^{k+1} - x^k\|^2 \leq \alpha_k^2 \|\nabla f(x^k) + \xi^k\|^2$, taking into account $F = f + g$, and reorganizing terms, we obtain a refined inequality of (55):

$$\|x^{k+1} - x^*\|^2 + 2\alpha_k(F(x^k) - F_*) + \frac{\alpha_k}{L} \|\nabla f(x^k) - \nabla f(x^*)\|^2 \leq \|x^k - x^*\|^2 + \alpha_k^2 \|\nabla f(x^k) + \xi^k\|^2. \quad (79)$$

Then, by using (79) in place of (55) in the proof of Lemma 5.5, we can derive

$$V_{k+1} + \frac{\alpha_k}{L} \|\nabla f(x^k) - \nabla f(x^*)\|^2 \leq U_k \leq V_k, \quad (80)$$

which holds for all $k \geq 1$. Telescoping this inequality leads to

$$\sum_{k=1}^{\infty} \alpha_k \|\nabla f(x^k) - \nabla f(x^*)\|^2 \leq V_1 < +\infty. \quad (81)$$

Since $\alpha_k \geq c > 0$ from Proposition 5.2 (ii), (81) implies that $\lim_{k \rightarrow \infty} \nabla f(x^k) = \nabla f(x^*)$. We next split the proof into three cases, which cover all possibilities:

- (a). $\liminf_{k \rightarrow \infty} \alpha_k = \limsup_{k \rightarrow \infty} \alpha_k = +\infty$, i.e., $\lim_{k \rightarrow \infty} \alpha_k = +\infty$;
- (b). $\limsup_{k \rightarrow \infty} \alpha_k < +\infty$, i.e., there exists a constant $C > 0$, such that $\alpha_k \leq C$ for all k ;
- (c). $\limsup_{k \rightarrow \infty} \alpha_k = +\infty$, yet $\liminf_{k \rightarrow \infty} \alpha_k < +\infty$, i.e., there exists an infinite set K such that its complementary set $\bar{K} := \{1, 2, \dots\} \setminus K$ is also infinite, $\lim_{K \ni k \rightarrow \infty} \alpha_k = +\infty$ and, for some $C > 0$, $\alpha_k \leq C$ for all $k \in \bar{K}$.

First, we assume case (c) holds and show that $\lim_{K \ni k \rightarrow \infty} F(x^k) = \lim_{\bar{K} \ni k \rightarrow \infty} F(x^k) = F_*$, and thus $\lim_{k \rightarrow \infty} F(x^k) = F_*$. Recall that $\{x^k\}$ is bounded and $\|\nabla f(x^{k+1}) - \nabla f(x^k)\| \rightarrow 0$ as $k \rightarrow \infty$ since $\{\nabla f(x^k)\}$ converges. By setting $x = x^*$ in (72) and then taking the limit “ $K \ni k \rightarrow \infty$ ” on both sides of (72), we derive $0 \leq \lim_{K \ni k \rightarrow \infty} (F(x^{k+1}) - F_*) \leq 0$ and thus $\lim_{K \ni k \rightarrow \infty} F(x^{k+1}) = F_*$. On the other hand, by taking sum over $k = 1, 2, \dots$ on both sides of (70), using $\alpha_k \leq C$ for all $k \in \bar{K}$, $F(x^k) \geq F_*$ for all $k \geq 0$, and (81), we obtain

$$\frac{1}{C} \sum_{k \in \bar{K}} \|x^{k+1} - x^k\|^2 \leq \sum_{k=1}^{\infty} \frac{1}{\alpha_k} \|x^{k+1} - x^k\|^2 \leq 4\sqrt{2}V_1 + 4V_1 + 2(F(x^1) - F_*) < \infty. \quad (82)$$

Note that \bar{K} is infinite. Thus, (82) implies that $\lim_{\bar{K} \ni k \rightarrow \infty} \|x^{k+1} - x^k\| = 0$. Moreover, by applying Cauchy-Schwartz inequality to (72) with $x = x^*$, we derive

$$0 \leq F(x^{k+1}) - F_* \leq \frac{1}{\alpha_k} \|x^{k+1} - x^k\| \|x^* - x^{k+1}\| + \|\nabla f(x^k) - \nabla f(x^{k+1})\| \|x^* - x^{k+1}\|. \quad (83)$$

Again, $\{x^k\}$ is bounded and $\alpha_k \geq c > 0$ from Proposition 5.2 (ii). Then, by taking the limit “ $\bar{K} \ni k \rightarrow \infty$ ” on both sides of (83) and noting $\lim_{\bar{K} \ni k \rightarrow \infty} \|x^{k+1} - x^k\| = 0$ and $\lim_{k \rightarrow \infty} \|\nabla f(x^{k+1}) - \nabla f(x^k)\| = 0$, we obtain $\lim_{\bar{K} \ni k \rightarrow \infty} F(x^{k+1}) = F(x^*)$. In summary, we have shown that $\lim_{k \rightarrow \infty} F(x^{k+1}) = F(x^*)$ in case (c), which confirms that all limit points of $\{x^k\}$ belong to \mathcal{X}^* . It follows from (80) that $U_{k+1} \leq U_k$, where U_k is defined in (66). Using Lemma 3.5 with $a_k := 2B_k \alpha_k^2 \|\nabla f(x^{k-1}) + \xi^{k-1}\|^2 + 2E_k \alpha_k^2 (F(x^{k-1}) - F_*)$ and $\mathcal{X} = \mathcal{X}^*$, we derive the convergence of whole sequence $\{x^k\}$ to an element in \mathcal{X}^* . This completes the proof for case (c). The proofs for cases (a) and (b) are much simpler and thus are omitted. In summary, we have shown that the sequence $\{x^k\}$ generated by Algorithm 3 converges to an optimal solution of (51). \square

6 Numerical Experiments

In this section, we apply our AdaBB algorithms to two representative problems: logistic regression, where f is convex and L -smooth, and cubic regularization, where f is convex and locally smooth. We will first compare the four algorithms implied by Algorithm 2 by choosing different options in (Case ii) and (Case iii). These are given in Table 2.

	AdaBB	AdaBB1	AdaBB2	AdaBB3
Case ii	Option II	Option I	Option I	Option II
Case iii	Option II	Option I	Option II	Option I

Table 2: Four AdaBB Variants.

Moreover, we will also compare the four algorithms in Table 2 with the following algorithms: GD (2) with $\alpha = 1/L$, AdGD [MM23, Algorithm 1], and AdaPGM [Lat+23]. This comparison will help demonstrate the efficiency of our proposed method. For initial points, we set $x^0 = 0$ for all situations. For adaptive methods: AdGD, AdaPGM and the four AdaBB variants in Table 2, we choose $\alpha_0 = 10^{-10}$ as recommended in [MM20]. This ensures that x^1 will be close to x^0 , and provides a reliable estimate of α_1 . Since λ_1 is very likely to be greater than α_0 , it is more likely that $1 \in I_1$. Hence, for the four AdaBB variants in Table 2, we set θ_0 as defined in (36). In the numerical experiments, we also set $\theta_1 = 1 \ll \frac{\alpha_1}{\alpha_0}$ to prevent excessive α_2 values due to the small value of $\alpha_0 = 10^{-10}$.

Our codes were written in Python 3.11.0 and used the framework provided by Malitsky and Mishchenko [MM20]. All numerical experiments were conducted on a personal computer with an AMD Ryzen 7 5800H processor, Radeon Graphics, and 16GB memory. Additionally, the experiments utilized the mushrooms, w8a, and covtype datasets from LIBSVM [CL11].

6.1 Logistic Regression

In this subsection, we consider the logistic regression problem

$$\min_{x \in \mathbb{R}^n} f(x) = -\frac{1}{m} \sum_{i=1}^m (y_i \log(s(a_i^\top x)) + (1 - y_i) \log(1 - s(a_i^\top x))) + \frac{\gamma}{2} \|x\|^2, \quad (84)$$

where $a_i \in \mathbb{R}^n$, $y_i \in \{0, 1\}$. Here, $s(z) = 1/(1 + \exp(-z))$ denotes the sigmoid function, m represents the number of observations, and γ serves as a regularization parameter. For this problem, the gradient of f is given by $\nabla f(x) = \frac{1}{m} \sum_{i=1}^m a_i (s(a_i^\top x) - y_i) + \gamma x$. This means that f is a L -smooth function with $L = \frac{1}{4} \lambda_{\max}(A^\top A) + \gamma$, where $A = (a_1^\top, \dots, a_m^\top)^\top$ and $\lambda_{\max}(A^\top A)$ denotes the largest eigenvalue of matrix $A^\top A$ [MM20]. In this experiment, we run all algorithms for a fixed number of iterations, denoted by MaxIter in Table 3. We use f_* to denote the lowest objective function value obtained among all tested algorithms.

Details of the data sets and the parameters are given in Table 3.

We first present the numerical performances for the four AdaBB variants given in Table 2. Figure 1 shows that AdaBB and AdaBB3 are more efficient than AdaBB1 and AdaBB2. This indicates that choosing Option II in (Case ii) of Algorithm 2 is more preferable. This further implies that when the BB stepsize λ_k is not too large and not too small, then it gives superior performance by choosing λ_k as the stepsize. In the rest of this subsection, we only compare AdaBB and AdaBB3 with other popular optimization algorithms.

	m	n	L	γ	MaxIter
mushrooms	8124	112	2.59	$\frac{L}{m} = 3.18 \times 10^{-5}$	1000
w8a	49749	300	0.66	$\frac{L}{m} = 1.32 \times 10^{-6}$	3000
covtype	581012	54	5.04×10^6	$\frac{L}{10m} = 8.68$	10000

Table 3: Parameters settings for different datasets.

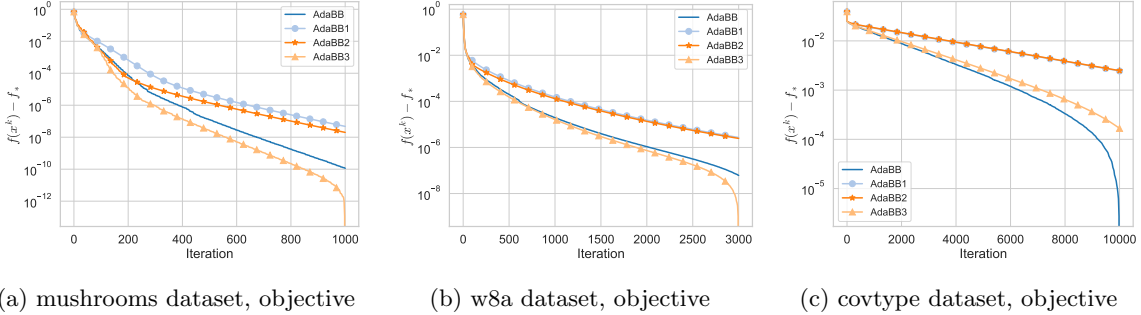


Figure 1: Results for the logistic regression problem via AdaBB, AdaBB1, AdaBB2, AdaBB3 concerning the function value residual.

In Figure 2, we compare AdaBB and AdaBB3 with GD, AdGD and AdaPGM. In subfigures (a), (b), and (c), we show the function value error, and in subfigures (d), (e) and (f), we show the norm of the gradient. From these figures we see that AdaBB and AdaBB3 both perform very well and are usually better than the other three algorithms – AdGD is comparable sometimes.

We also compare AdaBB and AdaBB3 with line-search methods that do not require prior knowledge of L , including line search for GD (with Armijo) [Arm66], and BB stepsize with line search [Ray97]. The results are shown in Figure 3. This time the x -axis denotes the number of matrix-vector multiplications. The results indicate that both AdaBB and AdaBB3 usually perform better than the two line search methods.

At the end of this subsection, we show the stepsizes generated in the first 100 iterations of AdGD and AdaBB. The results are shown in Figure 4. From Figure 4 (a), (b) and (c) we see that the stepsizes produced by both AdGD and AdaBB have a fractal-like nature, and AdaBB usually produces larger stepsizes comparing with AdGD. Figure 4 (d), (e) and (f) illustrate the pattern of the stepsizes generated by AdaBB. We see with excessively large α_k , it is more likely that the next stepsize will be very small, i.e., $(k+1) \in I_3$. Conversely, when α_k is too small, AdaBB automatically opts for $(k+1) \in I_1$ to enlarge the stepsize, and opting for the BB stepsize λ_k is rational when the stepsize is moderate.

6.2 Subproblem of cubic regularized Newton method

The cubic regularized Newton method [NP06] requires solving the following subproblem:

$$\min_{x \in \mathbb{R}^n} f(x) = g^\top x + \frac{1}{2} x^\top H x + \frac{M}{6} \|x\|^3, \quad (85)$$

in each iteration, where $g \in \mathbb{R}^n$, $H \in \mathbb{R}^{n \times n}$, and $M > 0$ is a given regularization parameter. For this problem, the gradient of f is given by $\nabla f(x) = g + Hx + \frac{M\|x\|}{2}x$. Note that there is no value of L

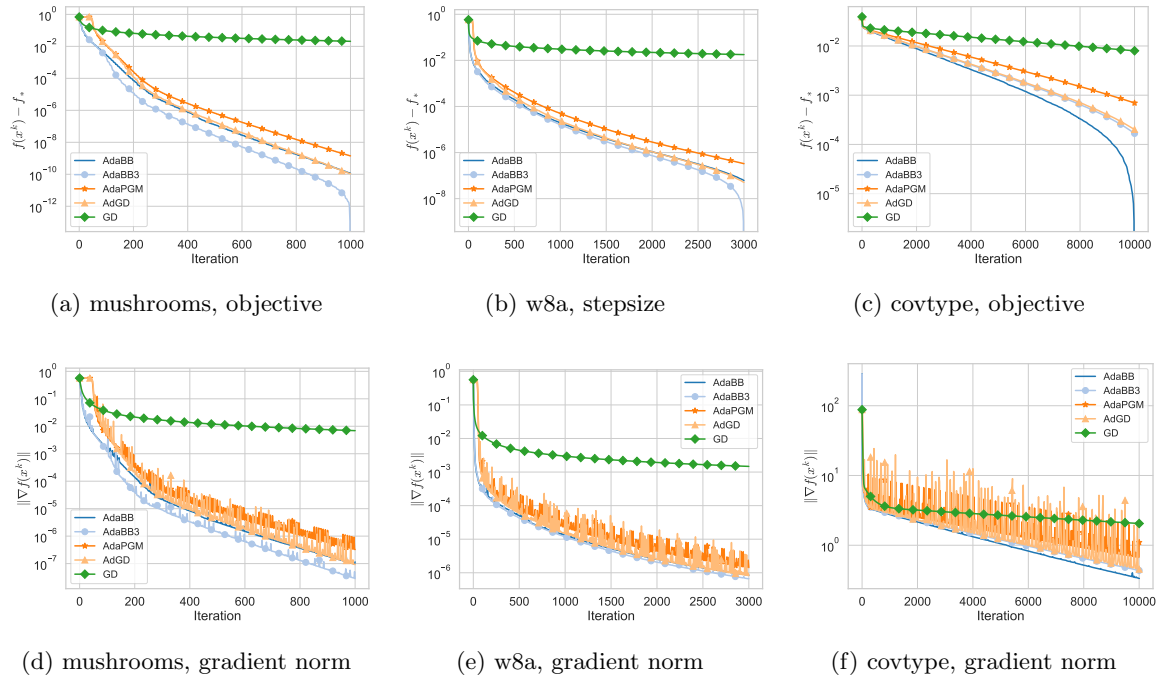


Figure 2: Results for the logistic regression problem via GD, AdGD, AdaPGM, and AdaBB concerning the function value residual and gradient norm.

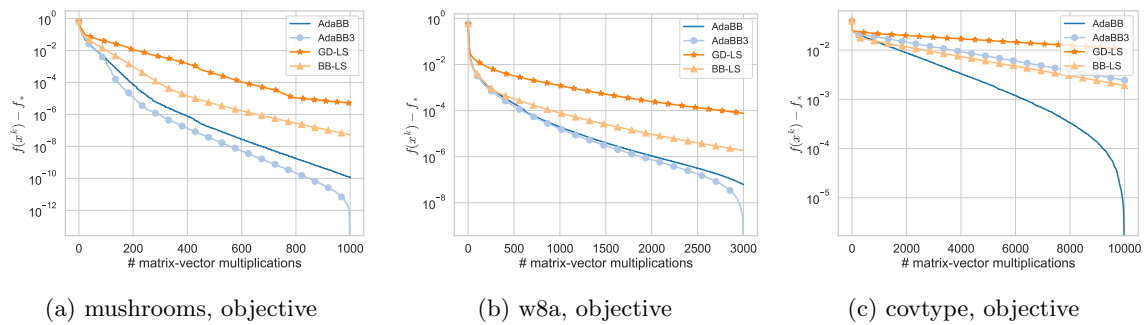


Figure 3: Results for the logistic regression problem via line search methods, and AdaBB concerning the function value residual.

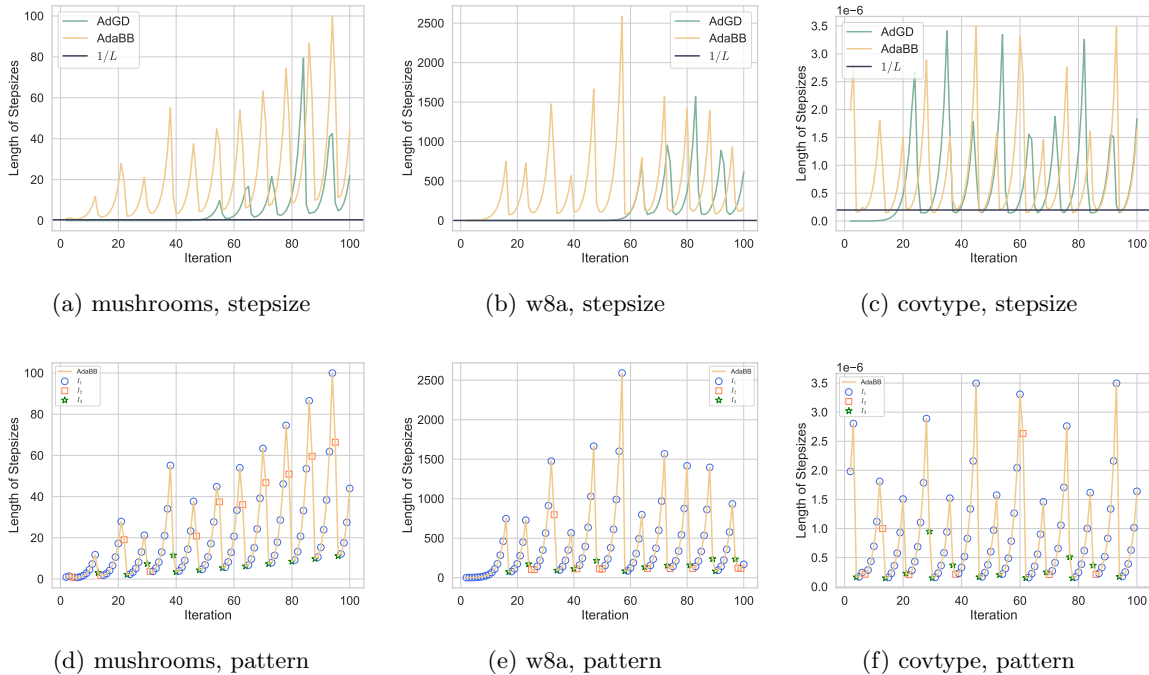


Figure 4: Stepsizes generated by AdGD and AdaBB.

that can guarantee $\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$ for all x and y in \mathbb{R}^n . This implies that f is only smooth locally. Therefore, it becomes challenging to determine the stepsizes for GD. To solve this issue, we adopt a trial-and-error approach [MM20] to fine-tune the stepsizes for these two methods. Specifically, we designate the values from an array of 10 numbers evenly spaced on a logarithmic scale between 10^{-1} and 10 , as potential stepsizes. By computing $f(x^k)$ with $k = \text{MaxIter}/2$ for each of these stepsizes, we select the largest number for which $f(x^k)$ is evaluated, i.e., not NaN, to be our fine-tuned stepsize.

In this experiment, we assume that the problem (85) is the subproblem of the cubic regularized Newton method for solving the logistic regression problem (84). We thus generate the gradient g and the Hessian H for logistic regression problem evaluated at 0 for different values of M . Specifically, we consider $M = \{10, 15\}$ for all three datasets. The dimension n stays the same as in Table 3. Moreover, the stopping criterion is similar to the one outlined in Subsection 6.1. We run all algorithms for a given number of iterations. Specifically, MaxIter is set to 30 for mushrooms and w8a datasets, and to 1000 for the covtype dataset, because the covtype dataset is around 90 times larger than the mushrooms dataset and 30 times larger than the w8a dataset and it requires more iterations to solve. Note that we here set a limited number of iterations. To avoid AdGD and AdaPGM requiring several initial iterations to allow for the small $\alpha_0 = 10^{-10}$ to grow to an appropriate step, we first calculate x^1 and L_1 defined in (5) for the given α_0 and then reset α_0 to $1/(\sqrt{2}L_1)$.

We first compare the four AdaBB variants in Table 2. The results are shown in Figure 5, which again confirms that AdaBB and AdaBB3 are usually better than the other two variants, and thus indicates the effectiveness of opting for Option II in (Case ii).

In Figure 6, we show the comparison of AdaBB and AdaBB3 with GD, AdGD and AdaPGM, from which we see again that AdaBB and AdaBB3 are usually better than the other three algorithms. Moreover, we also draw the stepsizes generated by AdGD and AdaBB in Figure 7.

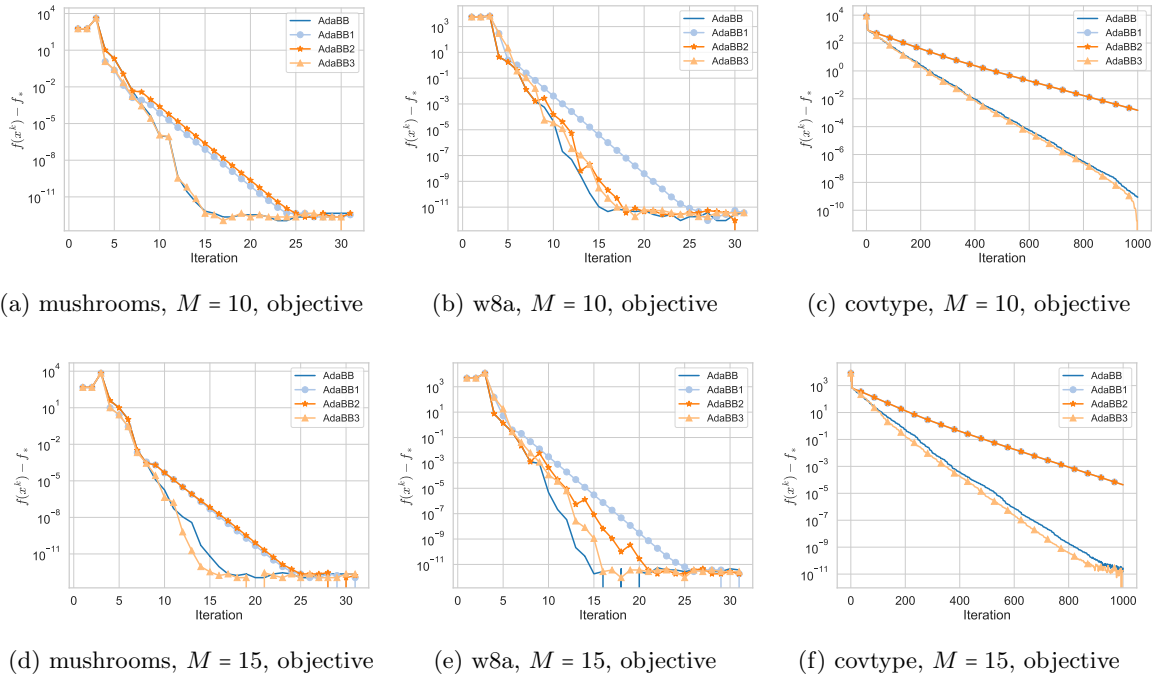


Figure 5: Results for the cubic regulation problem for AdaBB, AdaBB1, AdaBB2, AdaBB3 concerning the function value residual.

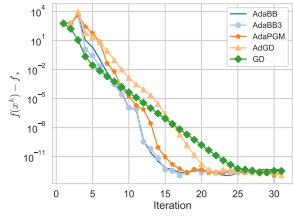
From Figure 7 we obtain similar observations as the ones in the previous subsection.

7 Concluding Remarks

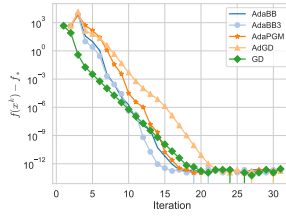
In this paper, we proposed an adaptive BB method for solving unconstrained smooth convex optimization. The proposed AdaBB algorithm is line-search-free and parameter-free. It essentially provides a convergent variant of the BB method for general unconstrained convex optimization. Comparing other adaptive algorithms along the same line of research, our AdaBB achieves the best lower bounds on the stepsize and the average of the stepsizes, which also affirmatively answers an open question posed by Malitsky and Mishchenko [MM23]. Our numerical results show the superior performance of two versions of AdaBB which takes the BB stepsize directly when it is not too large and not too small. This confirms the great potential of using BB stepsize in practice, under the condition that some safeguard procedure ensuring the convergence is taken, just like our AdaBB algorithm.

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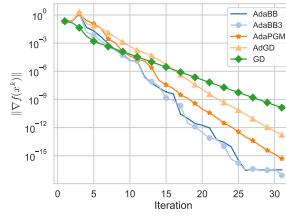
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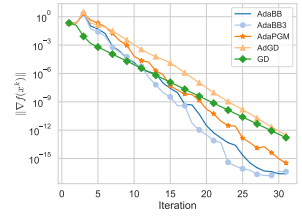
(a) mushrooms, $M = 10$



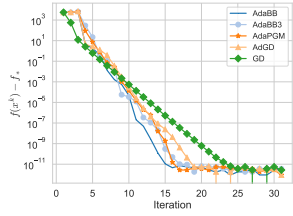
(b) mushrooms, $M = 15$



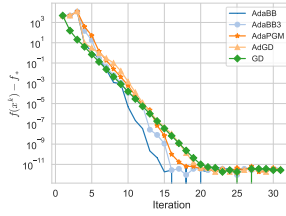
(c) mushrooms, $M = 10$



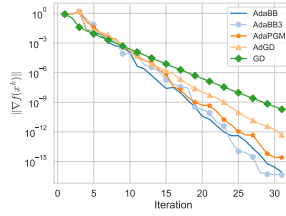
(d) mushrooms, $M = 15$



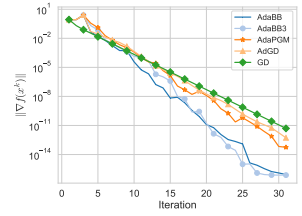
(e) w8a, $M = 10$



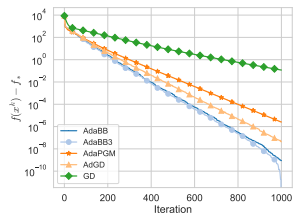
(f) w8a, $M = 15$



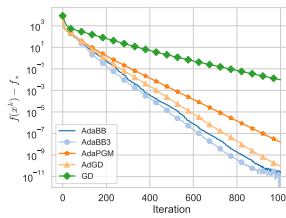
(g) w8a, $M = 10$



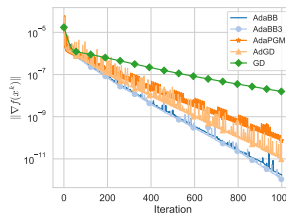
(h) w8a, $M = 15$



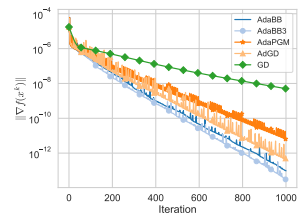
(i) covtype, $M = 10$



(j) covtype, $M = 15$



(k) covtype, $M = 10$



(l) covtype, $M = 15$

Figure 6: Results for the cubic regulation problem for GD, AdGD, AdaPGM, and AdaBB concerning the function value residual and gradient norm.

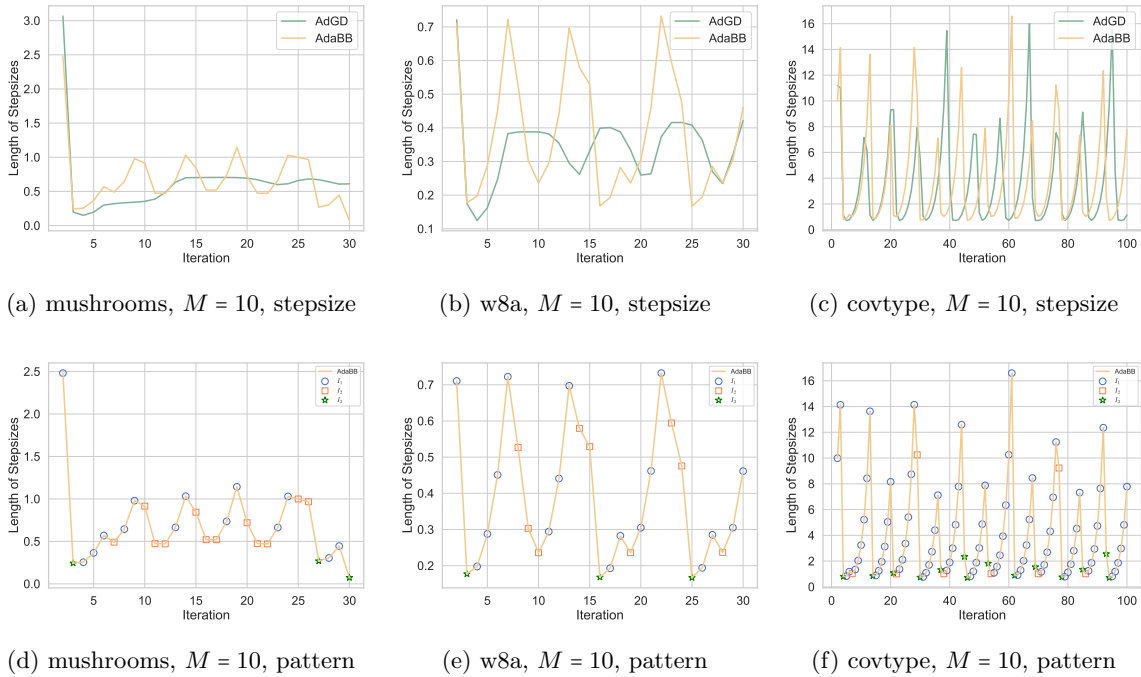


Figure 7: Stepsizes generated by AdGD and AdaBB.

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