ON RANK-MONOTONE GRAPH OPERATIONS AND MINIMAL OBSTRUCTION GRAPHS FOR THE LOVÁSZ–SCHRIJVER SDP HIERARCHY

YU HIN (GARY) AU AND LEVENT TUNÇEL

Abstract. We study the lift-and-project rank of the stable set polytopes of graphs with respect to the Lovász–Schrijver SDP operator LS_+, with a particular focus on finding and characterizing the smallest graphs with a given LS_+-rank (the least number of iterations of the LS_+ operator on the fractional stable set polytope to compute the stable set polytope). We introduce a generalized vertex-stretching operation that appears to be promising in generating LS_+-minimal graphs and study its properties. We also provide several new LS_+-minimal graphs, most notably the first known instances of 12-vertex graphs with LS_+-rank 4, which provides the first advance in this direction since Escalante, Montelar, and Nasini's discovery of a 9-vertex graph with LS_+-rank 3 in 2006.

1. Introduction

Given a simple, undirected graph $G = (V(G), E(G))$, we say that $S \subseteq V(G)$ is a *stable set* if no two vertices in S are joined by an edge. The *(maximum) stable set problem*, which aims to find a stable set of maximum cardinality in a given graph G, is one of the most well-studied problems in combinatorial optimization. While this problem is $\mathcal{NP}$-hard, a standard approach for tackling the problem is to associate stable sets of G with points in $\mathbb{R}^{V(G)}$, and model it as a convex optimization problem. Given a set $S \subseteq V(G)$, its *incidence vector* $\chi_S \in \{0,1\}^{V(G)}$ is defined so that $[\chi_S]_i = 1$ if $i \in S$, and $[\chi_S]_i = 0$ otherwise. Then we define the *stable set polytope* of a given graph G to be the convex hull of the incidence vectors of stable sets of G:

$$\text{STAB}(G) \coloneqq \text{conv}\left(\{\chi_S : \ S \subseteq V(G) \text{ is a stable set of } G\}\right).$$

Observe that if we let $\alpha(G)$ be the cardinality of a maximum stable set in G, then

$$\alpha(G) = \max\left\{\sum_{i \in V(G)} x_i : x \in \text{STAB}(G)\right\}. \tag{1}$$

While (1) is a linear program, considering again that the underlying combinatorial problem is $\mathcal{NP}$-hard, it is a difficult task to find an explicit description (e.g., via listing its facets) of $\text{STAB}(G)$ for a general graph G. This naturally leads to the pursuit of "nice" convex relaxations of $\text{STAB}(G)$. Below we list several desirable characteristics of such a convex relaxation P:

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Yu Hin (Gary) Au: Corresponding author. Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, Saskatchewan, S7N 5E6 Canada. E-mail: gary.au@usask.ca.

Levent Tunçel: Research of this author was supported in part by an NSERC Discovery Grant. Department of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1 Canada. E-mail: levent.tuncel@uwaterloo.ca.

- $P \cap \{0,1\}^{V(G)} = \text{STAB}(G) \cap \{0,1\}^{V(G)}$. That is, a 0-1 vector is in P if and only if it is the incidence vector of a stable set in G.
- P is tractable. That is, one can optimize a linear function over P with arbitrary precision in polynomial time.
- P is a "strong" relaxation. This is a relatively subjective measure, and can mean that important families of valid inequalities of STAB(G) are also valid for P, and/or that $\max \left\{ \sum_{i \in V(G)} x_i : x \in P \right\}$ is "close" to $\alpha(G)$.

One of the simplest convex relaxations of STAB(G) is the *fractional stable set polytope*

$$\text{FRAC}(G) \coloneqq \left\{ x \in [0,1]^{V(G)} : x_i + x_j \leq 1, \forall \{i,j\} \in E(G) \right\}.$$

While FRAC(G) is both a correct and tractable relaxation, it is rather weak in general. For a stronger relaxation, we call $C \subseteq V(G)$ a *clique* if every pair of vertices in C is joined by an edge. Then notice that, for every clique C, the inequality

$$\sum_{i \in C} x_i \leq 1$$

is valid for STAB(G). Thus, if we define the *clique polytope*

$$\text{CLIQ}(G) \coloneqq \left\{ x \in [0,1]^{V(G)} : \sum_{i \in C} x_i \leq 1 \text{ for every clique } C \subseteq V(G) \right\},$$

then $\text{STAB}(G) \subseteq \text{CLIQ}(G) \subseteq \text{FRAC}(G)$ for every graph G. (For the second containment. observe that every edge is a clique of size 2.) However, while CLIQ(G) is a stronger relaxation than FRAC(G), it is not tractable in general.

In this manuscript, we focus on semidefinite relaxations of STAB(G) produced by LS_+, a lift-and-project operator devised by Lovász and Schrijver [LS91] which we will fully define in Section 2. (The operator has also been referred to as N_+ in the literature.) Given a graph G, the LS_+ operator generates a sequence of relaxations $\text{LS}_+^k(G)$ which satisfies

$$\text{FRAC}(G) =: \text{LS}_+^0(G) \supseteq \text{LS}_+^1(G) \supseteq \text{LS}_+^2(G) \supseteq \cdots \supseteq \text{LS}_+^{|V(G)|}(G) = \text{STAB}(G).$$

(We will usually refer to $\text{LS}_+^1(G)$ as simply $\text{LS}_+(G)$.) When $k \in O(1)$, $\text{LS}_+^k(G)$ can be described as the feasible region of a semidefinite program whose number of variables and constraints are polynomial in the size of the number of vertices and edges in G, and thus the relaxation is indeed tractable in this case. Moreover, the first relaxation $\text{LS}_+(G)$ already satisfies many well-known families of valid inequalities of STAB(G), including (among others) the aforementioned clique inequalities, odd hole and odd antihole constraints, odd wheel constraints, and orthogonality constraints imposed by the Lovász theta body [Lov79].

The hierarchy of relaxations generated by LS_+ gives rise to the notion of the LS_+*-rank* of a graph G, which is defined to be the smallest integer k where $\text{LS}_+^k(G) = \text{STAB}(G)$, and gives us a measure of how difficult the stable set problem is for the LS_+ operator. It is well-known that a graph G has LS_+-rank 0 (i.e., satisfies $\text{FRAC}(G) = \text{STAB}(G)$) if and only if G is bipartite. Some families of graphs that are known to have LS_+-rank 1 (i.e., satisfies $\text{LS}_+(G) = \text{STAB}(G)$) include — but are not limited to — odd cycles, odd antiholes, odd wheels, and perfect graphs (which are defined to be graphs where $\text{CLIQ}(G) = \text{STAB}(G)$). In the last decade, considerable progress has been made in finding a combinatorial characterization of graphs with LS_+-rank 1 — see, for instance, [BENT13, BENT17, Wag22, BENW23].

Nevertheless, since the maximum stable problem is $\mathcal{NP}$-hard, there has to be graphs with unbounded LS_+-rank. The first family of graphs that have unbounded LS_+-rank was obtained by Stephen and the second author [ST99], who showed that the line graph of the complete graph on $2k+1$ vertices has LS$_+$-rank k, giving a family of graphs G whose LS$_+$-rank is asymptotically $\Omega(\sqrt{|V(G)|})$. On the other hand, Lipták and the second author [LT03] showed the following:

Theorem 1. *For every graph G, the* LS$_+$*-rank of G is at most* $\left\lfloor \frac{|V(G)|}{3} \right\rfloor$.

This begs the natural question: For every integer $\ell \geq 1$, is there a graph on 3ℓ vertices which has LS$_+$-rank ℓ? If these graphs exist, their extremal nature (in terms of being the smallest possible graphs with a given LS$_+$-rank) may help reveal the critical structures that expose the limitations of these LS$_+$-relaxations. This understanding could be extremely helpful when it comes to analyzing other convex relaxations of the maximum stable set problem, particularly those which are produced by other lift-and-project methods.

This direction of investigation was already set in the seminal paper [LS91] and questions about the behaviour of LS$_+$-rank under simple graph operations were also raised in [GT01]. In the same general direction of research, Laurent [Lau02] analyzed the LS$_+$-rank and related ranks in the context of the maximum cut problem by establishing nice behaviour (only in the context of maximum cut problems) of the underlying lift-and-project operators under graph minor operations; also see [Lau03] for an analysis of the Lasserre operator. However, as it was illustrated in some depth in [LT03], the LS$_+$-rank of a graph does not behave in a nice, uniform way under the usual graph minor operations for the stable set problem. Therefore, a deeper investigation is necessary to construct the kind of graph operations which would be helpful in discovering and understanding minimal obstructions to tractable convex relaxations of the stable set polytope obtained by LS$_+$ or other convex optimization based lift-and-project hierarchies. Overall, the importance of the quest to understand minimal obstructions to families of SDP relaxations in particular — and convex relaxations in general — has been raised by many others. For example, Knuth, in his well-known survey "The Sandwich Theorem" [Knu94] poses six open problems in the general context of Lovász theta function. Two of the six open problems concern LS$_+$(FRAC(G)). One of them asks for finding what we call below a 2-minimal graph (answered in [LT03]).

We say that a graph G is *ℓ-minimal* if $|V(G)| = 3\ell$ and G has LS$_+$-rank ℓ. It is known that ℓ-minimal graphs exist for $\ell \in \{1, 2, 3\}$. For $\ell = 1$, it is easy to see that the 3-cycle is the only 1-minimal graph. The first 2-minimal graph ($G_{2,1}$ in Figure 1) was found by Lipták and the second author [LT03], who also conjectured that ℓ-minimal graphs exist for all $\ell \in \mathbb{N}$. Subsequently, Escalante, Montelar, and Nasini [EMN06] showed that there is only one other 2-minimal graph ($G_{2,2}$ in Figure 1), while providing the first example of a 3-minimal graph ($G_{3,1}$ in Figure 1). (The logic behind the seemingly odd choice of vertex labels in the figures of this section will be explained in Section 4 when we introduce the vertex-stretching operation.)

In producing the first 3-minimal graph, Escalante et al. [EMN06] also showed that there does not exist an ℓ-minimal graph for any $\ell \geq 4$ if we restrict ourselves to graphs that can be obtained by starting with a complete graph and replacing every edge by a path of length at least 1. (Let K_n denote the complete graph on n vertices. Notice that $G_{2,1}$ and $G_{3,1}$ can be respectively obtained from K_4 and K_5 by replacing some edges with paths of length 3.)

Recently, the authors [AT23] discovered several family of graphs G for which the LS$_+$-rank of G is $\Omega(|V(G)|)$. One of them is the family of graphs H_k, which is defined as follows. Given $k \in \mathbb{N}$, let $[k]$ denote the set $\{1, 2, \ldots, k\}$. For every $k \geq 3$, let

$$V(H_k) := \{i_0, i_1, i_2 : i \in [k]\}$$

and

$$E(H_k) := \{\{i_1, i_0\}, \{i_0, i_2\} : i \in [k]\} \cup \{i_1, j_2 : i, j \in [k], i \neq j\}.$$

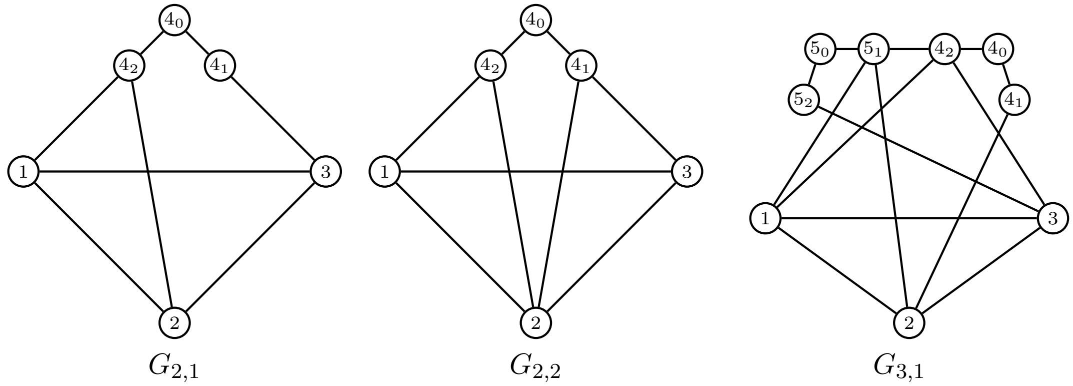

FIGURE 1. Known 2- and 3-minimal graphs due to [LT03] and [EMN06]

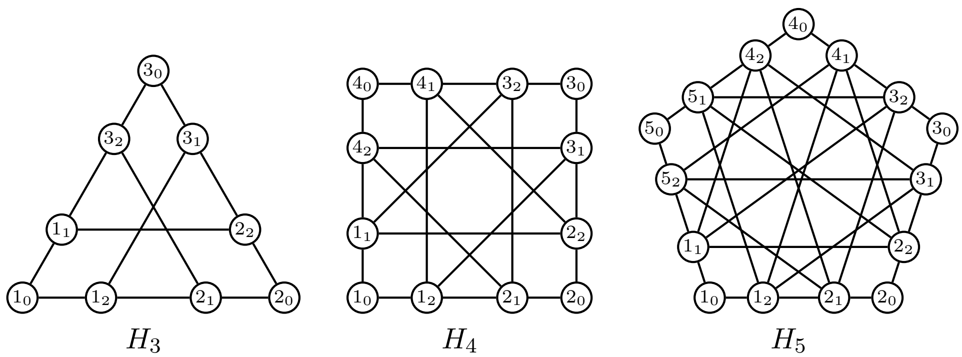

FIGURE 2. Several graphs in the family H_k

Figure 2 illustrates the graphs H_k for $k = 3, 4, 5$. (Note that our vertex labels for H_k are different from that in [AT23].) The authors [AT23, Theorem 2] proved the following.

Theorem 2. *For every $k \geq 3$, The* LS_+*-rank of H_k is at least $\frac{3k}{16}$.*

Theorem 2 (and other results in [AT23]) ended a 17-year lull in new hardness results for LS_+-relaxations of the stable set problem, and provides renewed hope that ℓ-minimal graphs do exist for $\ell \geq 4$. Indeed, one of the main contributions of this work is the discovery of what we believe to be the first known instance of a 4-minimal graph ($G_{4,1}$ in Figure 3).

This paper is organized as follows. In Section 2, we define the LS_+ operator and introduce some of the tools and notation we will need for our subsequent analysis. Then, in Section 3, we discuss what we call star-homomorphism between graphs, and provide a template for constructing graph operations that are LS_+-rank non-decreasing. Using this template, we define our vertex-stretching operation in Section 4, which generalizes similar graph operations studied previously [LT03, AEF14, BENT17]. We then show in Section 5 that every ℓ-minimal graph for $\ell \geq 2$ must be obtained from applying our vertex-stretching operation to a smaller graph, and in particular study the LS_+-ranks of graphs obtained from stretching the vertices of a complete graph. In Section 6, we prove that $G_{4,1}$ indeed has LS_+-rank 4 and discuss some of the immediate consequences of the result, which includes the discovery of several other new 3- and 4-minimal graphs. We then revisit the aforementioned families of graphs H_k in Section 7, and apply our results on vertex stretching to show that there exists a family of graphs G with maximum degree

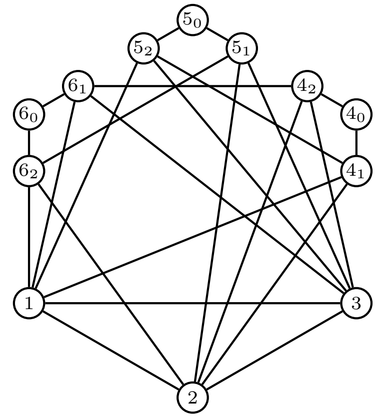

FIGURE 3. $G_{4,1}$, a 12-vertex graph with LS$_+$-rank 4

3 and LS$_+$-rank $\Omega(\sqrt{|V(G)|})$. Finally, we conclude our paper in Section 8 by mentioning some natural future research directions.

2. Preliminaries

In this section, we define the lift-and-project operator LS$_+$ due to Lovász and Schrijver [LS91] and the convex relaxations of STAB(G) it produces, as well as go over the basic tools we will use in subsequent sections to analyze the LS$_+$-rank of graphs.

2.1. **The LS$_+$-operator.**

Given a set $P \subseteq [0,1]^n$, we define the *homogenized cone* of P to be

$$\operatorname{cone}(P) := \left\{ \begin{bmatrix} \lambda \\ \lambda x \end{bmatrix} : \lambda \geq 0, x \in P \right\}.$$

Notice that $\operatorname{cone}(P) \subseteq \mathbb{R}^{n+1}$, and we will refer to the new coordinate with index 0. Next, given a vector x and an index i, we may refer to the i-entry in x by x_i or $[x]_i$. All vectors are column vectors by default, so $x^\top$, the transpose of a vector x, is a row vector. Next, let $\mathbb{S}_+^n$ denote the set of n-by-n real symmetric positive semidefinite matrices, and $\operatorname{diag}(Y)$ be the vector formed by the diagonal entries of a square matrix Y. We also let e_i be the i^{th} unit vector.

Given $P \subseteq [0,1]^n$, the operator LS$_+$ first *lifts* P to the following set of matrices:

$$\widehat{\text{LS}}_+(P) := \left\{ Y \in \mathbb{S}_+^{n+1} : Ye_0 = \operatorname{diag}(Y), Ye_i, Y(e_0 - e_i) \in \operatorname{cone}(P)\ \forall i \in [n] \right\}.$$

It then *projects* the set back down to the following set in $\mathbb{R}^n$:

$$\text{LS}_+(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \widehat{\text{LS}}_+(P), Ye_0 = \begin{bmatrix} 1 \\ x \end{bmatrix} \right\}.$$

Given $x \in \text{LS}_+(P)$, we say that $Y \in \widehat{\text{LS}}_+(P)$ is a *certificate matrix* for x if $Ye_0 = \begin{bmatrix} 1 \\ x \end{bmatrix}$. Also, given a set $P \subseteq [0,1]^n$, we define

$$P_I := \operatorname{conv}\left(P \cap \{0,1\}^n\right)$$

to be the *integer hull* of P. The following is a well-known and foundational property of LS$_+$.

Lemma 3. *For every set $P \subseteq [0,1]^n$, $P_I \subseteq \text{LS}_+(P) \subseteq P$.*

Proof. For the first containment, let $x \in P \cap \{0,1\}^n$. Observe that $Y := \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^\top \in \widehat{\mathrm{LS}}_+(P)$, and so $x \in \mathrm{LS}_+(P)$. For the second containment, let $x \in \mathrm{LS}_+(P)$, and $Y \in \widehat{\mathrm{LS}}_+(P)$ be a certificate matrix for x. Since $Ye_0 = Ye_i + Y(e_0 - e_i)$ for any index $i \in [n]$ and that $\widehat{\mathrm{LS}}_+$ imposes that $Ye_i, Y(e_0 - e_i) \in \mathrm{cone}(P)$, it follows that $Ye_0 = \begin{bmatrix} 1 \\ x \end{bmatrix} \in \mathrm{cone}(P)$, and thus $x \in P$. □

Therefore, $\mathrm{LS}_+(P)$ contains the same set of integral solutions as P. Also, if P is tractable, then so is $\mathrm{LS}_+(P)$, and it is known that $P \supset \mathrm{LS}_+(P)$ unless $P = P_I$. Thus, $\mathrm{LS}_+(P)$ offers a tractable relaxation of P_I that is tighter than the initial relaxation P.

Furthermore, we can apply LS_+ multiple times to obtain yet tighter relaxations. Given $k \in \mathbb{N}$, let $\mathrm{LS}_+^k(P)$ be the set obtained from applying k successive LS_+ operations to P. (We also let $\mathrm{LS}_+^0(P) := P$.) Then it is well known that

$$P_I = \mathrm{LS}_+^n(P) \subseteq \mathrm{LS}_+^{n-1}(P) \subseteq \cdots \subseteq \mathrm{LS}_+(P) \subseteq P.$$

Thus, LS_+ generates a hierarchy of progressively tighter convex relaxations which converge to P_I in no more than n iterations. The reader may refer to Lovász and Schrijver [LS91] for a proof of this fact and some other properties of the LS_+ operator.

2.2. **Analyzing the LS_+-rank of a graph.** Recall that $\mathrm{FRAC}(G)$, the fractional stable set polytope of a graph G, offers a simple and tractable convex relaxation of $\mathrm{STAB}(G)$. Thus, we could apply LS_+ to obtain stronger relaxations of $\mathrm{STAB}(G)$ than $\mathrm{FRAC}(G)$. Given an integer $k \geq 0$, define

$$\mathrm{LS}_+^k(G) := \mathrm{LS}_+^k(\mathrm{FRAC}(G)),$$

and let $r_+(G)$ denote the LS_+-rank of G (which, again, is the smallest integer k where $\mathrm{LS}_+^k(G) = \mathrm{STAB}(G)$).

To show that a graph G has LS_+-rank at least p, the standard approach is to find a point $\bar{x}$ where $\bar{x} \notin \mathrm{STAB}(G)$ and $\bar{x} \in \mathrm{LS}_+^{p-1}(G)$ — this is the approach we will take when verifying that $r_+(G_{4,1}) \geq 4$. We do remark that verifying $\bar{x} \in \mathrm{LS}_+^{p-1}(G)$ tends to get progressively more challenging as p increases, unless the symmetries of G allow for an inductive argument (which is the case for the line graphs of odd cliques [ST99], and to a lesser extent for H_k and related graphs [AT23]). Given $p \in \mathbb{N}$, we also define

$$\alpha_{\mathrm{LS}_+^p}(G) := \max\left\{ \bar{e}^\top x : x \in \mathrm{LS}_+^p(G) \right\},$$

where $\bar{e}$ denotes the vector of all-ones. Notice that if $\alpha_{\mathrm{LS}_+^p}(G) > \alpha(G)$, then $r_+(G) \geq p+1$.

Next, the following is a well-known property of LS_+.

Lemma 4. *Let $P \subseteq [0,1]^n$ be a polyhedron, and F be a face of $[0,1]^n$. Then*

$$\mathrm{LS}_+(P \cap F) = \mathrm{LS}_+(P) \cap F.$$

It follows from Lemma 4 that if $\bar{x} \in \mathrm{LS}_+^p(G)$ and G' is an induced subgraph of G, then the vector obtained from $\bar{x}$ by removing entries not in $V(G')$ is in $\mathrm{LS}_+^p(G')$. This in turn implies that $r_+(G') \leq r_+(G)$ — see, for instance, [AT23, Lemma 5] for a proof.

We next mention several other ways of bounding $r_+(G)$ using the LS_+-rank of graphs that are related to G. Given a graph G and $S \subseteq V(G)$, we let $G - S$ denote the subgraph of G induced by the vertices $V(G) \setminus S$, and call $G - S$ the graph obtained by the *deletion* of S. (When $S = \{i\}$ for some vertex i, we simply write $G - i$ instead of $G - \{i\}$.) Next, given $i \in V(G)$, let

$$\Gamma_G(i) := \{j \in V(G) : \{i,j\} \in E(G)\}$$

be the *open neighborhood* of i in G, and $\Gamma_G[i] \coloneqq \Gamma_G(i) \cup \{i\}$ be the *closed neighborhood* of i in G. Then the graph obtained from the *destruction* of i in G is defined as

$$G \ominus i \coloneqq G - \Gamma[i].$$

Then we have the following.

Theorem 5. *For every graph G,*

(i) [LS91, Corollary 2.16] $r_+(G) \leq \max\{r_+(G \ominus i) : i \in V(G)\} + 1$*;*
(ii) [LT03, Theorem 36] $r_+(G) \leq \min\{r_+(G - i) : i \in V(G)\} + 1$.

Recall that $r_+(G) = 0$ if and only if G is bipartite (in which case $\mathrm{FRAC}(G) = \mathrm{STAB}(G)$). Thus, it follows immediately from Theorem 5(ii) that if G is non-bipartite but $G - i$ is bipartite for some $i \in V(G)$, then $r_+(G) = 1$ — an example for such graphs is the odd cycles. Likewise, if G is non-bipartite while $G \ominus i$ is bipartite for every $i \in V(G)$, then $r_+(G) = 1$ as well — such as when G is an odd antihole (i.e., the graph complement of an odd cycle of length at least 5), or an odd wheel (i.e., the graph obtained from joining a vertex to every vertex of an odd cycle of length at least 5).

We say that a graph G is *perfect* if $\mathrm{CLIQ}(G) = \mathrm{STAB}(G)$. In terms of forbidden subgraphs, G is perfect if and only if it does not contain an induced subgraph that is an odd hole (i.e., an odd cycle of length at least 5) or an odd antihole [CRST06]. Since $\mathrm{LS}_+(G) \subseteq \mathrm{CLIQ}(G)$ in general [LS91], it follows that $r_+(G) \leq 1$ if G is perfect.

The following is a restatement of [LT03, Lemma 5].

Proposition 6. *Let G be a graph, and $S_1, S_2, C \subseteq V(G)$ are mutually disjoint subsets such that*

- $S_1 \cup S_2 \cup C = V(G)$*;*
- *C induces a clique in G;*
- *There is no edge $\{i,j\} \in E(G)$ where $i \in S_1, j \in S_2$.*

Then $r_+(G) = \max\{r_+(G - S_1), r_+(G - S_2)\}$.

Thus, if G has a cut clique (i.e., a clique C where $G - C$ has multiple components), then the LS_+-rank of G is equal to that of one of its proper subgraphs.

Finally, it is clear from the definition of LS_+ that if $P_1 \subseteq P_2$, then $\mathrm{LS}_+(P_1) \subseteq \mathrm{LS}_+(P_2)$. This implies the following.

Lemma 7. *Given graphs G, H where $V(G) = V(H)$ and $E(G) \subseteq E(H)$,*

(i) *If $a^\top x \leq \beta$ is valid for $\mathrm{LS}_+^p(G)$, then $a^\top x \leq \beta$ is valid for $\mathrm{LS}_+^p(H)$;*
(ii) *If $a^\top x \leq \beta$ is not valid for $\mathrm{LS}_+^p(H)$, then $a^\top x \leq \beta$ is not valid for $\mathrm{LS}_+^p(G)$.*

3. Star-homomorphic graphs

In this section, we introduce the notion of two graphs being star-homomorphic, and describe how the LS_+-relaxations of such a pair of graphs are related. Given a graph $G = (V(G), E(G))$, we define the graph $G^\dagger$ where

$$V(G^\dagger) \coloneqq \left\{ i, \bar{i} : i \in V(G) \right\},$$
$$E(G^\dagger) \coloneqq E(G) \cup \left\{ \{i, \bar{i}\} : i \in V(G) \right\}.$$

In other words, we obtain $G^\dagger$ from G by adding a new vertex $\bar{i}$ for every $i \in V(G)$, and then adding an edge between $\bar{i}$ and i. Figure 4 provides an example of constructing $G^\dagger$ from G.

Also, given graphs G and H, we say that $g : V(H) \to V(G)$ is a *homomorphism* if, for all $i, j \in V(H)$,

$$\{i,j\} \in E(H) \Rightarrow \{g(i), g(j)\} \in E(G).$$

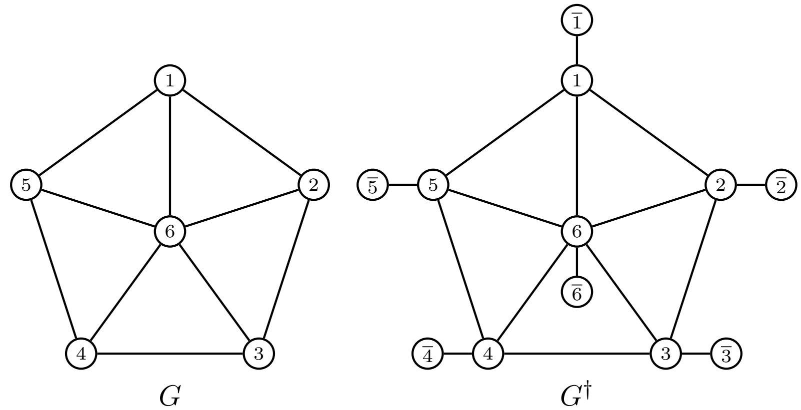

FIGURE 4. Constructing $G^\dagger$ from G

Next, given graphs G and H, if there exists a homomorphism $g : V(H) \to V(G^\dagger)$, then we say that H is *star-homomorphic to G under g*.

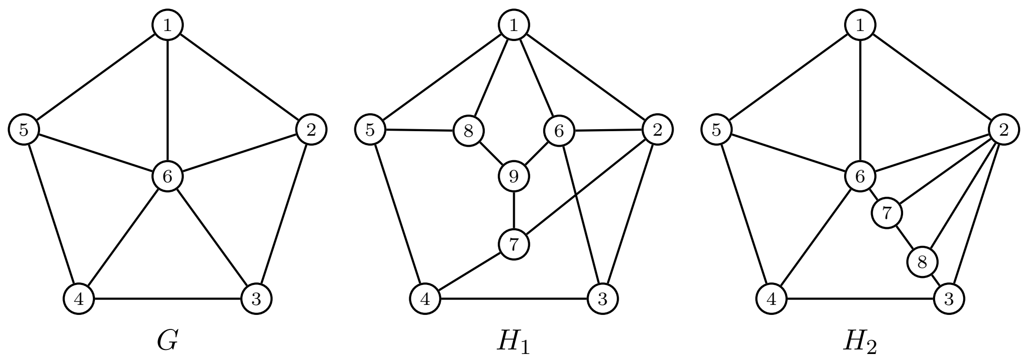

FIGURE 5. Two graphs H_1, H_2 that are star-homomorphic to G

Example 8. Consider the graphs in Figure 5. Then H_1 is star-homomorphic to G under g_1 where

$j \in V(H_1)$	1	2	3	4	5	6	7	8	9
$g_1(j) \in V(G^\dagger)$	1	2	3	4	5	6	6	6	$\bar{6}$

Likewise, H_2 is star-homomorphic to G under g_2 where

$j \in V(H_2)$	1	2	3	4	5	6	7	8
$g_2(j) \in V(G^\dagger)$	1	2	3	4	5	6	3	6

Given that H is star-homomorphic to G under g and $x \in \mathbb{R}^{V(G)}$, we let $\tilde{g}(x) \in \mathbb{R}^{V(H)}$ be the vector where

$$[\tilde{g}(x)]_j \coloneqq \begin{cases} x_i & \text{if } g(j) = i; \\ 1 - x_i & \text{if } g(j) = \bar{i}. \end{cases}$$

While the following property of $\tilde{g}$ follows readily from the definition of star-homomorphism, it is worth stating explicitly.

Lemma 9. *Suppose H is star-homomorphic to G under g, and $x \in \mathbb{R}^{V(G)}$. If $x \in \text{FRAC}(G)$, then $\tilde{g}(x) \in \text{FRAC}(H)$.*

Proof. First, it is easy to see that $x \in [0,1]^{V(G)}$ implies $\tilde{g}(x) \in [0,1]^{V(H)}$. Now given edge $\{j_1.j_2\} \in E(H)$, $\{g(j_1), g(j_2)\}$ is either an edge in G or $\{i, \bar{i}\}$ for some $i \in V(G)$. In both cases, we see that $[\tilde{g}(x)]_{j_1} + [\tilde{g}(x)]_{j_2} \leq 1$. □

In fact, the implication in Lemma 9 is preserved under applications of LS_+.

Proposition 10. *Suppose H is star-homomorphic to G under g, and $x \in \mathbb{R}^{V(G)}$. If $x \in \mathrm{LS}_+^p(G)$, then $\tilde{g}(x) \in \mathrm{LS}_+^p(H)$.*

Proof. Suppose $x \in \mathrm{LS}_+^p(G)$. We prove that $\tilde{g}(x) \in \mathrm{LS}_+^p(H)$ by induction on p. The base case $p = 0$ reduces to Lemma 9. Next, suppose $p \geq 1$, and let $Y \in \widehat{\mathrm{LS}}_+^p(G)$ be a certificate matrix. For convenience, we also extend the function $\tilde{g}$ as follows: Given a real number $k \geq 0$, define $\tilde{g}_k : \mathbb{R}^{\{0\} \cup V(G)} \to \mathbb{R}^{\{0\} \cup V(H)}$ such that

$$[\tilde{g}_k(x)]_j := \begin{cases} x_0 & \text{if } j = 0; \\ x_i & \text{if } j \in V(H) \text{ and } g(j) = i; \\ k - x_i & \text{if } j \in V(H) \text{ and } g(j) = \bar{i}. \end{cases}$$

Notice that the function $\tilde{g}_k$ satisfies

$$\tilde{g}_\lambda\left(\begin{bmatrix} \lambda \\ \lambda x \end{bmatrix}\right) = \begin{bmatrix} \lambda \\ \lambda \tilde{g}(x) \end{bmatrix}$$

for every $\lambda \geq 0$ and $x \in \mathbb{R}^{V(G)}$.

Since $Y \in \widehat{\mathrm{LS}}_+^p(G)$, $Ye_i, Y(e_0 - e_i) \in \operatorname{cone}\left(\mathrm{LS}_+^{p-1}(G)\right)$ for every $i \in V(G)$. Thus, applying the inductive hypothesis we have $\tilde{g}_{x_i}(Ye_i), \tilde{g}_{1-x_i}(Y(e_0 - e_i)) \in \operatorname{cone}\left(\mathrm{LS}_+^{p-1}(H)\right)$.

Next, define the matrix $U \in \mathbb{R}^{(\{0\} \cup V(G)) \times (\{0\} \cup V(H))}$ where

$$Ue_j := \begin{cases} e_0 & \text{if } j = 0; \\ e_i & \text{if } j \in V(H) \text{ and } g(j) = i; \\ e_0 - e_i & \text{if } j \in V(H) \text{ and } g(j) = \bar{i}. \end{cases}$$

Then, given $z \in \mathbb{R}^{\{0\} \cup V(G)}$, $U^\top z = \tilde{g}_{z_0}(z)$.

Next, we claim that the matrix $Y' := U^\top Y U \in \widehat{\mathrm{LS}}_+^p(H)$. First, since $Y = Y^\top$, $\operatorname{diag}(Y) = Ye_0$, and $Y \succeq 0$, it is easy to see that the corresponding properties also hold for Y'. We next show that $Y'e_j, Y'(e_0 - e_j) \in \operatorname{cone}\left(\mathrm{LS}_+^{p-1}(H)\right)$ for every $j \in V(H)$. First, if $g(j) = i \in V(G)$, then $Y'e_j = \tilde{g}_{x_i}(Ye_i)$, and

$$Y'(e_0 - e_j) = Y'e_0 - Y'e_j = \tilde{g}_1(Ye_0) - \tilde{g}_{x_i}(Ye_i) = \tilde{g}_{1-x_i}(Y(e_0 - e_i)).$$

To see the last equality, notice that

$$\begin{aligned} &[\tilde{g}_1(Ye_0) - \tilde{g}_{x_i}(Ye_i)]_\ell = [\tilde{g}_{1-x_i}(Y(e_0 - e_i))]_\ell \\ =&\begin{cases} 1 - x_i & \text{if } \ell = 0; \\ x_{i'} - Y[i', i] & \text{if } g(\ell) = i' \in V(G); \\ 1 - x_{i'} - x_i + Y[i', i] & \text{if } g(\ell) = \bar{i'} \text{ for some } i' \in V(G). \end{cases} \end{aligned}$$

Likewise, if $g(j) = \bar{i}$ for some $i \in V(G)$, then $Y'e_j = \tilde{g}_{1-x_i}(Y(e_0 - e_i))$ and $Y'(e_0 - e_j) = \tilde{g}_{x_i}(Ye_i)$. In all cases, it follows from the inductive hypothesis that $Y'e_j, Y'(e_0 - e_j) \in \operatorname{cone}\left(\mathrm{LS}_+^{p-1}(H)\right)$. Therefore, $Y' \in \widehat{\mathrm{LS}}_+^p(H)$. Since $Y'e_0 = \tilde{g}_1(Ye_0) = \begin{bmatrix} 1 \\ \tilde{g}(x) \end{bmatrix}$, it follows that $\tilde{g}(x) \in \mathrm{LS}_+^p(H)$. □

Proposition 10 helps establish a framework for bounding the LS_+-rank of a graph by that of another.

Lemma 11. *Given graphs G and H where H is star-homomorphic to G under g, if*

$$x \notin \mathrm{STAB}(G) \Rightarrow \tilde{g}(x) \notin \mathrm{STAB}(H).$$

then $r_+(H) \geq r_+(G)$.

Proof. Suppose $r_+(G) = p \geq 1$ (the claim is trivial if $p = 0$). Then there exists $x \in \mathrm{LS}_+^{p-1}(G) \setminus \mathrm{STAB}(G)$. Then by the hypothesis and Proposition 10, $\tilde{g}(x) \in \mathrm{LS}_+^{p-1}(H) \setminus \mathrm{STAB}(H)$, showing that $r_+(H) \geq p$. □

Finally, while our focus for this paper is the LS_+ operator, we remark that the framework of star homomorphism can be extended to analyze relaxations generated by other lift-and-project operators. Again, let H be a graph that is star-homomorphic to G under g. Then notice that the function $\tilde{g} : \mathbb{R}^{V(G)} \to \mathbb{R}^{V(H)}$ can be expressed as a composition of the following four elementary operations:

(1) Deleting a coordinate. E.g., $L : \mathbb{R}^n \to \mathbb{R}^{n-1}$ where
$$L(x_1, x_2, \ldots, x_n) = L(x_2, \ldots, x_n).$$
(2) Swapping two coordinates. E.g., $L : \mathbb{R}^n \to \mathbb{R}^n$ where
$$L(x_1, x_2, x_3, \ldots, x_n) = L(x_2, x_1, x_3, \ldots, x_n).$$
(3) Cloning a coordinate. E.g., $L : \mathbb{R}^n \to \mathbb{R}^{n+1}$ where
$$L(x_1, x_2, \ldots, x_n) = L(x_1, x_1, x_2, \ldots, x_n).$$
(4) Flipping a coordinate. E.g., $L : \mathbb{R}^n \to \mathbb{R}^n$ where
$$L(x_1, x_2, \ldots, x_n) = L(1 - x_1, x_2, \ldots, x_n).$$

Now, let $\mathcal{L}$ be a lift-and-project operator. If one can show that

$$x \in \mathcal{L}(P) \Rightarrow L(x) \in \mathcal{L}(L(P)), \tag{2}$$

for every function L that belongs to one of the four categories above (where $L(P)$ denotes $\{L(z) : z \in P\}$), then one can prove the analogous version of Proposition 10 for $\mathcal{L}$. For instance, the ideas from the proof of Proposition 10 show that (2) holds for $\mathcal{L} \in \{\mathrm{LS}_+, \mathrm{LS}, \mathrm{LS}_0\}$ (where $\mathrm{LS}, \mathrm{LS}_0$ [LS91] are operators that generate linear relaxations which are generally weaker than LS_+). Also, it has been shown [AT18, Proposition 1] that the Lasserre operator Las [Las01] commutes with all automorphisms of the unit hypercube, a property that is also shared by the Sherali–Adams operator SA [SA90] and one of its PSD variants SA_+ [AT16]. Thus, these operators satisfy (2) as well, and much of what we show for LS_+ in this section and the next section also applies to these operators.

4. The (generalized) vertex-stretching operation

In this section, we introduce a graph operation that shows promise in producing relatively small graphs with high LS_+-ranks, and study some of its properties. Given a graph G, vertex $v \in V(G)$, and non-empty sets $A_1, \ldots, A_p \subset \Gamma_G(v)$ where $\bigcup_{\ell=1}^{p} A_\ell = \Gamma_G(v)$, we define the *stretching* of v in G by applying the following sequence of transformations to G:

- Replace v by $p + 1$ vertices: $v_0, v_1, \ldots, v_p$;
- For every $\ell \in [p]$, Join v_ℓ to v_0, as well as to all vertices in A_ℓ.

We will also refer to the operation as *p-stretching* when we would like to specify p (which is necessarily at least 2). For example, Figure 6 shows the graph obtained from 2-stretching vertex 5 in K_5 (with $A_1 = \{2,3,4\}$ and $A_2 = \{1,2,3\}$). For another example, observe that in Figure 5, the graph H_1 can be obtained by 3-stretching vertex 6 in G.

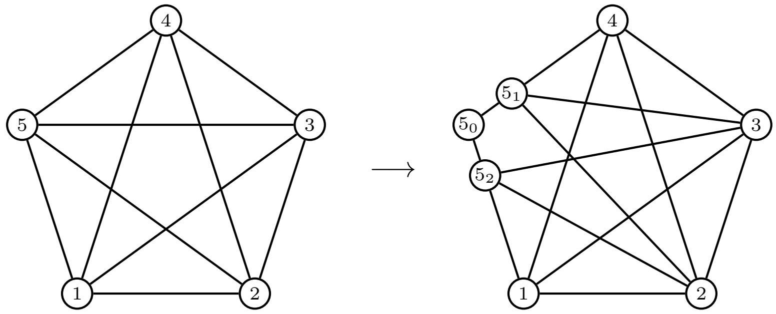

FIGURE 6. Demonstrating the vertex-stretching operation

We remark that our vertex stretching operation is a generalization of the type (i) stretching operation described in [LT03] and later studied in [AEF14] (which further requires that $p = 2$ and $A_1 \cap A_2 = \emptyset$), as well as the k-stretching operation described in [BENT17] (which further requires that $p = 2$ and the vertices $A_1 \cap A_2$ induce a clique of size k in G, with $k = 0$ allowed). Also, when $A_1, \ldots, A_p$ are mutually disjoint and at most one of these p sets has size greater than 1, our vertex stretching specializes to an instance of type (ii) stretching from [LT03].

Given a graph G and an integer $p \geq 2$, we define $\mathcal{S}_p(G)$ to be the set of graphs that can be obtained from G by p-stretching one vertex. Notice that every graph $H \in \mathcal{S}_p(G)$ is star-homomorphic to G under g where

$$g(j) \coloneqq \begin{cases} j & \text{if } j \in V(H \ominus v_0); \\ v & \text{if } j \in \{v_1, \ldots, v_p\}; \\ \overline{v} & \text{if } j = v_0. \end{cases} \tag{3}$$

We also define $\mathcal{S}(G) \coloneqq \bigcup_{p \geq 2} \mathcal{S}_p(G)$, and will show that $r_+(H) \geq r_+(G)$ for all $H \in \mathcal{S}(G)$ using Lemma 11. First, we need a tool that uses valid inequalities of $\text{STAB}(G)$ to generate potential valid inequalities for $\text{STAB}(H)$.

Lemma 12. *Let $H \in \mathcal{S}(G)$ be a graph obtained from G by p-stretching vertex $v \in V(G)$, and $a^\top x \leq \beta$ be a valid inequality* $\text{STAB}(G)$ *where $a \geq 0$. If $d \in \mathbb{R}^p_+$ satisfies $\sum_{\ell=1}^p d_\ell \geq a_v$ and*

$$\max\left\{a^\top x : x \in \text{STAB}(G), x_i = 0 \ \forall i \in \{v\} \cup \bigcup_{\ell \in T} A_\ell\right\} \leq \beta - a_v + \sum_{\ell \notin T} d_\ell \tag{4}$$

for all $\emptyset \subset T \subset [p]$, then

$$\sum_{i \in V(H \ominus v_0)} a_i x_i + \sum_{\ell=1}^p d_\ell x_{v_\ell} + \left(\left(\sum_{\ell=1}^p d_\ell\right) - a_v\right) x_{v_0} \leq \beta - a_v + \sum_{\ell=1}^p d_\ell \tag{5}$$

is valid for $\text{STAB}(H)$.

Proof. Suppose $S \subseteq V(H)$ is an inclusion-wise maximal stable set in H. We define $S' \subseteq V(G)$ where

$$S' := \begin{cases} S \setminus \{v_0\} & \text{if } v_0 \in S; \\ (S \setminus \{v_1, \ldots, v_p\}) \cup \{v\} & \text{if } \{v_1, \ldots, v_p\} \subseteq S; \\ S \setminus \{v_1, \ldots, v_p\} & \text{if } 1 \leq |\{v_1, \ldots, v_p\} \cap S| \leq p-1. \end{cases}$$

In all cases, S' is a stable set in G, and $a^\top \chi_{S'} \leq \beta$ implies that χ_S satisfies (5). Note that the third case is when we require the assumption (4) with $T := \{v_1, \ldots, v_p\} \cap S$. □

Due to its similarity with the aforementioned vertex-stretching operations studied in [LT03], our vertex-stretching operation shares some similar structural properties, which we point out below.

Proposition 13. *Let $H \in \mathcal{S}(G)$ be a graph obtained from G by p-stretching vertex $v \in V(G)$. Then we have the following.*

(i) *If $a^\top x \leq \beta$ is valid for* STAB(G) *where $a \geq 0$. Then*

$$\sum_{i \in V(H \ominus v_0)} a_i x_i + \sum_{\ell=1}^{p} a_v x_{v_\ell} + (p-1) a_v x_{v_0} \leq \beta + (p-1) a_v \tag{6}$$

is valid for STAB(H).

(ii) *Let g be as defined in* (3). *If $x \notin$ STAB(G), then $\tilde{g}(x) \notin$ STAB(H).*

(iii) $r_+(H) \geq r_+(G)$.

Proof. First, (i) follows readily from Lemma 12 with $d_\ell := a_v$ for every $\ell \in [p]$. Here, the condition (4) holds as the right hand side is at least β for all non-empty $T \subset [p]$.

For (ii), first suppose $x \notin$ STAB(G). If $x \notin [0,1]^{V(G)}$, then $\tilde{g}(x) \notin [0,1]^{V(H)}$ and the claim follows. Otherwise, there is a facet $a^\top x \leq \beta$ of STAB(G) where $a \geq 0$ that is violated by x. Now notice that for every $j \in V(H)$,

$$[\tilde{g}(x)]_j = \begin{cases} x_j & \text{if } j \in V(H \ominus v_0); \\ x_v & \text{if } j \in \{v_1, \ldots, v_p\}; \\ 1 - x_v & \text{if } j = v_0. \end{cases}$$

Then $\tilde{g}(x)$ violates (6), and thus does not belong to STAB(H).

Finally, as we have shown that H is star-homomorphic to G under the function g as defined in (3), (iii) follows directly from Lemma 11. □

We remark that Proposition 13 is a generalization of the corresponding results on types (i) and (ii) stretching from [LT03], and our proof uses many of the same ideas from similar arguments therein.

Next, we prove a result somewhat similar to Proposition 13(i) that derives some facets of the stable set polytope of the stretched graph.

Proposition 14. *Let $H \in \mathcal{S}(G)$ be a graph obtained from G by p-stretching vertex $v \in V(G)$, and suppose $a^\top x \leq \beta$ is a facet of* STAB(G) *where $a \geq 0$. For every $\ell \in [p]$, define $A_\ell := \Gamma_H(v_\ell) \setminus \{v_0\}$ and*

$$d_\ell := a_v - \beta + \max \left\{ a^\top x : x \in \text{STAB}(G), x_i = 0 \ \forall i \in \{v\} \cup \bigcup_{j \in [p], j \neq \ell} A_j \right\}. \tag{7}$$

If the inequality (5) *is valid for* STAB(H), *then it is a facet of* STAB(H).

Proof. For convenience, let $n := |V(G)|$. Since $a^\top x \leq \beta$ is a facet of STAB(G), there exist stable sets $S_1, \ldots, S_n \subseteq V(G)$ whose incidence vectors are affinely independent and all satisfy $a^\top x \leq \beta$ with equality. Also, for every $\ell \in [p]$, let D_ℓ be a stable set that attains the maximum in the definition of d_ℓ in (7). We then define $S'_1, \ldots, S'_{n+p}$ as follows. For every $i \in [n]$,

$$S'_i := \begin{cases} S_i \cup \{v_0\} & \text{if } v \notin S_i; \\ (S_i \setminus \{v\}) \cup \{v_1, \ldots, v_p\} & \text{if } v \in S_i. \end{cases}$$

We also define

$$S'_{n+i} := D_i \cup \{v_j : j \in [p], j \neq i\}$$

for all $i \in [p]$.

Observe that $S'_1, \ldots, S'_{n+p}$ must all be stable sets in H. Also, using the fact that incidence vectors of $S_1, \ldots, S_n$ are affinely independent and satisfy $a^\top x \leq \beta$ with equality, we see that the incidence vectors of $S'_1, \ldots, S'_{n+p}$ are affinely independent and all satisfy (5) with equality.

Thus, if we know that (5) is valid for STAB(H), it must be a facet. □

The special case of $p = 2$ in Proposition 14 is particularly noteworthy:

Corollary 15. *Let $H \in \mathcal{S}_2(G)$ be a graph obtained from G by 2-stretching vertex $v \in V(G)$, and suppose $a^\top x \leq \beta$ is a facet of* STAB(G) *where $a \geq 0$. Define $A_1 := \Gamma_H(v_1) \setminus \{v_0\}, A_2 := \Gamma_H(v_2) \setminus \{v_0\}$, as well as the quantities*

$$d_1 := a_v - \beta + \max\left\{a^\top x : x \in \text{STAB}(G), x_i = 0 \ \forall i \in \{v\} \cup A_2\right\},$$
$$d_2 := a_v - \beta + \max\left\{a^\top x : x \in \text{STAB}(G), x_i = 0 \ \forall i \in \{v\} \cup A_1\right\}.$$

If $d_1 + d_2 \geq a_v$, then (5) *is a facet of* STAB(H).

Proof. This is largely a specialization of Proposition 14 to the case $p = 2$. Notice that the additional assumption of (5) being valid is not necessary in this case because $\{1, 2\}$ has exactly two non-empty and proper subsets, and so the definition of d_1, d_2 herein are enough to guarantee that the assumption (4) is met. □

Finally, we close this section by mentioning a "reverse" implication of Proposition 13. Notice that given graph H, if any vertex $v_0 \in H$ has the property that $\Gamma_H(v_0)$ is a stable set, then there exists a graph G where $H \in \mathcal{S}(G)$. In particular, we can obtain this graph G by contracting the set of vertices $\Gamma_H[v_0]$. Thus, Proposition 13 implies the following.

Corollary 16. *Given a graph H and vertex $v_0 \in V(H)$ where $\Gamma_H(v_0)$ is a stable set, let G be the graph obtained from H by contracting the set of vertices $\Gamma_H[v_0]$. Then $r_+(G) \leq r_+(H)$.*

Example 17. In general, it is possible that contracting the closed neighborhood of a vertex results in an increase in the graph's LS$_+$-rank. For example, notice that the graph H in Figure 7 is the union of two LS$_+$-rank-1 graphs whose intersection is the cut clique $\{7, 8\}$, and thus it follows from Proposition 6 that $r_+(H) = 1$. However, contracting $\Gamma_H[6]$ in H results in G, which is isomorphic to $G_{2,1}$, and so $r_+(G) = 2$.

5. LS$_+$-minimal graphs via 2-stretching cliques

In this section, we are interested in studying graphs with the fewest number of vertices with a given LS$_+$-rank. Given $\ell \in \mathbb{N}$, define $n_+(\ell)$ to be the minimum number of vertices on which there exists a graph G with $r_+(G) = \ell$. It follows immediately from Theorem 1 that $n_+(\ell) \geq 3\ell$

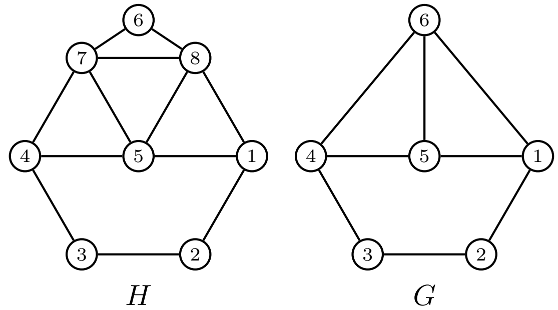

FIGURE 7. An example in which contracting the closed neighborhood of a vertex increases the graph's LS_+-rank

for every $\ell \in \mathbb{N}$. On the other hand, Theorem 2 implies that $n_+(\ell) \leq 16\ell$. Thus, we know that $n_+(\ell) = \Theta(\ell)$ asymptotically.

Recall that a graph G is ℓ-minimal if $r_+(G) = \ell$ and $|V(G)| = 3\ell$. The following result establishes a close connection between ℓ-minimal graphs and 2-stretching vertices.

Theorem 18. *Let H be an ℓ-minimal graph where $\ell \geq 2$. Then there exists a graph G where $H \in \mathcal{S}_2(G)$.*

Proof. First, since $r_+(H) = \ell$, there exists a vertex v_0 where $r_+(H \ominus v_0) = \ell - 1$. This implies that $|V(H \ominus v_0)| \geq 3\ell - 3$, and thus $\deg(v_0) \leq 2$. If $\deg(v_0) = 1$, H would contain a cut vertex, and Proposition 6 implies there would be a proper subgraph of H with the same LS_+-rank as H. Thus, we obtain that $\deg(v_0) = 2$. Let v_1, v_2 denote the two neighbours of v_0. If $\{v_1, v_2\} \in E(H)$, then the edge would form a cut clique in H, and Proposition 6 again implies that there would a proper subgraph of H with the same LS_+-rank as H. Thus, $\{v_1, v_2\}$ must be a stable set in H.

Next, define G to be the graph obtained from H by contracting $\{v_0, v_1, v_2\}$, and label the new vertex v. We claim that $H \in \mathcal{S}_2(G)$, and to prove that it only remains to show that both $\Gamma_H(v_1) \setminus \Gamma_H(v_2)$ and $\Gamma_H(v_2) \setminus \Gamma_H(v_1)$ are non-empty.

Let $a^\top x \leq \beta$ be a facet of $\mathrm{STAB}(H)$ of LS_+-rank ℓ. Then there must be stable sets $S_1, \ldots, S_{3\ell} \subseteq V(H)$ whose incidence vectors are affinely independent and all satisfy $a^\top x \leq \beta$ with equality. Also, since H is ℓ-minimal, a must have full support. Therefore, S_i must be inclusion-wise maximal for all $i \in [3\ell]$, and hence belongs to one of the following cases:

(1) $S_i \cap \{v_0, v_1, v_2\} = \{v_0\}$;
(2) $S_i \cap \{v_0, v_1, v_2\} = \{v_1\}$;
(3) $S_i \cap \{v_0, v_1, v_2\} = \{v_2\}$;
(4) $S_i \cap \{v_0, v_1, v_2\} = \{v_1, v_2\}$.

Since $\chi_{S_1}, \ldots, \chi_{S_{3\ell}}$ are affinely independent, one of these stable sets contains v_0 and belongs to Case (1), so assume without loss of generality that $v_0 \in S_1$. Now consider the matrix A formed by the row vectors $(\chi_{S_2} - \chi_{S_1})^\top, (\chi_{S_3} - \chi_{S_1})^\top, \ldots, (\chi_{S_{3\ell}} - \chi_{S_1})^\top$. Since $S_1, \ldots, S_{3\ell}$ are affinely independent, A must have linearly independent rows. This means that, if we focus on the submatrix A' of A which consists of just the three columns corresponding to v_0, v_1, and v_2, A' must have rank 3. This implies that there must exist at least one S_i belonging to each of Cases (2), (3), and (4).

Now consider a stable set S_i that belongs to Case (2). Since S_i is inclusion-wise maximal, it must contain a vertex that is adjacent to v_2 and not v_1, and so we have found a vertex that belongs to $\Gamma_H(v_2) \setminus \Gamma_H(v_1)$. The same argument applied to an S_i from Case (3) gives a vertex that belongs to $\Gamma_H(v_1) \setminus \Gamma_H(v_2)$. This finishes the proof. $\square$

Thus, for the remainder of this section, we will focus on 2-stretching vertices, and study when that helps (and does not help) in generating ℓ-minimal graphs. Since we will be studying graphs obtained from applying a sequence of 2-stretching operations, we recursively define

$$\mathcal{S}_2^k(G) \coloneqq \bigcup_{G' \in \mathcal{S}_2^{k-1}(G)} \mathcal{S}_2(G')$$

for every graph G and integer $k \geq 1$. That is, $\mathcal{S}_2^k(G)$ is the set of graphs that can be obtained from G by a sequence of k 2-stretching operations. We also let $\mathcal{S}_2^0(G) \coloneqq \{G\}$. The following is a basic property of the graphs in $\mathcal{S}_2^k(G)$.

Lemma 19. *Let G be a graph, and let $H \in \mathcal{S}_2^k(G)$. Then $\alpha(H) = \alpha(G) + k$.*

Proof. Let $H \in \mathcal{S}_2(G)$ be a graph obtained from G by 2-stretching vertex $v \in V(G)$. To prove our claim, it suffices to show that $\alpha(H) = \alpha(G) + 1$. Consider a set of vertices $S \subseteq V(G)$. If $v \in S$, then S is a stable set in G if and only if $(S \setminus \{v\}) \cup \{v_1, v_2\}$ is a stable set in H. If $v \notin S$, then S is a stable set in G if and only if $S \cup \{v_0\}$ is a stable set in H. Thus, we see that $\alpha(H) = \alpha(G) + 1$. □

Recall the graphs $G_{2,1}, G_{2,2}$, and $G_{3,1}$. In Figure 1, we labelled the vertices of these graphs to highlight the fact that all three graphs can be obtained from applying a number of 2-stretching operations to a complete graph. In fact, every known ℓ-minimal graph to date — the 3-cycle and the three graphs in Figure 1 — belongs to $\mathcal{S}^{\ell-1}(K_{\ell+2})$. Thus, for the remainder of this section, we focus on graphs obtained from 2-stretching vertices of a complete graph, and prove some results about the LS_+-ranks of these graphs. Some of our subsequent arguments rely on the positive semidefiniteness of some specific matrices, and so we first provide a framework for easily and reliably verifying such claims. Given a symmetric matrix $Y \in \mathbb{R}^{n \times n}$, we say that $U, V \in \mathbb{Z}^{n \times n}$ is a *UV-certificate* of Y if

- $kY = U^\top U + V$ for some $k \in \mathbb{N}$, and,
- V is diagonally dominant.

Observe that the existence of a UV-certificate implies that $Y \succeq 0$ (these certificates are sum-of-squares certificates, and every rational matrix $Y \in \mathbb{S}_+^n$ admits such certificates). Given a UV-certificate to verify that Y is PSD, it suffices to

(i) form $U^\top U + V$ and check that it is equal to kY for some integer k
(ii) check that $V_{ii} \geq \sum_{j \neq i} |V_{ij}|$ for every $i \in [n]$.

Since every entry in U and V is an integer, verifying (i) and (ii) only involve elementary numerical operations on whole numbers.

Next, we show that if we 2-stretch a vertex in a complete graph, the result is always a graph with LS_+-rank 2.

Proposition 20. *Let $n \geq 4$. Then $r_+(H) = 2$ for all $H \in \mathcal{S}_2(K_n)$.*

Proof. Let $H \in \mathcal{S}_2(K_n)$, and assume without loss of generality that H is obtained from K_n by 2-stretching vertex n. Also, let G_n be the graph obtained from 2-stretching vertex n in K_n with $A_1 \coloneqq \{2, 3, \ldots, n\}$ and $A_2 \coloneqq [n-1]$. (For example, G_4 is the graph $G_{2,2}$ from Figure 1 and G_5 is shown in Figure 6.) Then H must be isomorphic to a subgraph of G_n.

Next, we show that $\alpha_{\mathrm{LS}_+^2}(G_4) > 2$. Consider the certificate matrix

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} & 1 & 2 & 3 & 4_1 & 4_0 & 4_2 \end{array} \\
Y := & \begin{bmatrix}
200 & 78 & 12 & 78 & 78 & 78 & 78 \\
78 & 78 & 0 & 0 & 39 & 39 & 0 \\
12 & 0 & 12 & 0 & 0 & 12 & 0 \\
78 & 0 & 0 & 78 & 0 & 39 & 39 \\
78 & 39 & 0 & 0 & 78 & 0 & 39 \\
78 & 39 & 12 & 39 & 0 & 78 & 0 \\
78 & 0 & 0 & 39 & 39 & 0 & 78
\end{bmatrix}
\end{array}
$$

Note that the columns of Y are labelled by the vertices in G_4 they correspond to (the rows of Y follow the same order of indexing). Observe that $Y \succeq 0$ — a UV-certificate for Y is

$$
U := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-73 & -26 & -141 & -26 & 60 & 132 & 61 \\
0 & 124 & 0 & -124 & -200 & 0 & 200 \\
27 & -181 & 247 & -181 & 51 & 159 & 51 \\
0 & -527 & 0 & 527 & -326 & 0 & 326 \\
1 & -166 & -73 & -166 & 449 & -556 & 449 \\
1224 & 482 & 60 & 482 & 482 & 485 & 482
\end{bmatrix}, \quad
V := \begin{bmatrix}
2765 & 917 & 91 & 917 & 316 & -11 & 389 \\
917 & 1308 & 3 & -212 & -136 & 10 & 29 \\
91 & 3 & 601 & 3 & -280 & 71 & -139 \\
917 & -212 & 3 & 1308 & 3 & 10 & -110 \\
316 & -136 & -280 & 3 & 1328 & -155 & -45 \\
-11 & 10 & 71 & 10 & -155 & 664 & -287 \\
389 & 29 & -139 & -110 & -45 & -287 & 1207
\end{bmatrix},
$$

which gives $7535Y = U^\top U + V$. One can also check that $Ye_i, Y(e_0 - e_i) \in \text{FRAC}(G_4)$ for every $i \in V(G_4)$. This shows that $\bar{x} := \frac{1}{200}(78, 12, 78, 78, 78, 78)^\top \in \text{LS}_+(G_4)$. Now since $\bar{e}^\top \bar{x} = 2.01 > 2 = \alpha(G_4)$, we see that $r_+(G_4) \geq 2$. Since G_n contains G_4 as an induced subgraph for all $n \geq 4$, we conclude that $\alpha_{\text{LS}_+}(G_n) > 2$. Then Lemma 7(ii) implies that $\alpha_{\text{LS}_+}(H) > 2$. Since $\alpha(H) = 2$, it follows that $r_+(H) \geq 2$.

Finally, notice that $H - n_0$ must be a perfect graph, so $r_+(H - n_0) \leq 1$ and consequently $r_+(H) \leq 2$. Thus, we conclude that $r_+(H) = 2$. □

We remark that in the proof for $r_+(G_{2,2}) \geq 2$ in [EMN06], the following certificate matrix was given:

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} & 4_2 & 1 & 3 & 4_1 & 4_0 & 2 \end{array} \\
\frac{1}{2688} & \begin{bmatrix}
2688 & 769 & 769 & 769 & 769 & 769 & 1538 \\
769 & 769 & 0 & 336 & 413\frac{7}{13} & 0 & 0 \\
769 & 0 & 769 & 0 & 336 & 384 & 0 \\
769 & 336 & 0 & 769 & 0 & 384 & 0 \\
769 & 413\frac{7}{13} & 336 & 0 & 769 & 0 & 896 \\
769 & 0 & 384 & 384 & 0 & 769 & 0 \\
1538 & 0 & 0 & 0 & 896 & 0 & 1538
\end{bmatrix}
\end{array}
$$

However, the certificate is incorrect: $Y[2, 4_0] = \frac{896}{2688} = \frac{1}{3} > Y[0, 4_0]$, and thus violates $Ye_{4_0} \in \text{cone}(\text{FRAC}(G_{2,2}))$. In fact, since the vector $\frac{1}{2688}(769, 769, 769, 769, 769, 1538)^\top$ contains only one entry greater than $\frac{1}{3}$, any certificate matrix for this vector cannot contain the entry $\frac{1}{3}$ (which would have to appear in at least 2 columns in the certificate). Still, the claim that $r_+(G_{2,2}) = 2$ is correct, as shown in the proof of Proposition 20.

Next, while all graphs in $\mathcal{S}_2(K_n)$ have LS_+-rank 2, we show that not all graphs in $\mathcal{S}_2^2(K_n)$ have LS_+-rank 3. Given a graph G, we say that a path in G is *sparse* if at most one of the vertices in the path has degree greater than 2 in G. For example, in Figure 8, the graph on the left contains a sparse path $4_0, 4_1, 5_1, 5_0, 5_2$ of length 4, while the graph on the right also contains a sparse path $4_0, 4_2, 3, 5_2, 5_0$ of length 4. Then we have the following.

Proposition 21. *Let $n \geq 4$. If $G \in \mathcal{S}_2^{n-3}(K_n)$ contains a sparse path of length at least 3, then G is not ℓ-minimal.*

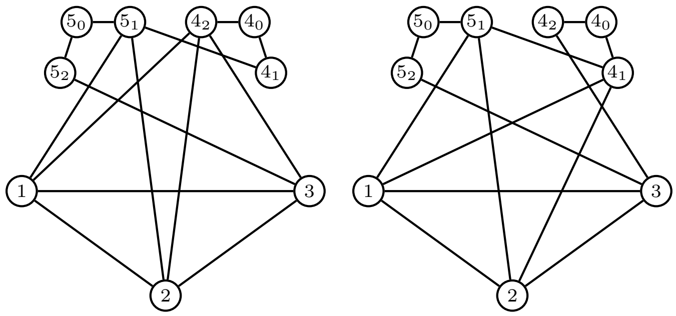

FIGURE 8. Two graphs in $\mathcal{S}_2^2(K_5)$ with sparse paths

Proof. Suppose $u_1, u_2, \ldots, u_m$ form a sparse path in G where $m \geq 4$, and let i be the index such that $\deg(u_j) = 2$ for all $j \in [m], j \neq i$. Then Proposition 6 implies that $r_+(G - u_i) = r_+(G - U)$, where $U \coloneqq \{u_1, \ldots, u_m\}$. Since $m \geq 4$, $G - U$ has $3(n-2) - |U| < 3(n-3)$ vertices, and so $r_+(G - u_i) = r_+(G - U) \leq n - 4$. As a result, $r_+(G) \leq n - 3$, and G is not ℓ-minimal. ☐

Thus, both graphs in Figure 8 have LS$_+$-rank at most 2. (In fact, they have rank 2 as they both contain $G_{2,1}$ as an induced subgraph.) Next, we show that if we 2-stretch a vertex in K_n, and then 2-stretch one of the three new vertices in the stretched graph, the resulting graph cannot have LS$_+$-rank 3.

Proposition 22. *Let $n \geq 4$. Suppose $G_1 \in \mathcal{S}_2(K_n)$ is obtained by stretching vertex n in K_n, and $G_2 \in \mathcal{S}_2^2(K_n)$ is obtained by stretching vertex n_0, n_1, or n_2 in G_1. Then $r_+(G_2) = 2$.*

Proof. First, if G_2 is obtained from G_1 by stretching n_0, then n_{00}, n_{01}, n_{02} all have degree 2. Notice that $G_2 - n_{00}$ must be a perfect graph, and so $r_+(G_2 - n_{00}) \leq 1$, which implies $r_+(G_2) \leq 2$ in this case.

Otherwise, assume without loss of generality that G_2 is obtained from G_1 by stretching n_1, and that $\{n_{12}, n_0\} \in E(G_2)$. (Note that $\{n_{11}, n_0\}$ may or may not be an edge.) Now notice that $G_2 - n_{12}$ is a perfect graph, and thus, $r_+(G_2) \leq 2$ in this case as well. Finally, since $r_+(G_1) = 2$ (from Proposition 20) and $G_2 \in \mathcal{S}(G_1)$, Proposition 13(iii) implies that $r_+(G_2) \geq 2$. ☐

Thus, to obtain a graph with LS$_+$-rank 3 in $\mathcal{S}_2^2(K_n)$, it is necessary that we stretch two of the original vertices of K_n. (That is not sufficient though, as shown for the graphs in Figure 8.)

Next, observe that if G is an ℓ-minimal graph, then it is necessary that STAB(G) has a facet with full support (or G would have a proper subgraph with the same LS$_+$-rank). We provide more circumstantial evidence that 2-stretching a number of original vertices of a complete graph is a promising approach for generating ℓ-minimal graphs by showing that the stable set polytope of these graphs all have a full-support facet.

Proposition 23. *Let k, ℓ be integers where $\ell \geq 3$ and $\ell \geq k \geq 0$. Suppose $H \in \mathcal{S}_2^k(K_\ell)$ is obtained from K_ℓ by 2-stretching k vertices in K_ℓ. Then $\sum_{i \in V(H)} x_i \leq k + 1$ is a facet of* STAB(H).

Proof. We prove our claim by induction on k. When $k = 0$, $H = K_\ell$, and the claim obviously holds. Next, assume $1 \leq k \leq \ell$. Let $T \subseteq [n]$ be the vertices in K_ℓ that were stretched to obtain H, and let $G \in \mathcal{S}_2^{k-1}(K_\ell)$ be a graph such that $H \in \mathcal{S}_2(G)$. (So there exists $v \in T$ where H is obtained from G by stretching v.)

By the inductive hypothesis, $\sum_{i \in V(G)} x_i \leq k$ is a facet of STAB(G). To prove our claim, we make use of Proposition 14 and show that $d_1 = d_2 = 1$. To do so, let $A_1 \coloneqq \Gamma_H(v_1) \setminus \{v_0\}, A_2 \coloneqq$

$\Gamma_H(v_2) \setminus \{v_0\}$,

$$c_1 := \max\left\{a^\top x : x \in \text{STAB}(G), x_\ell = 0 \ \forall \ell \in \{v\} \cup A_2\right\},$$
$$c_2 := \max\left\{a^\top x : x \in \text{STAB}(G), x_\ell = 0 \ \forall \ell \in \{v\} \cup A_1\right\}.$$

Then it suffices to prove that $c_1 = c_2 = k$, which would then imply that $d_1 = d_2 = 1$.

First, it is obvious that $c_1, c_2 \leq k$ since $\alpha(G) = k$. Next, consider $\Gamma(v_1) \subseteq V(H)$. By the definition of the vertex-stretching operation, one of the following must hold:

- There exists an index $j \in [n], j \neq v$ where $j \not\in T$ (so $j \in V(H)$) and $j \not\in \Gamma(v_1)$. Then
$$S := \{j\} \cup \{p_0 : p \in T, p \neq v\}$$
is a stable set that gives $c_1 = k$.
- There exists an index $j \in [n], j \neq v$ where $j \in T$ (so $j_0, j_1.j_2 \in V(H)$) and $j_0, j_1, j_2 \not\in \Gamma(v_1)$. Then
$$S := \{j_1, j_2\} \cup \{p_0 : p \in T, p \neq v, j\}$$
is a stable set that gives $c_1 = k$.

The same argument shows that $c_2 = k$, and this finishes the proof. □

We remark that the assumption of stretching only the original vertices of K_ℓ in Proposition 23 is necessary, as shown in the following example.

Example 24. Recall the graph $G_{2,2}$ from Figure 1. Observe that $G_{2,2} \in \mathcal{S}_2(K_4)$, and that $\bar{e}^\top x \leq 2$ is a facet of $\text{STAB}(G_{2,2})$. Now, we 2-stretch the vertex $4_2 \in V(G_{2,2})$ to obtain $H \in \mathcal{S}_2^2(K_4)$ as shown in Figure 9 (right). Observe that the subgraph of H induced by vertices $1, 2, 3, 4_1, 4_0, 4_{21}$ is isomorphic to $G_{2,1}$ from Figure 1. Thus,

$$x_1 + x_2 + x_3 + x_{4_1} + x_{4_0} + x_{4_{21}} \leq 2 \tag{8}$$

is valid for $\text{STAB}(H)$. (In fact, one can show that it is a facet of $\text{STAB}(G)$ using Proposition 14).) This implies that $\sum_{i \in V(H)} x_i \leq 3$, which is the sum of (8) and the edge inequality $x_{4_{20}} + x_{4_{22}} \leq 1$, is not a facet of $\text{STAB}(H)$.

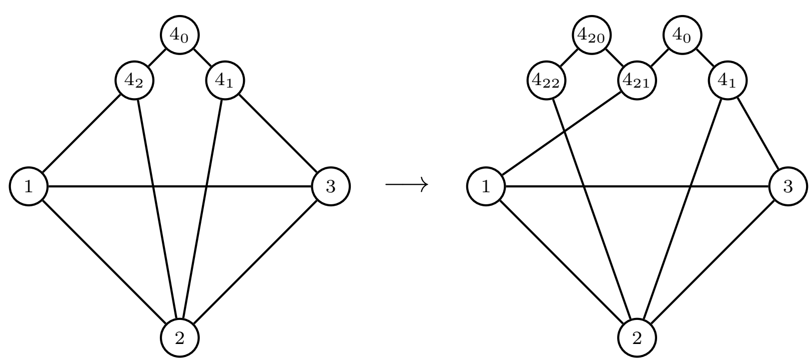

FIGURE 9. A graph in $H \in \mathcal{S}^2(K_4)$ (right) where $\bar{e}^\top x \leq \alpha(H)$ is not a facet of $\text{STAB}(H)$

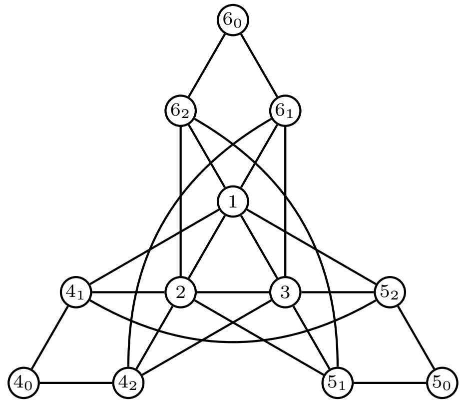

FIGURE 10. An alternative drawing of $G_{4,1}$ to highlight its automorphisms

6. EXISTENCE OF 4-MINIMAL GRAPHS

Recall the graph $G_{4,1}$ (Figure 3), which was introduced in Section 1. We show in this section that $r_+(G_{4,1}) = 4$, providing what we believe to be the first known example of a 4-minimal graph (and the first advance in this direction since 2006 [EMN06]). Observe from its drawing in Figure 3 that $G_{4,1} \in \mathcal{S}_2^3(K_6)$, and is obtained from stretching three of the original vertices in K_6. We also point out two important automorphisms of $G_{4,1}$ that will be useful in simplifying our analysis of its LS$_+$-rank. Consider the alternative drawing of $G_{4,1}$ in Figure 10, and define the functions $f_1, f_2 : V(G_{4,1}) \to V(G_{4,1})$ as follows:

i	1	2	3	4_1	4_0	4_2	5_1	5_0	5_2	6_1	6_0	6_2
$f_1(i)$	2	3	1	5_1	5_0	5_2	6_1	6_0	6_2	4_1	4_0	4_2
$f_2(i)$	1	3	2	5_2	5_0	5_1	4_2	4_0	4_1	6_2	6_0	6_1

Visually, f_1 corresponds to rotating the graph $G_{4,1}$ in Figure 10 counterclockwise by $\frac{2\pi}{3}$, and f_2 corresponds to reflecting the figure along the centre vertical line. Now we are ready to prove the main result of this section.

Theorem 25. *The* LS$_+$*-rank of* $G_{4,1}$ *is* 4.

Proof. For convenience, let $G := G_{4,1}$ throughout this proof. Since G has 12 vertices, by Theorem 1 it suffices to show that $r_+(G) \geq 4$. Consider the matrix Y_0 defined as follows:

$$
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4_1 & 4_0 & 4_2 & 5_1 & 5_0 & 5_2 & 6_1 & 6_0 & 6_2
\end{array}
$$

$$
Y_0 := \begin{bmatrix}
100000 & 25340 & 25340 & 25340 & 16500 & 75020 & 16500 & 16500 & 75020 & 16500 & 16500 & 75020 & 16500 \\
25340 & 25340 & 0 & 0 & 0 & 17502 & 7838 & 7838 & 17502 & 0 & 0 & 25340 & 0 \\
25340 & 0 & 25340 & 0 & 0 & 25340 & 0 & 0 & 17502 & 7838 & 7838 & 17502 & 0 \\
25340 & 0 & 0 & 25340 & 7838 & 17502 & 0 & 0 & 25340 & 0 & 0 & 17502 & 7838 \\
16500 & 0 & 0 & 7838 & 16500 & 0 & 8073 & 589 & 15911 & 0 & 589 & 15419 & 1081 \\
75020 & 17502 & 25340 & 17502 & 0 & 75020 & 0 & 15419 & 51150 & 15911 & 15911 & 51150 & 15419 \\
16500 & 7838 & 0 & 0 & 8073 & 0 & 16500 & 1081 & 15419 & 589 & 0 & 15911 & 589 \\
16500 & 7838 & 0 & 0 & 589 & 15419 & 1081 & 16500 & 0 & 8073 & 589 & 15911 & 0 \\
75020 & 17502 & 17502 & 25340 & 15911 & 51150 & 15419 & 0 & 75020 & 0 & 15419 & 51150 & 15911 \\
16500 & 0 & 7838 & 0 & 0 & 15911 & 589 & 8073 & 0 & 16500 & 1081 & 15419 & 589 \\
16500 & 0 & 7838 & 0 & 589 & 15911 & 0 & 589 & 15419 & 1081 & 16500 & 0 & 8073 \\
75020 & 25340 & 17502 & 17502 & 15419 & 51150 & 15911 & 15911 & 51150 & 15419 & 0 & 75020 & 0 \\
16500 & 0 & 0 & 7838 & 1081 & 15419 & 589 & 0 & 15911 & 589 & 8073 & 0 & 16500
\end{bmatrix}.
$$

Again, the columns of Y_0 are labelled by the vertices in G they correspond to. with the rows of Y_0 following the same order of indexing.

We prove our claim by showing that $Y_0 \in \widehat{\mathrm{LS}}_+^3(G)$. First, one can check that $Y_0 \succeq 0$ (a UV-certificate is provided in Table 1). Moreover, observe that for all $i, j \in V(G)$,

$$Y_0[i,j] = Y_0[f_1(i), f_1(j)] = Y_0[f_2(i), f_2(j)],$$

and thus the entries of Y_0 exhibit the same symmetries of the graph that are exposed by the automorphisms f_1 and f_2. Hence, to show that $Y_0 \in \widehat{\mathrm{LS}}_+^3(G)$, it suffices to verify the conditions $Y_0e_i, Y_0(e_0 - e_i) \in \mathrm{cone}(\mathrm{LS}_+^2(G))$ for $i \in \{1, 4_1, 6_0\}$, since for every other vertex j there is an automorphism of G that would map j to one of these three vertices.

Next, notice that

$$\begin{aligned}
Y_0e_1 &\leq 17502 \begin{bmatrix} 1 \\ \chi_{\{1,4_0,5_0,6_0\}} \end{bmatrix} + 7838 \begin{bmatrix} 1 \\ \chi_{\{1,4_2,5_1,6_0\}} \end{bmatrix}, \\
Y_0e_{4_1} &\leq 7838 \begin{bmatrix} 1 \\ \chi_{\{3,4_1,5_0,6_0\}} \end{bmatrix} + 589 \begin{bmatrix} 1 \\ \chi_{\{4_1,4_2,5_1,6_0\}} \end{bmatrix} + 6992 \begin{bmatrix} 1 \\ \chi_{\{4_1,4_2,5_0,6_0\}} \end{bmatrix} \\
&\quad + 492 \begin{bmatrix} 1 \\ \chi_{\{4_1,4_2,5_0,6_2\}} \end{bmatrix} + 589 \begin{bmatrix} 1 \\ \chi_{\{4_1,5_0,6_1,6_2\}} \end{bmatrix}, \\
Y_0(e_0 - e_{6_0}) &\leq 7366 \begin{bmatrix} 1 \\ \chi_{\{2,4_0,5_0,6_1\}} \end{bmatrix} + 476 \begin{bmatrix} 1 \\ \chi_{\{2,4_0,5_2,6_1\}} \end{bmatrix} + 476 \begin{bmatrix} 1 \\ \chi_{\{3,4_1,5_0,6_2\}} \end{bmatrix} \\
&\quad + 7366 \begin{bmatrix} 1 \\ \chi_{\{3,4_0,5_0,6_2\}} \end{bmatrix} + 605 \begin{bmatrix} 1 \\ \chi_{\{4_1,4_2,5_0,6_2\}} \end{bmatrix} + 605 \begin{bmatrix} 1 \\ \chi_{\{4_0,5_1,5_2,6_1\}} \end{bmatrix} \\
&\quad + 8058 \begin{bmatrix} 1 \\ \chi_{\{4_0,5_0,6_1,6_2\}} \end{bmatrix} + 28 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\end{aligned}$$

Since all incidence vectors above correspond to stable sets in G, we obtain that $Y_0e_1, Y_0e_{4_1}, Y_0(e_0 - e_{6_0}) \in \mathrm{cone}(\mathrm{STAB}(G)) \subseteq \mathrm{cone}(\mathrm{LS}_+^2(G))$. The details for $Y_0e_{6_0}, Y_0(e_0 - e_1), Y_0(e_0 - e_{4_1}) \in \mathrm{cone}(\mathrm{LS}_+^2(G))$ are provided, respectively, in the proofs of Lemmas 39, 38, and 40 in Appendix A.

Finally, let $\bar{x}$ be the vector such that $Y_0e_0 = 100000 \begin{bmatrix} 1 \\ \bar{x} \end{bmatrix}$. Since $Y_0 \in \widehat{\mathrm{LS}}_+^3(G)$, we have $\bar{x} \in \mathrm{LS}_+^3(G)$. Thus, we see that

$$\alpha_{\mathrm{LS}_+^3}(G) \geq \bar{e}^\top \bar{x} = 4.0008 > 4 = \alpha(G).$$

Thus, $r_+(G) \geq 4$. ☐

Notice that $G_{4,1}$ contains 24 edges. By Proposition 23, the inequality $\bar{e}^\top x \leq 4$ is a facet of $\mathrm{STAB}(G)$ for every graph G in $\mathcal{S}_2^3(K_6)$, which contains $G_{4,1}$. Thus, by Lemma 7(ii) it follows that every graph in $\mathcal{S}_2^3(K_6)$ which is a subgraph of $G_{4,1}$ (which can have as few as 21 edges) also has LS_+-rank 4, giving more examples of 4-minimal graphs. The six non-isomorphic proper subgraphs of $G_{4,1}$ that belong to $\mathcal{S}_2^3(K_6)$ are listed in Figure 11.

Moreover, the fact that $G_{4,1}$ is 4-minimal also provides some new examples of 3-minimal graphs.

Corollary 26. *Let $G_{3,2} \coloneqq G_{4,1} \ominus 6_0$. Then $G_{3,2}$ is a 3-minimal graph.*

Proof. Since $r_+(G_{4,1}) = 4$, there exists vertex $i \in V(G_{4,1})$ where $G \ominus i$ has LS_+-rank 3, which implies that $\deg(i) = 2$, and so $i \in \{4_0, 5_0, 6_0\}$. Now observe that $G_{4,1} \ominus 4_0, G_{4,1} \ominus 5_0, G_{4,1} \ominus 6_0$ are all isomorphic to each other. Thus, $G_{3,2}$ is 3-minimal. ☐

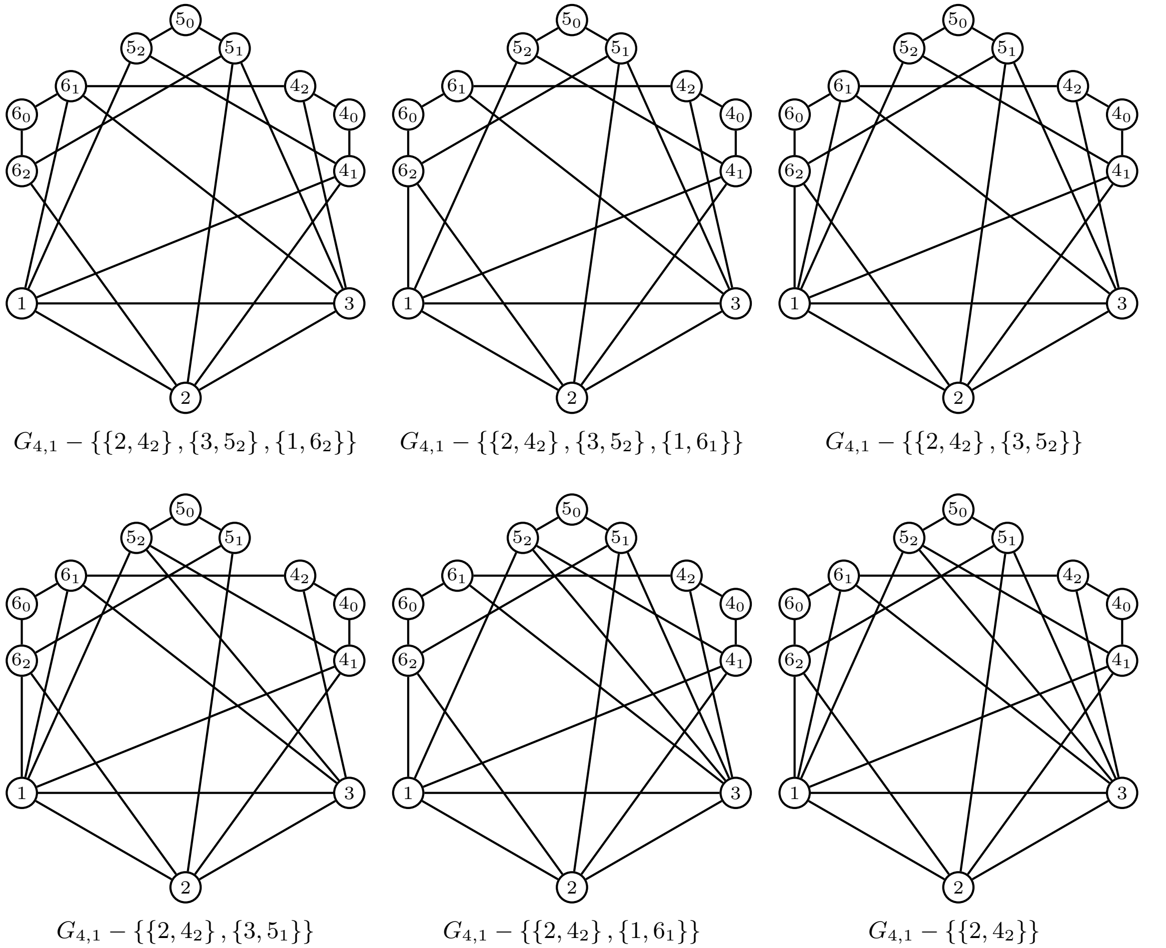

FIGURE 11. The six non-isomorphic proper subgraphs of $G_{4,1}$ that belong to $S_2^3(K_6)$ (and thus are 4-minimal)

By Lemma 7(ii) again, every graph in $\mathcal{S}_2^2(K_5)$ that is a subgraph of $G_{3,2}$ is 3-minimal. Figure 12 illustrates $G_{3,2}$ (top left) and its five non-isomorphic proper subgraphs that belong to $\mathcal{S}_2^2(K_5)$. Notice that one of these graphs (top right of Figure 12) is isomorphic to $G_{3,1}$, the first 3-minimal graph discovered in [EMN06].

We close the section by showing that there are no 3-minimal graphs with fewer edges than $G_{3,1}$.

Proposition 27. *Suppose G is a 3-minimal graph. Then $|E(G)| \geq 14$.*

Proof. Since G is 3-minimal, there must exist vertex v_0 where $r_+(G \ominus v_0) = 2$. This implies that $|V(G \ominus v_0)| \geq 6$, and thus $\deg(v_0) \leq 2$. Since ℓ-minimal graphs cannot have cut vertices, we see that $\deg(v_0) = 2$ and $|V(G \ominus v)| = 6$, and so $G \ominus v$ is isomorphic to either $G_{2,1}$ (8 edges) or $G_{2,2}$ (9 edges).

Let v_1, v_2 be the two neighbours of v_0, and let $A := \{v_0, v_1, v_2\}$ and $B := V(G) \setminus A$. Observe that

$$|E(G)| = \delta(A) + \delta(B) + \delta(A, B). \tag{9}$$

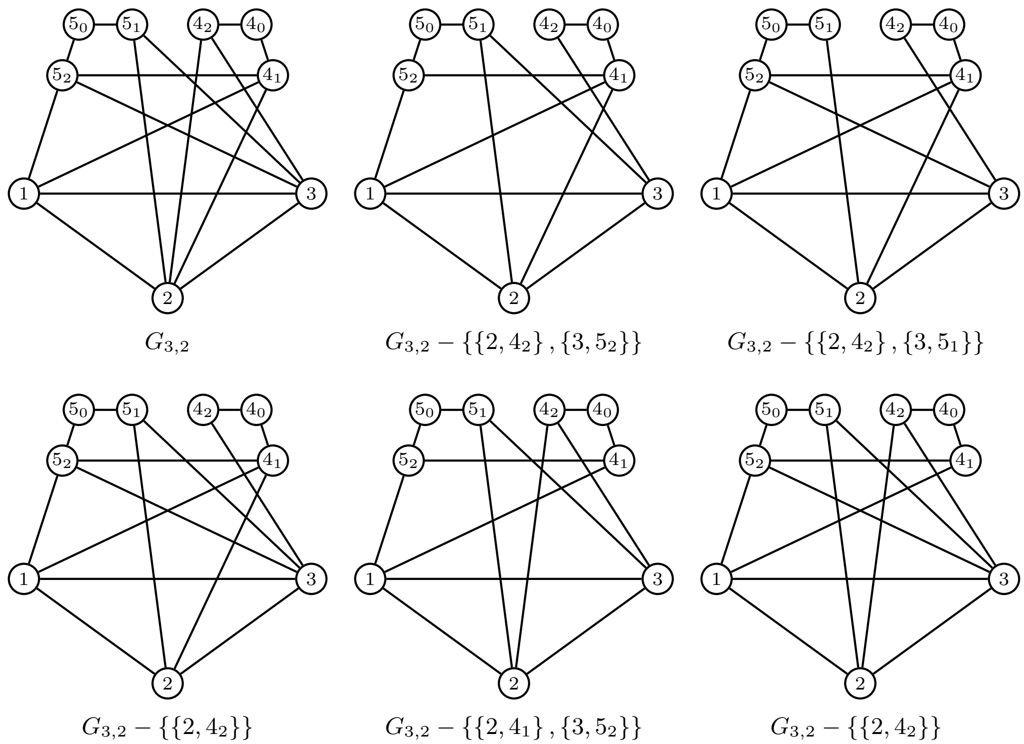

FIGURE 12. The graph $G_{3,2}$ (top left) and its five non-isomorphic proper subgraphs that belong to $S_2^2(K_5)$ (and thus are 3-minimal)

Since $|E(G)| \leq 13, \delta(A) = 2$, and $\delta(B) \geq 8$, we obtain $\delta(A, B) \leq 3$. Again, G being 3-minimal implies that $\deg(v_1), \deg(v_2) \geq 2$, and so we obtain that $2 \leq \delta(\{v_1\}, B) + \delta(\{v_2\}, B) \leq 3$. Thus, we may assume without loss of generality that $\delta(\{v_1\}, B) = 1$, and let u be the only neighbour of v_1 in B.

If $\delta(\{v_2\}, B) = 1$, then u, v_1, v_0, v_2 form a sparse path of length 3 (with $\deg(v_1) = \deg(v_0) = \deg(v_2) = 2$), and Proposition 21 implies that G is not 3-minimal. Now suppose $\delta(\{v_2\}, B) = 2$. This means that $\delta(A, B) = 3$, and so from (9) we know that $|E(G)| = 13, \delta(B) = 8$, and $G - A$ is indeed isomorphic to $G_{2,1}$ and not $G_{2,2}$.

Next, since $r_+(G) = 3$, we obtain that $r_+(G - u) \geq 2$. However, notice that v_0 is a cut vertex in $G - u$. Thus, if we let $A' \coloneqq \{u, v_1, v_0\}$ and $B' \coloneqq V(G) \setminus A'$, then we see that $r_+(G - A') \geq 2$. Since $G - A'$ has 6 vertices, it must be isomorphic to $G_{2,1}$ or $G_{2,2}$. Thus, we see that $\delta(B') \geq 8$. Also, $\delta(A') = 2$ and

$$\delta(A', B') = \delta(\{v_0\}, B') + \delta(\{u\}, B') = 1 + (\deg(u) - 1) = \deg(u).$$

Since $13 = |E(G)| = \delta(A') + \delta(A', B') + \delta(B')$, we obtain that $\deg(u) = \delta(A', B') = 3$, and $\delta(B') = 8$. Thus, $G - A'$ is also isomorphic to $G_{2,1}$ and not $G_{2,2}$. For both $G - A$ and $G - A'$ to be isomorphic to $G_{2,1}$, v_2 must be adjacent to the two neighbours of u in $G \ominus v_0$. Thus, G is isomorphic to the graph shown in Figure 13.

However, notice that $G - w$ has LS$_+$-rank 1, which contradicts $r_+(G) = 3$. This completes the proof. $\square$

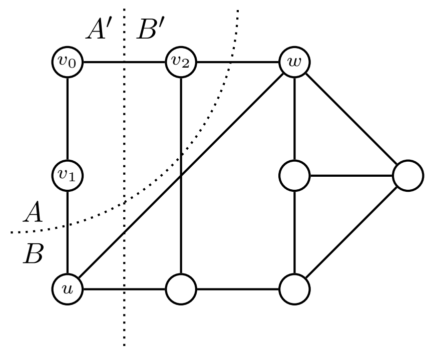

FIGURE 13. Illustrating the proof of Proposition 27

7. REVISITING H_k AND CONSTRUCTING SPARSE GRAPHS WITH HIGH LS$_+$-RANK

In this section, we revisit the graphs H_k defined in Section 1, and obtain other related graphs with high LS$_+$-ranks by applying some of our results on vertex stretching. First, we point out that the LS$_+$-rank lower bound in Theorem 2 also applies to some particular subgraphs of H_k. For every $k \geq 3$, define

$$H'_k \coloneqq H_k - \{1_0, 1_2, 2_0, 2_1\}.$$

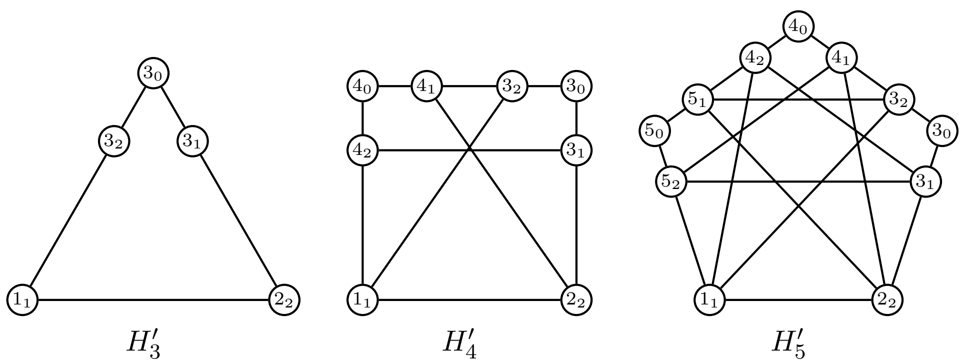

FIGURE 14. Several graphs in the family H'_k

Figure 14 illustrates the graphs H'_k for $k = 3, 4, 5$. Notice that $H'_k \in \mathcal{S}_2^{k-2}(K_k)$ for all $k \geq 3$ — this is apparent if one takes the drawings of H'_k from Figure 14 and relabels the vertices 1_1 and 2_2 by 1 and 2 respectively. Then we have the following.

Proposition 28. *For every $k \geq 3$, $r_+(H'_k) \geq \frac{3k}{16}$.*

Proof. For convenience, let $p \coloneqq \left\lceil \frac{3k}{16} \right\rceil - 1$ throughout this proof. Also, given $a, b \in \mathbb{R}$, we define the vector $w_k(a, b) \in \mathbb{R}^{V(H_k)}$ such that

$$[w_k(a,b)]_j \coloneqq \begin{cases} a & \text{if } j \in \{i_1, i_2 : i \in [k]\}; \\ b & \text{if } j \in \{i_0 : i \in [k]\}. \end{cases}$$

In [AT23], it was shown that there exists real numbers a, b where $w_k(a,b)$ is contained in $\mathrm{LS}_+^p(H_k)$ and violates the inequality

$$w_k(k-1, k-2)^\top x \leq k(k-1), \tag{10}$$

which is valid for $\mathrm{STAB}(H_k)$ [AT23, Lemma 8].

Now let $w_k(a,b)' \in \mathbb{R}^{V(H_k')}$ be the vector obtained from $w_k(a,b)$ by removing the four entries that correspond to vertices which are not in H_k'. Then by Lemma 4, we have $w_k(a,b)' \in \mathrm{LS}_+^p(H_k')$. On the other hand, the fact that $w_k(a,b)$ violates (10) implies that $(k-1)(2ka) + (k-2)(kb) > k(k-1)$, which implies that

$$\alpha_{\mathrm{LS}_+^p}(H_k') \geq \bar{e}^\top w_k(a,b)' = (2k-2)a + (k-2)b > k-1.$$

However, since $H_k' \in \mathcal{S}_2^{k-2}(K_k)$, it follows from Lemma 19 that $\alpha(H_k') = k-1$. This implies that $w_k(a,b)' \in \mathrm{LS}_+^p(H_k') \setminus \mathrm{STAB}(H_k')$, and that $r_+(H_k') \geq p+1 \geq \frac{3k}{16}$. ☐

In fact, we can use the argument above to find many subgraphs of H_k' for which the LS_+-rank lower bound given in Proposition 28 applies.

Proposition 29. *Let $G \in \mathcal{S}^{k-2}(K_k)$ be a subgraph of H_k'. Then $r_+(G) \geq \frac{3k}{16}$.*

Proof. Again, let $p := \lceil \frac{3k}{16} \rceil - 1$, and let $G \in \mathcal{S}^{k-2}(K_k)$ be a subgraph of H_k'. Since $\alpha_{\mathrm{LS}_+^p}(H_k') > k-1$ (as shown in the proof of Proposition 28, Lemma 7(ii) implies that $\alpha_{\mathrm{LS}_+^p}(G) > k-1$. But then Lemma 19 implies that $\alpha(G) = k-1$. Thus, $r_+(G) \geq p+1 \geq \frac{3k}{16}$. ☐

Given a graph G, define the *edge density* of G to be $d(G) := \frac{|E(G)|}{\binom{|V(G)|}{2}}$. For instance, $d(G) = 1$ for complete graphs, and $d(G) = 0$ for empty graphs. An interesting contrast that has emerged in the study of lift-and-project relaxations of the stable set polytope of graphs is that dense graphs tend to have high lift-and-project ranks with respect to operators that produce polyhedral relaxations, whereas graphs from both ends of the density spectrum tend to be of small lift-and-project ranks with respect to semidefinite operators. Thus, it is interesting to note that

$$d(H_k') = \frac{k^2-k-1}{\binom{3k-4}{2}} = \frac{2}{9} + o(k).$$

It was pointed to us by a reviewer that the family of graphs H_k' coincide with the family of graphs G_k in [DV15, page 675]. It is very interesting that the families of graphs H_k' have been considered as challenging instances for other but related convex relaxations of the stable set polytope. These graphs are also related to four graphs G_8, G_{11}, G_{13} and G_{17} considered as minimal obstructions in [PnVZ07] to the hierarchies discussed there which are related to the hierarchy proposed in [dKP02]. The latter four graphs are related to our family H_k'' below. These connections raise some more hope that some of our techniques and approaches in this paper may be useful for analyzing other lift-and-project operators.

Moreover, it follows from Proposition 29 that the LS_+-rank lower bound we showed for H_k' also applies for many subgraphs of H_k' with lower edge densities. For an example, given $k \geq 3$, we define the graph H_k'' where $V(H_k'') := V(H_k')$, with $E(H_k'')$ consisting of the following edges:

(i) $\{1_1, 2_2\}$;
(ii) $\{1_1, i_2\}, \{2_2, i_1\}, \{i_0, i_1\}$, and $\{i_0, i_2\}$ for every $i \in \{3, \ldots, k\}$;
(iii) $\{i_2, j_1\}$ for all $i, j \in \{3, \ldots, k\}$ where $(j-i) \bmod (k-2) < \frac{k-2}{2}$;
(iv) $\{i_2, j_1\}$ for all $i, j \in \{3, \ldots, k\}$ where $j - i = \frac{k-2}{2}$.

Observe that (iv) only contributes edges when k is even. Also, for every $k \geq 3$, notice that H_k'' is a subgraph of H_k', and that $H_k'' \in \mathcal{S}^{k-2}(K_k)$ (see Figures 15 and 16, respectively, for drawings of H_5'' and H_6''). Furthermore, H_k'' has the fewest edges among all graphs in $\mathcal{S}^{k-2}(K_k)$. To see this, suppose we start with a complete graph K_k with vertex labels $1_1, 2_2, 3, 4, \ldots, k$, and stretch the vertices $3, 4, \ldots, k$ to obtain a graph $G \in \mathcal{S}^{k-2}(K_k)$. If we define the sets $S_1 := \{1_1\}, S_2 := \{2_2\}$, and $S_i := \{i_0, i_1, i_2\}$ for all $i \in \{3, \ldots, k\}$, then there must be at least one edge in G joining S_i and S_j for all distinct $i, j \in [k]$. To minimize the number of edges in G, one can ensure that the sets A_1, A_2 are disjoint in each vertex stretching operation. This would result in a graph with exactly one edge joining S_i, S_j for all distinct $i, j \in [k]$, which is indeed the case for H_k''.

It is easy to check that $|E(H_k'')| = \frac{k^2+3k-8}{2}$, and thus $d(H_k'') = \frac{1}{9} + o(k)$. Thus, we see that there are many subgraphs of H_k' with edge densities between $\frac{1}{9}$ and $\frac{2}{9}$ for which the rank lower bound in Proposition 29 applies.

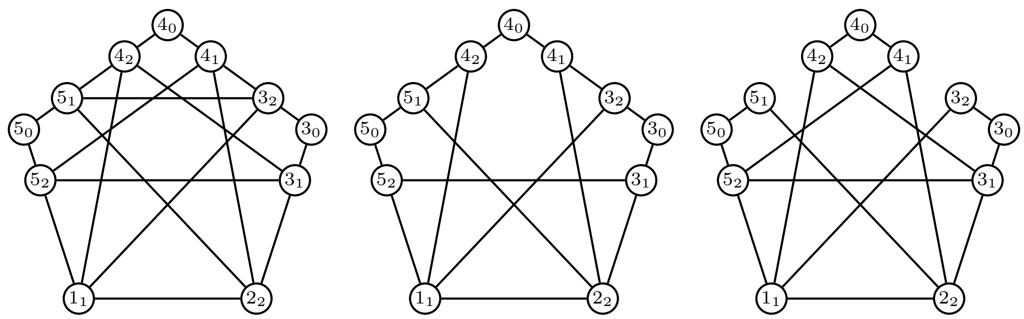

FIGURE 15. H_5' (left), H_5'' (centre), and another subgraph of H_5' in $\mathcal{S}^3(K_5)$ with the fewest possible edges (right)

Finally, we point out that we can further stretch the vertices of H_k'' to obtain very sparse graphs with arbitrarily high LS_+-ranks. Given a graph G and a vertex $v \in V(G)$, define

$$w(v) := \begin{cases} 1 & \text{if } \deg(v) \leq 3; \\ \deg(v) - 1 & \text{if } \deg(v) \geq 4. \end{cases}$$

We also define $w(G) := \sum_{v \in V(G)} w(v)$. Then we have the following.

Lemma 30. *For every graph G, there exists a graph H that can be obtained from G by a sequence of vertex stretching operations where* $\deg(v) \leq 3$ *for all $v \in V(H)$, and* $|V(H)| \leq w(G)$.

Proof. First, if every vertex in G has degree at most 3, then $H = G$ suffices, so we now assume there exists $v \in V(G)$ with $\deg(v) \geq 4$. Next, we define

$$w_1(G) := |\{v \in V(G) : \deg(v) \geq 4\}|,$$
$$w_2(G) := \sum_{i \in V(G)} \max\{\deg(i) - 3, 0\}.$$

Then $w(G) = |V(G)| + w_1(G) + w_2(G)$ for every graph G. It is helpful to think of $w_2(G)$ as the total "excess" vertex degree in G, and $w_2(G) = 0$ if and only if $\deg(v) \leq 3$ for all $v \in V(G)$. Now notice that

- If $v \in V(G)$ has $\deg(v) = 4$, we can 2-stretch it with $|A_1| = |A_2| = 2$. In this case, we obtain $H \in \mathcal{S}(G)$ with $|V(H)| = |V(G)| + 2$, $w_1(H) = w_1(G) - 1$, and $w_2(H) = w_2(G) - 1$.

- If $v \in V(G)$ has $\deg(v) = 5$, we can 3-stretch it with $|A_1| = |A_2| = 2$ and $|A_3| = 1$. In this case, we obtain $H \in \mathcal{S}(G)$ with $|V(H)| = |V(G)| + 3$, $w_1(H) = w_1(G) - 1$, and $w_2(H) = w_2(G) - 2$.
- If $v \in V(G)$ with $\deg(v) = p \geq 6$, we can 3-stretch it with $|A_1| = |A_2| = 2$ and $|A_3| = p-4$. In this case, we obtain $H \in \mathcal{S}(G)$ with $|V(H)| = |V(G)|+3$, $w_1(H) \leq w_1(G)$, and $w_2(H) = w_2(G) - 3$. (More precisely, notice that $w_1(H) = w_1(G) - 1$ if $p = 6$ and $w_1(H) = w_1(G)$ if $p \geq 7$.)

In all cases, we see that given a graph G with $w_2(G) > 0$, we can apply a stretching operation to obtain $H \in \mathcal{S}(G)$ such that $w(H) \leq w(G)$ and $w_2(H) < w_2(G)$. Iterating this process would result in a graph H with $w_1(H) = w_2(H) = 0$, which would satisfy $|V(H)| = w(H) \leq w(G)$. $\square$

Then we have the following.

Theorem 31. *For every $k \geq 5$, there exists a graph G on $k^2 - 4$ vertices such that* $\deg(i) \leq 3$ *for every $i \in V(G)$, and $r_+(G) \geq r_+(H''_k)$.*

Proof. Given $k \geq 5$, consider the graph H''_k. Notice that $\deg(1_1) = \deg(2_2) = k - 1$. Moreover, for every $i \in \{3, \ldots, k\}$, we have $\deg(i_0) = 2$, $\deg(i_1), \deg(i_2) \geq 3$, and $\deg(i_1) + \deg(i_2) = k + 1$.

Thus, using notation from the proof of Lemma 30, we obtain that $|V(H''_k)| = 3k-4$, $w_1(H''_k) \leq 2k - 2$, and $w_2(H''_k) = k^2 - 5k + 2$ (as each of $1_1, 2_2$ contributes $k - 4$ to the sum, while i_1 and i_2 together contribute $k - 5$ for every $i \in \{3, \ldots, k\}$). Therefore, $w(H''_k) \leq k^2 - 4$. Thus, we can apply Lemma 30 to obtain a graph G from stretching vertices of H''_k where $|V(G)| \leq k^2 - 4$ and $\deg(v) \leq 3$ for all $v \in V(G)$. Since stretching a vertex cannot decrease the LS_+-rank of a graph (Proposition 13), the claim follows. $\square$

Note that the bound $w_1(H''_k) \leq 2k-2$ is not tight for $k = 5$ and $k = 6$. In those cases, we can obtain a yet better bound as $w(H''_5) = 15$ and $w(H''_6) = 28$. Figure 16 illustrates H''_6 (left), and a stretched graph with $w(H''_6) = 28$ vertices which has maximum degree 3 (right). Note that we suppressed the vertex labels in this figure to reduce cluttering.

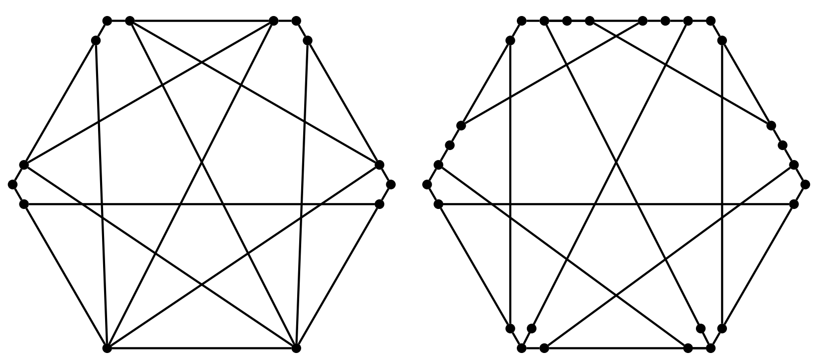

FIGURE 16. H''_6 (left), and a 28-vertex graph with maximum degree 3 obtained from stretching H''_6 (right)

For example, Figure 16 illustrates H''_6 and one possible graph obtained from successively stretching vertices of degree greater than 3 until there are no such vertices. Note that we suppressed the vertex labels in this figure to reduce cluttering.

Also, since $r_+(H''_k) = \Theta(k)$, it follows from Theorem 31 that there exists a family of graphs G with maximum degree 3 where $r_+(G) = \Omega(\sqrt{|V(G)|})$. This bound asymptotically matches the previously known bound achieved by line graphs of odd cliques, whose vertex degrees grow without bound.

8. Some Future Research Directions

In this section, we mention some follow-up questions to our work in this manuscript that could lead to interesting future research.

Problem 32. Is there an ℓ-minimal graph G in $\mathcal{S}^{\ell-1}(K_{\ell+2})$ for all $\ell \in \mathbb{N}$?

Results from [LT03, EMN06] show that the answer is "yes" for $\ell \in \{1, 2, 3\}$. Our 4-minimal graph $G_{4,1}$ shows that this is also true for $\ell = 4$. Does the pattern continue for larger ℓ? And more importantly, how can we verify the LS_+-rank of these graphs analytically, as opposed to primarily relying on specific numerical certificates?

Problem 33. Given $\ell \in \mathbb{N}$, what are the maximum and minimum possible edge densities of ℓ-minimal graphs?

Given $\ell \in \mathbb{N}$, let $d^+(\ell)$ (resp. $d^-(\ell)$) be the maximum (resp. minimum) possible edge density of an ℓ-minimal graph. It was previously known that $d^+(1) = d^-(1) = 1$ (attained by the 3-cycle), $d^+(2) = \frac{3}{5}$ ($G_{2,2}$), $d^-(2) = \frac{8}{15}$ ($G_{2,1}$), and $d^-(3) \leq \frac{7}{18}$ ($G_{3,1}$). In this work we showed that $d^-(3) = \frac{7}{18}$ (Proposition 27) and $d^+(3) \geq \frac{4}{9}$ ($G_{3,2}$). For $\ell = 4$, the discovery of $G_{4,1}$ and the other 4-minimal graphs presented in Figure 11 show that $d^-(4) \leq \frac{7}{22}$ and $d^+(4) \geq \frac{4}{11}$. Can we prove tight bounds for $d^+(\ell)$ and/or $d^-(\ell)$ in general?

Problem 34. How many non-isomorphic ℓ-minimal graphs are there for each $\ell \geq 1$?

Given $\ell \in \mathbb{N}$, let $c(\ell)$ denote the number of non-isomorphic ℓ-minimal graphs. We know that $c(1) = 1$ (the triangle) and $c(2) = 2$ ($G_{2,1}$ and $G_{2,2}$). We showed in Section 6 that $c(3) \geq 6$ ($G_{3,2}$ and its subgraphs in Figure 12) and $c(4) \geq 7$ ($G_{4,1}$ and its subgraphs in Figure 11). Does $c(\ell)$ grow without bound as ℓ increases? If so, at what rate asymptotically?

Problem 35. What is the fastest growing function f such that there exist graphs G with maximum degree at most three and $r_+(G) = \Theta(f(|V(G)|))$?

Problem 36. What is the fastest growing function f such that there exist cubic graphs G with $r_+(G) = \Theta(f(|V(G)|))$?

We proved in Section 7 that there exist very sparse graphs (maximum degree at most three) with $r_+(G) = \Theta(\sqrt{|V(G)|})$. Since all graphs G with maximum degree at most two satisfy $r_+(G) \leq 1$, Problems 35 and 36 are really about the sparsest graphs with high LS_+-ranks.

Problem 37. What can we say about the lift-and-project ranks of graphs for other positive semidefinite lift-and-project operators? To start with some concrete questions for this research problem, what are the solutions of Problems 32-36 when we replace LS_+ with $\text{Las}, \text{BZ}_+, \Theta_k$, or SA_+? (For Problem 32, we may have different sets $\mathcal{S}$, based on different graph operations, for different lift-and-project operators.)

After LS_+, many stronger semidefinite lift-and-project operators (such as Las [Las01], BZ_+ [BZ04], Θ_k [GPT10], and SA_+ [AT16]) have been proposed. While these stronger operators are capable of producing tighter relaxations than LS_+, these SDP relaxations can also be more computationally challenging to solve. For instance, while the LS_+^k-relaxation of a set $P \subseteq [0,1]^n$ involves $O(n^k)$ PSD constraints of order $O(n)$, the operators $\text{Las}^k, \text{BZ}_+^k$ and SA_+^k all impose one (or more) PSD constraint of order $\Omega(n^k)$ in their formulations. We have already briefly mentioned at the end of Section 3 that some of our tools for analyzing LS_+ relaxations can be extended to these other operators. More generally, it would be interesting to determine the corresponding properties of graphs which are minimal with respect to these stronger lift-and-project operators.

Declarations

Conflict of interest: The authors declare that they have no conflict of interest.

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Appendix A. Proofs of Lemmas 38, 39, and 40

The following lemmas provide the deferred technical details from the proof of Theorem 25. To reduce cluttering, given $S \subseteq [n]$ we will let $\hat{\chi}_S$ denote the vector $\begin{bmatrix} 1 \\ \chi_S \end{bmatrix} \in \mathbb{R}^{n+1}$.

Lemma 38. *Let Y_0 be as defined in the proof of Theorem 25. Then $Y_0(e_0 - e_1) \in \text{cone}(\text{LS}_+^2(G_{4,1}))$.*

Proof. First, notice that $[Y_0(e_0 - e_1)]_1 = 0$. Thus, let $G' := G_{4,1} - 1$ and v be the restriction of $Y_0(e_0 - e_1)$ to the coordinates indexed by $\text{cone}(\text{LS}_+^2(G'))$. Then, by Lemma 4, it suffices to show that $v \in \text{cone}(\text{LS}_+^2(G'))$. Consider the matrix

$$Y_2 := \begin{array}{c} \begin{array}{cccccccccccc} & 2 & 3 & 4_1 & 4_0 & 4_2 & 5_1 & 5_0 & 5_2 & 6_1 & 6_0 & 6_2 \end{array} \\ \begin{bmatrix} 74660 & 25340 & 25340 & 16500 & 57518 & 8662 & 8662 & 57518 & 16500 & 16500 & 49680 & 16500 \\ 25340 & 25340 & 0 & 0 & 25340 & 0 & 0 & 17166 & 8174 & 8363 & 16977 & 0 \\ 25340 & 0 & 25340 & 8174 & 17166 & 0 & 0 & 25340 & 0 & 0 & 16977 & 8363 \\ 16500 & 0 & 8174 & 16500 & 0 & 8320 & 342 & 16158 & 0 & 0 & 14067 & 2433 \\ 57518 & 25340 & 17166 & 0 & 57518 & 0 & 7678 & 41360 & 16158 & 16494 & 34971 & 14067 \\ 8662 & 0 & 0 & 8320 & 0 & 8662 & 984 & 7678 & 342 & 0 & 8662 & 0 \\ 8662 & 0 & 0 & 342 & 7678 & 984 & 8662 & 0 & 8320 & 0 & 8662 & 0 \\ 57518 & 17166 & 25340 & 16158 & 41360 & 7678 & 0 & 57518 & 0 & 14067 & 34971 & 16494 \\ 16500 & 8174 & 0 & 0 & 16158 & 342 & 8320 & 0 & 16500 & 2433 & 14067 & 0 \\ 16500 & 8363 & 0 & 0 & 16494 & 0 & 0 & 14067 & 2433 & 16500 & 0 & 8137 \\ 49680 & 16977 & 16977 & 14067 & 34971 & 8662 & 8662 & 34971 & 14067 & 0 & 49680 & 0 \\ 16500 & 0 & 8363 & 2433 & 14067 & 0 & 0 & 16494 & 0 & 8137 & 0 & 16500 \end{bmatrix} \end{array}.$$

We claim that $Y_2 \in \widehat{\text{LS}}_+^2(G')$. First, one can verify that $Y_2 \succeq 0$ (a UV-certificate is provided in Table 1). Also, notice that the function f_2 (restricted to $V(G')$) is an automorphism of G'. Moreover, observe that for all $i, j \in V(G')$, $Y_2[i,j] = Y_2[f_2(i), f_2(j)]$. Thus, by symmetry, it only remains to prove the conditions $Y_2 e_i, Y_2(e_0 - e_i) \in \text{cone}(\text{LS}_+(G'))$ for $i \in \{2, 4_1, 4_0, 4_2, 6_1, 6_0\}$.

First, notice that

- $[Y_2 e_{4_0}]_0 = [Y_2 e_{4_0}]_{4_0}$, $[Y_2 e_{4_0}]_{4_1} = [Y_2 e_{4_0}]_{4_2} = 0$, and that the following matrix certifies that $Y_2 e_{4_0}$ (with the entries corresponding to vertices $4_1, 4_0, 4_2$ removed) belongs to $\text{cone}(\text{LS}_+(G' \ominus 4_0))$.

$$Y_{21} := \begin{array}{c} \begin{array}{ccccccccc} & 2 & 3 & 5_1 & 5_0 & 5_2 & 6_1 & 6_0 & 6_2 \end{array} \\ \begin{bmatrix} 57518 & 25340 & 17164 & 7678 & 41360 & 16158 & 16496 & 34970 & 14068 \\ 25340 & 25340 & 0 & 0 & 19057 & 6283 & 5860 & 19444 & 0 \\ 17164 & 0 & 17164 & 0 & 17164 & 0 & 0 & 12010 & 5117 \\ 7678 & 0 & 0 & 7678 & 0 & 7678 & 3125 & 4516 & 0 \\ 41360 & 19057 & 17164 & 0 & 41360 & 0 & 10718 & 26585 & 10400 \\ 16158 & 6283 & 0 & 7678 & 0 & 16158 & 5778 & 8385 & 3668 \\ 16496 & 5860 & 0 & 3125 & 10718 & 5778 & 16496 & 0 & 8910 \\ 34970 & 19444 & 12010 & 4516 & 26585 & 8385 & 0 & 34970 & 0 \\ 14068 & 0 & 5117 & 0 & 10400 & 3668 & 8910 & 0 & 14068 \end{bmatrix} \end{array}$$

- $[Y_2 e_{6_0}]_0 = [Y_2 e_{6_0}]_{6_0}$, $[Y_2 e_{6_0}]_{6_1} = [Y_2 e_{6_0}]_{6_2} = 0$, and that the following matrix certifies that $Y_2 e_{6_0}$ (with the entries corresponding to vertices $6_1, 6_0, 6_2$ removed) belongs to $\text{cone}(\text{LS}_+(G' \ominus 6_0))$.

$$
Y_{22} := \begin{array}{c}
\begin{array}{ccccccccc}
 & 2 & 3 & 4_1 & 4_0 & 4_2 & 5_1 & 5_0 & 5_2
\end{array} \\
\begin{bmatrix}
49680 & 16977 & 16977 & 14068 & 34970 & 8662 & 8662 & 34970 & 14068 \\
16977 & 16977 & 0 & 0 & 16977 & 0 & 0 & 11129 & 5848 \\
16977 & 0 & 16977 & 5848 & 11129 & 0 & 0 & 16977 & 0 \\
14068 & 0 & 5848 & 14068 & 0 & 8220 & 442 & 13626 & 0 \\
34970 & 16977 & 11129 & 0 & 34970 & 0 & 7578 & 21344 & 13626 \\
8662 & 0 & 0 & 8220 & 0 & 8662 & 1084 & 7578 & 442 \\
8662 & 0 & 0 & 442 & 7578 & 1084 & 8662 & 0 & 8220 \\
34970 & 11129 & 16977 & 13626 & 21344 & 7578 & 0 & 34970 & 0 \\
14068 & 5848 & 0 & 0 & 13626 & 442 & 8220 & 0 & 14068
\end{bmatrix}
\end{array}
$$

- $[Y_2(e_0 - e_2)]_2 = 0$, and that the following matrix certifies that $Y_2(e_0 - e_2)$ (with the entry corresponding to vertex 2 removed) belongs to $\text{cone}(\text{LS}_+(G' - 2))$.

$$
Y_{23} := \begin{array}{c}
\begin{array}{ccccccccccc}
 & 3 & 4_1 & 4_0 & 4_2 & 5_1 & 5_0 & 5_2 & 6_1 & 6_0 & 6_2
\end{array} \\
\begin{bmatrix}
49320 & 25340 & 16500 & 32178 & 8662 & 8662 & 40354 & 8324 & 8137 & 32703 & 16500 \\
25340 & 25340 & 6118 & 19222 & 0 & 0 & 25340 & 0 & 0 & 19368 & 5972 \\
16500 & 6118 & 16500 & 0 & 8107 & 595 & 15905 & 0 & 2465 & 10494 & 6006 \\
32178 & 19222 & 0 & 32178 & 0 & 7688 & 24409 & 7769 & 5672 & 21928 & 10250 \\
8662 & 0 & 8107 & 0 & 8662 & 974 & 7688 & 555 & 0 & 5933 & 2729 \\
8662 & 0 & 595 & 7688 & 974 & 8662 & 0 & 8067 & 70 & 8592 & 0 \\
40354 & 25340 & 15905 & 24409 & 7688 & 0 & 40354 & 0 & 7763 & 24111 & 16243 \\
8324 & 0 & 0 & 7769 & 555 & 8067 & 0 & 8324 & 374 & 7950 & 257 \\
8137 & 0 & 2465 & 5672 & 0 & 70 & 7763 & 374 & 8137 & 0 & 8067 \\
32703 & 19368 & 10494 & 21928 & 5933 & 8592 & 24111 & 7950 & 0 & 32703 & 0 \\
16500 & 5972 & 6006 & 10250 & 2729 & 0 & 16243 & 257 & 8067 & 0 & 16500
\end{bmatrix}
\end{array}
$$

- $[Y_2(e_0 - e_{4_1})]_{4_1} = 0$, and that the following matrix certifies that $Y_2(e_0 - e_{4_1})$ (with the entry corresponding to vertex 4_1 removed) belongs to $\text{cone}(\text{LS}_+(G' - 4_1))$.

$$
Y_{24} := \begin{array}{c}
\begin{array}{ccccccccccc}
 & 2 & 3 & 4_0 & 4_2 & 5_1 & 5_0 & 5_2 & 6_1 & 6_0 & 6_2
\end{array} \\
\begin{bmatrix}
58160 & 25340 & 17164 & 57518 & 342 & 8320 & 41360 & 16500 & 16496 & 35612 & 14068 \\
25340 & 25340 & 0 & 25228 & 0 & 0 & 19229 & 6068 & 5788 & 19552 & 0 \\
17164 & 0 & 17164 & 17063 & 0 & 0 & 17055 & 0 & 0 & 12187 & 4977 \\
57518 & 25228 & 17063 & 57518 & 0 & 7946 & 41199 & 16198 & 16378 & 35219 & 13979 \\
342 & 0 & 0 & 0 & 342 & 340 & 1 & 190 & 0 & 340 & 1 \\
8320 & 0 & 0 & 7946 & 340 & 8320 & 0 & 8190 & 3046 & 5274 & 0 \\
41360 & 19229 & 17055 & 41199 & 1 & 0 & 41360 & 0 & 10763 & 26612 & 10417 \\
16500 & 6068 & 0 & 16198 & 190 & 8190 & 0 & 16500 & 5653 & 8919 & 3601 \\
16496 & 5788 & 0 & 16378 & 0 & 3046 & 10763 & 5653 & 16496 & 0 & 9091 \\
35612 & 19552 & 12187 & 35219 & 340 & 5274 & 26612 & 8919 & 0 & 35612 & 0 \\
14068 & 0 & 4977 & 13979 & 1 & 0 & 10417 & 3601 & 9091 & 0 & 14068
\end{bmatrix}
\end{array}
$$

Also, notice that $Y_{21}e_0 = Y_{21}(e_{5_0} + e_{5_2})$. Thus, if we let Y'_{21} be the matrix obtained from Y_{21} by removing the 0^{th} row and column, then we see that $Y'_{21} \succeq 0 \Rightarrow Y_{21} \succeq 0$. The UV-certificates of Y'_{21}, Y_{22}, Y_{23}, and Y_{24} are provided in Table 1.

Next, observe that

$$
\begin{aligned}
Y_2 e_2 &\leq 8291\hat{\chi}_{\{2,4_0,5_0,6_1\}} + 8873\hat{\chi}_{\{2,4_0,5_0,6_0\}} + 72\hat{\chi}_{\{2,4_0,5_2,6_1\}} + 8104\hat{\chi}_{\{2,4_0,5_2,6_0\}}, \\
Y_2 e_{4_1} &\leq 6365\hat{\chi}_{\{3,4_1,5_0,6_0\}} + 1811\hat{\chi}_{\{3,4_1,5_0,6_2\}} + 342\hat{\chi}_{\{4_1,4_2,5_1,6_0\}} + 7361\hat{\chi}_{\{4_1,4_2,5_0,6_0\}} \\
&\quad + 617\hat{\chi}_{\{4_1,4_2,5_0,6_2\}} + 4\hat{\chi}_{\{4_1,5_0,6_1,6_2\}}, \\
Y_2 e_{4_2} &\leq 642\hat{\chi}_{\{4_1,4_2,5_1,6_0\}} + 7678\hat{\chi}_{\{4_1,4_2,5_0,6_0\}} + 342\hat{\chi}_{\{4_2,5_1,5_2,6_0\}}, \\
Y_2 e_{6_1} &\leq 6764\hat{\chi}_{\{2,4_0,5_0,6_1\}} + 1599\hat{\chi}_{\{2,4_0,5_2,6_1\}} + 4\hat{\chi}_{\{4_1,5_0,6_1,6_2\}} + 7300\hat{\chi}_{\{4_0,5_0,6_1,6_2\}} \\
&\quad + 833\hat{\chi}_{\{4_0,5_2,6_1,6_2\}},
\end{aligned}
$$

$$\begin{aligned}
Y_2(e_0 - e_{4_0}) \leq{}& 7254\hat{\chi}_{\{3,4_1,5_0,6_0\}} + 1004\hat{\chi}_{\{3,4_1,5_0,6_2\}} + 489\hat{\chi}_{\{4_1,4_2,5_1,6_0\}} + 6472\hat{\chi}_{\{4_1,4_2,5_0,6_0\}} \\
&+ 1291\hat{\chi}_{\{4_1,4_2,5_0,6_2\}} + 137\hat{\chi}_{\{4_1,5_0,6_1,6_2\}} + 495\hat{\chi}_{\{4_2,5_1,5_2,6_0\}}, \\
Y_2(e_0 - e_{4_2}) \leq{}& 832\hat{\chi}_{\{2,4_0,5_0,6_1\}} + 3414\hat{\chi}_{\{2,4_0,5_0,6_0\}} + 496\hat{\chi}_{\{2,4_0,5_2,6_1\}} + 919\hat{\chi}_{\{2,4_0,5_2,6_0\}} \\
&+ 5480\hat{\chi}_{\{3,4_1,5_0,6_0\}} + 2094\hat{\chi}_{\{3,4_1,5_0,6_2\}} + 2827\hat{\chi}_{\{3,4_0,5_0,6_0\}} + 1634\hat{\chi}_{\{3,4_0,5_0,6_2\}} \\
&+ 700\hat{\chi}_{\{4_1,5_0,6_1,6_2\}} + 526\hat{\chi}_{\{4_0,5_1,5_2,6_1\}} + 1070\hat{\chi}_{\{4_0,5_15_2,6_0\}} + 690\hat{\chi}_{\{4_0,5_0,6_1,6_2\}} \\
&+ 455\hat{\chi}_{\{4_0,5_2,6_1,6_2\}} + 126\hat{\chi}_{\{4_2,5_1,5_2,6_0\}} + \frac{44735}{57518}Y_2e_{4_0}, \\
Y_2(e_0 - e_{6_1}) \leq{}& 275\hat{\chi}_{\{2,4_0,5_0,6_0\}} + 186\hat{\chi}_{\{3,4_1,5_0,6_0\}} + 2333\hat{\chi}_{\{3,4_1,5_0,6_2\}} + 186\hat{\chi}_{\{3,4_0,5_0,6_0\}} \\
&+ 5933\hat{\chi}_{\{3,4_0,5_0,6_2\}} + 140\hat{\chi}_{\{4_1,4_2,5_0,6_2\}} + 227\hat{\chi}_{\{4_0,5_1,5_2,6_0\}} + \frac{48880}{49680}Y_2e_{6_0}, \\
Y_2(e_0 - e_{6_0}) \leq{}& 7474\hat{\chi}_{\{2,4_0,5_0,6_1\}} + 978\hat{\chi}_{\{2,4_0,5_2,6_1\}} + 978\hat{\chi}_{\{3,4_1,5_0,6_2\}} + 7474\hat{\chi}_{\{3,4_0,5_0,6_2\}} \\
&+ 1454\hat{\chi}_{\{4_1,5_0,6_1,6_2\}} + 5168\hat{\chi}_{\{4_0,5_0,6_1,6_2\}} + 1454\hat{\chi}_{\{4_0,5_2,6_1,6_2\}}.
\end{aligned}$$

Since all incidence vectors above correspond to stable sets in G', and we already showed earlier that $Y_2e_{4_0}, Y_2e_{6_0} \in \mathrm{cone}(\mathrm{LS}_+(G'))$, we obtain that all the vectors above belong to $\mathrm{cone}(\mathrm{LS}_+(G'))$. Thus, we conclude that $Y_0(e_0 - e_1) \in \mathrm{cone}(\mathrm{LS}_+^2(G_{4,1}))$. □

Lemma 39. *Let Y_0 be as defined in the proof of Theorem 25. Then $Y_0e_{6_0} \in \mathrm{cone}(\mathrm{LS}_+^2(G_{4,1}))$.*

Proof. First, notice that $[Y_0e_{6_0}]_0 = [Y_0e_{6_0}]_{6_0}$, and $[Y_0e_{6_0}]_{6_1} = [Y_0e_{6_0}]_{6_2} = 0$. Thus, let $G' := G_{4,1} \ominus 6_0$ and v be the restriction of $Y_0e_{6_0}$ to the coordinates indexed by $\mathrm{cone}(\mathrm{LS}_+^2(G'))$. Then, by Lemma 4, it suffices to show that $v \in \mathrm{cone}(\mathrm{LS}_+^2(G'))$. Consider the matrix

$$Y_1 := \begin{array}{c} \\ \end{array}\begin{matrix} & 1 & 2 & 3 & 4_1 & 4_0 & 4_2 & 5_1 & 5_0 & 5_2 \end{matrix}$$
$$Y_1 := \begin{bmatrix}
75020 & 25340 & 17502 & 17502 & 15419 & 51150 & 15911 & 15911 & 51150 & 15419 \\
25340 & 25340 & 0 & 0 & 0 & 17400 & 7940 & 7940 & 17400 & 0 \\
17502 & 0 & 17502 & 0 & 0 & 17502 & 0 & 0 & 9571 & 7931 \\
17502 & 0 & 0 & 17502 & 7931 & 9571 & 0 & 0 & 17502 & 0 \\
15419 & 0 & 0 & 7931 & 15419 & 0 & 7488 & 396 & 14993 & 0 \\
51150 & 17400 & 17502 & 9571 & 0 & 51150 & 0 & 15485 & 27920 & 14993 \\
15911 & 7940 & 0 & 0 & 7488 & 0 & 15911 & 396 & 15485 & 396 \\
15911 & 7940 & 0 & 0 & 396 & 15485 & 396 & 15911 & 0 & 7488 \\
51150 & 17400 & 9571 & 17502 & 14993 & 27920 & 15485 & 0 & 51150 & 0 \\
15419 & 0 & 7931 & 0 & 0 & 14993 & 396 & 7488 & 0 & 15419
\end{bmatrix}.$$

We claim that $Y_1 \in \widehat{\mathrm{LS}}_+^2(G')$. First, one can verify that $Y_1 \succeq 0$ (a UV-certificate is provided in Table 1). Also, notice that the function f_2 (restricted to $V(G')$) is an automorphism of G'. Moreover, observe that for all $i, j \in V(G')$, $Y_1[i,j] = Y_1[f_2(i), f_2(j)]$. Thus, by symmetry, it only remains to prove the conditions $Y_1e_i, Y_1(e_0 - e_i) \in \mathrm{cone}(\mathrm{LS}_+(G'))$ for $i \in \{1, 2, 4_1, 4_0, 4_2\}$.

First, notice that $[Y_1e_{4_0}]_0 = [Y_1e_{4_0}]_{4_0}$, $[Y_1e_{4_0}]_{4_1} = [Y_1e_{4_0}]_{4_2} = 0$, and that the following matrix certifies that $Y_1e_{4_0}$ (with the entries corresponding to vertices $4_1, 4_0, 4_2$ removed) belongs to $\mathrm{cone}(\mathrm{LS}_+(G' \ominus 4_0))$. (See Table 1 for a UV-certificate.)

$$\begin{matrix} & 1 & 2 & 3 & 5_1 & 5_0 & 5_2 \end{matrix}$$
$$Y_{11} := \begin{bmatrix}
51150 & 17400 & 17502 & 9571 & 15485 & 27920 & 14993 \\
17400 & 17400 & 0 & 0 & 7544 & 9856 & 0 \\
17502 & 0 & 17502 & 0 & 0 & 10450 & 7052 \\
9571 & 0 & 0 & 9571 & 0 & 9571 & 0 \\
15485 & 7544 & 0 & 0 & 15485 & 0 & 7941 \\
27920 & 9856 & 10450 & 9571 & 0 & 27920 & 0 \\
14993 & 0 & 7052 & 0 & 7941 & 0 & 14993
\end{bmatrix}$$

Now consider the following vectors:

	1	2	3	4_1	4_0	4_2	5_1	5_0	5_2	
$z^{(1)} :=$ [	51150	17400	17502	9571	0	51150	0	5485	27920	14933	$]^\top$
$z^{(2)} :=$ [	51150	17502	17400	9571	0	51150	0	14933	27920	15485	$]^\top$
$z^{(3)} :=$ [	57518	25340	0	17164	14068	34970	16496	16158	41360	7678	$]^\top$
$z^{(4)} :=$ [	49680	0	16977	169771	40683	4970	8662	8662	34970	14068	$]^\top$

Notice that $z^{(1)} \in \operatorname{cone}(\mathrm{LS}_+(G'))$ follows from $Y_1 e_{4_0} \in \operatorname{cone}(\mathrm{LS}_+(G'))$ as shown above. Then it follows from the symmetry of G' that $z^{(2)} \in \operatorname{cone}(\mathrm{LS}_+(G'))$ as well. $z^{(3)}, z^{(4)} \in \operatorname{cone}(\mathrm{LS}_+(G'))$ follows respectively from $Y_2 e_{4_0}, Y_2 e_{6_0} \in \operatorname{cone}(\mathrm{LS}_+(G_{4,1} - 1))$, as shown in Lemma 38. Next, observe that

$$
\begin{aligned}
Y_1 e_1 &\leq 17400\hat{\chi}_{\{1,4_0,5_0\}} + 7940\hat{\chi}_{\{1,4_2,5_1\}},\\
Y_1 e_2 &\leq 9571\hat{\chi}_{\{2,4_0,5_0\}} + 7931\hat{\chi}_{\{2,4_0,5_2\}},\\
Y_1 e_{4_1} &\leq 7931\hat{\chi}_{\{3,4_1,5_0\}} + 396\hat{\chi}_{\{4_1,4_2,5_1\}} + 7092\hat{\chi}_{\{4_1,4_2,5_0\}},\\
Y_1 e_{4_2} &\leq 414\hat{\chi}_{\{1,4_0,5_1\}} + 7523\hat{\chi}_{\{2,4_0,5_0\}} + 7974\hat{\chi}_{\{3,4_0,5_0\}},\\
Y_1(e_0 - e_1) &\leq 874\hat{\chi}_{\{2,4_0,5_0\}} + 3160\hat{\chi}_{\{2,4_0,5_2\}} + 3160\hat{\chi}_{\{3,4_1,5_0\}} + 874\hat{\chi}_{\{3,4_0,5_0\}} + 1100\hat{\chi}_{\{4_1,4_2,5_0\}}\\
&\quad + 1100\hat{\chi}_{\{4_0,5_1,5_2\}} + \frac{39412}{49680}z^{(4)},\\
Y_1(e_0 - e_2) &\leq 1729\hat{\chi}_{\{1,4_0,5_1\}} + 6165\hat{\chi}_{\{1,4_0,5_0\}} + 626\hat{\chi}_{\{1,4_2,5_1\}} + 1749\hat{\chi}_{\{1,4_2,5_0\}}\\
&\quad + 4298\hat{\chi}_{\{3,4_1,5_0\}} + 2999\hat{\chi}_{\{3,4_0,5_0\}} + 1009\hat{\chi}_{\{4_1,4_2,5_1\}} + 1751\hat{\chi}_{\{4_1,4_2,5_0\}}\\
&\quad + 1959\hat{\chi}_{\{4_0,5_1,5_2\}} + 971\hat{\chi}_{\{4_2,5_1,5_2\}} + \frac{34235}{57518}z^{(3)} + 27\hat{\chi}_{\emptyset},\\
Y_1(e_0 - e_{4_1}) &\leq 498\hat{\chi}_{\{1,4_0,5_1\}} + 2500\hat{\chi}_{\{1,4_0,5_0\}} + 639\hat{\chi}_{\{1,4_2,5_1\}} + 6832\hat{\chi}_{\{1,4_2,5_0\}}\\
&\quad + 1613\hat{\chi}_{\{2,4_0,5_0\}} + 1022\hat{\chi}_{\{2,4_0,5_2\}} + 1421\hat{\chi}_{\{3,4_0,5_0\}} + 478\hat{\chi}_{\{4_0,5_1,5_2\}}\\
&\quad + 952\hat{\chi}_{\{4_2,5_1,5_2\}} + \frac{20799}{51150}z^{(1)} + \frac{22819}{51150}z^{(2)} + 28\hat{\chi}_{\emptyset},\\
Y_1(e_0 - e_{4_0}) &\leq 452\hat{\chi}_{\{1,4_0,5_1\}} + 7504\hat{\chi}_{\{2,4_0,5_0\}} + 7931\hat{\chi}_{\{3,4_1,5_0\}} + 7955\hat{\chi}_{\{3,4_0,5_0\}} + 28\hat{\chi}_{\emptyset},\\
Y_1(e_0 - e_{4_2}) &\leq 234\hat{\chi}_{\{1,4_0,5_1\}} + 195\hat{\chi}_{\{2,4_0,5_2\}} + 7935\hat{\chi}_{\{3,4_1,5_0\}} + 93\hat{\chi}_{\{3,4_0,5_0\}}\\
&\quad + \frac{46278}{51150}z^{(1)} + \frac{4354}{51150}z^{(2)} + 20\hat{\chi}_{\emptyset}.
\end{aligned}
$$

Since all incidence vectors above correspond to stable sets in G', we obtain that all the vectors above belong to $\operatorname{cone}(\mathrm{LS}_+(G'))$. Thus, we conclude that $Y_0 e_{6_0} \in \operatorname{cone}(\mathrm{LS}_+^2(G_{4,1}))$. □

Lemma 40. *Let* Y_0 *be as defined in the proof of Theorem 25. Then* $Y_0(e_0 - e_{4_1}) \in \operatorname{cone}(\mathrm{LS}_+^2(G_{4,1}))$.

Proof. For convenience, let $G := G_{4,1}$ throughout this proof. Using $Y_0 e_{6_0} \in \operatorname{cone}(\mathrm{LS}_+^2(G))$ from Lemma 39 and the symmetry of G, we know that the vector

	1	2	3	4_1	4_0	4_2	5_1	5_0	5_2	6_1	6_0	6_2	
$z :=$ [	75020	17502	25340	17502	0	75020	0	15419	51150	15911	15911	51150	15419	$]^\top$

belongs to $\operatorname{cone}(\mathrm{LS}_+^2(G))$. Now observe that

$$\begin{aligned}
Y_0(e_0 - e_{4_1}) \leq \frac{2}{3}z + \frac{1}{3}\Big(&7726\hat{\chi}_{\{1,4_0,5_1,6_0\}} + 17105\hat{\chi}_{\{1,4_0,5_0,6_0\}} + 16187\hat{\chi}_{\{1,4_2,5_0,6_0\}} \\
&+ 8509\hat{\chi}_{\{2,4_0,5_0,6_1\}} + 8324\hat{\chi}_{\{2,4_0,5_0,6_0\}} + 8509\hat{\chi}_{\{2,4_0,5_2,6_0\}} + 9486\hat{\chi}_{\{3,4_0,5_0,6_0\}} \\
&+ 8017\hat{\chi}_{\{3,4_0,5_0,6_2\}} + 7403\hat{\chi}_{\{4_0,5_0,6_1,6_2\}} + 9170\hat{\chi}_{\{4_2,5_1,5_2,6_0\}} + 24\hat{\chi}_{\emptyset}\Big).
\end{aligned}$$

Notice that all incidence vectors above correspond to stable sets in G. Since $\operatorname{cone}(\mathrm{LS}_+^2(G))$ is a lower-comprehensive convex cone, it follows that $Y_0(e_0 - e_{4_1}) \in \operatorname{cone}(\mathrm{LS}_+^2(G))$. □

Finally, we provide in Table 1 the UV-certificates of all PSD matrices used in Theorem 25 and Lemmas 38, 39, and 40.

Y_0 — U

−2	1	1	1	1	1	1	1	1	1	1	1	1
0	114	−8	−107	152	27	−114	202	411	348	−353	−440	−234
0	−57	128	−71	−320	−492	−335	292	270	69	29	221	268
−361	−710	−711	−711	−88	434	−88	−88	434	−88	−89	434	−87
0	−135	521	−387	477	230	31	395	−171	−771	−872	−60	741
0	525	−145	−379	731	−65	−873	−779	−167	409	48	231	462
0	0	0	0	−984	0	984	−984	0	984	−984	0	984
820	−787	−788	−787	1222	−569	1222	1221	−569	1221	1222	−569	1222
0	1693	−2745	1052	−901	857	−628	1238	−329	−652	−337	−529	1280
0	−2192	−370	2562	909	116	−1116	326	−801	1102	−1235	685	14
0	1845	−595	−1250	308	−990	1042	1707	−2079	1128	−2014	3069	−2170
0	−378	1787	−1409	−2148	2972	−1904	1341	−2343	1854	808	−629	50
8215	2154	2154	2154	1246	6313	1246	1246	6313	1246	1246	6313	1246

Y_0 — V

11050	1142	1601	781	−196	621	−196	624	621	624	−557	621	165
1142	15917	−509	−499	−1882	3390	−156	−27	432	−915	−1711	−214	1694
1601	−509	12805	−677	958	−1473	1373	−587	−70	59	−379	235	227
781	−499	−677	14384	1269	58	20	75	1698	−1891	−868	901	703
−196	−1882	958	1269	13455	−739	−38	−706	1544	−297	−936	409	−415
621	3390	−1473	58	−739	11866	−186	540	−364	186	−530	381	544
−196	−156	1373	20	−38	−186	8568	−1358	220	469	156	−1144	−163
624	−27	−587	75	−706	540	−1358	10158	1776	−61	476	−1550	172
621	432	−70	1698	1544	−364	220	1776	11890	−1026	1209	198	23
624	−915	59	−1891	−297	186	469	−61	−1026	13347	−1927	−193	1281
−557	−1711	−379	−868	−936	−530	156	476	1209	−1927	11530	−292	43
621	−214	235	901	409	381	−1144	−1550	198	−193	−292	9703	1337
165	1694	227	703	−415	544	−163	172	23	1281	43	1337	8773

Y_2 — U

−2	1	1	0	0	1	1	0	0	1	1	0
0	1	−2	72	70	−5	5	−70	−72	0	0	0
−67	−50	−51	82	87	−35	−35	88	82	−57	−58	−56
−56	−129	−129	35	−44	−250	−250	−44	35	142	278	143
0	−110	110	−110	141	384	−384	−141	111	52	0	−52
0	216	−216	−18	25	125	−125	−25	18	−567	0	567
−144	−423	−423	−424	255	253	252	255	−424	−48	263	−48
362	−556	−556	538	−221	531	531	−221	538	546	−248	546
0	928	−928	442	−429	492	−492	429	−442	429	0	−429
−62	82	82	−588	568	−470	−470	568	−588	1034	−1489	1034
0	−1020	1021	1031	−1020	432	−432	1020	−1031	−371	0	371
3377	1203	1203	673	2686	338	338	2686	673	717	2294	717

Y_2 — V

4918	41	−26	−35	−29	−89	−233	38	−35	−130	576	−9
41	6219	172	148	1281	−511	448	−1143	908	−675	173	−80
−26	172	4074	−79	−16	399	−532	329	1079	54	115	−922
−35	148	−79	5725	−112	974	−28	−20	201	−548	953	−344
−29	1281	−16	−112	4852	−1238	−471	−31	−79	−27	−22	−1183
−89	−511	399	974	−1238	5976	−273	−691	12	−533	26	−469
−233	448	−532	−28	−471	−273	6481	−948	934	−803	289	−295
38	−1143	329	−20	−31	−691	−948	4677	−53	−1083	36	−14
−35	908	1079	201	−79	12	934	−53	5504	−279	953	−613
−130	−675	54	−548	−27	−533	−803	−1083	−279	4546	77	−53
576	173	115	953	−22	26	289	36	953	77	9521	−142
−9	−80	−922	−344	−1183	−469	−295	−14	−613	−53	−142	4375

Y'_{21} — U

−44	−1	0	−20	−1	0	0	0
233	22	465	229	20	−410	−389	86
385	793	−241	−149	305	244	−460	−545
439	137	−877	−360	468	−700	−18	959
1565	−1333	−1075	528	−1287	508	−788	−327
655	−1159	930	−530	1923	1995	−641	1022
−1375	1105	−499	1255	−949	1180	−1505	1780
2466	1620	307	4002	775	986	3444	835

Y'_{21} — V

3758	−873	−519	−1	636	9	−429	−479
−873	7118	−18	2774	−91	−423	−1334	−135
−519	−18	1822	17	282	−78	70	−402
−1	2774	17	8485	−2504	−708	22	687
636	−91	282	−2504	6178	−107	586	225
9	−423	−78	−708	−107	3903	−635	−904
−429	−1334	70	22	586	−635	5449	602
−479	−135	−402	687	225	−904	602	5052

Y_{22} — U

5	−3	−3	−1	−1	−3	−3	−1	−1
−1	−12	9	8011	8038	−189	190	−8035	−8009
−7048	−7604	−7605	6746	7345	−5664	−5664	7345	6752
−1	8338	−8339	10389	−11032	−29236	29235	11032	−10389
5767	23052	23052	28635	−19951	−26290	−26290	−19952	28635
0	82739	−82739	16909	−15913	35612	−35612	15913	−16909
32757	−40385	−40385	61612	−35034	61878	61878	−35034	61613
0	55744	−55744	−88585	87206	−48485	48485	−87206	88585
239799	88332	88332	60363	177671	34664	34664	177671	60363

Y_{22} — V

245290	67346	43642	33983	37	8790	67640	11795	6716
67346	262681	−47235	−60196	1386	−15331	−35125	−4652	21255
43642	−47235	230858	−23610	−7313	−78942	460	18674	−1875
33983	−60196	−23610	293133	−56794	−5587	28035	3646	3783
37	1386	−7313	−56794	116143	891	−8115	−5728	−26068
8790	−15331	−78942	−5587	891	159853	−29478	−14768	−1859
67640	−35125	460	28035	−8115	−29478	217945	−6899	−36239
11795	−4652	18674	3646	−5728	−14768	−6899	124459	2908
6716	21255	−1875	3783	−26068	−1859	−36239	2908	120960

Y_{23} — U

−1	−1	0	0	−1	0	0	0	−1	0	0
3	0	−10	−5	−6	−1	11	−3	−6	23	24
−1	7	26	29	7	−50	−61	−4	4	18	20
−56	−71	75	74	−61	−45	63	55	−78	−15	−13
−20	−29	71	77	−26	125	−34	−186	−29	4	12
3	−6	219	−154	−525	−3	−17	16	490	160	−165
71	395	245	−200	−264	256	−195	248	−296	−219	283
−382	1053	−281	−46	−387	−664	309	−650	−488	157	−530
−61	135	−946	906	−666	−101	25	−28	493	−776	714
25	−59	764	−724	359	−748	769	−732	442	−855	883
2625	1525	824	1782	403	373	2243	358	385	1800	822

Y_{23} — V

3268	−126	−24	−112	−237	198	−265	583	192	−200	631
−126	5487	−1306	−32	22	−488	501	−233	−286	8	753
−24	−1306	3523	−177	4	802	29	9	−228	−99	75
−112	−32	−177	2815	−3	−4	−57	928	47	597	−258
−237	22	4	−3	4247	119	−1336	82	−822	−708	8
198	−488	802	−4	119	3240	189	165	337	−120	68
−265	501	29	−57	−1336	189	4825	96	−29	796	459
583	−233	9	928	82	165	96	3654	262	374	−122
192	−286	−228	47	−822	337	−29	262	4115	−118	199
−200	8	−99	597	−708	−120	796	374	−118	4024	−139
631	753	75	−258	8	68	459	−122	199	−139	4828

Y_{24} — U

−23	−1	−1	0	0	−1	−1	−1	0	0	−1
20	7	6	−12	−25	0	−15	−19	0	5	10
−4	−1	2	44	43	−2	−42	−42	−2	−2	0
72	31	31	−119	136	21	3	6	13	7	20
−41	−70	16	−41	−1	−164	−61	26	192	188	−6
−19	−131	−304	−36	25	107	83	−101	−95	203	256
−13	−170	−62	−28	16	369	166	−186	294	26	−360
82	−767	605	68	15	358	−253	330	−375	269	44
33	510	−537	20	15	373	−620	652	−252	506	−562
−222	245	337	−215	−3	−333	402	−620	−773	659	−567
2058	955	634	2045	7	242	1551	501	533	1338	451

Y_{24} — V

3039	256	−4	11	33	616	259	212	195	517	213
256	2748	−376	−1	−168	−556	192	−60	147	−215	−197
−4	−376	2699	−164	9	−114	−33	−861	7	513	−52
11	−1	−164	3056	−981	−238	125	−117	651	−6	−116
33	−168	9	−981	2948	−11	−251	5	−464	102	−130
616	−556	−114	−238	−11	3982	71	516	145	253	−36
259	192	−33	125	−251	71	4861	−1304	80	1182	218
212	−60	−861	−117	5	516	−1304	3960	18	−14	8
195	147	7	651	−464	145	80	18	2459	90	−12
517	−215	513	−6	102	253	1182	−14	90	5059	1081
213	−197	−52	−116	−130	−36	218	8	−12	1081	2689

Y_1 — U

−3	1	1	1	1	1	1	1	1	1
0	0	312	−312	−663	−1314	−1080	1080	1313	665
639	262	1834	1834	−1199	−1101	1007	1007	−1100	−1201
−573	−2813	−954	−953	−1543	1304	1872	1872	1305	−1544
0	0	1938	−1939	3269	304	−1822	1822	−303	−3269
0	0	6254	−6254	−1331	−1182	4069	−4069	1182	1331
−2501	2818	1055	1055	−5112	3150	−4138	−4138	3151	−5112
−123	−9594	4954	4954	2208	1240	−3751	−3751	1240	2208
0	0	3904	−3904	−6195	8605	−5528	5528	−8605	6195
22183	7784	5318	5318	4055	15768	4304	4304	15768	4055

Y_1 — V

15782	3040	944	1517	−961	810	−1074	−1074	3245	−256
3040	28790	−3737	−924	−1274	561	−1625	−1625	294	−3563
944	−3737	23948	−126	−1931	2745	203	−3361	−4007	3326
1517	−924	−126	21978	6048	−1446	−7055	153	828	−318
−961	−1274	−1931	6048	37807	−12613	−459	−3449	188	3617
810	561	2745	−1446	−12613	36247	852	−2975	6408	−2004
−1074	−1625	203	−7055	−459	852	18359	322	−974	2597
−1074	−1625	−3361	153	−3449	−2975	322	18359	1369	1267
3245	294	−4007	828	188	6408	−974	1369	32772	−4348
−256	−3563	3326	−318	3617	−2004	2597	1267	−4348	27264

Y_{11} — U

−2	1	1	1	1	1	1
1218	1284	1319	2378	−1339	−3175	−1273
11	2338	−2256	−81	−4392	40	4398
−981	5772	5824	−8802	−2137	−362	−2372
169	−10840	10788	−537	−4810	−823	6490
2335	1031	−1134	−5111	10842	−11453	9705
26748	9331	9439	5079	7256	15899	7008

Y_{11} — V

32150	−109	12752	4226	506	−460	13236
−109	21193	568	−2619	−6911	−795	−2712
12752	568	20891	130	−600	863	−5224
4226	−2619	130	19072	−4217	463	−1160
506	−6911	−600	−4217	20400	2412	−2691
−460	−795	863	463	2412	32551	−3917
13236	−2712	−5224	−1160	−2691	−3917	37492

TABLE 1. UV-certificates for matrices in the proofs of Theorem 25 and Lemmas 38, 39, and 40