

Shadow splitting methods for nonconvex optimisation: epi-approximation, convergence and saddle point avoidance

Felipe Atenas*

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Abstract

We propose the shadow Davis-Yin three-operator splitting method to solve nonconvex optimisation problems. Its convergence analysis is based on a merit function resembling the Moreau envelope. We explore variational analysis properties behind the merit function and the iteration operators associated with the shadow method. By capitalising on these results, we establish convergence of a damped version of the shadow method via sufficient descent of the merit function, and obtain (almost surely) guarantees of avoidance of strict saddle points of weakly convex semialgebraic optimisation problems. We perform numerical experiments for a nonconvex variable selection problem to test our findings.

Keywords. Nonconvex optimisation, weak convexity, Davis-Yin splitting, Douglas-Rachford splitting, Moreau envelope, strict saddle point

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1 Introduction

We consider structured nonconvex optimisation problems of the form

$$\min_{x \in \mathbb{R}^d} \varphi(x) := f(x) + g(x) + h(x), \quad (1)$$

where $f, h : \mathbb{R}^d \rightarrow \mathbb{R}$ are smooth functions, and $g : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is a *prox-friendly* function, that is, a function whose proximal operator prox_g (see (4)) is readily available or easy to compute. Our main interest in this work is an iterative method to solve problems of the form (1) that exploits the separable nature of the objective function by iteratively performing separate operations in the model components. In particular, we explore a variant of the Davis-Yin (DY) three-operator splitting method with damping. The original form of the DY method can be stated as follows: given an initial point $z^0 \in \mathbb{R}^d$, a stepsize parameter $\gamma > 0$, a relaxation parameter $\lambda > 0$, and an iteration counter $k \geq 1$, the k -th iteration of the DY method is given by

$$\begin{cases} x^k &= \text{prox}_{\gamma f}(z^k) \\ y^k &= \text{prox}_{\gamma g}(2x^k - z^k - \gamma \nabla h(x^k)) \\ z^{k+1} &= z^k + \lambda(y^k - x^k). \end{cases} \quad (2)$$

*School of Mathematics and Statistics, University of Melbourne, felipe.atenas@unimelb.edu.au

The iterative scheme in (2) reduces to two well-known two-operator splitting methods [15, 23]: when $f = 0$, it corresponds to the proximal gradient or *forward-backward* (FB) method, and when $h = 0$, it is the *Douglas-Rachford* (DR) method. Problems with this type of structure appear in several applications in statistics, signal processing, and machine learning [10, 29, 32].

The DY method was originally introduced in [14] to solve convex optimisation problems in the form of (1), under the assumption that both f and g are convex, and h is smooth and convex with cocoercive gradient. In [2, 9], the authors established convergence of the method in a nonconvex setting by means of suitable *merit functions* that satisfy a sufficient descent condition alongside iterations, where f and h are smooth with Lipschitz continuous gradients and g is prox-friendly nonconvex. Under similar lenses, another three-operator splitting method for nonconvex optimisation problems is examined in [1]. Moreover, the authors of [24] analysed the method in (2) as a variable metric gradient method for the so-called *Davis-Yin envelope*, and obtained conditions for the FB and the DR methods to avoid *strict saddle points* with high probability in the fully smooth case, namely, assuming in addition that f, g and h are C^2 functions.

Our goal is to investigate the reach of the Davis-Yin envelope and its corresponding solution map in nonconvex optimisation. In [24], this envelope is tailored to the fixed-point sequence $(z_k)_{k \geq 1}$ defined in (2). Instead, the envelope in this work is defined in such a way that its argument is associated with the sequence $(x_k)_{k \geq 1}$ in (2), the so-called *shadow sequence*. Namely, our Davis-Yin envelope (DYE) is defined for all $x \in \mathbb{R}^d$ as

$$\varphi_\gamma^{\text{DY}}(x) := \inf_{y \in \mathbb{R}^d} \left\{ f(x) + \langle \nabla f(x), y - x \rangle + h(x) + \langle \nabla h(x), y - x \rangle + g(y) + \frac{1}{2\gamma} \|y - x\|^2 \right\}. \quad (3)$$

We investigate smoothing and approximation properties of the DYE in (3), and continuity properties of the associated solution map. Additionally, we also examine how the DY method avoids solutions of “poor quality”, that is, (strict) saddle points, when nonsmoothness appears in a structured manner. For the latter, we establish convergence of a damped version of (2), presented in Algorithm 1, and state conditions under which this method converges to local minimisers, almost surely. We discover that the shadow sequence plays a central role in the avoidance of strict saddle points. Characterising this behaviour is particularly relevant in nonconvex optimisation, since different from the convex case, stationary points may not always be global minimisers.

Input: Choose a starting point $x^0 \in \mathbb{R}^d$, a stepsize $\gamma > 0$, a damping parameter $\alpha \in (0, 1)$, and a relaxation parameter $\lambda > 0$.

for $k = 1, 2, \dots$ **do**

Step 1. Proximal gradient step. Define

$$y^k = \text{prox}_{\gamma g}(x^k - \gamma \nabla(f + h)(x^k)).$$

Step 2. Damped shadow step. Define

$$x^{k+1} = (1 - \alpha)x^k + \alpha \text{prox}_{\gamma f}((1 - \lambda)x^k + \gamma \nabla f(x^k) + \lambda y^k).$$

end

Algorithm 1: Damped shadow Davis-Yin splitting for solving (1).

In Algorithm 1, Step 1 is a proximal gradient step applying the proximal operator to g and a gradient step to the smooth function $f + h$. As for Step 2, observe that f and h do not play symmetric roles: although Step 1 is performed for $f + h$ as one smooth term,

the argument of the proximal operation in Step 2 is not $(1 - \lambda)x^k + \gamma \nabla(f + h)(x^k) + \lambda y^k$. Hence, Algorithm 1 is conceptually different from a damped version of the proximal gradient method to minimise $(f + h) + g$. We shall see in Section 4 that Algorithm 1 is essentially equivalent to the scheme in (2) when $\alpha = 1$.

Our analysis is based on the shadow sequence, an approach that has been largely unexplored for splitting in nonconvex optimisation. In [12] the shadow Douglas-Rachford method is introduced for monotone inclusions, and thus applicable to convex optimisation. For nonconvex optimisation, in [2, 9, 36], the authors examine splitting methods using envelopes that take the fixed-point sequence $(z_k)_{k \geq 1}$ as input. The relevance of the latter sequence comes from the convex optimisation case (or, more generally, for the maximally monotone case for operators), as splitting methods are known to be equivalent to fixed-point iterations of an appropriate nonexpansive operator (i.e. Lipschitz continuous with constant $L = 1$), see [4, 16, 25] and references therein. In the convex optimisation case, the base of the analysis is usually *Fejér monotonicity* with respect to the set of fixed-points of the iteration operator (or some generalisation), a property that may fail to hold in the nonconvex setting. Although useful, the fixed-point sequence does not seem to reveal new insights on splitting methods for nonconvex optimisation, and thus we propose to centre the analysis around the shadow sequence. Another fact that supports our proposed approach is that, in both convex and nonconvex case, the customary argument is the following: establish (subsequential) convergence of the fixed-point sequence $(z_k)_{k \geq 1}$, from where solutions of the problem in (1) are recovered by applying $\text{prox}_{\gamma f}$ to the limits of $(z_k)_{k \geq 1}$, which are in turn limits of the shadow sequence $(x_k)_{k \geq 1}$. This line of reasoning suggests that $(z_k)_{k \geq 1}$ is an artifact that facilitates the analysis, but it does not directly provide solutions to the optimisation problem, and a different perspective might be available. This is exactly what we explore in this work: we skip the fixed-point sequence $(z_k)_{k \geq 1}$, and directly inspect the shadow sequence $(x_k)_{k \geq 1}$.

The rest of this paper is organised as follows. In Section 2, we present some background needed for our analysis, including facts from variational analysis and a brief discussion on envelopes. In Section 3, we investigate the iteration operators of the (damped) shadow DY method (Algorithm 1), and approximation and smoothing properties of the DYE. Section 4 is dedicated to the analysis of Algorithm 1 using the shadow sequence, establishing its asymptotic behaviour. We show in Section 5 that the damped shadow DY method converges to local minimisers almost surely under structural properties of the objective function in problem (1). Section 6 illustrates our results for the nonconvex elastic net variable selection problem. In Section 7, we conclude with some final remarks and future research directions.

2 Notation and preliminaries

A function $\phi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be proper when its domain is nonempty, that is, $\text{dom}(\phi) := \{x \in \mathbb{R}^d : \phi(x) < +\infty\} \neq \emptyset$. The epigraph of ϕ is the set $\text{epi}(\phi) := \{(x, t) \in \mathbb{R}^d \times \mathbb{R} : \phi(x) \leq t\}$. We say that ϕ is level-bounded if it has bounded level sets, that is, if for any $\alpha \in \mathbb{R}$, the set $\text{lev}_\phi(\alpha) := \{x \in \mathbb{R}^d : \phi(x) \leq \alpha\}$ is bounded. A function ϕ is locally Lipschitz continuous, if for all $\bar{x} \in \text{dom}(\phi)$, there exists a neighbourhood U of $\bar{x}$, and a constant $L = L(U) > 0$, such that for all $x, y \in U$, $|\phi(x) - \phi(y)| \leq L\|x - y\|$. We say ϕ is (globally) Lipschitz continuous if the latter estimate holds for $U = \mathbb{R}^d$ with a uniform constant $L > 0$ over the whole space. A function ϕ is ρ -weakly convex ($\rho \geq 0$) whenever $\phi(\cdot) + \frac{\rho}{2}\|\cdot\|^2$ is convex. For a differentiable function ϕ , we say ϕ is L_ϕ -smooth if its gradient $\nabla\phi$ is Lipschitz continuous with parameter $L_\phi \geq 0$. In particular, such a function ϕ is ρ -weakly convex, where ρ is bounded below by the smallest Lipschitz constant of $\nabla\phi$ [5, Proposition 2.4]. If ϕ is a L_ϕ -smooth C^2 function, then for all $x \in \mathbb{R}^d$, $\|\nabla^2\phi(x)\|_{\text{op}} \leq L_f$, where $\|\cdot\|_{\text{op}}$ denotes the operator norm, and $I \pm \gamma \nabla^2\phi(x)$ is positive definite for $\gamma \in (0, L_\phi^{-1})$. If ϕ is ρ -

weakly convex and twice continuously differentiable, then $\mathbb{R}^d \ni x \mapsto \text{prox}_{\gamma\phi}(x)$ is well-defined and continuously differentiable for $\gamma \in (0, \rho^{-1})$ [24, Lemma 4.2].

2.1 Variational analysis

For a set $C \subseteq \mathbb{R}^d$, and a sequence of sets $(C^k)_{k \geq 1}$ in $\mathbb{R}^d$, the inner limit of $(C^k)_{k \geq 1}$, denoted $\liminf C^k$, is the set of all limit points of sequences $(x^k)_{k \geq 1}$ such that $x^k \in C^k$ for all $k \geq 1$, that is,

$$\liminf C^k = \{x \in \mathbb{R}^d \mid \exists C^k \ni x^k \rightarrow x\}.$$

The outer limit of $(C^k)_{k \geq 1}$, denoted $\limsup C^k$, is the set of all cluster points of sequences $(x^k)_{k \geq 1}$ with elements in C^k throughout the respective convergent subsequence, that is,

$$\limsup C^k = \{x \in \mathbb{R}^d \mid \exists (x_k)_{k \geq 1} \subseteq \mathbb{R}^d, \text{ and a subsequence } C^{k_j} \ni x^{k_j} \rightarrow x \text{ as } j \rightarrow +\infty\}.$$

Note that $\liminf C^k \subseteq \limsup C^k$. We say $(C^k)_{k \geq 1}$ set-converges to C , denoted $C^k \xrightarrow{s} C$, if

$$\limsup C^k \subseteq C \subseteq \liminf C^k.$$

A set-valued map $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is said to be outer semicontinuous at $\bar{x}$ if $\limsup S(x^k) \subseteq S(\bar{x})$ for any sequence $x^k \rightarrow \bar{x}$. For a function $\phi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$, we say a sequence of extended-valued functions $(\phi^k)_{k \geq 1}$ *epi-converges* to ϕ , denoted $\phi^k \xrightarrow{e} \phi$, if $\text{epi}(\phi^k) \xrightarrow{s} \text{epi}(\phi)$.

For a lower semicontinuous function ϕ , we say $s_\phi : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is an *epi-smoothing* function [11] for ϕ if (i) $s_\phi(\cdot, \gamma_k)$ epi-converges to ϕ for all $\gamma_k \downarrow 0$, and (ii) $s_\phi(\cdot, \gamma)$ is continuously differentiable for all $\gamma > 0$. An example of an epi-smoothing function is the Moreau envelope of a proper lower semicontinuous convex function (see [11]), which we proceed to define. Given a point $x \in \mathbb{R}^d$ and a proper function $\phi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$, the proximal operator of ϕ at x with prox-parameter $\gamma > 0$ is defined as

$$\text{prox}_{\gamma\phi}(x) := \underset{y \in \mathbb{R}^d}{\text{argmin}} \left\{ \phi(y) + \frac{1}{2\gamma} \|y - x\|^2 \right\}. \quad (4)$$

The optimal value of this minimisation problem defines the Moreau envelope at x ,

$$\phi_\gamma^M(x) := \inf_{y \in \mathbb{R}^d} \left\{ \phi(y) + \frac{1}{2\gamma} \|y - x\|^2 \right\}.$$

The proximal operator and the Moreau envelope are well-defined whenever ϕ is prox-bounded with threshold $\gamma_\phi > 0$, namely, when $\phi(\cdot) + \frac{1}{2\gamma_\phi} \|\cdot\|^2$ is bounded from below. In this case, $\text{prox}_{\gamma\phi}$ is outer semicontinuous [34, Example 5.23(b)]. If ϕ is ρ -weakly convex (and thus prox-bounded with threshold ρ^{-1}), then for any $\gamma \in (0, \rho^{-1})$, $\text{prox}_{\gamma\phi}$ is single-valued and Lipschitz continuous with constant $\frac{\rho}{\rho-\gamma}$ [34, Proposition 12.19], and ϕ_γ^M is continuously differentiable with gradient given by

$$\nabla \phi_\gamma^M(x) = \frac{1}{\gamma} (x - \text{prox}_{\gamma\phi}(x)), \quad (5)$$

such that $\nabla \phi_\gamma^M$ is Lipschitz continuous with constant $\max\{\gamma^{-1}, \frac{\rho}{1-\rho\gamma}\}$ [20, Corollary 3.4].

For a proper lower semicontinuous function $\phi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$, the (limiting) subdifferential of ϕ at $\bar{x} \in \text{dom}(\phi)$ is defined as the outer limit of the Fréchet subdifferential in the ϕ -attentive sense [34, Definition 8.3], namely,

$$\partial\phi(\bar{x}) = \limsup_{\substack{x \rightarrow \bar{x} \\ \phi(x) \rightarrow \phi(\bar{x})}} \left\{ s \in \mathbb{R}^d : \liminf_{\substack{y \rightarrow x \\ y \neq x}} \frac{\phi(y) - \phi(x) - \langle s, y - x \rangle}{\|y - x\|} \geq 0 \right\}.$$

Observe that $\partial\phi$ is outer semicontinuous from construction. When ϕ is convex, $\partial\phi$ coincides with the subdifferential of convex analysis. In the latter case, the subdifferential is known to be a maximally monotone operator [34, Theorems 12.17 & 12.25]. A point $x \in \mathbb{R}^d$ such that $0 \in \partial\phi(x)$ is said to be a stationary point of ϕ . For the problem in (1), x is a stationary point of φ if $0 \in \partial g(x) + \nabla(f + h)(x)$ in view of the subdifferential sum rule [34, Exercise 8.8(c)]. We say $\phi^* \in \mathbb{R}$ is a stationary value of the function ϕ if there exists a stationary point x^* of ϕ such that $\phi(x^*) = \phi^*$. For a set-valued map $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$, D^*S denotes the coderivative of S [34, Definition 8.33].

2.2 Envelopes and iteration operators

From (5), it directly follows that $\text{prox}_{\gamma\phi} = I - \gamma\nabla\phi_\gamma^M$ when ϕ is weakly convex. Hence, the proximal point method [26, 33], whose iterates are defined via $x^{k+1} = \text{prox}_{\gamma\phi}(x^k)$ for all $k \geq 1$, can be interpreted as the gradient descent method on ϕ_γ^M . For this reason, similar merit functions have been proposed in the literature [24, 36] to analyse splitting methods as methods of descent. In particular, in [24] the following merit function is introduced: for all $z \in \mathbb{R}^d$,

$$V_\gamma(z) := f_\gamma^M(z) - \gamma\|\nabla f_\gamma^M(z)\|^2 - \gamma\langle \nabla h(\text{prox}_{\gamma f}(z)), \nabla f_\gamma^M(z) \rangle + h(\text{prox}_{\gamma f}(z)) \\ - \frac{\gamma}{2}\|\nabla h(\text{prox}_{\gamma f}(z))\|^2 + g_\gamma^M\left(z - 2\gamma\nabla f_\gamma^M(z) - \gamma\nabla h(\text{prox}_{\gamma f}(z))\right).$$

This definition is inspired by the Douglas-Rachford envelope introduced in [30]. One of the advantages of the above formulation is its explicit relation with the Moreau envelope, and thus properties of the latter can be exploited directly. Nevertheless, it lacks interpretability. Here, we reformulate it similarly to the merit functions presented in [2, 36]. After some algebra manipulations, one yields to

$$V_\gamma(z) = \inf_{y \in \mathbb{R}^d} \left\{ f(\text{prox}_{\gamma f}(z)) + \langle \nabla f(\text{prox}_{\gamma f}(z)), y - \text{prox}_{\gamma f}(z) \rangle \right. \\ \left. + h(\text{prox}_{\gamma f}(z)) + \langle \nabla h(\text{prox}_{\gamma f}(z)), y - \text{prox}_{\gamma f}(z) \rangle \right. \\ \left. + g(y) + \frac{1}{2\gamma}\|y - \text{prox}_{\gamma f}(z)\|^2 \right\}. \quad (6)$$

The formulation in (6) suggests that V_γ is the value function associated with a model of the objective function in (1), where f and h are replaced by their first-order Taylor approximations. The first-order optimality condition of the problem in the second line in (2) reveals that $y = y^k$ solves the problem in (6) when $z = z^k$. Observe that the argument of the merit function in (6) is related to the sequence $(z_k)_{k \geq 1}$ in (2). The merit function in (6) coincides with the one defined in [2, eq. (3.7)], when $p = 0$ therein.

In this work, we propose the DYE introduced in (3), in such a way that its argument is linked to the sequence $(x_k)_{k \geq 1}$ in (2). Observe that $\varphi_\gamma^{\text{DY}} \circ \text{prox}_{\gamma f} = V_\gamma$. This seemingly naive change of variables has a direct impact in the study of saddle point avoidance (see Section 5), as it explicitly relates the DYE with the merit function known as the *forward-backward envelope*, without the need of the composition with $\text{prox}_{\gamma f}$ as above. More precisely, given $x \in \mathbb{R}^d$, the forward-backward envelope

$$\varphi_{\gamma, f, g}^{\text{FB}}(x) := \inf_{y \in \mathbb{R}^d} \left\{ f(x) + \langle \nabla f(x), y - x \rangle + g(y) + \frac{1}{2\gamma}\|y - x\|^2 \right\}, \quad (7)$$

satisfies

$$\varphi_\gamma^{\text{DY}}(x) = \varphi_{\gamma, f+h, g}^{\text{FB}}(x). \quad (8)$$

In the following, we state our blanket assumptions and properties of the DYE (3).

Assumption 1. Consider the optimisation problem in (1), and suppose that

- (i) $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is L_f -smooth, and that $h : \mathbb{R}^d \rightarrow \mathbb{R}$ is L_h -smooth,
- (ii) $g : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper lower semicontinuous and prox-bounded with threshold $\gamma_g > 0$,
- (iii) the problem in (1) has a nonempty set of minimisers. In particular, φ is bounded below.

Lemma 1. Let Assumption 1 hold, and $\gamma \in (0, \min\{\gamma_g, (L_f + L_h)^{-1}\})$. Then,

- (i) $\varphi_\gamma^{\text{DY}}$ is locally Lipschitz continuous
- (ii) $\inf \varphi = \inf \varphi_\gamma^{\text{DY}}$
- (iii) φ is level bounded if and only if $\varphi_\gamma^{\text{DY}}$ is level bounded.

Proof. Given that $\varphi_\gamma^{\text{DY}} \circ \text{prox}_{\gamma f} = V_\gamma$, then item (i) can be deduced similarly to [36, Proposition 3.2], item (ii) can be obtained from [24, Theorem 4.1 1.], and item (iii) is analogous to [36, Theorem 3.4]. $\square$

Remark 1. In [9], a different merit function is introduced to analyse the DY method. In the context of the iterative scheme (2), said merit function, denoted $\mathfrak{D}_\gamma$, satisfies $\mathfrak{D}_\gamma(x^k, y^k, z^k) = V_\gamma(z^k) - \frac{1}{\gamma} \|z^{k+1} - z^k\|^2$ for all $k \geq 1$. Hence, $\mathfrak{D}_\gamma(x^k, y^k, z^k) \leq V_\gamma(z^k) \leq \varphi(x^k)$, where the second inequality follows from taking $z = z^k$ and $y = x^k$ in (6). In this manner, the merit function V_γ (and thus, the DYE in (3)) is a tighter approximation of φ than $\mathfrak{D}_\gamma$. As a consequence, the range of stepsizes for which the method in (2) is convergent is larger when V_γ is used in the analysis instead of $\mathfrak{D}_\gamma$ (see, e.g. [2, Figure 1]). This fact has a direct consequence in the numerical performance of the method.

2.3 Active manifolds

Given a map $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$, we say T is a *lipeomorphism* if T is Lipschitz continuous, and its inverse T^{-1} exists and is Lipschitz continuous as well. A point $\bar{z}$ is a fixed-point of T if $\bar{z} = T(\bar{z})$. If T is a C^1 map around some fixed-point $\bar{z}$, we say $\bar{z}$ is an *unstable fixed-point* of T if the Jacobian $\nabla T(\bar{z})$ has at least one eigenvalue with magnitude strictly greater than one.

We say that a set $\mathcal{M} \subseteq \mathbb{R}^d$ is a C^2 -smooth manifold around $\bar{x} \in \mathcal{M}$ [13, Definition 2.2], if there exist an open neighbourhood U of $\bar{x}$, and a C^2 function G defined on $\mathbb{R}^d$, such that $\nabla G(\bar{x})$ has full rank, and $\mathcal{M} \cap U = \{x \in \mathbb{R}^d : G(x) = 0\}$. Intuitively, such $\mathcal{M}$ can be locally described around $\bar{x}$ as the solution of smooth equations with linearly independent gradients at $\bar{x}$. The system of equations $G = 0$ is called the local defining equations for $\mathcal{M}$ around $\bar{x}$, and $T_{\mathcal{M}}(\bar{x}) := \ker(\nabla G(\bar{x}))$ is the tangent space to $\mathcal{M}$ at $\bar{x}$.

Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous ρ -weakly convex function, $\bar{x} \in \mathbb{R}^d$ be a stationary point of ϕ , and $\mathcal{M} \ni \bar{x}$. The set $\mathcal{M}$ is said to be an *active C^2 -smooth manifold* of ϕ around $\bar{x}$ [13, Definition 2.6], if there exists a neighborhood U of $\bar{x}$, such that the following two properties hold: (i) (smoothness) $\mathcal{M} \cap U$ is a C^2 -smooth manifold around $\bar{x}$, such that ϕ is a C^2 function on $\mathcal{M} \cap U$, and (ii) (sharpness) $\inf\{\|s\| : s \in \partial\phi(x), x \in U \setminus \mathcal{M}\} > 0$. The sharpness condition essentially states that normal to the manifold, the function cannot be flat, that is, the norm of subgradients at points outside $\mathcal{M}$ are bounded away from 0. Naturally, these two conditions holds whenever ϕ is a C^2 function.

We say that a stationary point $\bar{x}$ of a weakly convex function $\phi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is a *strict saddle* (point) of ϕ [13, Definition 2.7], if there exists an active C^2 -smooth manifold $\mathcal{M}$

of ϕ at $\bar{x}$, such that for some vector $u \in T_{\mathcal{M}}(\bar{x})$, the parabolic subderivative [34, Definition 13.59] satisfies $d^2\phi_{\mathcal{M}}(\bar{x})(u) < 0$. A geometric interpretation of strict saddle points is that the function ϕ restricted to $\mathcal{M}$ has negative curvature at such points. Moreover, we say ϕ satisfies the *strict saddle property*, if any stationary point of ϕ is either a strict saddle point or a local minimiser. As shown in [13, Theorem 2.9], the strict saddle property is “generic”, in the sense that it holds for perturbations of ϕ drawn from a set of full Lebesgue measure, when ϕ is lower semicontinuous weakly convex and semialgebraic. A function is semialgebraic if its graph can be described as the solution of a finite system of polynomial inequalities. The semialgebraic assumptions is now state-of-the-art due to the seminal work [8]. Examples of weakly convex semialgebraic functions are nonconvex regularisers used in statistics, for instance, the smoothly clipped absolute deviation and the minimax concave penalty [17, 38]. The strict saddle property is the key assumption in Section 5.

3 Properties of the DYE and solution maps

We start the analysis of the DYE from a variational point of view. In this section, we establish that $\varphi_{\gamma}^{\text{DY}}$ is an epi-smoothing function of the objective function φ in (1), and deduce continuity properties for the associated solution map. These properties provide an intuitive interpretation of why envelopes help explain the behaviour of splitting methods (see [3, Section 3]).

3.1 Epi-approximation and epi-smoothing of the DYE

We first establish that $\varphi_{\gamma}^{\text{DY}}$ *epi-approximates* φ (cf. [3, Corollary 3.2.1]). The following result is a direct consequence of (8) and [3, Theorem 3.2].

Proposition 1 (DYE epigraphic convergence). Let Assumption 1 hold, and suppose $g : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper lower semicontinuous ρ -weakly convex. Then, for any $\gamma_k \downarrow 0$, $\varphi_{\gamma_k}^{\text{DY}} \xrightarrow{e} \varphi$.

For the DYE to be an epi-smoothing function, we also need to establish when it is continuously differentiable. First, we characterise its subdifferential. For that, we denote the *forward-backward map*, the solution map of the problem in (7), by

$$\mathbb{R}^d \times \mathbb{R}_{++} \ni (x, \gamma) \mapsto Y_{f,g}^{\text{FB}}(x, \gamma) := \text{prox}_{\gamma g} \left(x - \gamma \nabla f(x) \right). \quad (9)$$

The following result follows directly from (8) and [3, Lemma 3.3].

Lemma 2 (Subdifferential of DYE). Let Assumption 1 hold. Suppose that f and h are twice continuously differentiable with Lipschitz continuous Hessians. Then, for any $\gamma \in (0, \min\{\gamma_g, (L_f + L_h)^{-1}\})$ and $x \in \mathbb{R}^d$,

$$\partial \varphi_{\gamma}^{\text{DY}}(x) = \gamma^{-1} \left(I - \gamma \nabla^2(f + h)(x) \right) \left[x - \text{conv} \left(Y_{f+h,g}^{\text{FB}}(x) \right) \right],$$

where $\text{conv}(\cdot)$ denotes the convex hull, and the sum (subtraction) in the right-most brackets should be understood in the Minkowski sense.

We are now ready to state the epi-smoothing property of the DYE.

Theorem 1 (DYE as an epi-smoothing function). Let Assumption 1 hold. Suppose that f and h are twice continuously differentiable with Lipschitz continuous Hessians, and $g : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper lower semicontinuous ρ -weakly convex. Then, $\mathbb{R}^d \times \mathbb{R}_{++} \ni (x, \gamma) \mapsto \varphi_{\gamma}^{\text{DY}}(x)$ is an epi-smoothing of φ .

Proof. Epiconvergence of the DYE to φ is Proposition 1. Moreover, from Lemma 2, the gradient of the DYE (cf. [24, eq. (17)]) is given by

$$\nabla \varphi_\gamma^{\text{DY}}(x) = \gamma^{-1} \left(I - \gamma \nabla^2(f + h)(x) \right) \left(x - Y_{f+h,g}^{\text{FB}}(x) \right).$$

which is continuously differentiable for any $\gamma \in (0, \min\{\rho^{-1}, L_f^{-1}\})$, in view of (9) and continuous differentiability of the proximal operator. $\square$

Remark 2 (On the redefinition of the DYE). In (3), we defined the DYE as a function of the “shadow” variable x , instead of the “fixed-point” variable z . Its direct consequence in Proposition 1, Lemma 2, and Theorem 1, is that in contrast to [3, Corollary 3.2.1 & Corollary 3.2.2], the proximal operator $\text{prox}_{\gamma g}$ does not play a role in any of these results. This fact is key in Section 5.

3.2 Solution maps: continuity properties

In this section, we study continuity properties of the forward-backward solution map (9) and its consequences for the *DY map*, defined as

$$Y_{f,h,g}^{\text{DY}} := Y_{f+h,g}^{\text{FB}}. \quad (10)$$

The relationship between the DYE and the forward-backward envelope (and, by extension, with the Moreau envelope), suggests that not only epi-properties are inherited as discussed above, but also properties of their solution maps, even in nonconvex optimisation. By construction, any property of the forward-backward map will also hold for the DY map, under appropriate regularity assumptions. In the following, we list these properties for DY map, and then we prove them for the forward-backward map.

Proposition 2 (Properties of the DY solution map). Suppose $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is L_f -smooth, $h : \mathbb{R}^d \rightarrow \mathbb{R}$ is L_h -smooth, and $g : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper lower semicontinuous ρ -weakly convex. Then, the following statements hold true.

- (i) $Y_{f,h,g}^{\text{DY}}(\cdot, \gamma)$ is Lipschitz continuous for $\gamma \in (0, \rho^{-1})$. More specifically, for all $u_1, u_2 \in \mathbb{R}^d$,

$$\|Y_{f,h,g}^{\text{DY}}(u_1, \gamma) - Y_{f,h,g}^{\text{DY}}(u_2, \gamma)\| \leq \left(\frac{1 + \gamma(L_f + L_h)}{1 - \gamma\rho} \right) \|u_1 - u_2\|.$$

- (ii) $Y_{f,h,g}^{\text{DY}}(u^k, \gamma_k) \rightarrow \text{proj}_{\overline{\text{dom}(\varphi)}}(\bar{u})$ whenever $u^k \rightarrow \bar{u}$ and $\gamma_k \downarrow 0$.
- (iii) For any $x \in \text{dom}(\varphi)$, $\|Y_{f,h,g}^{\text{DY}}(x, \gamma) - x\| = O(\gamma)$ as $\gamma \downarrow 0$.
- (iv) If, in addition, f and h are C^2 functions, then $Y_{f,h,g}^{\text{DY}}$ is a locally Lipschitz continuous function at any $(\bar{x}, \bar{\gamma}) \in \text{int}(\text{dom}(g)) \times (0, \min\{\rho^{-1}, (L_f + L_h)^{-1}\})$.

Proof. Items (i), (ii), (iii), and (iv) follow from (8) and Proposition 3, Proposition 4, Proposition 5, and Proposition 6 below, respectively. $\square$

Remark 3. Note that in view of Proposition 2 (ii) and (iv), $Y_{f,h,g}^{\text{DY}}$ can be extended continuously to $\text{dom}(g) \times [0, \min\{\rho^{-1}, (L_f + L_h)^{-1}\})$ (cf. [4, Proposition 3.4]).

We now proceed to prove each statement in Proposition 2 for $h = 0$.

Proposition 3. Suppose $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is L_f -smooth, and $g : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper lower semicontinuous ρ -weakly convex. Then, the operator $Y_{f,g}^{\text{FB}}(\cdot, \gamma)$ is Lipschitz continuous for $\gamma \in (0, \rho^{-1})$. More specifically, for all $u_1, u_2 \in \mathbb{R}^d$,

$$\|Y_{f,g}^{\text{FB}}(u_1, \gamma) - Y_{f,g}^{\text{FB}}(u_2, \gamma)\| \leq \left(\frac{1 + \gamma L_f}{1 - \gamma \rho} \right) \|u_1 - u_2\|.$$

Proof. From the first-order optimality condition of (9), for $i = 1, 2$, we have:

$$\gamma^{-1}(u_i - \gamma \nabla f(u_i)) + (\rho - \gamma^{-1})Y_{f,g}^{\text{FB}}(u_i, \gamma) \in \left(\partial g(\cdot) + \rho I \right) \left(Y_{f,g}^{\text{FB}}(u_i, \gamma) \right).$$

Since $\partial g(\cdot) + \rho I$ is monotone, then

$$\begin{aligned} 0 \leq & \langle Y_{f,g}^{\text{FB}}(u_1, \gamma) - Y_{f,g}^{\text{FB}}(u_2, \gamma), \gamma^{-1}(u_1 - \gamma \nabla f(u_1)) - \gamma^{-1}(u_2 - \gamma \nabla f(u_2)) \rangle \\ & + (\rho - \gamma^{-1}) \langle Y_{f,g}^{\text{FB}}(u_1, \gamma) - Y_{f,g}^{\text{FB}}(u_2, \gamma), Y_{f,g}^{\text{FB}}(u_1, \gamma) - Y_{f,g}^{\text{FB}}(u_2, \gamma) \rangle, \end{aligned}$$

which implies

$$\begin{aligned} (\gamma^{-1} - \rho) \|Y_{f,g}^{\text{FB}}(u_1, \gamma) - Y_{f,g}^{\text{FB}}(u_2, \gamma)\|^2 & \leq \gamma^{-1} \langle Y_{f,g}^{\text{FB}}(u_1, \gamma) - Y_{f,g}^{\text{FB}}(u_2, \gamma), u_1 - u_2 \rangle \\ & \quad + \langle Y_{f,g}^{\text{FB}}(u_1, \gamma) - Y_{f,g}^{\text{FB}}(u_2, \gamma), \nabla f(u_2) - \nabla f(u_1) \rangle \\ & \leq (\gamma^{-1} + L_f) \|Y_{f,g}^{\text{FB}}(u_1, \gamma) - Y_{f,g}^{\text{FB}}(u_2, \gamma)\| \|u_1 - u_2\|, \end{aligned}$$

where the last line follows from Lipschitz continuity of ∇f . Since $\gamma^{-1} > \rho$, then the desired estimate follows. $\square$

In the previous section, we examined the behaviour of the DYE as the stepsize approaches zero. In the next result, we investigate the behaviour of $Y_{f,g}^{\text{FB}}$ as $\gamma \downarrow 0$ (cf. [18, Proposition 5]).

Proposition 4. Suppose $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is L_f -smooth, and $g : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper lower semicontinuous ρ -weakly convex, with $\varphi = f + g$. Then, $Y_{f,g}^{\text{FB}}(u^k, \gamma_k) \rightarrow \text{proj}_{\overline{\text{dom}(\varphi)}}(\bar{u})$ whenever $u^k \rightarrow \bar{u}$ and $\gamma_k \downarrow 0$.

Proof. Consider any $\gamma_k \downarrow 0$, and $u^k \rightarrow \bar{u}$. Define

$$\begin{aligned} \phi_k(u) &:= \gamma_k \left(f(u^k) + \langle \nabla f(u^k), u - u^k \rangle + g(u) \right) + \frac{1}{2} \|u - u^k\|^2 \\ &= \gamma_k \left(f(u^k) + \langle \nabla f(u^k), u - u^k \rangle \right) + \frac{1}{2} (1 - \rho \gamma_k) \|u\|^2 - \langle u^k, u \rangle + \frac{1}{2} \|u^k\|^2 \\ &\quad + \gamma_k \left(g(u) + \frac{\rho}{2} \|u\|^2 \right). \end{aligned}$$

Observe that $Y_{f,g}^{\text{FB}}(u^k, \gamma_k) = \text{argmin } \phi_k$. Next, note that

$$\gamma_k \left(f(u^k) + \langle \nabla f(u^k), \cdot - u^k \rangle \right) + \frac{1}{2} (1 - \rho \gamma_k) \|\cdot\|^2 - \langle u^k, \cdot \rangle + \frac{1}{2} \|u^k\|^2 \xrightarrow{c} \frac{1}{2} \|\cdot - \bar{u}\|^2,$$

where $\xrightarrow{c}$ denotes continuous convergence of maps [34, Definition 5.41]. Furthermore, from [18, Proposition 4(b)],

$$\gamma_k \left(g(\cdot) + \frac{\rho}{2} \|\cdot\|^2 \right) \xrightarrow{e} \iota_{\overline{\text{dom}(\varphi)}}(\cdot).$$

Therefore, in view of [34, Theorem 7.46(b)],

$$\phi_k \xrightarrow{e} \phi := \iota_{\overline{\text{dom}(\varphi)}}(\cdot) + \frac{1}{2} \|\cdot - \bar{u}\|^2.$$

Since ϕ_k is (strongly) convex and ϕ is strongly convex, and thus level-bounded, [34, Exercise 7.32(c)] implies that $(\phi_k)_{k \geq 1}$ is eventually level-bounded. Hence, [34, Theorem 7.33] implies that

$$Y_{f,g}^{\text{FB}}(u^k, \gamma_k) = \operatorname{argmin} \phi_k \rightarrow \operatorname{argmin} \phi = \operatorname{proj}_{\operatorname{dom}(\varphi)}(\bar{u}).$$

□

The result above characterises the behaviour of $Y_{f,g}^{\text{FB}}(x, \gamma)$ when both arguments x and γ vary. In the following result, for a fixed $x \in \operatorname{dom}(g)$, we establish the speed of convergence of $Y_{f,g}^{\text{FB}}(x, \gamma)$ to x as $\gamma \downarrow 0$.

Proposition 5. Suppose $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is L_f -smooth, and $g : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper lower semicontinuous ρ -weakly convex, with $\varphi = f + g$. Then, for any $x \in \operatorname{dom}(g)$, $\|Y_{f,g}^{\text{FB}}(x, \gamma_k) - x\| = O(\gamma)$ as $\gamma \downarrow 0$.

Proof. We first show the property for the proximal operator of g . Fix $x \in \operatorname{dom}(g)$. For all $\gamma \in (0, \rho^{-1})$, define $g_{\rho,x}(\cdot) = g(\cdot) + \frac{\rho}{2} \|\cdot - x\|^2$, in such a way that $\operatorname{prox}_{\gamma g}(x) = \operatorname{prox}_{(\gamma^{-1} - \rho)^{-1} g_{\rho,x}}(x)$. Define $A_x = \partial g_{\rho,x}$, which is maximally monotone as $g_{\rho,x}$ is convex. Since $(\gamma^{-1} - \rho)^{-1} = \frac{\gamma}{1 - \rho\gamma} \downarrow 0$ as $\gamma \downarrow 0$, then in view of [6, Remark 3.32], for $u \in \operatorname{dom}(g)$, the *Moreau-Yosida approximation* of A_x at u with parameter $(\gamma^{-1} - \rho)^{-1} > 0$, namely,

$$\frac{1}{(\gamma^{-1} - \rho^{-1})^{-1}}(u - \operatorname{prox}_{(\gamma^{-1} - \rho)^{-1} g_{\rho,x}}(u)),$$

converges to the minimal norm subgradient in $A_x(u)$. Furthermore, for $u = x$, the Moreau-Yosida approximation above coincides with $\nabla(g_{\rho,x})_{\gamma^{-1} - \rho^{-1}}^M(x)$ in view of (5), which is thus bounded as $\gamma \downarrow 0$. Moreover, since $x - \operatorname{prox}_{\gamma g}(x)$ is bounded as $\gamma \downarrow 0$ (take $f = 0$ and $u^k = x$ in Proposition 4), then

$$\frac{1}{\gamma}(x - \operatorname{prox}_{\gamma g}(x)) = \nabla(g_{\rho,x})_{(\gamma^{-1} - \rho)^{-1}}(x) + \rho^{-1}(x - \operatorname{prox}_{\gamma g}(x))$$

is also bounded as $\gamma \downarrow 0$. Hence $\|\operatorname{prox}_{\gamma g}(x) - x\| = O(\gamma)$. Next, since

$$\frac{1}{\gamma}(Y_{f,g}^{\text{FB}}(x, \gamma) - x) = \frac{1}{\gamma}(Y_{f,g}^{\text{FB}}(x, \gamma) - \operatorname{prox}_{\gamma g}(x)) + \frac{1}{\gamma}(\operatorname{prox}_{\gamma g}(x) - x),$$

it is sufficient to prove that $\frac{1}{\gamma}(Y_{f,g}^{\text{FB}}(x, \gamma) - \operatorname{prox}_{\gamma g}(x))$ remains bounded as $\gamma \downarrow 0$ to show the desired property of the forward-backward map. In view of Lipschitz continuity of $\operatorname{prox}_{\gamma g}$, it holds that

$$\begin{aligned} \frac{1}{\gamma}\|Y_{f,g}^{\text{FB}}(x, \gamma) - \operatorname{prox}_{\gamma g}(x)\| &= \frac{1}{\gamma}\|\operatorname{prox}_{\gamma g}(x - \gamma \nabla f(x)) - \operatorname{prox}_{\gamma g}(x)\| \\ &\leq \frac{1}{\gamma} \left(\frac{\rho}{1 - \gamma\rho} \right) \|x - \gamma \nabla f(x) - x\| \\ &= \left(\frac{\rho}{1 - \gamma\rho} \right) \|\nabla f(x)\|. \end{aligned}$$

This concludes the proof.

□

To close this section, we show that as a function of the joint variable (x, γ) , the operator $Y_{f,g}^{\text{FB}}(x, \gamma)$ is locally Lipschitz continuous (cf. [18, Theorem 3 (b)] and [4, Proposition 3.4]).

Proposition 6. Suppose $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is L_f -smooth and a C^2 function, and $g : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper lower semicontinuous ρ -weakly convex. Then, $\mathbb{R}^d \times \mathbb{R}_{++} \ni (x, \gamma) \mapsto Y_{f,g}^{\text{FB}}(x, \gamma)$ is a locally Lipschitz continuous function at any $(\bar{x}, \bar{\gamma}) \in \text{int}(\text{dom}(g)) \times (0, \rho^{-1})$.

Proof. The result will be a consequence of an implicit map theorem [34, Theorem 9.56 (a)], as in the proof of [18, Theorem 3 (b)]. Given $(\bar{x}, \bar{\gamma}) \in \mathbb{R}^d \times \mathbb{R}_{++}$, let $\bar{y} = Y_{f,g}^{\text{FB}}(\bar{x}, \bar{\gamma})$. Define the set-valued map

$$S(x, \gamma, y) := \nabla f(x) + \partial g(y) + \gamma^{-1}(y - x).$$

We study local continuity properties of the generalised equation $S(x, \gamma, y) \ni 0$ with parameter (x, γ) and variable y , with solution map given by $Y_{f,g}^{\text{FB}}(x, \gamma) = \{y \in \mathbb{R}^d : S(x, \gamma, y) \ni 0\}$. Let

$$\begin{aligned} S_0(x, \gamma, y) &:= \partial g(y) + \rho(y - x), \\ F(x, \gamma, y) &:= \nabla f(x) + L_f(x - y) + (\gamma^{-1} - \rho + L_f)(y - x), \\ \text{so that } S &= S_0 + F. \end{aligned}$$

From [34, Exercise 10.43(b)],

$$D^*S(\bar{x}, \bar{\gamma}, \bar{y}|0)(y) = D^*S_0(\bar{x}, \bar{\gamma}, \bar{y}| - F(\bar{x}, \bar{\gamma}, \bar{y}))(y) + \nabla F(\bar{x}, \bar{\gamma}, \bar{y})y,$$

where

$$\begin{aligned} D^*S_0(\bar{x}, \bar{\gamma}, \bar{y}| - F(\bar{x}, \bar{\gamma}, \bar{y}))(y) &= \{-\rho y\} \times \{0\} \times D^*(\partial g(\cdot) + \rho I)(\bar{y}| - F(\bar{x}, \bar{\gamma}, \bar{y}))(y), \\ \nabla F(\bar{x}, \bar{\gamma}, \bar{y})y &= \left(\nabla^2 f(\bar{x})y - (\gamma^{-1} - \rho)y, \bar{\gamma}^{-2} \langle \bar{x} - \bar{y}, y \rangle, (\gamma^{-1} - \rho)y \right). \end{aligned}$$

In order to apply [34, Theorem 9.56 (a)], we need to prove that if $(u, \gamma, 0) \in D^*S(\bar{x}, \bar{\gamma}, \bar{y}|0)(y)$, then $y = u = 0$ and $\gamma = 0$. Let $(u, \gamma, 0) \in D^*S(\bar{x}, \bar{\gamma}, \bar{y}|0)(y)$, which is equivalent to

$$\begin{cases} u &= \nabla^2 f(\bar{x})y - \bar{\gamma}^{-1}y \\ \gamma &= \bar{\gamma}^{-2} \langle \bar{x} - \bar{y}, y \rangle \\ -(\gamma^{-1} - \rho)y &\in D^*(\partial g(\cdot) + \rho I)(\bar{y}| - F(\bar{x}, \bar{\gamma}, \bar{y}))(y) \end{cases}$$

The third inclusion together with [18, Lemma 3] implies $-(\gamma^{-1} - \rho)\|y\|^2 \geq 0$, hence $y = 0$. The second identity thus implies $\gamma = 0$, and the first one yields $u = 0$. In view of [34, Theorem 9.56 (a)], $Y_{f,g}^{\text{FB}}$ has the Aubin property at $(\bar{x}, \bar{\gamma})$ for $\bar{y}$ [34, Definition 9.36]. Since it is a single-valued map, then $Y_{f,g}^{\text{FB}}$ is locally Lipschitz continuous around $(\bar{x}, \bar{\gamma})$. $\square$

In this section, we explored the variational analysis properties behind solution maps of splitting methods in nonconvex optimisation. In the next section, we examine the asymptotic behaviour of the sequences generated by these solution maps.

4 Damped Davis-Yin splitting method

Now we turn the analysis of splitting methods to convergence via envelopes. It is known that the DY method in (2) asymptotically produces stationary points of φ in the nonconvex problem (1), under now standard regularity assumptions, see e.g. [2, 9]. However, these convergence results are not enough in the context of saddle point avoidance. As we shall see in Section 5, avoidance of strict saddle points can be guaranteed, almost surely, for a damped version of the method in (2). This section is devoted to the convergence analysis of this variant of the method using the shadow sequence.

4.1 The shadow sequence

In this section, we first show that the method in (2) is equivalent to Algorithm 1 when $\alpha = 1$. As mentioned above, the shadow DY sequence is the input of the DYE in (3). We start by introducing the associated iteration operators.

Proposition 7 (Iteration operators). Let Assumption 1 hold. Let $\rho_f \geq 0$ be the constant of weak convexity of f , $\gamma \in (0, \min\{\gamma_g, \rho_f^{-1}\})$,

$$Y_\gamma := Y_{f,h,g}^{\text{DY}},$$

and $x \in \mathbb{R}^d$. Consider the shadow DY iteration operator

$$T_\gamma(x) := \text{prox}_{\gamma f} \left((1 - \lambda)x + \gamma \nabla f(x) + \lambda Y_\gamma(x) \right). \quad (11)$$

Then,

$$\gamma^{-1} \left((1 - \lambda)x + \gamma \nabla f(x) + \lambda Y_\gamma(x) - T_\gamma(x) \right) = \nabla f(T_\gamma(x)). \quad (12)$$

Proof. Since f is ρ_f -weakly convex, then $T_\gamma(x)$ is well-defined, and (12) corresponds to the first-order optimality condition of the optimisation problem in (11). $\square$

Remark 4. Under the assumptions of Proposition 7, the operator $\mathbb{R}^d \ni x \mapsto Y_\gamma(x)$ is set-valued. Hence, in (12) (and throughout this section), we abuse the notation and understand $Y_\gamma(x)$ therein as a selection of a proximal point of γg centred at $x - \gamma \nabla(f + h)(x)$ (unless stated otherwise).

Proposition 8 (Shadow DY splitting). Let Assumption 1 hold, $\rho_f \geq 0$ be the constant of weak convexity of f , and $\gamma \in (0, \min\{\gamma_g, \rho_f^{-1}\})$. Then, the shadow DY sequence $(x_k)_{k \geq 1}$ generated by (2) is equivalently determined by

$$x^{k+1} = \text{prox}_{\gamma f} \left((1 - \lambda)x^k + \gamma \nabla f(x^k) + \lambda \text{prox}_{\gamma g} \left(x^k - \gamma \nabla(f + h)(x^k) \right) \right). \quad (13)$$

In other words, the shadow sequence $(x_k)_{k \geq 1}$ generated by (2) is the same as the one generated via Algorithm 1 with $\alpha = 1$, whenever initialised at the same point.

Proof. From the optimality condition of the f -proximal step in (2), it holds that $z^k = x^k + \gamma \nabla f(x^k)$. Substituting this identity in the g -proximal step in (2) yields $y^k = \text{prox}_{\gamma g} \left(x^k - \gamma \nabla(f + h)(x^k) \right)$. Therefore, the third step in (2) is equivalent to

$$z^{k+1} = \gamma \nabla f(x^k) + (1 - \lambda)x^k + \lambda \text{prox}_{\gamma g} \left(x^k - \gamma \nabla(f + h)(x^k) \right).$$

Using this identity and the fact that $x^{k+1} = \text{prox}_{\gamma f}(z^{k+1})$, we conclude (13). $\square$

The reformulation in Proposition 8 is not only convenient for the convergence analysis, but also to characterise stationary points of φ in terms of fixed-points of the shadow iteration operator.

Proposition 9 (Davis-Yin splitting: stationary points are fixed points). Let Assumption 1 hold, and $\rho_f \geq 0$ be the constant of weak convexity of f . Then, for any $\gamma \in (0, \min\{\gamma_g, \rho_f^{-1}\})$, the following statements are equivalent.

- (i) $\bar{x}$ is a fixed-point of T_γ .

(ii) $\bar{x}$ is a fixed-point of Y_γ .

(iii) $\bar{x}$ is a stationary point of $\varphi = f + g + h$.

Proof. Let $x \in \mathbb{R}^d$, and define $x^+ = T_\gamma(x)$. From Proposition 7, x^+ is the unique point satisfying

$$\begin{aligned} \nabla f(x^+) &= \gamma^{-1} \left((1 - \lambda)x + \gamma \nabla f(x) + \lambda Y_\gamma(x) - x^+ \right) \\ \iff Y_\gamma(x) &= \frac{1}{\lambda} \left(x^+ + \gamma \left(\nabla f(x^+) - \nabla f(x) \right) - (1 - \lambda)x \right). \end{aligned}$$

and

$$\frac{1}{\gamma} \left(x - \gamma \nabla(f + h)(x) - Y_\gamma(x) \right) \in \partial g(S_\gamma(x))$$

If $T_\gamma(\bar{x}) = \bar{x}$, then

$$Y_\gamma(x) = \frac{1}{\lambda} \left(\bar{x} + \gamma \left(\nabla f(\bar{x}) - \nabla f(\bar{x}) \right) - (1 - \lambda)\bar{x} \right) = \bar{x},$$

and

$$-\nabla(f + h)(\bar{x}) = \frac{1}{\gamma} \left(\bar{x} - \gamma \nabla(f + h)(\bar{x}) - Y_\gamma(\bar{x}) \right) \in \partial g(\bar{x}),$$

that is, $\bar{x}$ is a stationary point of φ . The converse follows similarly, since $T_\gamma(\bar{x})$ and $Y_\gamma(\bar{x})$ satisfy their corresponding first-order optimality conditions. Indeed, if $\bar{x}$ is a stationary point of φ , then

$$\frac{1}{\gamma} \left(\bar{x} - \gamma \nabla(f + h)(\bar{x}) - \bar{x} \right) \in \partial g(\bar{x}),$$

which is equivalent to $\bar{x} = Y_\gamma(\bar{x})$. In turn, this identity also implies

$$\frac{1}{\gamma} \left((1 - \lambda)\bar{x} + \gamma \nabla f(\bar{x}) + \lambda \bar{x} - \bar{x} \right) = \nabla f(\bar{x}),$$

which, from Proposition 7, implies $\bar{x} = T_\gamma(\bar{x})$. □

4.2 Convergence of damped shadow Davis-Yin splitting

Consider a stepsize $\gamma > 0$ and a damping parameter $\alpha \in (0, 1)$. Define the damped shadow DY iteration operator as

$$\mathbb{R}^d \ni x \mapsto \text{DY}_{\gamma, \alpha}(x) := (1 - \alpha)x + \alpha T_\gamma(x), \quad (14)$$

where T_γ is defined in (11). In this manner, the damped shadow DY sequence $(x^k)_{k \geq 1}$ is, for all $k \geq 1$,

$$x^{k+1} = \text{DY}_{\gamma, \alpha}(x^k). \quad (15)$$

Hence, the sequence generated via (15) corresponds to the one in Algorithm 1. In order to show convergence of the sequence generated by this algorithm, we revisit a strategy introduced by Opial in [27] to analyse fixed-point iterations, namely, *asymptotic regularity*. We say an operator $S : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is asymptotically regular if for any $u^0 \in \mathbb{R}^d$, the sequence $(u^k)_{k \geq 1}$ generated by $u^{k+1} = S(u^k)$ for all $k \geq 1$ satisfies $u^k - S(u^k) \rightarrow 0$. Opial's theorem [27, Theorem 1] holds true for nonexpansive operators, suitable for convex optimisation. In the next result, we explore this result when the operator is not necessarily nonexpansive, which is the case in nonconvex optimisation.

Lemma 3. Let Assumption 1 hold, and $\rho_f \geq 0$ be the constant of weak convexity of f , $\gamma \in (0, \min\{\gamma_g, \rho_f^{-1}\})$ be a stepsize, $\alpha \in (0, 1)$ a dumping parameter, and $\lambda > 0$ a relaxation parameter. Suppose that the damped shadow DY sequence $(x_k)_{k \geq 1}$ in (15) has a nonempty set of cluster points, and that $x^k - \text{DY}_{\gamma, \alpha}(x^k) \rightarrow 0$. Then any cluster point of $(x_k)_{k \geq 1}$ is a stationary point of φ .

Proof. Asymptotic regularity implies that $T_\gamma(x^k) - x^k \rightarrow 0$. Since ∇f is Lipschitz continuous, then $\nabla f(T_\gamma(x^k)) - \nabla f(x^k) \rightarrow 0$. Let x^* be a cluster point of $(x_k)_{k \geq 1}$, such that, up to a subsequence, $x^k \rightarrow x^*$. Then, treating $x \mapsto Y_\gamma(x)$ as a set-valued map, it follows from (12) that

$$\frac{\gamma}{\lambda}(\nabla f(T_\gamma(x^k)) - \nabla f(x^k)) + \frac{1}{\lambda}(T_\gamma(x^k) - x^k) + x^k \in Y_\gamma(x^k).$$

As $x \mapsto \text{prox}_{\gamma g}(x)$ is outer semicontinuous, continuity of $\nabla(f + h)$ implies that $x \mapsto Y_\gamma(x)$ is outer semicontinuous as well. Then, taking the limit in the above inclusion yields $x^* \in \limsup Y_\gamma(x^k) \subseteq Y_\gamma(x^*)$. The first-order optimality condition associated with the DY map (10) implies that $0 \in \partial g(x^*) + \frac{1}{\gamma}(x^* - (x^* - \gamma \nabla(f + h)(x^*))) = \partial g(x^*) + \nabla(f + h)(x^*)$, and thus x^* is a stationary point of φ . $\square$

In this manner, the convergence analysis of Algorithm 1 relies on verifying that the conditions in Lemma 3 are satisfied. The core of the argument is based on a sufficient descent condition for the DYE. Different from [1, 2, 9, 36], this sufficient condition needs to hold alongside the damped shadow DY sequence, and not for the traditional fixed-point sequence $(z_k)_{k \geq 1}$ from the iteration scheme (2). Its proof is inspired by [2, Theorem 3.5, Theorem 3.9], which in turn was motivated by [36]. For convenience, we define

$$Q_{\gamma(f+h)}(y; x) := (f + h)(x) + \langle \nabla(f + h)(x), y - x \rangle + \frac{1}{2\gamma} \|y - x\|^2, \quad (16)$$

in such a way that

$$\varphi_\gamma^{\text{DY}}(x) = \inf_{y \in \mathbb{R}^d} \{Q_{\gamma(f+h)}(y; x) + g(y)\}. \quad (17)$$

From the definition of the map Y_γ , $y = Y_\gamma(x^k)$ solves the optimisation problem in (17).

We start the proof of the sufficient descent condition with two technical results.

Lemma 4. Suppose Assumption 1 holds. Let $\rho_f \geq 0$ such that f is ρ_f -weakly convex. Let $(x_k)_{k \geq 1}$ be the damped shadow DY sequence of Algorithm 1. Then for any $L \geq L_f$ and all $k \geq 1$,

$$\varphi_\gamma^{\text{DY}}(x^{k+1}) \leq Q_{\gamma(f+h)}(Y_\gamma(x^k); x^{k+1}) + g(Y_\gamma(x^k)),$$

and

$$\begin{aligned} \varphi_\gamma^{\text{DY}}(x^k) &\geq Q_{\gamma(f+h)}(Y_\gamma(x^k); x^{k+1}) + g(Y_\gamma(x^k)) + q_1^k + q_2^k \\ &\quad + \frac{1}{2(L - \rho_f)} \|\nabla f(x^k) - \nabla f(x^{k+1})\|^2 \\ &\quad - \left(\frac{L_h}{2} + \frac{1}{2\gamma} + \frac{\rho_f L}{2(L - \rho_f)} \right) \|x^k - x^{k+1}\|^2, \end{aligned}$$

where

$$q_1^k := \left\langle \nabla f(x^{k+1}) - \nabla f(x^k) - \frac{1}{\gamma}(x^{k+1} - x^k), x^k - Y_\gamma(x^k) \right\rangle \quad (18)$$

and

$$q_2^k := \left\langle \nabla h(x^k) - \nabla h(x^{k+1}), Y_\gamma(x^k) - x^{k+1} \right\rangle. \quad (19)$$

Proof. The first estimate follows directly from (16)-(17) by taking $y = Y_\gamma(x^k)$. The second estimate is [2, Lemma 3.4] with $x = x^k$, $y^+ = Y_\gamma(x^k)$, $\hat{x} = x^{k+1}$, $p = 0$, $\rho_p = 0$ and $\alpha \leftarrow \gamma$ (therein, α is the stepsize). $\square$

Lemma 5. In the context of Lemma 4, the following hold for all $k \geq 1$:

$$q_1^k \geq \left[-\frac{2L_f}{\lambda\alpha} - \frac{L_f^2\gamma}{\lambda} \left(\frac{1-\alpha}{\alpha} \right) + \frac{1}{\lambda\alpha\gamma} \right] \|x^k - x^{k+1}\|^2 - \frac{\gamma}{\lambda} \|\nabla f(x^k) - \nabla f(x^{k+1})\|^2$$

and

$$q_2^k \geq -\frac{L_h}{\lambda\alpha} (|1 - \lambda\alpha| + L_f\gamma) \|x^k - x^{k+1}\|^2.$$

Proof. From (14) and (15),

$$x^k - T_\gamma(x^k) = \frac{1}{\alpha}(x^k - x^{k+1}), \quad (20)$$

which combined with (12) yields

$$\begin{aligned} x^k - Y_\gamma(x^k) &= \frac{1}{\lambda} \left[x^k - T_\gamma(x^k) + \gamma(\nabla f(x^k) - \nabla f(T_\gamma(x^k))) \right] \\ &= \frac{1}{\lambda\alpha}(x^k - x^{k+1}) + \frac{\gamma}{\lambda}(\nabla f(x^k) - \nabla f(T_\gamma(x^k))) \end{aligned} \quad (21)$$

By substituting (21) in (18), we get

$$\begin{aligned} q_1^k &= \frac{1}{\lambda\alpha} \langle \nabla f(x^{k+1}) - \nabla f(x^k), x^k - x^{k+1} \rangle \\ &\quad + \frac{\gamma}{\lambda} \langle \nabla f(x^{k+1}) - \nabla f(x^k), \nabla f(x^k) - \nabla f(T_\gamma(x^k)) \rangle \\ &\quad + \frac{1}{\lambda\alpha\gamma} \|x^k - x^{k+1}\|^2 - \frac{1}{\lambda} \langle x^{k+1} - x^k, \nabla f(x^k) - \nabla f(T_\gamma(x^k)) \rangle \\ &\geq -\frac{1}{\lambda\alpha} \|\nabla f(x^{k+1}) - \nabla f(x^k)\| \|x^k - x^{k+1}\| - \frac{\gamma}{\lambda} \|\nabla f(x^{k+1}) - \nabla f(x^k)\|^2 \\ &\quad - \frac{\gamma}{\lambda} \|\nabla f(x^{k+1}) - \nabla f(x^k)\| \|\nabla f(x^{k+1}) - \nabla f(T_\gamma(x^k))\| \\ &\quad + \frac{1}{\lambda\alpha\gamma} \|x^k - x^{k+1}\|^2 - \frac{1}{\lambda} \|x^{k+1} - x^k\| \|\nabla f(x^k) - \nabla f(T_\gamma(x^k))\| \\ &\geq -\frac{L_f}{\lambda\alpha} \|x^k - x^{k+1}\|^2 - \frac{\gamma}{\lambda} \|\nabla f(x^{k+1}) - \nabla f(x^k)\|^2 - \frac{\gamma L_f^2}{\lambda} \|x^{k+1} - x^k\| \|x^{k+1} - T_\gamma(x^k)\| \\ &\quad + \frac{1}{\lambda\alpha\gamma} \|x^k - x^{k+1}\|^2 - \frac{L_f}{\lambda} \|x^{k+1} - x^k\| \|x^k - T_\gamma(x^k)\|, \end{aligned}$$

where the first inequality follows from adding and subtracting $\nabla f(x^{k+1})$ in the inner product in the second line and apply the Cauchy-Schwarz inequality, and the second inequality follows from using Lipschitz continuity of ∇f . Moreover, from (20), $T_\gamma(x^k) - x^{k+1} = (1-\alpha)(T_\gamma(x^k) - x^k) = \left(\frac{1-\alpha}{\alpha}\right)(x^{k+1} - x^k)$, then

$$\begin{aligned} q_1^k &\geq -\frac{L_f}{\lambda\alpha} \|x^k - x^{k+1}\|^2 - \frac{\gamma}{\lambda} \|\nabla f(x^{k+1}) - \nabla f(x^k)\|^2 - \frac{\gamma L_f^2}{\lambda} \left(\frac{1-\alpha}{\alpha} \right) \|x^{k+1} - x^k\|^2 \\ &\quad + \frac{1}{\lambda\alpha\gamma} \|x^k - x^{k+1}\|^2 - \frac{L_f}{\lambda\alpha} \|x^{k+1} - x^k\|^2, \end{aligned}$$

from where the first estimate follows. Next, in view of (21),

$$Y_\gamma(x^k) - x^{k+1} = \left(\frac{1}{\lambda\alpha} - 1\right)(x^{k+1} - x^k) + \frac{\gamma}{\lambda}(\nabla f(T_\gamma(x^k)) - \nabla f(x^k)),$$

then substituting this identity in (19) yields

$$\begin{aligned} q_2^k &= \left(\frac{1}{\lambda\alpha} - 1\right) \langle \nabla h(x^k) - \nabla h(x^{k+1}), x^{k+1} - x^k \rangle \\ &\quad + \frac{\gamma}{\lambda} \langle \nabla h(x^k) - \nabla h(x^{k+1}), \nabla f(T_\gamma(x^k)) - \nabla f(x^k) \rangle \\ &\geq - \left| \frac{1}{\lambda\alpha} - 1 \right| L_h \|x^k - x^{k+1}\|^2 - \frac{\gamma}{\lambda} L_h L_f \|x^k - x^{k+1}\| \|\nabla f(T_\gamma(x^k)) - \nabla f(x^k)\| \\ &= - \left| \frac{1}{\lambda\alpha} - 1 \right| L_h \|x^k - x^{k+1}\|^2 - \frac{\gamma}{\lambda\alpha} L_h L_f \|x^k - x^{k+1}\|^2, \end{aligned}$$

where the inequality is obtained by using Lipschitz continuity of ∇h and ∇f , and the fact that $\beta \geq -|\beta|$ for all $\beta \in \mathbb{R}$, and the equality follows from (20). Thus we obtain the estimate for q_2^k . $\square$

In the following result, we show that the DYE satisfies a sufficient descent condition alongside the damped shadow DY sequence in Algorithm 1.

Theorem 2 (DY splitting: damped shadow sufficient descent). *Let Assumption 1 hold. Let $\alpha \in (0, 1)$ be a damping parameter, and $\lambda > 0$ a relaxation parameter such that $\lambda\alpha < 2$. Consider the damped shadow DY sequence $(x^k)_{k \geq 1}$ generated via Algorithm 1. Then, for all sufficiently small $\gamma > 0$, there exists $c > 0$, such that for all $k \geq 1$,*

$$\varphi_\gamma^{\text{DY}}(x^{k+1}) + \frac{c}{\gamma} \|x^{k+1} - x^k\|^2 \leq \varphi_\gamma^{\text{DY}}(x^k). \quad (22)$$

Proof. Let $\rho_f \geq 0$ be a constant of weak convexity of f , and $L > \rho_f$. By subtracting the two estimates in Lemma 4, and using Lemma 5, we get

$$\begin{aligned} \varphi_\gamma^{\text{DY}}(x^k) - \varphi_\gamma^{\text{DY}}(x^{k+1}) &\geq \left[-\frac{\gamma}{\lambda} + \frac{1}{2(L - \rho_f)} \right] \|\nabla f(x^k) - \nabla f(x^{k+1})\|^2 \\ &\quad + C(\gamma, L) \|x^k - x^{k+1}\|^2, \end{aligned} \quad (23)$$

where

$$\begin{aligned} C(\gamma, L) &:= - \left(\frac{L_f^2}{\lambda} \left(\frac{1 - \alpha}{\alpha} \right) + \frac{L_f L_h}{\lambda\alpha} \right) \gamma - \left(\frac{2L_f}{\lambda\alpha} + \left(\frac{1}{2} + \frac{|1 - \lambda\alpha|}{\lambda\alpha} \right) L_h + \frac{\rho_f L}{2(L - \rho_f)} \right) \\ &\quad + \left(\frac{1}{\lambda\alpha} - \frac{1}{2} \right) \frac{1}{\gamma}. \end{aligned} \quad (24)$$

We separate the analysis in two cases, depending on the sign of the constant multiplying $\|\nabla f(x^k) - \nabla f(x^{k+1})\|^2$ above.

First, suppose that $(2 - \lambda)L_f - 2\rho_f \geq \lambda L_h$. Then $L_f - \rho_f \geq \frac{\lambda}{2}(L_h + L_f) > 0$. In this case, we may take $L = L_f$, so that $C(\gamma, L_f)$ is well-defined. Furthermore, choose $\hat{\gamma} \geq \frac{\lambda}{2(L_f - \rho_f)}$. Then, for all $\gamma \in (0, \hat{\gamma}]$,

$$-\frac{\gamma}{\lambda} + \frac{1}{2(L_f - \rho_f)} \geq -\frac{\hat{\gamma}}{\lambda} + \frac{1}{2(L_f - \rho_f)}, \text{ where } -\frac{\hat{\gamma}}{\lambda} + \frac{1}{2(L_f - \rho_f)} \leq 0.$$

Hence,

$$\left[-\frac{\gamma}{\lambda} + \frac{1}{2(L_f - \rho_f)}\right] \|\nabla f(x^k) - \nabla f(x^{k+1})\|^2 \geq \left[-\frac{\hat{\gamma}}{\lambda} + \frac{1}{2(L_f - \rho_f)}\right] L_f^2 \|x^k - x^{k+1}\|^2.$$

In this manner, it suffices to show that $\hat{C}_1(L_f) := \hat{\gamma}C(\hat{\gamma}, L_f) + \left[-\frac{\hat{\gamma}}{\lambda} + \frac{1}{2(L_f - \rho_f)}\right] L_f^2 \hat{\gamma}$ is positive. This is indeed the case, as

$$\begin{aligned} \hat{C}_1(L_f) = & -\left(\frac{L_f^2}{\lambda\alpha} + \frac{L_f L_h}{\lambda\alpha}\right) \hat{\gamma}^2 + \frac{1}{\lambda\alpha} - \frac{1}{2} \\ & - \left(\frac{2L_f}{\lambda\alpha} + \left(\frac{1}{2} + \frac{|1 - \lambda\alpha|}{\lambda\alpha}\right) L_h + \frac{\rho_f L_f}{2(L - \rho_f)} - \frac{L_f^2}{2(L_f - \rho_f)}\right) \hat{\gamma} \end{aligned}$$

defines a concave quadratic function with one strictly positive root $\hat{\gamma}^+$, as the intercept satisfies $\frac{1}{\lambda\alpha} - \frac{1}{2} > 0$ from the assumption $\lambda\alpha < 2$. Therefore, $\hat{C}_1(L_f) > 0$ as long as $\hat{\gamma} < \hat{\gamma}^+$. Hence, (22) holds in this case for $\gamma \in (0, \min\{\hat{\gamma}^+, \hat{\gamma}\})$ and $c = \hat{C}_1(L_f)$.

Secondly, suppose now that $(2 - \lambda)L_f - 2\rho_f < \lambda L_h$, and $L > \rho_f$. Define $\hat{\gamma}(L) := \frac{\lambda}{2(L - \rho_f)} > 0$, and consider $\gamma \in (0, \hat{\gamma}(L))$. Then

$$-\frac{\gamma}{\lambda} + \frac{1}{2(L - \rho_f)} \geq -\frac{\hat{\gamma}(L)}{\lambda} + \frac{1}{2(L - \rho_f)} = 0.$$

Hence, we can bound below the term associated with $\|\nabla f(x^k) - \nabla f(x^{k+1})\|^2$ in (23) by zero. In this case, we define

$$\begin{aligned} \hat{C}_2(L) := & -\left(\frac{L_f^2}{\lambda} \left(\frac{1 - \alpha}{\alpha}\right) + \frac{L_f L_h}{\lambda\alpha}\right) \hat{\gamma}^2 - \left(\frac{2L_f}{\lambda\alpha} + \left(\frac{1}{2} + \frac{|1 - \lambda\alpha|}{\lambda\alpha}\right) L_h + \frac{\rho_f L}{2(L - \rho_f)}\right) \hat{\gamma} \\ & + \frac{1}{\lambda\alpha} - \frac{1}{2} \quad (25) \end{aligned}$$

such that $\hat{C}_2(L) = C(\hat{\gamma}(L), L)$. Then $\hat{C}_2(L) = \frac{1}{[2(L - \rho_f)]^2} p(L - \rho_f)$, where

$$p(\eta) := 4 \left(\frac{1}{\lambda\alpha} - \frac{1}{2}\right) \eta^2 - 2 \left(\frac{2L_f}{\alpha} + \left(\frac{\lambda}{2} + \frac{|1 - \lambda\alpha|}{\alpha}\right) L_h\right) \eta - \lambda \left(\rho_f L + L_f^2 \left(\frac{1 - \alpha}{\alpha}\right) + \frac{L_f L_h}{\alpha}\right).$$

Observe that p is a convex quadratic function with one strictly positive root η^+ , since $\lambda\alpha < 2$. Furthermore, its intercept is negative as $\alpha \in (0, 1)$. Then $\eta = \eta^+$ is the smallest positive value for which $p(\eta) \geq 0$. Hence, $L^+ := \max\{L_f, \eta^+ + \rho_f\}$ is the smallest L such that $C(\hat{\gamma}(L), L) \geq 0$. Furthermore, from (25), we see that $\mathbb{R}_{++} \ni \gamma \mapsto C(\gamma, L^+)$ is a concave quadratic with positive intercept $\frac{1}{\lambda\alpha} - \frac{1}{2}$. Therefore, (22) holds with $c = C(\gamma, L^+) > C(\hat{\gamma}(L^+), L^+) \geq 0$ for $\gamma \in (0, \hat{\gamma}(L^+))$. This completes the proof. $\square$

Remark 5. For ease of presentation, the bounds for $\gamma > 0$ and the value of the constant $c > 0$ for which the sufficient decrease condition in Theorem 2 holds are not explicitly stated. However, they can be easily obtained from the proof: $\gamma \in (0, \min\{\hat{\gamma}^+, \hat{\gamma}, \hat{\gamma}(L^+)\})$, and $c = \min\{\hat{C}_1(L_f), C(\gamma, L^+)\}$.

As we have shown that the damped shadow DY method behaves as a method of descent for the DYE, we are now ready to prove (subsequential) convergence leveraging on Lemma 3.

Corollary 1 (Subsequential convergence of the damped shadow DY sequence). Let Assumption 1 hold, such that f is ρ_f -weakly convex, and that φ in (1) is level-bounded. Let $\gamma \in (0, \min\{\gamma_g, (L_f + L_h)^{-1}\})$ satisfy Remark 5. Then, $\text{DY}_{\gamma, \alpha}$ is asymptotically regular, the damped shadow DY sequence $(x_k)_{k \geq 1}$ of Algorithm 1 is bounded and any of its cluster points is a stationary point of φ . Furthermore, $\lim \varphi_\gamma^{\text{DY}}(x^k)$ is a stationary value.

Proof. From Theorem 2, $(\varphi_\gamma^{\text{DY}}(x^k))_{k \geq 1}$ is nonincreasing, and since $(x_k)_{k \geq 1} \subseteq \text{lev}_{\varphi_\gamma^{\text{DY}}}(\varphi_\gamma^{\text{DY}}(x^0))$, then $(x_k)_{k \geq 1}$ is a bounded sequence due to Lemma 1(iii) and the assumption that φ is level-bounded. Therefore, the set of cluster points of $(x_k)_{k \geq 1}$ is nonempty. Furthermore, in view of Lemma 1(ii), $\varphi_\gamma^{\text{DY}}$ is bounded below, and thus so is the sequence $(\varphi_\gamma^{\text{DY}}(x^k))_{k \geq 1}$. Hence, there exists $\varphi^* \geq \inf \varphi$ such that $\varphi_\gamma^{\text{DY}}(x^k) \rightarrow \varphi^*$. From Theorem 2 again, we conclude then that $x^{k+1} - x^k \rightarrow 0$, so that $\text{DY}_{\gamma, \alpha}$ is asymptotically regular. Lemma 3 implies that any cluster point of $(x_k)_{k \geq 1}$ is a stationary point of φ . This proves the first part of the statement. Next, let x^* be an arbitrary cluster point of $(x_k)_{k \geq 1}$. From (14) and (15), we also have $T_\gamma(x^k) - x^k \rightarrow 0$, and since ∇f is Lipschitz continuous, then $\nabla f(T_\gamma(x^k)) - \nabla f(x^k) \rightarrow 0$. Hence (21) implies $x^k - Y_\gamma(x^k) \rightarrow 0$, and thus $Y_\gamma(x^k) \rightarrow x^*$. Moreover, due to [24, eq. (8)], for some constant $K(\gamma) > 0$, there holds

$$\varphi(Y_\gamma(x^k)) \leq \varphi_\gamma^{\text{DY}}(x^k) - K(\gamma)\|Y_\gamma(x^k) - x^k\|^2.$$

Taking the limit, as φ is lower semicontinuous, it follows that

$$\varphi(x^*) \leq \liminf \varphi(Y_\gamma(x^k)) \leq \limsup \varphi(Y_\gamma(x^k)) \leq \lim \varphi_\gamma^{\text{DY}}(x^k) = \varphi(x^*),$$

where the last equality follows from $\varphi^* = \varphi(x^*)$, since continuity of $\varphi_\gamma^{\text{DY}}$ and uniqueness of the limit of $(\varphi_\gamma^{\text{DY}}(x^k))_{k \geq 1}$ imply $\varphi^* = \varphi(x^*)$. This concludes the proof. $\square$

Remark 6 (Global convergence and rates of convergence). Under now standard assumptions in nonconvex optimisation (see, e.g. [5, 8]), namely, the Kurdyka-Łojasiewicz inequality, global convergence of the damped shadow DY sequence to a stationary point of $\varphi_\gamma^{\text{DY}}$ can be shown using [8, Theorem 2.9], for which one needs to verify [8, (H1)–(H3)]. First, (H1) is the sufficient descent condition in (22). (H2) is a subgradient estimate we proceed to deduce. Suppose, in addition to the assumptions in Corollary 1, that both f and h have Lipschitz continuous Hessians. Suppose $\mathbb{R}^d \ni x \mapsto Y_\gamma(x)$ is single-valued (e.g. when g is weakly convex). In view of boundedness of $(x_k)_{k \geq 1}$, Lemma 2, (21), and (20), there exists $M > 0$, such that for all $k \geq 1$ and $w^k \in \partial \varphi_\gamma^{\text{DY}}(x^k)$, there holds

$$\begin{aligned} \|w^k\| &= \frac{1}{\gamma} \|I - \gamma \nabla^2(f + h)(x^k)\|_{\text{op}} \|x^k - Y_\gamma(x^k)\| \\ &\leq \frac{M}{\gamma} \left(\frac{1}{\lambda \alpha} \|x^k - x^{k+1}\| + \frac{\gamma L_f}{\lambda} \|x^k - T_\gamma(x^k)\| \right) \\ &= \frac{M}{\lambda \alpha \gamma} (1 + L_f \gamma) \|x^k - x^{k+1}\|. \end{aligned}$$

The last condition (H3) holds because $(x_k)_{k \geq 1}$ has a nonempty set of cluster points (Corollary 1) and $\varphi_\gamma^{\text{DY}}$ is locally Lipschitz continuous (Lemma 1(i)). In this manner, $(x_k)_{k \geq 1}$ converges to a stationary point x^* of $\varphi_\gamma^{\text{DY}}$. Since $I - \gamma \nabla^2(f + g)(x^*)$ is invertible for $\gamma \in (0, (L_f + L_h)^{-1})$, from Lemma 2 then $x^* = Y_\gamma(x^*)$. Proposition 9 thus implies that x^* is a stationary point of φ . Rates of convergence can also be obtained whenever the Kurdyka-Łojasiewicz exponent is known [7, Theorem 3.4]. Finally, the assumption on the Hessians can be weakened if we follow an approach similar to [3, Section 4.1].

5 Saddle point avoidance of shadow DY method

In the nonconvex smooth optimisation case, first-order methods for the minimisation of C^2 functions avoid stationary points with negative curvature when randomly initialised [21, 28]. In [13], this result is extended to methods of proximal type for nonconvex problems with structured nonsmoothness, namely, under the assumption of the strict saddle property. The methods included in the latter work are the proximal point algorithm, the proximal gradient and prox-linear methods. The heart of the analysis therein is the *Centre-Stable manifold theorem* [35, Theorem III.7]. In this section, for the same type of model structures, we show that the same result holds for splitting methods.

The saddle point avoidance result of the damped shadow DY splitting method will follow from the analogous results of the proximal gradient method, in view of (8). The next proposition summarises the results we need, retrieved from [13, Theorem 4.1] and its proof.

Proposition 10 (Proximal-gradient map properties). Consider the optimisation problem (1), and let $\bar{x}$ be a stationary point of $\varphi = f + g + h$. Suppose that f is L_f -smooth, h is L_h -smooth, and g is a proper lower semicontinuous ρ -weakly convex function that admits a C^2 -smooth active manifold $\mathcal{M}$ at $\bar{x}$ with local defining equations $G = 0$. Then, for any $\gamma \in (0, \rho^{-1})$,

- (i) $Y_\gamma(x) \in \mathcal{M}$ for all x near $\bar{x}$
- (ii) if W denotes the orthogonal projection onto $T_{\mathcal{M}}(\bar{x})$, then the linear map $\overline{H}_{yy} := WH_{yy}W$, where $H_{yy} := \nabla \hat{g}(\bar{x}) + \gamma^{-1}I + \sum_{i=1}^p \bar{\lambda}_i \nabla^2 G_i(\bar{x})$, is (symmetric) positive definite
- (iii) $Y_\gamma(\cdot)$ is a C^1 map around $\bar{x}$, such that for all $h \in T_{\mathcal{M}}(\bar{x})$,

$$\nabla Y_\gamma(\bar{z})h = -\overline{H}_{yy}^{-1}\overline{H}_{xy}h$$

where $\overline{H}_{xy} := WH_{xy}W$, with $H_{xy} := \nabla^2(f + h)(\bar{x}) - \gamma^{-1}I$, is symmetric

- (iv) if, in addition, $\bar{x}$ is a strict saddle point, then $\overline{H}_{xy} + \overline{H}_{yy}$ has a strictly negative eigenvalue.

Proof. Apply [13, Theorem 4.1] to the $(L_f + L_h)$ -smooth function $f + g$ and the ρ -weakly convex function g . $\square$

5.1 Strict saddle point avoidance: Davis-Yin

In this section, we show that the DY method converges to local minimisers almost surely, whenever the strict saddle property holds. We first state two preliminary results.

Lemma 6 ([13, Lemma 2.14]). Consider a map $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and a damping parameter $\alpha \in (0, 1)$. Define the map $R := (1 - \alpha)I + \alpha T$. Then

- (i) the fixed-points of R and T coincide,
- (ii) if T is continuous and the sequence $(x_k)_{k \geq 1}$ generated via $x^{k+1} = R(x^k)$ for all $k \geq 1$, converge to a point $\bar{x}$, then $\bar{x}$ is a fixed-point of T ,
- (iii) if T is differentiable and the Jacobian $\nabla T(\bar{x})$ has a real eigenvalue strictly greater than one, then $\bar{x}$ is an unstable fixed-point of R ,
- (iv) if the map $I - T$ is Lipschitz continuous with constant $L > 0$, then R is a lipeomorphism for any $\alpha > 0$ such that $\alpha L < 1$.

Lemma 7 ([13, Corollary 2.12]). Let $Q : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a lipeomorphism and let $\mathcal{U}_Q$ consist of all unstable fixed-points x of Q at which the Jacobian $\nabla Q(x)$ is invertible. Then the set of initial conditions attracted by such fixed points

$$W := \left\{ x \in \mathbb{R}^d : \lim_{k \rightarrow \infty} Q^k(x) \in \mathcal{U}_Q \right\}$$

has zero Lebesgue measure.

Remark 7. From Lemma 6(iv) and Lemma 7, it is clear we need the damping term in Algorithm 1. From Lemma 6(ii), we also observe it is necessary to establish convergence of the damped version of the method, which we did in Section 4.2.

The main result of this section will be a consequence of the two lemmas above. We start by checking that the conditions in Lemma 6(iii) are satisfied.

Proposition 11 (Unstable fixed-points of DY iteration operator). Consider the optimisation problem (1), and let $\bar{x}$ be a strict saddle point of $\varphi = f + g + h$. Suppose that f is L_f -smooth, h is L_h -smooth, and both f and h are C^2 functions. Additionally, suppose g is a proper lower semicontinuous ρ -weakly convex function that admits a C^2 -smooth active manifold $\mathcal{M}$ at the stationary point $\bar{x}$ of φ . Then, for any $\gamma \in (0, \min\{\rho^{-1}, L_f^{-1}\})$, $\nabla T_\gamma(\bar{x})$ has a real eigenvalue that is strictly greater than one.

Proof. In view of (11), continuous differentiability of $\text{prox}_{\gamma f}$, and Proposition 10(iii), T_γ is a C^1 map locally around $\bar{x}$. Furthermore, from (12), for all x sufficiently close to $\bar{x}$,

$$\gamma \nabla^2 f(T_\gamma(x)) \nabla T_\gamma(x) + \nabla T_\gamma(x) = (1 - \lambda)I + \lambda \nabla Y_\gamma(x) + \gamma \nabla^2 f(x).$$

Then, for $x = \bar{x}$, Proposition 9 and Proposition 10(iii) yield, for all $h \in T_{\mathcal{M}}(\bar{x})$,

$$\begin{aligned} (I + \gamma \nabla^2 f(\bar{x})) \nabla T_\gamma(\bar{x}) h &= ((1 - \lambda)I + \lambda \nabla Y_\gamma(x) + \gamma \nabla^2 f(x)) h \\ &= (1 - \lambda)h - \lambda \bar{H}_{yy}^{-1} \bar{H}_{xy} h + \gamma \nabla^2 f(\bar{x}) h. \end{aligned}$$

As $I + \gamma \nabla^2 f(\bar{x})$ is invertible, $\mu \in \mathbb{R}$ is a real eigenvalue of $\nabla T_\gamma(\bar{x})$ with an associated eigenvector $h \in T_{\mathcal{M}}(\bar{x})$ if and only if

$$\begin{aligned} & \nabla T_\gamma(\bar{x}) h = \mu h \\ \iff & ((1 - \lambda)I - \lambda \bar{H}_{yy}^{-1} \bar{H}_{xy} + \gamma \nabla^2 f(\bar{x})) h = \mu (I + \gamma \nabla^2 f(\bar{x})) h \\ \iff & (-\lambda \bar{H}_{yy}^{-1} \bar{H}_{xy} + \gamma(1 - \mu) \nabla^2 f(\bar{x}) + (1 - \lambda - \mu)I) h = 0 \\ \iff & H(\mu) h = 0, \end{aligned}$$

where

$$H(\mu) := (I + \gamma \nabla^2 f(\bar{x})) (\lambda \bar{H}_{xy} + (\lambda - 1 + \mu) \bar{H}_{yy} + \gamma(\mu - 1) \bar{H}_{yy} \nabla^2 f(\bar{x})).$$

In this manner, the claim is equivalent to $H(\mu)$ being singular for some $\mu > 1$. On the one hand, from Proposition 10(iv), the minimal eigenvalue of $H(\mu = 1) = \lambda (I + \gamma \nabla^2 f(\bar{x})) (\bar{H}_{xy} + \bar{H}_{yy})$ is strictly negative. On the other hand, let

$$\begin{aligned} H(\mu) &= H_1 + H_2(\mu), \text{ where} \\ H_1 &:= (I + \gamma \nabla^2 f(\bar{x})) (\lambda \bar{H}_{xy} + (\lambda - 1) \bar{H}_{yy} - \gamma \bar{H}_{yy} \nabla^2 f(\bar{x})), \text{ and} \\ H_2(\mu) &:= \mu (I + \gamma \nabla^2 f(\bar{x})) \bar{H}_{yy} (I + \gamma \nabla^2 f(\bar{x})). \end{aligned}$$

In view of Proposition 10(ii), for $\mu > 1$, the direction $H_2(\mu)$ of the ray is positive definite. Then the minimum eigenvalue of $H(\mu)$, which is bounded below by the minimum eigenvalues of H_1 and $H_2(\mu)$, is strictly positive for sufficiently large $\mu > 1$. Hence, resorting to the continuity of the minimal eigenvalue of $H(\mu)$ as a function of μ , there exists $\mu > 1$ such that the minimal eigenvalue of $H(\mu)$ is zero. Therefore, $H(\mu)$ is singular for some $\mu > 1$, from where we can conclude. $\square$

We now state and prove the main result of this section, avoidance of strict saddle points for the damped shadow DY sequence generated by Algorithm 1.

Theorem 3 (Davis-Yin splitting: global escape). *Let Assumption 1 hold, in such a way that g is ρ -weakly convex. Suppose $\varphi = f + g + h$ has the strict saddle property. Choose a stepsize $\gamma \in (0, \rho^{-1})$, a relaxation parameter $\lambda > 0$, and a damping parameter $\alpha \in (0, 1)$ such that $\alpha L_2 < 1$, where*

$$\begin{aligned} L_2 &= (1 + \gamma L_f + \lambda L_1) \max \left\{ 1, \frac{\gamma L_f}{1 - \gamma L_f} \right\} + \lambda L_1 + \gamma L_f, \text{ and} \\ L_1 &= 1 + \left(\frac{1 + \gamma(L_f + L_h)}{1 - \gamma \rho} \right). \end{aligned}$$

Consider the damped shadow DY sequence $(x_k)_{k \geq 1}$ generated by Algorithm 1. Then, for almost all initial points $x^0 \in \mathbb{R}^d$, if the limit x^ of $(x_k)_{k \geq 1}$ exists, then x^* is a local minimiser of φ .*

Proof. First, in view of

- (i) Lemma 6(i) and (ii) and Proposition 9, the fixed-points of $DY_{\gamma, \alpha}$ coincide with the stationary points of φ , and if $(x_k)_{k \geq 1}$ converges to some x^* , then x^* is a stationary point of φ ,
- (ii) Lemma 6(iii) and Proposition 11, if x^* is a strict saddle point of φ , such that φ admits a C^2 -smooth active manifold $\mathcal{M}$ at x^* , then x^* is an unstable fixed-point of $DY_{\gamma, \alpha}$,
- (iii) Lemma 6(iv), $DY_{\gamma, \alpha}$ is a lipeomorphism, since

$$\begin{aligned} I - T_\gamma &= (1 - \lambda)I + \gamma \nabla f + \lambda Y_\gamma - \text{prox}_{\gamma f} \circ ((1 - \lambda)I + \gamma \nabla f + \lambda Y_\gamma) \\ &\quad + \lambda I - \gamma \nabla f - \lambda Y_\gamma \\ &= \gamma \nabla f_\gamma^M \circ ((1 - \lambda)I + \gamma \nabla f + \lambda Y_\gamma) + \lambda I - \gamma \nabla f - \lambda Y_\gamma \\ &= \gamma \nabla f_\gamma^M \circ (I + \gamma \nabla f - \lambda(I - Y_\gamma)) + \lambda(I - Y_\gamma) - \gamma \nabla f. \end{aligned}$$

Note that $I - Y_\gamma$ is Lipschitz continuous with constant L_1 , in view of Proposition 2(i). Then $I - T_\gamma$ is Lipschitz continuous with constant L_2 .

In this manner, given $x^0 \in \mathbb{R}^d$, suppose $x^k \rightarrow x^*$. In view of (i), x^* is a stationary point of φ . If x^* is not a local minimiser, then (ii) implies it is an unstable fixed-point of $DY_{\gamma, \alpha}$. Using (iv) and applying Proposition 7 with $Q = DY_{\gamma, \alpha}$ yields a contradiction, almost surely. $\square$

Remark 8. Global escape (almost surely) of the damped shadow DY sequence can be guaranteed whenever $\gamma > 0$ also satisfies the bound in Remark 6, as in this case the limit of this sequence is guaranteed to exist. Furthermore, the redefinition of the DYE in (3) omits the explicit use of $\text{prox}_{\gamma f}$ that appear in the relation $\varphi_\gamma^{\text{DY}} \circ \text{prox}_{\gamma f} = V_\gamma$, where V_γ is the merit function in (6). Otherwise, the Jacobian of $\text{prox}_{\gamma f}$ would appear in the analysis, breaking the analysis in Proposition 11 and the symmetric nature of the matrices in Proposition 10.

5.2 Strict saddle point avoidance: Douglas-Rachford

As an application of Theorem 3 with $h = 0$, we obtain the avoidance of strict saddles of the shadow Douglas-Rachford method.

Corollary 2 (Douglas-Rachford splitting: global escape). Let Assumption 1 hold with $h = 0$, in such a way that g is ρ -weakly convex. Suppose $\varphi = f + g$ has the strict saddle property. Choose a stepsize $\gamma \in (0, \rho^{-1})$, a relaxation parameter $\lambda > 0$, and a damping parameter $\alpha \in (0, 1)$ such that $\alpha L_2 < 1$, where

$$\begin{aligned} L_2 &= \gamma L_f + \lambda L_1 + (1 + \gamma L_f + \lambda L_1) \max \left\{ 1, \frac{\gamma L_f}{1 - \gamma L_f} \right\} \\ L_1 &= 1 + \left(\frac{1 + \gamma L_f}{1 - \gamma \rho} \right). \end{aligned}$$

Consider the damped shadow Douglas-Rachford sequence

$$x^{k+1} := (1 - \alpha)x^k + \alpha T_\gamma(x^k),$$

where $T_\gamma(x) = \text{prox}_{\gamma f}((1 - \lambda)x + \gamma \nabla f(x) + \lambda \text{prox}_{\gamma g}(x - \gamma \nabla f(x)))$. Then, for almost all initial points $x^0 \in \mathbb{R}^d$, if the limit x^* of $(x_k)_{k \geq 1}$ exists, then x^* is a local minimiser of φ .

Remark 9. A few remarks are in order regarding Theorem 3 and Corollary 2.

1. Corollary 2 extends [24], where the authors prove a similar result for the fully smooth case (where f , h and g are C^2 functions) of the Douglas-Rachford method [24, Theorem 5.4]. Theorem 3 also includes as a special case the fully smooth case of the forward-backward method in [24, Theorem 5.4]. Therein, these two methods are treated separately. Hence, our analysis unifies and extends the latter two. Furthermore, Theorem 3 for $h = 0$ corresponds to the structured nonsmooth case in [13, Theorem 4.2]
2. If $h = 0$ and $\lambda = 2$ in Theorem 3, we obtain strict saddle point avoidance for the Peaceman–Rachford splitting method [22, 31]. As far as the author knows, this is the first result of this type for this method.

6 Numerical experiments

In this section, we test Algorithm 1 in a nonconvex variable selection problem. As mentioned in [3], some modern penalties used in statistics fall under the category of weakly convex semi-algebraic functions, including the minimax concave penalty (MCP) [38], and the smoothly clipped absolute deviation (SCAD) [17] (whose closed forms and respective proximal operators can be found in [3, Section 5]). It has been observed empirically (see, e.g. [3, Figure 5]) that models with nonconvex regularisers can display a superior numerical performance in terms of the fixed-point residual, which motivates the analysis in this section.

The elastic net problem [39] is a regularised linear modelling technique for variable selection that can handle collinear features. Consider a matrix $A \in \mathbb{R}^{m \times d}$ with possibly highly correlated columns, and a vector $b \in \mathbb{R}^m$. The optimisation model of the elastic net is the following regularised minimisation of residuals problem:

$$\min_{x \in \mathbb{R}^d} \frac{1}{2} \|Ax - b\|_2^2 + \mu \left(\nu P(x) + \frac{1 - \nu}{2} \|x\|_2^2 \right), \quad (26)$$

where $\mu \geq 0$ controls the level of regularisation, P is a sparsity-inducing penalisation (traditionally, the ℓ_1 -norm), $\|\cdot\|_2^2$ induces grouped selection of correlated predictors, and $\nu \in [0, 1]$

controls the weight of each component in the hybrid penalty term. For the classical choice $P = \|\cdot\|_1$ the problem is convex, thus any stationary point of this problem is a global minimiser. In this case, when $\nu = 1$, the model reduces to the LASSO [37], while for $\nu = 0$, it becomes the ridge regression [19]. The advantage of the elastic net over the LASSO is that the latter may fail to select variables corresponding to highly correlated features, as it only induces sparsity via the ℓ_1 -norm.

For nonconvex penalties like MCP and SCAD, values larger than a predetermined threshold are not penalised, and thus sparsity only applies to variables with “small values” (see Figure 1). This effect reduces the *shrinking bias* effect of large coefficients known for the ℓ_1 -norm. It is also known, as Figure 2 demonstrates, that nonconvex penalties can be more aggressive in inducing sparsity in the solution. In particular, Figure 2c shows that SCAD does a better job at recovering the zero components of the true vector (blue dots).

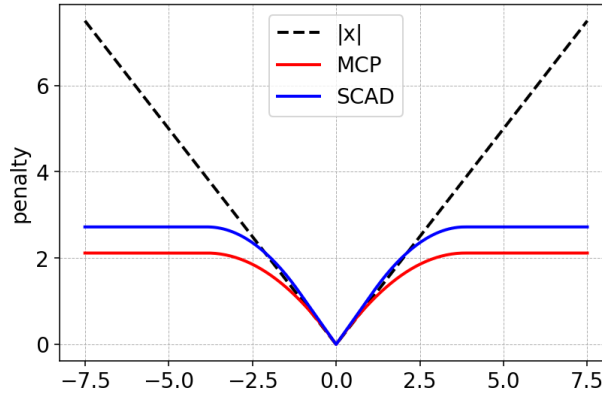


Figure 1: Comparison close to 0 between nonconvex penalties and the absolute value.

In order to assess the quality of the solutions obtained by Algorithm 1 in relation to avoiding stationary points of “poor quality”, we individually test the cases $P \in \{\text{MCP}, \text{SCAD}\}$. Figure 3 displays the final objective value when applying Algorithm 1 to solve problem (26). We choose $m = 100$, $d = 50$, and randomly generate the matrix A which is later modified to produce collinear columns. We set $\mu = 0.5$, $\nu = \frac{2}{3}$, and randomly choose 1,000 initial points $x^0 \in \mathbb{R}^d$. For Algorithm 1, we set $\lambda = 1$, α to be 0.9 of its upper bound in Theorem 3, and $\gamma = 0.9 \cdot \min\{(L_f + L_h)^{-1}, \rho^{-1}\}$. The stopping criterion is the relative residual test $\frac{\|x^{k+1} - x^k\|}{\|x^k\|} \leq 10^{-6}$. The orange horizontal lines in Figures 3a and 3b represent the target value $\varphi(x_{\text{true}})$, where x_{true} is the true vector to recover (randomly generated).

Table 1 summarises the results of Figure 3. The table shows the percentage of instances that achieve at most the target value $\varphi(x_{\text{true}})$ (third column), and at most the 10% above the value $\varphi(x_{\text{true}})$ (fourth column).

Penalty	$\varphi(x_{\text{true}})$	$\% \leq \varphi(x_{\text{true}})$	$\% \leq 1.1 \cdot \varphi(x_{\text{true}})$
SCAD	378.6107	39.1	75.1
MCP	366.4684	51.1	96

Table 1: Percentage of instances that achieve the target value $\varphi(x_{\text{true}})$, and 10% above the target value, for 1,000 randomly initialised instances.

We observe from Figure 3 and Table 1 that the majority of the stationary points found by Algorithm 1 do not stray too far away from the target value $\varphi(x_{\text{true}})$, given the nonconvex penalty chosen. This metric may not be too informative in reality, as x_{true} is usually not known. Further research is necessary to define (local) optimality certificates that can be implemented to test the quality of solutions.

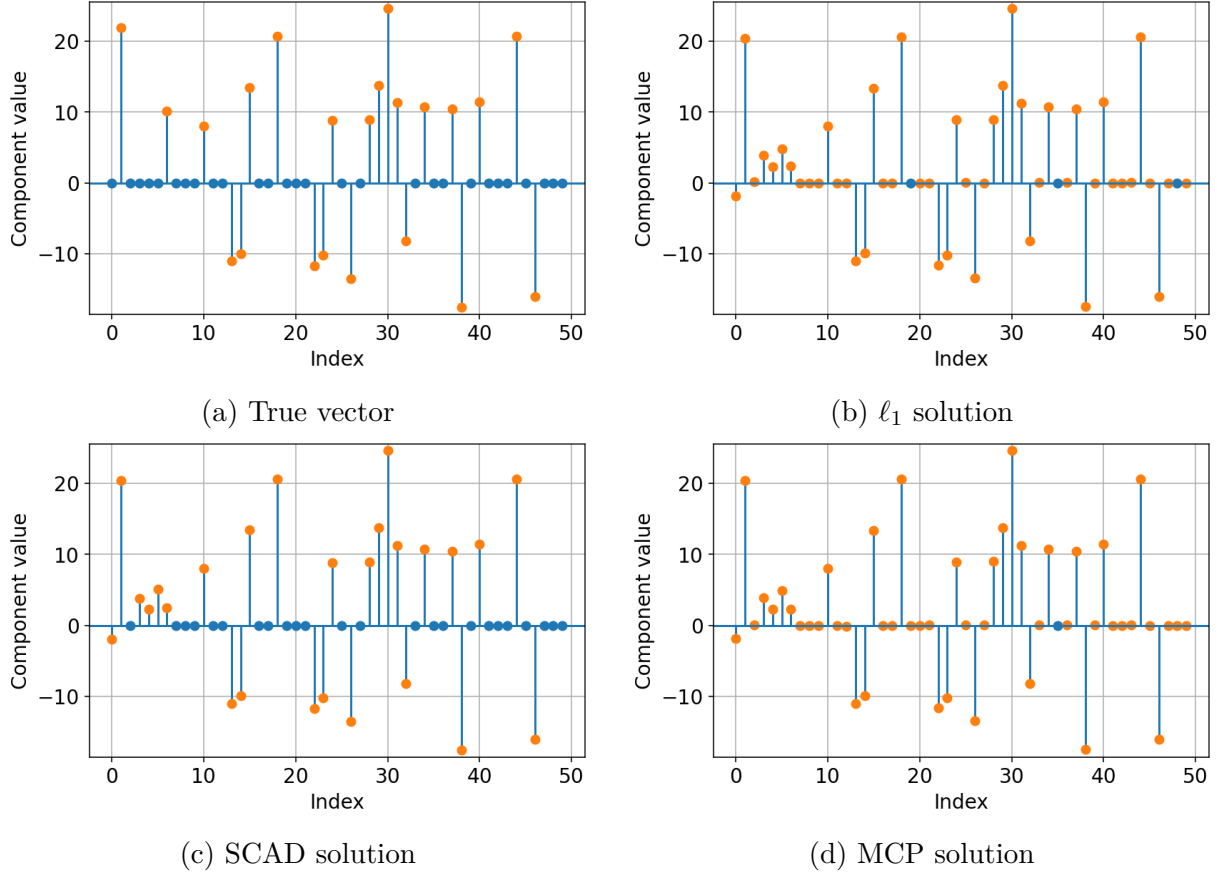


Figure 2: Sample solutions of the elastic net model with convex and nonconvex penalisation, obtained using Algorithm 1.

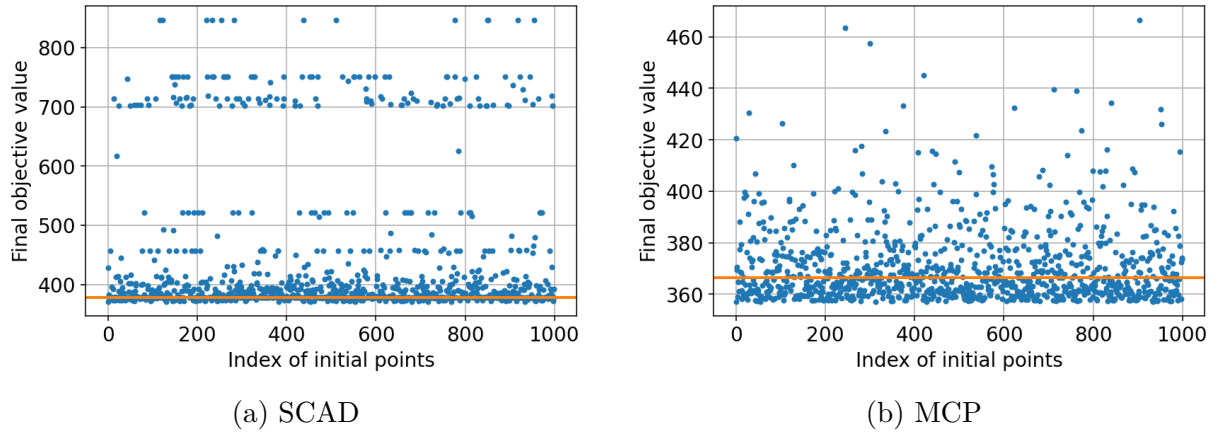


Figure 3: Stationary values of the elastic net model with nonconvex penalisation, obtained using Algorithm 1 for 1,000 randomly chosen initial points.

7 Conclusion

In this work, we used the Davis-Yin envelope to explain the asymptotic behaviour of the Davis-Yin splitting method when solving nonconvex optimisation problems. In particular, we showed epigraphic properties of the envelope, continuity properties of the solution maps, and convergence with high probability to local minimisers of a damped version of the algorithm.

Future research directions can explore relaxing the assumption of having two of the functions in problem (1) smooth. This would allow treating constraints directly via indicator

functions, instead of via its Moreau envelope (the squared distance to the set, which is a smooth function), so that feasibility can be guaranteed in each iteration. This would also expand the applicability of multioperator splitting methods in nonconvex optimisation, as the expected generalisation given by the current theory is that problems of the form $\min \sum_{i=1}^{n-1} f_i + g$ can only be solved via splitting methods if $f_1, \dots, f_{n-1}$ are smooth and g is nonconvex.

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Data availability and replication of results. The datasets and code used in this work to run the experiments are available at github.com/fatenasm/nonconvex-elastic-net.

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