T-semidefinite programming relaxation with third-order tensors for constrained polynomial optimization

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Abstract

We study T-semidefinite programming (SDP) relaxation for constrained polynomial optimization problems (POPs). T-SDP relaxation for unconstrained POPs was introduced by Zheng, Huang and Hu in 2022. In this work, we propose a T-SDP relaxation for POPs with polynomial inequality constraints and show that the resulting T-SDP relaxation formulated with third-order tensors can be transformed into the standard SDP relaxation with block-diagonal structures. The convergence of the T-SDP relaxation to the optimal value of a given constrained POP is established under moderate assumptions as the relaxation level increases. Additionally, the feasibility and optimality of the T-SDP relaxation are discussed. Numerical results illustrate that the proposed T-SDP relaxation enhances numerical efficiency.

Key words. T-SDP relaxation, constrained polynomial optimization, third-order tensors, convergence to the optimal value, block-diagonal structured SDP relaxation, numerical efficiency.

AMS Classification. 90C22, 90C25, 90C26.

1 Introduction

We consider constrained polynomial optimization problems (POPs):

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && g_i(x) \geq 0, \quad i = 1, \ldots, r, \end{aligned} \tag{$\mathbb{P}_K$}$$

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where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_1, \ldots, g_r : \mathbb{R}^n \to \mathbb{R}$ are real-valued polynomials. Without loss of generality, we assume that no constant exists in the objective function. The feasible set of $(\mathbb{P}_K)$ is denoted as $K := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, \ \ i = 1, \ldots, r\}$.

Constrained POP $(\mathbb{P}_K)$ has been widely studied in the field of optimization such as 0-1 integer programming and non-convex quadratic optimization, and has various applications in medical bioengineering [4, 31], signal processing [10, 20], materials engineering [24], and computer vision [5].

As POP $(\mathbb{P}_K)$ is known to be NP-hard, finding a global optimal solution of $(\mathbb{P}_K)$ is a significant challenge. A widely adopted strategy in solving $(\mathbb{P}_K)$ has been the use of semidefinite programming (SDP) relaxation based on Putinar's Positivstellensatz [19] by Lasserre [9]. Parrilo also proposed the SDP relaxation problem through the sum-of-squares (SOS) polynomial relaxation [18], which is viewed as the dual of Lasserre's SDP relaxation. While Nesterov and Nemirovskii [15] theoretically showed that SDP problems can be solved in polynomial time, the size of SDP relaxation problems for POPs becomes increasingly large [16] with the degree and the number of variables of POPs. Thus solving the SDP problems is challenging due to computational costs [25]. In fact, for POPs of degree d with n variables, the size of the variable matrix in the SDP problem amounts to $s(d) := \binom{n+d}{d}$. To address this challenge, various approaches have been proposed, such as exploiting the sparse structures of $(\mathbb{P}_K)$ [27, 28, 21] and employing weaker relaxations as diagonally-dominant-sum-of-squares relaxation [1].

For the difficulty in solving large-sized SDP relaxations of constrained POPs, our objective is to employ a third-order tensor and propose a relaxation method for constrained POP $(\mathbb{P}_K)$. The third-order tensor has been studied in many applications [17, 26]. Furthermore, Kilmer, Martin and Perrone [7] introduced the product in third-order tensors, an extension of the product of matrices, and third-order tensors have become gradually popular in recent years [2, 11, 23].

Applying the product in third-order tensors, Zheng et al. [33] introduced semidefiniteness and SDP in third-order tensors (referred to as semidefinite tensor and T-SDP, respectively). Semidefinite tensor and T-SDP serve as extensions of semidefinite matrices and SDPs, respectively. The entire set of semidefinite tensors is nonempty, closed, convex, pointed cone, and self-dual. Under these properties, duality theorems and optimality conditions have been shown for T-SDP as well as for SDP. While various application problems can be formulated as T-SDP, no solver has currently been developed to directly solve the T-SDP problem. Therefore, it is common to use the structural advantage of third-order tensors to convert the problem into an equivalent SDP problem and solve it by applying existing solvers such as [13, 29, 30].

As the close relationship between SOS relaxation and SOS polynomials, semidefinite third-order tensor and T-SDP are based on the SOS polynomial with a block circulant structure, commonly referred to as block circulant SOS polynomials [32]. This block circulant SOS polynomial can be viewed as an extension of SOS polynomials.

The purpose of this paper is to introduce a relaxation by the block circulant SOS polynomial instead of the SOS relaxation for constrained POPs in [8] to increase the efficiency of solving SDP relaxations of $(\mathbb{P}_K)$. The SOS relaxation proposed in [8] is called the SOS relaxation or the basic SOS relaxation in this paper. Let the degree of $f(x)$ and $g_i(x)$ be d and w_i, respectively, and a positive integer N, the relaxation level, such that $2N \geq d$

and $2N \geq \max_i\{w_i\}$, and $\tilde{w}_i = \lceil w_i/2 \rceil,\ i = 1, \dots, r$. With positive integers m_0, l_0, m_i, and l_i such that $s(N) = m_0 l_0,\ s(N - \tilde{w}_i) = m_i l_i,\ i = 1, \dots, r$, we employ $2N$-degree l_0-block circulant SOS polynomials and $2(N - \tilde{w}_i)$-degree l_i-block circulant SOS polynomials for our block-circulant SOS relaxation. If we choose $l_i = 1,\ i = 0, \dots, r$, the proposed block circulant SOS relaxation coincides with the basic SOS relaxation. The proposed block circulant SOS relaxation relies on the existence of $2N$-degree l_0 and $2(N - \tilde{w}_i)$-degree l_i-block circulant SOS polynomials for $(\mathbb{P}_K)$. Thus, the choice of m_i and $l_i, i = 0, \dots, r$ is essential for numerical efficiency, as shown in Section 4.1.

We first show that the block circulant SOS relaxation problem can be described as an equivalent T-SDP problem. Then, the T-SDP relaxation problem is equivalently transformed into an SDP problem of smaller size than the SDP problem from the basic SOS relaxation. Therefore, the T-SDP relaxation is expected to reduce the computational time. Indeed, we observe the computational advantage through numerical experiments presented in Section 4.1. For instance, the computational time for a constrained POP is reduced from 1264 seconds by the basic SOS relaxation to 633 seconds by the proposed relaxation based on circulant SOS polynomials with appropriate block sizes, under a mild assumption.

The main contributions of this paper are:

- We propose a T-SDP relaxation for constrained POPs, extending the result in [33] for unconstrained POPs.
- By transforming the T-SDP relaxation problem derived from a block circulant SOS relaxation for a constrained POP into the equivalent SDP problem of smaller size with the block-diagonal structure, the computational efficiency of solving the constrained POP is increased.
- We analyze the feasibility and global optimality of the T-SDP relaxation problem, and prove the convergence of the T-SDP relaxation sequence to the optimal value of $(\mathbb{P}_K)$ under an assumption that extends Putinar's Positivstellensatz.

The remaining of our paper is organized as follows. In Section 2, we describe the basic definitions of third-order tensor and block diagonalization, which are important for the equivalent transformation from T-SDP to SDP. Based on these definitions, we introduce the concepts of semidefiniteness and the T-SDP problem over third-order tensors and show that the T-SDP problem can be transformed into an equivalent SDP. We also define the block circulant SOS polynomial, an extension of the SOS polynomial, and introduce an important theorem that holds for semidefinite tensors. In Section 3, we propose a block circulant SOS relaxation method that extends the basic SOS relaxation method for constrained POPs, and show that the relaxation problem can be described by a T-SDP form. We also show that it has an advantage in problem size compared to the SDP relaxation. In addition, we provide theoretical analysis such as feasibility and global optimality for the T-SDP relaxation problem. In Section 4, we compare the proposed T-SDP relaxation method with the SDP relaxation using a set of test problems for constrained POPs. Finally, we conclude in Section 5.

2 Preliminaries

2.1 Notation

Throughout this paper, $\mathbb{R}^n$ denotes the set consisting of all real vectors of size n, $\mathbb{C}^n$ the set consisting of all complex vectors of size n, and $\mathbb{N}^n$ the set of nonnegative integers. We use $\mathbb{R}^{m\times n}$ to denote the set of $m \times n$ real matrices, $\mathbb{C}^{m\times n}$ the set of $m \times n$ complex matrices, $\mathbb{R}^{m\times n\times l}$, which is an array with one dimension added to the matrix, the set of $m \times n \times l$ real third-order tensors. We consider only real third-order tensors in this paper. The nth-order identity matrix is denoted by I_n and the nth-order zero matrix by O_n, or I and O. The zero tensor is denoted by $\mathcal{O}$. The identity tensor is defined in Section 2.2.

The transpose of a matrix A is denoted by A^T, the conjugate by $\overline{A}$, and the conjugate transpose by A^H. We denote the set consisting of all real symmetric (complex hermitian) matrices of size $n \times n$ as $\mathbb{S}^{n\times n}$ ($\mathbb{H}^{n\times n}$). $A \succ (\succeq) O$ denotes that a matrix A is positive definite (positive semidefinite), while the set of all real symmetric positive definite (positive semidefinite) matrices of size $n \times n$ is denoted by $\mathbb{S}^{n\times n}_{++}(\mathbb{S}^{n\times n}_{+})$. Similarly, we denote the set of all complex hermitian positive definite (positive semidefinite) matrices of size $n \times n$ as $\mathbb{H}^{n\times n}_{++}$ ($\mathbb{H}^{n\times n}_{+}$).

The inner product of two matrices in $\mathbb{R}^{m\times n}$ is defined as $\langle A, B\rangle := \mathrm{Tr}(A^\mathsf{T} B) = \sum_{i=1}^m \sum_{j=1}^n [A]_{(i,j)}[B]_{(i,j)}$, where $[A]_{(i,j)}$, $[B]_{(i,j)}$ denote the (i, j)th elements of A, B, respectively. The inner product of two third-order tensors $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{R}^{m\times n\times l}$ is defined as $\langle \mathcal{A}, \mathcal{B}\rangle := \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l [\mathcal{A}]_{(i,j,k)}[\mathcal{B}]_{(i,j,k)}$, where $[\mathcal{A}]_{(i,j,k)}$, $[\mathcal{B}]_{(i,j,k)}$ denote the (i, j, k)th elements of $\mathcal{A}$, $\mathcal{B}$, respectively. A symbol $\otimes$ denotes the Kronecker product of two matrices.

2.2 Third-order tensor

For a third-order tensor $\mathcal{A} \in \mathbb{R}^{m\times n\times l}$, we consider $\mathcal{A}$ as a stack of frontal slices $A^{(i)} \in \mathbb{R}^{m\times n}, i = 1, \ldots, l$, as proposed in [7] where several operators on $\mathcal{A} \in \mathbb{R}^{m\times n\times l}$ were introduced as follows:

$$
\mathrm{bcirc}(\mathcal{A}) := \begin{bmatrix} A^{(1)} & A^{(l)} & A^{(l-1)} & \ldots & A^{(2)} \\ A^{(2)} & A^{(1)} & A^{(l)} & \ldots & A^{(3)} \\ A^{(3)} & A^{(2)} & A^{(1)} & \ldots & A^{(4)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A^{(l)} & A^{(l-1)} & A^{(l-2)} & \ldots & A^{(1)} \end{bmatrix}, \quad \mathrm{unfold}(\mathcal{A}) := \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ A^{(3)} \\ \vdots \\ A^{(l)} \end{bmatrix},
$$
$$
\mathrm{bcirc}_l^{-1}(\mathrm{bcirc}(\mathcal{A})) := \mathcal{A}, \quad \mathrm{fold}_l(\mathrm{unfold}(\mathcal{A})) := \mathcal{A}.
$$

Let $\mathrm{bcirc} : \mathbb{R}^{m\times n\times l} \to \mathbb{R}^{ml\times nl}$ be the operator that arranges each frontal slice of $\mathcal{A}$ into a block circulant matrix, $\mathrm{unfold} : \mathbb{R}^{m\times n\times l} \to \mathbb{R}^{ml\times n}$ be the operator that arranges each frontal slice of $\mathcal{A}$ in columns. bcirc_l^{-1} and fold_l are operators that are the inverse operations of bcirc and unfold with l, respectively. From the definition of *bcirc*, it clearly holds that

$$
\langle \mathcal{A}, \mathcal{B}\rangle = \frac{1}{l} \langle \mathrm{bcirc}(\mathcal{A}), \mathrm{bcirc}(\mathcal{B})\rangle . \tag{1}
$$

It is known that any circulant matrix $A \in \mathbb{R}^{n\times n}$ can be diagonalized with the normalized discrete Fourier transform (DFT) matrix, where the DFT matrix is the Fourier matrix of size $n \times n$ defined as

$$F_n := \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{(n-1)2} & \dots & \omega^{(n-1)(n-1)} \end{bmatrix}, \quad \omega := e^{\frac{2\pi i}{n}}.$$

Using this, the block circulant matrix $\mathrm{bcirc}(\mathcal{A})$ can be block-diagonalized with the Kronecker product.

Lemma 1. *[6] Any third-order tensor $\mathcal{A} \in \mathbb{R}^{m\times n\times l}$ can be block-diagonalized as*

$$(F_l^{\mathsf{H}} \otimes I_m)\mathrm{bcirc}(\mathcal{A})(F_l \otimes I_n) = Diag(A_1, A_2, \dots, A_l) = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_l \end{bmatrix},$$

where $A_1 \in \mathbb{R}^{m\times n}$, $A_i \in \mathbb{C}^{m\times n}$ and $A_i = \overline{A_{l+2-i}}$, $i = 2, \dots, l$.

In what follows, we define the product and transposition of third-order tensors and extend the semidefiniteness to the space of third-order tensors, and introduce some theorems on semidefinite tensors.

Definition 2. *[7, Definition 4.1] Let $\mathcal{A} \in \mathbb{R}^{m\times n\times l}$ and $\mathcal{B} \in \mathbb{R}^{n\times p\times l}$ be two third-order tensors. Then, the product of third order tensors $\mathcal{A} * \mathcal{B} \in \mathbb{R}^{m\times p\times l}$ is defined by*

$$\mathcal{A} * \mathcal{B} := \mathrm{fold}_l\left(\mathrm{bcirc}(\mathcal{A})\mathrm{unfold}(\mathcal{B})\right).$$

In the case of $l = 1$, the third-order tensors $\mathcal{A} \in \mathbb{R}^{m\times n\times l}$, $\mathcal{B} \in \mathbb{R}^{n\times p\times l}$ are the matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, respectively, and $\mathrm{bcirc}(A) = A$ and $\mathrm{unfold}(B) = B$ hold for $l = 1$, thus the product in the third-order tensors $\mathcal{A} * \mathcal{B}$ is the matrix product AB.

We call a third-order tensor whose first frontal slice is the identity matrix and the other frontal slices are the zero matrix the identity tensor $\mathcal{I}$. The identity tensor $\mathcal{I}$ is defined as a third-order tensor whose first frontal slice is the identity matrix and the other frontal slices are the zero matrix. For a third-order tensor $\mathcal{A} \in \mathbb{R}^{n\times n\times l}$ and the identity tensor $\mathcal{I} \in \mathbb{R}^{n\times n\times l}$, we have $\mathcal{A} * \mathcal{I} = \mathcal{I} * \mathcal{A} = \mathcal{A}$.

Definition 3. *[7, Definition 4.7] If $\mathcal{A} \in \mathbb{R}^{m\times n\times l}$ is a third-order tensor, then the transpose $\mathcal{A}^{\mathsf{T}}$ is obtained by transposing each of the frontal slices $A^{(i)}$ and then reversing the order of transposed frontal slices 2 through l. Furthermore, $\mathcal{A} \in \mathbb{R}^{m\times m\times l}$ is called a symmetric sensor if $\mathcal{A}^{\mathsf{T}} = \mathcal{A}$. We denote the set of all real symmetric tensors of size $m \times m \times l$ as $\mathbb{S}^{m\times m\times l}$.*

Definition 4. *[33, Definition 6] Let $\mathcal{A} \in \mathbb{S}^{m\times m\times l}$ be a symmetric tensor. We say $\mathcal{A}$ is symmetric positive (semi)definite, if and only if,*

$$\langle \mathcal{X}, \mathcal{A} * \mathcal{X} \rangle > (\geq) 0$$

holds for any $\mathcal{X} \in \mathbb{R}^{m\times 1\times l} \setminus \{\mathcal{O}\}$(for any $\mathcal{X} \in \mathbb{R}^{m\times 1\times l}$).

We denote $\mathcal{A} \succ_{\mathcal{T}} (\succeq_{\mathcal{T}})\ \mathcal{O}$ if $\mathcal{A} \in \mathbb{S}^{m\times m\times l}$ is positive (semi)definite, and $\mathbb{S}_{++}^{m\times m\times l}$ ($\mathbb{S}_{+}^{m\times m\times l}$) denotes the set of all real symmetric positive (semi)definite tensors of size $m \times m \times l$. The set $\mathbb{S}_{+}^{m\times m\times l}$ is a nonempty, closed, convex, and pointed cone [33, Proposition 3]. Moreover, $\mathbb{S}_{+}^{m\times m\times l} = (\mathbb{S}_{+}^{m\times m\times l})^*$ holds for the dual cone $(\mathbb{S}_{+}^{m\times m\times l})^*$ of $\mathbb{S}_{+}^{m\times m\times l}$ (the self-duality) [33, Theorem 9].

Theorem 5. *[33, Theorem 4] Let $\mathcal{A} \in \mathbb{S}^{m\times m\times l}$ be a symmetric tensor. Then, the following statements are equivalent:*

(i) $\mathcal{A} \in \mathbb{S}_{+}^{m\times m\times l}$.

(ii) $\mathrm{bcirc}(\mathcal{A}) \in \mathbb{S}_{+}^{ml\times ml}$.

(iii) The block-diagonal matrix $Diag(A_1, \ldots, A_l)$, which is a block diagonalization of A, is a hermitian positive semidefinite matrix. In other words, $A_1 \in \mathbb{S}_{+}^{m\times m}$, $A_i \in \mathbb{H}_{+}^{m\times m}$, $i = 2, \ldots, l$.

2.3 SDP in third-order tensor space

We discuss an SDP problem in the space of third order tensors (T-SDP) in this section using the semidefiniteness in third-order tensor defined in Section 2.2. We also describe that a T-SDP problem can be converted into an SDP problem in complex space. The transformed SDP can be solved using existing SDP solvers, for instance Mosek [14], which is used to solve the SDP in our numerical experiments in Section 4.1.

An SDP problem is defined as:

$$\text{minimize } \langle C, X\rangle \text{ subject to } \quad \langle A_i, X\rangle = b_i, \quad i = 1, \ldots, r, \quad X \succeq O,$$

where $X \in \mathbb{S}^{m\times m}$ is the decision variable, $C, A_i \in \mathbb{S}^{m\times m}$ and $b \in \mathbb{R}^r$ are given symmetric matrices and a vector, respectively. Its dual problem is described as:

$$\text{maximize } \sum_{i=1}^{r} b_i y_i \text{ subject to } \quad \sum_{i=1}^{r} y_i A_i + S = C, \quad S \succeq O,$$

where $y \in \mathbb{R}^r$ is the decision variable and $S \in \mathbb{S}^{m\times m}$ is the slack variable.

Now, we extend the SDP problem in the space of symmetric matrices to the space of third-order tensors by extending the real symmetric matrices C, A_i, X to real symmetric tensors $\mathcal{C}, \mathcal{A}_i, \mathcal{X} \in \mathbb{S}^{m\times m\times l}$ and the semidefinite constraint $X \succeq O$ to $\mathcal{X} \succeq_{\mathcal{T}} \mathcal{O}$. A T-SDP problem is defined as:

$$\text{minimize } \langle \mathcal{C}, \mathcal{X}\rangle \text{ subject to } \quad \langle \mathcal{A}_i, \mathcal{X}\rangle = b_i, \quad i = 1, \ldots, r, \quad \mathcal{X} \succeq_{\mathcal{T}} \mathcal{O}. \tag{PTSDP}$$

Similarly, its dual problem is defined as:

$$\text{maximize } \sum_{i=1}^{r} b_i y_i \text{ subject to } \quad \sum_{i=1}^{r} y_i \mathcal{A}_i + \mathcal{S} = \mathcal{C}, \quad \mathcal{S} \succeq_{\mathcal{T}} \mathcal{O}. \tag{DTSDP}$$

Clearly, an SDP problem can be considered as a special case of T-SDP at $l = 1$.

The following duality theorem holds for T-SDP as well as SDP.

Theorem 6. *[33, Theorem 11] Let $F(P)$, $F(D)$, p^*, and d^* be defined as follows:*

$$F(P) = \left\{ \mathcal{X} \in \mathbb{S}^{m \times m \times l} \mid \langle \mathcal{A}_i, \mathcal{X} \rangle = b_i,\ i = 1, \dots, r,\ \mathcal{X} \succeq_{\mathcal{T}} \mathcal{O} \right\},$$
$$F(D) = \left\{ (y, \mathcal{S}) \in \mathbb{R}^r \times \mathbb{S}^{m \times m \times l} \;\middle|\; \sum_{i=1}^{r} y_i \mathcal{A}_i + \mathcal{S} = \mathcal{C},\ \mathcal{S} \succeq_{\mathcal{T}} \mathcal{O} \right\},$$
$$p^* = \inf \left\{ \langle \mathcal{C}, \mathcal{X} \rangle \mid \mathcal{X} \in F(P) \right\}, \quad d^* = \sup \left\{ \langle b, y \rangle \mid (y, \mathcal{S}) \in F(D) \right\}.$$

Suppose that $\mathcal{X} \in F(P)$ and $(y, \mathcal{S}) \in F(D)$. Then, $\langle b, y \rangle \le \langle \mathcal{C}, \mathcal{X} \rangle$. In addition, if one of (PTSDP) or (DTSDP) is bounded below and strictly feasible, then the other is solvable and $p^* = d^*$.

Based on [33], we briefly describe a method for solving the T-SDP problem by transforming it into an equivalent SDP problem over the complex space. For every third-order tensor $\mathcal{C} \in \mathbb{S}^{m \times m \times l}$, $\mathrm{bcirc}(\mathcal{C})$ can be block-diagonalized as

$$\mathrm{bcirc}(\mathcal{C}) = (F_l \otimes I_m) Diag(C_1, C_2, \dots, C_l)(F_l^{\mathsf{H}} \otimes I_m).$$

Thus, for the objective function of (PTSDP), we obtain

$$\begin{aligned}
\langle \mathcal{C}, \mathcal{X} \rangle &= \frac{1}{l} \langle \mathrm{bcirc}(\mathcal{C}), \mathrm{bcirc}(\mathcal{X}) \rangle = \frac{1}{l} \mathrm{Tr}(\mathrm{bcirc}(\mathcal{C}) \mathrm{bcirc}(\mathcal{X})) \\
&= \frac{1}{l} \mathrm{Tr}((F_l \otimes I_m) Diag(C_1, \dots, C_l)(F_l^{\mathsf{H}} \otimes I_m)(F_l \otimes I_m) Diag(X_1, \dots, X_l)(F_l^{\mathsf{H}} \otimes I_m)) \\
&= \frac{1}{l} \mathrm{Tr}(Diag(C_1, \dots, C_l) Diag(X_1, \dots, X_l)) = \frac{1}{l} \sum_{k=1}^{l} \langle C_k, X_k \rangle,
\end{aligned}$$

using (1) for the first equality and the commutative property of trace for the fourth equality.

Similarly, using the block-diagonalized matrix $Diag(A_1^i, \dots, A_l^i)$ of $\mathcal{A}_i$, we have $\langle \mathcal{A}_i, \mathcal{X} \rangle = \frac{1}{l} \sum_{k=1}^{l} \langle A_k^i, X_k \rangle$. From Theorem 5, $\mathcal{X} \succeq_{\mathcal{T}} \mathcal{O}$ is equivalent to $X_1, \dots, X_l \succeq O$. Since l is an integer such that $l \ge 1$, we can replace $\frac{1}{l} X_k$ by X_k without loss of generality. Consequently, (PTSDP) and (DTSDP) are transformed into SDP problems in the space of complex matrices:

$$\min \sum_{k=1}^{l} \langle C_k, X_k \rangle \text{ sub. to } \sum_{k=1}^{l} \langle A_k^i, X_k \rangle = b_i, \quad i = 1, \dots, r, \quad X_1, \dots, X_l \succeq O, \quad \text{(PCSDP)}$$

$$\max \sum_{i=1}^{r} b_i y_i \text{ sub. to } \sum_{i=1}^{r} y_i A_k^i + S_k = C_k, \quad k = 1, \dots, l, \quad S_1, \dots, S_l \succeq O, \quad \text{(DCSDP)}$$

where $X_k \in \mathbb{H}^{m \times m}$, $y \in \mathbb{R}^r$ and $S_k \in \mathbb{H}^{m \times m}$ are the decision variables, and C_k, $A_k^i \in \mathbb{H}^{m \times m}$ are obtained from the diagonalization.

Since Lemma 1 holds for block diagonal matrices as described in Section 2.2, (PCSDP) can be transformed into an equivalent problem of smaller size depending on the values of l, even or odd.

Case 1: l is even

Let C_k, A_k^i, X_k, $k = 1, \ldots, l$ be the blocks in the block diagonalized matrix of $\mathcal{C}, \mathcal{A}_i, \mathcal{X}$ in (PTSDP). Then, the following relation holds for C_k, A_k^i, X_k by (1):

$$
\begin{aligned}
&C_1 \in \mathbb{S}^{m\times m},\ C_{\frac{l+2}{2}} \in \mathbb{S}^{m\times m},\ C_k \in \mathbb{H}^{m\times m},\ C_k = \overline{C_{l+2-k}},\\
&A_1^i \in \mathbb{S}^{m\times m},\ A^i_{\frac{l+2}{2}} \in \mathbb{S}^{m\times m},\ A_k^i \in \mathbb{H}^{m\times m},\ A_k^i = \overline{A^i_{l+2-k}}, \quad i = 1.\ldots, r,\\
&X_1 \in \mathbb{S}^{m\times m},\ X_{\frac{l+2}{2}} \in \mathbb{S}^{m\times m},\ X_k \in \mathbb{H}^{m\times m},\ X_k = \overline{X_{l+2-k}}.
\end{aligned}
$$

For the objective function of (PCSDP), we obtain

$$
\begin{aligned}
\sum_{k=1}^{l} \langle C_k, X_k\rangle &= \langle C_1, X_1\rangle + \left\langle C_{\frac{l+2}{2}}, X_{\frac{l+2}{2}}\right\rangle + \sum_{k=2}^{\frac{l}{2}} \left(\langle C_k, X_k\rangle + \langle \overline{C_k}, \overline{X_k}\rangle\right)\\
&= \langle C_1, X_1\rangle + \left\langle C_{\frac{l+2}{2}}, X_{\frac{l+2}{2}}\right\rangle + 2\sum_{k=2}^{\frac{l}{2}} \langle C_k, X_k\rangle,
\end{aligned}
$$

using the fact that the inner product of the two hermitian matrices is equal to the inner product of their conjugates in the second equality. Similarly, we obtain

$$
\sum_{k=1}^{l} \left\langle A_k^i, X_k\right\rangle = \left\langle A_1^i, X_1\right\rangle + \left\langle A^i_{\frac{l+2}{2}}, X_{\frac{l+2}{2}}\right\rangle + 2\sum_{k=2}^{\frac{l}{2}} \left\langle A_k^i, X_k\right\rangle, \quad i = 1, \ldots, r.
$$

Therefore, (PCSDP) is equivalent to the following problem:

$$
\begin{aligned}
&\text{minimize} && \langle C_1, X_1\rangle + 2\sum_{k=2}^{\frac{l}{2}} \langle C_k, X_k\rangle + \left\langle C_{\frac{l+2}{2}}, X_{\frac{l+2}{2}}\right\rangle\\
&\text{subject to} && \left\langle A_1^i, X_1\right\rangle + 2\sum_{k=2}^{\frac{l}{2}} \left\langle A_k^i, X_k\right\rangle + \left\langle A^i_{\frac{l+2}{2}}, X_{\frac{l+2}{2}}\right\rangle = b_i, \quad i = 1, \ldots, r,\\
& && X_1, \ldots, X_{\frac{l+2}{2}} \succeq O.
\end{aligned}
\tag{P'CSDP}
$$

Case 2: l is odd

As in Case 1, the following relation holds for C_k, A_k^i, X_k by (1):

$$
\begin{aligned}
&C_1 \in \mathbb{S}^{m\times m},\ C_k \in \mathbb{H}^{m\times m},\ C_k = \overline{C_{l+2-k}},\\
&A_1^i \in \mathbb{S}^{m\times m},\ A_k^i \in \mathbb{H}^{m\times m},\ A_k^i = \overline{A^i_{l+2-k}}, \quad i = 1.\ldots, r,\\
&X_1 \in \mathbb{S}^{m\times m},\ X_k \in \mathbb{H}^{n\times n},\ X_k = \overline{X_{l+2-k}}.
\end{aligned}
$$

Therefore, (PCSDP) is equivalent to the following problem:

$$
\begin{aligned}
&\text{minimize} && \langle C_1, X_1\rangle + 2\sum_{k=2}^{\frac{l+1}{2}} \langle C_k, X_k\rangle\\
&\text{subject to} && \left\langle A_1^i, X_1\right\rangle + 2\sum_{k=2}^{\frac{l+1}{2}} \left\langle A_k^i, X_k\right\rangle = b_i, \quad i = 1, \ldots, r,\\
& && X_1, \ldots, X_{\frac{l+1}{2}} \succeq O.
\end{aligned}
\tag{P"CSDP}
$$

We have shown that (PTSDP) with a third-order tensor of size $m \times m \times l$ can be transformed into (PCSDP) which includes the sum of l third-order tensors of size $m \times m$ matrices. Furthermore, (PCSDP) can be transformed into (P'CSDP) and (P"CSDP) which include the sum of $\frac{l+2}{2}$ or $\frac{l+1}{2}$ matrices of size $m \times m$, respectively, depending on even or odd value of l. These properties will be used to propose a T-SDP relaxation for constrained POPs.

2.4 Block circulant SOS polynomials

In this section, we first define the SOS polynomial with block circulant structure, referred to as block circulant SOS polynomial, and then explain the relationship between the block circulant SOS polynomial and semidefinite tensor.

We first describe some notation. A real-valued polynomial $f(x)$ of degree d, $d \in \mathbb{N}$, is expressed as

$$f(x) = \sum_{0 \le |\alpha| \le d} b_\alpha x^\alpha \text{ with } x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},\ b_\alpha := b_{\alpha_1 \alpha_2 \cdots \alpha_n} \in \mathbb{R} \text{ and } |\alpha| := \sum_{i=1}^{r} \alpha_i \le d, \tag{2}$$

for $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$. Let

$$[x]_d := [1, x_1, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_n^2, \ldots, x_1^d, \ldots, x_n^d]^\top,$$

and let $s(d) = \binom{n+d}{d}$ be the dimension of $[x]_d$. Let $\mathbb{R}[x]_d$ be the set of all d-degree real-valued polynomials. If $f(x) \in \mathbb{R}[x]_d$ and $f(x) \ge 0$ for any $x \in \mathbb{R}^n$, then f is called a nonnegative polynomial. The set of all nonnegative polynomials is denoted by $\mathbb{R}_+[x]_d$. We say that q is an SOS polynomial if the real-valued polynomial $q : \mathbb{R}^n \to \mathbb{R}$ can be expressed as $q(x) = \sum_i \bar{q}_i^2(x)$ with some real-valued polynomial $\bar{q}_i : \mathbb{R}^n \to \mathbb{R}$. It is well-known that a nonnegative polynomial is not necessarily an SOS polynomial except for special cases [22].

Definition 7. *[32, Definition 3] Let $q : \mathbb{R}^n \to \mathbb{R}$ be a real-valued polynomial of degree $2d$, and m and l be positive integers such that $s(d) = m \cdot l$. Then we say q is an l-block circulant SOS polynomial if*

$$q(x) = \sum_{i=1}^{r} \sum_{j=1}^{l} \left(\left(q_j^i\right)^\top [x]_d \right)^2,$$

where q_j^i is the j-th column vector of $\mathrm{bcirc}(\mathrm{fold}_l(q^i))$ *for some generators $\{q^1, \ldots, q^r\} \subset \mathbb{R}^{s(d)}$.*

Example 8. *(3-block circulant SOS polynomial)*
A polynomial q in 2 variables (x_1 and x_2) defined by

$$q(x) = 21 + 8x_2 + 9x_1^2 + 20x_1x_2 + 33x_2^2 + 8x_1x_2^2 + 9x_1^4 + 12x_1^3x_2 + 21x_1^2x_2^2 + 9x_2^4$$

is a 3-block circulant SOS polynomial. Two generators q^1, q^2 for $q(x)$ are

$$q^1 = [2, 0, 0, 0, 0, 3]^\top, \quad q^2 = [4, 0, 1, 0, 0, 0]^\top.$$

We see that

$$\mathrm{bcirc}(\mathrm{fold}_3(q^1)) = \left[\begin{array}{c:c:c} 2 & 0 & 0 \\ 0 & 3 & 0 \\ \hdashline 0 & 2 & 0 \\ 0 & 0 & 3 \\ \hdashline 0 & 0 & 2 \\ 3 & 0 & 0 \end{array}\right], \quad \mathrm{bcirc}(\mathrm{fold}_3(q^2)) = \left[\begin{array}{c:c:c} 4 & 0 & 1 \\ 0 & 0 & 0 \\ \hdashline 1 & 4 & 0 \\ 0 & 0 & 0 \\ \hdashline 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right].$$

From $[x]_2 := [1, x_1, x_2, x_1^2, x_1x_2, x_2^2]^\top$, *we have*

$$\sum_{i=1}^{2}\sum_{j=1}^{3}((q_j^i)^\top [x]_2)^2 = (2+3x_2^2)^2 + (3x_1+2x_2)^2 + (3x_1^2+2x_1x_2)^2 \\ + (4+x_2)^2 + (4x_2+x_1x_2)^2 + (1+4x_1x_2)^2,$$

which is equivalent to q.

If $l = 1$ in Definition 7, then $\mathrm{bcirc}(\mathrm{fold}_l(q^i)) = q^i$, thus q_j^i is q^i itself. Thus, a 1-block circulant SOS polynomial is an SOS polynomial.

Theorem 9. *[32] Let* $q : \mathbb{R}^n \to \mathbb{R}$ *be a real-valued polynomial of degree* $2d$*, and* m *and* l *be positive integers such that* $s(d) = m \cdot l$*. Then, the following statements are equivalent:*

(i) q *is an* l*-block circulant SOS polynomial*

(ii) There exists an l*-block circulant matrix* $A \in \mathbb{S}_+^{ml\times ml}$ *such that each block* $A^{(1)}, \ldots, A^{(l)} \in \mathbb{R}^{m\times m}$ *and* q *can be expressed by*

$$q(x) = \left\langle A, [x]_d[x]_d^\top \right\rangle$$

(iii) There exists a semidefinite tensor $\mathcal{A} \in \mathbb{S}_+^{m\times m\times l}$ *and* $[\mathcal{X}]_d^l = \mathrm{fold}_l([x]_d)$ *such that* $\bar{f}$ *can be expressed by*

$$q(x) = \left\langle \mathcal{A}, [\mathcal{X}]_d^l * \left([\mathcal{X}]_d^l\right)^\top \right\rangle.$$

In the case of $l = 1$ in Theorem 9, the semidefinite 1-block circulant matrix in *(ii)* is a semidefinite matrix. Moreover, since the third-order tensor $\mathcal{A}$ in *(iii)* becomes a matrix and $[\mathcal{X}]_d^l$ a vector $[x]_d$, *(ii)* and *(iii)* are equivalent. For $l \geq 2$, the l-block circulant matrix A in *(ii)* can be regarded as a 1-block circulant matrix when the entire matrix is considered as a single block. As a result, an l-block circulant SOS polynomial is also an SOS polynomial. Consequently, we have the following relation:

$$\Sigma^l[x]_d\,(l \geq 2) \subseteq \Sigma^1[x]_d \subseteq \mathbb{R}_+[x]_d, \tag{3}$$

where $\Sigma^l[x]_d$ denotes the set of l-block circulant SOS polynomials of degree d.

3 T-SDP relaxations for constrained POPs

In this section, we propose a T-SDP relaxation method for the constrained POP $(\mathbb{P}_K)$, which is an extension of the SDP relaxation method in Section 2.3. Then, we show that the proposed T-SDP relaxation can be reduced to a smaller SDP using the properties of T-SDP than the SDP problem from the basic SOS relaxation. We also discuss the feasibility and global optimality of the proposed method.

3.1 SDP relaxation in the third-order tensor space

To derive T-SDP relaxation for $(\mathbb{P}_K)$, we let $\tilde{w}_i = \lceil w_i/2 \rceil$, where w_i is the degree of $g_i(x)$ and define a positive integer N called the relaxation level such that $2N \geq d$ and $2N \geq \max_i\{w_i\}$. We also determine certain positive integers m_0, l_0, m_i, and l_i, $i = 1, \ldots, r$ such that

$$s(N) = m_0 l_0, \quad s(N - \tilde{w}_i) = m_i l_i, \quad i = 1, \ldots, r. \tag{4}$$

For the discussion of the optimal value of $(\mathbb{P}_K)$, we introduce the following assumptions. We should mention that the convexity and connectivity of the feasible set K are not assumed here.

Assumption 10. *[9, Assumption 4.1] The set $K = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, \ \ i = 1, \ldots, r\}$ is compact and there exists a real-valued polynomial $u : \mathbb{R}^n \to \mathbb{R}$ such that $\{x \in \mathbb{R}^n \mid u(x) \geq 0\}$ is compact, then*

$$u(x) = u_0(x) + \sum_{i=1}^{r} g_i(x) u_i(x) \quad \textit{for all } x \in \mathbb{R}^n, \tag{5}$$

where $u_i(x)$, $i = 0, \ldots, r$ are SOS polynomials.

Assumption 11. *The set $K = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, \ \ i = 1, \ldots, r\}$ is compact. If $p_K^* = \min_{x \in K} f(x)$, for any $\epsilon > 0$,*

$$f(x) - p_K^* + \epsilon = q_0(x) + \sum_{i=1}^{r} g_i(x) q_i(x) \quad \textit{for all } x \in K, \tag{6}$$

where $q_0(x)$ is a $2N$-degree l_0-block circulant SOS polynomial, $q_i(x)$, $i = 1, \ldots, r$ are $2(N - \tilde{w}_i)$-degree l_i-block circulant SOS polynomials.

We note that Assumption 10 is commonly used and Assumption 11 is an extension of Putinar's Positivstellensatz [19], which was used in the SDP relaxation [9] for the objective function f. In Assumption 11, q_0 and q_i need to be l_0- and l_i-block circulant SOS polynomials instead of SOS polynomials. Thus, Assumption 11 always holds if $l_0 = 1$ and $l_i = 1$ under the condition for Putinar's Positivstellensatz.

We extend the basic SOS relaxation method based on the generalized Lagrange function by Kim et al. [8] to SOS relaxations using block circulant SOS polynomials. The generalized Lagrangian function for $(\mathbb{P}_K)$ is defined as

$$L(x, \phi_1, \ldots, \phi_r) := f(x) - \sum_{i=1}^{r} g_i(x) \phi_i(x), \quad \phi_i(x) \in \Sigma^{l_i}[x]_{2(N - \tilde{w}_i)}.$$

Here, the l_i-block circulant SOS polynomials ϕ_i is used. Then, the Lagrangian dual problem for $(\mathbb{P}_K)$ can be expressed as

$$\max_{\phi_i} \ \min_{x} \ L(x, \phi_1, \ldots, \phi_r).$$

Now, we consider the following problem with fixed $\phi_i(x)$:

$$\min_{x} \ L(x, \phi_1, \ldots, \phi_r), \tag{7}$$

which can be regarded as an unconstrained problem and also be written as

$$\max \ \gamma \text{ s.t. } L(x, \phi_1, \ldots, \phi_r) - \gamma \geq 0. \tag{8}$$

Then, we can apply l_0-block circulant SOS relaxation [32] to (7), or equivalently to (8):

$$\max_{\gamma, \phi_i} \ \gamma \quad \text{s.t. } L(x, \phi_1, \ldots, \phi_r) - \gamma \in \Sigma^{l_0}[x]_{2N}. \tag{9}$$

In the problem (9), the constraint $L(x, \phi_1, \ldots, \phi_r) - \gamma \in \Sigma^{l_0}[x]_{2N}$ is equivalent to the existence of some $\phi_0(x) \in \Sigma^{l_0}[x]_{2N}$ such that $L(x, \phi_1, \ldots, \phi_r) - \gamma = \phi_0(x)$. Thus, $(\mathbb{P}_K)$ can be relaxed to the following problem:

$$\begin{aligned} &\text{maximize} && \gamma \\ &\text{subject to} && f(x) - \gamma \in \Gamma := \left\{ \phi_0 + \sum_{i=1}^{r} g_i(x)\phi_i(x) \;\middle|\; \phi_0 \in \Sigma^{l_0}[x]_{2N},\ \phi_i \in \Sigma^{l_i}[x]_{2(N-\tilde{w}_i)} \right\}. \end{aligned} \tag{$\tilde{\mathbb{P}}_K$}$$

By Assumption 11, $(\tilde{\mathbb{P}}_K)$ has a feasible solution. Moreover, the optimal value of $(\tilde{\mathbb{P}}_K)$ is not greater than the optimal value of $(\mathbb{P}_K)$, based on the inclusion relations between the set of block circulant SOS polynomials and the set of nonnegative polynomials (3), i.e., $\max(\tilde{\mathbb{P}}_K) \leq \min(\mathbb{P}_K)$.

Next, we show that the relaxation problem $(\tilde{\mathbb{P}}_K)$ of $(\mathbb{P}_K)$ can be transformed into an equivalent problem using the T-SDP discussed in Section 2.3. With a monomial vector $[x]_N$, we define monomial tensors by

$$\mathcal{X}_0 := \text{fold}_{l_0}([x]_N), \quad \mathcal{X}_i := \text{fold}_{l_i}([x]_{N-\tilde{w}_i}), \quad i = 1, \ldots, r.$$

We determine third-order tensors $\mathcal{A}_\alpha \in \mathbb{R}^{m_0 \times m_0 \times l_0}$, $\mathcal{D}_{i\alpha} \in \mathbb{R}^{m_i \times m_i \times l_i}$ such that

$$\mathcal{X}_0 * \mathcal{X}_0^{\top} = \sum_{0 \leq |\alpha| \leq 2N} \mathcal{A}_\alpha x^\alpha \tag{10}$$

$$g_i(x)\mathcal{X}_i * \mathcal{X}_i^{\top} = \sum_{0 \leq |\alpha| \leq 2N} \mathcal{D}_{i\alpha} x^\alpha, \quad i = 1, \ldots, r. \tag{11}$$

If $l_0 = 1$ and $l_i = 1$ in (4), then $\text{fold}_1([x]_N) = [x]_N$ and $\text{fold}_1([x]_{N-\tilde{w}_i}) = [x]_{N-\tilde{w}_i}$ hold and the monomial tensors $\mathcal{X}_0$ and $\mathcal{X}_i$ become the monomial vectors, and the third-order tensors $\mathcal{A}_\alpha, \mathcal{D}_{i\alpha}$ to be determined become the matrices $A_\alpha \in \mathbb{R}^{m_0 l_0 \times m_0 l_0}$, $D_{i\alpha} \in \mathbb{R}^{m_i l_i \times m_i l_i}$. By Theorem 9, the condition $f(x) - \gamma \in \Gamma$ in $(\tilde{\mathbb{P}}_K)$ can be expressed as

$$
\begin{aligned}
f(x) - \gamma &= \left\langle \mathcal{Z}_0, \mathcal{X}_0 * \mathcal{X}_0^\mathsf{T} \right\rangle + \sum_{i=1}^{r} g_i(x) \left\langle \mathcal{Z}_i, \mathcal{X}_i * \mathcal{X}_i^\mathsf{T} \right\rangle \\
&= \sum_{0 \le |\alpha| \le 2N} \langle \mathcal{Z}_0, \mathcal{A}_\alpha \rangle x^\alpha + \sum_{i=1}^{r} \sum_{0 \le |\alpha| \le 2N} \langle \mathcal{Z}_i, \mathcal{D}_{i\alpha} \rangle x^\alpha, \qquad (12)
\end{aligned}
$$

where $\mathcal{Z}_0 \succeq_\mathcal{T} \mathcal{O}$, $\mathcal{Z}_i \succeq_\mathcal{T} \mathcal{O}$ are semidefinite tensors. Since the objective function $f(x)$ has no constant term ($f(0) = 0$), by comparing the coefficients on both sides of (12) using (2), we obtain

$$
\begin{aligned}
-\gamma &= \langle \mathcal{Z}_0, \mathcal{A}_0 \rangle + \sum_{i=1}^{r} \langle \mathcal{Z}_i, \mathcal{D}_{i0} \rangle \\
b_\alpha &= \langle \mathcal{Z}_0, \mathcal{A}_\alpha \rangle + \sum_{i=1}^{r} \langle \mathcal{Z}_i, \mathcal{D}_{i\alpha} \rangle, \quad 0 < |\alpha| \le 2N.
\end{aligned}
$$

Consequently, we obtain the T-SDP problem that is equivalent to problem $(\tilde{\mathbb{P}}_K)$:

$$
\begin{aligned}
&\text{maximize} \quad -\langle \mathcal{Z}_0, \mathcal{A}_0 \rangle - \sum_{i=1}^{r} \langle \mathcal{Z}_i, \mathcal{D}_{i0} \rangle \\
&\text{subject to} \quad \langle \mathcal{Z}_0, \mathcal{A}_\alpha \rangle + \sum_{i=1}^{r} \langle \mathcal{Z}_i, \mathcal{D}_{i\alpha} \rangle = b_\alpha, \quad 0 < |\alpha| \le 2N, \qquad (\mathbb{Q}_N^{l_0, l_1, \dots, l_r}) \\
&\qquad\qquad \mathcal{Z}_0 \succeq_\mathcal{T} \mathcal{O}, \ \mathcal{Z}_i \succeq_\mathcal{T} \mathcal{O}, \quad i = 1, \dots, r.
\end{aligned}
$$

The above $(\mathbb{Q}_N^{l_0, l_1, \dots, l_r})$ is an extension of the SDP relaxation problem derived from Parrilo's SOS relaxation [18] to a T-SDP problem. The dual problem $(\mathbb{Q}_N^{l_0, l_1, \dots, l_r})^*$ of $\mathbb{Q}_N^{l_0, l_1, \dots, l_r}$ is expressed as

$$
\begin{aligned}
&\text{minimize} \quad \sum_{0 < |\alpha| \le 2N} b_\alpha y_\alpha \\
&\text{subject to} \quad \sum_{0 < |\alpha| \le 2N} \mathcal{A}_\alpha y_\alpha \succeq_\mathcal{T} -\mathcal{A}_0, \qquad (\mathbb{Q}_N^{l_0, l_1, \dots, l_r})^* \\
&\qquad\qquad \sum_{0 < |\alpha| \le 2N} \mathcal{D}_{i\alpha} y_\alpha \succeq_\mathcal{T} -\mathcal{D}_{i0}, \quad i = 1, \dots, r.
\end{aligned}
$$

We notice that if we let $\mathcal{M}_N^{l_0}(y) := \mathcal{A}_0 + \sum_{0 < |\alpha| \le 2N} \mathcal{A}_\alpha y_\alpha$, $\mathcal{M}_{N - \tilde{w}_i}^{l_i}(g_i y) := \mathcal{D}_{i0} + \sum_{0 < |\alpha| \le 2N} \mathcal{D}_{i\alpha} y_\alpha$, then the constraints of $(\mathbb{Q}_N^{l_0, l_1, \dots, l_r})^*$ can be written as $\mathcal{M}_N^{l_0}(y) \succeq_\mathcal{T} \mathcal{O}$, $\mathcal{M}_N^{l_i}(g_i y) \succeq_\mathcal{T} \mathcal{O}$, $i = 1, \dots, r$. For the primal problem $(\mathbb{Q}_N^{l_0, l_1, \dots, l_r})$, its dual $(\mathbb{Q}_N^{l_0, l_1, \dots, l_r})^*$ is an extension of the SDP relaxation problem derived by Lasserre's SDP relaxation [9] to a T-SDP problem. The positive integers l_0 and l_i determined to satisfy (4) are the parameters in the block circulant SOS relaxation. By choosing a relaxation level N, l_0 and l_i, the relaxation problems $(\mathbb{Q}_N^{l_0, l_1, \dots, l_r})$ and $(\mathbb{Q}_N^{l_0, l_1, \dots, l_r})^*$ are uniquely determined.

From (4), (10) and (11), the size of the T-SDP relaxation problem $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$ is given with $\mathcal{A}_\alpha, \mathcal{Z}_0 \in \mathbb{S}^{m_0\times m_0\times l_0}$ and $\mathcal{D}_{i\alpha}, \mathcal{Z}_i \in \mathbb{S}^{m_i\times m_i\times l_i}$. As discussed in Section 2.3, $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$ can be transformed into an equivalent SDP of smaller size, depending on even or odd l_0, l_i, which includes hermitian matrices of size determined by the sum of $\frac{l_0+2}{2}$ or $\frac{l_0+1}{2}$ of $m_0 \times m_0$ size and the sum of $\frac{l_i+2}{2}$ or $\frac{l_i+1}{2}$ of $m_i \times m_i$ size. In this case, we need to transform the SDP formulated with complex matrices to real-symmetric matrices [13].

For the case of $l_0 = 1$ and $l_i = 1$, the T-SDP relaxation problem $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$ coincides with the SDP relaxation problem which involves real symmetric matrices of size $m_0 l_0$ and $m_i l_i$. Therefore, the T-SDP relaxation problem can be reduced to a smaller problem than the SDP relaxation in [9], as shown in Table 1.

Table 1: Comparing the number and size of positive semidefinite (PSD) matrices for the basic SOS relaxation and the block circulant SOS relaxation

Relaxation	number and size of PSD matrices in Nth-level relaxation
Basic SOS	[$1{*}(\binom{n+N}{N} \times \binom{n+N}{N})$ PSD matrix, $1{*}(\sum_{i=1}^r \binom{n+N-\tilde{w}_i}{N-\tilde{w}_i} \times \binom{n+N-\tilde{w}_i}{N-\tilde{w}_i})$ PSD matrices]
Block circulant SOS	[$(\frac{l_0+2}{2}$ or $\frac{l_0+1}{2}) * \left(\frac{\binom{n+N}{N}}{l_0} \times \frac{\binom{n+N}{N}}{l_0}\right)$ PSD matrices, $\left(\sum_{i=1}^r \frac{l_i+2}{2} \text{ or } \sum_{i=1}^r \frac{l_i+1}{2}\right) * \left(\frac{\binom{n+N-\tilde{w}_i}{N-\tilde{w}_i}}{l_i} \times \frac{\binom{n+N-\tilde{w}_i}{N-\tilde{w}_i}}{l_i}\right)$ PSD matrices]

3.2 Feasibility and global optimality

We present theoretical analysis of the T-SDP relaxation problems $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$ and $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})^*$ for the constrained POPs proposed in Section 3.1. We also show that the sequence of relaxation problems $\left\{(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})^*\right\}$ for a relaxation level N converges to the optimal value of $(\mathbb{P}_K)$ when $N \to \infty$.

Using the relation between block circulant SOS polynomials and semidefinite block circulant matrices described in Section 2.4, we present the following theorem, which serves as a necessary and sufficient condition for the feasibility of $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$.

Theorem 12. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued polynomial of degree d with the zero constant term. Assume that there exist the positive integers $m_0, l_0, m_i, l_i, i = 1, \ldots, r$ such that $s(N) = m_0 l_0$, $s(N - \tilde{w}_i) = m_i l_i$. Then $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$ has a feasible solution if and only if there exists some $\gamma \in \mathbb{R}$ and third-order tensors $\mathcal{Z}_0 \in \mathbb{S}_+^{m_0\times m_0\times l_0}$, $\mathcal{Z}_i \in \mathbb{S}_+^{m_i\times m_i\times l_i}, i = 1, \ldots, r$ such that*

$$f(x) - \gamma = \left\langle \mathcal{Z}_0, \mathcal{X}_0 * \mathcal{X}_0^\mathsf{T} \right\rangle + \sum_{i=1}^r g_i(x) \left\langle \mathcal{Z}_i, \mathcal{X}_i * \mathcal{X}_i^\mathsf{T} \right\rangle .$$

Proof. Assume that $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_r)$ is a feasible solution of $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$. More precisely,

$$\langle \mathcal{Z}_0, \mathcal{A}_\alpha \rangle + \sum_{i=1}^r \langle \mathcal{Z}_i, \mathcal{D}_{i\alpha} \rangle = b_\alpha, \quad 0 < |\alpha| \leq 2N$$
$$\mathcal{Z}_0 \succeq_\mathcal{T} \mathcal{O},\ \mathcal{Z}_i \succeq_\mathcal{T} \mathcal{O}, \quad i = 1, \ldots, r.$$

Since the objective function in $(\mathbb{P}_K)$ has no constant term, we have

$$\begin{aligned} f(x) &= \sum_{0<|\alpha|\leq 2N} b_\alpha x^\alpha = \sum_{0<|\alpha|\leq 2N} \left(\langle \mathcal{Z}_0, \mathcal{A}_\alpha \rangle x^\alpha + \sum_{i=1}^r \langle \mathcal{Z}_i, \mathcal{D}_{i\alpha} \rangle x^\alpha \right) \\ &= \sum_{0<|\alpha|\leq 2N} \langle \mathcal{Z}_0, \mathcal{A}_\alpha \rangle x^\alpha + \sum_{i=1}^r \sum_{0<|\alpha|\leq 2N} \langle \mathcal{Z}_i, \mathcal{D}_{i\alpha} \rangle x^\alpha \\ &= \left\langle \mathcal{Z}_0, \mathcal{X}_0 * \mathcal{X}_0^\mathsf{T} \right\rangle - \langle \mathcal{Z}_0, \mathcal{A}_0 \rangle + \sum_{i=1}^r \left(g_i(x) \left\langle \mathcal{Z}_i, \mathcal{X}_i * \mathcal{X}_i^\mathsf{T} \right\rangle - \langle \mathcal{Z}_i, \mathcal{D}_{i0} \rangle \right), \end{aligned}$$

where (10) and (11) are used in the last equality. Since $\mathcal{A}_0 \in \mathbb{R}^{m_0 \times m_0 \times l_0}$, $\mathcal{D}_{i0} \in \mathbb{R}^{m_i \times m_i \times l_i}$ are all 0 except for the $(1,1,1)$th element, which is 1 by its definition,

$$f(x) + [\mathcal{Z}_0]_{(1,1,1)} + \sum_{i=1}^r [\mathcal{Z}_i]_{(1,1,1)} = \left\langle \mathcal{Z}_0, \mathcal{X}_0 * \mathcal{X}_0^\mathsf{T} \right\rangle + \sum_{i=1}^r g_i(x) \left\langle \mathcal{Z}_i, \mathcal{X}_i * \mathcal{X}_i^\mathsf{T} \right\rangle.$$

The desired result follows by setting $\gamma = -[\mathcal{Z}_0]_{(1,1,1)} - \sum_{i=1}^r [\mathcal{Z}_i]_{(1,1,1)}$.

On the other hand, assume that $\gamma \in \mathbb{R}$ and $\mathcal{Z}_0 \in \mathbb{S}_+^{m_0 \times m_0 \times l_0}$, $\mathcal{Z}_i \in \mathbb{S}_+^{m_i \times m_i \times l_i}$, $i = 1, \dots, r$ exist such that

$$f(x) - \gamma = \left\langle \mathcal{Z}_0, \mathcal{X}_0 * \mathcal{X}_0^\mathsf{T} \right\rangle + \sum_{i=1}^r g_i(x) \left\langle \mathcal{Z}_i, \mathcal{X}_i * \mathcal{X}_i^\mathsf{T} \right\rangle.$$

Then, from (10) and (11), we have

$$f(x) = \sum_{0<|\alpha|\leq 2N} \langle \mathcal{Z}_0, \mathcal{A}_\alpha \rangle x^\alpha + \langle \mathcal{Z}_0, \mathcal{A}_0 \rangle + \sum_{0<|\alpha|\leq 2N} \sum_{i=1}^r \langle \mathcal{Z}_i, \mathcal{D}_{i\alpha} \rangle x^\alpha + \sum_{i=1}^r \langle \mathcal{Z}_i, \mathcal{D}_{i0} \rangle + \gamma.$$

Therefore, by comparing the coefficients for each monomial on both sides, we obtain

$$\begin{aligned} \langle \mathcal{Z}_0, \mathcal{A}_0 \rangle + \sum_{i=1}^r \langle \mathcal{Z}_i, \mathcal{D}_{i0} \rangle + \gamma = 0 \\ \langle \mathcal{Z}_0, \mathcal{A}_\alpha \rangle + \sum_{i=1}^r \langle \mathcal{Z}_i, \mathcal{D}_{i\alpha} \rangle = b_\alpha, \quad 0 < \alpha \leq 2N. \end{aligned}$$

Thus, $\mathcal{Z}_0 \in \mathbb{S}_+^{m_0 \times m_0 \times l_0}$, $\mathcal{Z}_i \in \mathbb{S}_+^{m_i \times m_i \times l_i}$, $i = 1, \dots, r$ are feasible solutions of $(\mathbb{Q}_N^{l_0, l_1, \dots, l_r})$. □

We show the relation between the SDP relaxation problem by Parrilo [18] and the T-SDP relaxation problem of the proposed method in the following theorem. In particular, we discuss the necessary and sufficient conditions under which the optimal values of SDP relaxation and T-SDP relaxation are equivalent.

Theorem 13. *Let $(\mathbb{Q}_N^{1,1,\dots,1})$ be the SDP relaxation problem for $(\mathbb{P}_K)$ with $l_0 = 1$, $l_1 = 1, \dots, l_r = 1$ and $(\mathbb{Q}_N^{l_0, l_1, \dots, l_r})$ be the T-SDP relaxation problem with l_0, $l_1, \dots, l_r$ such that $\max\{l_0, l_1, \dots, l_r\} \geq 2$. Let the optimal values of the relaxation problem be $\max(\mathbb{Q}_N^{1,1,\dots,1})$, $\max(\mathbb{Q}_N^{l_0, l_1, \dots, l_r})$, respectively. Then $\max(\mathbb{Q}_N^{1,1,\dots,1}) = \max(\mathbb{Q}_N^{l_0, l_1, \dots, l_r})$ if and only if there exist optimal solutions Z_0^*, Z_i^* of $(\mathbb{Q}_N^{1,1,\dots,1})$ represented by l_0-block circulant and l_i-block circulant matrices, respectively.*

Proof. From Theorem 12, the constraints of $(\mathbb{Q}_N^{1,1,\ldots,1})$ is equivalent to

$$f(x) - \gamma = \left\langle Z_0, [x]_N * [x]_N^\mathsf{T} \right\rangle + \sum_{i=1}^r g_i(x) \left\langle Z_i, [x]_{N-\tilde{w}_i} * [x]_{N-\tilde{w}_i}^\mathsf{T} \right\rangle,$$

and the constraints of $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$ is equivalent to

$$f(x) - \gamma = \left\langle \mathcal{Z}_0, \mathcal{X}_0 * \mathcal{X}_0^\mathsf{T} \right\rangle + \sum_{i=1}^r g_i(x) \left\langle \mathcal{Z}_i, \mathcal{X}_i * \mathcal{X}_i^\mathsf{T} \right\rangle.$$

Then,

$$\begin{aligned}
&\left\langle \mathcal{Z}_0, \mathcal{X}_0 * \mathcal{X}_0^\mathsf{T} \right\rangle + \sum_{i=1}^r g_i(x) \left\langle \mathcal{Z}_i, \mathcal{X}_i * \mathcal{X}_i^\mathsf{T} \right\rangle \\
&= \frac{1}{l_0} \mathrm{Tr}\left(\mathrm{bcirc}(\mathcal{X}_0)^\mathsf{T} \mathrm{bcirc}(\mathcal{Z}_0) \mathrm{bcirc}(\mathcal{X}_0) \right) + \sum_{i=1}^r g_i(x) \frac{1}{l_i} \mathrm{Tr}\left(\mathrm{bcirc}(\mathcal{X}_i)^\mathsf{T} \mathrm{bcirc}(\mathcal{Z}_i) \mathrm{bcirc}(\mathcal{X}_i) \right) \\
&= \frac{1}{l_0} \left\langle \mathrm{bcirc}(\mathcal{X}_0), \mathrm{bcirc}(\mathcal{Z}_0) \mathrm{bcirc}(\mathcal{X}_0) \right\rangle + \sum_{i=1}^r g_i(x) \frac{1}{l_i} \left\langle \mathrm{bcirc}(\mathcal{X}_i), \mathrm{bcirc}(\mathcal{Z}_i) \mathrm{bcirc}(\mathcal{X}_i) \right\rangle \\
&= \left\langle \mathrm{unfold}(\mathcal{X}_0), \mathrm{bcirc}(\mathcal{Z}_0) \mathrm{unfold}(\mathcal{X}_0) \right\rangle + \sum_{i=1}^r g_i(x) \left\langle \mathrm{unfold}(\mathcal{X}_i), \mathrm{bcirc}(\mathcal{Z}_i) \mathrm{unfold}(\mathcal{X}_i) \right\rangle \\
&= \left\langle \mathrm{bcirc}(\mathcal{Z}_0), [x]_N * [x]_N^\mathsf{T} \right\rangle + \sum_{i=1}^r g_i(x) \left\langle \mathrm{bcirc}(\mathcal{Z}_i), [x]_{N-\tilde{w}_i} * [x]_{N-\tilde{w}_i}^\mathsf{T} \right\rangle.
\end{aligned}$$

Hence, we have $\max(\mathbb{Q}_N^{1,1,\ldots,1}) \geq \max(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$ since a feasible solution $(\mathcal{Z}_0^*, \mathcal{Z}_1^*, \ldots, \mathcal{Z}_r^*)$ of $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$ becomes a feasible solution of $(\mathbb{Q}_N^{1,1,\ldots,1})$.

Now, we assume that $\max(\mathbb{Q}_N^{1,1,\ldots,1}) = \max(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$. Let $(\mathcal{Z}_0^*, \mathcal{Z}_1^*, \ldots, \mathcal{Z}_r^*)$ be an optimal solution of $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$. Then $(\mathrm{bcirc}(\mathcal{Z}_0^*), \mathrm{bcirc}(\mathcal{Z}_1^*), \ldots, \mathrm{bcirc}(\mathcal{Z}_r^*))$ is an optimal solution of $(\mathbb{Q}_N^{1,1,\ldots,1})$. Clearly, $\mathrm{bcirc}(\mathcal{Z}_0^*)$ and $\mathrm{bcirc}(\mathcal{Z}_i^*)$ are l_0-block circulant and l_i-block circulant matrices, respectively. On the other hand, we can take $\mathrm{bcirc}_{l_0}^{-1}(Z_0^*), \mathrm{bcirc}_{l_i}^{-1}(Z_i^*)$, assuming that there exists an optimal solution Z_0^* and Z_i^* of $(\mathbb{Q}_N^{1,1,\ldots,1})$ represented as l_0-block circulant and l_i-block circulant matrices, respectively, which is also a feasible solution of $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$. Consequently, $\max(\mathbb{Q}_N^{1,1,\ldots,1}) \leq \max(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$ holds, and together with $\max(\mathbb{Q}_N^{1,1,\ldots,1}) \geq \max(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$, we have $\max(\mathbb{Q}_N^{1,1,\ldots,1}) = \max(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$. □

Now, we discuss the global optimality of the T-SDP relaxation problem and the convergence of the relaxation problem sequence $\left\{ (\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})^* \right\}$ with respect to the relaxation level N, extending the Lasserre's hierarchy in [9].

Theorem 14. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued polynomial of degree d and K be the compact set. Let Assumption 11 hold, and let $p_K^* := \min_{x \in K} f(x)$. Let x^* be a global optimal solution of $(\mathbb{P}_K)$ and*

$$y^* = [x_1^*, \ldots, x_n^*, (x_1^*)^2, x_1^* x_2^*, \ldots, (x_n^*)^2, \ldots, (x_1^*)^{2N}, \ldots, (x_n^*)^{2N}]^\mathsf{T}.$$

Then,
(a) For fixed parameters l_0, l_i, we have

$$\inf(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r}) \uparrow p_K^*,$$

as $N \to \infty$. Moreover, for N sufficiently large, there is no duality gap between $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$ and its dual $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})^*$ if K has a nonempty interior.
(b) If $f(x) - p_K^*$ can be represented in the form (6), i.e.,

$$f(x) - p_K^* = q_0(x) + \sum_{i=1}^{r} g_i(x)q_i(x)$$

for a block circulant SOS polynomial $q_0(x)$ of degree at most $2N$, and some block circulant SOS polynomials $q_i(x)$ of degree at most $2(N - \tilde{w}_i)$, then

$$\min(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})^* = p_K^* = \max(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$$

and y^* is a global minimizer of $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})^*$.

Proof. (a) Let x^* be a global optimal solution of $(\mathbb{P}_K)$, and

$$y^* = [x_1^*, \ldots, x_n^*, (x_1^*)^2, x_1^* x_2^*, \ldots, (x_n^*)^2, \ldots, (x_1^*)^{2N}, \ldots, (x_n^*)^{2N}]^\mathsf{T}.$$

Then, $\mathcal{M}_N^{l_0}(y^*) = \text{fold}_{l_0}(y^*) * (\text{fold}_{l_0}(y^*))^\mathsf{T}$, $\mathcal{M}_{N-\tilde{w}_i}^{l_i}(g_i y^*) = g_i(x^*)\text{fold}_{l_i}(y^*) * \text{fold}_{l_i}(y^*)^\mathsf{T}$, $i = 1, \ldots, r$ are semidefinite tensors. Moreover, since b_α are the coefficients corresponding to the monomials x^α, $\sum_{0<\alpha\leq 2N} b_\alpha y_\alpha^*$ is equal to p_K^*. Therefore, y^* is a feasible solution of $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})^*$ with the objective function value p_K^*, hence, $\inf(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})^* \leq p_K^*$.

Now, for fixed $l_0, l_1, \ldots, l_r$, we consider any $N' \geq N$ such that $s(N) = m_0 l_0, s(N - \tilde{w}_i) = m_i l_i$ and $s(N') = m'_0 l_0, s(N' - \tilde{w}_i) = m'_i l_i$. In this case, since $s(N') - s(N)$ and $s(N' - \tilde{w}_i) - s(N - \tilde{w}_i)$ obviously have factors l_0 and l_i, respectively, it is possible to represent $\mathcal{M}_N^{l_0}(y)$ and $\mathcal{M}_{N-\tilde{w}_i}^{l_i}(g_i y)$ to be subtensors of $\mathcal{M}_{N'}^{l_0}(y)$ and $\mathcal{M}_{N'-\tilde{w}_i}^{l_i}(g_i y)$, respectively, by arranging the monomial vectors appropriately when creating the T-SDP relaxation problem. Specifically, if $\mathcal{M}_{N'}^{l_0}(y) \succeq_{\mathcal{T}} \mathcal{O}$ and $\mathcal{M}_{N'-\tilde{w}_i}^{l_i}(g_i y) \succeq_{\mathcal{T}} \mathcal{O}$, then $\mathcal{M}_N^{l_0}(y)$, $\mathcal{M}_{N-\tilde{w}_i}^{l_i}(g_i y)$ can be formed such that $\mathcal{M}_N^{l_0}(y) \succeq_{\mathcal{T}} \mathcal{O}$, $\mathcal{M}_{N-\tilde{w}_i}^{l_i}(g_i y) \succeq_{\mathcal{T}} \mathcal{O}$, respectively (the details are described in the subsequent discussion and Example 15). Thus, for any solution y of $(\mathbb{Q}_{N'}^{l_0,l_1,\ldots,l_r})^*$, the adequate truncated vector y' is a feasible solution of $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})^*$. Since $2N' \geq 2N \geq d$, b_α corresponding to $|\alpha| > \binom{n+d}{d}$ is 0, therefore, $\sum_{0<|\alpha|\leq 2N} b_\alpha y_\alpha = \sum_{0<|\alpha|\leq 2N'} b_\alpha y'_\alpha$ holds. Then the objective value in the feasible solution y of $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})^*$ is equal to that in the feasible solution y' of $(\mathbb{Q}_{N'}^{l_0,l_1,\ldots,l_r})^*$. Therefore, $\inf(\mathbb{Q}_{N'}^{l_0,l_1,\ldots,l_r})^* \geq \inf(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})^*$ for $N' \geq N$.

From Assumption 11,

$$f(x) - p_K^* + \epsilon = q_0(x) + \sum_{i=1}^{r} g_i(x)q_i(x),$$

where $q_0(x)$ is an l_0-block circulant SOS polynomial of degree $2N$ and $q_i(x)$ is an l_i-block circulant SOS polynomials of degree $2(N - \tilde{w}_i)$. In addition, from the definition of the block circulant SOS polynomial, there exist t_0 and t_k, $k = 1, \ldots, r$, q_0, and q_i that can be expressed as

$$q_0(x) = \sum_{j=1}^{t_0} \sum_{k=1}^{l_0} \left((u_{jk}^0)^\top [x]_N\right)^2$$

$$q_i(x) = \sum_{j=1}^{t_i} \sum_{k=1}^{l_i} \left((u_{jk}^i)^\top [x]_{N-\tilde{w}_i}\right)^2, \quad i = 1, \ldots, r.$$

We define

$$\mathcal{Z}_0 := \mathrm{bcirc}_{l_0}^{-1}(\sum_{j=1}^{t_0} \sum_{k=1}^{l_0} u_{jk}^0 (u_{jk}^0)^\top), \quad \mathcal{Z}_i := \mathrm{bcirc}_{l_i}^{-1}(\sum_{j=1}^{t_i} \sum_{k=1}^{l_i} u_{jk}^i (u_{jk}^i)^\top), \quad i = 1, \ldots, r.$$

Then, from Theorem 5, we have $\mathcal{Z}_0, \mathcal{Z}_k \succeq_{\mathcal{T}} \mathcal{O}$. Since

$$\begin{aligned} f(x) - p_K^* + \epsilon &= q_0(x) + \sum_{i=1}^r g_i(x) q_i(x) \\ &= \left\langle \sum_{j=1}^{t_0} \sum_{k=1}^{l_0} u_{jk}^0 (u_{jk}^0)^\top, [x]_N [x]_N^\top \right\rangle + \sum_{i=1}^r g_i(x) \left\langle \sum_{j=1}^{t_i} \sum_{k=1}^{l_i} u_{jk}^i (u_{jk}^i)^\top, [x]_{N-\tilde{w}_i} [x]_{N-\tilde{w}_i}^\top \right\rangle \\ &= \left\langle \mathcal{Z}_0, \mathcal{X}_0 * \mathcal{X}_0^\top \right\rangle + \sum_{i=1}^r g_i(x) \left\langle \mathcal{Z}_i, \mathcal{X}_i * \mathcal{X}_i^\top \right\rangle, \end{aligned}$$

holds from Theorem 12, $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_r)$ is a feasible solution of $(\mathbb{Q}_N^{l_0, l_1, \ldots, l_r})$ and $[\mathcal{Z}_0]_{(1,1,1)} + \sum_k [\mathcal{Z}_k]_{(1,1,1)} = -(p_K^* - \epsilon)$. Thus, we obtain

$$p_K^* - \epsilon \leq \sup(\mathbb{Q}_N^{l_0, l_1, \ldots, l_r}) \leq \inf\,(\mathbb{Q}_N^{l_0, l_1, \ldots, l_r})^* \leq p_K^*. \tag{13}$$

Next, we prove that there is no duality gap between $(\mathbb{Q}_N^{l_0, l_1, \ldots, l_r})$ and its dual $(\mathbb{Q}_N^{l_0, l_1, \ldots, l_r})^*$ for any N such that $N \geq N_0$ where N_0 is an initial relaxation level. Let μ be a probability measure with uniform distribution in K, which has a strictly positive density f with respect to the Lebesgue measure and satisfies that

$$y_\alpha = \int x^\alpha d\mu < +\infty, \quad \text{for all } \alpha \text{ such that } 0 \leq |\alpha| \leq 2N_0.$$

Then, from the discussion in Theorem 6 in [32], it follows that $\mathcal{M}_N^{l_0}(y_\mu) \succ_{\mathcal{T}} \mathcal{O}$ and $\mathcal{M}_{N-\tilde{w}_i}^{l_i}(g_i y_\mu) \succ_{\mathcal{T}} \mathcal{O}$, $i = 1, \ldots, r$ for $y_\mu = \{y_\alpha\}$, which is a feasible interior point of $(\mathbb{Q}_N^{l_0, l_1, \ldots, l_r})^*$. In addition, $(\mathbb{Q}_N^{l_0, l_1, \ldots, l_r})^*$ is bounded below by (13). By strong duality of T-SDP in Theorem 6, this indicates that $(\mathbb{Q}_N^{l_0, l_1, \ldots, l_r})$ is solvable, together with $(\mathbb{Q}_N^{l_0, l_1, \ldots, l_r})$ is feasible, and there is no duality gap between $(\mathbb{Q}_N^{l_0, l_1, \ldots, l_r})$ and $(\mathbb{Q}_N^{l_0, l_1, \ldots, l_r})^*$.

(b) If $f(x) - p_K^*$ can be expressed as in (6), then we can construct matrices $\mathcal{Z}_0 \succeq_{\mathcal{T}} \mathcal{O}$, $\mathcal{Z}_i \succeq_{\mathcal{T}} \mathcal{O}$, $i = 1, \ldots, r$ to be a feasible solution of $(\mathbb{Q}_N^{l_0, l_1, \ldots, l_r})$ whose objective value is $-[\mathcal{Z}_0]_{(1,1,1)} - \sum_k [\mathcal{Z}_k]_{(1,1,1)} = p_K^*$ with polynomials $q_0(x)$ and $q_i(x)$ of degree at most $2N$ and

$2(N-\tilde{w}_i)$, respectively, as in the proof of (a). As a result, from $p_K^* \leq \sup(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r}) \leq \inf(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})^* \leq p_K^*$, we have $\max(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r}) = p_K^* = \min(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})^*$, and $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_k)$ is an optimal solution of $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})$. Furthermore, since b_α are the coefficients corresponding to each monomial, we clearly see that y^* is a global optimal solution of $(\mathbb{Q}_N^{l_0,l_1,\ldots,l_r})^*$. □

For any two third-order tensors $\mathcal{A} \in \mathbb{R}^{m\times n\times l}$ and $\mathcal{B} \in \mathbb{R}^{m'\times n'\times l}$, $\mathcal{A}$ is called a subtensor of $\mathcal{B}$ if each frontal slice $\mathcal{A}^{(i)}$ of $\mathcal{A}$ is a principal submatrix of each frontal slice $\mathcal{B}^{(i)}$ of $\mathcal{B}$, respectively.

We illustrate the proof of (a) in Theorem 14 with the following examples.

Example 15. *We let $n = 2$, the relaxation level $N = 2$ and $N' = 3$. We fix $l_0 = 2$.*

In this case, $s(N) = 6$. We derive T-SDP relaxation by a monomial vector

$$[x]_2 = [1, x_1, x_2, x_1^2, x_1x_2, x_2^2]^\top.$$

Then $\mathcal{M}_2^2(y)$ is a third-order tensor of size $3 \times 3 \times 2$ whose frontal slices are given by

$$\mathcal{M}_2^2(y)^{(1)} = \begin{bmatrix} 1+y_{40} & y_{10}+y_{31} & y_{01}+y_{22} \\ y_{10}+y_{31} & y_{20}+y_{22} & y_{11}+y_{13} \\ y_{01}+y_{22} & y_{11}+y_{13} & y_{02}+y_{04} \end{bmatrix}, \quad \mathcal{M}_2^2(y)^{(2)} = \begin{bmatrix} 2y_{20} & y_{11}+y_{30} & y_{02}+y_{21} \\ y_{11}+y_{30} & 2y_{21} & 2y_{12} \\ y_{21}+y_{02} & 2y_{12} & 2y_{03} \end{bmatrix}.$$

With $s(N') = 10$, T-SDP relaxation can be derived using a monomial vector

$$[x]_3 = [1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3]^\top.$$

Then, $\mathcal{M}_3^2(y)$ is a third-order tensor of size $5 \times 5 \times 2$ whose frontal slices are given by

$$\mathcal{M}_3^2(y)^{(1)} = \begin{bmatrix} 1+y_{04} & y_{10}+y_{32} & y_{01}+y_{23} & y_{20}+y_{14} & y_{11}+y_{05} \\ y_{10}+y_{32} & y_{20}+y_{60} & y_{11}+y_{51} & y_{30}+y_{42} & y_{21}+y_{33} \\ y_{01}+y_{23} & y_{11}+y_{51} & y_{02}+y_{42} & y_{21}+y_{33} & y_{12}+y_{24} \\ y_{20}+y_{14} & y_{30}+y_{42} & y_{21}+y_{33} & y_{40}+y_{24} & y_{32}+y_{15} \\ y_{11}+y_{05} & y_{21}+y_{33} & y_{12}+y_{24} & y_{31}+y_{15} & y_{22}+y_{06} \end{bmatrix},$$

$$\mathcal{M}_3^2(y)^{(2)} = \begin{bmatrix} 2y_{02} & y_{12}+y_{30} & y_{03}+y_{21} & y_{12}+y_{22} & y_{30}+y_{13} \\ y_{12}+y_{30} & 2y_{40} & 2y_{31} & y_{22}+y_{50} & y_{13}+y_{42} \\ y_{03}+y_{21} & 2y_{31} & 2y_{22} & y_{13}+y_{41} & y_{04}+y_{32} \\ y_{12}+y_{22} & y_{22}+y_{50} & y_{13}+y_{42} & 2y_{32} & 2y_{23} \\ y_{03}+y_{13} & y_{13}+y_{42} & y_{04}+y_{32} & 2y_{23} & 2y_{14} \end{bmatrix}.$$

Obviously, $\mathcal{M}_2^2(y)$ is not a subtensor of $\mathcal{M}_3^2(y)$. However, if we derive the T-SDP relaxation problem by a monomial vector

$$[x]_3 = [1, x_1, x_2, x_1^3, x_1^2x_2, x_1^2, x_1x_2, x_2^2, x_1x_2^2, x_2^3]^\top,$$

then each frontal slice of $\mathcal{M}_3^2(y)$ is given by

$$\mathcal{M}_3^2(y)^{(1)} = \left[\begin{array}{ccc:cc} 1+y_{40} & y_{10}+y_{31} & y_{01}+y_{22} & y_{30}+y_{32} & y_{21}+y_{23} \\ y_{10}+y_{32} & y_{20}+y_{22} & y_{11}+y_{13} & y_{40}+y_{23} & y_{31}+y_{14} \\ y_{01}+y_{22} & y_{11}+y_{13} & y_{02}+y_{04} & y_{31}+y_{14} & y_{22}+y_{05} \\ \hdashline y_{30}+y_{32} & y_{40}+y_{23} & y_{31}+y_{14} & y_{24}+y_{60} & y_{51}+y_{15} \\ y_{21}+y_{23} & y_{31}+y_{14} & y_{22}+y_{05} & y_{51}+y_{15} & y_{42}+y_{06} \end{array}\right],$$

$$\mathcal{M}_3^2(y)^{(2)} = \left[\begin{array}{ccc:cc} 2y_{20} & y_{11}+y_{30} & y_{02}+y_{21} & y_{12}+y_{50} & y_{03}+y_{41} \\ y_{11}+y_{30} & 2y_{21} & 2y_{12} & y_{22}+y_{41} & y_{13}+y_{32} \\ y_{02}+y_{21} & 2y_{12} & 2y_{03} & y_{13}+y_{432} & y_{04}+y_{23} \\ \hdashline y_{12}+y_{50} & y_{22}+y_{41} & y_{13}+y_{32} & 2y_{42} & 2y_{33} \\ y_{03}+y_{41} & y_{13}+y_{32} & y_{04}+y_{23} & 2y_{33} & 2y_{24} \end{array}\right],$$

where $\mathcal{M}_2^2(y)$ is a subtensor of $\mathcal{M}_3^2(y)$.

Example 15 illustrates that for any N and N' such that $N' \geq N$, if l_0 is fixed, $\mathcal{M}_N^{l_0}(y)$ can be formed to be a subtensor of $\mathcal{M}_{N'}^{l_0}(y)$ by arranging the monomial vector $[x]_{N'}$ for the derivation of T-SDP relaxation with relaxation level N' such that the monomial tensor $\mathrm{fold}_{l_0}([x]_N)$ is a subtensor of $\mathrm{fold}_{l_0}([x]_{N'})$. Similarly, if l_i corresponding to each constraint is fixed, $\mathcal{M}_N^{l_i}(g_i y)$ can be formed to be a subtensor of $\mathcal{M}_{N'}^{l_i}(g_i y)$ by appropriately arranging the monomial vectors $[x]_{N'-\tilde{w}_i}$. For example, consider a constrained POP in two variables ($n = 2$) with the objective function of degree $d = 6$, the number of constraints $r = 2$, and degree $w_i = 1$, $i = 1, 2$ for each constraint. Let $(\mathbb{Q}_N^{5,2,2})^*$ denote the T-SDP relaxation obtained with the relaxation level $N = 3$. Then $s(N) = 10$, $s(N - \tilde{w}_i) = 6$, $l_0 = 5$ and $l_i = 2$. Furthermore, if the relaxation level $N' = 4$, then $s(N') = 15$, $s(N' - \tilde{w}_i) = 10$, and if $m'_0 = 3$, $m'_i = 5$, then $s(N') = m'_0 l_0$ and $s(N' - \tilde{w}_i) = m'_i l_i$. Thus, we can also derive the T-SDP relaxation with $l_0 = 5$ and $l_i = 2$, denoted by $(\mathbb{Q}_{N'}^{5,2,2})^*$. In (a) of Theorem 14, we have claimed that the relation $\inf(\mathbb{Q}_N^{5,2,2})^* \leq \inf(\mathbb{Q}_{N'}^{5,2,2})^*$ holds. Conversely, if $N'' = 5$, then $s(N'') = 21$ and $s(N'' - \tilde{w}_i) = 15$, but there are no positive integers m''_0 and m''_i such that $s(N'') = m''_0 l_0$ and $s(N'' - \tilde{w}_i) = m''_i l_i$. Therefore, any T-SDP relaxation with relaxation level $N'' = 5$ has no relation to $(\mathbb{Q}_N^{5,2,2})^*$ or $(\mathbb{Q}_{N'}^{5,2,2})^*$. Clearly, if we fix $l_0 = 1$ and $l_i = 1$, then $\inf(\mathbb{Q}_N^{1,1,1})^* \leq \inf(\mathbb{Q}_{N'}^{1,1,1})^*$ for any $N' \geq N$.

We have mentioned that "if $\mathcal{M}_{N'}^{l_0}(y) \succeq_{\mathcal{T}} \mathcal{O}$ and $\mathcal{M}_{N'-\tilde{w}_i}^{l_i}(g_i y) \succeq_{\mathcal{T}} \mathcal{O}$, then $\mathcal{M}_N^{l_0}(y)$, $\mathcal{M}_{N-\tilde{w}_i}^{l_i}(g_i y)$ can be constructed such that $\mathcal{M}_N^{l_0}(y) \succeq_{\mathcal{T}} \mathcal{O}$, $\mathcal{M}_{N-\tilde{w}_i}^{l_i}(g_i y) \succeq_{\mathcal{T}} \mathcal{O}$, respectively" in the proof of (a) of Theorem 14. As discussed earlier, $\mathcal{M}_N^{l_0}(y)$, $\mathcal{M}_{N-\tilde{w}_i}^{l_i}(g_i y)$ can be constructed to be subtensors of $\mathcal{M}_{N'}^{l_0}(y)$, $\mathcal{M}_{N'-\tilde{w}_i}^{l_i}(g_i y)$, respectively, in the T-SDP relaxation with relaxation level N such that $N' \geq N$, by appropriately arranging the monomial vectors. Therefore, from the definition of subtensor, $\mathrm{bcirc}\left(\mathcal{M}_N^{l_0}(y)\right)$ and $\mathrm{bcirc}\left(\mathcal{M}_{N-\tilde{w}_i}^{l_i}(g_i y)\right)$ are the principal submatrices of $\mathrm{bcirc}\left(\mathcal{M}_{N'}^{l_0}(y)\right)$ and $\mathrm{bcirc}\left(\mathcal{M}_{N'-\tilde{w}_i}^{l_i}(g_i y)\right)$, respectively. Now, if we assume $\mathcal{M}_{N'}^{l_0}(y) \succeq_{\mathcal{T}} \mathcal{O}$ and $\mathcal{M}_{N'-\tilde{w}_i}^{l_i}(g_i y) \succeq_{\mathcal{T}} \mathcal{O}$, then we obtain $\mathrm{bcirc}\left(\mathcal{M}_{N'}^{l_0}(y)\right) \succeq O$, $\mathrm{bcirc}\left(\mathcal{M}_{N'-\tilde{w}_i}^{l_i}(g_i y)\right) \succeq O$ from Theorem 5. A matrix A is positive semidefinite if and only if the determinants of all principal minors of A are nonnegative, so that the determinant of all principal minors of $\mathrm{bcirc}\left(\mathcal{M}_N^{l_0}(y)\right)$, $\mathrm{bcirc}\left(\mathcal{M}_{N-\tilde{w}_i}^{l_i}(g_i y)\right)$ also are nonnegative, respectively. Therefore, $\mathrm{bcirc}\left(\mathcal{M}_N^{l_0}(y)\right) \succeq O$ and $\mathrm{bcirc}\left(\mathcal{M}_{N-\tilde{w}_i}^{l_i}(g_i y)\right) \succeq O$. Thus, by Theorem 5,

$\mathcal{M}_N^{l_0}(y) \succeq \mathcal{O}$ and $\mathcal{M}_{N-\tilde{w}_i}^{l_i}(g_i y) \succeq \mathcal{O}$ follows.

4 Numerical experiments

We compare the basic SOS relaxation [8] with the T-SDP relaxation proposed in Section 3.1 for constrained POPs with ten test problems and show that the proposed T-SDP relaxation is more efficient than the basic SOS relaxation. The test problems are presented in detail in Appendix, some of which were from [3, 33, 12], and the number of variables of the test problems ranges from 2 to 19 and the degree from 2 to 40, as shown in Table 2.

For the experiments, the basic SOS relaxation method and the proposed block circulant SOS relaxation method were applied to obtain the SDP relaxation problem and the T-SDP relaxation problem, respectively. Then, the T-SDP relaxation problem was transformed into an equivalent SDP problem as described in Section 2.3. For computation, we used Julia 1.7.3 with Mosek [14] on a PC (Intel(R) Core(TM) i7-1185G7 @ 3.00GHz, 16GB, windows 10 Pro).

Problems 1 [3], 2, 3, 4 [33, Example 2] in Section 4.1 demonstrate that the block circulant SOS relaxation takes shorter computational time than the basic SOS relaxation method. Problem 5 [12, st_bpaf1b] illustrates a case where the block circulant SOS relaxation problem is not feasible. With problem 6 [3, Problem 2.9.1], we demostrate that the block circulant SOS relaxation can be an alternative approach to the basic SOS relaxation for the SDP relaxation problem with numerical instability, which can be viewed as an additional benefit of the proposed T-SDP relaxation to its capability of handling larger-sized problems described in Section 3.1. For problem 7 [12, st_e34], we compare the basic SOS relaxation with the proposed block circulant SOS relaxation for large size problem. Problems 8, 9, 10 show the numerical efficiency of the proposed T-SDP relaxation.

Table 2: Test problems. n: the number of variable, d: the degree of f, $\max w_i$: the maximum of the degree of constraints, N: the relaxation level used to generate the SDP relaxation and the T-SDP relaxation, $s(N) = \binom{n+N}{N}$, and $s(N-\tilde{w}_i) = \binom{n+N-\tilde{w}_i}{N-\tilde{w}_i}$.

No.	n	d	$\max_i\{w_i\}$	N	$s(N)$	$s(N-\tilde{w}_i)$
1 [3]	10	2	1	2	66	11
2	2	40	2	20	231	210
3	2	20	2	10	281	220
4 [33, Example 2]	2	58	2	29	465	435
5 [12, st_bpaf1b]	10	2	1	3	286	66
6 [3, Problem 2.9.1]	3	1	2	6	84	56
7 [12, st_e34]	6	1	1	5	462	210
8	11	6	2	3	364	78
9	19	4	2	2	210	20
10	19	4	2	2	210	20

4.1 Numerical results

Tables 3 and 4 report the numerical results on problems 1-5 and 6-10, respectively. In the tables, "Relax." denotes the SDP relaxation or T-SDP relaxation. In this subsection, the SDP relaxation and the T-SDP relaxation correspond to the basic SOS relaxation and the proposed block circulant SOS relaxation, respectively. "Pn" (pattern) describes the block sizes, more precisely, the sizes are arranged with the following columns "(m_0, l_0)" and "(m_i, l_i)". For instance, $(m_0, l_0) = (66, 1)$ and $(m_0, l_0) = (11, 1)$ indicates that $\phi_0 \in \Sigma^1[x]_{2N}$ and $\phi_i \in \Sigma^1[x]_{2(N-\tilde{w}_i)}$ in the feasible set Γ of the relaxation problem $(\tilde{\mathbb{P}}_K)$ which results in the T-SDP relaxation equivalent to the basic SOS relaxation. For $(m_0, l_0) = (11, 6)$ and $(m_i, l_i) = (11, 1)$, we have $\phi_0 \in \Sigma^6[x]_{2N}$ and $\phi_i \in \Sigma^1[x]_{2(N-\tilde{w}_i)}$ in Γ. More precisely, the 6-block circulant SOS polynomial is employed to ϕ_0 and an SOS polynomial to ϕ_i. Tables 3 and 4 also include the numbers and sizes of positive semidefinite matrices based on Table 1. "# of var." and "# of nnz" indicate the number of decision variables in the variable matrix and the number of nonzero elements of the SDP to be solved, respectively, and "Opt.val" denotes its optimal value. "CPU1", "CPU2", and "Tot." denote the time for generating the SDP problem to be solved, the computational time for solving the SDP problem, which are added to show the total time in seconds, respectively.

4.1.1 Problem 1

In Table 3, we see that the number of decision variables for the proposed T-SDP relaxation method is 1804 is smaller than 3003 for the SDP relaxation method, resulting in shorter CPU1 (the computational time to generate the SDP problem). However, the T-SDP relaxation took slightly longer computation time in CPU2. This has been caused by the larger number of nonzero elements shown in the column of "# of nnz.". Nevertheless, we can see in the "Tot." column that the T-SDP relaxation consumed shorter overall computing time.

4.1.2 Problem 2

For problem 2, Table 3 displays the results by the SDP relaxation (Pn 1) and two T-SDP relaxations (Pn 2 and Pn 3). The difference between Pn 2 and Pn 3 is that the block circulant SOS relaxation was applied only to ϕ_0 for T-SDP (Pn 2) in the feasible set Γ of $(\tilde{\mathbb{P}}_K)$ whereas the block circulant SOS relaxation was applied to both ϕ_0 and ϕ_i for T-SDP (Pn 3).

We observe in Table 3 that the number of decision variables is significantly reduced to 97464 and 53364 in T-SDP (Pn 2) and T-SDP (Pn 3), respectively, from 115416 in the SDP relaxation (Pn 1). As a result, the time for generating the SDP problem to be solved for both T-SDP relaxations is reduced. However, the computational cost of solving the resulting T-SDP (Pn 2) is slightly higher than the SDP relaxation for the same reason mentioned in problem 1. On the other hand, the number of decision variables in T-SDP (Pn 3) is much smaller than that in T-SDP (Pn 2), and thus the computational time for solving T-SDP (Pn 3) was shorter than that for the SDP relaxation, even when the number of nonzero elements is taken into account. Hence, the total time for both T-SDP (Pn 2) and T-SDP (Pn 3) is less that that of the SDP relaxation (Pn 1).

Table 3: Comparison of SDP and T-SDP relaxation for Problems 1-5.

Relax.	Pn.	(m_0, l_0)	(m_i, l_i)	No. * size of PSD matrix	# of var.	# of nnz	Opt.val	CPU1(s)	CPU2(s)	Tot.(s)
Problem 1 (the known optimal value is 0.375)										
SDP	1	$(66, 1)$	$(11, 1)$	$[1 * (66 \times 66), 12 * (11 \times 11)]$	3003	8228	0.37500	2.99	0.42	3.41
T-SDP	2	$(11, 6)$	$(11, 1)$	$[4 * (22 \times 22), 12 * (11 \times 11)]$	1804	45884	0.37500	2.22	0.66	2.88
Problem 2 (the known optimal value is 14)										
SDP	1	$(231, 1)$	$(210, 1)$	$[1 * (231 \times 231), 4 * (210 \times 210)]$	115416	406161	14.00000	66.66	13.14	79.80
T-SDP	2	$(33, 7)$	$(210, 1)$	$[4 * (66 \times 66), 4 * (210 \times 210)]$	97464	931200	14.00000	53.84	15.89	69.73
T-SDP	3	$(33, 7)$	$(105, 2)$	$[4 * (66 \times 66), 8 * 105 \times 105]$	53364	1269808	14.00000	32.61	4.64	37.25
Problem 3 (the known optimal value is 1)										
SDP	1	$(286, 1)$	$(220, 1)$	$[1*(286 \times 286), 6*(220 \times 220)]$	186901	662596	1.00000	187.55	20.16	207.71
T-SDP	2	$(143, 2)$	$(220, 1)$	$[2*(143 \times 143), 6*(220)]$	166452	743413	1.00000	186.36	19.30	205.66
T-SDP	3	$(286, 1)$	$(110, 2)$	$[1*(286 \times 286), 12*(110 \times 110)]$	114301	1229508	1.00000	110.60	14.92	125.52
T-SDP	4	$(143, 2)$	$(110, 2)$	$[2*(143 \times 143), 12*(110 \times 110)]$	93852	1310326	1.00000	99.61	12.75	112.36
T-SDP	5	$(26, 11)$	$(220, 1)$	$[6*(52 \times 52), 6*(220 \times 220)]$	154128	2082916	1.00000	176.65	25.88	202.53
T-SDP	6	$(26, 11)$	$(110, 2)$	$[6*(52), 12*(110 \times 110)]$	81528	2649828	1.00000	88.60	15.97	104.57
Problem 4 (the known optimal value is 1)										
SDP	1	$(465, 1)$	$(435, 1)$	$[1*(465 \times 465), 2*(435 \times 435)]$	298005	1351575	1.00000	329.86	64.16	394.02
T-SDP	2	$(93, 5)$	$(435, 1)$	$[3*(186 \times 186), 2*(435 \times 435)]$	241833	2972864	1.00000	306.78	63.17	369.95
T-SDP	3	$(31, 15)$	$(435, 1)$	$[8*(62 \times 62), 2*(435 \times 435)]$	205284	5633042	1.00000	221.96	52.91	274.87
Problem 5 (the known optimal value is -42.96256)										
SDP	1	$(286, 1)$	$(66, 1)$	$[1*(286 \times 286), 30*(66 \times 66)]$	107371	438988	-42.96256	509.51	421.66	931.17
T-SDP	2	$(143, 2)$	$(66, 1)$	$[2*(143 \times 143), 30*(66 \times 66)]$	86922	520338	-42.96255	428.96	449.61	878.57
T-SDP	3	$(143, 2)$	$\left\{ \begin{array}{l} (66, 1) \\ (33, 2) \end{array} \right.$	$[2 * (143 \times 143), 10 * (66 \times 66), 40 * (33 \times 33)]$	65142	648402	-42.96256	334.09	455.33	782.42
T-SDP	4	$(143, 2)$	$(33, 2)$	$[2*(143 \times 143), 60*(33 \times 33)]$	54252	869670	infeasible	-	-	-

Table 4: Comparison of SDP and T-SDP relaxation for Problems 6-10.

Relax.	Pn.	(m_0, l_0)	(m_i, l_i)	No. * size of PSD matrix	# of var.	# of nnz	Opt.val	CPU1(s)	CPU2(s)	Tot.(s)
Problem 6 (the known optimal value is -4)										
SDP	1	$(84, 1)$	$(56, 1)$	[1*(84×84), 8*(56×56]	16338	82320	-3.99972	4.30	1.63	5.93
T-SDP	2	$(42, 2)$	$(56, 1)$	[2*(42×42), 8*(56×56)]	14754	89192	-3.99998	4.09	1.45	5.54
T-SDP	3	$(42, 2)$	$(28, 2)$	[2*(42×42), 16*(28×28)]	8302	160448	-3.99999	3.04	1.47	4.51
Problem 7										
SDP	1	$(462, 1)$	$(210, 1)$	[1*(462×462), 16*(210×210)]	461433	2462544	1.56195×10^{-2}	1877.58	437.64	2315.22
T-SDP	2	$(231, 2)$	$(210, 1)$	[2*(1231×1231), 16*(210×210)]	408072	2675042	1.56195×10^{-2}	1225.32	246.63	1471.95
T-SDP	3	$(231, 2)$	$\{(210, 1), (105, 2)$	$[2 * (231 \times 231), 4 * (210 \times 210), 24 * (105 \times 105)]$	275772	4897672	1.56195×10^{-2}	927.21	305.72	1232.93
T-SDP	4	$(42, 11)$	$\{(210, 1), (105, 2)$	$[6 * (84 \times 84), 4 * (210 \times 210), 24 * (105 \times 105)]$	243600	8057606	1.56195×10^{-2}	999.07	387.34	1386.41
Problem 8										
SDP	1	$(364, 1)$	$(78, 1)$	[1*(364×364), 24*(78×78)]	140374	546208	-2.71766	964.06	300.58	1264.64
T-SDP	2	$(182, 2)$	$(78, 1)$	[2*(91×91), 24*(78×78)]	107250	678036	-2.71766	542.06	227.47	769.53
T-SDP	3	$(182, 2)$	$\{(78, 1), (39, 2)$	$[2 * (91 \times 91), 2 * (78 \times 78), 44 * (39 \times 39)]$	73788	928920	-2.71766	302.05	331.75	633.80
T-SDP	4	$(182, 2)$	$(39, 2)$	[2*(91×91), 48*(39×39)]	70746	1067368	-2.71766	374.34	368.06	742.40
Problem 9										
SDP	1	$(210, 1)$	$(20, 1)$	[1*(210×210), 40*(20×20)]	30555	90500	-4.46273	160.40	46.52	206.92
T-SDP	2	$(105, 2)$	$(20, 1)$	[2*(105×105), 40*(20×20)]	19530	134062	-8.27671	64.45	57.25	121.70
T-SDP	3	$(105, 2)$	$(10, 2)$	[2*(105×105), 80*(10×10)]	15530	177874	infeasible	-	-	-
T-SDP	4	$(42, 5)$	$(20, 1)$	[3*(84×84), 40*(20×20)]	19110	439292	infeasible	-	-	-
Problem 10										
SDP	1	$(210, 1)$	$(20, 1)$	[1*(210×210), 40*(20×20)]	30555	90500	-1.00000	91.22	80.02	171.24
T-SDP	2	$(105, 2)$	$(20, 1)$	[2*(105×105), 40*(20×20)]	19530	134062	-1.00000	78.67	51.16	129.83
T-SDP	3	$(42, 5)$	$(20, 1)$	[3*(84×84), 40*(20×20)]	19110	439292	-1.00000	71.22	36.36	107.58
T-SDP	4	$(21, 10)$	$(20, 1)$	[6*(42×42), 40*(20×20)]	13818	810308	-1.00000	52.20	53.69	105.89
T-SDP	5	$(42, 5)$	$(10, 2)$	[3*(84×84), 80*(10×10)]	15110	483104	-1.00000	52.68	39.89	92.57

4.1.3 Problem 3

The results for problem 3 show that the number of variables in the T-SDP relaxations is less than that of the SDP relaxation. Consequently, it took less time to solve the T-SDP relaxations than the SDP relaxation as shown in the last column. As the number of variables in the T-SDP relaxations decreases, the number of nonzero elements increases from 662596 in the SDP relaxation to 2649828 for T-SDP in the last row. The total times for all T-SDP relaxations are shorter than that for the SDP relaxation.

4.1.4 Problem 4

As shown in the column CPU1 on problem 4, generating T-SDP relaxation problems consumed shorter computational time than the SDP relaxation. Also, two T-SDP relaxations, (Pn 2) and (Pn 3), spent less computational time than that of the SDP relaxation, despite the increase in the number of nonzero elements.

4.1.5 Problem 5

T-SDP relaxation problems may become infeasible depending on the values of m_i and l_i, as illustrated with this problem. In Table 3, the T-SDP relaxation (Pn 4) is obtained by using the 2-block circulant SOS polynomial for ϕ_0 and the 2-block circulant SOS polynomial for ϕ_i which corresponds to the constraints for upper and lower bounds on each variable $x_1, \ldots, x_{10}$, and SOS polynomials to ϕ_i which is associated with the other inequality constraints in the feasible set Γ. This is described as $(66, 1)$ and $(33, 2)$ in the (m_i, l_i) column.

As discussed in Section 3, Assumption 11 must hold for the T-SDP relaxation problem to have a feasible solution, but here we see that Assumption 11 no longer holds. The T-SDP relaxation (Pn 4) becomes infeasible, as shown in the last row of Table 3, when the number of blocks l_0 or l_i is increased in the block circulant SOS relaxation. However, the T-SDP relaxations (Pn 2) and (Pn 3) are still feasible and computationally less expensive than the SDP relaxation.

4.1.6 Problem 6

We observe the effectiveness of the T-SDP relaxation methods with this problem. It is known that the optimal value obtained by the SDP relaxation with relaxation order $N = 4$ is -4 [9]. The optimal value with $N = 6$, however, is -3.99972, which may contain numerical error. If the basic SOS relaxations with increasing the degree of the SOS polynomials is employed, then the optimal value by the SDP relaxation with a larger relaxation level approaches the optimal value of the original POP. We mention that the results in Table 4 do not follow the theoretical result, which may be caused by the numerical instability of the SDP relaxation problem. In such cases, the T-SDP relaxation is still effective, and by changing parameters such as l_0 and l_i, another T-SDP relaxation problem can be generated. In particular, the smaller positive semidefinite matrices may improve the numerical accuracy.

4.1.7 Problem 7

The results by the T-SDP relaxation (Pn 3) and (Pn 4) were obtained by applying the 2-block and 11-block circulant SOS polynomial, respectively. The 2-block circulant SOS polynomial for ϕ_i corresponds to the constraints for upper and lower bounds on each variable $x_1, \dots, x_6$, and the SOS polynomial for ϕ_i is associated with the other inequality constraints in the feasible set K of $(\tilde{\mathbb{P}}_K)$, as in problem 5.

In the rows for problem 7 in Table 4, we see that the number of variables in the T-SDP relaxation is smaller than that of the SDP relaxation, which results in faster computational time for CPU1 in all T-SDP cases. In particular, when the block circulant SOS polynomial was used to ϕ_i in the feasible set Γ, a significant reduction in the number of decision variables and CPU1 can be observed. For CPU2, as in the previous problems, there exists a trade-off between the increase in the number of nonzero elements and the decrease in the number of decision variables. Hence, increasing the parameters l_0 and l_i does not necessarily reduce the computational time. We observe that all T-SDP relaxations outperform the SDP relaxation in terms of the total time.

4.1.8 Problem 8

We see that the T-SDP relaxation efficiently provides an accurate optimal value for large-sized problems for this problem. The number of decision variables in the T-SDP relaxation is smaller that that of the SDP relaxation, which leads to shorter CPU1 time in all cases. In particular, CPU1 consumed by the T-SDP relaxation (Pn 3) is less than $\frac{1}{3}$ of that of the SDP relaxation. For CPU2, only the T-SDP relaxation (Pn 2) took less time compared to the SDP relaxation, but all T-SDP relaxations outperform the SDP relaxation in terms of the total time.

4.1.9 Problem 9

We discuss the performance of the T-SDP relaxation with two similar problems, 9 and 10. The objective functions of two problems are similar in that they have the same number of variables $n = 19$ and degree $d = 4$. The constraints of the problems are equivalent. The T-SDP relaxation becomes infeasible for problem 9, while it provides an accurate optimal value for problem 10.

In Table 4, the optimal value of the T-SDP relaxation (Pn 2) is smaller than that of the SDP relaxation, indicating that a weak lower bound was attained. In addition, the T-SDP relaxation (Pn 3) and (Pn 4) were infeasible. Thus, the T-SDP relaxation may not work effectively for some problems, especially for the cases where the objective function does not satisfy Assumption 11 on the block circulant SOS polynomials.

4.1.10 Problem 10

We discuss cases where the T-SDP relaxation performs well for the problems with the same number of variables and degree as problem 9. The objective function of problem 10 contains a block circulant SOS polynomial.

In the results for problem 10 in Table 4, we see that the performance of the T-SDP relaxation is better than the SDP relaxation, unlike problem 9. For CPU1, all T-SDP relaxations spent shorter computational time than the SDP relaxation. Furthermore, all T-SDP relaxations perform superior to the SDP relaxation for CPU2.

5 Concluding remarks

We have proposed a relaxation method for constrained POPs using block circulant SOS polynomials. The proposed block circulant SOS relaxation method is an extension of the SOS relaxation method proposed by Parrilo [18], and also an extension of T-SDP relaxation for unconstrained POPs [33] in the sense that it can be transformed into an equivalent T-SDP using the third-order tensor. Theoretical analysis of the feasibility and global optimality of the T-SDP relaxation problem have been studied, and the convergence of the sequence of T-SDP relaxations with increasing the relaxation level has been established.

The T-SDP relaxation method has shown to perform superior in computation to the SDP relaxation method from the basic SOS relaxation as it induces smaller-sized problem. The computational results in Section 4 show that the T-SDP relaxation method provides the same quality of optimal values as the SDP relaxation method with shorter computational time under Assumption 11.

For future study, it will be interesting to exploit sparsity for further improving the numerical efficiency. As in [27], the chordal sparsity of the block circulant SOS relaxation method can be developed for further reducing the computational cost. Another issue is to develop an SDP solver that directly solves T-SDP, instead of converting the T-SDP relaxation to an equivalent SDP relaxation. Then the computational time for the transformation can be saved. However, as mentioned earlier, the advantage of reducing the size of the relaxation problem when transforming from T-SDP to SDP may not be achieved in some case, which may require careful investigation.

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Appendix

Problem 1 [3]

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{9} x_i x_{i+1} + \sum_{i=1}^{9} x_i x_{i+2} + x_1x_7 + x_1x_9 + x_1x_{10} + x_2x_{10} + x_4x_7 \\ \text{subject to} \quad & \sum_{i=1}^{10} x_i = 1, \\ & x_i \geq 0, \quad i = 1, \ldots, 10. \end{aligned}$$

Problem 2

$$\begin{aligned} \text{minimize} \quad & x_1^{40} + x_1^{18}x_2^{20} + x_1^{14}x_2^{24} + x_1^{26}x_2^{8} + x_1^{22}x_2^{12} + x_1^{26}x_2^{4} \\ & + x_1^{22}x_2^{8} + x_1^{18}x_2^{8} + x_1^{14}x_2^{12} + x_1^{2}x_2^{20} + x_1^{20} + x_1^{12}x_2^{2} + x_1^{8}x_2^{6} + x_2^{2} \\ \text{subject to} \quad & x_1, x_2 \in \{-1, +1\} \end{aligned}$$

Problem 3

$$\begin{aligned} \text{minimize} \quad & x_1^{20} + x_2^2x_3^2 - 2.0x_2x_3^3 + x_3^4 - 4.0x_2x_3^2 + 4.0x_3^3 + 4.0x_3^2 \\ \text{subject to} \quad & x_1, x_2, x_3 \in \{-1, +1\} \end{aligned}$$

Problem 4 [33, Example 2]

$$\begin{aligned} \text{maximize} \quad & x_1^{10}x_2^{4} + x_1^{8}x_2^{12} + x_1^{24}x_2^{2} + x_1^{24}x_2^{6} + x_1^{32}x_2^{2} + x_1^{8}x_2^{28} + x_1^{28}x_2^{12} \\ & + x_1^{10}x_2^{32} + x_1^{42}x_2^{4} + x_1^{30}x_2^{18} + x_1^{20}x_2^{30} + x_1^{12}x_2^{40} + x_1^{6}x_2^{48} + x_1^{2}x_2^{54} + x_2^{58} \\ \text{subject to} \quad & x_1^2 + x_2^2 = 1. \end{aligned}$$

Problem 5 [12, st_bpaf1b]

$$\begin{aligned} \text{minimize} \quad & x_1x_6 + 2x_1 - 2x_6 + x_2x_7 + 4x_2 - x_7 + x_3x_8 \\ & + 8x_3 - 2x_8 + x_4x_9 - x_4 - 4x_9 + x_5x_{10} - 3x_5 + 5x_{10} \\ \text{subject to} \quad & -8x_1 - 6x_3 + 7x_4 - 7x_5 \leq 1, \\ & -6x_1 + 2x_2 + 3x_3 - 9x_4 - 3x_5 \leq 3, \\ & 6x_1 - 7x_3 - 8x_4 + 2x_5 \leq 5, \\ & -x_1 + x_2 - 8x_3 - 5x_5 \leq 4, \\ & 4x_1 - 7x_2 + 4x_3 + 5x_4 + x_5 \leq 0, \\ & 5x_7 - 4x_8 + 9x_9 - 7x_{10} \leq 0, \\ & 7x_6 + 4x_7 + 3x_8 + 7x_9 + 5x_{10} \leq 7, \\ & 6x_6 + x_7 - 8x_8 + 8x_9 \leq 3, \\ & -3x_6 + 2x_7 + 7x_8 + x_{10} \leq 6, \\ & -2x_6 - 3x_7 + 8x_8 + 5x_9 - 2x_{10} \leq 2, \\ & 0 \leq x_i \leq 20, \quad i = 1, \ldots, 10. \end{aligned}$$

Problem 6 [3, Problem 2.9.1]

$$\begin{aligned} \text{minimize} \quad & -2x_1 + x_2 - x_3 \\ \text{subject to} \quad & x^\mathsf{T} B^\mathsf{T} B x - 2r^\mathsf{T} B x + ||r||^2 - 0.25||b - v||^2 \geq 0, \\ & x_1 + x_2 + x_3 \leq 4, \\ & x_1 \leq 2, \\ & x_3 \leq 3, \\ & 3x_2 + x_3 \leq 6, \\ & x_i \geq 0, \quad i = 1, \ldots, 3. \end{aligned}$$

Here,

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -2 & 1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \quad v = \begin{bmatrix} 0 \\ -1 \\ -6 \end{bmatrix}, \quad r = \begin{bmatrix} 1.5 \\ -0.5 \\ -5 \end{bmatrix}.$$

Problem 7 [12, st_e34]

$$\begin{array}{ll} \text{minimize} & 4.3x_1 + 31.8x_2 + 63.3x_3 + 15.8x_4 + 68.5x_5 + 4.7x_6 \\ \text{subject to} & 17.1x_1 - 169x_1x_3 + 204.2x_3 - 3580x_3x_5 + 623.4x_5 - 3810x_4x_6 \\ & +212.3x_4 + 1495.5x_6 - 18500x_4x_6 + 38.2x_2 \geq 4.97, \\ & 17.9x_1 - 139x_1x_3 + 113.9x_3 - 2450x_4x_5 + 169.7x_4 + 337.8x_5 \\ & -16600x_4x_6 + 1385.2x_6 - 17200x_5x_6 + 36.8x_2 \geq -1.88, \\ & 26000x_4x_5 - 70x_4 - 819x_5 - 273x_2 \geq -69.08, \\ & 159.9x_1 - 14000x_1x_6 + 2198x_6 - 311x_2 + 587x_4 + 391x_5 \geq -118.02, \\ & 0 \leq x_1 \leq 0.31, \\ & 0 \leq x_2 \leq 0.046, \\ & 0 \leq x_3 \leq 0.068, \\ & 0 \leq x_4 \leq 0.042, \\ & 0 \leq x_5 \leq 0.028, \\ & 0 \leq x_6 \leq 0.0134. \end{array}$$

Problem 8

$$\begin{array}{ll} \underset{x \in \mathbb{R}^{11}}{\text{minimize}} & \sum_{0<\alpha\leq 6} \text{rand}(0:1)_\alpha x^\alpha \\ \text{subject to} & \sum_{i=1}^{11} x_i^2 = 1, \\ & -1 \leq x_i \leq 1, \quad i = 1, \ldots, 11, \end{array}$$

where $\text{rand}(0:1)_\alpha$ is a random number of 0 or 1.

Problem 9

$$\begin{array}{ll} \underset{x \in \mathbb{R}^{19}}{\text{minimize}} & \sum_{0<\alpha\leq 4} \text{rand}(0:1)_\alpha x^\alpha \\ \text{subject to} & \sum_{i=1}^{19} x_i^2 = 1, \\ & -1 \leq x_i \leq 1, \quad i = 1, \ldots, 19, \end{array}$$

where $\text{rand}(0:1)_\alpha$ is a random number of 0 or 1 as in Problem 8.

Problem 10

$$\begin{array}{ll} \text{minimize} & x_1^4 + x_1^2x_{12}^2 + 2.0x_1^2x_{12}x_{17} + x_1^2x_{17}^2 + 2.0x_1x_2x_3x_{12} + 2.0x_1x_2x_3x_{17} + x_2^2x_3^2 \\ & + x_2^2x_4^2 + x_2^2x_{15}^2 + 2.0x_2x_3^2x_{15} + 2.0x_2x_3x_7x_{15} + x_3^4 + 2.0x_3x_7 + x_3^2x_7^2 + x_3^2x_8^2 \\ & + x_3^2x_{19}^2 + 2.0x_3x_4x_8x_{19} + 2.0x_3x_4x_{12}x_{19} + x_4^2x_8^2 + 2.0x_4^2x_8x_{12} + x_4^2x_{12}^2 + x_4^2x_{13}^2 \\ & + x_5^2x_9^2 + 2.0x_5^2x_9x_{14} + 2.0x_5^2x_9x_{18} + x_5^2x_{14}^2 + 2.0x_5^2x_{14}x_{18} + x_5^2x_{18}^2 + x_5^2x_{19}^2 \\ & + x_6^2x_{16}^2 + 2.0x_6x_7x_8x_{16} + 2.0x_6x_7x_{12}x_{16} + x_7^2x_8^2 + 2.0x_7^2x_8x_{12} + x_7^2x_{12}^2 + x_7^2x_{13}^2 \\ & + x_8^2x_{12}^2 + 2.0x_8^2x_{12}x_{17} + x_8^2x_{17}^2 + 2.0x_8x_9x_{10}x_{12} + 2.0x_8x_9x_{10}x_{17} + x_9^2x_{10}^2 \\ & + x_9^2x_{11}^2 + x_{10}^2x_{12}^2 + 2.0x_{10}^2x_{12}x_{17} + x_{10}^2x_{17}^2 + 2.0x_{10}x_{11}x_{12}^2 + 2.0x_{10}x_{11}x_{12}x_{17} \\ & + x_{11}^2x_{12}^2 + x_{11}^2x_{13}^2 + x_{12}^2x_{16}^2 + 2.0x_{12}x_{13}x_{14}x_{16} + 2.0x_{12}x_{13}x_{16}x_{18} + x_{13}^2x_{14}^2 \\ & + 2.0x_{13}^2x_{14}x_{18} + x_{13}^2x_{18}^2 + x_{13}^2x_{19}^2 + x_{15}^2x_{19}^2 + 2.0x_{15}x_{17}^2x_{19} + 2.0x_{15}x_{18}x_{19}^2 \\ & + x_{17}^4 + 2.0x_{17}^2x_{18}x_{19} + x_{18}^2x_{19}^2 + x_{19}^4 + x_{11}^2 + 2.0x_{11}x_{16} + x_{16}^2 + 2.0x_{11} + 2.0x_{16} \\ \text{subject to} & \sum_{i=1}^{19} x_i^2 = 1, \\ & -1 \leq x_i \leq 1, \quad i = 1, \ldots, 19, \end{array}$$