The Maximum Singularity Degree for Linear and Semidefinite Programming

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Abstract

The singularity degree plays a crucial role in understanding linear and semidefinite programming, providing a theoretical framework for analyzing these problems. It is defined as the minimum number of facial reduction (FR) steps needed to reach strict feasibility for a convex set. On the other hand, the maximum singularity degree (MSD) is the maximum number of steps required. Recent progress in the applications of MSD has motivated us to explore its fundamental properties in this paper.

For semidefinite programming, we establish a necessary condition for an FR sequence to be the longest. Additionally, we propose an upper bound for MSD, which can be computed more easily. By leveraging these findings, we prove that computing MSD is NP-hard. This complexity result complements the existing algorithms for computing the singularity degree found in the literature. For linear programming, we provide a characterization for the longest FR sequences, which also serves as a polynomial-time algorithm for constructing such a sequence. In addition, we introduce two operations that ensure the longest FR sequences remain the longest. Lastly, we prove that MSD is equivalent to a novel parameter called the implicit problem singularity.

Key Words: semidefinite program, linear program, facial reduction, singularity degree, exposing vector, implicit problem singularity

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1 Introduction

Let F be a convex set defined as the intersection of an affine subspace and a convex cone $\mathcal{K}$. The way in which F is described is important for the efficiency of numerical algorithms in solving optimization problems constrained by F. When F is poorly described, these algorithms are prone to encountering numerical issues, leading to unreliable outcomes. A theoretical framework known as the *Facial Reduction Algorithm (FRA)*, introduced by Borwein and Wolkowicz, offers insights into addressing numerical challenges that emerge due to the absence of strict feasibility in the problem formulation, see [1–3]. Since its proposal, the FRA has enhanced the speed and stability of computations across numerous *Semidefinite Programming (SDP)* problems and has shed light on the geometric properties of the convex sets involved.

From a computational perspective, the efficacy of the FRA in numerically solving SDP problems is initially demonstrated through the quadratic assignment problem in the study by Zhao et al. [23]. Recent advancements in first-order methods, especially when integrated with the FRA, have enhanced the ability of solving SDP relaxations for large-scale problems, see [5, 7, 17, 22, 24].

The FRA is also an important tool in theory. Sturm [21] introduces a parameter called *singularity degree* which is the smallest number of facial reduction steps to recover strict feasibility; and it provides an important error bound for F. This parameter has been used to obtain many exciting new results for different choices of $\mathcal{K}$. For example, Drusvyatskiy, Li and Wolkowicz [4] derive convergence rate of the alternate projection method for semidefinite feasibility problems that do not satisfy Slater's condition; Pataki and Touzov [18] elucidate the occurrence of exponentially large solutions in SDP by using singularity degree. Lourenço [14] develops error bound result that does not require constraint qualification. Lindstrom, Lourenço and Pong [13] compute tight error bounds for the exponential cone feasibility problem.

A closely related parameter, known as *the maximum singularity degree (MSD)*, which represents the maximum number of facial reduction steps required to restore strict feasibility, has also been studied recently. In their work [12], Ito and Lourenço establish an upper bound for the Carathéodory number of convex cones in relation to MSD. Nishijima [16] explores the longest chain of faces of completely positive and copositive cones. Im and Wolkowicz [11] use MSD to derive an improved Barvinok-Pataki bound on SDP rank. Lourenço, Muramatsu and Tsuchiya [15] introduce a new concept called distance to polyhedrality. Im and Wolkowicz [10] provide numerical evidence that MSD can negatively affect the behavior of numerical algorithms in *linear programming (LP)*.

We provide new theoretical results about MSD for both LP and SDP. Our main contributions, presented in the sequence they appear within the paper, include:

- A necessary condition for an FR sequence to be one of the longest for SDP.
- An upper bound for MSD that can be computed more easily.
- NP-hardness of the MSD for SDP.
- Two useful operations for manipulating the longest FR sequences for LP.
- A characterization of the longest FR sequences for LP.
- The equivalence between MSD and the implicit problem singularity for LP.

The paper is organized as follows. In Section 2, we introduce the notations, and review the FRA in a general setting and formally define MSD. In Section 3.1, we examine the FRA for SDP. In

Section 3.2, we introduce and analyze the minimal FR sequences for SDP. In Section 3.3, we provide an upper bound for MSD. In Section 3.4, we show that computing MSD for SDP is NP-hard. In Section 4.1, we revisit the FRA for LP. In Section 4.2, we explore the minimal FR sequences for LP. In Section 4.3, we investigate operations that preserve minimality of FR sequences for LP. In Section 4.4, we apply these operations to characterize the longest FR sequences through the minimal FR sequences for LP. In Section 4.5, we establish the equivalence between the MSD and the implicit problem singularity, discussing their relationship with degeneracy and potential applications.

2 Notation and Preliminaries

Throughout, $\mathbb{S}^n$ is the set of *symmetric matrices* of order n. $\mathbb{S}^n_+$ is the set of *positive semidefinite matrices* of order n. Given $X, Y \in \mathbb{S}^n$, $\langle X, Y\rangle = \operatorname{tr}(XY)$ is the *trace inner product* between X and Y. For any $M \in \mathbb{R}^{n\times n}$, the *range space* of M is $\operatorname{Im}(M) := \{Mx \mid x \in \mathbb{R}^n\}$; the *kernel* of M is $\ker(M) := \{x \mid Mx = 0\}$. Denote by e_i the i-th unit vector, with the dimension determined by the context. The $n \times n$ identity matrix is denoted by I_n. For any positive semidefinite matrix $W \in \mathbb{S}^n_+$ of rank r, the set of orthornormal matrices whose columns span the null space of W is denoted by

$$\mathcal{N}(W) = \left\{P \in \mathbb{R}^{n\times(n-r)} \mid P^TP = I_{n-r},\ \operatorname{Im}(P) = \ker(W)\right\}.$$

For any vector $w \in \mathbb{R}^n$, the i-th entry of w is denoted by $w(i)$. If $S \subseteq \{1, \dots, n\}$, then $w(S)$ is the subvector of w consisting of entries in $w(i)$ for $i \in S$. The support of w is the subset $\operatorname{supp}(w) := \{i \mid w(i) \neq 0\} \subseteq \{1, \dots, n\}$.

Let $\mathcal{K}$ be a closed convex cone in a finite-dimensional real Euclidean space $\mathcal{V}$. Let $\mathcal{A} : \mathcal{K} \to \mathbb{R}^m$ be a linear operator, and $b \in \mathbb{R}^m$ an m-dimensional vector. Define the convex set

$$F(\mathcal{K}) := \mathcal{K} \cap \mathcal{L} \text{ where } \mathcal{L} := \{X \in \mathcal{V} \mid \mathcal{A}(X) = b\}. \tag{1}$$

Throughout, we assume that $F(\mathcal{K}) \neq \emptyset$, and we often write F for $F(\mathcal{K})$ to simplify the notation when $\mathcal{K}$ is clear from the context. We say *Slater's condition* holds for $F(\mathcal{K})$ if it contains a feasible solution in the relative interior of $\mathcal{K}$. When Slater's condition is not met, we can apply a conceptual method called the *Facial Reduction Algorithm (FRA)* to compute the minimal face of $\mathcal{K}$ containing the feasible region $F(\mathcal{K})$. To find this minimal face, we attempt to identify an element in the set

$$\mathcal{K}^* \cap \mathcal{L}^\perp. \tag{2}$$

Any element W in this set is an *exposing vector* of $\mathcal{K}$ containing the feasible region $F(\mathcal{K})$. Let $\mathcal{K}_1$ be the face of $\mathcal{K}$ exposed by W. If W additionally satisfies

$$W \notin \mathcal{K}^\perp, \tag{3}$$

then $\mathcal{K}_1$ is a proper face of $\mathcal{K}$. In this case, we can equivalently reformulate $F(\mathcal{K})$ as a *reduced feasible set* $F(\mathcal{K}_1)$. It is a fundamental result that if $F(\mathcal{K})$ fails to meet Slater's condition, then there always exists an element W satisfying the above conditions. Utilizing this fact, the FRA achieves a sequence of proper faces of $\mathcal{K}$ containing $F(\mathcal{K})$ through an iterative process, which converges to the minimal face in a finite number of steps. Specifically, the FRA follows this iterative scheme with $\mathcal{K}_1 := \mathcal{K}$,

1. Find $W_i \in \mathcal{K}_i^* \cap \mathcal{L}^\perp$ such that $W_i \notin \mathcal{K}_i^\perp$. If it does not exist, then we stop.

2. Let $\mathcal{K}_{i+1} \leftarrow \mathcal{K}_i \cap W_i^{\perp}$.

At the end of the FRA, we obtain the so-called *FR sequence* $\mathcal{K}_1 \supsetneq \cdots \supsetneq \mathcal{K}_{d+1}$ such that $\mathcal{K}_{d+1}$ is the smallest face of $\mathcal{K}$ containing $F(\mathcal{K})$. An FR step is a completed iteration in the FRA, i.e., both lines are executed. The length of the FR sequence is d which is the number of FR steps. The associated sequence of exposing vectors $(W_1, \ldots, W_d)$ comprises exactly d terms. For $i < d+1$, the subsequence $\mathcal{K}_1 \supsetneq \cdots \supsetneq \mathcal{K}_i$ is referred to as a *partial FR sequence* since the final face $\mathcal{K}_i$ is not yet the minimal face.

The *singularity degree SD*(F) is the length of the shortest FR sequence for F. A shortest FR sequence can be obtained by picking an exposing vector in the relative interior of $\mathcal{K}_i^*$ at each FR step. Generally, we have $\mathrm{SD}(F) \geq 1$ and the inequality can be strict for $\mathcal{K} = \mathbb{S}_+^n$; however, $\mathrm{SD}(F) = 1$ always holds for $\mathcal{K} = \mathbb{R}^n$.

The *maximum singularity degree (MSD)* of F, denoted by $\mathrm{MSD}(F)$, is the length of the longest FR sequences for F. In contrast to the singularity degree, numerous fundamental questions about MSD remain unanswered, such as how to identify the longest FR sequences for both LP and SDP. This paper aims to address these questions.

3 Maximum Singularity Degree for Semidefinite Programming

3.1 FRA for SDP

Let $\mathcal{K} = \mathbb{S}_+^n$ be the set of positive semidefinite matrices of order n. Given symmetric matrices $A_1, \ldots, A_m \in \mathbb{S}^n$, we define the linear operator $\mathcal{A} : \mathbb{S}_+^n \to \mathbb{R}^m$ by

$$\mathcal{A}(X) := \begin{bmatrix} \langle A_1, X \rangle \\ \vdots \\ \langle A_m, X \rangle \end{bmatrix} \in \mathbb{R}^m.$$

Let $b \in \mathbb{R}^m$ be given. Then (1) becomes a *semidefinite programming* (SDP) problem, i.e.,

$$F(\mathbb{S}_+^n) := \mathbb{S}_+^n \cap \mathcal{L} \text{ where } \mathcal{L} := \{X \in \mathbb{S}^n \mid \mathcal{A}(X) = b\}. \tag{4}$$

To provide a more explicit description of the FRA applied to SDP, we include the following well-known results.

- Note that $\mathcal{L}$ is an affine subspace, and its orthogonal complement is given by

  $$\mathcal{L}^* = \left\{\mathcal{A}^*(y) \in \mathbb{S}^n \mid b^T y = 0\right\},$$

  where $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$.

- Any non-empty face $\mathcal{K}_1$ of $\mathbb{S}_+^n$ can be described by a matrix $V \in \mathbb{R}^{n \times r}$ with orthonormal columns, as expressed through the relation

  $$\mathcal{K}_1 = \left\{X \in \mathbb{S}_+^n \mid \operatorname{Im} X \subseteq \operatorname{Im} V\right\}.$$

  The matrix V is termed a *facial range vector* associated with the face $\mathcal{K}_1$. Thus, we can write V and $\mathcal{K}_1$ interchangeably without causing confusion. For example, $F(V)$ is understood to mean $F(\mathcal{K}_1)$. The dual cone of the face $\mathcal{K}_1$ is

  $$\mathcal{K}_1^* = \left\{W \in \mathbb{S}^n \mid V^T W V \in \mathbb{S}_+^r\right\}.$$

Now we are prepared to detail the FRA for SDP more explicitly. At the i-th iteration, the face $\mathcal{K}_i$ is associated with a specific facial range vector $V_i \in \mathbb{R}^{r_i}$. As previously mentioned, we will refer to $\mathcal{K}_i$ using its facial range vector V_i for clarity. Consequently, V_1 is always assumed to be the $n \times n$ identity matrix I_n for SDP. The set of exposing vectors $\mathcal{K}_i^* \cap \mathcal{L}^\perp$ is

$$D(V_i) := \left\{ \mathcal{A}^*(y) \in \mathbb{S}^n \mid V_i^T \mathcal{A}^*(y) V_i \in \mathbb{S}_+^{r_i} \text{ and } b^T y = 0 \text{ for some } y \in \mathbb{R}^m \right\}. \tag{5}$$

The exposing vector W_i exposes a proper face of $\mathcal{K}_i$ if $W_i \notin \mathcal{K}_i^\perp$ which can be written as

$$V^T W_i V \neq 0. \tag{6}$$

To obtain $\mathcal{K}_{i+1}$, or equivalently, its facial range vector V_{i+1}, we use the following formula

$$V_{i+1} = V_i P_i \in \mathbb{R}^{n \times r_{i+1}} \text{ for any } P_i \in \mathcal{N}(V_i^T W_i V_i).$$

In the subsequent discussion, upon introducing an FR sequence f, we will directly use W_i, V_i, P_i, and r_i without redefining them for f. The length of f is denoted by $|f|$. The i-th FR step in f is denoted by f_i. If there is another FR sequence g, then $f_i = g_i$ indicates that the i-th FR steps in both f and g are identical, in that the facial range vectors share the same range space and the exposing vectors expose the same proper face. To specify an FR sequence, it suffices to list the corresponding sequence of exposing vectors or facial range vectors. To validate that f_i is an FR step, we usually need to verify the following three conditions:

1. W_i is an exposing vector of V_i containing $F(V_i)$.
2. W_i exposes a proper face of V_i containing $F(V_i)$.
3. The proper face exposed by W_i is V_{i+1}.

To keep the presentation concise, we will simply state "W_i is an exposing vector of V_i" in the proof, without referencing the set F.

When studying FR sequences, there are a few tricks that help make proofs and notation easier. They are outlined below:

Fact 3.1. *When we prove the i-th FR step satisfies some conditions, it is often possible to assume $i = 1$ without loss of generality. This is because the reduced feasible set $F(\mathcal{K}_i)$ can be viewed as the initial problem. As V_1 is the identity matrix, this approach makes the proof simpler to write and understand.*

Fact 3.2. *The length of the FR sequence is independent of the choice $P_i \in \mathcal{N}(V_i^T W_i V_i)$ at each FR step. This means we can pick any P_k that makes the proof easier, as shown in [8, 20].*

3.2 The Minimal FR Sequences

Let V be a given facial range vector. We say an exposing vector $W \in D(V)$ is *minimal* for the face V, if there does not exist an $M \in D(V)$ such that

$$0 \neq \operatorname{Im}(V^T M V) \subsetneq \operatorname{Im}(V^T W V).$$

An FR sequence f is called *minimal* if every exposing vector $W_i \in D(V_i)$ is minimal. In this section, we prove that if f is one of the longest FR sequence, then f must be minimal. We also provide examples showing that the converse direction does not hold in general, even if we impose a more restrictive rank assumption. Additionally, we introduce some technical lemmas related to minimal FR sequences.

The first technical result shows that minimal exposing vectors are unique. Its special case for LP is a key result to be used later.

Lemma 3.1. *Let V be a facial range vector. Let $W \in D(V)$ be minimal, and $M \in \mathcal{L}$. Define $\bar{M} := V^T M V$ and $\bar{W} := V^T W V$. If $0 \neq \operatorname{Im}(\bar{M}) \subseteq \operatorname{Im}(\bar{W})$, then $\bar{M} = \bar{W}$ up to some non-zero scaling.*

Proof. Let U be an orthonormal matrix such that $\operatorname{Im}(U) = \operatorname{Im}(\bar{W})$. Since $\bar{W}$ and $\bar{M}$ have the same range space, we have $\bar{W} = UDU^T$ and $\bar{M} = U\Lambda U^T$ for some diagonal matrices D and Λ. In addition, the diagonal entries of D are strictly positive as $\bar{W}$ is positive semidefinite, and $\Lambda \neq 0$ as $\operatorname{Im}(\bar{M}) \neq 0$. Assume Λ contains a strictly negative diagonal entry. (The proof is similar if every diagonal entry of Λ is positive.) Denote by d_i and λ_i the i-th diagonal entry of D and Λ, respectively. Define

$$\alpha := \min_i \left\{ -\frac{d_i}{\lambda_i} \mid \lambda_i < 0 \right\} > 0.$$

Then $D + \alpha\Lambda$ is positive semidefinite, but not positive definite. This implies that $\bar{W} + \alpha\bar{M} \in \mathbb{S}^r_+$ and thus $W + \alpha W' \in D(V)$. In addition, we have $\operatorname{Im}(\bar{W} + \alpha\bar{M}) \subsetneq \operatorname{Im}(\bar{W})$. Since W is minimal, we must have $\bar{W} + \alpha\bar{M} = 0$. □

Theorem 3.1. *If f is a longest FR sequence, then f is minimal.*

Proof. Let f be an FR sequence of length d. We prove that if f is not minimal, then it is not the longest. Assume $W_j \in D(V_j)$ is not minimal. We construct and prove that the sequence g in Figure 1 is an FR sequence.

iteration	1	$\cdots$	$j-1$	j	$j+1$	$j+2$	$\cdots$	$d+1$
face	V_1	$\cdots$	V_{j-1}	V_j	V_jP	V_{j+1}	$\cdots$	V_d
exposing vector	W_1	$\cdots$	W_{j-1}	M	W_j	W_{j+1}	$\cdots$	W_d

Figure 1: The sequence g obtained by splitting W_j into two parts.

Based on the construction, we can apply Fact 3.1 to assume $j = 1$. Recall that V_1 is the identity matrix. As $W_1 \in D(V_1)$ is not minimal, there exists an $M \in D(V_1)$ such that

$$0 \neq \operatorname{Im}(M) \subsetneq \operatorname{Im}(W_1). \tag{7}$$

As $M \in D(V_1)$ yields a proper face of V_1, we have g_1 is an FR step. The facial range vector in g_2 is $P \in \mathbb{R}^{n\times k}$ for any $P \in \mathcal{N}(M)$. Since $W_1 \in \mathbb{S}^n_+$ implies that $P^T W_1 P \in \mathbb{S}^k_+$, we have $W_1 \in D(P)$ is an exposing vector in g_2. Next we show that W_1 yields a proper face. From (7), we have $\ker(M) \supsetneq \ker(W_1)$ and thus $\operatorname{Im}(P) \supsetneq \operatorname{Im}(P_1)$ for any $P_1 \in \mathcal{N}(W_1)$. So there exists a vector $u \in \mathbb{R}^n$ such that $u \in \operatorname{Im}(P)$ and $u \notin \operatorname{Im}(P_1)$. Let $u = Pv$ for some $v \in \mathbb{R}^k$. Then

$$v^T(P^T W_1 P)v = u^T W_1 u \neq 0.$$

This shows that $P^T W_1 P \neq 0$, and thus W_1 yields a proper face. This proves that g_2 is an FR step.

The facial range vector in g_3 is obtained as follows. Let $G \in \mathcal{N}(P^T W_1 P)$. We show that $\mathrm{Im}(P_1) = \mathrm{Im}(PG)$. As $W_1 \in \mathbb{S}^n_+$, we have $\mathrm{Im}(PG) \subseteq \ker(W_1) = \mathrm{Im}(P_1)$. Conversely, let $u \in \mathrm{Im}(P_1)$. As $\mathrm{Im}(P_1) \subsetneq \mathrm{Im}(P)$, we have $u \in \mathrm{Im}(P)$ and $u = Pv$ for some vector v. Since $P_1 \in \mathcal{N}(W_1)$, we have $W_1 u = W_1 P v = 0$ and thus $v \in \ker(P^T W_1 P)$. As $G \in \mathcal{N}(P^T W_1 P)$, we have $v = Gw$ for some vector w. This shows that $u = PGw$ and thus $u \in \mathrm{Im}(PG)$. This proves $\mathrm{Im}(PG) = \mathrm{Im}(P_1)$. By Fact 3.2, we can take P and G such that $PG = P_1$ to obtain $V_1 PG = V_1 P_1 = V_2$. This allows us to conclude that $g_i = f_{i-1}$ for $i = 3, \dots, d+1$. Putting together, we prove that g is also an FR sequence. Since $|g| = |f| + 1$, this shows f is not a longest FR sequence. □

It turns out that a minimal FR sequence may not be a longest one for SDP. We demonstrate this in the next example.

Example 3.1. *Consider the SDP problem* (4) *given by*

$$A_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 := \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_3 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } b := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3.$$

The only feasible solution is zero. The set of exposing vectors $D(V_1)$ defined in (5) *with variable $y = (y(1), y(2), y(3))$ is given by*

$$\mathcal{A}^*(y) = \begin{bmatrix} y(1) - y(2) & y(2) & 0 \\ y(2) & y(2) + y(3) & 0 \\ 0 & 0 & y(3) \end{bmatrix} \succeq 0.$$

The sequence of exposing vectors (A_3, A_1) is a minimal FR sequence, and it has length 2. However it is not a longest FR sequence, as the minimal FR sequence (A_1, A_2, A_3) has length 3.

In addition, this example also shows that after taking a minimal FR step, there can be more than one redundant equality constraints if we reformulate the data matrices correspondingly. After taking A_3 as the exposing vector at the first FR step, the three data matrices become 1, -1 and 0, and thus there are two redundant constraints. This phenomenon is different for LP and SDP, see Theorem 4.2.

In fact, even if we pick an exposing vector of minimal rank at each FR step, then it still does not always yield a longest FR sequence. We illustrate this with the following example.

Example 3.2. *Consider the SDP problem with data matrices*

$$A_1 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A_2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A_3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } b := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3. \tag{8}$$

In the first FR step, all the possible exposing vectors and their ranks can be listed as follows

exposing vector	*rank*
A_3	2
$A_1 + \alpha A_2$ *for* $\alpha \in \{0, 1\}$	3
$A_1 + \alpha A_2$ *for* $\alpha \in (0, 1)$	4
$A_1 + \alpha A_2 + \beta A_3 \succeq 0$ *for* $\alpha \in \{\frac{1-\sqrt{1+4\beta}}{2}, \frac{1+\sqrt{1+4\beta}}{2}\}, \beta > 0$	4
$A_1 + \alpha A_2 + \beta A_3 \succeq 0$ *for* $\alpha \in (\frac{1-\sqrt{1+4\beta}}{2}, \frac{1+\sqrt{1+4\beta}}{2}), \beta > 0$	5

Based on the minimum rank rule, A_3 must be the exposing vector in the first FR step. This corresponds to removing the last two rows and columns. In the second FR step, there is only one possible exposing vector, namely, A_1. This yields an FR sequence (A_3, A_1) of length 2. However, (A_1, A_2, A_3) is a longer FR sequence, and A_1 does not have the minimum rank.

Example 3.2 suggests that computing MSD can be highly non-trivial, even without considering the difficulties associated with solving SDP alone. Indeed, we will prove the hardness of computing MSD. To achieve this goal, we provide a useful upper bound for MSD next.

3.3 An Upper Bound for MSD

Given that computing MSD can be challenging, we provide an upper bound for MSD that may be easier to compute in certain cases. For example, we will use the upper bound in this section to derive the hardness result about computing MSD later. Assume $F = \{0\}$. Let $U \in \mathbb{R}^{n\times r}$ be any facial range vector. Let f be a given FR sequence for F. We are interested in constructing an FR sequence for the reduced feasible set $F(U)$ based on f. To this end, we derive the following technical result.

Lemma 3.2. *Let $U \in \mathbb{R}^{n\times k}$ and $V \in \mathbb{R}^{n\times r}$ be facial range vectors, and $W \in \mathbb{S}^n$. Define $\hat{W} = U^T W U$ and $\bar{W} = V^T W V$. Assume $\operatorname{Im}(U) \subseteq \operatorname{Im}(V)$ and $\bar{W} \in \mathbb{S}^r_+$. Then $\hat{W} \in \mathbb{S}^k_+$, and $\operatorname{Im}(UQ) \subseteq \operatorname{Im}(VP)$ for any $Q \in \mathcal{N}(\hat{W})$ and $P \in \mathcal{N}(\bar{W})$.*

Proof. By the assumption $\operatorname{Im}(U) \subseteq \operatorname{Im}(V)$, we have $U = VR$ for some matrix $R \in \mathbb{R}^{r\times k}$. This yields $\hat{W} = R^T \bar{W} R \in \mathbb{S}^k_+$ as $\bar{W} \in \mathbb{S}^r_+$. Since $Q \in \mathcal{N}(\hat{W})$, we have $0 = Q^T \hat{W} Q = Q^T R^T \bar{W} R Q$. This shows that $\operatorname{Im}(RQ) \subseteq \ker(\bar{W}) = \operatorname{Im}(P)$ as $P \in \mathcal{N}(\bar{W})$. Thus $\operatorname{Im}(UQ) = \operatorname{Im}(VRQ) \subseteq \operatorname{Im}(VP)$. □

Define a sequence of facial range vectors $(U_1, \ldots, U_d)$ with $U_1 := U$ for the reduced feasible set $F(U)$ as follows. For $i = 1, \ldots, d-1$, we set

$$U_{i+1} = U_i Q_i \text{ for any } Q_i \in \mathcal{N}(U_i^T W_i U_i). \tag{9}$$

Since $V_1 = I_n$, we always have $\operatorname{Im}(U_1) \subseteq \operatorname{Im}(V_1)$. Applying Lemma 3.2 to the sequence $(U_1, \ldots, U_d)$ in an iteratively manner, we see that $\operatorname{Im}(U_i) \subseteq \operatorname{Im}(V_i)$ and $U_i^T W_i U_i$ is positive semidefinite for every $i = 1, \ldots, d$. This shows that W_i is an exposing vector of the face U_i. However, W_i may not expose a proper face. Thus, if we remove U_i whenever $\operatorname{Im}(U_i) = \operatorname{Im}(U_{i-1})$ for $i \geq 2$, then this yields an FR sequence g for the reduced feasible set $F(U)$ with the corresponding sequence of facial range vectors

$$(U_{n_1}, \ldots, U_{n_k}), \tag{10}$$

for some positive integers $1 = n_1 < \cdots < n_k$. We call *g the subsequence of f with respect to U.* Now we are ready to present the upper bound for MSD.

Theorem 3.2. *Let $\begin{bmatrix} U_1 & \cdots & U_k \end{bmatrix} \in \mathbb{R}^{n\times n}$ be any orthogonal matrix such that $U_j \in \mathbb{R}^{n\times n_j}$ with $\sum_{j=1}^k n_j = n$. Assume $F = \{0\}$. It holds that*

$$\mathit{MSD}(F) \leq \sum_{j=1}^{k} \mathit{MSD}(F(U_j)).$$

Proof. Let f be an FR sequence. For each $j = 1, \ldots, k$, define $U_{j,1} := U_j$ and let

$$(U_{j,1}, \ldots, U_{j,d})$$

be the sequence of facial range vectors defined as in (9) for $F(U_{j,1})$. For each $i = 1, \ldots, d$, define the matrix

$$H_i = \begin{bmatrix} U_{1,i} & \cdots & U_{k,i} \end{bmatrix}. \tag{11}$$

We prove that if $\operatorname{Im}(V_i) = \operatorname{Im}(H_i)$, then

- There exists at least one index j such that W_i exposes a proper face of $U_{j,i}$.
- $\operatorname{Im}(V_{i+1}) = \operatorname{Im}(H_{i+1})$

Since W_i exposes a proper face of V_i, we have $V_i^T W_i V_i \neq 0$ is positive semidefinite matrix, see (6). As $\operatorname{Im}(V_i) = \operatorname{Im}(H_i)$, we have $H_i^T W_i H_i \neq 0$ is also positive semidefinite. By construction (11), there exists an index j such that $U_{j,i}^T W_i U_{j,i} \neq 0$ is positive semidefinite matrix. This shows that W_i exposes a proper face of $U_{j,i}$.

To show $\operatorname{Im}(V_{i+1}) = \operatorname{Im}(H_{i+1})$, define the block-diagonal matrix $\tilde{Q}_i$ as

$$\tilde{Q}_i := \begin{bmatrix} Q_{1,i} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & Q_{k,i} \end{bmatrix} \text{ where } Q_{j,i} \in \mathcal{N}(U_{j,i}^T W_i U_{j,i}) \text{ for } j = 1, \ldots, k.$$

By construction, we have $\operatorname{Im}(\tilde{Q}_i) = \ker(H_i^T W_i H_i)$. The assumption $\operatorname{Im}(H_i) = \operatorname{Im}(V_i)$ implies that $V_i R = H_i$ for some invertible matrix $R \in \mathbb{R}^{r_i \times r_i}$. For any $P_i \in \mathcal{N}(V_i^T W_i V_i)$, it holds that

$$\operatorname{Im}(\tilde{Q}_i) = \ker(H_i^T W_i H_i) = \ker(R^T V_i^T W_i V_i R) = \operatorname{Im}(R^{-1} P_i).$$

This shows the relation

$$\operatorname{Im}(H_{i+1}) = \operatorname{Im}(H_i \tilde{Q}_i) = \operatorname{Im}(H_i R^{-1} P_i) = \operatorname{Im}(V_i P_i) = \operatorname{Im}(V_{i+1}).$$

As $V_1 = I_n$, we always have $\operatorname{Im}(V_1) = \operatorname{Im}(H_1)$. Thus the above relation implies that, at each FR step f_i, there exists at least one index j such that W_i exposes a proper face of $U_{j,i}$. Denote by g^j the subsequence of f with respect to U_j as defined in (10). We obtain that

$$|f| \leq \sum_{j=1}^{k} |g^j| \leq \sum_{j=1}^{k} \operatorname{MSD}(F(U_j)).$$

In particular, if f is a longest FR sequence, then $|f| = \operatorname{MSD}(F)$ and thus $\operatorname{MSD}(F) \leq \sum_{j=1}^{k} \operatorname{MSD}(F(U_j))$. $\square$

Note that as $\operatorname{SD}(F) \leq \operatorname{MSD}(F)$, Theorem 3.2 is also an upper bound for $\operatorname{SD}(F)$. When we partition the space into proper subspaces $U_1, \ldots, U_k$, it can be easier to compute $\operatorname{MSD}(F(U_i))$ and thus obtaining an upper bound for $\operatorname{MSD}(F)$. For example, if we take $U_1 = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}$ and $U_2 = \begin{bmatrix} e_4 & e_5 \end{bmatrix}$ for (8), then $\operatorname{MSD}(F(U_1)) = 1$ and $\operatorname{MSD}(F(U_2)) = 2$. This yields a tight upper bound as $\operatorname{MSD}(F) = 3$.

We further clarify the power of the upper bound in Theorem 3.2 by constructing a set F in Example 3.3 and computing upper bounds for $\mathrm{MSD}(F)$ in Lemmas 3.3 and 3.4. This example and its upper bounds are not artificial. In fact, it is an essential part in proving one of the main results in Theorem 3.3.

Example 3.3. *Let $M \in \{0,1\}^{p\times q}$ be a zero-one matrix with p rows and q columns. Define the set F associated with M as follows. Let n be the number of ones in M. The matrix variable X in F is of order n. The rows and columns of X are indexed by the indices (i,j) such that $M_{ij} = 1$. There are $m := p+q$ linear constraints. The first p constraints are associated with the rows in M. For each row $i = 1,\dots,p$, the matrix A_i is a diagonal matrix with ones in the diagonal entries associated with $\{(i,j) \mid M_{i,j} = 1 \text{ for some } j\}$. For each column $j = 1,\dots,q$, the matrix A_{p+j} is a diagonal matrix with ones in the diagonal entries associated with $\{(i,j) \mid M_{i,j} = 1 \text{ for some } i\}$. Let $b := 0 \in \mathbb{R}^{p+q}$ be the all-zeros vector of length $p+q$.*

In Example 3.3, the only feasible solution of F is zero. The data matrices defining F can be a zero matrix if the corresponding row or column in M are zeros, and we keep them only to ease the notation. Note that F is actually a polyhedron. In fact, we have a characterization of the longest FR sequences for polyhedra, together with a polynomial-time algorithm for finding such a sequence in Section 4.4. However, the algorithm relies on solving LP problems, and thus the upper bound in Theorem 3.2 remains meaningful for LP. We first consider the case when M is the all-ones matrix.

Lemma 3.3. *Let $M \in \{0,1\}^{p\times q}$ be the all-ones matrix. Define the set F associated with M as in Example 3.3. Then $MSD(F) = p+q-1$.*

Proof. As F is a polyhedron, any minimal FR sequence is also a longest one, see Theorem 4.1. It is clear that $A_1,\dots,A_{p+q}$ are exposing vectors. We distinguish two cases based on p and q.

- Assume $p=1$ or $q=1$. If $q=1$, then A_i is minimal for $i=1,\dots,p$, and thus $(A_1,\dots,A_p)$ is a longest FR sequence. This shows that $\mathrm{MSD}(F) = p$. Similarly, if $p=1$, then $\mathrm{MSD}(F) = q$.
- Assume $p > 2$ and $q > 2$. We show that all exposing vectors are minimal in the first FR step. Assume A_1 is not minimal. Then there exists a vector y such that $W := \mathcal{A}^*(y) \neq 0$ exposes a proper face of $\mathbb{S}^n_+$ such that
$$\emptyset \neq S := \{(i,j) \mid \text{The } (i,j)\text{-th diagonal entry of } W \text{ is nonzero}\} \subsetneq \{(1,j) \mid j = 1,\dots,q\}. \tag{12}$$
Without loss of generality, we assume that $(1,1) \notin S$. Assume $y(1) = \lambda$. Then $y(p+1) = -\lambda$ as $(1,1) \notin S$. It follows from the containment in (12) that $y(2),\dots,y(p) = \lambda$, and thus $y(p+2),\dots,y(p+q) = -\lambda$. But then $\mathcal{A}^*(y) = 0$ which is a contradiction.

  Thus, we can apply the above argument inductively to conclude that the exposing vector at any minimal FR step must be one of the data matrices. Therefore, a longest FR sequence can be obtained as follows: we use $A_1,\dots,A_{p-1}$ as the first $p-1$ FR steps; after that the problem reduced to the first case above, and thus we can use $A_{p+1},\dots,A_{p+q}$ as the last q FR steps. This yields an FR sequence $(A_1,\dots,A_{p-1},A_{p+1},\dots,A_{p+q})$ of length $p+q-1$.

□

Based on the upper bound related to the all-ones matrix above, we provide an upper bound for $\mathrm{MSD}(F)$ when M contains duplicated columns.

Lemma 3.4. *Assume $M = \begin{bmatrix} M_1 & \cdots & M_k \end{bmatrix}$ where the submatrix $M_j = \begin{bmatrix} v_j & \cdots & v_j \end{bmatrix} \in \{0,1\}^{p\times q_j}$ consists of q_j copies of the same non-zero column vector $v_j \in \{0,1\}^p$. Define the set F associated with M as in Example 3.3. Then*

$$MSD(F) \leq \sum_{j=1}^{k} \mathbf{1}^T v_j + q_j - 1.$$

Proof. Let F_j be the polyhedron associated with the submatrix M_j as defined in Example 3.3. Let $U = \begin{bmatrix} U_1 & \cdots & U_k \end{bmatrix}$ be a square orthogonal matrix such that $F(U_j) = F_j$. Applying Theorem 3.2 and Lemma 3.3, we have $\mathrm{MSD}(F) \leq \sum_{j=1}^{k} \mathrm{MSD}(F(U_j)) = \sum_{j=1}^{k} \mathbf{1}^T v_j + q_j - 1$. □

3.4 NP-hardness

In this section, we show that finding a longest FR sequence for SDP is NP-hard. We will construct a polynomial-time transformation from the well-known NP-complete problem 3SAT.

3SATISFIABILITY (3SAT)
INSTANCE: A finite set $U = \{u_1, \ldots, u_p\}$ of variables, and $C = \{c_1, \ldots, c_q\}$ a collection of clauses on U such that $|c_i| = 3$ for $i = 1, \ldots, q$.
QUESTION: Is there a truth assignment for U such that all the clauses in C are satisfiable.

To facilitate our analysis later, we can simplify a given 3SAT instance in the following way. If a clause $c_k = (u_i, \bar{u}_i, u_j)$ contains both the positive and negative literal of the same variable u_i, then c_k is always true. Thus we can remove this clause c_k from the problem. If the positive literal u_i never appear in any clause, then it is always possible to assume u_i is assigned false. This allows us to remove u_i and all clauses containing the negative literal $\bar{u}_i$. A similar approach is applied if the negative literal $\bar{u}_i$ is absent from all clauses. After simplification, we obtain an equivalent 3SAT instance which satisfies the following two properties.

Definition 3.1. *We say an instance of 3SAT is simplified if it satisfies*

- *For each variable u_i, we have $u_i \in c_j$ and $\bar{u}_i \in c_k$ for some j and k.*
- *If $u_i \in c_j$, then $\bar{u}_i \notin c_j$. And if $\bar{u}_i \in c_j$, then $u_i \notin c_j$.*

The simplification can clearly be implemented in polynomial time, ensuring that the simplified 3SAT instance preserves the complexity of the original problem. Next, we define the MSD problem for SDP as decision problems.

MAXIMUM SINGULARITY DEGREE for SDP (MSD-SDP)
INSTANCE: An SDP problem (4) with $A_1, \ldots, A_m \in \mathbb{S}^n$, $b \in \mathbb{R}^m$ and a positive integer d.
QUESTION: Does (4) have an FR sequence of length d or more?

Theorem 3.3. *MSD-SDP is NP-hard.*

Proof. We transform 3SAT to MSD-SDP. Let $U = \{u_1, \ldots, u_p\}$ and $\tilde{C} = \{c_1, \ldots, c_{\tilde{q}}\}$ be any simplified instance of 3SAT, see Definition 3.1. For technical reasons, it is necessary to introduce certain redundancy into the simplified 3SAT instance as follows. We duplicate the clauses in $\tilde{C}$ so that each clause $c_i \in \tilde{C}$ is replicated to have $2\tilde{q}$ copies, including the original. Let $q := 2\tilde{q}^2$. This results in a 3SAT instance that retains the same set of variables U, with the collection of clauses being $C = \{c_1, \ldots, c_q\}$ such that $c_i = c_j$ if $i \equiv j \pmod{\tilde{q}}$.

We construct an MSD-SDP instance with $n := 3q + p + 1$ and $m := 2p + q$. The rows and columns of the matrix variable X are indexed by the n elements in $\left(\bigcup_{i \in U} \mathcal{T}_i \cup \mathcal{F}_i\right) \cup \{0, 1, \ldots, p\}$, where

$$\begin{aligned} \mathcal{T}_i &= \{(i,j) \mid u_i \in c_j\}, \\ \mathcal{F}_i &= \{(i,j) \mid \bar{u}_i \in c_j\}. \end{aligned}$$

Note that there are $3q$ elements in $\bigcup_{i \in U} \mathcal{T}_i \cup \mathcal{F}_i$, and $p+1$ elements in $\{0, 1, \ldots, p\}$.

Next, we construct the data matrices $A_1, \ldots, A_m$ which can be classified into two different components as follows.

- For each variable $u_i \in U$, we make two diagonal matrices $A_i, A_{n+i} \in \{0,1\}^{n \times n}$ as a truth-setting component to force a choice between assigning u_i true and assigning u_i false. The diagonal entries of A_i and A_{n+i} associated with $\mathcal{T}_i \cup \{i\}$ and $\mathcal{F}_i \cup \{i\}$ are ones, respectively. All other entries are zeros.

  Note that A_i is an exposing vector as it is positive semidefinite for $i = 1, \ldots, 2p$. We intent to build a correspondence between the way using A_i or A_{n+i} in the FR sequence and the truth-false setting of variable u_i.

- For each clause $c_j \in C$, we make one matrix $A_{2p+j} \in \{0,1\}^{n \times n}$ to serve as a satisfaction testing component. Define

  $$\mathcal{C}_j = \{(i,j) \mid (i,j) \in \mathcal{T}_i \cup \mathcal{F}_i\}.$$

  By Definition 3.1, the set $\mathcal{C}_j$ is well-defined as each variable u_i appears in each clause at most once, either as a positive or negative literal. In addition, $|\mathcal{C}_j| = 3$ as each clause contains exactly 3 literals. The entries of A_{2p+j} are specified as follows. The diagonal entries of A_{2p+j} associated with $\mathcal{C}_j$ are ones. In addition, the principal submatrix of A_{2p+j} associated with $\{0, \ldots, p\}$ is the matrix

  $$\begin{bmatrix} 0 & \mathbf{1}_p^T \\ \mathbf{1}_p & 0 \end{bmatrix} \in \mathbb{S}^{p+1}, \tag{13}$$

  where $\mathbf{1}_p$ is the all-ones column vector of length p. All other entries are zeros. The principal submatrix (13) is indefinite and it has rank $p-1$. The two non-zero eigenvalues are $\sqrt{p}$ and $-\sqrt{p}$ corresponding to the eigenvectors $\begin{bmatrix} \sqrt{p} & 1_p^T \end{bmatrix}^T$ and $\begin{bmatrix} -\sqrt{p} & 1_p^T \end{bmatrix}^T$, respectively. Since none of the data matrices contains a non-zero diagonal entry associated with $\{0\}$, A_{2p+j} is not an exposing vector unless the rows and columns associated with $\{1, \ldots, p\}$ are all removed from the matrix variable X by the FRA.

Let $b := 0 \in \mathbb{R}^m$ be the all-zeros vector of length m, and $d := p + q$. This defines an MSD-SDP instance, and the construction can be accomplished in polynomial time. It is not difficult to see that if X is feasible for the constructed SDP problem, then all rows and columns are zeros except the one associated with $\{0\}$.

All that remains to be shown is that 3SAT is satisfiable if and only if MSD-SDP has an FR sequence of length at least d. Assume there exists a satisfying assignment for the 3SAT instance. We construct an FR sequence f of length d by specifying the exposing vector W_i at each FR step as follows.

- The first p steps are based on the truth-assignment of the p variables. For $i = 1, \ldots, p$, the exposing vector at the i-th FR step is given by

$$W_i = \begin{cases} A_i & \text{if } u_i \text{ is false} \\ A_{n+i} & \text{if } u_i \text{ is true.} \end{cases} \tag{14}$$

  By construction, these are clearly FR steps.

- The last q steps are given by

$$W_{p+j} = A_{2p+j} \text{ for } j = 1, \ldots, q.$$

  As discussed already, from the $(p+1)$-th FR step, the matrices A_{2p+j} for $j = 1, \ldots, q$ become exposing vectors. We only need to show that A_{2p+j} exposes a proper face at each iteration. By construction, this is equivalent to show that the principal submatrix of $A_{2p+j} - \sum_{i=1}^{p} W_i$ associated with the indices

$$\mathcal{E} := \{\mathcal{T}_i \mid u_i \text{ is true}\} \cup \{\mathcal{F}_i \mid u_i \text{ is false}\}$$

  is non-zero. Recall that A_{2p+j} contains exactly three positive diagonal entries corresponding to the literals in the clause c_j. Since we are given a satisfying assignment for the 3SAT instance, the clause c_j contains a true literal. Assume the positive literal u_i in c_j is true. Then $W_i = A_{n+i}$ based on the construction in (14) and thus, $(i, j) \in \mathcal{E}$. As the (i, j)-th diagonal entry is one in A_{2p+j} and zero in $\sum_{i=1}^{p} W_i$, this proves that A_{2p+j} exposes a proper face. A similar argument can be used when the negative literal $\bar{u}_i$ in c_j is true. Therefore the last q steps are also FR steps.

This yields an FR sequence of length $p + q$.

Conversely, assume that the 3SAT instance is not satisfiable. Let f be any FR sequence for the constructed SDP problem. We prove that $|f| < p + q$. Denote by l the smallest positive integer such that the rows and columns associated with $\{1, \ldots, p\}$ are removed after the l-th FR step. We show that the first l FR steps either satisfy or can be assumed to satisfy the following properties.

- Let $W = \mathcal{A}^*(y)$ be any exposing vector for some $y \in \mathbb{R}^m$ at one of the first l FR steps. It follows from the discussion after (13) that

$$y(2p + j) = 0 \text{ for } j = 1, \ldots, q. \tag{15}$$

  Thus the exposing vectors in the first l FR steps do not involve matrices $A_{2p+1}, \ldots, A_{2p+q}$.

- Applying the first condition in Definition 3.1, we see that the exposing vectors A_i for $i = 1, \ldots, 2p$ are minimal, and there exists no other minimal exposing vectors obtained from their linear combinations. As we are interested in finding an FR sequence with longer length, it is possible to assume that each of the first l FR steps is minimal by applying Theorem 3.1, i.e.,

$$W_i \in \{A_1, \ldots, A_{2p}\} \text{ for } i = 1, \ldots, l.$$

  In addition, either A_i or A_{p+i} is an exposing vector in the first l FR steps, and we are free to permute the first l FR steps. Therefore, after suitable permutation, we can assume that $l = p$ and the first p FR steps satisfy

$$W_i \in \{A_i, A_{p+i}\} \text{ for } i = 1, \ldots, p. \tag{16}$$

By assuming f is in the form of (16), it induces unambiguously a truth-assignment via the relation

$$u_i = \begin{cases} \text{false} & \text{if } W_i = A_i \\ \text{true} & \text{if } W_i = A_{n+i}. \end{cases} \tag{17}$$

After the p-th FR step, only the rows and columns associated with $\tilde{\mathcal{E}} \cup \{0\}$ are left, where

$$\tilde{\mathcal{E}} := \{\mathcal{T}_i \mid W_i = A_{p+i}\} \cup \{\mathcal{F}_i \mid W_i = A_i\}$$

The reduced SDP problem after removing rows and columns not in $\tilde{\mathcal{E}} \cup \{0\}$ is equivalent to the SDP problem defined in Example 3.3 associated with the zero-one matrix $M \in \{0,1\}^{p \times q}$ defined as follows. The (i,j)-th entry $M(i,j)$ is given by

$$M(i,j) := \begin{cases} 1 & \text{if } u_i \text{ is false and } \bar{u}_i \in c_j \\ 1 & \text{if } u_i \text{ is true and } u_i \in c_j \\ 0 & \text{otherwise.} \end{cases}$$

Up to some permutation of the columns, M is the same as the matrix $\begin{bmatrix} M_1 & \cdots & M_{\tilde{q}} \end{bmatrix}$ where each submatrix $M_j \in \{0,1\}^{p \times 2\tilde{q}}$ are columns consisting of the $2\tilde{q}$ duplication associated with the clause c_j for $j = 1, \ldots, \tilde{q}$.

Let U_j be a facial range vector corresponding to the non-zero entries in M_j. Consider the truth-assignment in (17). If c_j is unsatisfied, then M_j is an all-zeros matrix and thus $\mathrm{MSD}(F(U_j)) = 0$. If c_j is satisfied, then M_j is a matrix with at least one row of ones. Since M_j contains at most three rows of ones, we have $\mathrm{MSD}(F(U_j)) = 2\tilde{q} + 2$ by Lemma 3.3. Since the given 3SAT instance is not satisfiable, there exists at least one unsatisfied clause. It follows from Lemma 3.4 that

$$\sum_{j=1}^{\tilde{q}} \mathrm{MSD}(F(U_j)) \leq (2\tilde{q} + 2)(\tilde{q} - 1). \tag{18}$$

Applying Theorem 3.2, the remaining number of FR steps after the p-th FR step can be upper bounded by (18). This means the length of f satisfies

$$|f| \leq p + (2\tilde{q} + 2)(\tilde{q} - 1) = 2\tilde{q}^2 + p - 2 = p + q - 2 < d.$$

Therefore the constructed MSD-SDP instance has no FR sequences of length d or more. $\square$

4 Maximum Singularity Degree for Linear Programming

4.1 FRA for LP

While LP is a special case of SDP, it is pretty cumbersome to derive theories and algorithms following the same notations used for SDP. Thus we describe the FRA for LP with some new notations to simplify the presentation. The most notable difference is that we use subsets in (23) to represent exposing vectors in LP.

Let $\mathcal{K} = \mathbb{R}^n$. Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ be given. Then (1) becomes a *linear programming* (LP) problem, i.e.,

$$F(\mathbb{R}^n) := \mathbb{R}^n \cap \mathcal{L} \text{ where } \mathcal{L} := \{x \in \mathbb{R}^n \mid Ax = b\}. \tag{19}$$

As before, we provide $\mathcal{L}^\perp$, the faces of $\mathbb{R}^n_+$ and its dual faces in a more concrete form.

- The orthogonal complement of $\mathcal{L}$ is given by

$$\mathcal{L}^* = \left\{ A^T y \in \mathbb{R}^n \mid b^T y = 0 \right\}.$$

- The non-empty faces $\mathcal{K}_1$ of the non-negative orthant $\mathbb{R}^n_+$ can be characterized by subsets $V \subseteq \{1, \ldots, n\}$ via the relation

$$\mathcal{K}_1 = \left\{ x \in \mathbb{R}^n_+ \mid x(i) = 0 \text{ for } i \notin V \right\}.$$

  For LP, the subset V is a *facial range vector* associated with the face $\mathcal{K}_1$. Similar to SDP, we write V and $\mathcal{K}_1$ interchangeably without causing confusion. The dual cone of the face $\mathcal{K}_1$ is

$$\mathcal{K}_1^* = \{ w \in \mathbb{R}^n \mid w(V) \geq 0 \}. \tag{20}$$

The FRA applied to LP can be described as follows. At the i-th iteration, the face $\mathcal{K}_i$ is associated with some facial range vector $V_i \subseteq \{1, \ldots, n\}$. In particular, we always set $V_1 := \{1, \ldots, n\}$. The set of exposing vectors $\mathcal{K}_i^* \cap \mathcal{L}^\perp$ is

$$D(V_i) := \left\{ w \in \mathbb{R}^n \mid w = A^T y, w(V_i) \geq 0 \text{ and } b^T y = 0 \text{ for some } y \in \mathbb{R}^m \right\}. \tag{21}$$

The exposing vector $w_i \in D(V_i)$ exposes a proper face of $\mathcal{K}_i$, if $w_i \notin \mathcal{K}_i^\perp$ which can be written as

$$w_i(V_i) \neq 0. \tag{22}$$

To simplify the presentation, we define a subset associated with w_i. The exposing vector $w_i \in D(V_i)$ induces a unique subset

$$S_i = \operatorname{supp}(w_i) \cap V_i \subseteq \{1, \ldots, n\}. \tag{23}$$

We abuse the notation and call both the vector w_i and the subset S_i exposing vectors of V_i. As we can apply standard set operations like intersection and exclusion straightforwardly, this greatly eases the presentation. For example, the update of the facial range vector becomes

$$V_{i+1} = V_i \backslash S_i.$$

We follow the same convention as for SDP. For example, after introducing an FR sequence f, we will use V_i, w_i, S_i directly without defining them again.

4.2 Minimal FR sequences for LP

The notion of minimality can also be simplified for LP as follows. Let $V \subseteq \{1, \ldots, n\}$ be a given facial range vector. We say an exposing vector $S \subseteq V$ is *minimal* for the face V, if there does not exist an exposing vector S' of V such that $\emptyset \neq S' \subsetneq S$. Let f be an FR sequence for (19). If S_i is minimal in V_i, then we say the i-th FR step is minimal. If all FR steps are minimal, then we say f is minimal.

In this section, we state some useful technical lemmas for LP as special cases of the established lemmas for SDP. We also highlight new lemmas that apply only to LP and do not hold for SDP. The next two results are special cases of Lemmas 3.1 and 3.2 for SDP. They play a key role in the analysis of our results for LP. A self-contained proof is provided for Lemma 4.2.

Lemma 4.1. *Let S be an exposing vector of V, and u its associated vector in* (23)*. Assume S is minimal. For any $u \in \mathcal{L}^\perp$ such that $u(S) \neq 0$ and $u(V\backslash S) = 0$, we have $w(S) = u(S)$ up to some non-zero scaling.*

Lemma 4.2. *Let S be exposing vector of V. For any $U \subseteq V$, the intersection $S \cap U$ is an exposing vector of U.*

Proof. Let w be the vector corresponding to S. Then $w(S) > 0$ and $w(V\backslash S) = 0$. Since $S \cap U \subseteq S$, we have that $w(S \cap U) > 0$. In addition, $U \subseteq V$ implies that $U\backslash(S \cap U) = U\backslash S \subseteq V\backslash S$ and thus $w(U\backslash(S \cap U)) = 0$. This shows that $S \cap U$ is an exposing vector of U. □

Corollary 4.1. *Let f be an FR sequence. If S be an exposing vector of V_i, then $S \cap V_j$ is also an exposing vector of V_j for any $j > i$.*

Proof. Since $V_j \subsetneq V_k$ for any $j > k$, the statement follows by applying Lemma 4.2. □

If an exposing vector S of V_1 has size one, i.e., $|S| = 1$, then any minimal FR sequence must use S as an exposing vector in one of its FR step. This is stated formally in the next result.

Lemma 4.3. *Let f be a minimal FR sequence. If $\{t\}$ is an exposing vector of V_k, then $S_j = \{t\}$ for some $j \geq k$.*

Proof. The subset $\{t\}$ is an exposing vector of V_k implies that $j \geq k$. Since $t \in S_j \subseteq V_j$, we have $\{t\}$ is an exposing vector of V_j Corollary 4.1. As f is minimal, we must have $S_j = \{t\}$. □

Lemma 4.3 is a special property of FR sequences for LP, if exposing vectors of size one for LP are generalized as rank one exposing vectors for SDP. Consider the spectrahedron F given by

$$A_1 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_3 := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } b := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3.$$

The exposing vector A_1 is rank one and thus minimal in the first FR step. However, the sequence (A_3, A_2) is a minimal FR sequence and it does not contain A_1.

As LP is a special case of SDP, minimality is also a necessary condition for being the longest FR sequence in LP, see Theorem 3.1. We sketch a self-contained proof here for Theorem 4.1. Let f be an FR sequence of length d. If S_j is not minimal in V_j, then there exists an exposing vector I of V_j such that $\emptyset \neq I \subsetneq S_j$. We claim that it is possible to split S_j into I and $S_j\backslash I$ as two FR steps, and thus obtain a longer FR sequence. To be more precise, we define the sequence g as in Figure 2. The j-th iteration is an FR step as I is an exposing vector of V_j by assumption. Since S_j is an exposing vector of V_j, we have $S_j \cap (V_j\backslash I)$ is an exposing vector of $V_j\backslash I$ by Lemma 4.2. Since $S_j \subseteq V_j$ and $I \subsetneq S_j$, we have $S_j \cap (V_j\backslash I) = S_j\backslash I$ is non-empty. This shows that the $(j+1)$-th iteration is also an FR step. The facial range vector at the $(j+2)$-th iteration is $(V_j\backslash I)\backslash(S_j\backslash I) = V_j\backslash S_j$ as $I \subsetneq S_j$. This yields a longer FR sequence g.

iteration	1	$\cdots$	$j-1$	j	$j+1$	$j+2$	$\cdots$	$d+1$
face	V_1	$\cdots$	V_{j-1}	V_j	$V_j\backslash I$	V_{j+1}	$\cdots$	V_d
exposing vector	S_1	$\cdots$	S_{j-1}	I	$S_j\backslash I$	S_{j+1}	$\cdots$	S_d

Figure 2: The FR sequence g obtained by splitting S_j into I and $S_j\backslash I$.

4.3 Operations that Preserve Minimality

In this section, we present two operations called *swapping* and *deletion* which preserve the minimality of FR sequences. These operations only work for LP, which is part of the reasons why computing MSD for LP is easier than SDP. We provide an application of these operations in Section 4.4 showing that minimal FR sequences are also the longest for LP.

4.3.1 Swapping FR Steps

In this section, we study the operation of swapping adjacent steps S_j and S_{j+1} in a minimal FR sequence f of length d. After swapping, this yields a new sequence g. We will present a sufficient condition for g to remain a minimal FR sequence. And when this sufficient condition is satisfied, the faces associated with g expressed in terms of faces in f can be summarized as follows.

iteration	1	$\cdots$	$j-1$	j	$j+1$	$j+2$	$\cdots$	d
face	V_1	$\cdots$	V_{j-1}	V_j	$V_j \backslash S_{j+1}$	V_{j+2}	$\cdots$	V_d
exposing vector	S_1	$\cdots$	S_{j-1}	S_{j+1}	S_j	S_{j+2}	$\cdots$	S_d

Figure 3: The sequence g obtained by swapping S_j and S_{j+1}.

Lemma 4.4 (Swapping)**.** *Let f be a minimal FR sequence for* (19)*. Let g be the sequence in Figure 3 obtained by swapping S_j and S_{j+1}. If any one of the following conditions is met, then g is also a minimal FR sequence for* (19)*.*

- $|S_j| = 1$.
- $|S_{j+1}| = 1$ *and S_{j+1} is an exposing vector of V_j.*

Proof. In both cases, the sequences f and g are identical in the first $(j-1)$ steps. Applying Fact 3.1, we can assume $j = 1$ without loss of generality. Next we discuss two different conditions separately.

- Assume the first condition $|S_1| = 1$ holds. Up to some labeling, we can assume that $S_1 = \{1\}$ and $S_2 = \{2, \ldots, k\}$ for some $k \geq 2$. As f is an FR sequence, S_1 and S_2 are exposing vectors in V_1 and V_2, respectively. Let w and u be the vectors associated with S_1 and S_2 as in (23), respectively. Then w and u satisfy

  $$\begin{array}{rclrclrcl} w(1) & = & 1, & w(2), \ldots, w(k) & = & 0, & w(k+1), \ldots, w(n) & = & 0, \\ u(1) & \in & \mathbb{R}, & u(2), \ldots, u(k) & > & 0, & u(k+1), \ldots, u(n) & = & 0. \end{array}$$

  Define $h := u - u(1)w$. Then the following holds

  - $h(1) = u(1) - u(1)w(1) = u(1) - u(1) = 0$.
  - For $i \in \{2, \ldots, k\}$, we have $h(i) = u(i) - u(1)w(i) = u(i) > 0$.
  - For $i \geq k+1$, we have $h(i) = u(i) - u(1)w(i) = 0$.

  We obtain that $h(S_2) > 0$ and $h(V_1 \backslash S_2) = 0$, which means $S_2 \neq \emptyset$ is an exposing vector of V_1. Thus the g_1 is an FR step.

  The subset S_2 is also minimal for V_1. If not, there exists an exposing vector S for V_1 such that $\emptyset \neq S \subsetneq S_2$. As $1 \notin S$, we have $S \cap V_2 = S$ and thus S is also an exposing vector of V_2

by Lemma 4.2. But this means S_2 is not minimal for V_2, and this is a contradiction to the minimality of f. Thus g_1 is a minimal FR step.

The facial range vector in g_2 is $V_1 \backslash S_2$. By Lemma 4.2, we have S_1 is an exposing vector of $V_1 \backslash S_2$ in g_2, as $S_1 \cap (V_1 \backslash S_2) = S_1$. In addition, $|S_1| = 1$ and thus the g_2 is a minimal FR step. The facial range vector in g_3 is $V_1 \backslash (S_1 \cup S_2) = V_3$.

- For the second condition, we assume that $S_1 = \{1, \ldots, k\}$ for some $k \geq 1$ and $S_2 = \{n\}$ up to some relabeling. By assumption, S_2 exposes a proper face for V_1. As $|S_2| = 1$, we have g_1 is a minimal FR step. The facial range vector at g_2 is $V_1 \backslash \{n\} = \{1, \ldots, n-1\}$. It remains to prove that g_2 is a minimal FR step.

  By Lemma 4.2, we have S_1 is an exposing vector of $V_1 \backslash \{n\}$ as $S_1 \cap (V_1 \backslash \{n\}) = S_1$. Thus we only need to show that S_1 is minimal in $V_1 \backslash \{n\}$. Assume this is not the case. There exists an exposing vector S for $V_1 \backslash \{n\}$ such that $S = \{1, \ldots, l\} \subsetneq S_1$ for some $1 < l < k$. Let w be the vector associated with S as in (23), i.e.,

  $$w(1), \ldots, w(l) > 0, w(l+1), \ldots, w(n-1) = 0, w(n) \in \mathbb{R}. \tag{24}$$

  If $w(n) = 0$, then $w(l+1), \ldots, w(n) = 0$ and thus $S \subsetneq S_1$ is an exposing vector of V_1 as well. This is a contradiction to S_1 is minimal in V_1 in f.

  Assume $w(n) \neq 0$. By assumption, $S_2 = \{n\}$ is an exposing vector of V_1, and let u be the associated vector as in (23). Up to some positive scaling, the vector u is

  $$u(1), \ldots, u(n-1) = 0, u(n) = 1. \tag{25}$$

  Define $h := w - w(n)u$. Then the vector h satisfies

  $$h(S) > 0 \text{ and } h(V_1 \backslash S) = 0. \tag{26}$$

  This shows that S is an exposing vector of V_1 such that $\emptyset \neq S \subsetneq S_1$, and thus S_1 is not minimal in V_1. This is a contradiction. We conclude that S_1 is minimal in $V_1 \backslash \{n\}$ which shows that g_2 is also a minimal FR step. The facial range vector at g_3 is $(V_1 \backslash S_2) \backslash S_1 = V_3$.

As $g_i = f_i$ for $i \geq 3$ are minimal FR steps, this shows that g is a minimal FR sequence in both cases. □

4.3.2 Deleting Variables

In this section, we discuss the effects of removing a variable from the linear system on its FR sequences. Let f be a minimal FR sequence of length d for the set F in (19). For any fixed $t \in \{1, \ldots, n\}$, let g be the subsequence of f with respect to $U := V_1 \backslash \{t\}$, see (10). Recall that g is an FR sequence for the system $F(U)$, and the system $F(U)$ can be viewed as the linear system obtained from removing the variable x_t. We study necessary and sufficient conditions for g to be a minimal FR sequence for $F(U)$.

Let S_j be the exposing vector containing t in the sequence f. We distinguish two cases based on the size of S_j. The first case is when $|S_j| = 1$. Assume $S_j = \{t\}$. Then the FR sequence g has the form in (4). As S_j is removed, the length of g is $d-1$ which is one less than the length of f. In Lemma 4.5, we provide a necessary and sufficient condition for g to be minimal.

iteration	1	$\cdots$	$j-1$	j	$\cdots$	$d-1$
face	$V_1\backslash\{t\}$	$\cdots$	$V_{j-1}\backslash\{t\}$	V_{j+1}	$\cdots$	V_d
exposing vector	S_1	$\cdots$	S_{j-1}	S_{j+1}	$\cdots$	S_d

Figure 4: The sequence g obtained by deleting $S_j = \{t\}$.

Lemma 4.5. *(Deletion-Case1) Let f be a minimal FR sequence for* (19). *Assume $S_j = \{t\}$. Let g be the subsequence of f with respect to $V_1\backslash\{t\}$. It holds that g is minimal if and only if $\{t\}$ is an exposing vector of V_1.*

Proof. If $\{t\}$ is an exposing vector of V_1, then we can assume that $S_1 = \{t\}$ in f by applying Lemmas 4.3 and 4.4. In this case, it is clear that g is minimal. Conversely, assume $\{t\}$ is not an exposing vector of V_1. Then $S_j = \{t\}$ for some $j \geq 2$. We will show that S_{j-1} is not minimal in g and thus g is not minimal. In view of Fact 3.1, we can assume without loss of generality that $j = 2$. Up to some relabeling, suppose that $S_1 = \{1, \ldots, k\}$ for some $k \geq 1$, and $S_2 = \{n\}$ with $t = n$. Since $S_2 = \{n\}$ is an exposing vector of $V_2 = V_1\backslash S_1 = \{k+1, \ldots, n\}$ in f, the associated vector w for S_2 in (23) satisfies

$$w(1), \ldots, w(k) \in \mathbb{R}, w(k+1), \ldots, w(n-1) = 0, w(n) > 0.$$

In addition, if $w(S_1) = 0$, then w implies that $\{n\}$ is an exposing vector of V_1 in f which is a contradiction. Thus the vector w also satisfies

$$w(S_1) \neq 0.$$

Since S_1 is an exposing vector of V_1 in f, the associated vector u satisfies

$$u(1), \ldots, u(k) > 0, u(k+1), \ldots, u(n-1) = 0, u(n) = 0.$$

Suppose for the sake of contradiction that S_1 is minimal for $V_1\backslash\{n\}$ in g. But then, since $w(S_1) \neq 0$ and $w(V_1\backslash(S_1 \cup \{n\})) = u(V_1\backslash(S_1 \cup \{n\})) = 0$, we can apply Lemma 4.1 to get $w(S_1) = \alpha u(S_1)$ for some non-zero constant $\alpha \neq 0$. Define $h := w - \alpha u$. Then we have

$$h(1), \ldots, h(n-1) = 0, h(n) > 0.$$

This implies that h, or equivalently, $\{n\}$ is an exposing vector of V_1. This is a contradiction. Thus S_1 is not minimal in $V_1\backslash\{n\}$, i.e., the FR step g_1 is not minimal. □

From Lemma 4.5, we know that g is not a minimal FR sequence if $\{t\}$ is not an exposing vector of V_1. However it is not hard to see that the FR step g_i remains minimal for every $i \geq k$, where k is the smallest positive integer such that $\{t\}$ is an exposing vector of V_k. Thus at most $k-1$ the FR steps in g are not minimal.

In the second case, we assume $t \in S_j$ and $|S_j| \geq 2$. The FR sequence g is described in Figure 5. As $|S_j| > 2$, the length of g is the same as the length of f. We will show that g is always minimal.

iteration	1	$\cdots$	$j-1$	j	$j+1$	$\cdots$	d
face	$V_1\backslash\{t\}$	$\cdots$	$V_{j-1}\backslash\{t\}$	$V_j\backslash\{t\}$	V_{j+1}	$\cdots$	V_d
exposing vector	S_1	$\cdots$	S_{j-1}	$S_j\backslash\{t\}$	S_{j+1}	$\cdots$	S_d

Figure 5: The sequence g obtained by deleting the variable x_t, assuming $t \in S_j$.

Lemma 4.6. *(Deletion-Case2) Let f be a minimal FR sequence for* (19)*. Assume $t \in S_j$ and $|S_j| \geq 2$. Let g be the subsequence of f with respect to $V_1\backslash\{t\}$. Then g is minimal and $|g| = |f|$.*

Proof. Assume $S_i\backslash\{t\}$ is not minimal in $V_i\backslash\{t\}$ for some $i \leq j$. Then there exists an exposing vector S of $V_i\backslash\{t\}$ such that $\emptyset \neq S \subsetneq (S_i\backslash\{t\})$. Let w be the vector associated with S as in (23). Then w satisfies

$$w(S) > 0, w(V_i\backslash(S \cup \{t\})) = 0, w(t) \in \mathbb{R}. \tag{27}$$

If $w(t) = 0$, then $w(S) > 0$ and $w(V_i\backslash S) = 0$. This means that S is also an exposing vector of V_i in f, which is a contradiction to S_i is minimal for V_i in f. Thus $w(t) \neq 0$. We distinguish two cases based on i.

- Assume $i < j$. Then $V_j \subsetneq V_i\backslash S_j \subsetneq V_i\backslash S$ and thus $V_j\backslash\{t\} \subsetneq V_i\backslash(S \cup \{t\})$. The last inclusion and (27) imply that $w(V_j\backslash\{t\}) = 0$. Multiply w by -1 if necessary, we can assume $w(t) > 0$. Then w shows that $\{t\}$ is an exposing vector of V_j in f, see (23). As $t \in S_j$, this means S_j is not minimal which is a contradiction.

- Assume $i = j$. Recall that S_j is an exposing vector of V_j in f. Let u be the vector associated with S_j as in (23). Then u satisfies

$$u(S_j) > 0, u(V_j\backslash S_j) = 0.$$

  Since $S \subsetneq S_j\backslash\{t\}$, we have $S \cup \{t\} \subsetneq S_j$ and thus $V_j\backslash(S \cup \{t\}) \supsetneq V_j\backslash S_j$. The last inclusion and (27) imply that $w(V_j\backslash S_j) = 0$. As S_j is minimal for V_j, we can apply Lemma 4.1 to obtain $w(S_j) = \alpha u(S_j)$ for some constant $\alpha \neq 0$. Let $k \in S_j\backslash\{t\}$ and $k \notin S$. We have $0 = w(k) = \alpha u(k) \neq 0$ which is a contradiction.

It is clear that g_i is a minimal FR step for $i > j$. Thus g is a minimal FR sequence.

□

4.4 The Longest FR Sequences

In this section, we show a striking difference between LP and SDP in the characterization of the longest FR sequences. For LP, any minimal FR sequence is also one of the longest FR sequences.

Theorem 4.1. *Let f be an FR sequence for* (19)*. We have f is minimal if and only if f is one of the longest FR sequences.*

Proof. If f is one of the longest FR sequences, then f is minimal by Theorem 3.1. Conversely, we show that any two minimal FR sequences f and g have the same length. Assume this is not the case, we pick a counterexample with the smallest number of variables. Denote by S_i and T_i the exposing vector at the i-th FR step in f and g, respectively. We prove that the sequences f and g have certain special properties in this counterexample below.

- Property 1. If $k \in S_i \cap T_j \neq \emptyset$ for some i and j, then $S_i = \{k\}$ or $T_j = \{k\}$. If this is not case, then $S_i\backslash\{k\} \neq \emptyset$ and $T_j\backslash\{k\} \neq \emptyset$. By Lemma 4.6, when we remove the variable x_k from the linear system, the corresponding subsequences are still minimal FR sequences and their lengths do not change. This yields a smaller counterexample and thus a contradiction.

- Property 2. The subset $\{k\}$ is not an exposing vector of V_1 for any $k = 1, \dots, n$. Assume this is not the case. By Lemmas 4.3 and 4.4, we can assume that $S_1 = \{k\}$ and $T_1 = \{k\}$. Applying Lemma 4.5, we can remove x_k to obtain a smaller linear system. The corresponding subsequences remain minimal, and their lengths are both decreased by one and thus different. This is a contradiction.

Without loss of generality, assume that $|f| > |g|$. If the exposing vector at each FR step in g has size one, then $|f| > |g| = \text{MSD}(F)$ which is not possible. Thus, there exists an index j such that $|T_j| \geq 2$ at the j-th FR step in g. Let $S_i = \{t\}$ be the exposing vector at the i-th FR step in f such that $t \in T_j$, see Property 1. Assume we delete the variable x_t from the linear system (19) to obtain the smaller linear system $F(U)$ with $U = V_1 \backslash \{t\}$. Let $\tilde{f}$ and $\tilde{g}$ be the subsequences of f and g with respect U as in Lemma 4.5 and Lemma 4.6, respectively. Then $\tilde{f}$ and $\tilde{g}$ are FR sequences for $F(U)$ satisfying $|\tilde{f}| = |f| - 1$ and $|\tilde{g}| = |g|$. As $|f| > |g|$, we have $|\tilde{f}| \geq |\tilde{g}|$.

By Lemma 4.6, the FR sequence $\tilde{g}$ remains minimal for $F(U)$. We prove that $\tilde{f}$ is also minimal for $F(U)$. Assume this is not the case. Then we can iteratively split the non-minimal exposing vectors as described in the paragraph before Figure 2 until it becomes minimal. But after each splitting, the length of $\tilde{f}$ is increased by one. This results in two minimal FR sequences with different lengths for the smaller linear system $F(U)$, which is not possible. Thus $\tilde{f}$ is also minimal. Applying Lemma 4.5, we have $\{t\}$ is an exposing vector of V_1. This is a contradiction to Property 2. Thus the counterexample does not exist. □

Based on Theorem 4.1, we can generate a longest FR sequence by ensuring the exposing vector at each FR step is minimal. This can be achieved easily by solving a number of auxiliary LP problems. Thus we also obtain a polynomial-time algorithm for generating the longest FR sequences.

4.5 Implicit Problem Singularity and Degeneracy

In this section, we investigate a parameter called *Implicit Problem Singularity (IPS)* for a given set (1) introduced by Im and Wolkowicz in [10]. We show an equivalence between IPS and MSD for LP, and provide applications of this equivalence. The parameter IPS is motivated by the observation that there exists at least one redundant constraint after each FR step, see [20]. To be more precise, let $I \subseteq \{1, \dots, n\}$ be a given fixed subset. Denote by A_I the submatrix of A consisting of the columns associated with elements in I. Similarly, x_I denotes the entries in x associated with I.[1] Let F be as given in (19). Recall that if $S \subseteq V := \{1, \dots, n\}$ is an exposing vector of V, then there exists a vector y such that $w = A^T y$ and

$$w(S) > 0, w(V \backslash S) = 0 \text{ and } y^T b = 0. \tag{28}$$

Let $I := V \backslash S$. Then $A_I x_I = b, x_I \geq 0$ is the reduced linear system after the FR step with exposing vector S. The equality constraints $A_I x_I = b$ in the reduced system is linearly dependent, as the vector y in (28) yields a non-trivial linear combination of them equal to zero, i.e., $y^T A_I = 0$ and $y^T b = 0$. This means there exists at least one redundant equality constraints after each FR step.

Assume that at the end of an FR sequence, the final reduced linear system is $A_I x_I = b$ for some $I \subseteq \{1, \dots, n\}$. Then the implicit singularity degree for F is defined as

$$\text{IPS}(F) := m - \text{rank } A_I. \tag{29}$$

[1]The notation x_I is the same as $x(I)$, but it is more convenient for the discussion here.

The next result shows that after taking a minimal FR step, the number of redundant equality constraints in the reduced system is exactly one.

Theorem 4.2. *Assume $A \in \mathbb{R}^{m\times n}$ has full row rank. Let S be minimal for $V := \{1,\dots,n\}$, and $I := V\backslash S$. Then there is exactly one redundant equality constraint in the linear system $A_I x_I = b$.*

Proof. Assume there are more than one redundant equality constraints in $A_I x_I = b$. Without loss of generality, we assume the $(m-1)$-th and the m-th equations are linear combinations of the first $m-2$ equality constraints, respectively. Thus there exist non-zero vectors y and z in $\mathbb{R}^m$ such that

$$\begin{array}{c} y^T A_I = 0, y^T b = 0, y(m-1) = 1, y(m) = 0. \\ z^T A_I = 0, z^T b = 0, z(m-1) = 0, z(m) = 1. \end{array}$$

Note that y and z are linearly independent. As A has full row rank, the vectors $w := A^T y$ and $u := A^T z$ are both non-zero and linearly independent of each other. However, as $w, u \in \mathcal{L}^\perp$ with $w(S), u(S) \neq 0$ and $w(V\backslash S) = u(V\backslash S) = 0$, it follows from Lemma 4.1 that w and u are linearly dependent. This is a contradiction. □

It is worth to note that Theorem 4.2 only works for LP. In Example 3.1, we show that there can be more than one redundant constraints for SDP. Next we note that it is always possible to assume that A has full row rank in the study of MSD.

Lemma 4.7. *Let $F := \{x \in \mathbb{R}^n_+ \mid Ax = b\} \neq \emptyset$ for some $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$. Assume $r :=$ rank $A < m$. Let $\tilde{A}x = \tilde{b}$ be any r linearly independent constraints in $Ax = b$. Define $\tilde{F} := \{x \in \mathbb{R}^n_+ \mid \tilde{A}x = \tilde{b}\}$. Then MSD($F$) = MSD($\tilde{F}$).*

Proof. By assumption, there exists $Q \in \mathbb{R}^{r\times m}$ such that $A^T = \tilde{A}^T Q$ and $b^T = \tilde{b}Q$. Denote by $D = \left\{A^T y \geq 0 \mid b^T y = 0\right\}$ and $\tilde{D} := \left\{\tilde{A}^T \tilde{y} \geq 0 \mid b^T \tilde{y} = 0\right\}$ the set of exposing vectors for F and $\tilde{F}$, respectively. In view of Fact 3.1, it suffices to show that $D = \tilde{D}$ for the first FR step. It is trivial to see that $\tilde{D} \subseteq D$. For the converse direction, let $w \in D$. Then $w = A^T y$ and $b^T y = 0$ for some vector $y \in \mathbb{R}^m$. Define $\tilde{y} := Qy \in \mathbb{R}^r$. We have $w = A^T y = \tilde{A}^T Q y = \tilde{A}^T \tilde{y}$ and $0 = b^T y = \tilde{b}^T Q y = \tilde{b}^T \tilde{y}$. Thus $w \in \tilde{D}$. □

Theorem 4.3. *Let F be as given in* (19)*. Let $I = \{i \mid x_i > 0, x \in F\}$ be the set of irredundant variables. Then MSD(F) = rank A − rank A_I.*

Proof. We remove any redundant constraints in $Ax = b$ so that A has full row rank. Let f be a minimal FR sequence for F, see Lemma 4.7. After the first FR step, we find exactly one redundant equation in the reduced system. Applying Lemma 4.7 again, we can remove the redundant constraint in the reduced system without affecting the remaining FR steps. We repeat this procedure for all FR steps.

At the beginning, we remove $m -$ rank A constraints to meet the full row rank assumption. After each FR step, we remove a single redundant equation. In total, we remove $m -$ rank $A + |f|$ constraints which must be equal to the total number of redundant equations in $A_I x_I = b$, i.e., $m -$ rank $A + |f| = m -$ rank A_I. This yields $|f| =$ rank $A -$ rank A_I. By Theorem 4.1, we have $|f| = \text{MSD}(F)$ and this finishes the proof. □

In fact, Theorem 4.3 provides an alternative proof for Theorem 4.1 as it shows that the minimal FR sequences for (19) have the same length via the relation $|f| = \text{rank } A - \text{rank } A_I$. As a direct corollary of Theorem 4.3, we see that $\text{MSD}(F) = \text{IPS}(F)$ whenever the data matrix A has full row rank. This result is stated below.

Corollary 4.2. *Let F be as given in* (19)*. It holds that $MSD(F) = IPS(F) - m + \text{rank } A$. In particular, if A has full row rank, then $MSD(F) = IPS(F)$.*

Proof. It follows directly from Theorem 4.3 and the definition of IPS in (29). □

Next, we discuss the interplay between FRA and degeneracy. There are three sources of degeneracy, namely, *weakly redundant constraints*, *implicit equalities* and *geometry*, see details in [6]. Im and Wolkowicz [10] show that the loss of strict feasibility implies that every basic feasible solution is degenerate, and this can be viewed as degeneracy from implicit equalities.

Denote by $\nu(x)$ the number of different ways to use n linearly independent active constraints for identifying x as a basic feasible solution. If a basic feasible solution x is degenerate, then $\nu(x) > 1$ and thus the simplex algorithm may stuck at the same basic feasible solution without making any progress. A large MSD may imply that $\nu(x)$ is huge. To be more precise, any FR sequence yields a lower bound for $\nu(x)$ as follows. Let f be an FR sequence of length d for (19), and S_i the exposing vector at the i-th FR step. It holds that

$$\nu(x) \geq \prod_{i=1}^{d} |S_i|. \tag{30}$$

Clearly, we expect to obtain a stronger lower bound from (30) if f is a longer FR sequence, e.g., $d = \text{MSD}(F)$. Unfortunately the lower bound also depends on the choice of f, and it can vary significantly for different FR sequences. We provide an exponential separation between the lower bounds (30) obtained from different FR sequences next. Let I_p be the $p \times p$ identity matrix. Define

$$A := \begin{bmatrix} 0 & 1_{2p}^T \\ I_{2p} & M \end{bmatrix} \in \mathbb{R}^{(2p+1)\times 4p} \text{ where } M := I_p \otimes \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \in \mathbb{R}^{2p\times 2p}.$$

Let $b := 0 \in \mathbb{R}^{2p+1}$. Note that $\text{MSD}(F) \leq 2p + 1$. We construct two different FR sequences below.

- The FR sequence f has $2p - 1$ exposing vectors of size 2, and two exposing vectors of size 1; they are given by $S_1 = \{1, 2\}, \ldots, S_{p-1} = \{2p - 3, 2(p - 1)\}$, $S_p = \{2p - 1\}, S_{p+1} = \{2p\}$ and $S_{p+2} = \{2p + 1, 2p + 2\}, \ldots, S_{2p+1} = \{4p - 1, 4p\}$.
- The FR sequence g has 1 exposing vector of size $2p$, and $2p$ exposing vectors of size 1; they are given by $T_1 = \{2p + 1, \ldots, 4p\}$, $T_i = \{i - 1\}$ for $i = 2, \ldots, 2p + 1$.

The lengths of f and g are both $2p + 1$, and thus they are the longest FR sequences. But f yields an exponential lower bound $\nu(x) \geq 2^{2p-1}$, and g only yields a linear bound $\nu(x) \geq 2p$.

Finally, we discuss potential applications of MSD related to FRA and degeneracy. Implementing FRA to regularize LP and SDP problems is challenging because it involves solving an optimization problem of about the same size at each FR step. Therefore, a primary concern in applying FRA is how to efficiently achieve a partial FR sequence at reasonable costs. For SDP, numerous studies have tackled this challenge, as indicated by research from Permenter, Zhu, Hu, and others, see

[9, 19, 25]. Im and Wolkowicz [10] propose a strategy of applying FRA for LP. Let $k := \mathrm{MSD}(F)$, and $\mathcal{S} := \{i \mid x_i = 0 \text{ for every } x \in P\}$ be the set of variables fixed at zero. Let x be any basic feasible solution with the set of basic variables $\mathcal{B}$. They show that $|\mathcal{B} \cap \mathcal{S}| \geq k$, implying that every basic feasible solution is degenerate whenever $k > 1$. Hence, if x is degenerate, at least one basic variable must be zero. They employ specific pivoting steps to determine whether any zero basic variables are in $\mathcal{S}$, and if so, eliminate them from the problem. This approach can be integrated with the two-phase simplex algorithm. In our work, we additionally discover from Theorem 4.2 that $|\mathcal{B} \cap \mathcal{S}| = k$ under the assumption that implicit equalities in $\mathcal{S}$ are the sole cause of degeneracy at x. In such instances, we can remove all zero basic variables without incurring extra computational costs. We understand that this assumption is strict and leave its further investigation for future research.

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