The Maximum Singularity Degree for Linear and Semidefinite Programming

Hao Hu*

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Abstract

Facial reduction (FR) is an important tool in linear and semidefinite programming, providing both algorithmic and theoretical insights into these problems. The maximum length of an FR sequence for a convex set is referred to as the maximum singularity degree (MSD). We observe that the behavior of certain FR algorithms can be explained through the MSD. Combined with recent applications of the MSD in the literature, this motivates our study of its fundamental properties in this paper.

In this work, we show that an FR sequence has the longest length implies that it satisfies a certain minimal property. For linear programming (LP), we introduce two operations for manipulating the longest FR sequences. These operations enable us to characterize the longest FR sequences for LP problems. To study the MSD for semidefinite programming (SDP), we provide several useful tools including simplification and upper-bounding techniques. By leveraging these tools and the characterization for LP problems, we prove that finding a longest FR sequence for SDP problems is NP-hard. This complexity result highlights a striking difference between the shortest and the longest FR sequences for SDP problems.

Key Words: semidefinite program, linear program, facial reduction, singularity degree, maximum singularity degree, exposing vector

*School of Mathematical and Statistical Sciences, Clemson University, Clemson, USA; Emai: hhu2@clemson.edu;
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1 Introduction

Let $L \cap K$ be a convex set defined as the intersection of an affine subspace L and a closed convex cone K. The description of $L \cap K$ is important in efficiently solving optimization problems constrained by $L \cap K$. A poor description of $L \cap K$ can lead to numerical issues, resulting in unreliable outcomes. To address these challenges, Borwein and Wolkowicz introduced a theoretical framework known as the *Facial Reduction Algorithm* (FRA) in their works [1–3]. The FRA resolves numerical difficulties arising from the loss of strict feasibility in the problem formulation. Since its inception, the FRA has significantly improved the computational efficiency and stability of solving numerous *semidefinite programming* (SDP) problems, while also providing deeper insights into the geometric properties of the associated convex sets.

From a computational perspective, the effectiveness of the FRA in numerically solving SDP problems was first demonstrated through its application to the quadratic assignment problem in the study by Zhao et al. [33]. Recent advancements in first-order methods, especially when combined with the FRA, have enhanced the ability to solve SDP relaxations for large-scale problems, as shown in [10, 11, 21, 32, 34].

From a theoretical perspective, the lengths of FR sequences offer valuable insights into various important questions. Sturm [26] introduces a parameter called the *singularity degree* (SD) which is the minimum length among all FR sequences for $L \cap K$. The SD provides an important error bound for $L \cap K$, and has been instrumental in deriving significant theoretical results for different convex cones, as discussed in [8, 16, 18, 23, 27]. The maximum length among all FR sequences for $L \cap K$ is called the *maximum singularity degree* (MSD). Im and Wolkowicz [14] use the MSD to derive an improved Barvinok-Pataki bound on SDP rank. Im and Wolkowicz [13] show that the MSD can negatively affect the performance of numerical algorithms for *linear programming (LP)*. Some more related works can be found in [15, 19]. Additionally, we provide a new application of the MSD in Section 2.4 to further motivate our studies.

The main contribution of this work is to address the problem of constructing the longest FR sequences for linear and semidefinite programming. While the construction of the shortest FR sequences has been thoroughly investigated, extending this analysis to the longest FR sequence provides a natural and important direction for further investigation.

The paper is organized as follows. We provide some preliminary materials in Section 2. We also discuss a novel application of the MSD in Section 2.4, which serves to further motivate our study. In Section 3, we demonstrate that an FR sequence having the longest length necessarily satisfies a minimal property, and we establish several key facts about minimal FR sequences. In Section 4, we investigate the longest FR sequences for LP problems. Specifically, we introduce two operations for manipulating these sequences. We apply these operations to prove that an FR sequence has the longest length if and only if it satisfies the minimal property for LP problems. In Section 5, we provide counterexamples showing that, for SDP problems, an FR sequence satisfying the minimal property does not necessarily have the longest length. Furthermore, we prove that finding one of the longest FR sequences for SDP problems is NP-hard, a result that highlights a striking difference between these two parameters, the MSD and the SD.

Notation: Let $\mathbb{R}^n$ denote the n-dimensional real space, and let $\mathbb{R}^n_+$ represent the n-dimensional real space with nonnegative entries. Consider a finite set $\mathcal{N}$ with n elements. Let $w \in \mathbb{R}^n$. When the entries of w are indexed by the elements of $\mathcal{N}$, we may alternatively write $w \in \mathbb{R}^{\mathcal{N}}$. For any subset $S \subseteq \mathcal{N}$, let $w(S)$ denote the subvector of w consisting of entries indexed by the elements in S. For example, if $\mathcal{N} = \{1, \ldots, n\}$ follows the standard labeling and $n \geq 3$, and if $S = \{1, 2, 3\}$, then $w(S)$ is the vector in $\mathbb{R}^3$ containing the first three entries of w.

Let F and G be subsets of $\mathbb{R}^n$. The *support* of F, denoted by $\mathrm{supp}(F)$, is the subset of $\{1,\dots,n\}$ such that $i \in \mathrm{supp}(F)$ if and only if there exist some $x \in F$ such that the i-th entry of x is nonzero. The set difference $F \setminus G$ is defined by $F \setminus G := \{x \mid x \in F \text{ and } x \notin G\}$.

Let $\mathbb{S}^n$ denote the set of $n \times n$ *symmetric matrices*, and let $\mathbb{S}^n_+$ represent the set of $n \times n$ *positive semidefinite matrices*. Given $X, Y \in \mathbb{S}^n$, the *trace inner product* between X and Y is defined as $\langle X, Y\rangle = \mathrm{tr}(XY)$, where $\mathrm{tr}(\cdot)$ denotes the trace of a matrix. If the rows and columns of $X \in \mathbb{S}^n$ are indexed by the elements of a finite set $\mathcal{N}$, we may alternatively write $X \in \mathbb{S}^{\mathcal{N}}$. For any subset $S \subseteq \mathcal{N}$, the principal submatrix of X corresponding to S is denoted by $X(S,S)$.

For any set L, the orthogonal complement of L is denoted by $L^\perp$. For singleton sets, we simplify the notation by writing $w^\perp$ instead of $\{w\}^\perp$.

2 Preliminaries and Motivation

2.1 Facial reduction algorithm (FRA)

Let K be a nonempty closed convex cone in a finite dimensional Euclidean space. The *dual cone* of K is $K^* = \{y \mid \langle y, x\rangle \geq 0, \forall x \in K\}$. We say F is a *face* of K, denoted by $F \trianglelefteq K$, if $x, y \in K$ and $x + y \in F$ imply that $x, y \in F$. If F is a face of K such that F is nonempty and $F \neq K$, then we say F is a *proper face*. A face F of K is called *exposed* if it is of the form $F = K \cap v^\perp$ for some $v \in K^*$. The vector v is then called an *exposing vector*. We say K is *exposed* if all of its faces are exposed. The conjugate face of F is $F^\triangle := K^* \cap F^\perp$. For any $x \in \mathrm{ri}\, F$, we have $F^\triangle = K^* \cap x^\perp$. For any $S \subseteq K$, the smallest of face of K containing S is denoted by $\mathrm{face}(S, K)$. Note that $\mathrm{face}(S, K) = \mathrm{face}(\mathrm{conv}(S), K)$. In particular, for any $x \in \mathrm{ri}(\mathrm{conv}(S))$, we have $\mathrm{face}(S, K) = \mathrm{face}(x, K)$.

Let L be an affine subspace such that $L \cap K \neq \emptyset$. We say *Slater's condition* holds for $L \cap K$ if it contains a feasible solution in the relative interior of K, i.e., $L \cap \mathrm{ri}\, K \neq \emptyset$. The smallest face of K containing $L \cap K$ is called the *minimal cone* of $L \cap K$, see [17]. To find the minimal cone of $L \cap K$, FRA exploits the following theorem of alternative

$$L \cap \mathrm{ri}\, K = \emptyset \quad \Leftrightarrow \quad L^\perp \cap (K^* \setminus K^\perp) \neq \emptyset.$$

We describe FRA applied to $L \cap K$ below, and its proof of convergence can also be found in some recent works, see [17, 22, 30].

Algorithm 1 Facial Reduction Algorithm (FRA)

1: **Initialization:** Let $F_0 = K$, $i = 1$.
2: **while** we can pick $w_i \in L^\perp \cap (F^*_{i-1} \setminus F^\perp_{i-1})$ **do**
3: Set $F_i \leftarrow F_{i-1} \cap w_i^\perp$.
4: Set $i \leftarrow i + 1$.
5: **end while**

If $L \cap \mathrm{ri}\, K = \emptyset$, then FRA applied to $L \cap K$ generates a sequence of faces $(F_0, \dots, F_d)$ satisfying

$$F_0 \supsetneq F_1 \supsetneq \cdots \supsetneq F_d,$$

where F_d is the minimal cone of $L \cap K$. It also generates a sequence of exposing vectors $(w_1, \ldots, w_d)$ satisfying

$$w_i \in L^\perp \cap (F_{i-1}^* \setminus F_{i-1}^\perp) \text{ for } i = 1, \ldots, d.$$

We call $(w_1, \ldots, w_d)$ an FR sequence for $L \cap K$, corresponding to the sequence of faces $(F_0, \ldots, F_d)$. The positive integer d is called the length of the FR sequence. If $L \cap \operatorname{ri} K \neq \emptyset$, then FRA terminates immediately, and we set $d = 0$.

The *singularity degree of* $L \cap K$ is defined as the minimum length among all FR sequences for $L \cap K$. For linear and semidefinite programming problems, a shortest FR sequence can be obtained by selecting an exposing vector w_i from the relative interior of F_{i-1}^* at each FR step. For $K = \mathbb{S}_+^n$, the singularity degree can be any integer between 0 and $n-1$. For $K = \mathbb{R}_+^n$, the singularity degree is either 0 or 1.

The *maximum singularity degree (MSD)* of $L \cap K$, denoted by $\mathrm{MSD}(L \cap K)$, is the maximum length among all FR sequences for $L \cap K$. In contrast to the singularity degree, numerous fundamental questions about the MSD remain open, such as how to identify the longest FR sequences for both LP and SDP problems. This paper aims to address these questions.

Throughout, we make the following assumptions to avoid trivial cases in our analysis. Unless otherwise stated or ambiguity arises, the symbols K and L will consistently denote a closed convex cone and an affine subspace, respectively, without repeated definitions.

Assumption 2.1. *Let L be an affine subspace and K a closed convex cone such that:*

1. *K and K^* are both nonempty and exposed.*
2. *$L \cap K \neq \emptyset$.*

In addition, we rely on the following well-known results, which serve as essential tools for analyzing the relationships between convex sets and their faces.

Lemma 2.1. 1. *Let $\emptyset \neq S \subseteq K$. Then $F = face(S, K)$ if and only if $ri(conv(S)) \subseteq ri(F)$.*

2. *Let $F \trianglelefteq K$. Then F is an exposed face if and only if $F^{\triangle\triangle} = F$.*
3. *Let $F, G \trianglelefteq K$ be exposed. If $F \subsetneq G$, then $F^\triangle \supsetneq G^\triangle$.*
4. *Let $F, G \trianglelefteq K$. If $F \subsetneq G$, then $\dim F < \dim G$.*

Proof. 1. This is a well-known result; see, e.g., Proposition 2.2.5 in [7].

2. This result appears in Proposition 3.1, Item 2, of [29].
3. We have $F^\triangle \supseteq G^\triangle$ from the definition of the conjugate face. If $F^\triangle = G^\triangle$, then we have $F = (F^\triangle)^\triangle = (G^\triangle)^\triangle = G$. This contradiction proves the result.
4. See Corollary 5.5 in [4].

□

2.2 FRA for LP

Let $K = \mathbb{R}^n_+$. Define the affine subspace $L := \{x \in \mathbb{R}^n \mid Ax = b\}$ for some $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then $L \cap \mathbb{R}^n_+$ is a polyhedron, and we obtain an LP problem. The entries of x are indexed by $\{1, \dots, n\}$. Recall that, for any subset $S \subseteq \{1, \dots, n\}$, $x(S) \in \mathbb{R}^S$ denotes the subvector of x indexed by S. A set F is a nonempty face of the nonnegative orthant $\mathbb{R}^n_+$ if and only if there exists a subset $S \subseteq \{1, \dots, n\}$ such that

$$F = \left\{x \in \mathbb{R}^n_+ \mid x(S) = 0\right\}. \tag{1}$$

In the FRA, the set $L^\perp \cap F^*$ is also a polyhedron. To see this, note that the orthogonal complement of L and the dual cone of the face F are given by

$$\begin{array}{rcl} L^\perp & = & \left\{A^T y \in \mathbb{R}^n \mid b^T y = 0\right\}, \\ F^* & = & \{w \in \mathbb{R}^n \mid w(\{1, \dots, n\} \setminus S) \geq 0\}. \end{array} \tag{2}$$

2.3 FRA for SDP

Let $K = \mathbb{S}^n_+$. Let $A_1, \dots, A_m \in \mathbb{S}^n$ and $b \in \mathbb{R}^m$ be given. Consider the following SDP problem

$$L \cap \mathbb{S}^n_+ \text{ where } L := \{X \in \mathbb{S}^n \mid \langle A_i, X\rangle = b_i \text{ for } i = 1, \dots, m\}. \tag{3}$$

A set F is a nonempty face of $\mathbb{S}^n_+$ if and only if there exists a linear subspace $\mathcal{V}$ of $\mathbb{R}^n$ such that

$$F = \left\{X \in \mathbb{S}^n_+ \mid \operatorname{range}(X) \subseteq \mathcal{V}\right\}. \tag{4}$$

Here, range(X) denotes the range space of X. In the FRA, the set $L^\perp \cap F^*$ also defines an SDP problem. Let V be any matrix such that range$(V) = \mathcal{V}$. Then

$$\begin{array}{rcl} L^\perp & = & \left\{\sum_{i=1}^m A_i y_i \in \mathbb{S}^n \mid b^T y = 0\right\}, \\ F^* & = & \left\{W \in \mathbb{S}^n \mid V^T W V \text{ is positive semidefinite}\right\}. \end{array}$$

A special case is when all the matrices in the face F have a block-diagonal structure, possibly after some reordering. Assume the rows and columns of X are indexed by the elements of a finite set $\mathcal{N}$. Let $S \subseteq \mathcal{N}$. The set of positive semidefinite matrices whose rows and columns corresponding to elements in $\mathcal{N} \setminus S$ are zero is a face, i.e.,

$$F = \left\{X \in \mathbb{S}^n_+ \mid X(\mathcal{N} \setminus S, \mathcal{N} \setminus S) = 0\right\}. \tag{5}$$

Note that $X(\mathcal{N} \setminus S, \mathcal{N} \setminus S)$ is the principal submatrix of X corresponding $\mathcal{N} \setminus S$. For example, let $\mathcal{N} = \{1, \dots, n\}$ be the standard labeling of the rows and columns from 1 to n. Let $S = \{1, \dots, r\}$. Then F can be written as

$$F = \left\{X \in \mathbb{S}^n_+ \mid X = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \text{ with } R \in \mathbb{S}^r_+\right\}.$$

In the above case, the dual cone F^* admits a simple characterization: $W \in F^*$ if and only if the leading $r \times r$ principal submatrix of W is positive semidefinite.

In our main theorem, we construct an SDP problem with a special structure, ensuring that all faces in any FR sequences exhibit a block-diagonal structure. This allows us to leverage the simple structures of F and F^*.

2.4 An application of the MSD

In this section, we provide a new application of the MSD to further motivate our study. A central question in the implementation of FRA is how to find an element in $L^\perp \cap (K^* \setminus K^\perp)$ efficiently. For instance, when $K = \mathbb{S}^n_+$, finding an element in $L^\perp \cap \mathbb{S}^n_+$ becomes an SDP problem, making it a challenging task. This question is very important, and thus, many special FRAs have been developed to achieve efficient implementations in practice; see [7, 9, 12, 24, 35].

Here, we provide a discussion on a special FRA based on the well-known low-rank approach for SDP problems, highlighting how the MSD offers valuable insights into the behavior of this algorithm as an application. The low-rank approach for SDP problems, also known as the Burer-Monteiro SDP method, was introduced in [5], and it has demonstrated significant practical success and has received considerable attention over the past decade.

As mentioned in Section 2.3, finding an element in $L^\perp \cap (\mathbb{S}^n_+ \setminus \{0\})$ is equivalent to searching for a vector $y = (y_i)_{i=1}^m \in \mathbb{R}^m$ such that

$$\sum_{i=1}^m A_i y_i \in \mathbb{S}^n_+, \operatorname{tr}\left(\sum_{i=1}^m A_i y_i\right) = 1 \text{ and } b^T y = 0. \tag{6}$$

To overcome the difficulties in solving (6), we introduce a new matrix variable $V \in \mathbb{R}^{n\times r}$ for some positive integer $r \le n$ and consider the following non-linear system in the variable $(y, V) \in \mathbb{R}^m \times \mathbb{R}^{n\times r}$,

$$\sum_{i=1}^m A_i y_i = VV^T, \operatorname{tr}\left(\sum_{i=1}^m A_i y_i\right) = 1 \text{ and } b^T y = 0. \tag{7}$$

In particular, if r is much smaller than n, the number of variables in the above non-linear system is significantly smaller than that in (6) as we do not need the matrix variable in $\mathbb{S}^n$ for the positive semidefinite constraint. In this case, the system (7) is an overdetermined system of equations. Thus, we can try to apply the Gauss–Newton algorithm or Levenberg-Marquardt algorithm to find a solution if it exists. Whenever we find a solution (y^*, V^*), it is straightforward to see that y^* is feasible for (6). Thus, we obtain an exposing vector $\sum_{i=1}^m A_i y_i^*$ for $L \cap \mathbb{S}^n_+$ efficiently.

However, if r is too small, the non-linear system (7) may fail to capture any exposing vectors for $L \cap \mathbb{S}^n_+$. Specifically, let r^* denote the smallest rank of exposing vectors from (6) for $L \cap \mathbb{S}^n_+$. If $r < r^*$, then there is no feasible solution for (7), and consequently, it fails to detect any exposing vectors. The behavior of the Burer-Monteiro SDP method has been investigated from many different aspects; see, e.g., [6, 20, 31]. In the next paragraph, we provide an example to show that the MSD can provide sufficient conditions for the solution set of the non-linear system (7) to contain at least one non-trivial exposing vector.

The following problem is a well-known example for clarifying the numerical issues of ill-conditioned SDP problems, see [24, 25, 28]. Define the affine subspace

$$L = \{X \in \mathbb{S}^n \mid X_{11} = 1,\ X_{22} = 0,\ X_{k+1,k+1} = X_{1,k} \text{ for } k = 2, \dots, n-1\}.$$

The singularity degree of $L \cap \mathbb{S}^n_+$, with L defined above, is $n-1$, representing the worst possible scenario. However, the MSD of $L \cap \mathbb{S}^n_+$ is also $n-1$. This proves that the non-linear system (7) is guaranteed to capture an exposing vector with $r = 1$. Thus, assuming we successfully find a nonzero solution in (7), the non-linear formulation (7) with $r = 1$ is guaranteed to identify the minimal cone of $L \cap \mathbb{S}^n_+$ in $n-1$ FR steps. In our numerical experiments, this non-linear approach efficiently finds the minimal cone for $L \cap \mathbb{S}^n_+$.

3 The minimal FR sequences

Let K be a closed convex cone and L an affine subspace. Let $F \trianglelefteq K$. We say w is *minimal* for $L \cap F$ if $w \in L^\perp \cap (F^* \setminus F^\perp)$ and there does not exist $u \in L^\perp \cap (F^* \setminus F^\perp)$ such that

$$F \cap w^\perp \subsetneq F \cap u^\perp. \tag{8}$$

Let $f = (w_1, \ldots, w_d)$ be an FR sequence for $L \cap K$, and $(F_0, \ldots, F_d)$ the corresponding sequence of faces. We call f *minimal* if w_i is minimal for $L \cap F_{i-1}$ for all $i = 1, \ldots, k$. Our first result is that any longest FR sequence is necessarily minimal.

Theorem 3.1. *If f is one of the longest FR sequences for $L \cap K$, then f is minimal.*

Proof. Let $f = (w_1, \ldots, w_d)$ be an FR sequence for $L \cap K$, and $(F_0, \ldots, F_d)$ the corresponding sequence of faces. Assume that w_i is not minimal for $L \cap F_{i-1}$. By definition, there exists a $u \in L^\perp \cap (F_{i-1}^* \setminus F_{i-1}^\perp)$ that

$$F_i \subsetneq G \subsetneq F_{i-1} \text{ with } G := F_{i-1} \cap u^\perp. \tag{9}$$

Since taking duals reverses the inclusion order, the above inclusion implies that

$$w_i \in F_{i-1}^* \subseteq G^*.$$

In addition, the two inclusions in (9) also imply that

$$G \cap w_i^\perp = G \cap (F_{i-1} \cap w_i^\perp) = G \cap F_i = F_i.$$

This means $w_i \in L^\perp \cap (G^* \setminus G^\perp)$. Thus $(w_1, w_2, \ldots, w_{i-1}, u, w_i, \ldots, w_k)$ is an FR sequence for $L \cap K$ of length $d+1$. Thus, f is not one of the longest FR sequences for $L \cap K$. □

For LP and SDP problems, we provide two simple sufficient conditions ensuring that an exposing vector w is minimal for $L \cap F$. This result is required in Lemma 5.4 for $\mathbb{R}^n_+$.

Lemma 3.1. *Let $F \trianglelefteq K$ and $w \in L^\perp \cap (F^* \setminus F^\perp)$ be an exposing vector for $L \cap F$.*

1. *If $K = \mathbb{R}^n_+$ and $\dim(F) - \dim(F \cap w^\perp) = 1$, then w is minimal.*
2. *If $K = \mathbb{S}^n_+$ and $\max\{\text{rank}\,(X) \mid X \in F\} - \max\{\text{rank}\,(X) \mid X \in F \cap w^\perp\} = 1$, then w is minimal.*

Proof. The result follows directly from Item 4 in Lemma 2.1, along with the characterization of the faces of $\mathbb{R}^n_+$ and $\mathbb{S}^n_+$ in (1) and (4). □

Suppose w and v are both minimal for $L \cap F$ and satisfy $F \cap w^\perp = F \cap v^\perp$. We show that w and v must be equivalent in some sense. For example, if F is full-dimensional, then $w = v$ up to some positive scaling. We formally describe and prove this result in a slightly more general setting below, as it is needed in Lemma 4.2.

Lemma 3.2. *Let $F \trianglelefteq K$ be nonempty. Assume $w \in L^\perp \cap (F^* \setminus F^\perp)$ is minimal for $L \cap K$. Let $G = \text{face}(w, K^*) \trianglelefteq F^*$. For any vector $d \in L^\perp \cap (\text{span}(G) \setminus F^\perp)$, we have $w - \alpha d \in F^\perp$ for some nonzero constant $\alpha \in \mathbb{R}$. In particular, if F is full-dimensional, then $w = \alpha d$.*

Proof. By applying Item 1 in Lemma 2.1, we have $w \in \mathrm{ri}(G)$. As $d \in \mathrm{span}(G)$, there exists $\epsilon > 0$ such that $w - \epsilon d \in G$ and $w + \epsilon d \in G$. Since $d \notin F^\perp$ and $F \neq \emptyset$, there exists $x \in F$ such that $\langle x, d\rangle \neq 0$. Assume $\langle x, d\rangle > 0$. Then, for sufficiently large $\alpha > 0$, we have $\langle x, w - \alpha d\rangle < 0$ and thus $w - \alpha d \notin F^*$. Since G is a face of F^*, it follows that $w - \alpha d \notin G$. Similarly, if $\langle x, d\rangle < 0$, then $w + \alpha d \notin G$ for sufficiently large $\alpha > 0$. Thus, the following maximization problem is bounded above and attains its maximum $\tilde{\alpha}$ since its feasible region is closed.

$$\tilde{\alpha} := \arg\max\{\alpha : \alpha \geq 0, w - \alpha d \in G \text{ and } w + \alpha d \in G\}.$$

Observe that $\tilde{\alpha} > 0$. Without loss of generality, assume that $u := w - \tilde{\alpha} d \in G$ and $w - (\tilde{\alpha} + \delta)d \notin G$ for any $\delta > 0$. This implies $u \notin \mathrm{ri}(G)$ and $\mathrm{face}(u, G) \subsetneq G$. As $\mathrm{face}(u, G)$ and G are faces of F^*, applying Item 3 in Lemma 2.1 yields

$$F \cap w^\perp = G^\triangle \subsetneq \mathrm{face}(u, G)^\triangle = F \cap u^\perp. \tag{10}$$

Note that $u \in L^\perp \cap F^*$. Thus, $u \in F^\perp$, as otherwise w would not be minimal for $L \cap F$ by (10). $\square$

4 Maximum Singularity Degree for Linear Programming

In this section, we prove that an FR sequence is minimal for a polyhedron $L \cap \mathbb{R}^n_+$ if and only if it is one of the longest FR sequences. This result will be used for proving our main result for SDP in Theorem 5.1. Necessity follows directly from Theorem 3.1 as a special case. To prove sufficiency, we introduce two operations called *swapping* and *removal*. The swapping operation allows us to manipulate the minimal FR sequences, and the removal operation is for simplifying the underlying set $L \cap \mathbb{R}^n_+$.

4.1 Swapping

Let $f = (w_1, w_2)$ be a minimal FR sequence for $L \cap \mathbb{R}^n_+$, and (F_0, F_1, F_2) the corresponding sequence of faces. Define $S_i := \mathrm{supp}(F_{i-1}) \setminus \mathrm{supp}(F_i)$ for $i = 1, 2$. In the first FR step, we reformulate $L \cap F_0$ as $L \cap F_1$. Intuitively, this corresponds to setting the variables associated with S_1 to zero in F_0. Similarly, in the second FR step, reformulating $L \cap F_1$ as $L \cap F_2$ corresponds to setting the variables associated with S_2 to zero in F_1.

We ask whether it is possible to swap the order in which we set these variables to zero. For example, can we find a minimal FR sequence that first sets the variables associated with S_2 to zero and then sets those associated with S_1 to zero? Formally, this requires the existence of a minimal FR sequence (v_1, v_2) corresponding to the sequence of faces (G_0, G_1, G_2) given by

$$\begin{array}{lcl} G_0 & := & \mathbb{R}^n_+, \\ G_1 & := & \{x \in G_0 \mid x(S_2) = 0\}, \\ G_2 & := & \{x \in G_1 \mid x(S_1) = 0\}. \end{array} \tag{11}$$

We provide sufficient conditions ensuring the existence of a minimal FR sequence for $L \cap \mathbb{R}^n_+$ corresponding to the sequence of faces (G_0, G_1, G_2).

Lemma 4.1 (Swapping)**.** *Let $f = (w_1, w_2)$ be a minimal FR sequence for $L \cap \mathbb{R}^n_+$, and (F_0, F_1, F_2) the corresponding sequence of faces. Let $S_i = \mathrm{supp}(F_{i-1}) \setminus \mathrm{supp}(F_i)$ for $i = 1, 2$. Let (G_0, G_1, G_2) be the sequence of faces defined in* (11)*. If any of the following conditions hold, then there exists a minimal FR sequence for $L \cap \mathbb{R}^n_+$ corresponding to (G_0, G_1, G_2).*

1. $|S_1| = 1$.

2. $|S_2| = 1$ *and there exists* $v_1 \in L^\perp \cap (\mathbb{R}^n_+ \setminus \{0\})$ *such that* $\mathbb{R}^n_+ \cap v_1^\perp = G_1$.

Proof. Note that, by construction, G_2 is the minimal face of $\mathbb{R}^n_+$ containing $L \cap \mathbb{R}^n_+$. For each case, we construct a minimal FR sequence (v_1, v_2) for $L \cap \mathbb{R}^n_+$ such that it corresponds to the sequence of faces (G_0, G_1, G_2).

1. Without loss of generality, we can assume that $S_1 = \{1\}$ and $S_2 = \{2, \dots, k\}$ for some $k \geq 2$. Based on the expression of the dual cone in (2), we can assume the entries in the exposing vectors w_1 and w_2 satisfy

$$\begin{array}{rclrclrcl} w_1(\{1\}) & = & 1, & w_1(\{2,\dots,k\}) & = & 0, & w_1(\{k+1,\dots,n\}) & = & 0, \\ w_2(\{1\}) & \in & \mathbb{R}, & w_2(\{2,\dots,k\}) & > & 0, & w_2(\{k+1,\dots,n\}) & = & 0. \end{array}$$

   Let $\alpha = w_2(\{1\}) \in \mathbb{R}$. Then the vector $v_1 := w_2 - \alpha w_1$ satisfies

$$v_1(\{1\}) \;=\; 0, \quad v_1(\{2,\dots,k\}) \;>\; 0, \quad v_1(\{k+1,\dots,n\}) \;=\; 0.$$

   Thus, $v_1 \in L^\perp \cap (\mathbb{R}^n_+ \setminus \{0\})$ is an exposing vector for $L \cap \mathbb{R}^n_+$, and $G_1 = \mathbb{R}^n_+ \cap v_1^\perp$.

   To show that v_1 is minimal for $L \cap \mathbb{R}^n_+$, assume that there exists $u \in L^\perp \cap (\mathbb{R}^n_+ \setminus \{0\})$ such that

$$\mathbb{R}^n_+ \cap v_1^\perp \subsetneq \mathbb{R}^n_+ \cap u^\perp.$$

   This means $\mathrm{supp}(u) \subsetneq \mathrm{supp}(v_1) = S_2$. As $1 \notin S_2$, we have $1 \notin \mathrm{supp}(u)$ and thus $u \notin F_1^\perp$. This proves that $u \in L^\perp \cap (F_1^* \setminus F_1^\perp)$ is an exposing vector for $L \cap F_1$. Since

$$F_1 \cap w_2^\perp \subsetneq F_1 \cap u^\perp,$$

   this implies that w_2 is not minimal for $L \cap F_1$. This is a contradiction to f is a minimal FR sequence. Thus v_1 is minimal for $L \cap \mathbb{R}^n_+$.

   Let $v_2 = w_1$. From the definition of w_1 and Lemma 3.1, it is easy to see that $v_2 \in L^\perp \cap (G_1^* \setminus G_1^\perp)$ is minimal for G_1. In addition, $G_2 = G_1 \cap v_2^\perp$.

   This proves that (v_1, v_2) is a minimal FR sequence corresponding to (G_0, G_1, G_2).

2. Up to some relabeling, assume that $S_1 = \{1, \dots, k\}$ for some $1 \leq k < n$ and $S_2 = \{n\}$. By assumption, there exists an exposing vector $v_1 \in L^\perp \cap (\mathbb{R}^n_+ \setminus \{0\})$ such that $G_1 = \mathbb{R}^n_+ \cap v_1^\perp$. As $|S_2| = 1$, we have v_1 is minimal for $L \cap \mathbb{R}^n_+$ by Lemma 3.1.

   Let $v_2 := w_1$. Note that $w_1 \in L^\perp \cap (G_1^* \setminus G_1^\perp)$ and $G_2 = G_1 \cap w_1^\perp$. It remains to prove that $v_2 = w_1$ is minimal for $L \cap G_1$. Assume this is not the case. There exists an exposing vector $u \in L^\perp \cap (G_1^* \setminus G_1^\perp)$ such that

$$G_1 \cap w_1^\perp \subsetneq G_1 \cap u^\perp.$$

   Define $S := \mathrm{supp}(u) \cap \{1, \dots, n-1\}$. Up to some relabeling, the above inclusion implies that $S = \{1, \dots, l\} \subsetneq S_1$ for some $1 \leq l < k$. Then

$$u(\{1,\dots,l\}) > 0, u(\{l+1,\dots,n-1\}) = 0, u(\{n\}) \in \mathbb{R}. \tag{12}$$

After an appropriate scaling, the vector v_1 is simply a unit vector

$$v_1(\{1,\dots,n-1\}) = 0, v_1(\{n\}) = 1. \tag{13}$$

Define $h := u - \alpha v_1$ with $\alpha = u(\{n\}) \in \mathbb{R}$. Then the vector h satisfies

$$h(\{1,\dots,l\}) > 0 \text{ and } h(\{l+1,\dots,n\}) = 0. \tag{14}$$

As $S \subsetneq S_1$, this shows that h is an exposing vector of $L \cap \mathbb{R}^n_+$ such that

$$\mathbb{R}^n_+ \cap w_1^\perp \subsetneq \mathbb{R}^n_+ \cap h^\perp.$$

Thus w_1 is not minimal for $L \cap \mathbb{R}^n_+$, which is a contradiction to f is minimal. This shows that $v_2 = w_1$ is minimal for $L \cap G_1$.

Thus, (v_1, v_2) is a minimal FR sequence corresponding to (G_0, G_1, G_2).

□

The above result extends naturally to minimal FR sequences of arbitrary length.

Corollary 4.1 (Swapping)**.** *Let $f = (w_1, \dots, w_d)$ be a minimal FR sequence for $L \cap \mathbb{R}^n_+$ with $(F_0, \dots, F_d)$ the corresponding sequence of faces. Let $S_i = supp(F_{i-1}) \setminus supp(F_i)$ for $i = 1, \dots, d$. Fix an index $j \in \{1, \dots, d-1\}$. Define the sequence of faces as follows:*

$$G_i = \begin{cases} \mathbb{R}^n_+ & \text{if } i = 0, \\ \{x \in G_{i-1} \mid x(S_i) = 0\} & \text{if } i \notin \{0, j, j+1\}, \\ \{x \in G_{j-1} \mid x(S_{j+1}) = 0\} & \text{if } i = j, \\ \{x \in G_j \mid x(S_j) = 0\} & \text{if } i = j+1. \end{cases}$$

If any of the following conditions hold, then there exists a minimal FR sequence for $L \cap \mathbb{R}^n_+$ corresponding to $(G_0, \dots, G_d)$.

- *$|S_j| = 1$.*
- *$|S_{j+1}| = 1$ and there exists $v_j \in L^\perp \cap (F^*_{j-1} \setminus F^\perp_{j-1})$ such that $F_{j-1} \cap v_j^\perp = G_{j+1}$.*

Proof. Note that $G_i = F_i$ for $i < j$ or $i > j+1$. By the facial structure of $\mathbb{R}^n_+$, the statement follows from applying the same argument as in Lemma 4.1 to the j-th and $(j+1)$-th FR steps. □

4.2 Removal

In this section, we analyze the effects of removing a redundant variable from $L \cap \mathbb{R}^n_+$. Recall that the polyhedron $L \cap \mathbb{R}^n_+$ is defined as

$$L = \{x \in \mathbb{R}^n \mid Ax = b\}$$

for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. We obtain a smaller system by removing one of the redundant variables. Without loss of generality, we assume that the n-th variable in x is redundant, i.e., the

n-th entry of x is zero for every $x \in L \cap \mathbb{R}^n_+$. Let $\tilde{A} \in \mathbb{R}^{m\times(n-1)}$ be the first $n-1$ columns of A. Thus, removing the n-th variable yields an equivalent but lower-dimensional set $\tilde{L} \cap \mathbb{R}^{n-1}_+$, where

$$\tilde{L} = \{\tilde{x} \in \mathbb{R}^{n-1} \mid \tilde{A}\tilde{x} = b\}. \tag{15}$$

Let $f = (w_1, \dots, w_d)$ be a minimal FR sequence for $L \cap \mathbb{R}^n_+$. We aim to determine whether it is possible to construct a (minimal) FR sequence for the smaller system $\tilde{L} \cap \mathbb{R}^{n-1}_+$. Let $\tilde{w}_i \in \mathbb{R}^{n-1}$ be the vector consisting of the first $n-1$ entries of $w_i \in \mathbb{R}^n$. Since the n-th variable is a redundant variable, there exists an FR step where it is set to zero. Denote this as the j-th FR step, where $j \in \{1, \dots, d\}$. Let $S_j = \operatorname{supp}(F_j \setminus F_{j-1})$ represent the subset of variables being fixed to zero at the j-th FR step. Thus, $n \in S_j$. We distinguish two cases:

1. If the n-th variable of x is the only one set to zero at the j-th FR step, i.e., $S_j = \{n\}$, then the sequence $\tilde{f} = (\tilde{w}_1, \dots, \tilde{w}_{j-1}, \tilde{w}_{j+1}, \dots, \tilde{w}_d)$ with $\tilde{w}_j$ removed is still an FR sequence for $\tilde{L} \cap \mathbb{R}^{n-1}_+$. Furthermore, we determine the necessary and sufficient conditions for $\tilde{f}$ to be minimal.

2. If two or more variables are fixed to zero at the j-th FR step, i.e., $|S_j| \geq 2$, then the sequence $\tilde{f} = (\tilde{w}_1, \dots, \tilde{w}_d)$ is always a minimal FR sequence for $\tilde{L} \cap \mathbb{R}^{n-1}_+$.

We now present the proof of the above statements.

Lemma 4.2. *(Removal-Case1) Let $f = (w_1, \dots, w_d)$ be a minimal FR sequence for $L \cap \mathbb{R}^n_+$, and $(F_0, \dots, F_d)$ the corresponding sequence of faces. Define $S_i = supp(F_i) \setminus supp(F_{i-1})$ for $i = 1, \dots, d$. Let $\tilde{w}_i \in \mathbb{R}^{n-1}$ be the first $n-1$ entries of $w_i \in \mathbb{R}^n$. Assume that $n \in S_j$ and $|S_j| = 1$ for some $j \in \{1, \dots, d\}$. Then*

1. *$\tilde{f} = (\tilde{w}_1, \dots, \tilde{w}_{j-1}, \tilde{w}_{j+1}, \dots, \tilde{w}_d)$ is an FR sequence for $\tilde{L} \cap \mathbb{R}^{n-1}_+$, see* (15).
2. *$\tilde{f}$ is minimal for $\tilde{L} \cap \mathbb{R}^{n-1}_+$ if and only if the n-th unit vector $e_n \in \mathbb{R}^n$ is in $L^\perp$.*

Proof. It is straightforward to verify that $\tilde{f}$ is indeed an FR sequence for $\tilde{L} \cap \mathbb{R}^{n-1}_+$. Let $(\tilde{F}_0, \dots, \tilde{F}_{d-1})$ be the corresponding sequence of faces.

Assume $e_n \in L^\perp$. Then $e_n \in L^\perp \cap (\mathbb{R}^n_+ \setminus \{0\})$ is an exposing vector for $L \cap \mathbb{R}^n_+$. By applying Corollary 4.1, we can perform a series of swapping operations to f so that $j = 1$. This yields $F_1 = \{x \in \mathbb{R}^n_+ \mid x_n = 0\}$. In this case, it is clear that $\tilde{f}$ is minimal.

Conversely, assume that $e_n \notin L^\perp$. Then $j \geq 2$. We will show that $\tilde{w}_{j-1}$ is not minimal for $\tilde{L} \cap \tilde{F}_{j-2}$. By the facial structure of $\mathbb{R}^n_+$, we can assume $j = 2$ without loss of generality. Define $\tilde{S}_i = \operatorname{supp}(\tilde{F}_{i-1} \setminus \tilde{F}_i)$ for $i = 1, 2$. Up to some relabeling, suppose that $\tilde{S}_1 = \{1, \dots, k\}$ for some k such that $1 \leq k < n$, and $\tilde{S}_2 = \{n\}$. Recall that $\tilde{w}_i \in \mathbb{R}^{n-1}$ is the first $n-1$ entries of $w_i \in \mathbb{R}^n$. This implies that w_1 and w_2 have the following nonzero structure,

$$\begin{array}{llllllll} w_1(\{1,\dots,k\}) & > & 0, & w_1(\{k+1,\dots,n-1\}) & = & 0, & w_1(\{n\}) & = & 0, \\ w_2(\{1,\dots,k\}) & \in & \mathbb{R}^k, & w_2(\{k+1,\dots,n-1\}) & = & 0, & w_2(\{n\}) & > & 0. \end{array} \tag{16}$$

In addition, $e_n \notin L^\perp$ implies that $w_2(\{1, \dots, k\}) \neq 0$ and thus $\tilde{w}_2 \neq 0$.

Suppose for the sake of contradiction that $\tilde{w}_1$ is minimal for $\tilde{L} \cap \mathbb{R}^{n-1}_+$. Since $\tilde{w}_2 \neq 0$, the nonzero structure in (16) implies that $\tilde{w}_2 \in \tilde{L}^\perp \cap (\operatorname{aff}(\operatorname{face}(\tilde{w}_1, \mathbb{R}^{n-1}_+)) \setminus \{0\})$. Thus, we can apply Lemma 3.2 to get $\tilde{w}_2 = \alpha\tilde{w}_1$ for some nonzero constant $\alpha \neq 0$. Define $h := w_2 - \alpha w_1$. Then we have $h \in L^\perp$ is a positive multiple of e_n, which leads to a contradiction. Thus $\tilde{w}_1$ is not minimal for $\tilde{L} \cap \mathbb{R}^{n-1}_+$, and $\tilde{f}$ is not minimal. $\square$

Lemma 4.3. *(Removal-Case2) Let $f = (w_1, \ldots, w_d)$ be a minimal FR sequence for $L \cap \mathbb{R}^n_+$, and $(F_0, \ldots, F_d)$ the corresponding sequence of faces. Define $S_i = supp(F_i) \setminus supp(F_{i-1})$ for $i = 1, \ldots, d$. Let $\tilde{w}_i \in \mathbb{R}^{n-1}$ denote the vector consisting of the first $n-1$ entries of $w_i \in \mathbb{R}^n$. Assume $n \in S_j$ and $|S_j| \geq 2$ for some $j \in \{1, \ldots, d\}$. Then $\tilde{f} = (\tilde{w}_1, \ldots, \tilde{w}_d)$ is a minimal FR sequence for $\tilde{L} \cap \mathbb{R}^{n-1}_+$, see* (15).

Proof. It is straightforward to verify that $\tilde{f}$ is indeed an FR sequence for $\tilde{L} \cap \mathbb{R}^{n-1}_+$, and let $(\tilde{F}_0, \ldots, \tilde{F}_d)$ be its corresponding sequence of faces. To show that $\tilde{f}$ is minimal, suppose for contradiction that $\tilde{w}_i$ is not minimal for $\tilde{L} \cap \tilde{F}_{i-1}$. If $i > j$, then this immediately implies that w_i is not minimal for $L \cap F_{i-1}$ which is a contradiction. Thus, we consider the case $i \leq j$. By the facial structure of $\mathbb{R}^n_+$, we can assume that $i = 1$ without loss of generality. Then there exists $\tilde{u} \in \tilde{L}^\perp \cap (\mathbb{R}^{n-1}_+ \setminus \{0\})$ such that

$$\mathbb{R}^{n-1}_+ \cap \tilde{w}_1^\perp \subsetneq \mathbb{R}^{n-1}_+ \cap \tilde{u}^\perp. \tag{17}$$

Recall that $\tilde{w}_1$ consists of the first $n-1$ entries in w_1. In addition, $\tilde{u} \in \tilde{L}^\perp$ implies that $\tilde{u} = \tilde{A}^T y$ for some vector $y \in \mathbb{R}^m$. We can define $u := A^T y \in \mathbb{R}^n$ so that $\tilde{u}$ corresponds to the first $n-1$ entries of u. Up to some relabeling, the strict inclusion in (17) implies that vectors w_1 and u have the following nonzero structure,

$$\begin{array}{llllllll} w_1(\{1,\ldots,l\}) > 0, & w_1(\{l+1,\ldots,k\}) > 0, & w_1(\{k+1,\ldots,n-1\}) = 0, & w_1(\{n\}) \geq 0, \\ u(\{1,\ldots,l\}) > 0, & u(\{l+1,\ldots,k\}) = 0, & u(\{k+1,\ldots,n-1\}) = 0, & u(\{n\}) \in \mathbb{R}, \end{array} \tag{18}$$

for some positive integers l and k such that $1 \leq l < k < n$. If the n-th entry of u is zero, then $u \in L^\perp \cap (\mathbb{R}^n_+ \setminus \{0\})$ is an exposing vector for $L \cap \mathbb{R}^n_+$. This is a contradiction to w_1 is minimal for $L \cap \mathbb{R}^n_+$, as f is minimal. Thus $u(\{n\}) \neq 0$. We now consider two cases based on the value of j.

- Assume $j > 1$. In this case, $w_1(\{n\}) = 0$. If $u(\{n\}) > 0$, then $u \in L^\perp \cap (F_1^* \setminus F_1^\perp)$ is an exposing vector for $L \cap F_1$. As $n \in S_j$, we have u is minimal for $L \cap F_{j-1}$ by Lemma 3.1. In addition, as $|S_j| \geq 2$, we have
$$F_{j-1} \cap w_j^\perp \subsetneq F_{j-1} \cap u^\perp.$$
Thus, w_j is not minimal for $L \cap F_{j-1}$ and this is a contradiction to f is minimal. If $u(\{n\}) < 0$, then we can apply the same argument to $-u$.

- Assume $j = 1$. In this case, $w_1(\{n\}) > 0$. The nonzero structure in (18) implies that $u \in L^\perp \cap (\mathrm{aff}(\mathrm{face}(w_1, \mathbb{R}^n_+)) \setminus \{0\})$. In addition, as w_1 is minimal for $L \cap \mathbb{R}^n_+$, we can apply Lemma 3.2 to obtain $w = \alpha u$ for some constant $\alpha \neq 0$. But this is a contradiction to $w_1(\{l+1, \ldots, k\}) > 0$ and $u(\{l+1, \ldots, k\}) = 0$.

This shows that $\tilde{w}_i$ is minimal for $L \cap \tilde{F}_{i-1}$. Thus, $\tilde{f}$ is a minimal FR sequence for $\tilde{L} \cap \mathbb{R}^{n-1}_+$. $\square$

4.3 The Longest FR Sequences

By applying swapping and removal operations, we show that any minimal FR sequence is also one of the longest FR sequences for LP problems.

Theorem 4.1. *Let f be an FR sequence for the polyhedron $L \cap \mathbb{R}^n_+$. Then, f is minimal if and only if it is one of the longest FR sequences.*

Proof. If f is one of the longest FR sequences, then f is minimal by Theorem 3.1. Conversely, we show that any two minimal FR sequences for $L \cap \mathbb{R}^n_+$ have the same length. Suppose, for contradiction, that this is not the case. Then we choose a counterexample with the smallest number of variables n. Let $f = (w_1, \dots, w_d)$ and $g = (u_1, \dots, u_k)$ be two minimal FR sequences for $L \cap \mathbb{R}^n_+$ corresponding to the sequence of faces $(F_0, \dots, F_d)$ and $(G_0, \dots, G_k)$, respectively. Define $S_i := \operatorname{supp}(F_{i-1}) \setminus \operatorname{supp}(F_i)$ for $i = 1, \dots, d$ and $T_j := \operatorname{supp}(G_{j-1} \setminus G_j)$ for $j = 1, \dots, k$.

Without loss of generality, assume $d > k$. If $|T_j| = 1$ for every $j = 1, \dots, k$, then $d > k = \operatorname{MSD}(L \cap \mathbb{R}^n_+)$ which is a contradiction. Let us fix an index $j \in \{1, \dots, k\}$ such that $|T_j| \geq 2$. Without loss of generality, assume that $n \in T_j$, meaning the n-th variable is set to zero at the j-th FR step in g. Next, we choose the index $i \in \{1, \dots, d\}$ such that $n \in S_i$.

We remove the n-th variable from $L \cap \mathbb{R}^n_+$ to obtain the smaller set $\tilde{L} \cap \mathbb{R}^{n-1}_+$ as described in (15). Since $|T_j| \geq 2$, we can apply Lemma 4.3 to obtain a minimal FR sequence $\tilde{g}$ for $\tilde{L} \cap \mathbb{R}^{n-1}_+$, and the length of $\tilde{g}$ is also k. We now show that $|S_i| = 1$ and the n-th unit vector e_n satisfies $e_n \notin L^\perp$.

- If $|S_i| \geq 2$, then applying Lemma 4.3 yields a minimal FR sequence $\tilde{f}$ for $\tilde{L} \cap \mathbb{R}^{n-1}_+$ and $\tilde{f}$ has the same length d as f. This means $\tilde{L} \cap \mathbb{R}^{n-1}_+$ is a smaller counterexample as it has two minimal FR sequences $\tilde{f}$ and $\tilde{g}$ with different lengths, which is not possible.
- Assume $e_n \in L^\perp$. As $n \in T_j$, we have $e_n \in L^\perp \cap (G^*_{j-1} \setminus G^\perp_{j-1})$ is an exposing vector for $L \cap G_{j-1}$. Since $|T_j| \geq 2$, this means $G_{j-1} \cap u_j^\perp \subsetneq G_{j-1} \cap e_n^\perp$. This is a contradiction to u_j is minimal for $L \cap G_{j-1}$.

This proves that $|S_i| = 1$ and $e_n \notin L^\perp$. Now, applying Lemma 4.2, we obtain an FR sequence $\tilde{f}$ for $L \cap \mathbb{R}^{n-1}_+$. In addition, we have $\tilde{f}$ is not minimal and the length of $\tilde{f}$ is $d-1$. By Theorem 3.1, $\tilde{f}$ is not one of the longest FR sequences. Let $\bar{f}$ be one of the longest FR sequences for $\tilde{L} \cap \mathbb{R}^{n-1}_+$, and l be the length of $\bar{f}$. Then $l > d - 1 \geq k$. This means $\tilde{L} \cap \mathbb{R}^{n-1}_+$ is a smaller counterexample as it has two minimal FR sequences $\bar{f}$ and $\tilde{g}$ with different lengths. Therefore the counterexample does not exist. □

5 Maximum Singularity Degree for Semidefinite Programming

In this section, we demonstrate that minimal FR sequences may not always attain the longest length for SDP problems, i.e., the converse direction of Theorem 3.1 does not hold. In fact, we show that even if an exposing vector of minimum rank is chosen at each FR step, this may not yield one of the longest FR sequences. We prove that the complexity of finding one of the longest FR sequences is NP-hard. This observation highlights a fundamental difference between the shortest and the longest FR sequences.

5.1 Examples

Consider the SDP problem (3) defined by the following data matrices,

$$A_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 := \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_3 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } b := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3. \tag{19}$$

The only feasible solution is zero. The set of exposing vectors $L^\perp \cap \mathbb{S}^3_+$ is any positive semidefinite matrix of the following form

$$\begin{bmatrix} y_1 - y_2 & y_2 & 0 \\ y_2 & y_2 + y_3 & 0 \\ 0 & 0 & y_3 \end{bmatrix} \succeq 0 \text{ for some } y_1, y_2, y_3 \in \mathbb{R}.$$

It is straightforward to verify that the sequence of exposing vectors (A_3, A_1) is a minimal FR sequence, and it has length 2. However it is not one of the longest FR sequences, as the FR sequence (A_1, A_2, A_3) is minimal by Lemma 3.1 and it has length 3.

In fact, even selecting an exposing vector of minimal rank at each FR step, it does not always yield one of the longest FR sequences. We illustrate this with an example. Consider the SDP problem with data matrices,

$$A_1 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A_2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A_3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } b := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3. \tag{20}$$

In the first FR step, all possible exposing vectors and their ranks can be listed as follows

exposing vector	rank
A_3	2
$A_1 + \alpha A_2$ for $\alpha \in \{0, 1\}$	3
$A_1 + \alpha A_2$ for $\alpha \in (0, 1)$	4
$A_1 + \alpha A_2 + \beta A_3 \succeq 0$ for $\alpha \in \{\frac{1-\sqrt{1+4\beta}}{2}, \frac{1+\sqrt{1+4\beta}}{2}\}, \beta > 0$	4
$A_1 + \alpha A_2 + \beta A_3 \succeq 0$ for $\alpha \in (\frac{1-\sqrt{1+4\beta}}{2}, \frac{1+\sqrt{1+4\beta}}{2}), \beta > 0$	5

Based on the minimum rank rule, A_3 should be used as the exposing vector in the first FR step. In the second FR step, there is only one possible exposing vector, namely, A_1. This yields an FR sequence (A_3, A_1) of length 2. However, (A_1, A_2, A_3) is a longer FR sequence, and A_1 does not have the minimum rank.

5.2 Main ideas and tools

In this section, we introduce the key tools for proving the main result in Theorem 5.1. A central idea in the proof is to establish an upper bound for the MSD of a special SDP problem. To achieve this, we first reduce the given SDP problem to an LP problem with the same MSD. The MSD of this LP problem can then be upper bounded by the MSD of simplier LP problems. By applying Theorem 4.1, we derive a formula for the MSD of these simplier LP problems, leading to the desired upper bound.

To accomplish this, we present three key tools in this section.

5.2.1 Simplification

If the data matrices of an SDP problem have special structures, the problem can often be simplified, making its MSD easier to compute. The following results hold straightforwardly and are provided without proof.

Lemma 5.1. *Let $L = \{X \in \mathbb{S}^n \mid \langle A_i, X\rangle = b_i$ for $i = 1,\dots,m\}$ for some $A_i \in \mathbb{S}^n$ $(i = 1,\dots,m)$ and $b \in \mathbb{R}^m$.*

1. *Assume $F \trianglelefteq \mathbb{S}^n_+$ has a block-diagonal structure given by*
$$F = \left\{X \in \mathbb{S}^n_+ \mid X = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \text{ with } R \in \mathbb{S}^{k+1}_+\right\}. \tag{21}$$
Let $\tilde{A}_i$ be the $(k+1)$-th leading principal submatrix of A_i. Define
$$\tilde{L} := \{\tilde{X} \in \mathbb{S}^{k+1} \mid \langle \tilde{A}_i, \tilde{X}\rangle = b_i \text{ for } i = 1,\dots,m\}.$$
Then MSD$(L \cap F) =$ MSD$(\tilde{L} \cap \mathbb{S}^{k+1}_+)$.

2. *Asssume $\tilde{A}_1,\dots,\tilde{A}_m$ are diagonal matrices. Let $\tilde{a}_i = diag(A_i) \in \mathbb{R}^{k+1}$. Define*
$$\tilde{H} := \{\tilde{x} \in \mathbb{R}^{k+1} \mid \tilde{a}_i^T\tilde{x} = b_i \text{ for } i = 1,\dots,m\}.$$
Then MSD$(\tilde{L} \cap \mathbb{S}^{k+1}_+) =$ MSD$(\tilde{H} \cap \mathbb{R}^{k+1}_+)$.

3. *Assume that the last entry of $\tilde{a}_i$ is zero, i.e., $\tilde{a}_i = \begin{bmatrix} a_i \\ 0 \end{bmatrix} \in \mathbb{R}^{k+1}$ where $a_i \in \mathbb{R}^k$, for all $i = 1,\dots,m$. Define*
$$H := \{x \in \mathbb{R}^k \mid a_i^T x = b_i \text{ for } i = 1,\dots,m\}.$$
Then MSD$(\tilde{H} \cap \mathbb{R}^{k+1}_+) =$ MSD$(H \cap \mathbb{R}^k_+)$.

5.2.2 An upper bound

In this section, we establish an upper bound for the MSD. Intuitively, this bound allows us to decompose the problem into smaller, more manageable subproblems, whose MSD can be derived analytically. While the upper bound is presented in a general setting, for the proof of Theorem 5.1, we only require its special case when K is the nonnegative orthant.

Lemma 5.2. *Assume $L \cap K = \{0\}$. Let $G^1,\dots,G^p$ be faces of K such that*
$$\textit{face}(\{G^1 \cap F,\dots,G^p \cap F\}, F) = F \quad \forall F \trianglelefteq K. \tag{22}$$
Then
$$\text{MSD}(L \cap K) \le \sum_{i=1}^{p} \text{MSD}(L \cap G^i).$$

Proof. Let $f = (w_1,\dots,w_d)$ be an FR sequence for $L \cap K$, and $(F_0,\dots,F_d)$ be the corresponding sequence of faces. For any nonempty $G \trianglelefteq K$, we can construct an FR sequence for $L \cap G$ as follows. (Note that $L \cap K = \{0\}$, we have $L \cap G = L \cap K$.) Let $G_0 = G$. For $i = 1,\dots,d$, define $G_i = G_{i-1} \cap w_i^\perp$. Since $G_i \trianglelefteq F_i$, we have $F_i^* \subseteq G_i^*$. Thus, $w_i \in L^\perp \cap (F_{i-1}^* \setminus F_{i-1}^\perp)$ implies that $w_i \in L^\perp \cap G_{i-1}^*$. This implies that $(G_0,\dots,G_d)$ is a sequence of faces such that $G_i \trianglelefteq G_{i-1}$ for $i = 1,\dots,d$. However, it is possible that $w_i \in G_{i-1}^\perp$ for some i, and in this case, we have $G_i = G_{i-1}$.

Thus, if counting only the steps with strict containment $G_i \subsetneq G_{i-1}$, then it is an FR sequence and this also yields a lower bound for $\mathrm{MSD}(L \cap G)$, i.e.,

$$|\{i \in \{1,\dots,d\} : G_i \subsetneq G_{i-1}\}| \le \mathrm{MSD}(L \cap G).$$

For each $j \in \{1,\dots,p\}$, we construct the sequence of faces $(G_0^j,\dots,G_d^j)$ for $L \cap G^j$ in the way as described above. Note that $G_0^j = G^j$. For every $i = 1,\dots,d$, we have

$$G_i^j = G^j \cap w_1^\perp \cap \cdots \cap w_i^\perp = G^j \cap K \cap w_1^\perp \cap \cdots \cap w_i^\perp = G^j \cap F_i.$$

Thus, by (22), we have

$$\mathrm{face}(\{G_i^1,\dots,G_i^p\},F_i) = \mathrm{face}(\{G^1 \cap F_i,\dots,G^p \cap F_i\},F_i) = F_i. \tag{23}$$

Next we show that the above condition implies that, for any $i \in \{1,\dots,d\}$, there exists at least one index $j \in \{1,\dots,p\}$ such that $w_i \notin (G_{i-1}^j)^\perp$ and thus the containment below is strict

$$G_i^j = G_{i-1}^j \cap w_i^\perp \subsetneq G_{i-1}^j.$$

Let $x \in \mathrm{ri}\left(\mathrm{conv}(\{G_{i-1}^1,\dots,G_{i-1}^p\})\right) \subseteq \mathrm{ri}(F_{i-1})$, see (23) and Item 1 in Lemma 2.1. Then $x = \sum_{j=1}^p \lambda_j v_j$ for some nonzero $\lambda \ge 0$ and $v_j \in G_{i-1}^j$. As $w_i \in F_{i-1}^* \setminus F_{i-1}^\perp$, we have

$$0 < \langle x, w_i\rangle = \sum_{j=1}^p \lambda_j \langle v_j, w_i\rangle.$$

As $G_{i-1}^j \trianglelefteq F_{i-1}$, we have $F_{i-1}^* \subseteq (G_{i-1}^j)^*$ and thus $\langle v_j, w_i\rangle \ge 0$. This means $\langle v_j, w_i\rangle > 0$ for at least one index j. This shows that $w_i \notin (G_{i-1}^j)^\perp$ and thus $G_i^j \subsetneq G_{i-1}^j$. This yields the desired inequality as

$$\mathrm{MSD}(L \cap K) \le \sum_{j=1}^p |\{i \in \{1,\dots,d\} : G_i^j \subsetneq G_{i-1}^j\}| \le \sum_{j=1}^p \mathrm{MSD}(L \cap G^j).$$

□

In the proof of Theorem 5.1, we need to apply Lemma 5.2 to a polyhedron. We show that it suffices to find faces $G^1,\dots,G^p$ of $\mathbb{R}_+^n$ such that $\mathrm{face}(\{G^1,\dots,G^p\},\mathbb{R}_+^n) = \mathbb{R}_+^n$. Then the condition (22) holds automatically.

Lemma 5.3. *Assume $G^1,\dots,G^p$ are faces of $\mathbb{R}_+^n$ such that*

$$\mathit{face}(\{G^1,\dots,G^p\},\mathbb{R}_+^n) = \mathbb{R}_+^n. \tag{24}$$

Then face$(\{G^1 \cap F,\dots,G^p \cap F\},F) = F$ for any $F \trianglelefteq \mathbb{R}_+^n$.

Proof. By the facial structure of $\mathbb{R}_+^n$ in (1), there exist subsets $T, S^1,\dots,S^p \subseteq \{1,\dots,n\}$ such that

$$\begin{array}{rcl} F &=& \{x \in \mathbb{R}_+^n \mid x(T) = 0\},\\ G^i &=& \{x \in \mathbb{R}_+^n \mid x(S^i) = 0\} \text{ for } i = 1,\dots,p. \end{array}$$

We have $G^i \cap F = \{x \in \mathbb{R}_+^n \mid x(S^i \cup T) = 0\}$. The assumption (24) is equivalent to $\cap_{i=1}^p S^i = \emptyset$. This implies that $\cap_{i=1}^p (S^i \cup T) = T$ and thus $\mathrm{face}(\{G^1 \cap F,\dots,G^p \cap F\},F) = F$. □

We note that Lemma 5.3 does not hold for the cone of positive semidefinite matrices. Let $K = \mathbb{S}^3_+$. Recall that the nonempty faces of $\mathbb{S}^n_+$ is characterized by the linear subspaces of $\mathbb{R}^n$, see (4). Let G^1, G^2 be faces of $\mathbb{S}^3_+$ associated with the following linear subspaces, respectively.

$$\operatorname{span}\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix} \text{ and } \operatorname{span}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}.$$

Clearly, we have $\operatorname{face}(\{G^1, G^2\}, \mathbb{S}^3_+) = \mathbb{S}^3_+$. Let W be the all-ones matrix. Then $W \in K^* = \mathbb{S}^3_+$. The exposed face $F = \mathbb{S}^3_+ \cap W^\perp$ is associated with the linear subspace

$$\operatorname{span}\begin{bmatrix}1 & 1\\ -1 & 0\\ 0 & -1\end{bmatrix}.$$

However, the faces $G^1 \cap F$ and $G^2 \cap F$ are associated with the following linear subspaces, respectively.

$$\operatorname{span}\begin{bmatrix}1\\ -1\\ 0\end{bmatrix} \text{ and } \{0\}.$$

Clearly, the smallest face of $\mathbb{S}^3_+$ containing both $G^1 \cap F$ and $G^2 \cap F$ is just $G^1 \cap F$, but $G^1 \cap F \subsetneq F$. This shows $\operatorname{face}(\{G^1 \cap F, G^2 \cap F\}, F) \subsetneq F$.

5.2.3 The MSD of a special set

In this section, we derive the MSD of a special polyhedron. Let $M = (M_{ij}) \in \{0,1\}^{p \times q}$ be a binary matrix with p rows and q columns. Let $\mathcal{E} = \{(i,j) \mid M_{ij} = 1\}$ denote the set of indices corresponding to nonzero entries in M. We define the polyhedron $H \cap \mathbb{R}^{\mathcal{E}}_+$, where the associated affine subspace H is given by

$$H := \{x \in \mathbb{R}^{\mathcal{E}} \mid a_i^T x = 0 \text{ for } i = 1, \ldots, p+q\} \tag{25}$$

for some binary vectors $a_i \in \{0,1\}^{\mathcal{E}}$. Note that the entries in the vectors $a_1, \ldots, a_{p+q}$ and x are indexed by the elements in $\mathcal{E}$. For $i = 1, \ldots, p$, the ones in $a_i \in \mathbb{R}^{\mathcal{E}}$ are exactly the entries associated with the ones in the i-th row of M. For $j = 1, \ldots, q$, the ones in $a_{p+j} \in \mathbb{R}^{\mathcal{E}}$ are exactly the entries associated with the ones in the j-th column of M. For example, let

$$M = \begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix} \in \mathbb{R}^{2\times 3}.$$

Then $p = 2$ and $q = 3$. The index set of nonzero entries is $\mathcal{E} = \{(1,1), (1,3), (2,1), (2,2)\}$. The vectors $a_1, \ldots, a_5 \in \mathbb{R}^{\mathcal{E}}$ and the variable $x \in \mathbb{R}^{\mathcal{E}}$ are indexed by the elements in $\mathcal{E}$, and they can be expressed as follows:

$$x = \begin{bmatrix}x_{(1,1)}\\ x_{(1,3)}\\ x_{(2,1)}\\ x_{(2,3)}\end{bmatrix} \in \mathbb{R}^{\mathcal{E}}, a_1 = \begin{bmatrix}1\\ 1\\ 0\\ 0\end{bmatrix}, a_2 = \begin{bmatrix}0\\ 0\\ 1\\ 1\end{bmatrix}, a_3 = \begin{bmatrix}1\\ 0\\ 1\\ 0\end{bmatrix}, a_4 = \begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix} \text{ and } a_5 = \begin{bmatrix}0\\ 1\\ 0\\ 0\end{bmatrix}.$$

Note that a_1, a_2 correspond to the two rows of M, and a_3, a_4, a_5 correspond to the three columns of M.

When $M \neq 0$, the only feasible solution to $H \cap \mathbb{R}_+^{\mathcal{E}}$ is the zero vector. In general, a_i may be a zero vector if it corresponds to a zero row or zero column in M; and we retain such vectors only for notational convenience. We show that if M is an all-ones matrix, then the MSD of the corresponding polyhedron $H \cap \mathbb{R}_+^{\mathcal{E}}$ can be derived analytically. This result relies on the characterization of the longest FR sequences for polyhedra in Theorem 4.1.

Lemma 5.4. *Let $M \in \{0,1\}^{p \times q}$ be an all-ones matrix with $p, q \geq 1$. Let $H \cap \mathbb{R}_+^{\mathcal{E}}$ be the polyhedron associated with M as defined in* (25). *Then MSD$(H \cap \mathbb{R}_+^{\mathcal{E}}) = p + q - 1$.*

Proof. We construct a minimal FR sequence of length $p + q - 1$ for $H \cap \mathbb{R}_+^{\mathcal{E}}$. As $H \cap \mathbb{R}_+^{\mathcal{E}}$ is a polyhedron, any minimal FR sequence is also one of the longest FR sequences (see Theorem 4.1). Note that $a_1, \ldots, a_{p+q}$ are in the dual cone of any face of $\mathbb{R}_+^n$ as they are nonnegative. We distinguish two cases depending on the values of p and q.

1. Assume $p = 1$ or $q = 1$. If $q = 1$, then $(a_1, \ldots, a_p)$ is a minimal FR sequence of length p. Indeed, one can verify that $a_i \in L^\perp \cap (F_{i-1}^* \setminus F_{i-1}^\perp)$ with $F_0 = \mathbb{R}_+^{\mathcal{E}}$ and $F_i = F_{i-1} \cap a_i^\perp$. By Lemma 3.1, a_i is minimal as $\dim F_{i-1} - \dim F_i = 1$. This shows that $\mathrm{MSD}(H \cap \mathbb{R}_+^{\mathcal{E}}) = p$. Similarly, if $p = 1$, then $\mathrm{MSD}(H \cap \mathbb{R}_+^{\mathcal{E}}) = q$.

2. Assume $p \geq 2$ and $q \geq 2$. We show that $(a_1, \ldots, a_{p-1}, a_{p+1}, \ldots, a_{p+q})$ is a minimal FR sequence. It is straightforward to verify that it is indeed an FR sequence. We show that it is minimal. Let $(F_0, \ldots, F_{p+q-1})$ be the corresponding sequence of faces. Assume, for contradiction, that a_1 is not minimal for $F_0 = \mathbb{R}_+^{\mathcal{E}}$. By (2) and (8), there exist $y_1, \ldots, y_{p+q} \in \mathbb{R}$ such that $u := \sum_{i=1}^{p+q} a_i y_i \in \mathbb{R}_+^{\mathcal{E}} \setminus \{0\}$ exposes a proper face of $\mathbb{R}_+^{\mathcal{E}}$ and
$$\emptyset \neq S := \{(i,j) \mid \text{the } (i,j)\text{-th entry of } u \text{ is nonzero}\} \subsetneq \{(1,j) \mid j = 1, \ldots, q\}. \tag{26}$$
Without loss of generality, we assume that $(1,1) \notin S$. Assume $y_1 = \lambda$. Then $y_{p+1} = -\lambda$ as $(1,1) \notin S$. From the containment in (26), we conclude that $y_2, \ldots, y_p = \lambda$, and therefore, $y_{p+2}, \ldots, y_{p+q} = -\lambda$. This leads to $u = 0$, contradicting our assumption.

   For $i = 2, \ldots, p-1$, we can apply the above argument in the same way to conclude that a_i is minimal for F_{i-1}. After that the problem is essentially equivalent to the first case. Thus, for $j = 1, \ldots, q$, we have a_{p+j} is minimal for F_{p+j-2}.

□

Using the formula for $\mathrm{MSD}(H \cap \mathbb{R}_+^{\mathcal{E}})$ when H is defined by the all-ones matrix in Lemma 5.4, we provide an upper bound for $\mathrm{MSD}(H \cap \mathbb{R}_+^{\mathcal{E}})$ when H is defined by an arbitrary binary matrix containing many duplicated columns. We state this upper bound in Lemma 5.5 in a form that is convenient for the proof of Theorem 5.1.

Lemma 5.5. *Let p and $\tilde{q}$ be some positive integers. For $j = 1, \ldots, \tilde{q}$, let $v_j \in \{0,1\}^p$ be a binary vector, and define the binary matrix $M_j = \begin{bmatrix} v_j & \cdots & v_j \end{bmatrix} \in \{0,1\}^{p \times 2\tilde{q}}$ consisting of $2\tilde{q}$ copies of v_j. Define the binary matrix $M \in \{0,1\}^{p \times (2\tilde{q}^2)}$ with p rows and $2\tilde{q}^2$ columns as*
$$M = \begin{bmatrix} M_1 & \cdots & M_{\tilde{q}} \end{bmatrix}.$$
Let $H \cap \mathbb{R}_+^{\mathcal{E}}$ be the polyhedron associated with M as described in (25).

1. *If $v_1, \ldots, v_{\tilde{q}}$ are all nonzero, then MSD$(H \cap \mathbb{R}^{\mathcal{E}}_+) \leq \sum_{j=1}^{\tilde{q}} (\boldsymbol{1}^T v_j + 2\tilde{q} - 1)$.*

2. *If each column v_j contains at most 3 nonzero entries, and at least one column v_j is the zero vector, then MSD$(H \cap \mathbb{R}^{\mathcal{E}}_+) \leq (2\tilde{q} + 2)(\tilde{q} - 1)$.*

Proof. 1. Recall that $\mathcal{E}$ is the index set of nonzero entries in M. Let $\mathcal{E}_j$ be the index set of nonzero entries in M_j, the j-th submatrix of M. Define the face G^j of $\mathbb{R}^{\mathcal{E}}_+$ as

$$G^j := \{x \in \mathbb{R}^{\mathcal{E}}_+ \mid x(\mathcal{E} \setminus \mathcal{E}_j) = 0\}.$$

As $\cap_{j=1}^{\tilde{q}} (\mathcal{E} \setminus \mathcal{E}_j) = \emptyset$, it is clear that $\text{face}(\{G^1, \ldots, G^p\}, \mathbb{R}^{\mathcal{E}}_+) = \mathbb{R}^{\mathcal{E}}_+$. Applying Lemma 5.2 and Lemma 5.3 yields $\text{MSD}(H \cap \mathbb{R}^{\mathcal{E}}_+) \leq \sum_{j=1}^{\tilde{q}} \text{MSD}(H \cap G^j)$. As every column v_j is nonzero, M_j is an all-ones matrix with possibly some additional rows of zeros. Since the additional rows of zeros do not correspond to any variables, we can apply Lemma 5.4 to obtain $\text{MSD}(H \cap G^j) = \mathbf{1}^T v_j + 2\tilde{q} - 1$. This yields the first inequality.

2. If v_j is nonzero and contains at most 3 nonzeros, then we have $\text{MSD}(H \cap G^j) \leq 2\tilde{q} + 2$. If v_j is the zero vector, then $\text{MSD}(H \cap G^j) = 0$. Since there are at most $\tilde{q} - 1$ nonzero columns v_j, the second inequality follows.

□

5.3 NP-hardness

In this section, we show that the decision version of finding a longest FR sequence for SDP problems is NP-hard. We will construct a polynomial-time transformation from the well-known NP-complete problem 3SAT. These problems are formally defined below.

3SATISFIABILITY (3SAT)
INSTANCE: A set $U = \{u_1, \ldots, u_p\}$ of Boolean variables, and $C = \{c_1, \ldots, c_q\}$ a collection of clauses on U where each clause c_i contains exactly three literals.
QUESTION: Is there a truth assignment for U such that all the clauses in C are satisfied.

MAXIMUM SINGULARITY DEGREE for SDP (MSD-SDP)
INSTANCE: Given matrices $A_1, \ldots, A_m \in \mathbb{S}^n$, vector $b \in \mathbb{R}^m$ and a positive integer d.
QUESTION: Does $L \cap \mathbb{S}^n_+$, as defined in (3), admit an FR sequence of length at least d?

For our analysis later, we preprocess any given 3SAT instance so that it satisfies some additional assumptions. If a clause $c_k = (u_i, \bar{u}_i, u_j)$ contains both a variable u_i and its negation $\bar{u}_i$, then c_k is trivially satisfied. Thus we can remove this clause c_k from the problem. If the positive literal u_i never appears in any clause, then we can assume u_i is assigned false, allowing us to remove u_i and all clauses containing $\bar{u}_i$. Similarly, if the negative literal $\bar{u}_i$ never appears, we can assume u_i is true, removing all clauses containing u_i. Since we can implement this preprocessing in polynomial time, we can assume without loss of generality that the given 3SAT instance satisfies the following properties.

Assumption 5.1. *The 3SAT instance satisfies*

- *For each variable u_i, we have $u_i \in c_j$ and $\bar{u}_i \in c_k$ for some j and k.*

- *Each clause c_j contains at most one of u_i or $\bar{u}_i$, but not both.*

For any 3SAT instance with p variables and q constraints, we construct an MSD-SDP instance as follows. The order of the matrix variable is $n = 6q+1$ and the number of constraints is $m = 2p+q$. The rows and columns of the matrix variable X and the data matrices $A_1, \dots, A_m$ are indexed by elements of the set

$$\mathcal{N} := \mathcal{N}_1 \cup \mathcal{N}_2 \cup \{(0,0,0)\},$$

where

$$\mathcal{N}_k := \{(i,j,k) \mid u_i \in c_j \text{ or } \bar{u}_i \in c_j,\ i = 1, \dots, p,\ j = 1, \dots, q\} \text{ for } k = 1, 2.$$

Note that $(i,j,k) \in \mathcal{N}_k$ for some k if and only if the clause c_j contains the variable u_i. By Assumption 5.1, u_i is either a positive literal or a negative literal in c_j, but not both. The third index k is simply used for making two identical copies $\mathcal{N}_1$ and $\mathcal{N}_2$. The last element $\{(0,0,0)\}$ serves an auxiliary role in the analysis.

Next, we specify the data matrices $A_1, \dots, A_m$ which can be classified into two different components based on their roles.

1. For each $i \in \{1, \dots, p\}$, we define two diagonal binary matrices A_i and $A_{p+i} \in \mathbb{S}^n$ with entries in $\{0,1\}$. They serve as truth-setting components to enforce a choice between assigning the variable $u_i \in U$ to true or false. Define the subsets

$$\begin{aligned} \mathcal{T}_i &:= \left\{(i,j,1) \in \mathcal{N}_1 \mid u_i \in c_j \text{ for some } j\right\}, \\ \mathcal{F}_i &:= \left\{(i,j,1) \in \mathcal{N}_1 \mid \bar{u}_i \in c_j \text{ for some } j\right\}, \\ \mathcal{V}_i &:= \left\{(i,j,2) \in \mathcal{N}_2 \mid j = 1, \dots, q\right\}. \end{aligned} \tag{27}$$

   In all three sets above, the variable j serves as the running index, iterating over clause indices. Conceptually, these sets represent the following: the set $\mathcal{T}_i$ can be viewed as the set of clauses containing the variable u_i as a positive literal, while the set $\mathcal{F}_i$ can be viewed as the set of clauses containing the variable u_i as a negative literal. The set $\mathcal{V}_i$ can be viewed as all clauses containing the variable u_i as a positive or negative literal. Finally, the third index in each tuple distinguishes elements in $\mathcal{T}_i$ and $\mathcal{F}_i$ (which use index 1) from those in $\mathcal{V}_i$ (which use index 2).

   The diagonal entries of A_i corresponding to $\mathcal{T}_i \cup \mathcal{V}_i$, and those of A_{p+i} corresponding to $\mathcal{F}_i \cup \mathcal{V}_i$, are set to 1. All other entries are 0. Note that A_i and A_{p+i} are positive semidefinite.

2. For each $j \in \{1, \dots, q\}$, we define a symmetric matrix $A_{2p+j} \in \mathbb{S}^n$ with binary entries, which serves as a satisfaction testing component for the clause c_j. Define the subsets

$$\begin{aligned} \mathcal{C}_j &:= \left\{(i,j,1) \in \mathcal{N}_1 \mid i = 1, \dots, p\right\}, \\ \mathcal{D}_j &:= \left\{(i,j,2) \in \mathcal{N}_2 \mid i = 1, \dots, p\right\}. \end{aligned}$$

   The sets $\mathcal{C}_j$ and $\mathcal{D}_j$ represent all elements of $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively, that correspond to the clause c_j. Note that $|\mathcal{C}_j| = |\mathcal{D}_j| = 3$. The nonzero entries of A_{2p+j} are specified as follows: the diagonal entries corresponding to $\mathcal{C}_j$ are ones. For the off-diagonal entries, the $(0,0,0)$-th row and the columns corresponding to $\mathcal{D}_j$ are set to 1, with symmetry ensuring the corresponding transpose entries are also 1. For example, the principal submatrix of A_{2p+j} associated with

$\mathcal{D}_j \cup \{(0,0,0)\}$ is the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \in \mathbb{S}^4. \tag{28}$$

Note that the principal submatrix (28) is indefinite and it has rank 2. The two nonzero eigenvalues are $\pm\sqrt{3}$ corresponding to the eigenvectors $\begin{bmatrix} 1 & 1 & 1 & \sqrt{3} \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 1 & 1 & -\sqrt{3} \end{bmatrix}^T$, respectively.

Let $b = 0 \in \mathbb{R}^m$ be the all-zeros vector of length m, and $d := p + q$. This defines an MSD-SDP instance, which can be constructed in polynomial time.

As usual, the feasible region of the MSD-SDP instance is denoted by $L \cap \mathbb{S}^n_+$, where $L = \{X \in \mathbb{S}^n \mid \langle A_i, X\rangle = 0 \text{ for } i = 1, \dots, m\}$ is the affine subspace determined by the constructed data matrices. It is not difficult to see that if $X \in L \cap \mathbb{S}^n_+$, then all rows and columns of X corresponding to $\mathcal{N}_1 \cup \mathcal{N}_2$ are zero, and the $(0,0,0)$-th diagonal entry can be any nonnegative number. Thus, the smallest face of $\mathbb{S}^n_+$ containing $L \cap \mathbb{S}^n_+$ has a block-diagonal structure given by

$$\{X \in \mathbb{S}^n_+ \mid X(\mathcal{N}_1 \cup \mathcal{N}_2, \mathcal{N}_1 \cup \mathcal{N}_2) = 0\}. \tag{29}$$

The sparsity pattern of the matrices in $L^\perp$ is important in the subsequent analysis. Let $W = A_1 y_1 + \cdots + A_m y_m \in L^\perp$, where $y_1, \dots, y_m \in \mathbb{R}$. For any $(i,j,1) \in \mathcal{N}_1$, the $(i,j,1)$-th diagonal entry of W is given by $y_i + y_{2p+j}$ if u_i appears as a positive literal in clause c_j, and $y_{p+i} + y_{2p+j}$ if u_i appears as a negative literal in clause c_j. Formally,

$$W(\{(i,j,1)\}, \{(i,j,1)\}) = \begin{cases} y_i + y_{2p+j}, & \text{if } (i,j,1) \in \mathcal{T}_i, \\ y_{p+i} + y_{2p+j}, & \text{if } (i,j,1) \in \mathcal{F}_i. \end{cases} \tag{30}$$

For any $(i,j,2) \in \mathcal{N}_2$, the 2×2 principal submatrix of W corresponding to $\{(i,j,2),(0,0,0)\}$ is given by

$$\begin{array}{cc} \begin{array}{cc} (i,j,2) & (0,0,0) \end{array} & \\ \begin{bmatrix} y_i + y_{p+i} & y_{2p+j} \\ y_{2p+j} & 0 \end{bmatrix} & \begin{array}{c} (i,j,2) \\ (0,0,0) \end{array} \end{array} \tag{31}$$

All other entries of W are zero, except those explicitly defined in (30) and (31).

We clarify the construction and the sparsity pattern using the following concrete example.

Example 5.1. *Assume $U = \{u_1, u_2, u_3\}$ and $C = \{c_1\}$, where c_1 is the clause $u_1 \vee u_2 \vee \bar{u}_3$. Then $\mathcal{N}_1 = \{(1,1,1),(2,1,1),(3,1,1)\}$ and $\mathcal{N}_2 = \{(1,1,2),(2,1,2),(3,1,2)\}$. The matrix $W \in L^\perp$ has the following form for some $y_1, \dots, y_7 \in \mathbb{R}$.*

$$W = \begin{array}{cc} \begin{array}{ccccccc} (1,1,1) & (2,1,1) & (3,1,1) & (1,1,2) & (2,1,2) & (3,1,2) & (0,0,0) \end{array} & \\ \begin{bmatrix} y_1+y_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_2+y_7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_6+y_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_1+y_4 & 0 & 0 & y_7 \\ 0 & 0 & 0 & 0 & y_2+y_5 & 0 & y_7 \\ 0 & 0 & 0 & 0 & 0 & y_3+y_6 & y_7 \\ 0 & 0 & 0 & y_7 & y_7 & y_7 & 0 \end{bmatrix} & \begin{array}{c} (1,1,1) \\ (2,1,1) \\ (3,1,1) \\ (1,1,2) \\ (2,1,2) \\ (3,1,2) \\ (0,0,0) \end{array} \end{array}$$

Note that the 3SAT instance in this example does not satisfy Assumption 5.1, as it was chosen for illustrative purposes.

The key idea behind the proof of Theorem 5.1 is to establish a correspondence between the truth assignments of a given 3SAT instance and the FR sequences for $L \cap \mathbb{S}^n_+$ in the constructed MSD-SDP problem. For the forward direction, we provide an informal discussion with a concrete example to clarify the idea. For the backward direction, we show some auxiliary results about the structure of the FR sequences. Finally, we provide the formal proof of both directions in Theorem 5.1.

In the forward direction, given a truth assignment that satisfies all clauses in the 3SAT instance, we can construct an FR sequence of length $p+q$ as follows. First, for the truth-setting components, if u_i is assigned true, we include A_{p+i} in the FR sequence; if u_i is assigned false, we include A_i in the FR sequence. Then, for the satisfaction-testing components, we include $A_{2p+1}, \ldots, A_{2p+q}$ in the FR sequence. We formally prove the forward direction in Theorem 5.1, referencing (37) and (38). To make it easier for the reader to verify the formal proof, we illustrate the construction using a concrete example here.

To illustrate this construction, consider the example in Example 5.1, where $p = 3$ and $q = 1$. Suppose a truth assignment is given where u_1 is set to true and u_2, u_3 are set to false. In this case, the clause c_1 is satisfied. This truth assignment induces an FR sequence (A_4, A_2, A_3, A_7) of length 4, following the construction process described above. Let $(F_0, \ldots, F_4)$ be the corresponding sequence of faces. These faces exhibit a block-diagonal structure, see (5),

$$F_i = \{X \in \mathbb{S}^n_+ \mid X(\mathcal{N} \setminus S_i, \mathcal{N} \setminus S_i) = 0\},$$

where $S_0, \ldots, S_4$ are subsets of $\mathcal{N}$ given by

$$\begin{aligned}
S_0 &= \{(0,0,0), (1,1,1), (3,1,1), (3,1,2), (2,1,1), (2,1,2), (1,1,2)\},\\
S_1 &= \{(0,0,0), (1,1,1), (3,1,1), (3,1,2), (2,1,1), (2,1,2)\},\\
S_2 &= \{(0,0,0), (1,1,1), (3,1,1), (3,1,2)\},\\
S_3 &= \{(0,0,0), (1,1,1), (3,1,1)\},\\
S_4 &= \{(0,0,0)\}.
\end{aligned}$$

The more challenging direction is to demonstrate that if a 3SAT instance is not satisfiable, then no FR sequence of length $p + q$ or greater exists. To this end, we first establish that there exists a longest FR sequence for $L \cap \mathbb{S}^n_+$ in the constructed MSD-SDP instance such that it corresponds to a truth assignment of the variables of the 3SAT instance.

We show that the faces in any FR sequence for the constructed MSD-SDP have a block-diagonal structure (see (5)). Since the rows and columns of the matrix variable X are indexed by the elements in $\mathcal{N}$, each face in an FR sequence corresponds to a specific subset of $\mathcal{N}$.

Lemma 5.6. *Let $f = (W_1, \ldots, W_d)$ be an FR sequence for $L \cap \mathbb{S}^n_+$, and let $(F_0, \ldots, F_d)$ denote the corresponding sequence of faces. Define $S_0 = \mathcal{N}$. For $r = 1, \ldots, d$, define subsets as follows:*

$$S_r = \{(i,j,k) \in S_{r-1} \mid \textit{the } (i,j,k)\textit{-th diagonal entry of } W_r \textit{ is zero}\}. \tag{32}$$

For any $r \in \{0, \ldots, d\}$, the face F_r has a block-diagonal structure given by:

$$F_r = \{X \in \mathbb{S}^n_+ \mid X(\mathcal{N} \setminus S_r, \mathcal{N} \setminus S_r) = 0\}, \tag{33}$$

and the exposing vector $W_r = A_1 y_1 + \cdots + A_m y_m \in L^\perp \cap F_{r-1}^*$, with $y_1, \ldots, y_m \in \mathbb{R}$, satisfies the following property: [1]

$$(i, j, 2) \in S_{r-1} \implies y_{2p+j} = 0. \tag{34}$$

In addition, $S_d = \mathcal{N} \setminus \{(0,0,0)\}$.

Proof. We prove the statement by induction. As $W_1 \in L^\perp \cap \mathbb{S}_+^n$, it follows from (31) that $y_{2p+j} = 0$ for all $j \in \{1, \ldots, q\}$. Thus, W_1 is a diagonal matrix. Consequently, S_1 and F_1 must be in the forms given by (32) and (33), respectively.

Assume that F_{r-1} has a block-diagonal structure in (33) for some subset $S_{r-1} \subseteq \mathcal{N}$ satisfying (32). Then $W_r \in L^\perp \cap F_{r-1}^*$ implies that the principal submatrix $W_r(S_{r-1}, S_{r-1})$ is positive semidefinite. Since the smallest face of $\mathbb{S}_+^n$ containing $L \cap \mathbb{S}_+^n$ is given by (29), we have $(0,0,0) \in S_{r-1}$ for all r. Thus, for any $(i,j,2) \in S_{r-1}$, the 2×2 principal submatrix corresponding to $\{(i,j,2),(0,0,0)\}$ is positive semidefinite. By (31), this implies $y_{2p+j} = 0$ for any $(i,j,2) \in S_{r-1}$. Hence, $W_r(S_{r-1}, S_{r-1})$ is a diagonal matrix. This ensures that F_r also has the form in (33), with the subset S_r given by $S_r = \{(i,j,k) \in S_{r-1} \mid \text{the } (i,j,k)\text{-th diagonal entry of } W_r \text{ is zero}\}$.

Since the smallest face of $\mathbb{S}_+^n$ containing $L \cap \mathbb{S}_+^n$ is given by (29), we conclude that the final subset in the sequence satisfies $S_d = \mathcal{N} \setminus \{(0,0,0)\}$. ☐

Using the relation between the subsets of $\mathcal{N}$ and the faces in FR sequences, we establish key properties of FR sequences in the following result. The key observation is that any longest FR sequence uses exactly one of the truth-setting components, A_i or A_{p+i}, in one of its FR steps.

Lemma 5.7. *Let $f = (W_1, \ldots, W_d)$ be an FR sequence for $L \cap \mathbb{S}_+^n$, and $(F_0, \ldots, F_d)$ the corresponding sequence of faces. Fix $\alpha \in \{1, \ldots, p\}$. Let $\mathcal{T}_\alpha, \mathcal{F}_\alpha$ and $\mathcal{V}_\alpha$ be as defined in (27). Let $S_0, \ldots, S_d$ be subsets of $\mathcal{N}$ as defined in (32). There exists a unique integer $l \in \{1, \ldots, d\}$ such that:*

1. $\mathcal{T}_\alpha \cup \mathcal{F}_\alpha \cup \mathcal{V}_\alpha \subseteq S_{l-1}$.
2. $\mathcal{V}_\alpha \cap S_l = \emptyset$.
3. $\mathcal{T}_\alpha \cup \mathcal{F}_\alpha \not\subseteq S_l$.

Assume f is also a longest FR sequence. Then the following statements holds:

4. $W_l \in \{A_\alpha, A_{p+\alpha}\}$ *up to some positive scaling.*
5. $(W_l, W_1, \ldots, W_{l-1}, W_{l+1}, \ldots, W_d)$ *is an FR sequence.*

Proof. By Lemma 5.6, there exists $l \in \{1, \ldots, d\}$ such that $\mathcal{T}_\alpha \cup \mathcal{F}_\alpha \cup \mathcal{V}_\alpha \subseteq S_{l-1}$ and $\mathcal{T}_\alpha \cup \mathcal{F}_\alpha \cup \mathcal{V}_\alpha \not\subseteq S_l$. The uniqueness follows from the inclusion $S_0 \supsetneq \cdots \supsetneq S_d$. Let the exposing vector $W_l \in L^\perp \cap (F_{l-1}^* \setminus F_{l-1}^\perp)$ at the l-th FR step be expressed as $W_l = A_1 y_1 + \cdots + A_m y_m$ for some $y_1, \ldots, y_m \in \mathbb{R}$. Since $W_l(S_{l-1}, S_{l-1})$ is positive semidefinite and $\mathcal{V}_\alpha \subseteq S_{l-1}$, it follows from (34) that

$$y_{2p+j} = 0 \quad \text{for every } j \in \{1, \ldots, q\} \text{ such that } (\alpha, j, 2) \in \mathcal{V}_\alpha. \tag{35}$$

By Assumption 5.1, there exists a clause c_β containing the variable u_α as a positive literal, i.e., $(\alpha, \beta, 1) \in \mathcal{T}_\alpha$. Since $W_l(S_{l-1}, S_{l-1})$ is positive semidefinite and $\mathcal{T}_\alpha \subseteq S_{l-1}$, it follows from (30) that $y_\alpha + y_{2p+\beta} \geq 0$. Substituting $y_{2p+\beta} = 0$ from (35), this implies $y_\alpha \geq 0$. Similarly,

[1] The coefficients $y_1, \ldots, y_m$ depend on r, but for readability, we omit explicit dependence on r in the notation.

we can argue that there exists a clause c_γ containing u_α as a negative literal, yielding $y_{p+\alpha} \geq 0$. If $y_\alpha = y_{p+\alpha} = 0$, the principal submatrix of W_l corresponding to $\mathcal{T}_\alpha \cup \mathcal{F}_\alpha \cup \mathcal{V}_\alpha$ would be zero. Consequently, $\mathcal{T}_\alpha \cup \mathcal{F}_\alpha \cup \mathcal{V}_\alpha \subseteq S_l$, contradicting the choice of l. Hence, $y_\alpha + y_{p+\alpha} > 0$. Consequently, the principal submatrix $W_l(\mathcal{V}_\alpha, \mathcal{V}_\alpha)$ is diagonal with positive diagonal entries $y_\alpha + y_{p+\alpha} > 0$. This implies $\mathcal{V}_\alpha \cap S_l = \emptyset$ by (32). Additionally, if $y_\alpha > 0$, then $\mathcal{T}_\alpha \cap S_l = \emptyset$. Similarly, if $y_{p+\alpha} > 0$, then $\mathcal{F}_\alpha \cap S_l = \emptyset$. This establishes the first three claims.

To prove the fourth claim, define $\tilde{W} := A_\alpha y_\alpha + A_{p+\alpha} y_{p+\alpha}$. Since y_α and $y_{p+\alpha}$ are nonnegative with at least one strictly positive and $\mathcal{T}_\alpha \cup \mathcal{F}_\alpha \cup \mathcal{V}_\alpha \subseteq S_{l-1}$, it follows that $\tilde{W} \in L^\perp \cap (F_{l-1}^* \setminus F_{l-1}^\perp)$. By (35) and the sparsity structure of elements in $L^\perp$ described in (30) and (31), the principal submatrices of $\tilde{W}$ and $W_l - \tilde{W}$ corresponding to S_{l-1} have disjoint nonzero entries. This means $W_l - \tilde{W} \in L^\perp \cap F_{l-1}^*$. Thus, we must have $W_l - \tilde{W} \in F_{l-1}^\perp$. If not, the aforementioned nonzero pattern of $\tilde{W}$ and $W_l - \tilde{W}$ implies that $W_l - \tilde{W} \notin (F_{l-1} \cap \tilde{W}^\perp)^\perp$. Then $(W_1, \ldots, W_{l-1}, \tilde{W}, W_l - \tilde{W}, W_{l+1}, \ldots, W_d)$ is also an FR sequence whose length is $d+1$, which is a contradiction. Consequently,

$$F_l = F_{l-1} \cap W_l^\perp = F_{l-1} \cap (W_l - \tilde{W} + \tilde{W})^\perp = F_{l-1} \cap (W_l - \tilde{W})^\perp \cap \tilde{W}^\perp = F_{l-1} \cap \tilde{W}^\perp.$$

The third equality uses $\tilde{W}, W_l - \tilde{W} \in F_{l-1}^*$, and the fourth equality follows from $W_l - \tilde{W} \in F_{l-1}^\perp$. Thus, W_l and $\tilde{W}$ expose the same face of F_{l-1}, allowing us to assume $W_l = \tilde{W} = A_\alpha y_\alpha + A_{p+\alpha} y_{p+\alpha}$. If y_α and $y_{p+\alpha}$ are both positive, then $\mathcal{T}_\alpha \cup \mathcal{F}_\alpha \subseteq S_{l-1}$ and Assumption 5.1 imply that $(W_1, \ldots, W_{l-1}, A_\alpha, A_{p+\alpha}, W_{l+1}, \ldots, W_d)$ is an FR sequence of length $d+1$, which is a contradiction. Thus, either $y_\alpha > 0$ or $y_{p+\alpha} > 0$, but not both. This means we can assume $W_l \in \{A_\alpha, A_{p+\alpha}\}$ for the longest FR sequences.

For the last claim, note that the nonzero diagonal terms of A_α and $A_{p+\alpha}$ correspond to elements in $\mathcal{T}_\alpha \cup \mathcal{V}_\alpha$ and $\mathcal{F}_\alpha \cup \mathcal{V}_\alpha$, respectively. Since $\mathcal{T}_\alpha \cup \mathcal{F}_\alpha \cup \mathcal{V}_\alpha \subseteq S_{l-1} \subsetneq \cdots \subsetneq S_0$ and $W_l \in \{A_\alpha, A_{p+\alpha}\}$, it follows from Lemma 5.6 that $(W_l, W_1, \ldots, W_{l-1}, W_{l+1}, \ldots, W_d)$ is also an FR sequence. $\square$

We conclude that at least one of the longest FR sequences corresponds to a truth assignment.

Corollary 5.1. *There exists at least one longest FR sequence $(W_1, \ldots, W_d)$ for $L \cap \mathbb{S}_+^n$ such that $W_i \in \{A_i, A_{p+i}\}$ for $i = 1, \ldots, p$.*

Proof. By applying Lemma 5.7, we can reorder the FR steps to ensure that $W_i \in \{A_i, A_{p+i}\}$ for $i = 1, \ldots, p$. This reordering preserves the length and validity of the FR sequence, guaranteeing the existence of at least one longest FR sequence that satisfies the desired property. $\square$

We are now ready to prove the main result.

Theorem 5.1. *The MSD-SDP problem is NP-hard.*

Proof. Let $U = \{u_1, \ldots, u_p\}$ and $\tilde{C} = \{\tilde{c}_1, \ldots, \tilde{c}_{\tilde{q}}\}$ be a 3SAT instance satisfying Assumption 5.1. Note that $p \geq 3$. To achieve the desired outcome, we introduce redundancy as follows. Each clause $\tilde{c}_i \in \tilde{C}$ is duplicated to create $2\tilde{q}$ copies. This results in a new set of clauses given by:

$$\{\underbrace{\tilde{c}_1, \ldots, \tilde{c}_1}_{2\tilde{q}}, \ldots, \underbrace{\tilde{c}_{\tilde{q}}, \ldots, \tilde{c}_{\tilde{q}}}_{2\tilde{q}}\}. \tag{36}$$

Define $q := 2\tilde{q}^2$. We consider the new 3SAT instance with the same variable set U and an expanded clause set $C = \{c_1, \ldots, c_q\}$ as defined in (36). For $j = 1, \ldots, \tilde{q}$, it follows that $c_i = \tilde{c}_j$ for any $i \in \{1 + 2\tilde{q}(j-1), \ldots, 2\tilde{q}j\}$.

We then transform this new 3SAT instance into an instance of MSD-SDP following the transformation process outlined earlier in this section. We show that the new 3SAT instance is satisfiable if and only if the constructed MSD-SDP instance has an FR sequence of length at least $d = p + q$.

Now, suppose a satisfying assignment exists for the new 3SAT instance. We will demonstrate how to construct an FR sequence $f = (W_1, \dots, W_d)$ of length d associated with the sequence of faces $(F_0, \dots, F_d)$. Let $(S_0, \dots, S_d)$ be the corresponding sequence of subsets of $\mathcal{N}$ as defined in (32).

- The first p steps are based on the truth assignment of the variables $u_1, \dots, u_p$. For $i = 1, \dots, p$, the exposing vector at the i-th FR step is given by

$$W_i = \begin{cases} A_i & \text{if } u_i \text{ is false} \\ A_{p+i} & \text{if } u_i \text{ is true.} \end{cases} \tag{37}$$

  By construction, $W_i \in L^\perp \cap (F_{i-1}^* \setminus F_{i-1}^\perp)$ ensuring that these are valid FR steps.

- The final q FR steps are defined as follows:

$$W_{p+j} = A_{2p+j} \text{ for } j = 1, \dots, q. \tag{38}$$

  Since the first p FR steps in (37) ensure that $S_p \cap \mathcal{N}_2 = \emptyset$, it follows that the principal submatrix $W_{p+j}(S_{p+j-1}, S_{p+j-1})$ is positive semidefinite and $W_{p+j} \in L^\perp \cap F_{p+j-1}$ is an exposing vector for $L \cap F_{p+j-1}$ for $j = 1, \dots, q$. Thus, it remains to show that $W_{p+j} \notin F_{p+j-1}^\perp$ so that it exposes a proper face of F_{p+j-1}. By Lemma 5.6, this is equivalent to showing that the principal submatrix of W_{p+j} associated with S_{p+j-1} is nonzero.

  Let $j = 1$. Recall that A_{2p+j} contains exactly three positive diagonal entries corresponding to the literals in the clause c_j. Given a satisfying assignment for the new 3SAT instance, at least one literal in c_j is true. Assume the clause c_j contains a positive literal u_i, and u_i is assigned true. Then $W_i = A_{p+i}$, based on the choice in (37). Since the $(i, j, 1)$-th diagonal entry is zero in A_{p+i}, it follows that $(i, j, 1) \in S_{p+j-1}$. Moreover, the $(i, j, 1)$-th diagonal entry of A_{2p+j} is one, which implies that $A_{2p+j} \notin F_{2p+j-1}^\perp$. A similar argument holds if the clause c_j contains the negative literal $\bar{u}_i$, and the variable u_j is assigned false. This shows that W_{p+j} yields a valid FR step.

  Furthermore, note that $A_{2p+\beta}$ and $A_{2p+\gamma}$ do not have any common nonzero entries for distinct $\beta, \gamma \in \{1, \dots, q\}$. By repeating this reasoning for each clause, we conclude that $W_{p+2}, \dots, W_{p+q}$ are also valid FR steps.

This yields an FR sequence of length $p + q$.

Conversely, assume that the new 3SAT instance is not satisfiable. Let $f = (W_1, \dots, W_r)$ be one of the longest FR sequences for the constructed SDP problem, and let $(F_0, \dots, F_r)$ denote the corresponding sequence of faces. Let $(S_0, \dots, S_r)$ be the corresponding sequence of subsets of $\mathcal{N}$ as defined in (32). We prove that the length of f is strictly smaller than $p + q$.

By Corollary 5.1, we can assume that

$$W_i \in \{A_i, A_{p+i}\} \quad \text{for } i = 1, \dots, p. \tag{39}$$

This induces a truth assignment unambiguously via the relation

$$u_i = \begin{cases} \text{false} & \text{if } W_i = A_i, \\ \text{true} & \text{if } W_i = A_{p+i}. \end{cases} \tag{40}$$

The face F_p is given by

$$F_p = \left\{ X \in \mathbb{S}^n_+ \mid X(\mathcal{N} \setminus S_p) = 0 \right\},$$

where

$$S_p = \{(i,j,1) \in \mathcal{T}_i \mid u_i \text{ is true}\} \cup \{(i,j,1) \in \mathcal{F}_i \mid u_i \text{ is false}\} \cup \{(0,0,0)\}. \tag{41}$$

Since f is one of the longest FR sequences, we obtain the relation

$$\mathrm{MSD}(L \cap \mathbb{S}^n_+) = p + \mathrm{MSD}(L \cap F_p).$$

We apply the simplification in Lemma 5.1 to show that $L \cap F_p$ is equivalent to an LP problem, and they have the same maximum singularity degree. Since the face F_p has a block-diagonal structure, we can apply Item 1 in Lemma 5.1 to simplify the problem by removing the rows and columns corresponding to $\mathcal{N} \setminus S_p$. Formally, define the data matrices

$$\tilde{A}_i := \begin{cases} A_i(S_p, S_p) & \text{if } W_i = A_{p+i}, \\ A_{p+i}(S_p, S_p) & \text{if } W_i = A_i, \end{cases}$$

for $i = 1, \dots, p$, and

$$\tilde{A}_{p+j} = A_{2p+j}(S_p, S_p),$$

for $j = 1, \dots, q$. Let $\tilde{n} = |S_p|$. Define the affine subspace

$$\tilde{L} := \left\{ \tilde{X} \in \mathbb{S}^{\tilde{n}} \mid \langle \tilde{A}_i, \tilde{X} \rangle = 0 \text{ for } i = 1, \dots, p+q \right\}.$$

By Item 1 in Lemma 5.1, we have $\mathrm{MSD}(L \cap F_p) = \mathrm{MSD}(\tilde{L} \cap \mathbb{S}^{\tilde{n}}_+)$. Note that we discarded exactly half of the data matrices from $A_1, \dots, A_{2p}$ in this process. This does not cause any issues. Specifically, if $W_i = A_i$, then $A_i \in F_p^\perp$; if $W_i = A_{p+i}$, then $A_{p+i} \in F_p^\perp$. Thus, these matrices can be removed freely.

Since $\tilde{A}_1, \dots, \tilde{A}_{p+q}$ are diagonal matrices, we can further simplify $\tilde{L} \cap \mathbb{S}^{\tilde{n}}_+$ by applying Item 2 in Lemma 5.1. This yields an equivalent LP problem. Additionally, note that the diagonal entry of $\tilde{A}_i$ associated with $(0,0,0)$ is zero for all i. Therefore, Item 3 in Lemma 5.1 allows us to freely remove the corresponding entry from the LP problem. Define $\mathcal{E} := S_p \setminus \{(0,0,0)\}$, and let a_i be the vector of diagonal entries of $\tilde{A}_i(\mathcal{E}, \mathcal{E})$, i.e., $a_i = \mathrm{diag}(\tilde{A}_i(\mathcal{E}, \mathcal{E})) \in \mathbb{R}^{\mathcal{E}}$. The final LP problem is $H \cap \mathbb{R}^{\mathcal{E}}_+$, where

$$H := \left\{ x \in \mathbb{R}^{\mathcal{E}} \mid a_i^T x = 0 \text{ for } i = 1, \dots, p+q \right\}.$$

By Items 2 and 3 in Lemma 5.1, we have $\mathrm{MSD}(L \cap F_p) = \mathrm{MSD}(H \cap \mathbb{R}^{\mathcal{E}}_+)$. Note that the elements in $\mathcal{E}$ are uniquely defined by their first two indices i and j, as $(i,j,k) \in \mathcal{E}$ implies $k = 1$. By examining the definition of S_p in (41), it follows directly that $H \cap \mathbb{R}^{\mathcal{E}}_+$ corresponds to the polyhedron constructed in (25), which is associated with the following binary matrix $M = (M_{ij}) \in \{0,1\}^{p \times q}$:

$$M_{i,j} = \begin{cases} 1 & \text{if } u_i \text{ is true and } u_i \in c_j, \\ 1 & \text{if } u_i \text{ is false and } \bar{u}_i \in c_j, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that the set of clauses C contains $2\tilde{q}$ duplications of each clause in $\tilde{C}$, as shown in (36). The matrix M can be written as:

$$M = \begin{bmatrix} M_1 & \cdots & M_{\tilde{q}} \end{bmatrix},$$

where each submatrix $M_j \in \{0,1\}^{p \times 2\tilde{q}}$ corresponds to the $2\tilde{q}$ duplicates associated with the same clause $\tilde{c}_j$ for $j = 1, \ldots, \tilde{q}$.

Consider the truth assignment defined in (40):

1. If c_j is satisfied, then $M_j \neq 0$ and it has at least one row of ones. Additionally, M_j contains at most three rows of ones because each clause contains three literals.

2. Since the given 3SAT instance is not satisfiable, there exists at least one unsatisfied clause. If c_j is unsatisfied, then M_j is an all-zeros matrix. Thus, there is at least one submatrix M_j that is equal to zero.

Therefore, the matrix M satisfies the assumptions in Item 2 of Lemma 5.5. This yields

$$\mathrm{MSD}(H \cap \mathbb{R}_+^{\mathcal{E}}) \leq (2\tilde{q} + 2)(\tilde{q} - 1). \tag{42}$$

We now obtain an upper bound for the length of f,

$$\mathrm{MSD}(L \cap \mathbb{S}_+^n) = p + \mathrm{MSD}(L \cap F_p) \leq p + (2\tilde{q} + 2)(\tilde{q} - 1) = p + q - 2 < d.$$

Thus, the constructed MSD-SDP instance does not admit any FR sequences of length d or more.

□

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