A Polyhedral Characterization of Linearizable Quadratic Combinatorial Optimization Problems

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Abstract

We introduce a polyhedral framework for characterizing instances of quadratic combinatorial optimization problems (QCOPs) as being linearizable, meaning that the quadratic objective can be equivalently rewritten as linear in such a manner that preserves the objective function value at all feasible solutions. In particular, we show that an instance is linearizable if and only if the quadratic cost coefficients can be used to construct a linear equation, in a lifted variable space, that is valid for the affine hull of a specially structured discrete set. In addition to developing this result for general QCOPs, we illustrate its utility in the specific context of the quadratic minimum spanning tree problem (QMSTP). As a consequence of this new polyhedral perspective on the concept of linearizability, we are able to make progress on a recent open question regarding linearizable QMSTP instances defined on biconnected graphs.

Key Words: nonlinear combinatorial optimization, linearizable, quadratic minimum spanning tree

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1 Introduction

Quadratic combinatorial optimization problems (QCOPs) are among the most widely studied problems in operations research, due both to their interesting mathematical structures and the diversity of their practical applications. The objective of such problems is to find an optimal collection of objects from within some finite set, where the function to be optimized is quadratic. Specifically, let $E = \{1, 2, \ldots, n\}$ be a finite set of size n and let F be some family of subsets of E. Suppose that a cost c_i is associated with each $i \in E$,

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and furthermore, a cost q_{ij} is associated with each *pair* of elements $i, j \in E$ with $i < j$. The QCOP can be expressed as

$$\text{minimize} \left\{ \sum_{i \in S} c_i + \sum_{i \in S} \sum_{\substack{j \in S \\ j > i}} q_{ij} : S \in F \right\}. \tag{1}$$

Note that any subset $S \in F$ can be represented by its 0-1 incidence vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, where $x_i = 1$ if and only if $i \in S$. Thus the restriction $S \in F$ can be represented as $\boldsymbol{x} \in \mathbf{X}$, where the constraints of $\mathbf{X}$ enforce that $\boldsymbol{x}$ is the incidence vector of some subset $S \in F$. In this way, (1) can be equivalently formulated as the quadratic 0-1 program

$$\text{QCOP: minimize} \left\{ \sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} q_{ij} x_i x_j : \boldsymbol{x} \in \mathbf{X},\ \boldsymbol{x} \text{ binary} \right\}. \tag{2}$$

Without loss of generality, in (1) and (2) we assume that the costs q_{ij} have an upper-triangular structure; that is, we do not include any q_{ij} having $i \geq j$. Using appropriate transformations, one could rewrite these costs to have some different structure (e.g., positive/negative semidefinite, symmetric), producing a theoretically equivalent formulation that could potentially have different computational performance; we refer the interested reader to [30] for a comprehensive study.

Our paper focuses exclusively on those instances of (2) for which $\mathbf{X}$ can be expressed as a system of linear equations and inequalities that has at least one solution. Furthermore, we are only concerned with such problems that are NP-hard, but have a linear counterpart, minimize $\{\sum_{i=1}^{n} c_i x_i : \boldsymbol{x} \in \mathbf{X},\ \boldsymbol{x} \text{ binary}\}$, that can be solved efficiently. While these conditions may seem restrictive, they are satisfied by many important problems. For example, the widely studied quadratic assignment problem (QAP) is NP-hard, but its linear counterpart, the linear assignment problem, can be solved efficiently by algorithms such as Kuhn's Hungarian method. Other such examples include the quadratic minimum spanning tree problem (QMSTP), the quadratic shortest path problem (QSPP), the bilinear assignment problem, and the quadratic transportation problem.

Unfortunately, despite having been studied extensively, most cases of Problem QCOP exhibit a significant gap between the sizes of motivating applications and those instances that can be optimally solved. For example, state-of-the-art exact solution strategies for the QAP [1, 4, 21, 22, 23] are limited to problems having $n \leq 40$. Similarly, the most efficient algorithms for solving the QMSTP on a complete graph are limited to graphs with 50 or fewer vertices [28]. An active research direction therefore focuses on identifying, based on objective function structure, special problem instances that are solvable in polynomial time. Many such identifications are available in the literature, often considered independently and from vastly different perspectives; no overarching theory exists. For example, researchers have studied readily solvable special cases of the QAP [3, 7, 8, 9, 10, 12, 11, 15, 19, 20, 25, 27, 29, 34], the QMSTP [16, 18], the QSPP [13, 14, 24, 31], and the bilinear assignment problem [17].

Of particular interest to our study are readily solvable instances of Problem QCOP that are known as *linearizable*, meaning that the problem can be equivalently rewritten as an instance of its linear counterpart in such a way that the objective function value is preserved for all binary $\boldsymbol{x} \in \mathbf{X}$. Formally, an instance of Problem QCOP with objective coefficients $\boldsymbol{c} \in \mathbb{R}^n$ and $\boldsymbol{q} \in \mathbb{R}^{n(n-1)/2}$ is called linearizable if there exist a vector $\hat{\boldsymbol{c}} \in \mathbb{R}^n$ and a scalar κ so that

$$\sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} q_{ij} x_i x_j = \kappa + \sum_{i=1}^{n} \hat{c}_i x_i \tag{3}$$

for all $\boldsymbol{x} \in \mathbf{X}$, $\boldsymbol{x}$ binary. Clearly, every linearizable instance of Problem QCOP is readily solvable because it can be reduced to an instance of its linear counterpart which, according to our stated assumptions of (2), can be solved efficiently.

Although much research has been done to identify necessary and/or sufficient conditions for various specific problem classes to be linearizable (see references above), no unifying framework exists to identify such conditions for general instances of Problem QCOP. The primary contribution of this paper is the establishment of a direct connection between the concept of linearizability and polyhedral theory for general instances of Problem QCOP. Specifically, we show in our main result that an instance of Problem QCOP is linearizable if and only if the quadratic cost coefficients can be used to create a linear equation, in a lifted variable space, that is valid (in the sense of Definition 2.1) for the affine hull of a specially structured discrete set. While this condition has been implicitly used [33] in the context of the QAP, we develop it explicitly for general QCOP instances. The remainder of the paper is structured as follows. Section 2 establishes our main result and connects it to the previous QAP-focused work of [33]. Then Section 3 illustrates the implications of this result in the specific context of the QMSTP. Notably, we use our new polyhedral perspective on the concept of linearizability to make progress on a recent open question of [16] regarding which underlying graph structures cause a known sufficient condition for QMSTP linearizability (that the quadratic cost coefficients form a weak sum matrix) to be necessary as well.

2 A Polyhedral Characterization of Linearizable QCOPs

Given an instance of Problem QCOP of (2), define the set

$$P \equiv \left\{ (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^{n(n-1)/2} : \ \boldsymbol{x} \in \mathbf{X},\ \boldsymbol{x} \text{ binary},\ y_{ij} = x_i x_j \ \ \forall\, i < j \right\}, \tag{4}$$

and let aff$\{\bullet\}$ denote the affine hull of the set $\bullet$.

Definition 2.1. *We say that an equation is* valid *for a set if it holds true for every point in that set.*

Theorem 2.1. *An instance of Problem QCOP with objective function coefficients $\boldsymbol{c} \in \mathbb{R}^n$ and $\boldsymbol{q} \in \mathbb{R}^{n(n-1)/2}$ is linearizable if and only if there exist a vector $\overline{\boldsymbol{c}} \in \mathbb{R}^n$ and a scalar $\overline{\kappa}$ such that the equation*

$$\sum_{i=1}^{n} \overline{c}_i x_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} q_{ij} y_{ij} = \overline{\kappa} \tag{5}$$

is valid for aff$\{P\}$*. Furthermore, the linearized problem is then given by*

$$\text{minimize} \left\{ \overline{\kappa} + \sum_{i=1}^{n} (c_i - \overline{c}_i) x_i : \boldsymbol{x} \in \mathbf{X},\ \boldsymbol{x} \text{ binary} \right\}. \tag{6}$$

Proof. Consider a linearizable instance of Problem QCOP with objective function coefficients $\boldsymbol{c} \in \mathbb{R}^n$ and $\boldsymbol{q} \in \mathbb{R}^{n(n-1)/2}$. By the definition of linearizable, there exist a vector $\hat{\boldsymbol{c}} \in \mathbb{R}^n$ and a scalar κ so that the equation

$$\sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} q_{ij} x_i x_j = \kappa + \sum_{i=1}^{n} \hat{c}_i x_i \tag{7}$$

holds for all $\boldsymbol{x} \in \mathbf{X}$, $\boldsymbol{x}$ binary. Setting $\overline{\boldsymbol{c}} = \boldsymbol{c} - \hat{\boldsymbol{c}}$ and $\overline{\kappa} = \kappa$ gives us that

$$\sum_{i=1}^{n} \overline{c}_i x_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} q_{ij} x_i x_j = \overline{\kappa} \tag{8}$$

holds for all $\boldsymbol{x} \in \mathbf{X}$, $\boldsymbol{x}$ binary. Therefore, the equation (5) holds for all $\boldsymbol{x} \in \mathbf{X}$, $\boldsymbol{x}$ binary, where $y_{ij} = x_i x_j$, meaning that (5) holds for all $(\boldsymbol{x}, \boldsymbol{y}) \in P$, and is therefore valid for aff$\{P\}$, as desired.

Now consider any instance of Problem QCOP with objective function coefficients $\boldsymbol{c} \in \mathbb{R}^n$ and $\boldsymbol{q} \in \mathbb{R}^{n(n-1)/2}$, and suppose that there exist a vector $\overline{\boldsymbol{c}} \in \mathbb{R}^n$ and a scalar $\overline{\kappa}$ such that (5) is valid for aff$\{P\}$. Since (5) must therefore hold for every $(\boldsymbol{x}, \boldsymbol{y}) \in P$, we have that (8) must hold for all $\boldsymbol{x} \in \mathbf{X}$, $\boldsymbol{x}$ binary. Setting $\hat{\boldsymbol{c}} = \boldsymbol{c} - \overline{\boldsymbol{c}}$ and $\kappa = \overline{\kappa}$ gives us that (7) holds for all $\boldsymbol{x} \in \mathbf{X}$, $\boldsymbol{x}$ binary, meaning that Problem QCOP is linearizable.

Finally, note that subtracting (8) from the objective function of Problem QCOP gives the linearized problem (6). □

Two remarks relative to Theorem 2.1 are warranted. First, the "if" direction of the theorem provides a framework for generating linearizable QCOP instances from any collection of linear equations that are valid for aff$\{P\}$. Specifically, a linearizable instance can be generated by taking any linear combination of the equations (note that this linear combination will also be valid for aff$\{P\}$), and then choosing the resulting coefficients on the variables y_{ij} to be the objective function coefficients q_{ij}.

Second, recall that any affine set can be expressed as the solution set of a system of linear equations (see, for example, [6, Example 2.1, p. 22]), meaning that the linear equations valid for aff$\{P\}$ completely define that set. Therefore, the "only if" direction of Theorem 2.1 reveals that the set of linearizable instances is completely determined by the structure of the set aff$\{P\}$. Specifically, the *only* way that an instance can be linearizable is if the quadratic objective coefficients q_{ij} match the coefficients on the variables y_{ij} from some linear equation that is valid for aff$\{P\}$. Hence, the task of characterizing all linearizable instances of Problem QCOP is *exactly* the same task as characterizing the linear equations valid for aff$\{P\}$ that have a non-zero coefficient for at least one of the variables y_{ij}. Although not explicitly stated in the terms of Theorem 2.1, the paper [33, Theorem 3] accomplished this task in the specific context of the QAP by showing that all of the equations valid for aff$\{P\}$ are implied by the equality constraints of the well-known level-1 reformulation-linearization-technique (RLT) reformulation of the QAP first given by [2]. This result led directly to a new polyhedral-based necessary and sufficient condition for QAP linearizability, namely that an instance of the QAP is linearizable if and only if a relaxed version of the level-1 RLT form is bounded. In Section 3, we similarly investigate the set aff$\{P\}$ for the QMSTP and use the resulting insights to characterize linearizable instances.

3 A Polyhedral Characterization of Linearizable QMSTP Instances

In this section, we discuss the implications of the results of Section 2 when specifcally applied to the QMSTP. Section 3.1 introduces the QMSTP, and then Section 3.2 uses the polyhedral framework provided by Theorem 2.1 to establish a sufficient condition for QMSTP linearizability, which is given in Theorem 3.2. Interestingly, this sufficient linearizability condition could also be necessary, depending on the structure of the graph upon which the QMSTP instance is defined. The framework of Theorem 2.1 allows us to develop, in Theorem 3.3, a polyhedral characterization of those graph structures for which the linearizability condition is both necessary and sufficient. We then restate our results in terms of weak sum matrices in order to draw parallels with previous work of [16]. In Section 3.3, we illustrate the utility of our results by showing that the sufficient linearizability condition of Theorem 3.2 is *not* necessary for biconnected graphs containing a vertex of degree 2, but that the condition *is* necessary for complete graphs. Finally, in Section 3.4, we use our new polyhedral perspective to make progress on the open question of [16] regarding linearizable QMSTP instances that was mentioned at the end of Section 1.

3.1 QMSTP Background

Given an undirected, connected graph $G = (V, E)$ with $|V| = n$ vertices and $|E| = m$ edges, the QMSTP is an NP-hard [5] QCOP of the form (2) that is formulated as

$$\text{QMSTP: minimize} \left\{ \sum_{e \in E} c_e x_e + \sum_{e \in E} \sum_{\substack{f \in E \\ f > e}} q_{ef} x_e x_f : \boldsymbol{x} \in \mathbf{X}_2(G),\ \boldsymbol{x} \text{ binary} \right\},$$

where

$$\mathbf{X}_2(G) \equiv \left\{ \begin{array}{ll} \boldsymbol{x} \in \mathbb{R}^m : & \sum_{e \in E} x_e = n - 1, \\ & \sum_{e \in E(S)} x_e \le |S| - 1 \quad \forall\, S \subset V,\ |S| \ge 2, \\ & \mathbf{0} \le \boldsymbol{x} \le \mathbf{1} \end{array} \right\}, \tag{9}$$

and $E(S)$ is the set of all edges having both vertices in the set S. Note that the objective function assumes an ordering of the edges in E. The problem's name arises from the fact that the objective function is quadratic in the m binary variables $\boldsymbol{x}$, and the constraints of $\mathbf{X}_2(G)$ (known as the *spanning tree polytope*), together with the binary restrictions, define the set of all 0-1 incidence vectors of spanning trees of G. A detailed description of the spanning tree polytope and its properties can be found, for example, in [26, Section 6.3, p. 142].

The QMSTP was first introduced by Assad and Xu in 1992 [5]. One classical application involves a telecommunications company laying cable in a new neighborhood. In this context, each house is represented by a vertex in the graph G, and edges correspond to possible paths along which cable could be buried between the houses. Each edge $e \in E$ costs a certain amount c_e based on factors such as the length of the path between the houses and how far underground the cable must be buried. The goal of the QMSTP is to find the cheapest way to connect all of the houses, with the coefficients q_{ef} on the quadratic terms accounting for the cost of potential interference between pairs of cables. The QMSTP has also found applications in areas such as oil and water transmission networks (where edges represent pipes) and transportation networks (where edges represent roads). In these settings, it is common for the quadratic costs to represent the price of transferring from one pipe/road to another, and therefore these costs are often only non-zero for pairs of adjacent edges.

The number of feasible solutions to Problem QMSTP quickly becomes unmanageable as the size of the problem grows; a complete graph with n vertices has n^{n-2} spanning trees (known as Cayley's formula). As a result, the best-performing QMSTP exact solution methods are only capable of solving problems on complete graphs having up to approximately 50 vertices [28]. Therefore, researchers have recently begun to study conditions under which Problem QMSTP is in fact readily solvable [16, 18]. This includes the study of linearizable instances. Consistent with (3), an instance of Problem QMSTP on a graph $G = (V, E)$ with objective coefficients $\boldsymbol{c} \in \mathbb{R}^m$ and $\boldsymbol{q} \in \mathbb{R}^{m(m-1)/2}$ is said to be linearizable if there exist a vector $\hat{\boldsymbol{c}} \in \mathbb{R}^m$ and a scalar κ so that

$$\sum_{e\in E} c_e x_e + \sum_{e\in E} \sum_{\substack{f\in E\\ f>e}} q_{ef} x_e x_f = \kappa + \sum_{e\in E} \hat{c}_e x_e \tag{10}$$

for all $\boldsymbol{x} \in \mathbf{X}_2(G)$, $\boldsymbol{x}$ binary, with $\mathbf{X}_2(G)$ given by (9). An instance of Problem QMSTP whose objective function is expressible in the form (10) is polynomially solvable, because it is reducible to a linear minimum spanning tree problem which can be solved by efficient methods such as Kruskal's algorithm or Prim's algorithm (see, for example, [26, Section 6.1, pp. 134–135] for a description of these algorithms).

3.2 A Sufficient Condition for QMSTP Linearizability

Given an instance of Problem QMSTP defined on a graph $G = (V, E)$ with $|V| = n$ vertices and $|E| = m$ edges, define the set

$$P_2(G) \equiv \left\{ (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^m \times \mathbb{R}^{m(m-1)/2} : \boldsymbol{x} \in \mathbf{X}_2(G),\ \boldsymbol{x} \text{ binary},\ y_{ef} = x_e x_f \ \ \forall\, e < f \right\}, \tag{11}$$

which is consistent with (4). The notation $P_2(G)$ is used to emphasize that this set depends on the structure of G, since the set $\mathbf{X}_2(G)$ depends on the structure of G. In light of the first remark following the proof of Theorem 2.1, a natural first step toward trying to identify linearizable QMSTP instances is to find equations that are valid for aff$\{P_2(G)\}$. Consider the m equations

$$\sum_{\substack{e\in E\\ e<f}} y_{ef} + \sum_{\substack{e\in E\\ e>f}} y_{fe} = (n-2)x_f \quad \forall f \in E. \tag{12}$$

Lemma 3.1. *Given any graph $G = (V,E)$, the equations (12) hold for all $(\boldsymbol{x},\boldsymbol{y}) \in P_2(G)$, i.e., they are valid for* $\text{aff}\{P_2(G)\}$.

Proof. The equations (12) are precisely those obtained by multiplying the single equation of $\mathbf{X}_2(G)$ in (9) by each of the m binary variables x_f, and then making the substitutions implied by (11) that $x_f x_f = x_f$ for all $f \in E$ and $x_e x_f = y_{ef}$ for all $e < f$. □

Since we have identified a set of equations that are valid for $\text{aff}\{P_2(G)\}$, the following sufficient condition for QMSTP linearizability is a direct consequence of the first remark following the proof of Theorem 2.1.

Theorem 3.2. *Consider an instance of Problem QMSTP defined on a graph $G = (V,E)$ with objective function coefficients $\boldsymbol{c} \in \mathbb{R}^m$ and $\boldsymbol{q} \in \mathbb{R}^{m(m-1)/2}$. If there exists a vector $\overline{\boldsymbol{c}} \in \mathbb{R}^m$ such that the equation*

$$\sum_{e \in E} \overline{c}_e x_e + \sum_{e \in E} \sum_{\substack{f \in E \\ f > e}} q_{ef} y_{ef} = 0 \tag{13}$$

can be computed as a linear combination of the equations (12), then the instance is linearizable.

This result begs the following question: are there any circumstances, such as special structures in the graph G, that would cause the sufficient linearizability condition of Theorem 3.2 to be necessary as well? It turns out that Theorem 2.1 provides a framework for answering this question, as shown in the following theorem.

Theorem 3.3. *Consider an instance of Problem QMSTP defined on a graph $G = (V,E)$. The sufficient linearizability condition of Theorem 3.2 is also necessary if and only if the structure of G enforces that for any equation of the form*

$$\sum_{e \in E} \alpha_e x_e + \sum_{e \in E} \sum_{\substack{f \in E \\ f > e}} \beta_{ef} y_{ef} = \kappa \tag{14}$$

that is valid for $\text{aff}\{P_2(G)\}$*, there exists a vector $\overline{\boldsymbol{\alpha}} \in \mathbb{R}^m$ such that*

$$\sum_{e \in E} \overline{\alpha}_e x_e + \sum_{e \in E} \sum_{\substack{f \in E \\ f > e}} \beta_{ef} y_{ef} = 0 \tag{15}$$

can be computed as a linear combination of the equations (12).

Proof. We begin with the ‘if’ direction. Consider a linearizable instance of Problem QMSTP defined on the graph G, say with objective function coefficients $\boldsymbol{c} \in \mathbb{R}^m$ and $\boldsymbol{q} \in \mathbb{R}^{m(m-1)/2}$. By Theorem 2.1 there exist a vector $\overline{\boldsymbol{c}} \in \mathbb{R}^m$ and a scalar $\overline{\kappa}$ such that the equation

$$\sum_{e\in E} \overline{c}_e x_e + \sum_{e\in E} \sum_{\substack{f\in E \\ f>e}} q_{ef} y_{ef} = \overline{\kappa} \tag{16}$$

is valid for aff$\{P_2(G)\}$. Since (16) is of the form (14), there exists a vector $\overline{\boldsymbol{\alpha}} \in \mathbb{R}^m$ such that

$$\sum_{e\in E} \overline{\alpha}_e x_e + \sum_{e\in E} \sum_{\substack{f\in E \\ f>e}} q_{ef} y_{ef} = 0$$

can be computed as a linear combination of the equations (12). Therefore, the sufficient linearizability condition of Theorem 3.2 is also necessary.

For the 'only if' direction, suppose that there exists an equation of the form (14), say

$$\sum_{e\in E} \hat{\alpha}_e x_e + \sum_{e\in E} \sum_{\substack{f\in E \\ f>e}} \hat{\beta}_{ef} y_{ef} = \hat{\kappa}, \tag{17}$$

that is valid for aff$\{P_2(G)\}$. Consider an instance of Problem QMSTP with objective function coefficients given by the $\hat{\boldsymbol{\beta}}$ of (17) and some arbitrary $\boldsymbol{c} \in \mathbb{R}^m$. This QMSTP instance is clearly linearizable since

$$\sum_{e\in E} c_e x_e + \sum_{e\in E} \sum_{\substack{f\in E \\ f>e}} \hat{\beta}_{ef} y_{ef} = \hat{\kappa} + \sum_{e\in E} (c_e - \hat{\alpha}_e) x_e$$

for all $(\boldsymbol{x}, \boldsymbol{y}) \in P_2(G)$, meaning that

$$\sum_{e\in E} c_e x_e + \sum_{e\in E} \sum_{\substack{f\in E \\ f>e}} \hat{\beta}_{ef} x_e x_f = \hat{\kappa} + \sum_{e\in E} (c_e - \hat{\alpha}_e) x_e$$

for all $\boldsymbol{x} \in \mathbf{X}_2(G)$, $\boldsymbol{x}$ binary. Therefore, because of the hypothesis that the sufficient linearizability condition of Theorem 3.2 is also necessary, there exists a vector $\overline{\boldsymbol{\alpha}} \in \mathbb{R}^m$ such that the equation

$$\sum_{e\in E} \overline{\alpha}_e x_e + \sum_{e\in E} \sum_{\substack{f\in E \\ f>e}} \hat{\beta}_{ef} y_{ef} = 0 \tag{18}$$

can be computed as a linear combination of the equations (12). □

In order to draw parallels with previous results from the literature [16] that we will explore in Section

3.3, we can rewrite the sufficient linearizability condition of Theorem 3.2 (in the upcoming Theorem 3.5) by defining $\boldsymbol{W}$ to be an $m \times m$ symmetric matrix containing the objective function coefficients of Problem QMSTP, where the $(e,f)^{th}$ element of $\boldsymbol{W}$ is given by

$$w_{ef} = \begin{cases} \frac{1}{2}q_{ef}, & \text{if } e < f \\ \frac{1}{2}q_{fe}, & \text{if } e > f \\ c_e, & \text{if } e = f \end{cases} . \tag{19}$$

The symmetric matrix $\boldsymbol{W}$ is called a *weak sum matrix* if there exists a vector $\boldsymbol{a} \in \mathbb{R}^m$ such that $w_{ef} = a_e + a_f$ for all $e \neq f$.

Lemma 3.4. *Given an instance of Problem QMSTP defined on a graph $G = (V,E)$ with objective function coefficients $\boldsymbol{c} \in \mathbb{R}^m$ and $\boldsymbol{q} \in \mathbb{R}^{m(m-1)/2}$, the matrix $\boldsymbol{W}$ defined in (19) is a weak sum matrix if and only if there exists a vector $\overline{\boldsymbol{c}} \in \mathbb{R}^m$ such that (13) can be computed as a linear combination of the equations (12).*

Proof. First suppose that there exists a vector $\overline{\boldsymbol{c}} \in \mathbb{R}^m$ such that (13) can be computed as a linear combination of the equations (12). Let $\boldsymbol{\lambda} \in \mathbb{R}^m$ be the vector of coefficients of that linear combination. It follows that $q_{ef} = \lambda_e + \lambda_f$ for all $e < f$, and therefore $w_{ef} = \alpha_e + \alpha_f$ for all $e \neq f$, where $\boldsymbol{\alpha} = \frac{1}{2}\boldsymbol{\lambda}$. Thus, $\boldsymbol{W}$ is a weak sum matrix.

Now suppose that $\boldsymbol{W}$ is a weak sum matrix, meaning that there exists a vector $\boldsymbol{\alpha} \in \mathbb{R}^m$ such that $w_{ef} = \alpha_e + \alpha_f$ for all $e \neq f$. So $q_{ef} = 2(\alpha_e + \alpha_f)$ for all $e < f$. Using $\boldsymbol{\lambda} = 2\boldsymbol{\alpha}$ as the coefficients of a linear combination of the equations (12) results in an equation of the form (13). □

Theorem 3.5. *Consider an instance of Problem QMSTP defined on a graph $G = (V,E)$ with objective function coefficients $\boldsymbol{c} \in \mathbb{R}^m$ and $\boldsymbol{q} \in \mathbb{R}^{m(m-1)/2}$. If the matrix $\boldsymbol{W}$ defined in (19) is a weak sum matrix, then the instance is linearizable.*

Proof. This result is previously established in the literature (see, for example, the proof of the "if" direction of [16, Theorem 5]). However, we situate it here within the polyhedral framework developed in Section 2 by observing that it immediately follows from Theorem 3.2 and Lemma 3.4. □

Observe that Lemma 3.4 establishes the equivalence of the sufficient conditions of Theorems 3.2 and 3.5. We therefore end this section with the remark that the sufficient linearizability condition of Theorem 3.5 is also necessary in precisely the same circumstances as those described by Theorem 3.3.

3.3 Examples

In this section we illustrate the utility of Theorem 3.3 by applying it to two different graph structures upon which instances of Problem QMSTP can be defined. In the first example, we show that for biconnected graphs having at least four vertices and at least one vertex of degree two, the sufficient linearizability condition of Theorem 3.2 (or equivalently that of Theorem 3.5) is *not* necessary. In the second example, we show that the condition is both necessary and sufficient for complete graphs. Although these results are previously known [16], we approach the proofs from the new polyhedral perspective of Theorem 3.3. This new perspective allows us to make progress on an open question related to the linearizability of QMSTP instances defined on biconnected graphs, as discussed in Section 3.4.

Corollary 3.6. *Consider an instance of Problem QMSTP defined on a biconnected graph $G = (V, E)$ with $n = |V| \geq 4$ and at least one vertex of degree two. The sufficient linearizability condition of Theorem 3.2 (or equivalently that of Theorem 3.5) is not necessary.*

Proof. By Theorem 3.3, it is sufficient to find an equation of the form (14) that is valid for $\operatorname{aff}\{P_2(G)\}$, but for which there exists no vector $\overline{\boldsymbol{\alpha}} \in \mathbb{R}^m$ such that (15) can be computed as a linear combination of the equations (12).

Let v be a vertex of degree two, and suppose that the two edges adjacent to v are labeled i and j, with $i < j$. Note that a spanning tree of G could contain both of the edges i and j, or it could contain exactly one of them. Since the degree of v is two, it is impossible for a spanning tree of G to contain neither edge i nor edge j.

Consider any $(\boldsymbol{x}, \boldsymbol{y}) \in P_2(G)$. If $y_{ij} = 1$, we have that $x_i + x_j = 2$. On the other hand, if $y_{ij} = 0$, we have that $x_i + x_j = 1$, which means that the equation

$$x_i + x_j - y_{ij} = 1 \tag{20}$$

is valid for $\operatorname{aff}\{P_2(G)\}$. Note that (20) is an equation of the form (14).

All that remains to show is that there exists no vector $\overline{\boldsymbol{\alpha}} \in \mathbb{R}^m$ such that the equation

$$\sum_{e \in E} \overline{\alpha}_e x_e - y_{ij} = 0 \tag{21}$$

can be computed as a linear combination of the equations (12). Suppose that there *did* exist such a linear combination, and let $\boldsymbol{\lambda} \in \mathbb{R}^m$ be the multipliers of that linear combination. Since the coefficient of y_{ij} in

(21) is -1, we must have that $\lambda_i + \lambda_j = -1$, so at least one of λ_i and λ_j must be negative. Without loss of generality, assume that $\lambda_i < 0$. Choose any other two edges in the graph, say edge g and edge h (for notational convenience, assume that the edges are labeled such that $g < h < i < j$). Since the coefficient of y_{gi} in (21) is zero, it must be true that $\lambda_g = -\lambda_i$ (so $\lambda_g > 0$). Furthermore, since the coefficient of y_{gh} in (21) is zero, it must be true that $\lambda_h = -\lambda_g$ (so $\lambda_h < 0$). Hence, λ_h and λ_i must both be negative, but the coefficient of y_{hi} in (21) is zero, a contradiction. □

Corollary 3.7. *Consider an instance of Problem QMSTP defined on the complete graph K_n. The sufficient linearizability condition of Theorem 3.2 (or equivalently that of Theorem 3.5) is also necessary.*

Proof. By Theorem 3.3, it is sufficient to show that given any equation of the form (14) that is valid for $\text{aff}\{P_2(G)\}$, there exists a vector $\overline{\alpha} \in \mathbb{R}^m$ such that (15) can be computed as a linear combination of the equations (12).

For this proof, we will assume that the $m = \frac{n(n-1)}{2}$ edges of the graph are labeled with the set of numbers $\{1, 2, \ldots, m\}$. Without loss of generality, we will label the edges such that edges 1, $m-1$, and m are connected on a path of length three that doesn't form a cycle, with edge 1 in the middle (except in the case where $n = 3$ and this is not possible). We define the set of edge pairs S as

$$S \equiv \{(e,f) \in E \times E : e < f,\ e \in \{1, m-1\}\}, \tag{22}$$

so that $|S| = m$. For notational convenience, we define the set

$$R \equiv \{(e,f) \in E \times E : e < f,\ e \notin \{1, m-1\}\}, \tag{23}$$

so that S and R form a partition of the edge pairs that define the variables y_{ef} in (14).

Note that within the left-hand-side coefficient matrix of the m equations (12), the columns corresponding to those variables y_{ef} having $(e,f) \in S$ form an $m \times m$ submatrix of zeros and ones. This submatrix has full rank, since it is exactly the same as the node-arc incidence matrix of a connected, undirected graph containing an odd cycle [32]. Therefore, we can take a linear combination of the equations (12) so that the resulting coefficients on the variables y_{ef} having $(e,f) \in S$ are any values that we want them to be.

In particular, consider some arbitrary equation of the form (14), say

$$\sum_{e \in E} \hat{\alpha}_e x_e + \sum_{e \in E} \sum_{\substack{f \in E \\ f > e}} \hat{\beta}_{ef} y_{ef} = \hat{\kappa}, \tag{24}$$

that is valid for aff$\{P_2(G)\}$. Following the discussion in the preceding paragraph, there must exist a linear combination of the equations (12) that yields an equation of the form

$$\sum_{e\in E}\overline{\alpha}_e x_e + \sum_{(e,f)\in S}\hat{\beta}_{ef}y_{ef} + \sum_{(e,f)\in R}\overline{\beta}_{ef}y_{ef} = 0, \tag{25}$$

where the coefficients $\hat{\beta}_{ef}$ having $(e,f)\in S$ are the same as those found in (24). Note that by Lemma 3.1, equation (25) is valid for aff$\{P_2(G)\}$.

The proof is now to show one final fact, that given any equation of the form (14) that is valid for aff$\{P_2(G)\}$, each β_{ef} having $(e,f)\in R$ is uniquely defined in terms of the β_{ef} having $(e,f)\in S$. Because (24) and (25) are both of the form (14) and are both valid for aff$\{P_2(G)\}$, the coefficients $\hat{\beta}_{ef}$ of (24) and $\overline{\beta}_{ef}$ of (25) will then have $\hat{\beta}_{ef}=\overline{\beta}_{ef}$ for all $(e,f)\in R$, so that equation (25) is expressible as

$$\sum_{e\in E}\bar{\alpha}_e x_e + \sum_{e\in E}\sum_{\substack{f\in E\\ f>e}}\hat{\beta}_{ef}y_{ef} = 0,$$

as desired. The proof of this final fact, details of which can be found in the Appendix, is inductive on the number of vertices n in the graph. □

3.4 Progress on an Open Question regarding Linearizable QMSTP Instances

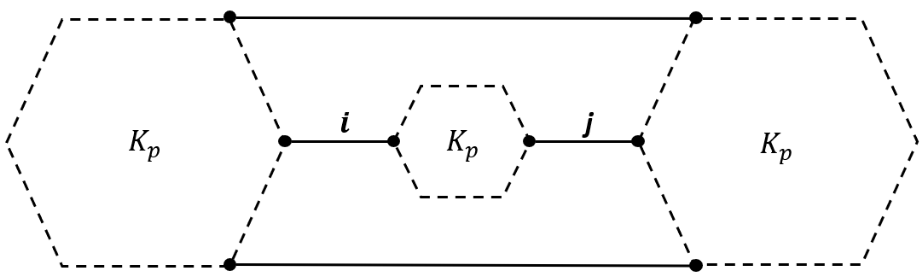
In Corollary 3.7 we saw that the sufficient linearizability condition of Theorem 3.2 (or equivalently that of Theorem 3.5) is also necessary for instances of Problem QMSTP defined on a complete graph. The paper [16] also shows that the condition is both necessary and sufficient for QMSTP instances defined on a complete bipartite graph K_{n_1,n_2} with $\min\{n_1,n_2\}\geq 3$. However, as shown in Corollary 3.6, there are biconnected graph structures for which the sufficient linearizability condition is not necessary. This naturally leads to the following open question, first posed [16] in the context of the weak sum matrices of Theorem 3.5.

Open Question 3.1. *For which biconnected graph structures is the sufficient QMSTP linearizability condition of Theorem 3.2 (or equivalently that of Theorem 3.5) also necessary?*

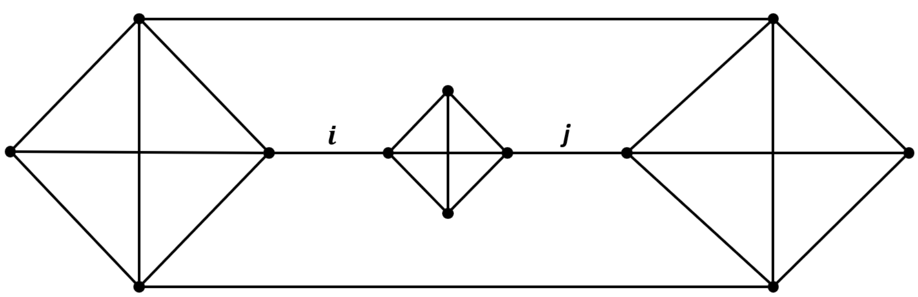
In their concluding remarks, the authors of [16] wonder whether it would be enough for a biconnected graph to have a minimum vertex degree of at least three in order to guarantee that the linearizability condition is both necessary and sufficient. By looking at this question through the new polyhedral lens of Theorem 3.3, we can deduce that there is no fixed lower bound on the minimum vertex degree that would guarantee the necessity of the linearizability condition.

Corollary 3.8. *There does not exist a positive integer K for which the following statement is true: the sufficient linearizability condition of Theorem 3.2 (or equivalently that of Theorem 3.5) is also necessary for any instance of Problem QMSTP defined on a biconnected graph with a minimum vertex degree of at least K.*

Proof. To prove the result we will construct a biconnected graph with arbitrarily large minimum vertex degree for which the sufficient linearizability condition of Theorem 3.2 (or equivalently that of Theorem 3.5) is not necessary. Consider the biconnected graph shown in Figure 1a, which is constructed from three copies of the complete graph on p vertices, K_p, connected by four additional edges, two of which are labeled i and j. Notice that the minimum vertex degree of this graph is $p-1$. So, for example, when $p=4$, this construction results in the graph shown in Figure 1b, which has minimum vertex degree three.



(a) Biconnected graph constructed from three copies of K_p, with minimum vertex degree $p-1$.



(b) The graph from Figure 1a when $p=4$, with minimum vertex degree three.

Figure 1: Graph structures used in the proof of Corollary 3.8.

By Theorem 3.3, it is sufficient to find an equation of the form (14) that is valid for aff$\{P_2(G)\}$, but for which there exists no vector $\overline{\alpha} \in \mathbb{R}^m$ such that (15) can be computed as a linear combination of the equations (12). We can accomplish this in exactly the same way as in the proof of Corollary 3.6 by forming equation (20), with edges i and j defined as shown in Figure 1. Recall the key component in the formation of (20) is that any spanning tree must contain either both of the edges i and j, or exactly one of them. $\square$

We end this section with an example that illustrates the result of Corollary 3.8. Consider an instance of Problem QMSTP defined on the graph depicted in Figure 1b having $n = 12$ vertices and $m = 22$ edges, where the the edges are labeled with the set of numbers $\{1,2,\ldots,22\}$, with edges i and j from the figure labeled 1 and 2, respectively. Suppose that the objective function coefficients are all equal to zero, with the single exception that $q_{12} = 1$ (i.e., the objective is to minimize $x_1 x_2$).

This QMSTP instance is clearly linearizable, since it can be equivalently rewritten in the sense of (10), using the idea of equation (20), as a *linear* minimum spanning tree problem defined on the same graph whose objective is to minimize $x_i + x_j - 1$. However, the corresponding 22×22 matrix $\boldsymbol{W}$ of (19), having all entries equal to zero except $w_{12} = w_{21} = \frac{1}{2}$, is not a weak sum matrix.

4 Conclusions and Future Research

This paper provides a polyhedral-based necessary and sufficient condition for characterizing general instances of Problem QCOP as being linearizable. Our general approach differs from previous literature on linearizable QCOPs, which considers specific problem classes (such as the QAP or the QMSTP) independently and from vastly different perspectives, in that our main result (Theorem 2.1) holds for any QCOP. This main result is of a primarily theoretical nature; its practical application to a specific QCOP requires a characterization of the set $\text{aff}\{P\}$ of (4), which is a non-trivial open question for many important problems.

This potential practical utility is illustrated in the specific context of the QMSTP, where we reestablish several previously known linearizability results using our new polyhedral framework, before using the insights gained to make progress on an open question of [16]. Specifically, we show that there is no fixed lower bound on the minimum vertex degree of a biconnected graph that would guarantee that the sufficient QMSTP linearizability condition of Theorem 3.2 (or equivalently that of Theorem 3.5) is also necessary.

Based on this work, two directions for future research naturally arise. First, we conjecture that the polyhedral framework of Theorem 3.3 can lead to further progress on Open Question 3.1. For example, perhaps the sufficient QMSTP linearizability condition of Theorem 3.2 is also necessary for instances defined on 3-connected graphs, or perhaps alternatively for instances defined on biconnected graphs where the minimum vertex degree is at least some fixed percentage of the total number of vertices. The proof of any such result would require an argument similar to that of the proof of Corollary 3.7. Equally interesting would be the establishment that such graph structures are *not* enough to guarantee the necessity of the linearizability condition of Theorem 3.2, which would require the construction of counterexamples similar to those found in the proofs of Corollaries 3.6 and 3.8. A second future research direction would be to investigate the implications of Theorem 2.1 to other QCOPs, such as the QSPP, the bilinear assignment problem, and the quadratic transportation problem.

5 Appendix

Corollary 3.7 – the proof's "final fact". *Consider an instance of Problem QMSTP defined on the complete graph K_n. Given any equation of the form (14) that is valid for* $\text{aff}\{P_2(G)\}$, *each β_{ef} having $(e,f) \in R$ is uniquely defined in terms of the β_{ef} having $(e,f) \in S$, where S and R are defined as in (22) and (23), respectively.*

Proof. The proof is inductive on the number of vertices n in the graph. For $n = 3$, we have that $R = \emptyset$, so the result trivially holds.

Base Case: $n = 4$

Consider an instance of Problem QMSTP defined on the complete graph with $n = 4$ vertices and $m = 6$ edges, as shown in Figure 2, and suppose that we have an equation of the form (14) that is valid for $\text{aff}\{P_2(G)\}$.

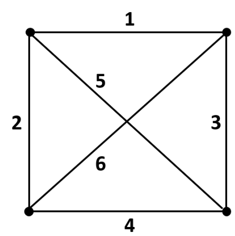


Figure 2: The complete graph with $n = 4$ vertices and $m = 6$ edges.

There are $4^{4-2} = 16$ solutions to $\boldsymbol{x} \in \mathbf{X}_2(G)$, $\boldsymbol{x}$ binary, meaning that $|P_2(G)| = 16$, with each of the 16 corresponding spanning trees containing three edges. Notationally, we distinguish each of these solutions with a row vector $\boldsymbol{\theta} \in \{1,2,3,4,5,6\}^3$ so that $\boldsymbol{x_\theta}$ denotes that binary solution having $x_{\theta(i)} = 1$ for all $i \in \{1,2,3\}$, and all other $x_f = 0$. For example, $\boldsymbol{x}_{(1,2,4)}$ has $x_1 = x_2 = x_4 = 1$, and $x_3 = x_5 = x_6 = 0$. Define $\boldsymbol{y_\theta}$ appropriately so that $(\boldsymbol{x_\theta}, \boldsymbol{y_\theta}) \in P_2(G)$. Since (14) is satisfied by all $(\boldsymbol{x}, \boldsymbol{y}) \in P_2(G)$, the 16 equations listed below must hold, where $E(\boldsymbol{x_\theta})$ denotes that linear equation obtained by setting $\boldsymbol{x} = \boldsymbol{x_\theta}$ and $\boldsymbol{y} = \boldsymbol{y_\theta}$ within (14).

(1) $E(\boldsymbol{x}_{(1,2,3)}) : \alpha_1 + \alpha_2 + \alpha_3 + \beta_{12} + \beta_{13} + \beta_{23} = \kappa$

(2) $E(\boldsymbol{x}_{(1,2,4)}) : \alpha_1 + \alpha_2 + \alpha_4 + \beta_{12} + \beta_{14} + \beta_{24} = \kappa$

(3) $E(\boldsymbol{x}_{(1,2,5)}) : \alpha_1 + \alpha_2 + \alpha_5 + \beta_{12} + \beta_{15} + \beta_{25} = \kappa$

(4) $E(\boldsymbol{x}_{(1,3,4)}) : \alpha_1 + \alpha_3 + \alpha_4 + \beta_{13} + \beta_{14} + \beta_{34} = \kappa$

(5) $E(\boldsymbol{x}_{(1,3,6)}) : \alpha_1 + \alpha_3 + \alpha_6 + \beta_{13} + \beta_{16} + \beta_{36} = \kappa$

(6) $E(\boldsymbol{x}_{(1,4,5)}) : \alpha_1 + \alpha_4 + \alpha_5 + \beta_{14} + \beta_{15} + \beta_{45} = \kappa$

(7) $E(\boldsymbol{x}_{(1,4,6)}) : \alpha_1 + \alpha_4 + \alpha_6 + \beta_{14} + \beta_{16} + \beta_{46} = \kappa$

(8) $E(\boldsymbol{x}_{(1,5,6)}) : \alpha_1 + \alpha_5 + \alpha_6 + \beta_{15} + \beta_{16} + \beta_{56} = \kappa$

(9) $E(\boldsymbol{x}_{(2,3,4)}) : \alpha_2 + \alpha_3 + \alpha_4 + \beta_{23} + \beta_{24} + \beta_{34} = \kappa$

(10) $E(\boldsymbol{x}_{(2,3,5)}) : \alpha_2 + \alpha_3 + \alpha_5 + \beta_{23} + \beta_{25} + \beta_{35} = \kappa$

(11) $E(\boldsymbol{x}_{(2,3,6)}) : \alpha_2 + \alpha_3 + \alpha_6 + \beta_{23} + \beta_{26} + \beta_{36} = \kappa$

(12) $E(\boldsymbol{x}_{(2,4,6)}) : \alpha_2 + \alpha_4 + \alpha_6 + \beta_{24} + \beta_{26} + \beta_{46} = \kappa$

(13) $E(\boldsymbol{x}_{(2,5,6)}) : \alpha_2 + \alpha_5 + \alpha_6 + \beta_{25} + \beta_{26} + \beta_{56} = \kappa$

(14) $E(\boldsymbol{x}_{(3,4,5)}) : \alpha_3 + \alpha_4 + \alpha_5 + \beta_{34} + \beta_{35} + \beta_{45} = \kappa$

(15) $E(\boldsymbol{x}_{(3,5,6)}) : \alpha_3 + \alpha_5 + \alpha_6 + \beta_{35} + \beta_{36} + \beta_{56} = \kappa$

(16) $E(\boldsymbol{x}_{(4,5,6)}) : \alpha_4 + \alpha_5 + \alpha_6 + \beta_{45} + \beta_{46} + \beta_{56} = \kappa$

Note that for $n = 4$, we have $S = \{(1,2), (1,3), (1,4), (1,5), (1,6), (5,6)\}$, and $R = \{(2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6)\}$. The following linear combinations of equations (1)–(16) above recursively express the nine coefficients β_{ef} having $(e,f) \in R$ in terms of the six coefficients β_{ef} having $(e,f) \in S$.

$$
\begin{aligned}
(2)-(3)-(12)+(13): &\quad \beta_{46} = \beta_{14} - \beta_{15} + \beta_{56} \\
(1)-(2)-(11)+(12): &\quad \beta_{36} = \beta_{13} - \beta_{14} + \beta_{46} = \beta_{13} - \beta_{14} + (\beta_{14} - \beta_{15} + \beta_{56}) \\
&\quad\phantom{\beta_{36}} = \beta_{13} - \beta_{15} + \beta_{56} \\
(4)-(7)-(14)+(16): &\quad \beta_{35} = \beta_{13} - \beta_{16} + \beta_{56} \\
(2)-(3)-(9)+(10): &\quad \beta_{34} = \beta_{14} - \beta_{15} + \beta_{35} = \beta_{14} - \beta_{15} + (\beta_{13} - \beta_{16} + \beta_{56}) \\
&\quad\phantom{\beta_{34}} = \beta_{13} + \beta_{14} - \beta_{15} - \beta_{16} + \beta_{56} \\
(4)-(5)-(14)+(15): &\quad \beta_{45} = \beta_{14} - \beta_{16} + \beta_{56} \\
(6)-(3)-(16)+(13): &\quad \beta_{26} = \beta_{12} - \beta_{14} + \beta_{46} = \beta_{12} - \beta_{14} + (\beta_{14} - \beta_{15} + \beta_{56}) \\
&\quad\phantom{\beta_{26}} = \beta_{12} - \beta_{15} + \beta_{56}
\end{aligned}
$$

$$
\begin{aligned}
(6)-(7)-(10)+(11): \quad \beta_{25} &= \beta_{15}-\beta_{16}+\beta_{26}-\beta_{35}+\beta_{36}+\beta_{45}-\beta_{46} \\
&= \beta_{15}-\beta_{16}+(\beta_{12}-\beta_{15}+\beta_{56})-(\beta_{13}-\beta_{16}+\beta_{56})+(\beta_{13}-\beta_{15}+\beta_{56}) \\
&\quad +(\beta_{14}-\beta_{16}+\beta_{56})-(\beta_{14}-\beta_{15}+\beta_{56}) \\
&= \beta_{12}-\beta_{16}+\beta_{56} \\
(7)-(8)-(12)+(13): \quad \beta_{24} &= \beta_{14}-\beta_{15}+\beta_{25} = \beta_{14}-\beta_{15}+(\beta_{12}-\beta_{16}+\beta_{56}) \\
&= \beta_{12}+\beta_{14}-\beta_{15}-\beta_{16}+\beta_{56} \\
(11)-(12)-(15)+(16): \beta_{23} &= \beta_{24}+\beta_{35}-\beta_{45} \\
&= (\beta_{12}+\beta_{14}-\beta_{15}-\beta_{16}+\beta_{56})+(\beta_{13}-\beta_{16}+\beta_{56})-(\beta_{14}-\beta_{16}+\beta_{56}) \\
&= \beta_{12}+\beta_{13}-\beta_{15}-\beta_{16}+\beta_{56}
\end{aligned}
$$

This completes the $n=4$ base case.

Now consider an instance of Problem QMSTP defined on a complete graph having $n \geq 5$ vertices, and suppose that we have an equation of the form (14) that is valid for aff $\{P_2(G)\}$. Choose a vertex, call it T, that is not adjacent to edges labeled $1, m-1$, or m. Define E_T as the set of edges that are adjacent to vertex T and define E_N as the set of edges that are *not* adjacent to vertex T. Note that $|E_T| = n-1$ and $|E_N| = \frac{(n-1)(n-2)}{2}$. For convenience, we will partition the index set R of (23) into three subsets. Let $R_1 = \{(e,f) \in R : e \in E_N, f \in E_N\}, R_2 = \{(e,f) \in R : e \in E_N, f \in E_T \text{ or } e \in E_T, f \in E_N\}$, and $R_3 = \{(e,f) \in R : e \in E_T, f \in E_T\}$.

First, we will use an inductive argument to show that each β_{ef} having $(e,f) \in R_1$ is uniquely defined in terms of the β_{ef} having $(e,f) \in S$. To that end, randomly select some edge $\ell \in E_T$ (which means that $\ell \notin \{1, m-1, m\}$), and consider the set $P_2' \subset P_2(G)$ consisting of all $(\boldsymbol{x}, \boldsymbol{y}) \in P_2(G)$ that correspond to spanning trees containing the edge ℓ and no other edge adjacent to vertex T. For every such spanning tree, we must have that $x_\ell = 1$ and $x_e = 0$ for all other edges $e \in E_T$, and therefore from (14) we have that

$$
\sum_{\substack{e \in E_N \\ e<\ell}} (\alpha_e + \beta_{e\ell}) x_e + \sum_{\substack{e \in E_N \\ e>\ell}} (\alpha_e + \beta_{\ell e}) x_e + \sum_{e \in E_N} \sum_{\substack{f \in E_N \\ f>e}} \beta_{ef} y_{ef} = \kappa - \alpha_\ell \tag{26}
$$

must be satisfied by all $(\boldsymbol{x}, \boldsymbol{y}) \in P_2'$. Note that (26) takes the exact form of (14) for an instance of Problem QMSTP defined on the complete graph having the $n-1$ vertices in the set $V - \{T\}$ and the reduced edge set E_N.

It is convenient to retain the original labels on the $\frac{(n-1)(n-2)}{2}$ edges in E_N, which still includes the edges labeled $1, m-1$, and m. Now, in a similar manner to how (22) and (23) were used to define the index sets S

and R, respectively, we can define index sets S_N and R_N, but this time in terms of the reduced edge set E_N. Specifically,

$$S_N = \{(e,f) \in E_N \times E_N : e \in \{1, m-1\}\}, \tag{27}$$

and R_N is the set of all edge pairs $(e,f) \in E_N \times E_N$ such that $(e,f) \notin S_N$.

By the inductive hypothesis, each coefficient β_{ef} having $(e,f) \in R_N$ is uniquely defined in terms of the β_{ef} having $(e,f) \in S_N$. Notice that $S_N \subseteq S$ and $R_N = R_1$, and therefore each coefficient β_{ef} having $(e,f) \in R_1$ is uniquely defined in terms of the β_{ef} having $(e,f) \in S$, as desired.

Next we will show that each β_{ef} having $(e,f) \in R_2$ is uniquely defined in terms of the β_{ef} having $(e,f) \in S$. Arbitrarily select some $(i,j) \in R_2$. Without loss of generality, assume that $i \in E_N$ and $j \in E_T$. Note that the edges $1, i$, and j, along with vertex T, must be arranged in one of the seven configurations displayed in Figure 3 (the six-vertex configuration is only possible when $n \geq 6$).

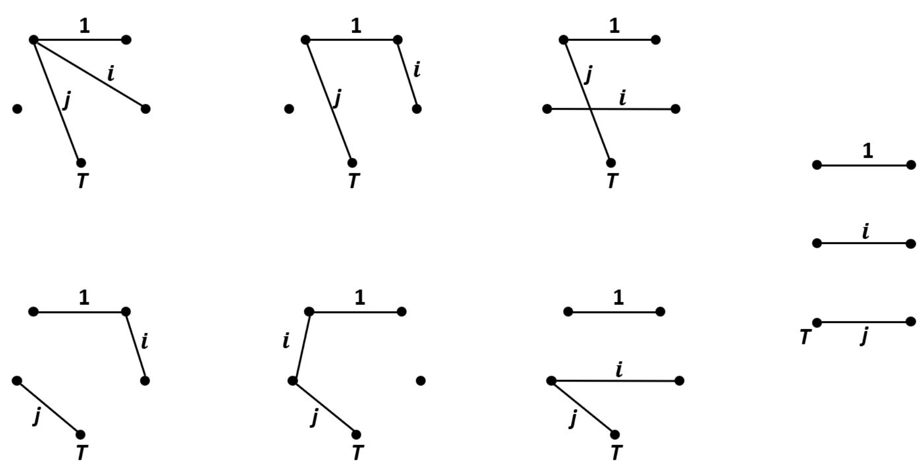


Figure 3: Possible edge configurations when $i \in E_N$ and $j \in E_T$.

For each of the seven configurations in Figure 3, we define edges a, b, and y as shown in Figure 4. For the configuration with six vertices, we define an additional edge c.

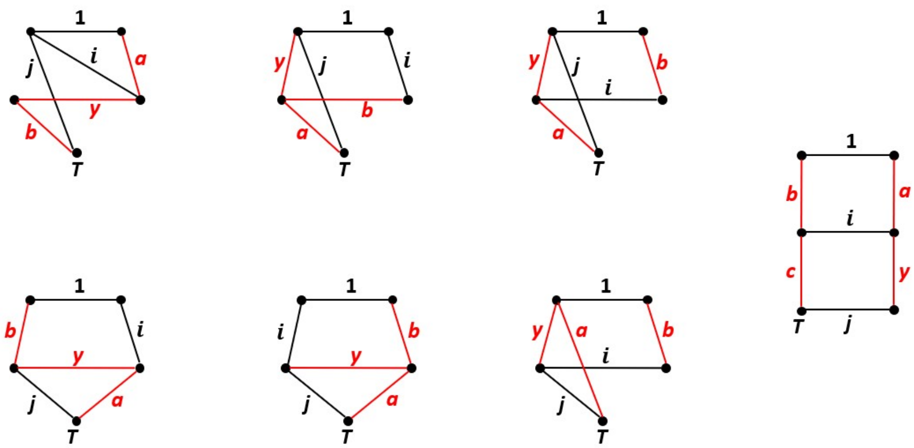


Figure 4: Additional edge definitions based on the configurations of Figure 3.

Now consider the four different spanning trees that result from selecting:

a. every edge that connects an undrawn vertex to T (if any such vertices exist),

b. edges a and b,

c. edge c in the case of the configuration with six vertices, and

d. one of the following pairs of edges:

   (i) edges i and j,

   (ii) edges i and y,

   (iii) edges 1 and j, or

   (iv) edges 1 and y.

Each of these four spanning trees, differentiated by options (i)–(iv) in part (d), corresponds to an $(\boldsymbol{x},\boldsymbol{y}) \in P_2(G)$ that can be plugged into (14) to generate an equation that relates the parameters α_e, β_{ef}, and κ. Taking the linear combination (i)-(ii)-(iii)+(iv) of these equations results in $\beta_{ij} = \beta_{iy} + \beta_{1j} - \beta_{1y}$ (if $i > y$, replace β_{iy} with β_{yi}). Since $(1,y) \in S, (1,j) \in S$ and $(i,y) \in R_1$ (unless edges i and y happen to be the edges labeled $m-1$ and m, in which case $(i,y) \in S$), we have that β_{ij} can be uniquely defined in terms of the β_{ef} having $(e,f) \in S$. Since i and j were arbitrarily chosen, the result holds true for any coefficient β_{ef} having $(e,f) \in R_2$.

Finally, we will prove that each β_{ef} having $(e,f) \in R_3$ is uniquely defined in terms of the β_{ef} having $(e,f) \in S$. Arbitrarily select some $(i,j) \in R_3$. Note that the edges $1, i$, and j, along with vertex T, must be arranged in one of the three configurations displayed in Figure 5 (it is unnecessary to show a fourth configuration that is equivalent to the third with i and j interchanged).

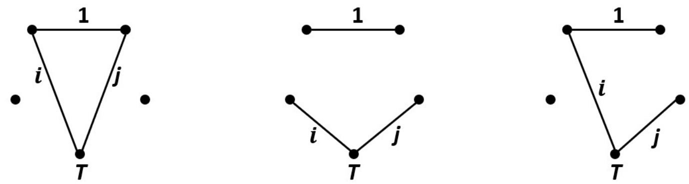


Figure 5: Possible edge configurations when $i \in E_T$ and $j \in E_T$.

For each of the three configurations in Figure 5, we define edges a, b, and y as shown in Figure 6.

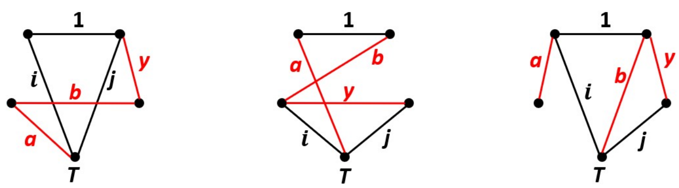


Figure 6: Additional edge definitions based on the configurations of Figure 5.

Now consider the four different spanning trees that result from selecting:

a. every edge that connects an undrawn vertex to T (if any such vertices exist),

b. edges a and b, and

c. one of the following pairs of edges:

   (i) edges i and j,

   (ii) edges i and y,

   (iii) edges 1 and j, or

   (iv) edges 1 and y.

Each of these four spanning trees, differentiated by options (i)–(iv) in part (d), corresponds to an $(\boldsymbol{x}, \boldsymbol{y}) \in P_2(G)$ that can be plugged into (14) to generate an equation that relates the parameters α_e, β_{ef}, and κ. Taking the linear combination (i)-(ii)-(iii)+(iv) of these equations results in $\beta_{ij} = \beta_{iy} + \beta_{1j} - \beta_{1y}$ (if $i > y$, replace β_{iy} with β_{yi}). Since $(1, j) \in S, (1, y) \in S$ and $(i, y) \in R_2$, we have that β_{ij} can be uniquely defined in terms of the β_{ef} having $(e, f) \in S$. Since i and j were arbitrarily chosen, the result holds true for any coefficient β_{ef} having $(e, f) \in R_3$. □

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