THE BEST APPROXIMATION PAIR PROBLEM RELATIVE TO TWO SUBSETS IN A NORMED SPACE

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Abstract. In the classical best approximation pair (BAP) problem, one is given two nonempty, closed, convex and disjoint subsets in a finite- or an infinite-dimensional Hilbert space, and the goal is to find a pair of points, each from each subset, which realizes the distance between the subsets. This problem, which has a long history, has found applications in science and technology. We discuss the problem in more general normed spaces and with possibly non-convex subsets, and focus our attention on the issues of uniqueness and existence of the solution to the problem. To the best of our knowledge these fundamental issues have not received much attention. In particular, we present several sufficient geometric conditions for the (at most) uniqueness of a BAP relative to these subsets. These conditions are related to the structure of the boundaries of the subsets, their relative orientation, and the structure of the unit sphere of the space. In addition, we present many sufficient conditions for the existence of a BAP, possibly without convexity. Our results allow us to significantly extend the horizon of the recent alternating simultaneous Halpern-Lions-Wittmann-Bauschke (A-S-HLWB) algorithm [Censor, Mansour and Reem, The alternating simultaneous Halpern-Lions-Wittmann-Bauschke algorithm for finding the best approximation pair for two disjoint intersections of convex sets, arXiv:2304.09600 (2023)] for solving the BAP problem.

1. Introduction

1.1. **Background.** The classical best approximation pair (BAP) problem is the following problem: there are two nonempty, disjoint, closed and convex subsets A and B in a finite- (i.e., Euclidean) or an infinite-dimensional real Hilbert space $(X, \|\cdot\|)$, and the goal is to find a pair of points, each from each subset, which realizes the distance between the subsets. In other words, the BAP problem is the following minimization problem: to find a pair $(\widetilde{a}, \widetilde{b}) \in A \times B$ such that

$$dist(A,B) := \inf\{\|a-b\| \mid a \in A, b \in B\} = \inf f(A \times B) = f(\widetilde{a}, \widetilde{b}) = \|\widetilde{a} - \widetilde{b}\| \quad (1.1)$$

where $f : X^2 \to [0,\infty)$ is defined by $f(x,y) := \|x-y\|$ for all $(x,y) \in X^2$. This problem has applications in science and technology: see, for instance, [**25**, **27**, **39**, **72**] and the references therein for applications in signal processing. It has a long history which goes back to the classical 1959 work of Cheney and Goldstein [**22**] (see also [**40**])

Date: March 27, 2024.

2020 *Mathematics Subject Classification.* 41A65, 46B20, 46N10, 90C25, 90C26.

Key words and phrases. Affine subspace, best approximation pair (BAP), boundary of a set, convex subset, existence of a BAP, normed space, parallel intervals, reflexive Banach space, strictly convex, uniqueness of a BAP, weakly sequentially compact.

and continues with various other works such as, e.g., [**1**, **4**–**6**, **9**, **10**, **20**, **30**, **39**, **44**, **48**, **49**, **51**, **70**, **72**]. See also [**15**, **35**, **36**, **41**, **42**] for the linear case, namely when both A and B are affine subspaces and the space is Euclidean, and [**31**, **65**] for the case where one of the subsets is a point (this is the so-called "best approximation problem", that is, the problem of projecting a point on another subset).

So far, most of the attention regarding this problem has been focused on the above-mentioned classical setting, but there are a few works which go beyond this setting, such as [**54**, **66**, **71**] (normed spaces beyond Hilbert spaces) and [**51**, **71**] (nonconvex sets), although with the exception of [**71**], their focus is not on the issues of existence or uniqueness, but rather on algorithmic or characterization aspects. We note that there is a related but somewhat different (and quite large) domain of research: the one concerning the so-called "best proximity pair/points theorems". Here one starts with some space X, subsets A and B of X, a mapping defined on X (possibly multi-valued, possibly with a non-full domain of definition), and one is interested in finding conditions on A, B, X and T which ensure the existence of some $x \in A$ such that $dist(x, Tx) = dist(A, B)$, or variations of this equation (our existence results might enlarge the pool of such sufficient conditions). For a very partial list of related works, see [**32**, **47**, **55**, **57**, **62**, **63**, **67**] and the references therein.

In this work we discuss the BAP problem in more general normed spaces and with possibly nonconvex subsets. We focus our attention on the fundamental issues of uniqueness and existence of the solution to the problem. To the best of our knowledge these issues have not received much attention, especially beyond the classical setting.

One of our main goals is to formulate conditions which will imply the (at most) uniqueness of the BAP. Our motivation comes from the recent work [**18**] in which we discussed the alternating simultaneous Halpern-Lions-Wittman-Bauschke (A-S-HLWB) algorithm for solving the BAP problem in the Euclidean space $\mathbb{R}^k$ ($k \in \mathbb{N}$), under the additional assumption that both A and B are finite intersections of closed and convex subsets, that is, $A = \cap_{i=1}^{m} A_i$ and $B = \cap_{j=1}^{n} B_j$ for some $m, n \in \mathbb{N}$ (this assumption leads to the computational advantage that one can orthogonally project iteratively onto the individual subsets A_i and B_j, $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$ instead of projecting directly onto A and B, a task which can be rather demanding from the computational point of view.). The practical importance of this scenario stems from its relevance to real-world situations, wherein the feasibility-seeking modelling is used and there are two disjoint constraints sets: one set, namely A, represents constraints which must be satisfied ("hard" constraints), while the other set (i.e., B) represents constraints which, hopefully, will be satisfied ("soft" constraints). In this scenario the goal is to find a point which satisfies all the hard constraints and located as close as possible to the intersection set B of the soft constraints. This goal leads to the problem of finding a BAP relative to these two sets: again, see, e.g., [**25**, **27**, **39**, **72**] and the references therein for applications in signal processing.

We showed in [**18**, Theorem 32] that the A-S-HLWB algorithm converges to a BAP whenever it is known in advance that there is a unique BAP. This naturally leads to the task of providing conditions, hopefully easy-to-verify, which ensure that there is a unique BAP. In [**18**, Proposition 16(iii)] we presented a sufficient condition for the uniqueness of the BAP problem: that both A and B are strictly convex, i.e., that their boundaries do not contain nondegenerate line segments, and that both A and B are compact. While this condition covers a large class of cases, there are many cases in which there is a unique BAP but the above-mentioned condition does not hold, and a simple example was given in [**18**, Figure 3.1].

We generalize [**18**, Proposition 16] to all normed spaces, and present various other sufficient (and sometimes necessary) geometric conditions for the (at most) uniqueness of a BAP in a wide class of normed spaces. These conditions are related to the structure of the boundaries of the subsets, their relative orientation, and the structure of the unit sphere of the space. Roughly speaking, one of these conditions (Corollary **4.3** below) says that if the unit sphere of the space does not contain nondegenerate intervals (that is, if $(X, \|\cdot\|)$ is strictly convex), and if either A or B is strictly convex, or if the boundaries of both of them contain nondegenerate intervals but no interval from the boundary of one subset is parallel to an interval contained in the boundary of the other subset, then there is at most one BAP. Our analysis, which is illustrated by various examples, also covers the case where each of the subsets A and B is a finite intersection of closed and convex subsets. As can be seen from this discussion, our results significantly extend the horizon of the A-S-HLWB algorithm.

In connection with the previous paragraph, we note the issue of the uniqueness of the BAP has been discussed so far only briefly: in [**66**, Theorem 1.1] (a general normed space, but the proof suffers from issues: see Remark **4.4** below), [**54**, Theorem 3.1] (a strictly convex normed space: see Remark **4.7(ii)** below), [**1**, Proof of Theorem 1] and [**18**, Proposition 16 and Theorem 32] (a Euclidean space in both cases).

The issue of existence is considered in Theorem **5.1** below, which presents many sufficient conditions for the existence of a BAP in various normed spaces and with possibly nonconvex subsets. This theorem generalizes most of the published results and adds many more new ones. See this theorem and Remark **5.2** below for more details regarding the known results (that we are aware of). The existence of a BAP is important also because without it various algorithms aimed at solving the BAP problem, such as the Dykstra algorithm [**6**], the alternating projection algorithm [**22**, Theorem 4], [**66**, Theorems 1.2–1.4] (inspired by von Neumann [**69**, Theorem 13.7, pp. 55–56]), and the A-S-HLWB algorithm [**18**, Theorem 32], may not converge.

As a final remark, we note that while the BAP problem is mainly concerned with the case where $A \cap B = \emptyset$, namely the inconsistent feasibility problem, the case where $A \cap B \neq \emptyset$ (namely the convex feasibility problem, or CFP for short, whenever $A \cap B$ is convex) is by itself interesting, has various applications, and has been investigated thoroughly: see, for example, the following (very partial) list of references, as well as the references therein: [**7**, **13**, **14**, **16**, **17**, **19**, **21**, **26**, **37**, **44**].

1.2. **Paper layout.** Section **2** presents our notation and recalls a few known concepts. Section **3** presents various auxiliary results. Section **4** presents several conditions which ensure the (at most) uniqueness of a BAP, and Section **5** presents many sufficient conditions for the existence of a BAP. Some of the results are illustrated by various examples and figures presented in Section **6**.

2. Preliminaries

In this section we present some terminology and recall several known concepts and results. Unless otherwise stated, our setting is a normed space $(X, \|\cdot\|)$, $X \neq \{0\}$, but since some of the notions below hold in a more general setting, such as metric spaces and vector spaces, we sometimes consider these settings too. We denote by X^* the dual of X. Given a subset $A \subseteq X$, we denote by $\overline{A}$ its closure, by ∂A its boundary, and by $\text{int}(A)$ its interior. Given another subset B of X, the distance between A and B is defined by $dist(A, B) := \inf\{\|a - b\| \,|\, a \in A, b \in B\}$ (if either A or B is empty, then $dist(A, B) := \infty$). We say that B is proximinal with respect to A if for every $a \in A$ there exists $b \in B$ such that $d(a, B) := d(\{a\}, B) = \|a - b\|$. We denote $A + B := \{a + b \,|\, a \in A, b \in B\}$ and $A - B := \{a - b \,|\, a \in A, b \in B\}$. The recession cone of A is the set $\{x \in X \,|\, \{x\} + A \subseteq A\}$. If A is a linear subspace, then we say that A is topologically complemented if A is closed and there exists a closed linear subspace F such that $A \oplus F = X$, that is, $A \cap F = \{0\}$ and $A + F = X$. In this case we denote by $\Pi_A : X \to A$ the linear projection onto A along F, that is, if $z \in X$ is (uniquely) represented as $z = z_1 + z_1$ for some $z_1 \in A$ and $z_2 \in F$, then $\Pi_A(z) = z_1$. Similarly, $\Pi_F : X \to F$ denotes the linear projection onto F along A. If F is finite dimensional, then we say that A has a finite codimension. We say that A is an affine subspace if $A = u + \widetilde{A}$ for some $u \in X$ and a linear subspace $\widetilde{A}$; in this case we say that $\widetilde{A}$ is the linear part of A, and the dimension/codimension of A is defined to be the dimension/codimension of $\widetilde{A}$. We say that A is polyhedral, or a polytope, if it is the intersection of finitely many closed halfspaces.

We say that $((a_k, b_k))_{k\in\mathbb{N}}$ is a distance minimizing sequence in $A \times B$ if

$$\lim_{k\to\infty} \|a_k - b_k\| = dist(A, B). \tag{2.1}$$

The definition of $dist(A, B)$ obviously implies the existence of at least one distance minimizing sequence when $A \neq \emptyset$ and $B \neq \emptyset$. We say that (A, B) satisfies the distance coercivity condition if $A \cup B$ is unbounded and

$$\lim_{\|(x,y)\|\to\infty, (x,y)\in A\times B} \|x - y\| = \infty, \tag{2.2}$$

where $\|(x, y)\| := \sqrt{\|x\|^2 + \|y\|^2}$ for all $(x, y) \in X^2$.

If a sequence $(x_k)_{k\in\mathbb{N}}$ converges weakly to $x \in X$, then we write $x = (w) \lim_{k\to\infty} x_k$. We say that A is weakly sequentially closed if for every $x \in X$ and $(x_k)_{k\in\mathbb{N}}$ in A, the condition $x = (w) \lim_{k\to\infty} x_k$ implies that $x \in A$. Note that if A is weakly closed, then it is weakly sequentially closed (since in general, if a subset is closed with respect to some topology, then it is sequentially closed with respect to that topology), but the converse is not necessarily true even in Hilbert spaces [**8**, Example 3.33, p. 60]. We say that A is weakly sequentially compact if every sequence in A has a subsequence which converges weakly to some $z \in A$ (this is a standard notion but occasionally, as in [**34**, Definition II.3.25, pp. 67–68], one requires less and gets less, namely that the limit z exists in X and not necessarily in A).

Given two points a_0 and a_1 in X, we denote by $[a_0, a_1] := \{a(t) := a_0 + t(a_1 - a_0) \,|\, t \in [0, 1]\}$ the interval, or line segment, connecting a_0 and a_1. This interval is said to be nondegenerate if $a_0 \neq a_1$. We denote by $(a_0, a_1) := \{a(t) \,|\, t \in (0,1)\}$ the open interval connecting a_0 and a_1, and by $[a_0, a_1) := \{a(t) \,|\, t \in [0,1)\}$ and $(a_0, a_1] := \{a(t) \,|\, t \in (0,1]\}$ the respective half-open intervals. We say that $A \subseteq X$ is strictly convex if for all distinct points $a_0, a_1 \in A$, the open interval (a_0, a_1) is contained in $\mathrm{int}(A)$. Any strictly convex set is obviously convex. Given two lines L and M in the space, we say that they are parallel if they are disjoint and there is a two-dimensional affine subspace in which they are located. Given two nondegenerate intervals $[a_0, a_1]$ and $[b_0, b_1]$, we say that they are parallel if they are located on parallel lines, namely there are two parallel lines L and M, such that $[a_0, a_1] \subset L$ and $[b_0, b_1] \subset M$.

Finally, we say that the underlying space X is strictly convex if its unit ball (and hence any other ball) is a strictly convex subset. Equivalently, the boundary of the ball does not contain nondegenerate intervals. Well-known examples of strictly convex spaces are Euclidean spaces, Hilbert spaces, the sequence spaces ℓ_p (sequences with possibly finitely many components) where $p \in (1, \infty)$, the function spaces $L_p(\Omega)$ where $p \in (1, \infty)$ and Ω is a Lebesgue measurable set in $\mathbb{R}^k$ for some $k \in \mathbb{N}$, uniformly convex spaces, and sums of strictly convex spaces with the $\|\cdot\|_p$ norm ($p \in (1, \infty)$). Well-known examples of spaces which are not strictly convex are the ℓ_1, ℓ_∞, $L_1(\Omega)$ and $L_\infty(\Omega)$ spaces. For more details and examples, see, for instance, [**11**,**24**,**34**,**38**,**50**,**56**].

3. Auxiliary results

In this section we formulate and prove several auxiliary results which we need in order to prove our main results. We start with the following two simple (and probably known) lemmata whose proofs are presented for the sake of completeness.

Lemma 3.1. *Given two nonempty subsets A and B in a metric space (X, d), one has $dist(A, B) = dist(\overline{A}, \overline{B})$, where $dist(A, B) := \inf\{d(a, b) \,|\, (a, b) \in A \times B\}$. Moreover, if $A \cap B = \emptyset$ and there exists a BAP with respect to (A, B), namely a pair $(a_0, b_0) \in A \times B$ such that $d(a_0, b_0) = dist(A, B)$, then $dist(A, B) > 0$.*

Proof. Let $a \in A$ and $b \in B$. Then $a \in \overline{A}$ and $b \in \overline{B}$, and hence, by the definition of $dist(\overline{A}, \overline{B})$, we have $dist(\overline{A}, \overline{B}) \leq d(a, b)$. Since $a \in A$ and $b \in B$ are arbitrary, we have $dist(\overline{A}, \overline{B}) \leq \inf\{d(a, b) \,|\, a \in A, b \in B\} = dist(A, B)$. On the other hand, let $\epsilon > 0$ be arbitrary. By the definition of $dist(\overline{A}, \overline{B})$ there are $\widetilde{a} \in \overline{A}$ and $\widetilde{b} \in \overline{B}$ such that $d(\widetilde{a}, \widetilde{b}) < dist(\overline{A}, \overline{B}) + 0.5\epsilon$. By the definition of $\overline{A}$ and $\overline{B}$ there are some $a \in A$ and $b \in B$ such that $d(\widetilde{a}, a) < 0.25\epsilon$ and $d(\widetilde{b}, b) < 0.25\epsilon$. Thus, by the triangle inequality and because $dist(A, B) \leq d(a, b)$, we have

$$\begin{aligned} dist(A, B) &\leq d(a, b) \leq d(a, \widetilde{a}) + d(\widetilde{a}, \widetilde{b}) + d(\widetilde{b}, b) \\ &< 0.25\epsilon + dist(\overline{A}, \overline{B}) + 0.5\epsilon + 0.25\epsilon = dist(\overline{A}, \overline{B}) + \epsilon. \end{aligned} \tag{3.1}$$

Since ϵ can be an arbitrarily small positive number, we conclude that $dist(A, B) \leq dist(\overline{A}, \overline{B})$ as well. Finally, suppose that $A \cap B = \emptyset$ and that there exists a BAP (a_0, b_0) with respect to (A, B). Then $a_0 \in A$, $b_0 \in B$ and $d(a_0, b_0) = dist(A, B)$, and since $a_0 \neq b_0$ (otherwise $a_0 \in A$ and $a_0 = b_0 \in B$, a contradiction to the assumption that $A \cap B = \emptyset$), we have $0 < d(a_0, b_0) = dist(A, B)$, as required. ∎

Lemma 3.2. *Suppose that $(X, \|\cdot\|)$ is a normed space and that A and B are nonempty and disjoint subsets of X. Then $dist(A, B) = dist(\partial A, \partial B)$.*

Proof. Let $\epsilon > 0$ be arbitrary. By the definition of $dist(A, B)$ there are $a \in A$ and $b \in B$ which satisfy $\|a - b\| < dist(A, B) + \epsilon$. Consider the function $g : [0, 1] \to X$ defined by $g(t) := a + t(b - a)$ for every $t \in [0, 1]$. We first show that $g(t) \in \partial A$ for some $t \in [0, 1]$. Indeed, if $g(0) \in \partial A$, then we are done. Otherwise, since $g(0) = a \in A$ and $g(0) \notin \partial A$, it follows that $g(0) \in \text{int}(A)$. Similarly, if $g(1) \in \partial A$, then we are done; otherwise, since $g(1) = b \in B$ and $B \cap A = \emptyset$, it follows that $g(1) \notin A$, and because $g(1) \notin \partial A$, one has $g(1) \notin \overline{A}$, namely $g(1) \in X \backslash \overline{A}$. Since $[0, 1]$ is connected and g is continuous, $[a, b] = g([0, 1])$ is connected. Since $X = \text{int}(A) \cup \partial A \cup (X \backslash \overline{A})$ and this union is disjoint, we can write $g([0, 1]) = [a, b] \cap X = ([a, b] \cap \text{int}(A)) \cup ([a, b] \cap \partial A) \cup ([a, b] \cap (X \backslash \overline{A}))$, where the first and the third subsets in this union are open subsets in the connected topological space $[a, b]$ (with the topology induced by the norm on X) because $\text{int}(A)$ and $X \backslash \overline{A}$ are open subsets in X. Moreover, $g(0) \in [a, b] \cap \text{int}(A)$ and $g(1) \in [a, b] \cap (X \backslash \overline{A})$. Therefore, if, for the sake of contradiction, $[a, b] \cap \partial A = \emptyset$, then $[a, b]$ can be represented as a disjoint union of two nonempty and open subsets, a contradiction to the fact that $[a, b]$ is connected. Hence $[a, b] \cap \partial A \neq \emptyset$, and therefore there is some $t_1 \in [0, 1]$ such that $g(t_1) \in \partial A$. In particular $\partial A \neq \emptyset$.

Similarly there is some $t_2 \in [0, 1]$ such that $g(t_2) \in \partial B$ and $\partial B \neq \emptyset$. These relations, the definition of $dist(\partial A, \partial B)$, and the fact that $g(t_1)$ and $g(t_2)$ are in the interval $[a, b]$, all imply that $dist(\partial A, \partial B) \leq \|g(t_1) - g(t_2)\| = |t_1 - t_2|\|a - b\| \leq \|a - b\| < dist(A, B) + \epsilon$. Since ϵ was an arbitrary positive number, we conclude that $dist(\partial A, \partial B) \leq dist(A, B)$. On the other hand, given $a \in \partial A$ and $b \in \partial B$, we have $a \in \overline{A}$ and $b \in \overline{B}$. Since $dist(A, B) = dist(\overline{A}, \overline{B})$ by Lemma **3.1** and since $dist(\overline{A}, \overline{B}) \leq \|a - b\|$ by the definition of $dist(\overline{A}, \overline{B})$, we have $d(A, B) = dist(\overline{A}, \overline{B}) \leq \|a - b\|$. Thus, since $a \in \partial A$ and $b \in \partial B$ were arbitrary, we have $dist(A, B) \leq \inf\{\|a - b\| \mid a \in \partial A,\, b \in \partial B\} = dist(\partial A, \partial B)$ as well. □

Remark 3.3. Lemma **3.2** does not hold in every path connected metric space. Indeed, suppose that X is the subset of $\mathbb{R}^2$ defined by $([-2, 2] \times \{-2\}) \cup (\{2\} \times [-2, 2]) \cup ([-2, 2] \times \{2\}) \cup (\{-2\} \times [1, 2]) \cup (\{-2\} \times [-2, -1])$, that is, X is the subset obtained by removing the line segment $\{-2\} \times (-1, 1)$ from the boundary (in $\mathbb{R}^2$) of the square $[-2, 2]^2$. Let $d : X^2 \to [0, \infty)$ be the metric induced by the Euclidean norm, namely $d((x_1, x_2), (y_1, y_2)) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ for all $(x_1, x_2), (y_1, y_2) \in X$. Let $A := \{-2\} \times [-2, -1]$ and $B := \{-2\} \times [1, 2]$. Then $\partial A = \{(-2, -2)\}$, $\partial B = \{(-2, 2)\}$, $A \cap B = \emptyset$ and $2 = dist(A, B) < dist(\partial A, \partial B) = 4$.

We continue with the following definition. See Section **6** below for illustrations.

Definition 3.4. *Let $(X, \|\cdot\|)$ be a normed space, let $a_0, a_1, b_0, b_1 \in X$ and let A and B be two nonempty subsets of X. We say that:*

(i) *(a_0, b_0) is a BAP with respect to (or relative to) (A, B) if $a_0 \in A$, $b_0 \in B$ and $\|a_0 - b_0\| = dist(A, B)$.*

(ii) *$([a_0, a_1], [b_0, b_1])$ is a BAP of intervals with respect to (or relative to) (A, B) if for all $t \in [0, 1]$ one has $a(t) \in A$, $b(t) \in B$ and $\|a(t) - b(t)\| = dist(A, B)$, where $a(t) := a_0 + t(a_1 - a_0)$ and $b(t) := b_0 + t(b_1 - b_0)$.*

(iii) $([a_0,a_1],[b_0,b_1])$ *is a nondegenerate BAP of intervals with respect to (or relative to)* (A,B) *if it is a BAP of intervals with respect to* (A,B) *and either* $a_0\neq a_1$ *or* $b_0\neq b_1$.

(iv) $([a_0,a_1],[b_0,b_1])$ *is a strictly nondegenerate BAP of intervals with respect to (or relative to)* (A,B) *if it is a BAP of intervals with respect to* (A,B) *and both* $a_0\neq a_1$ *and* $b_0\neq b_1$.

Lemma 3.5. *Suppose that* $(X,\|\cdot\|)$ *is a normed space and that* A *and* B *are nonempty and convex subsets of* X *such that* $a_0,a_1\in A$ *and* $b_0,b_1\in B$. *Then the following two conditions are equivalent:*

(i) (a_0,b_0) *and* (a_1,b_1) *are BAPs with respect to* (A,B).

(ii) $(a(t),b(t))$ *is a BAP with respect to* (A,B) *for all* $t\in[0,1]$, *where* $a(t):=a_0+t(a_1-a_0)$ *and* $b(t):=b_0+t(b_1-b_0)$.

Proof. Condition **(ii)** obviously implies Condition **(i)**, and hence from now on we assume that Condition **(i)** holds and show how it implies Condition **(ii)**. Since both (a_0,b_0) and (a_1,b_1) are BAPs with respect to (A,B), we have $\|a_0-b_0\|=dist(A,B)=\|a_1-b_1\|$. By the convexity of A and B and the fact that $a(t)=(1-t)a_0+ta_1$ and $b(t)=(1-t)b_0+tb_1$, we have $a_t\in A$ and $b_t\in B$ for each $t\in[0,1]$. Therefore, in particular, $dist(A,B)\leq\|a(t)-b(t)\|$. On the other hand, the triangle inequality and the assumption that (a_0,b_0) and (a_1,b_1) are BAPs imply that for all $t\in[0,1]$,

$$\|a(t)-b(t)\|\leq(1-t)\|a_0-b_0\|+t\|a_1-b_1\|=(1-t)dist(A,B)+tdist(A,B)=dist(A,B).$$

The last two inequalities imply that $\|a(t)-b(t)\|=dist(A,B)$ for all $t\in[0,1]$, namely $(a(t),b(t))$ is also a BAP with respect to (A,B) for all $t\in[0,1]$. □

The next lemma is related to, but definitely different from, [62, Proposition 3.1] (which, by the way, has a minor gap in its proof, where it is claimed without a proof that the line segment K defined there must intersect ∂A; in this connection, see the proof of Lemma **3.2** above).

Lemma 3.6. *Suppose that* $(X,\|\cdot\|)$ *is a normed space and that* A *and* B *are nonempty and disjoint subsets of* X. *If* (a_0,b_0) *is a BAP with respect to* (A,B), *then it is a BAP with respect to* $(\partial A,\partial B)$; *in particular,* $a_0\in\partial A$ *and* $b_0\in\partial B$. *Conversely, if* (a_0,b_0) *is a BAP with respect to* $(\partial A,\partial B)$ *and* A *and* B *are also closed (in addition to being nonempty and disjoint), then* (a_0,b_0) *is a BAP with respect to* (A,B).

Proof. Suppose first that (a_0,b_0) is a BAP with respect to (A,B). Then $\|a_0-b_0\|=dist(A,B)$, and because of Lemma **3.2**, we have $dist(A,B)=dist(\partial A,\partial B)$, namely, $\|a_0-b_0\|=dist(\partial A,\partial B)$. Therefore, in order to prove that (a_0,b_0) is a BAP with respect to $(\partial A,\partial B)$ it remains to show that $(a,b)\in\partial A\times\partial B$. Let $r>0$ be arbitrary and consider the open ball with radius r around a_0. We need to show that this ball contains points from A and points outside A. Obviously $a_0\in A$ is in the ball. In order to see that this ball contains points outside A, consider the line segment $[a_0,b_0]=\{a(t)\,|\,t\in[0,1]\}$ whose length is $\|a_0-b_0\|=dist(A,B)$. Let $t\in(0,\min\{r/(\|b_0-a_0\|+1),1\})$. Since $a(t)=a_0+t(b_0-a_0)$, it follows that $a(t)$ is both in $[a_0,b_0]$ and in the ball. Since $\|a(t)-b_0\|=(1-t)\|a_0-b_0\|<\|a_0-b_0\|=dist(A,B)$, it is impossible that $a(t) \in A$ because the distance between a point from A and a point from B is at least $dist(A,B)$ by the definition of $dist(A,B)$. Thus $a(t)$ is both outside A and in the ball. Since $r>0$ can be arbitrarily small $a_0 \in \partial A$. Similarly, $b_0 \in \partial B$.

Now assume that A and B are closed and that (a_0,b_0) is a BAP with respect to $(\partial A, \partial B)$. Then $a_0 \in \partial A \subseteq A$, $b_0 \in \partial B \subseteq B$ and $\|a_0-b_0\| = dist(\partial A, \partial B)$. Because $dist(\partial A, \partial B) = dist(A,B)$ by Lemma **3.2**, we also have $\|a_0-b_0\| = dist(A,B)$. Hence (a_0,b_0) is a BAP with respect to (A,B). $\square$

Lemma 3.7. *Let A and B be nonempty, convex and disjoint subsets of a normed space $(X, \|\cdot\|)$. Assume that $a_0, a_1 \in A$ and $b_0, b_1 \in B$. If (a_0,b_0) and (a_1,b_1) are BAPs with respect to (A,B), then $([a_0,a_1],[b_0,b_1])$ is a BAP of intervals with respect to both (A,B) and $(\partial A, \partial B)$. In particular, $[a_0,a_1] \subseteq \partial A$ and $[b_0,b_1] \subseteq \partial B$.*

Proof. Let $t \in [0,1]$ be arbitrary. Lemma **3.5** implies that $(a(t),b(t))$ is a BAP with respect to (A,B), and so $\|a(t)-b(t)\| = dist(A,B)$. Because $(a(t),b(t))$ is a BAP with respect to (A,B), Lemma **3.6** (with $(a(t),b(t))$ instead of (a_0,b_0)) implies that $(a(t),b(t))$ is also a BAP with respect to $(\partial A, \partial B)$. Thus, $a(t) \in \partial A$, $b(t) \in \partial B$ and $\|a(t)-b(t)\| = dist(\partial A, \partial B)$. Since $[a_0,a_1] = \{a(t) \,|\, t \in [0,1]\}$ and $[b_0,b_1] = \{b(t) \,|\, t \in [0,1]\}$, we have $[a_0,a_1] \subseteq \partial A$ and $[b_0,b_1] \subseteq \partial B$. Thus, Definition **3.4** implies that $([a_0,a_1],[b_0,b_1])$ is a BAP of intervals with respect to both (A,B) and $(\partial A, \partial B)$. $\square$

Lemma 3.8. *Let A and B be nonempty, convex and disjoint subsets of a strictly convex normed space $(X, \|\cdot\|)$. If $([a_0,a_1],[b_0,b_1])$ is a nondegenerate BAP of intervals with respect to (A,B), then it is a strictly nondegenerate BAP of intervals relative to both (A,B) and $(\partial A, \partial B)$. In particular, $[a_0,a_1]$ is a nondegenerate interval contained in ∂A, and $[b_0,b_1]$ is a nondegenerate interval contained in ∂B.*

Proof. Since $([a_0,a_1],[b_0,b_1])$ is a BAP of intervals with respect to (A,B), we have, in particular, that (a_0,b_0) and (a_1,b_1) are BAPs with respect to (A,B). Hence Lemma **3.7** implies that $([a_0,a_1],[b_0,b_1])$ is a BAP of intervals with respect to $(\partial A, \partial B)$. In particular, $[a_0,a_1] \subseteq \partial A$ and $[b_0,b_1] \subseteq \partial B$.

It remains to be shown that the pair $([a_0,a_1],[b_0,b_1])$ is strictly nondegenerate. Since this pair is a nondegenerate BAP of intervals with respect to (A,B), we have that either $a_0 \neq a_1$ or $b_0 \neq b_1$. Assume that $a_0 \neq a_1$: the case $b_0 \neq b_1$ can be treated similarly. If, for the sake of contradiction, $b_0 = b_1$, then this equality and the fact that (a_0,b_0) and (a_1,b_1) are BAPs with respect to (A,B) imply that $\|a_0-b_0\| = dist(A,B) = \|a_1-b_1\| = \|a_1-b_0\|$. This equality means that both a_0 and a_1 are located on the boundary of the ball whose center is b_0 and its radius is $dist(A,B)$, which is positive according to Lemma **3.1**. Since $a_0 \neq a_1$ and the space is strictly convex, the open interval (a_0,a_1) is strictly inside this ball, and so, in particular, $a_{0.5}$ is strictly inside this ball. Thus, $\|a_{0.5}-b_0\| < dist(A,B)$. On the other hand, since $a_{0.5} \in A$ by the convexity of A and since $b_0 \in B$, the definition of $dist(A,B)$ implies that $\|a_{0.5}-b_0\| \geq dist(A,B)$, a contradiction. Hence $b_0 \neq b_1$ and indeed $([a_0,a_1],[b_0,b_1])$ is a strictly nondegenerate BAP of intervals relative to (A,B) (and relative to $(\partial A, \partial B)$). $\square$

Lemma 3.9. *Suppose that $(X, \|\cdot\|)$ is a strictly convex normed space and that A and B are nonempty, convex and disjoint subsets of X. If (a_0,b_0) and (a_1,b_1) are two* *distinct BAPs with respect to (A,B), then both $[a_0,a_1]$ and $[b_0,b_1]$ are nondegenerate intervals, they are parallel, $[a_0,a_1]\subseteq\partial A$ and $[b_0,b_1]\subseteq\partial B$.*

Proof. Since (a_0,b_0) and (a_1,b_1) are BAPs with respect to (A,B), Lemma **3.7** implies that $([a_0,a_1],[b_0,b_1])$ is a BAP of intervals with respect to (A,B), and since $(a_0,b_0)\neq(a_1,b_1)$, either $[a_0,a_1]$ is nondegenerate or $[b_0,b_1]$ is nondegenerate. Thus $([a_0,a_1],[b_0,b_1])$ is a nondegenerate BAP of intervals with respect to (A,B), and since X is strictly convex Lemma **3.8** implies that both $[a_0,a_1]$ and $[b_0,b_1]$ are nondegenerate. In addition, Lemma **3.7** implies that $[a_0,a_1]\subseteq\partial A$ and $[b_0,b_1]\subseteq\partial B$.

It remains to be shown that $[a_0,a_1]$ and $[b_0,b_1]$ are parallel. Lemma **3.5** implies that $(a(t),b(t))$ is a BAP with respect to (A,B) for each $t\in[0,1]$. Thus, if we define $f:[0,1]\to[0,\infty)$ by

$$f(t):=\|a(t)-b(t)\|=\|a_0-b_0+(a_1-b_1-(a_0-b_0))t\|,\quad \forall t\in[0,1], \tag{3.2}$$

then we have $f(t)=dist(A,B)$ for all $t\in[0,1]$. In particular,

$$f(0.5)=dist(A,B). \tag{3.3}$$

Assume for the sake of contradiction that $a_0-b_0\neq a_1-b_1$. Since $\|a_0-b_0\|=f(0)=f(1)=\|a_1-b_1\|=dist(A,B)$ and since $dist(A,B)>0$ according to Lemma **3.1**, the distinct points a_0-b_0 and a_1-b_1 are located on the boundary of the ball of positive radius $dist(A,B)$ around the origin. Since $(X,\|\cdot\|)$ is strictly convex, $\|0.5(a_0-b_0)+0.5(a_1-b_1)\|<dist(A,B)$. But from **(3.2)** we have $f(0.5)=\|0.5(a_0-b_0)+0.5(a_1-b_1)\|$. Therefore $f(0.5)<dist(A,B)$, a contradiction to **(3.3)**. Thus $a_0-b_0=a_1-b_1$ and hence $v:=a_1-a_0=b_1-b_0$. Since we already know that $b_0\neq b_1$ (because $[b_0,b_1]$ is nondegenerate as we showed earlier), we also have $v\neq 0$.

Consider the lines $L:=\{a_0+tv:t\in\mathbb{R}\}$ and $M:=\{b_0+sv:s\in\mathbb{R}\}$. By letting $t,s\in[0,1]$ in the definitions of L and M, we see that $[a_0,a_1]\subset L$ and $[b_0,b_1]\subset M$. We claim that $b_0\notin L$. Indeed, suppose for the sake of contradiction that $b_0\in L$. Then $b_0=a_0+tv$ for some $t\in\mathbb{R}$ and hence either $t\in[0,1]$ or $t>1$ or $t\in[-1,0)$ or $t<-1$. If $t\in[0,1]$, then $b_0\in[a_0,a_1]\subseteq A$, a contradiction since we assume that $A\cap B=\emptyset$. If $t>1$, then $\|b_0-a_1\|=\|(a_0+tv)-(a_0+v)\|=(t-1)\|v\|<t\|v\|=\|b_0-a_0\|=dist(A,B)$, a contradiction to the minimality of $dist(A,B)$. If $t\in[-1,0)$, then $t+1\in[0,1)$, and so $b_1=b_0+v=a_0+(t+1)v\in[a_0,a_1]\subseteq A$, a contradiction to the assumption $A\cap B=\emptyset$. Therefore only the case $t<-1$ remains; in this case $t+1<0$, and since $b_1=b_0+v=a_0+(t+1)v$, we have $\|b_1-a_0\|=|t+1|\|v\|=-(t+1)\|v\|<-t\|v\|=\|b_0-a_0\|=dist(A,B)$, a contradiction to the minimality of $dist(A,B)$. As a result, indeed $b_0\notin L$.

It must be that v and b_0-a_0 are linearly independent. Indeed, if, for the sake of contradiction, $\lambda_1(b_0-a_0)+\lambda_2 v=0$ for a pair of scalars $(\lambda_1,\lambda_2)\neq(0,0)$, then $\lambda_1\neq 0$ since otherwise $\lambda_2 v=0$, and since the assumption that $(\lambda_1,\lambda_2)\neq(0,0)$ implies that $\lambda_2\neq 0$, we have $v=0$, a contradiction; thus $\lambda_1\neq 0$, and hence $b_0=a_0+(-\lambda_2/\lambda_1)v$, that is, $b_0\in L$, a contradiction to what has been proved in the previous paragraph. In addition, $L\cap M=\emptyset$, since otherwise $a_0+tv=b_0+sv$ for some $t,s\in\mathbb{R}$, and so $b_0=a_0+(t-s)v\in L$, a contradiction. Thus L and M are parallel since their intersection is the empty set and both of them are located on the same two-dimensional affine subspace (namely on $a_0 + \text{span}\{v, b_0 - a_0\}$), and hence $[a_0, a_1]$ and $[b_0, b_1]$ are parallel since they are located on the parallel lines L and M, respectively. $\square$

The following lemma, which actually holds in any topological vector space with essentially the same proof, might be known.

Lemma 3.10. *Let C be a nonempty and convex subset of a normed space $(X, \|\cdot\|)$. If x,y and z are three distinct points in ∂C satisfying $y \in [x, z]$, then $[x, z] \subseteq \partial C$.*

Proof. Let $w \in [x, z]$. Since x and z are in $\overline{C}$ and since $\overline{C}$ is convex (because C is convex, see, e.g., [68, Theorem 2.23(a), p. 28]), we have $w \in \overline{C}$. Since $[x, z] = [x, y] \cup [y, z]$, either $w \in [x, y]$ or $w \in [y, z]$. Suppose that the first case holds. If $w = x$ or $w = y$, then $w \in \partial C$ by our assumption on x and y. Otherwise, $w \in (x, y)$. Assume for the sake of contradiction that $w \notin \partial C$. This assumption and the fact that $w \in \overline{C}$ imply that $w \in \text{int}(C)$. Since, as is well known [68, Theorem 2.23(b), p. 28], the half-open line segment between an interior point of C and a point in $\overline{C}$ is contained in $\text{int}(C)$, we have $[w, z) \subseteq \text{int}(C)$. From the fact that $y \in [w, z)$ we conclude that $y \in \text{int}(C)$, a contradiction to our assumption that $y \in \partial C$. Hence w must be in ∂C, and since w is an arbitrary point in $[x, y]$, we conclude that $[x, y] \subseteq \partial C$. Similarly, $[y, z] \subseteq \partial C$. Thus $[x, z] \subseteq \partial C$. $\square$

The final two auxiliary assertions will be used in Section 5.

Lemma 3.11. *Let $(X, \|\cdot\|)$ be a normed space, A and B be nonempty subsets of X, and $((a_k, b_k))_{k\in\mathbb{N}}$ be a distance minimizing sequence in $A \times B$. Then:*

(i) Either both $(a_k)_{k\in\infty}$ and $(b_k)_{k\in\mathbb{N}}$ are bounded or both of them are unbounded.

(ii) If either $A \cup B$ is bounded, or $A \cup B$ is unbounded and (2.2) holds, then both $(a_k)_{k\in\infty}$ and $(b_k)_{k\in\mathbb{N}}$ are bounded.

(iii) If $((a_k, b_k))_{k\in\mathbb{N}}$ has a subsequence which converges weakly in X^2 to some point $(a, b) \in A \times B$, then (a, b) is a BAP relative to (A, B).

Proof. (i) If both $(a_k)_{k\in\infty}$ and $(b_k)_{k\in\mathbb{N}}$ are bounded, then we are done. Otherwise, one of these sequences, say $(a_k)_{k\in\infty}$, is unbounded. Hence there is an infinite subset N_1 of $\mathbb{N}$ such that $\lim_{k\to\infty, k\in N_1} \|a_k\| = \infty$. Since $(a_k - b_k)_{k\in\mathbb{N}}$ converges to the finite number $dist(A, B)$, this sequence is bounded. Since $b_k = a_k - (a_k - b_k)$ for all $k \in N_1$, it follows that $(b_k)_{k\in N_1}$ is a difference between an unbounded sequence and a bounded one, and therefore $\lim_{k\to\infty, k\in N_1} \|b_k\| = \infty$. Thus $(b_k)_{k\in\mathbb{N}}$ is unbounded too.

(ii) The assertion obviously holds if $A \cup B$ is bounded since then A and B are bounded, and so are any sequences contained in them. Now assume that $A \cup B$ is unbounded and (2.2) holds. If, say, $(a_k)_{k\in\infty}$ is unbounded, then by Part (i) there is an infinite subset N_1 of $\mathbb{N}$ such that $\lim_{k\to\infty, k\in N_1} \|a_k\| = \lim_{k\to\infty, k\in N_1} \|b_k\| = \infty$. But then (2.2) implies that $\lim_{k\to\infty, k\in N_1} \|a_k - b_k\| = \infty$, a contradiction to (2.1). Thus $(a_k)_{k\in\infty}$ is bounded, and from Part (i) also $(b_k)_{k\in\infty}$ is bounded.

(iii) Suppose that $(a, b) = (w) \lim_{k\to\infty, k\in N_1} (a_k, b_k)$ for some infinite subset N_1 of $\mathbb{N}$. Then $a = (w) \lim_{k\to\infty, k\in N_1} a_k$ and $b = (w) \lim_{k\to\infty, k\in N_1} b_k$, and hence $a - b = (w) \lim_{k\to\infty, k\in N_1} (a_k - b_k)$. Since the norm is weakly sequentially lower semicontinuous [34, II.3.27, p. 68], we have $\|a - b\| \leq \lim_{k\to\infty, k\in N_1} \|a_k - b_k\| =$

$dist(A, B)$. On the other hand $dist(A, B) \leq \|a - b\|$ since $(a, b) \in A \times B$. Hence $\|a - b\| = dist(A, B)$ and (a, b) is a BAP relative to (A, B).

$\square$

Lemma 3.12. *Suppose that C is a nonempty, convex, closed and locally weakly sequentially compact subset of a normed space $(X, \|\cdot\|)$, where local weak sequential compactness of C means that for every $x \in C$ there is a closed ball $D \subseteq X$ centered at x such that $D \cap C$ is weakly compact. Then any bounded sequence in C has a weakly convergent subsequence whose weak limit is in C.*

Proof. Let $(c_k)_{k\in\mathbb{N}}$ be an arbitrary bounded sequence in C. We need to show that there exists an infinite subset $K \subseteq \mathbb{N}$ and a point $c \in C$ such that $(w)\lim_{k\to\infty, k\in K} c_k = c$. Fix an arbitrary point $z \in C$. If $\epsilon := \inf\{\|c_k - z\| \,|\, k \in \mathbb{N}\} = 0$, then the definition of the infimum implies that there is an infinite subset $K \subseteq \mathbb{N}$ such that $\lim_{k\to\infty, k\in K} \|c_k - z\| = 0$. Since $(c_k)_{k\in K}$ converges strongly to z, it also converges weakly to z, and so we are done (with $c := z$). Otherwise $\epsilon > 0$, and so $\|c_k - z\| \geq \epsilon > 0$ for all $k \in \mathbb{N}$. Since C is closed and locally weakly sequentially compact, there is a closed ball D, centered at z, with radius $r \in (0, \epsilon)$, whose intersection with C is weakly sequentially compact. Since $(c_k)_{k\in\mathbb{N}}$ is bounded, there is some $\rho > 0$ such that $\|c_k\| < \rho$ for all $k \in \mathbb{N}$. Define $\alpha_k := 0.5r/\|c_k - z\|$. Then $\alpha_k \in [0.5r/(\rho + \|z\|), 0.5r/\epsilon]$ for all $k \in \mathbb{N}$ by the triangle inequality and the choice of r and ρ. Hence the compactness of the real-line interval $[0.5r/(\rho + \|z\|), 0.5r/\epsilon]$ implies that there is an infinite subset $S \subseteq \mathbb{N}$ and a real number $\alpha \in [0.5r/(\rho + \|z\|), 0.5r/\epsilon]$ such that $\lim_{k\to\infty, k\in S} \alpha_k = \alpha$.

Define $c'_k := z + \alpha_k(c_k - z)$ for all $k \in S$. Then $c'_k \in D$ for every $k \in S$. Moreover, $c'_k \in [z, c_k] \subseteq C$ because C is convex and $\alpha_k \in [0, 1]$. Hence $c'_k \in C \cap D$ for all $k \in S$, and therefore, since $C \cap D$ is weakly sequentially compact, there is a point $c' \in C \cap D$ and an infinite subset $K \subseteq S$ such that $(w)\lim_{k\to\infty, k\in K} c'_k = c'$. We claim that $(w)\lim_{k\to\infty, k\in K} c_k = c$, where $c := z + (1/\alpha)(c' - z)$. Indeed, given an arbitrary continuous linear functional $g \in X^*$, the triangle inequality, the definitions of c'_k and c, the linearity of g, and the definition of $\|g\|$, the limits $\lim_{k\to\infty, k\in K} \alpha_k = \alpha$ and $\lim_{k\to\infty, k\in K} g(c'_k - c') = 0$, and the fact that $\|c'_k - z\| = 0.5r$ for every $k \in K$, all imply that for each $k \in K$,

$$
\begin{aligned}
|g(c_k - c)| &= \left| g\left((z + \frac{1}{\alpha_k}(c'_k - z)) - (z + \frac{1}{\alpha}(c' - z)) \right) \right| \\
&= \left| g\left(\frac{1}{\alpha_k}(c'_k - z) - \frac{1}{\alpha}(c' - z) \right) \right| \\
&= \left| g\left(\left(\frac{1}{\alpha_k} - \frac{1}{\alpha} \right)(c'_k - z) \right) + g\left(\frac{1}{\alpha}(c'_k - z) - \frac{1}{\alpha}(c' - z) \right) \right| \\
&\leq \left| \frac{1}{\alpha_k} - \frac{1}{\alpha} \right| |g(c'_k - z)| + \frac{1}{\alpha}|g(c'_k - c')| \\
&\leq \left| \frac{1}{\alpha_k} - \frac{1}{\alpha} \right| \|g\| \cdot 0.5r + \frac{1}{\alpha}|g(c'_k - c')| \xrightarrow[k\to\infty, k\in K]{} 0. \quad (3.4)
\end{aligned}
$$

Because $g \in X^*$ was arbitrary, we have $c = (w)\lim_{k\to\infty,\, k\in K} c_k$. Since C is closed and convex, it is also weakly closed, and hence, because $(c_k)_{k\in K}$ is in C, also $c \in C$. □

Remark 3.13. A result related to (but definitely different from) Lemma **3.12** says that if (X, d) is a locally compact and almost complete geodesic metric space, then every infinite set in X has an accumulation point with respect to the topology induced by the geodesic metric: see [**52**, Theorem 4.3].

4. Uniqueness

This section presents our results regarding the (at most) uniqueness of the BAP.

Theorem 4.1. *Suppose that A and B are two nonempty, convex and disjoint subsets of a normed space $(X, \|\cdot\|)$. If there does not exist a nondegenerate BAP of intervals with respect to $(\partial A, \partial B)$, then there exists at most one BAP relative to (A, B). Conversely, if A and B are also closed and there exists at most one BAP with respect to (A, B), then there does not exist a nondegenerate BAP of intervals relative to $(\partial A, \partial B)$.*

Proof. Assume first that there does not exist a nondegenerate BAP of intervals with respect to $(\partial A, \partial B)$. If $dist(A, B)$ is not attained, then there does not exist any BAP relative to (A, B), and hence obviously there exists at most one BAP relative to (A, B). Otherwise $dist(A, B)$ is attained, and hence there exists at least one BAP (a_0, b_0) with respect to (A, B). If, for the sake of contradiction, there exists another (different) BAP (a_1, b_1) with respect to (A, B), then Lemma **3.7** implies that $([a_0, a_1], [b_0, b_1])$ is a BAP of intervals with respect to $(\partial A, \partial B)$, and $([a_0, a_1], [b_0, b_1])$ is nondegenerate since either $a_0 \neq a_1$ or $b_0 \neq b_1$. This contradicts the assumption that there does not exist a nondegenerate BAP of intervals with respect to $(\partial A, \partial B)$. Hence (a_0, b_0) is the unique BAP relative to (A, B). Conversely, suppose that A and B are also closed and that there exists at most one BAP with respect to (A, B). If, for the sake of contradiction, there exists a nondegenerate BAP of intervals $([a_0, a_1], [b_0, b_1])$ with respect to $(\partial A, \partial B)$, then either $a_0 \neq a_1$ or $b_0 \neq b_1$, and in both cases (a_0, b_0) and (a_1, b_1) are two distinct BAPs with respect to $(\partial A, \partial B)$ and hence, according to Lemma **3.6**, also with respect to (A, B). This is a contradiction to the assumption that there exists at most one BAP with respect to (A, B). □

Theorem 4.2. *Let $(X, \|\cdot\|)$ be a normed space. Suppose that m and n are natural numbers and that $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_n$ are nonempty and strictly convex subsets of X. If $A := \cap_{i=1}^m A_i$ and $B := \cap_{j=1}^n B_j$ are nonempty and disjoint, then there exists at most one BAP relative to (A, B). If, in addition, $dist(A, B)$ is attained, then there exists exactly one BAP relative to (A, B).*

Proof. Since, as can easily be proved, any finite intersection of strictly convex sets is strictly convex, it follows that A and B are strictly convex and, in particular, convex. Now, if $dist(A, B)$ is not attained, then obviously there exists at most one (actually zero) BAP relative to (A, B). Otherwise, there exists at least one BAP (a_0, b_0) relative to (A, B). Assume for the sake of contradiction that there exists another BAP $(a_1, b_1) \neq (a_0, b_0)$ relative to (A, B). Then either $[a_0, a_1]$ or $[b_0, b_1]$ is nondegenerate. Since the conditions of Lemma **3.7** hold, we conclude from it that $[a_0, a_1] \subseteq \partial A$ and $[b_0, b_1] \subseteq \partial B$. This contradicts the fact that both A and B are strictly convex and hence their boundaries do not contain any nondegenerate intervals. Therefore there exists exactly one BAP relative to (A, B), as claimed. $\square$

Corollary 4.3. *Let $(X, \|\cdot\|)$ be a normed space and $\emptyset \neq A$ and $\emptyset \neq B$ be strictly convex and disjoint subsets of X. Then there exists at most one BAP relative to (A, B), and there exists a unique such BAP if, in addition, $dist(A, B)$ is attained.*

Proof. This follows from Theorem **4.2** by letting $m := n := 1$. $\square$

Remark 4.4. In [**66**, Theorem 1.1] Stiles claimed that if A and B are two nonempty and disjoint subsets of a normed space $(X, \|\cdot\|)$ and if either A or B is strictly convex, then there exists at most one BAP relative to (A, B) (Stiles formulated this statement in the following somewhat different manner: "the distance between A and B is attained at most at one point"). This claim is false, as Example **6.2** below shows. The main mistake in Stiles' proof is the implicit assumption that the line segment $[P_B(x), P_B(y)]$ mentioned there is non-degenerate, and this is not necessarily true if X is not strictly convex even if B is strictly convex: again, see Example **6.2**. We also note that [**66**, Proof of Theorem 1.1] suffers from other issues, such as the unproven claim that if A and B are disjoint and if (a, b) is a BAP relative to (A, B), then $a \in \partial A$ and $b \in \partial B$ (this claim is true but requires a proof, as we showed in Lemma **3.6** above) and the somewhat ambiguous notations $P_B(x)$ and $P_B(y)$ (while it is clear from the proof that Stiles meant that both $(x, P_B(x))$ and $(y, P_B(y))$ are BAPs with respect to (A, B), when presenting the operator of best approximation projection onto B one needs to consider the issues of existence and uniqueness of this operator, and this has not been done in [**66**, Proof of Theorem 1.1]).

Theorem 4.5. *Given $m, n \in \mathbb{N}$, suppose that $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_n$ are nonempty and convex subsets of a strictly convex normed space $(X, \|\cdot\|)$ such that $A := \bigcap_{i=1}^{m} A_i$ and $B := \bigcap_{j=1}^{n} B_j$ are nonempty and that $A \cap B = \emptyset$. If for each pair $(i, j) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ either:*

(i) A_i is strictly convex, or
(ii) B_j is strictly convex, or
(iii) there does not exist a pair of two nondegenerate and parallel intervals such that one of them is contained in ∂A_i and the other is contained in ∂B_j,

then there exists at most one BAP with respect to (A, B). If, in addition, $dist(A, B)$ is attained, then there exists a unique BAP with respect to (A, B).

Proof. If $dist(A, B)$ is not attained, then obviously there exists at most one (in fact, zero) BAP with respect to (A, B). Otherwise, there exists at least one BAP (a_0, b_0) with respect to (A, B). Suppose by way of contradiction that there exists another BAP $(a_1, b_1) \neq (a_0, b_0)$ relative to (A, B). Then Lemma **3.9** implies that $[a_0, a_1] \subseteq \partial A$ and $[b_0, b_1] \subseteq \partial B$, that both $[a_0, a_1]$ and $[b_0, b_1]$ are nondegenerate, and that $[a_0, a_1]$ and $[b_0, b_1]$ are parallel. As is well known, and can easily be proved (see, for instance [**3**] for the case of two subsets; the case of any finite number of subsets follows immediately by induction from the case of two subsets), the boundary of a finite intersection of subsets is contained in the union of the boundaries of the subsets which induce the intersection. Therefore, $\partial A \subseteq \bigcup_{i=1}^m \partial A_i$. Hence, $[a_0, a_1] \subseteq \bigcup_{i=1}^m \partial A_i$, and so for each $t \in [0,1]$ there is at least one index $\phi(t) \in \{1,2,\ldots,m\}$ such that $a(t)a_0 + t(a_1 - a_0) \in \partial A_{\phi(t)}$. Then ϕ is a function from $[0,1]$ to $\{1,2,\ldots,m\}$, and hence we have $[0,1] = \cup_{i=1}^m \phi^{-1}(i)$.

If $\phi^{-1}(i)$ is finite for each $i \in \{1,2,\ldots,m\}$, then so is the finite union $\cup_{i=1}^m \phi^{-1}(i)$, namely $[0,1]$ is finite, a contradiction. Hence $\phi^{-1}(i)$ is infinite for some $i \in \{1,2,\ldots,m\}$, i.e., there is an infinite subset $T_i \subseteq [0,1]$ such that $a(t) \in \partial A_i$ for each $t \in T_i$. In particular, there are three points $t_1 < t_2 < t_3$ in T_i, and since $a(t) = a_0 + t(a_1 - a_0)$ for all $t \in [0,1]$, the points a_{t_1}, a_{t_2}, and a_{t_3} are three distinct points in ∂A_i. In addition, these points are contained in $[a_0, a_1]$, and since $t_1 < t_2 < t_3$ they satisfy $a_{t_2} \in [a_{t_1}, a_{t_3}]$. We conclude from Lemma **3.10** that $[a_{t_1}, a_{t_3}] \subseteq \partial A_i$. Hence A_i is not strictly convex, and so Assumption **(i)** in the formulation of the theorem does not hold. Similarly, there are some $j \in \{1,2,\ldots,n\}$ and t'_1 and t'_3 in $[0,1]$ such that $[b_{t'_1}, b_{t'_3}]$ is a nondegenerate interval contained in $[b_0, b_1] \cap \partial B_j$. Hence B_j is not strictly convex, and so Assumption **(ii)** in the formulation of the theorem does not hold. Since $[a_0, a_1]$ is parallel to $[b_0, b_1]$ and since $[a_{t_1}, a_{t_3}] \subseteq [a_0, a_1]$ and $[b_{t'_1}, b_{t'_3}] \subseteq [b_0, b_1]$, it follows that $[a_{t_1}, a_{t_3}]$ is a nondegenerate interval which is contained in ∂A_i and is parallel to the nondegenerate interval $[b_{t'_1}, b_{t'_3}]$ which is contained in ∂B_j, and this shows that also Assumption **(iii)** in the formulation of the theorem does not hold.

We conclude that none of the Assumptions **(i)**–**(iii)** in the formulation of the theorem holds, a contradiction. Consequently, the assumption that there exists more than one BAP with respect to (A, B) cannot hold, namely, there exists a unique BAP with respect to (A, B). □

From Theorem **4.5** with $m := n := 1$ we obtain the following corollary.

Corollary 4.6. *Suppose that A and B are two nonempty, convex and disjoint subsets of a strictly convex normed space $(X, \|\cdot\|)$. If either*

(i) A is strictly convex, or
(ii) B is strictly convex, or
(iii) there does not exist any pair of two nondegenerate and parallel intervals having the property that one of them is contained in the boundary of A and the other is contained in the boundary of B,

then there exists at most one BAP with respect to (A, B). If, in addition, dist(A, B) is attained, then there exists a unique BAP with respect to (A, B).

Remark 4.7. (i) The strict convexity of the norm in Theorem **4.5** and Corollary **4.6** is essential for uniqueness (when both A and B are not strictly convex): see Examples **6.2** and **6.3** below for counterexamples.

(ii) Theorem **4.5(iii)** significantly generalizes [**54**, Theorem 3.1] which says that if X is a uniformly convex Banach space (actually strict convexity is sufficient), A and B are closed and convex and $(A - A) \cap (B - B) = \{0\}$, then there exists at most one BAP relative to (A, B). Indeed, we can assume that $A \cap B = \emptyset$, since otherwise everything is trivial. Suppose that the above-mentioned condition holds and assume, for a contradiction, that Theorem **4.5(iii)** does not hold, namely that there are nondegenerate and parallel intervals $[a_1, a_2] \subseteq \partial A$ and $[b_1, b_2] \subseteq \partial B$. Then either $u := (a_2 - a_1)/\|a_2 - a_1\|$ and $v := (b_2 - b_1)/\|b_2 - b_1\|$ are equal, or $u = -v$. Assume that the first case holds: the proof in the second case is similar. Let $r := \min\{\|a_2 - a_1\|, \|b_2 - b_1\|\}$. Then $r > 0$, $a_1 + ru \in A$, $b_1 + rv \in B$, and $ru = (a_1 + ru) - a_1 \in A - A$, $ru = rv = (b_1 + rv) - b_1 \in B - B$. Hence ru is a nonzero vector in $(A - A) \cap (B - B)$, a contradiction which proves the assertion. We also note that the condition $(A - A) \cap (B - B) = \{0\}$ is frequently violated: indeed, just consider the case where both A and B have nonempty interior, as in the case of Figure **5**: in this case Theorem **4.5(iii)** holds but there are $r > 0$, $a \in A$ and $b \in B$ such that the open balls of radius r and centers a and b, respectively, are contained in A and B, respectively, and hence, given an arbitrary unit vector $u \in X$, we have $a' := a + 0.5ru \in A$, $b' := b + 0.5u \in B$, and $0 \neq 0.5ru = a' - a = b' - b \in (A - A) \cap (B - B)$.

5. Existence

In this section we present, in Theorem **5.1** below, many useful conditions which ensure the existence of a best approximating pair, and by doing this we significantly extend the known pool of such sufficient conditions. In particular, in some of these conditions we do not assume that A and B are convex. Most of these conditions are new, but some of them are known and we formulate them for the sake of completeness, and frequently provide some new information regarding them such as a new proof. In this connection, see Remark **5.2** below for various relevant comments, including a counterexample (Part **(i)**), a comparison with several published results (Parts **(i)**–**(v)**), and some extensions (Part **(vi)**).

Theorem 5.1. *Suppose that A and B are two nonempty subsets of a normed space $(X, \|\cdot\|)$. If at least one of the following conditions holds, then dist(A, B) is attained, namely there exists at least one BAP with respect to (A, B):*

- *(i)* $A \cap B \neq \emptyset$;
- *(ii)* **([54, pp. 58–59])** *$A - B$ is proximinal with respect to $\{0\}$. Equivalently, there is a minimal norm vector in $A - B$, that is, $\inf\{\|u\| \mid u \in A - B\}$ is attained.*
- *(iii) A is weakly sequentially compact and B is closed, convex and locally weakly sequentially compact;*
- *(iv) A is compact and B is closed, convex and locally compact;*
- *(v)* **([71, Theorem 4]):** *A is weakly sequentially compact and B is convex and proximinal with respect to A;*
- *(vi)* **[71, Corollary 1]** *A is compact and B is proximinal with respect to A;*
- *(vii) A is boundedly compact and B is bounded and proximinal with respect to A;*
- *(viii)* **[71, p. 322]** *A and B are boundedly compact and one of them is bounded;*
- *(ix) A and B are weakly sequentially compact;*
- *(x)* **(classic)** *A and B are compact;*
- *(xi) For all closed balls D in X^2 about the origin the intersection $D \cap (A \times B)$ is weakly sequentially compact, and either $A \cup B$ is bounded or it is unbounded and the coercivity condition* ***(2.2)*** *holds;*

(xii) *X is a reflexive Banach space, A and B are weakly sequentially closed, and there is at least one distance minimizing sequence $((a_k,b_k))_{k\in\mathbb{N}}$ such that $(a_k)_{k\in\mathbb{N}}$ has a bounded subsequence;*
(xiii) *X is a reflexive Banach space, A is weakly sequentially compact (alternatively, bounded and weakly sequentially closed), and B is weakly sequentially closed;*
(xiv) **([71, Corollary 2]):** *X is a reflexive Banach space, A is bounded and weakly closed, and B is closed and convex;*
(xv) **[66, Theorem 1.1]** *X is a reflexive Banach space, both A and B are convex and closed, and A is bounded;*
(xvi) *X is a reflexive Banach space, A and B are weakly sequentially closed, the union $A\cup B$ is unbounded, and the coercivity condition **(2.2)** holds;*
(xvii) *X is a reflexive Banach space, both A and B are convex and closed, $A\cup B$ is unbounded and the coercivity condition **(2.2)** holds;*
(xviii) *X is a reflexive Banach space and $A-B$ is weakly sequentially closed.*
(xix) *X is a reflexive Banach space, A and B are convex, and $A-B$ is closed;*
(xx) *X is a reflexive Banach space, $A=\widetilde{A}+\widehat{A}$ and $B=\widetilde{B}+\widehat{B}$, where both $\widetilde{A}$ and $\widetilde{B}$ are weakly sequentially compact, and $\widehat{A}-\widehat{B}$ is weakly sequentially closed;*
(xxi) *X is a reflexive Banach space, A is a closed affine subspace with a closed linear part $\widetilde{A}$ which is complemented by a closed linear subspace F, and B is an affine subspace with a linear part $\widetilde{B}$ such that $\Pi_F(\widetilde{B})$ is closed;*
(xxii) *X is a reflexive Banach space, A is a closed affine subspace, and B is a finite-dimensional affine subspace;*
(xxiii) *X is a reflexive Banach space, A is a closed affine subspace of finite codimension, and B is an affine subspace;*
(xxiv) *X is a reflexive Banach space, A and B are closed and convex, B is locally compact, and the intersection of the recessions cones of A and B is $\{0\}$;*
(xxv) *X is a reflexive Banach space, A and B are closed affine subspaces of X with linear parts $\widetilde{A}$ and $\widetilde{B}$, respectively, such that $\widetilde{A}\cap\widetilde{B}\neq\emptyset$, and there is some $\alpha>0$ such that $d(x,\widetilde{A}\cap\widetilde{B})\leq \alpha d(x,\widetilde{B})$ for each $x\in\widetilde{A}$;*
(xxvi) **(implicit in [4])** *X is a real Hilbert space and both A and B are polyhedral;*
(xxvii) *X is a Hilbert space, $A\subseteq X$ is weakly sequentially closed, $p\in X\backslash A$, and B is the Voronoi cell of $P:=\{p\}$ with respect to A, i.e., $B:=\{z\in X\,|\,\|z-p\|\leq d(z,A)\}$;*
(xxviii) *X is a Hilbert space, A is a closed hyperplane, $p\in X\backslash A$, and B is the full hyperparaboloid induced by p and A, that is, the set of all points in X whose distance to p is not greater than their distance to A;*
(xxix) *X is a Hilbert space and both A and B are hypercylinders.*
(xxx) *A and B are finite-dimensional affine subsapces;*
(xxxi) *X is finite-dimensional, A and B are closed, and either $A\cup B$ is bounded or $A\cup B$ is unbounded and the coercivity condition **(2.2)** holds;*
(xxxii) *X is finite-dimensional, A is closed, $\emptyset\neq P\subset X$ is bounded, and B is the Voronoi cell of P with respect to A, namely $B:=\{z\in X\,|\,d(z,P)\leq d(z,A)\}$;*
(xxxiii) **([22, Theorem 5], [70, the Theorem on p. 209])** *X is a finite-dimensional Euclidean space and both A and B are polyhedral.*

Proof. In what follows $((a_k, b_k))_{k\in\mathbb{N}}$ is a distance minimizing sequence in $A \times B$.

(i) Since $A \cap B \neq \emptyset$ there is $b := a \in A \cap B$, and hence $0 = \|a - b\| = dist(A, B)$ and (a, b) is a BAP relative (A, B).

(ii) By our assumption there is some $v \in A-B$ such that $\|v\| = \inf\{\|u\| \mid u \in A-B\}$, and so there is a pair $(a, b) \in A \times B$ such that $\|a - b\| = \inf\{\|u\| \mid u \in A - B\}$. It is immediate to verify that $dist(A, B) = d(0, A - B) = \inf\{\|u\| \mid u \in A - B\}$. Therefore $dist(A, B) = \|a - b\|$ and (a, b) is a BAP relative to (A, B).

(iii) Since A is weakly sequentially compact there is an infinite subset $N_1 \subseteq \mathbb{N}$ and a point $a \in A$ which satisfy $a = (w) \lim_{k\to\infty, k\in N_1} a_k$. Since any weakly convergent sequence is bounded [**34**, II.3.27, p. 68], it follows that $(a_k)_{k\in N_1}$ is bounded, and therefore, by Lemma **3.11(i)**, also $(b_k)_{k\in N_1}$ is bounded. Thus (Lemma **3.12**) $b = (w) \lim_{k\to\infty, k\in N_2} b_k$ for some infinite subset $N_2 \subseteq N_1$ and $b \in B$. Hence $(a, b) = (w) \lim_{k\to\infty, k\in N_2} (a_k, b_k)$ and Lemma **3.11(iii)** implies that (a, b) is a BAP relative to (A, B).

(iv) This is an immediate consequence of Part **(iii)** because a compact subset is sequentially compact and hence (strong convergence implies weak convergence) also weakly sequentially compact, and a locally compact subset is locally sequentially compact and hence also locally weakly sequentially compact.

(v) Since A is weakly sequentially compact there exist an infinite subset $N_1 \subseteq \mathbb{N}$ and a point $a \in A$ such that $(w) \lim_{k\to\infty, k\in N_1} a_k = a$. We claim that $d(a, B) = dist(A, B)$. Indeed, consider the function $g : X \to [0, \infty)$ defined for all $x \in X$ by $g(x) := d(x, B)$. As is well known, g is continuous (even Lipschitz continuous [**46**, p. 19]), and it is also convex since B is convex [**68**, Examples 5.18(b), p. 66]. Hence g is weakly lower semicontinuous [**12**, Corollary 3.9, p. 61]. In addition, since $(b_k)_{k\in\mathbb{N}}$ is in B, we have $d(a_k, B) \leq \|a_k - b_k\|$ for all $k \in N_1$ by the definition of $d(a_k, B)$. Hence $d(a, B) = g(a) \leq \liminf_{k\to\infty, k\in N_1} g(a_k) = \liminf_{k\to\infty, k\in N_1} d(a_k, B) \leq \liminf_{k\to\infty} \|a_k - b_k\| = dist(A, B)$. On the other hand $dist(A, B) \leq d(a, B)$ because $a \in A$. Thus $d(a, B) = dist(A, B)$. Since B is proximinal with respect to A there is $b \in B$ such that $d(a, B) = \|a - b\|$. Therefore $\|a - b\| = dist(A, B)$ and so (a, b) is a BAP relative to (A, B).

(vi) From the compactness of A there are $a \in A$ and an infinite subset $N_1 \subseteq \mathbb{N}$ such that $\lim_{k\to\infty, k\in N_1} a_k = a$. Since B is proximinal with respect to A there is $b \in B$ such that $\|a - b\| = d(a, B)$. Since $(b_k)_{k\in\mathbb{N}}$ is in B, we have $d(a_k, B) \leq \|a_k - b_k\|$ for all $k \in \mathbb{N}$. Hence, because the function $g : X \to [0, \infty)$ defined for all $x \in X$ by $g(x) := d(x, B)$ is continuous [**46**, p. 19]), we have $\|a - b\| = d(a, B) = \lim_{k\to\infty, k\in N_1} d(a_k, B) \leq \lim_{k\to\infty, k\in N_1} \|a_k - b_k\| = dist(A, B)$, where the last equality is by the assumption that $((a_k, b_k))_{k\in\mathbb{N}}$ is a distance minimizing sequence. Therefore $\|a - b\| \leq dist(A, B)$, and obviously $dist(A, B) \leq \|a - b\|$ since $(a, b) \in A \times B$. Thus (a, b) is a BAP relative to (A, B).

(vii) Since B is bounded, so is $(b_k)_{k\in\mathbb{N}}$. Hence Lemma **3.11(i)** ensures that $(a_k)_{k\in\mathbb{N}}$ is bounded too. Let C be a closed ball which contains both $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$. Since A is boundedly compact, $A \cap C$ is compact. Hence there are $a \in A \cap C$ and an infinite subset $N_1 \subseteq \mathbb{N}$ such that $\lim_{k\to\infty, k\in N_1} a_k = a$. From now on we continue word for word as in the proof of Part **(vi)** and conclude the existence of a BAP (a,b) relative to (A,B).

(viii) Suppose that A is bounded. The proof is similar if B is bounded. Then $(a_k)_{k\in\mathbb{N}}$ is bounded, and hence, as follows from Lemma **3.11(i)**, also $(b_k)_{k\in\mathbb{N}}$ is bounded. Thus there is a closed ball C such that both $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ are in C, and since both A and B are boundedly compact, the intersections $A\cap C$ and $B\cap C$ are compact. Thus $(A\cap C)\times(B\cap C)$ is a compact subset of X^2 which contains $((a_k,b_k))_{k\in\mathbb{N}}$, and so there are $(a,b)\in(A\cap C)\times(B\cap C)$ and an infinite subset $N_1\subseteq\mathbb{N}$ such that $\lim_{k\to\infty,k\in N_1}(a_k,b_k)=(a,b)$. Since the norm is continuous and since $((a_k,b_k))_{k\in\mathbb{N}}$ is a distance minimizing sequence, we have $\|a-b\|=\lim_{k\to\infty,k\in N_1}\|a_k-b_k\|=dist(A,B)$, and so (a,b) is a BAP relative to (A,B).

(ix) Since A and B are nonempty and weakly sequentially compact, so is their product $A\times B$, and so there is some $(a,b)\in A\times B$ which is the weak limit of a subsequence of $((a_k,b_k))_{k\in\mathbb{N}}$, that is $(a,b)=(w)\lim_{k\to\infty,k\in N_1}(a_k,b_k)$ for some infinite subset $N_1\subseteq\mathbb{N}$. Hence Lemma **3.11(iii)** implies that (a,b) is a BAP relative to (A,B).

(x) This is a consequence of Part **(ix)** because any compact set is also sequentially compact and hence (strong convergence implies weak convergence) weakly sequentially compact. Alternatively, one can show directly, using the continuity of the norm, that any accumulation point of $((a_k,b_k))_{k\in\mathbb{N}}$ (which exists because of the compactness of $A\times B$) is a BAP relative to (A,B).

(xi) Since either $A\cup B$ is bounded, or $A\cup B$ is unbounded and the coercivity condition **(2.2)** holds, Lemma **3.11(ii)** implies that $(a_k)_{k\in\infty}$ and $(b_k)_{k\in\mathbb{N}}$ are bounded. Hence $((a_k,b_k))_{k\in\mathbb{N}}$ is contained in some closed ball D of X^2 about the origin. Since $((a_k,b_k))_{k\in\mathbb{N}}$ is contained in $A\times B$, we conclude that $((a_k,b_k))_{k\in\mathbb{N}}$ is contained in $C:=D\cap(A\times B)$, which is a weakly sequentially compact subset by the assumption in the formulation of this part. Hence there is a pair $(a,b)\in C$ and an infinite subset $N_1\subseteq\mathbb{N}$ such that $(a,b)=(w)\lim_{k\to\infty,k\in N_1}(a_k,b_k)$. Consequently, Lemma **3.11(iii)** implies that (a,b) is a BAP relative to (A,B).

(xii) Let $((a_k,b_k))_{k\in\mathbb{N}}$ be a distance minimizing sequence with the property that $(a_k)_{k\in\mathbb{N}}$ has a bounded subsequence $(a_k)_{k\in N_1}$ for some infinite subset $N_1\subseteq\mathbb{N}$. Since X is reflexive, any bounded sequence in it has a weakly convergent subsequence [**34**, Theorem II.3.28, p. 68]. Hence there is some $a\in X$ and an infinite subset N_2 of N_1 such that $a=(w)\lim_{k\to\infty,k\in N_2}a_k$. Since A is weakly sequentially closed, we have $a\in A$. Since $(a_k)_{k\in N_2}$ is bounded, also $(b_k)_{k\in N_2}$ is bounded by Lemma **3.11(i)**. Hence the reflexivity of X implies that there is some $b\in X$ and an infinite subset N_3 of N_2 such that $b=(w)\lim_{k\to\infty,k\in N_3}b_k$. Thus $(a,b)=(w)\lim_{k\to\infty,k\in N_3}(a_k,b_k)$ and $b\in B$ since B is weakly sequentially closed. Hence Lemma **3.11(iii)** implies that (a,b) is a BAP relative to (A,B).

(xiii) Since A is weakly sequentially compact, it must be bounded (otherwise there is some sequence $(x_k)_{k\in\mathbb{N}}$ in A such that $\lim_{k\to\infty}\|x_k\|=\infty$, and hence $(x_k)_{k\in\mathbb{N}}$ cannot have a weakly convergent subsequence since any weakly convergent subsequence is bounded [**34**, II.3.27, p. 68]; thus not every sequence in A has a convergent subsequence, in contradiction with the assumption that A is weakly sequentially compact). In addition, as a weakly sequentially compact subset, A is evidently weakly sequentially closed. Hence for every distance minimizing sequence $((a_k, b_k))_{k\in\mathbb{N}}$, the sequence $(a_k)_{k\in\mathbb{N}}$ is automatically bounded. Thus the assertion follows from Part **(xii)**.

(xiv) Since A is weakly closed and bounded and since X is reflexive, A is weakly compact [**34**, Corollary V.4.8, p. 415], and hence weakly sequentially compact since any weakly compact subset of a normed space is weakly sequentially compact [**45**, Corollary in Section 18A, p. 146]. Since B is closed and convex, it is weakly closed [**34**, Theorem V.3.13, p. 422], and so weakly sequentially closed. Thus the assertion follows from Part **(xiii)**.

(xv) The result follows from either Part **(xiv)** or Part **(xiii)** because any nonempty, closed and convex subset is weakly closed and hence weakly sequentially closed, and any nonempty, closed, convex and bounded subset of a reflexive Banach space is weakly compact and hence weakly sequentially compact.

(xvi) Because $A \cup B$ is unbounded and the coercivity condition **(2.2)** holds, we conclude from Lemma **3.11(ii)** that $(a_k)_{k\in\infty}$ and $(b_k)_{k\in\mathbb{N}}$ are bounded for every distance minimizing sequence $((a_k, b_k))_{k\in\mathbb{N}}$. The assertion now follows from Part **(xii)**.

(xvii) The assertion follows from Part **(xvi)** because any closed and convex subset of a Banach space is weakly closed and hence weakly sequentially closed.

(xviii) Since $\lim_{k\to\infty} \|a_k - b_k\| = dist(A, B) < \infty$, if we denote $z_k := a_k - b_k$ for every $k \in \mathbb{N}$, then $(z_k)_{k\in\mathbb{N}}$ is bounded and hence (because X is reflexive) $(w)\lim_{k\to\infty, k\in N_1} z_k = z$ for some infinite subset $N_1 \subseteq \mathbb{N}$ and some $z \in X$. Because $A - B$ is weakly sequentially closed and $(z_k)_{k\in\mathbb{N}}$ is in $A - B$, we have $z \in A - B$. Thus $z = a - b$ for some $(a, b) \in A \times B$. In addition, since the norm is weakly sequentially lower semicontinuous [**34**, II.3.27, p. 68], we have $\|a - b\| = \|z\| \leq \liminf_{k\to\infty, k\in N_1} \|z_k\| = dist(A, B)$. Since $(a, b) \in A \times B$, we obviously have $dist(A, B) \leq \|a - b\|$. Hence $\|a - b\| = dist(A, B)$ and (a, b) is a BAP relative to (A, B).

(xix) By the assumptions on A and B we see that $A - B$ is closed and convex. Hence $A - B$ is weakly closed and therefore weakly sequentially closed. Since X is reflexive, the assertion follows from Part **(xviii)**.

(xx) Since $A = \widetilde{A} + \widehat{A}$ and $B = \widetilde{B} + \widehat{B}$, an immediate verification shows that $A - B = (\widetilde{A} - \widetilde{B}) + (\widehat{A} - \widehat{B})$. Hence by Part **(xiii)** it is sufficient to show that $\widetilde{A} - \widetilde{B}$ is weakly sequentially compact because we already assume that $\widehat{A} - \widehat{B}$ is weakly sequentially closed. This is immediate because both $\widetilde{A}$ and $\widetilde{B}$ are weakly sequentially compact (hence if $(x_k - y_k)_{k\in\mathbb{N}}$ is an arbitrary sequence in $\widetilde{A} - \widetilde{B}$ where $(x_k)_{k\in\mathbb{N}}$ is in $\widetilde{A}$ and $(y_k)_{k\in\mathbb{N}}$ is in $\widetilde{B}$, then we can find infinite subsets $N_2 \subseteq N_1 \subseteq \mathbb{N}$ and points $x \in \widetilde{A}$ and $y \in \widetilde{B}$ such that $x = (w)\lim_{k\to\infty, k\in N_1} x_k$ and $y = (w)\lim_{k\to\infty, k\in N_2} y_k$; thus $x - y = (w)\lim_{k\to\infty, k\in N_2}(x_k - y_k)$, namely $(x_k - y_k)_{k\in\mathbb{N}}$ has a subsequence which converges to a point in $\widetilde{A} - \widetilde{B}$).

(xxi) Our goal is to use Part **(xix)**. Since A and B are affine and hence convex, it remains to show that $A - B$ is closed. We can write $A = p_1 + \widetilde{A}$ and $B = p_2 + \widetilde{B}$

for some $p_1, p_2 \in X$ and linear subspaces $\widetilde{A}$ and $\widetilde{B}$ of X. Since $A - B = (p_1 - p_2) + (\widetilde{A} - \widetilde{B})$, it is sufficient to show that $\widetilde{A} - \widetilde{B}$ is closed.

We claim that $\widetilde{A} - \widetilde{B} = \widetilde{A} \oplus \Pi_F(\widetilde{B})$. Indeed, let $z \in \widetilde{A} - \widetilde{B}$ be arbitrary. Then $z = x - y$ for some $x \in \widetilde{A}$ and $y \in \widetilde{B}$. From our assumption that $X = \widetilde{A} \oplus F$ we can write $y = y_1 + y_2$, where $y_1 = \Pi_{\widetilde{A}}(y) \in \widetilde{A}$ and $y_2 = \Pi_F(\widetilde{B}) \in F$. Because $\widetilde{A}$, as a linear subspace, is closed under sums, we have $x - y_1 \in A$. In addition, $\Pi_F(\widetilde{B}) = -\Pi_F(\widetilde{B})$ since $\Pi_F(\widetilde{B})$ is a linear subspace. Hence $-y_2 \in \Pi_F(\widetilde{B})$ and $x - y = (x - y_1) + (-y_2) \in \widetilde{A} \oplus \Pi_F(\widetilde{B})$. Since $z \in \widetilde{A} - \widetilde{B}$ was arbitrary, we have $\widetilde{A} - \widetilde{B} \subseteq \widetilde{A} + \Pi_F(\widetilde{B})$. Because $\Pi_F(\widetilde{B}) \subseteq F$ and $\widetilde{A} \cap F = \{0\}$, we actually have $\widetilde{A} + \Pi_F(\widetilde{B}) = \widetilde{A} \oplus \Pi_F(\widetilde{B})$. Now let $z \in \widetilde{A} \oplus \Pi_F(\widetilde{B})$ be arbitrary. Then $z = x + w$ for some (unique) $x \in \widetilde{A}$ and $w \in \Pi_F(\widetilde{B})$. Since $w \in \Pi_F(\widetilde{B})$, there is some $y \in \widetilde{B}$ such that $w = \Pi_F(y)$. Hence $z = x + w = (x - \Pi_{\widetilde{A}}(y)) + (\Pi_{\widetilde{A}}(y) + \Pi_F(y)) = (x - \Pi_{\widetilde{A}}(y)) + y = (x - \Pi_{\widetilde{A}}(y)) - (-y)$. Because $\widetilde{A}$ and $\widetilde{B}$ are linear subspaces, we have $x - \Pi_{\widetilde{A}}(y) \in \widetilde{A}$ and $-y \in \widetilde{B}$. Therefore $z \in \widetilde{A} - \widetilde{B}$. Since $z \in \widetilde{A} \oplus \Pi_F(\widetilde{B})$ was arbitrary, we have $\widetilde{A} \oplus \Pi_F(\widetilde{B}) \subseteq \widetilde{A} - \widetilde{B}$, as required.

We claim that $A \oplus \Pi_F(\widetilde{B})$ is a closed subset of X. Indeed, let $(z_k)_{k\in\mathbb{N}}$ be any convergent sequence in $\widetilde{A} \oplus \Pi_F(\widetilde{B})$, and let $z \in X$ be its limit. Then for all $k \in \mathbb{N}$, one has $z_k = x_k + w_k$ for some (unique) $x_k \in \widetilde{A}$ and $w_k \in \Pi_F(\widetilde{B})$. Since $\widetilde{A}$ and $\widetilde{B}$ are topologically complemented in the Banach space X, the linear projection $\Pi_{\widetilde{A}}$ is continuous [**28**, Theorems 13.1, 13.2, p. 94]. Hence $\lim_{k\to\infty} x_k = \lim_{k\to\infty} \Pi_{\widetilde{A}}(z_k) = \Pi_{\widetilde{A}}(z) \in \widetilde{A}$. Thus $w := \lim_{k\to\infty} w_k = \lim_{k\to\infty}(z_k - x_k) = z - \Pi_{\widetilde{A}}(z)$. Because $(w_k)_{k\in\mathbb{N}}$ is in the closed subspace $\Pi_F(\widetilde{B})$, its limit w is in $\Pi_F(\widetilde{B})$. Hence $z = \Pi_{\widetilde{A}}(z) + w \in \widetilde{A} \oplus \Pi_F(\widetilde{B})$ and $\widetilde{A} \oplus \Pi_F(\widetilde{B})$ is closed. Since $\widetilde{A} - \widetilde{B} = A \oplus \Pi_F(\widetilde{B})$, also $\widetilde{A} - \widetilde{B}$ is closed, as required.

(xxii) Since the linear part $\widetilde{A}$ of A is closed (because so is A), and since we assume that the linear part $\widetilde{B}$ of B is finite dimensional, we conclude from [**46**, Proposition 20.1, p. 195] that $\widetilde{A} + \widetilde{B}$ is closed. Since obviously $\widetilde{B} = -\widetilde{B}$ because $\widetilde{B}$ is a linear subspace, we see that $\widetilde{A} - \widetilde{B} = \widetilde{A} + \widetilde{B}$ is closed, and hence so is its translated copy $A - B$. Therefore the assertion follows from Part **(xix)**.

(xxiii) By our assumption $X = \widetilde{A} \oplus F$ for some finite dimensional linear subspace F. Therefore $\Pi_F(\widetilde{B})$, which is a linear subspace of F, is also finite dimensional and hence closed [**46**, p. 196]. The assertion now follows from Part **(xxi)**.

(xxiv) It follows from [**45**, Lemma 15D, p. 104] that $A - B$ is closed, and hence the assertion follows from Part **(xix)**.

(xxv) Let $\widetilde{A}$ and $\widetilde{B}$ be the linear parts of A and B, respectively. It follows from [**12**, Ex. 2.16, p. 52] that $\widetilde{A} + \widetilde{B}$ is closed. Since $\widetilde{B}$ is a linear subspace, $\widetilde{B} = -\widetilde{B}$ and so $\widetilde{A} - \widetilde{B} = \widetilde{A} + \widetilde{B}$ is closed. Thus $A - B$, which is a translation of $\widetilde{A} - \widetilde{B}$, is closed, and hence the assertion follows from Part **(xix)**.

(xxvi) According to [**4**, Corollary 3.4.8], the infimum $\sigma := \inf\{\|z - P_B P_A z\| \mid z \in X\}$ is attained at some $b \in X$, namely $\sigma = \|b - P_B P_A b\|$, where P_A is the orthogonal projection on A and P_B is the orthogonal projection on B, which are well defined since A and B are nonempty, closed and convex. According to [4, Corollary 4.4.3, Fact 4.4.4 and Remark 4.4.6], since σ is attained, one has $\sigma := 0$. Thus $\|b - P_B P_A b\| = 0$, namely b is a fixed point of $P_B P_A$, and, in particular, $b \in B$. But according to [4, Fact 5.1.4(i)], which is actually [22, Theorem 2], any fixed point z of $P_B P_A$ satisfies $d(z, A) = dist(A, B)$. Since $d(z, A) = \|z - P_A z\|$ by the definition of P_A, if we let $z := b$ and $a := P_A b$, then $a \in A$ and $dist(A, B) = \|b - a\|$, that is, (a, b) is a BAP relative to (A, B).

(xxvii) Since $B = \cap_{a\in A} H(p, a)$, where $H(p, a) := \{z \in X \,|\, \|z - p\| \leq \|z - a\|\}$, it follows that B is an intersection of closed halfpsaces and hence closed and convex (because $p \notin A$, one has $p \neq a$ for all $a \in A$, and hence $H(p, a)$ is indeed a halfspace for each $a \in A$). Thus B is weakly closed and thus weakly sequentially closed. Therefore if B is bounded, then the assertion follows from Part **(xiii)** (where B and A are interchanged there). Otherwise, B is unbounded and so is $A \cup B$. We claim that **(2.2)** holds. Indeed, let $\mu > 0$ be arbitrary and denote $\rho := 3(\mu + \|p\|)$. Let (x, y) be an arbitrary pair in $A \times B$ which satisfies $\|(x, y)\| > \rho$. Either $\|y\| > \mu + \|p\|$ or $\|y\| \leq \mu + \|p\|$. In the first case the relations $y \in B$, $x \in A$ and the triangle inequality imply that $\mu < \|y\| - \|p\| \leq \|y - p\| \leq d(y, A) \leq \|y - x\|$. In the second case we must have $\|x\| > 2(\mu + \|p\|)$ because otherwise $\|(x, y)\|^2 = \|x\|^2 + \|y\|^2 \leq 5(\mu + \|p\|)^2 < 9(\mu + \|p\|)^2 = \rho^2$, a contradiction to what we assumed on (x, y). Thus the triangle inequality and the inequalities $\|y\| \leq \mu + \|p\|$ and $\|x\| > 2(\mu + \|p\|)$ imply that $\mu \leq \mu + \|p\| < \|x\| - \|y\| \leq \|y - x\|$. Therefore $\mu < \|x - y\|$ whenever $(x, y) \in A \times B$ satisfies $\|(x, y)\| > \rho$. Since ρ was an arbitrary positive number, the definition of the limit implies that **(2.2)** holds. Consequently, Part **(xvi)** implies the existence of a BAP with respect to (A, B).

(xxviii) B is nothing but the Voronoi cell of $P := \{p\}$ with respect to A, i.e., $B := \{z \in X \,|\, \|z - p\| \leq d(z, A)\}$. Since A is closed and convex, it is weakly closed and hence weakly sequentially closed. Thus the assertion follows from Part **(xxvii)**

(xxix) We first recall that a hypercylinder is a set of the form $C + L$, where L is a line (hence a one-dimensional affine subspace) and C is a closed ball contained in the orthogonal complement subspace $\widetilde{L}^{\perp}$ of the linear part $\widetilde{L}$ of L, having the origin of $\widetilde{L}$ as its center. Therefore $A = C_1 + L_1$ and $B = C_2 + L_2$ for two lines L_1 and L_2, and two balls $C_1 \subset \widetilde{L}_1^{\perp}$ and $C_2 \subset \widetilde{L}_2^{\perp}$. Since the linear part of $L_1 - L_2$ is the linear subspace $\widetilde{L}_1 - \widetilde{L}_2$ which is finite dimensional and hence closed [46, p. 196], also $L_1 - L_2$ is closed. Because $L_1 - L_2$ is also convex, it is weakly closed and hence weakly sequentially closed. Since C_1 and C_2 are closed, convex and bounded and X is reflexive, both C_1 and C_2 are weakly sequentially compact. The assertion now follows from Part **(xx)**.

(xxx) Since both A and B are finite-dimensional affine subspaces, so is their difference $A - B$. As is well known, the distance from any point x in a normed space X to a finite-dimensional affine subspace F of X is attained (this is immediate: let $(x_k)_{k\in\mathbb{N}}$ in F satisfy $\lim_{k\to\infty} \|x - x_k\| = d(x, F)$; then $(x_k)_{k\in\mathbb{N}}$ is bounded, and so has a convergent subsequence since F is finite-dimensional, and the limit z

of this subsequence satisfies $\|x-z\| = d(x,F)$ by the continuity of the norm). Hence $d(0,A-B)$ is attained and the assertion follows from Part **(ii)**.

(xxxi) Since any finite-dimensional normed space is a reflexive Banach space and since in finite-dimensional normed spaces a sequence converges weakly if and only if it converges strongly, the assertion follows from either Part **(xiii)** (if $A\cup B$ is bounded) or Part **(xvi)** (if $A\cup B$ is unbounded).

(xxxii) The proof is somewhat similar to the proof of Part **(xxvii)**, but because there are differences in the settings, some modifications are needed. First we observe that since the function $g: X\to\mathbb{R}$ defined by $g(z) := d(z,P)-d(z,A)$ for all $z\in X$ is continuous (even Lipschitz continuous) and B is its 0-level-set, it follows that B is closed and hence weakly sequentially closed since X is finite dimensional. We also observe that since $d(z,P)=d(z,\overline{P})$ for every $z\in X$ (as follows, from instance, from Lemma **3.1**), we have $B=\{z\in X\,|\,d(z,\overline{P})\leq d(z,A)\}$.

If B is bounded, then the assertion follows from Part **(xiii)** (where B and A are interchanged there). Otherwise, B is unbounded and so is $A\cup B$. We claim that **(2.2)** holds. Indeed, let $\mu>0$ be arbitrary. Since P is bounded, so is $\overline{P}$, and there is some $r>0$ such that $\overline{P}$ is contained in the ball of radius r about the origin. Denote $\rho := 3(\mu+r)$ and let (x,y) be an arbitrary pair in $A\times B$ which satisfies $\|(x,y)\|>\rho$. Either $\|y\|>\mu+r$ or $\|y\|\leq\mu+r$. Suppose that the first case holds. Since $\overline{P}$ is closed and X is finite dimensional, there is some $p\in\overline{P}$ (hence $\|p\|\leq r$) such that $\|y-p\|=d(y,\overline{P})$ (this also follows from Part **(xiii)**, where A there is replaced by $\{y\}$ and B there is replaced by $\overline{P}$). These facts, as well as the triangle inequality and the fact that $y\in B$, all imply that $\mu<\|y\|-r\leq\|y\|-\|p\|\leq\|y-p\|=d(y,\overline{P})\leq d(y,A)\leq\|y-x\|$. Now suppose that the second case holds, that is, $\|y\|\leq\mu+r$. It must be that $\|x\|>2(\mu+r)$ because otherwise $\|(x,y)\|^2=\|x\|^2+\|y\|^2\leq 5(\mu+r)^2<\rho^2$ by the definition of ρ, a contradiction to what we assumed on (x,y). Hence the inequalities $\|y\|\leq\mu+r$ and $\|x\|>2(\mu+r)$, as well as the triangle inequality, imply that $\mu<\mu+r<\|x\|-\|y\|\leq\|y-x\|$. Thus $\mu<\|x-y\|$ for all $(x,y)\in A\times B$ which satisfies $\|(x,y)\|>\rho$. Since ρ was an arbitrary positive number, the definition of the limit implies that **(2.2)** holds. Hence Part **(xvi)** implies that there is a BAP relative to (A,B).

(xxxiii) This is just a particular case of Part **(xxvi)**. Alternatively, since B is polyhedral, also $-B$ is polyhedral, and since the sum of finite-dimensional polyhedral sets is polyhedral by [61, Corollary 19.3.2] (see also [70, Lemma 2]), it follows that $A-B$ is polyhedral and hence closed because a polyhedral set is closed as an intersection of closed sets. The assertion now follows from Part **(xix)**.

□

Remark 5.2. Here are a few comments related to Theorem **5.1**:

(i) In general, existence of a BAP might not hold even in very simple settings. Indeed, let X be the Euclidean plane, $A:=\{(x_1,x_2)\in\mathbb{R}^2\,|\,x_2\geq e^{x_1}+1\}$ and $B:=\{(x_1,x_2)\in\mathbb{R}^2\,|\,x_2\leq -e^{x_1}-1\}$. Then A and B are closed, strictly convex and boundedly compact, but any $(a,b)\in A\times B$ satisfies $\|a-b\|>2=dist(A,B)$, and so there is no BAP relative to (A, B). Of course, in this case the coercivity condition **(2.2)** does not hold. This example shows that [**65**, Theorem 2.3, p. 385], which claims that there is a BAP relative to (A, B) whenever A and B are boundedly compact and closed, is incorrect, as observed before in [**71**, p. 322] using a different (counter)example. Nevertheless, by adding to [**65**, Theorem 2.3, p. 385] the assumption that either A or B is bounded, the assertion becomes correct, as claimed without a proof in [**71**, p. 322] and proved in Part **(viii)**.

(ii) Part **(i)** is, of course, well known and is mentioned in [**6**, p. 434 in Section 5] and [**9**, Fact 2.3(v)], in the context of Hilbert spaces. Part **(iv)** generalizes [**71**, the assertion after Theorem 3] (which by itself generalizes a result from [**64**]), where there X is restricted to be a Banach space and A (denoted by G there) is also assumed to be convex (the proof of this assertion suffers from a gap, namely the convexity of F there - denoted by B in Part **(iv)** - is crucial for the existence of a convergent subsequence in F, but is omitted from the proof). Part **(v)** generalizes [**54**, Theorem 3.1]. Part **(ix)** generalizes an assertion made in [**32**, between Theorem 2.4 and Definition 2.5] in a Banash space setting. Part **(xix)** generalizes [**6**, Lemma 2.1(ii) and p. 434 in Section 5] (see also [**4**, Theorem 5.4.3]) from the case where X is a real Hilbert spaces and A and B are closed and convex. Parts **(xxii)**–**(xxiii)** combined generalize [**6**, Facts 5.1(iii)] from the case where X is a real Hilbert spaces and A and B are closed and convex. Part **(xxiv)** is essentially mentioned in a real Hilbert setting in [**6**, Facts 5.1(iv)], which refers to the lemma in [**45**, Section 15] for the proof, as in we did; however, while [**6**, Facts 5.1(iv)] says that the intersection of the recessions cones can be linear, [**45**, Lemma 15D, p. 104] says that this intersection should be $\{0\}$. Part **(xxxi)** extends related existence results in the Euclidean case, such as [**15**, Corollary 4.16], [**36**, Proposition 2.2, Corollary 2.3], [**41**, Proposition 2.3] (without giving explicit formulae as done there; note that there a BAP is referred to as "the" BAP, although there can be several other BAPs).

(iii) Part **(vii)** is stated without a proof in [**71**, Corollary 1]. It is claimed there that the proof can be obtained from [**71**, The proof of Theorem 4], but this is not very clear since [**71**, The proof of Theorem 4] is based on the convexity of B (denoted by G there), which is not assumed in [**71**, Corollary 1].

(iv) A more general version of Part **(vii)** is claimed in [**71**, Corollary 1], again without a proof: that A is locally compact instead of being boundedly compact (the rest of the assumptions are the same as in Part **(vii)**). It is not clear to us whether this statement is correct, and it might be that the author of [**71**] actually meant "boundedly compact" instead of "locally compact".

(v) Part **(xxvi)** is claimed without a proof in [**6**, Fact 5.1(ii)]. It is said there that a proof will appear in a certain future work, but eventually that specific work neither presented the claim nor presented the proof.

(vi) One might wonder regarding possible extensions of Theorem **5.1**. This is definitely possible. For example, consider the case where X is the dual of a Banach space Y (e.g., $X = L_\infty$, $Y = L_1$). This case allows the use of well known properties of the dual space such as the compactness of the unit ball in the weak-star topology (the Banach-Alaoglu theorem [**34**, Theorem V.4.2, p. 424]), the fact that every bounded sequence has a weak-star convergent subsequence [**12**, Corollary 3.30, p. 76], and - if Y is also separable - the weak-star sequential lower semiontinuity of the (dual) norm [**12**, Proposition 3.13(iii), p. 63], in order to get corresponding existence results, such as the following modification of Part **(xiii)**: "X is the dual of a separable Banach space, A is weak-star sequentially compact, and B is weak-star sequentially closed."

(vii) Some of the assertions formulated in Theorem **5.1** hold, with essentially the same proofs, in metric spaces: these are Parts **(i)**, **(vi)**, **(vii)**, **(viii)** and **(x)**.

(viii) Given a collection $(P_k)_{k\in K}$ of nonempty subsets of the ambient space X, called sites or generators, the Voronoi cell V_k of P_k is the set $V_k := \{x \in X \,|\, d(x, P_k) \leq d(x, A_k)\}$, where $A_k := \cup_{j\in K\setminus\{k\}} P_j$. The collection $(V_k)_{k\in K}$ is the so-called Voronoi diagram of the given sites. Voronoi diagrams have numerous applications in science and technology: see, for example, [**2**,**23**,**29**,**33**,**43**,**53**,**58**–**60**] and the references therein. If all the sites are closed and either K is finite or K is infinite and the gap between the sites is positive (namely $0 < \inf\{dist(P_i, P_j) \,|\, i, j \in K, i \neq j\}$, as happens, e.g., if each site is a lattice point or a subset located in a small neighborhood of a lattice point), then A_k is closed for all $k \in K$ (in the first case this is obvious, and in the second case this follows from the fact that any sequence in A_k, which converges to some point in X, must belong to the same site P_j starting from some place because of the positive gap between the sites). Therefore, if, in addition, X is finite dimensional and all the sites are bounded, then Theorem **5.1(xxxii)** ensures that for every $k \in K$ the distance between the Voronoi cell of P_k and the union A_k of the other sites is attained.

6. Examples

This section presents several examples and counterexamples which illustrate some of the results established earlier.

Example 6.1. Let $(X, \|\cdot\|)$ be the Euclidean plane, $A := [-2, 2] \times [-2, 0]$ be a rectangle, and $B := \{(x_1, x_2) \in X \,|\, \frac{x_1^2}{4} + (x_2 - 2)^2 \leq 1\}$ be an ellipse. See Figure **1**. Then $(X, \|\cdot\|)$ is strictly convex, both A and B are nonempty, convex and compact, and B is actually strictly convex, and so, according to Corollary **4.6**, there is a unique BAP (a_0, b_0) with respect to (A, B). In fact, $a_0 = (0, 0)$ and $b_0 = (0, 1)$.

Example 6.2. Let $X := \mathbb{R}^2$ be the plane with the $\|\cdot\|_\infty$ norm $\|(x_1, x_2)\|_\infty := \max\{|x_1|, |x_2|\}$, $(x_1, x_2) \in X$, and let A and B be defined as in Example **6.1**. See Figure **2**. Since $(X, \|\cdot\|_\infty)$ is not strictly convex, the existence of a unique BAP with respect to (A, B) is not guaranteed. Indeed, now $([a_0, a_1], [b_0, b_1])$ is a nondegenerate (but not a strictly nondegenerate) BAP of intervals with respect to $(\partial A, \partial B)$, where $a_0 := (-1, 0)$, $a_1 := (1, 0)$, and $b_0 := (0, 1) =: b_1$, because $\|a(t) - b_0\|_\infty = 1 = dist(A, B)$ for all $t \in [0, 1]$.

Example 6.3. Let $X := \mathbb{R}^3$ with the $\|\cdot\|_\infty$ norm $\|(x_1, x_2, x_3)\|_\infty := \max\{|x_1|, |x_2|, |x_3|\}$, $(x_1, x_2, x_3) \in X$. Let $A := \{(x_1, x_2, x_3) \in X \,|\, x_1 \in [-1, 1], x_2 = 0, x_3 = 0\}$ and

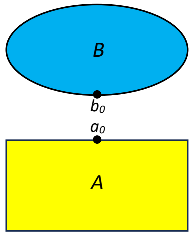

FIGURE 1. An ellipse and a rectangle in the Euclidean plane (Example **6.1**): a unique BAP.

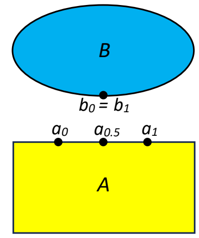

FIGURE 2. An ellipse and a rectangle in the plane with the $\|\cdot\|_\infty$ norm (Example **6.2**): many BAPs.

$B := \{(x_1, x_2, x_3) \in X \mid x_1 = 0, x_2 \in [-1, 1], x_3 = h\}$ for some fixed $h \geq 1$. Then A and B are nondegenerate intervals. Since any $x = (x_1, x_2, x_2) \in A$ and $y = (y_1, y_2, y_3) \in B$ satisfy $|x_1 - y_1| = |x_1| \leq 1$, $|x_2 - y_2| = |y_2| \leq 1$ and $|x_3 - y_3| = h \geq 1$, we have $\|x - y\|_\infty = h$ and $dist(A, B) = h$, namely (x, y) is a BAP relative to (A, B) for all $(x, y) \in A \times B$. Moreover, (A, B) is a strictly nondegenerate BAP of intervals with respect to (A, B) although A and B are not parallel.

Example 6.4. Let $X := \mathbb{R}^3$ with the Euclidean norm. Fix $\sigma_1, \sigma_2, h_1, h_2 \in (0, \infty)$ and let A be the elliptical cylinder defined by $A := \{(x_1, x_2, x_3) \in X \mid (x_1^2/\sigma_1^2) + (x_2^2/\sigma_2^2) \leq 1, x_3 \in [-h_1, 0]\}$. Let B be the ellipse defined by $B := \{(x_1, x_2, x_3) \in X \mid (x_1^2/\sigma_1^2) + ((x_2 - \sigma_2)^2/\sigma_2^2) \leq 1, x_3 = h_2\}$, namely B is a translated copy of the ellipse which generates A. See Figure **3**. Here $(X, \|\cdot\|)$ is strictly convex but both A and B are not (even though B is strictly convex in the affine hull that it spans, namely when restricted to the plane $\{(x_1, x_2, x_3) \in X : x_3 = h_2\}$) and indeed, $(S_1, \widetilde{S}_1)$ and $(S_2, \widetilde{S}_2)$ (see Figure **3**) are strictly nondegenerate BAPs of intervals with respect to (A, B). Of course, there are infinitely many other such pairs.

Example 6.5. Figure **4** presents a two-dimensional example in which there exists at least one BAP with respect to (A, B) because of Theorem **5.1(x)**, and this pair is unique because of Theorem **4.1** since there does not exist a nondegenerate BAP of intervals with respect to $(\partial A, \partial B)$. Nevertheless, there does exist an interval contained in ∂A which is parallel to an interval which is contained in ∂B.

Example 6.6. Figure **5** presents two compact shapes in the Euclidean plane whose boundaries contain intervals, but no interval contained in the boundary of one shape is parallel to an interval contained in the boundary of the other shape. Hence the conditions of Corollary **4.6** and of Theorem **5.1(x)** are satisfied, and thus there exists a unique BAP relative to (A, B).

Example 6.7. Figure **6** presents two disjoint and unbounded cylinders in the Euclidean space $\mathbb{R}^3$ whose main axes are not parallel. Therefore, despite the fact that the boundary of each cylinder contains intervals, no interval from one boundary is

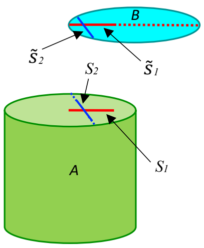

FIGURE 3. An ellipse and a cylinder in the Euclidean space $\mathbb{R}^3$ (Example **6.4**): two strictly nondegenerate BAPs of intervals (i.e., $(S_1, \widetilde{S}_1)$ and $(S_2, \widetilde{S}_2)$) are presented.

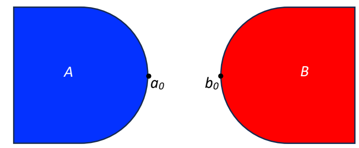

FIGURE 4. Two shapes in the Euclidean plane whose boundaries contain parallel intervals which satisfy the conditions of Theorem **4.1**, and hence induce a unique BAP (Example **6.5**).

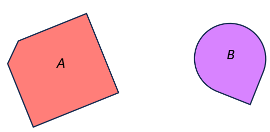

FIGURE 5. Two shapes in the Euclidean plane for which the conditions of Corollary **4.6** are satisfied (Example **6.6**).

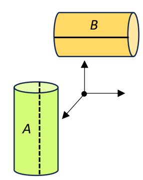

FIGURE 6. Two non-parallel unbounded cylinders (partly shown) in the Euclidean space $\mathbb{R}^3$ for which the conditions of Corollary **4.6** are satisfied (Example **6.7**).

parallel to an interval from the other boundary. Hence Corollary **4.6** ensures that there exists at most one BAP relative to (A, B), and Theorem **5.1(xxix)** ensures the existence of at least one BAP relative to (A, B), namely the pair is unique.

Example 6.8. An illustration of Theorem **4.2** in the plane, with any norm, is presented in Figure **7**. This theorem and Theorem **5.1(x)** imply that there is a unique BAP relative to (A, B).

Example 6.9. An illustration of Theorem **4.5** in the plane, with any strictly convex norm, is presented in Figure **8**. This theorem, as well as Theorem **5.1(x)**, ensure that there is a unique BAP relative to (A, B).

ACKNOWLEDGMENTS

This research is supported by the ISF Grant Number 2874/19 within the ISF-NSFC joint research program. The work of Y.C. was also supported by U. S. National

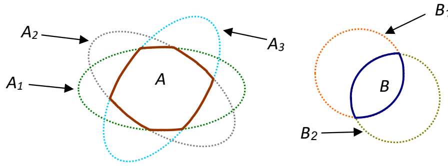

FIGURE 7. An illustration of Theorem **4.2**: two intersections (solid lines) of strictly convex subsets (dashed lines) in the plane with any norm (Example **6.8**). Only the boundaries of the subsets are shown.

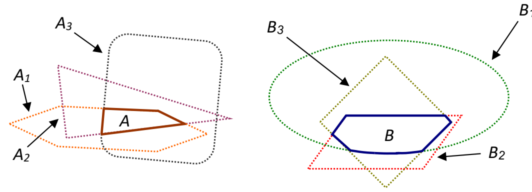

FIGURE 8. Two intersections (solid lines) of subsets (dashed lines) which satisfy the conditions of Theorem **4.5** and are located in the plane with any strictly convex norm (Example **6.9**). Only the boundaries of the sets are shown.

Institutes of Health Grant Number R01CA266467 and by the Cooperation Program in Cancer Research of the German Cancer Research Center (DKFZ) and the Israeli Ministry of Innovation, Science and Technology (MOST).

REFERENCES

[1] AHARONI, R., CENSOR, Y., AND JIANG, Z. Finding a best approximation pair of points for two polyhedra. *Comput. Optim. Appl. 71* (2018), 509–523.

[2] AURENHAMMER, F. Voronoi diagrams - a survey of a fundamental geometric data structure. *ACM Computing Surveys 3* (1991), 345–405.

[3] BANCEREK, G., AND PRIME.MOVER. Boundary of intersection is subset of union of boundaries. *Proof Wiki* (2016). (Updated: 2021, retrieved: July 18, 2023).

[4] BAUSCHKE, H., BORWEIN, J., AND LEWIS, A. The method of cyclic projections for closed convex sets in Hilbert space. *Contemp. Math. 204* (1997), 1–38.

[5] BAUSCHKE, H. H., AND BORWEIN, J. M. On the convergence of von Neumann's alternating projection algorithm for two sets. *Set-Valued Anal. 1* (1993), 185–212.

[6] BAUSCHKE, H. H., AND BORWEIN, J. M. Dykstra's alternating projection algorithm for two sets. *J. Approx. Theory 79* (1994), 418–443.

[7] BAUSCHKE, H. H., AND BORWEIN, J. M. On projection algorithms for solving convex feasibility problems. *SIAM Review 38* (1996), 367–426.

[8] BAUSCHKE, H. H., AND COMBETTES, P. L. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, 2 ed. CMS Books in Mathematics. Springer International Publishing, Cham, Switzerland, 2017.

[9] BAUSCHKE, H. H., COMBETTES, P. L., AND LUKE, D. R. Finding best approximation pairs relative to two closed convex sets in Hilbert spaces. *J. Approx. Theory 127* (2004), 178–192.

[10] BAUSCHKE, H. H., SINGH, S., AND WANG, X. Finding best approximation pairs for two intersections of closed convex sets. *Comput. Optim. Appl. 81* (2022), 289–308.

[11] BEAUZAMY, B. *Introduction to Banach Spaces and their Geometry*, vol. 68 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam-New York, 1982. Notas de Matemática [Mathematical Notes], 86.

[12] BREZIS, H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Universitext. Springer, New York, 2011.

[13] Butnariu, D., Censor, Y., Gurfil, P., and Hadar, E. On the behavior of subgradient projections methods for convex feasibility problems in Euclidean spaces. *SIAM J. OPTIM. 19* (2008), 786–807.

[14] Byrne, C., and Censor, Y. Proximity function minimization using multiple bregman projections, with applications to split feasibility and kullback–leibler distance minimization. *Annals of Operations Research 105* (2001), 77–98.

[15] Caseiro, R., Facas Vicente, M. A., and Vitória, J. Projection method and the distance between two linear varieties. *Linear Algebra Appl. 563* (2019), 446–460.

[16] Cegielski, A. *Iterative Methods for Fixed Point Problems in Hilbert Spaces*, vol. 2057 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012.

[17] Censor, Y., and Cegielski, A. Projection methods: an annotated bibliography of books and reviews. *Optimization 64* (2015), 2343–2358.

[18] Censor, Y., Mansour, R., and Reem, D. The alternating simultaneous Halpern-Lions-Wittmann-Bauschke algorithm for finding the best approximation pair for two disjoint intersections of convex sets. *arXiv:2304.09600 [math.OC]* (2023).

[19] Censor, Y., and Reem, D. Zero-convex functions, perturbation resilience, and subgradient projections for feasibility-seeking methods. *Mathematical Programming (Ser. A) 152* (2015), 339–380.

[20] Censor, Y., and Zaknoon, M. Algorithms and convergence results of projection methods for inconsistent feasibility problems: a review. *Pure Appl. Funct. Anal. 3* (2018), 565–586.

[21] Censor, Y., and Zenios, A. S. *Parallel Optimization: Theory, Algorithms, and Applications*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 1997. With a foreword by George B. Dantzig.

[22] Cheney, W., and Goldstein, A. A. Proximity maps for convex sets. *Proc. Amer. Math. Soc. 10* (1959), 448–450.

[23] Chiu, S. N., Stoyan, D., Kendall, W. S., and Mecke, J. *Stochastic Geometry and its Applications*, third ed. John Wiley & Sons, Chichester, UK, 2013.

[24] Clarkson, J. A. Uniformly convex spaces. *Trans. Amer. Math. Soc. 40* (1936), 396–414.

[25] Combettes, P. L. Inconsistent signal feasibility problems: least-squares solutions in a product space. *IEEE Transactions on Signal Processing 42* (1994), 2955–2966.

[26] Combettes, P. L. The convex feasibility problem in image recovery. vol. 95 of *Advances in Imaging and Electron Physics*. Elsevier, 1996, pp. 155–270.

[27] Combettes, P. L., and Bondon, P. Hard-constrained inconsistent signal feasibility problems. *IEEE Transactions on Signal Processing 47* (1999), 2460–2468.

[28] Conway, J. B. *A Course in Functional Analysis*, second ed., vol. 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1990. corrected fourth printing, 1997.

[29] Conway, J. H., and Sloane, N. J. A. *Sphere Packings, Lattices, and Groups*, third ed. Springer-Verlag, New York, 1999.

[30] Dax, A. The distance between two convex sets. *Linear Algebra Appl. 416* (2006), 184–213.

[31] Deutsch, F. *Best Approximation in Inner Product Spaces*, vol. 7 of *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC*. Springer-Verlag, New York, 2001.

[32] Digar, A., and Raju Kosuru, G. S. Existence of best proximity pairs and a generalization of Carathéodory theorem. *Numer. Funct. Anal. Optim. 41* (2020), 1901–1911.

[33] Du, Q., Faber, V., and Gunzburger, M. Centroidal Voronoi tessellations: applications and algorithms. *SIAM Rev. 41* (1999), 637–676.

[34] Dunford, N., and Schwartz, J. T. *Linear Operators. I. General Theory.* With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7. Interscience Publishers, Inc., New York; London, 1958.

[35] DuPré, A. M., and Kass, S. Distance and parallelism between flats in $\mathbf{R}^n$. *Linear Algebra Appl. 171* (1992), 99–107.

[36] Facas Vicente, M. A., Gonçalves, A., and Vitória, J. Euclidean distance between two linear varieties. *Appl. Math. Sci. (Ruse) 8*, 21-24 (2014), 1039–1043.

[37] Gholami, M., Tetruashvili, L., Ström, E., and Censor, Y. Cooperative wireless sensor network positioning via implicit convex feasibility. *IEEE Transactions on Signal Processing 61* (2013), 5830–5840.

[38] Goebel, K., and Reich, S. *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, vol. 83 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1984.

[39] Goldburg, M., and Marks, II, R. J. Signal synthesis in the presence of an inconsistent set of constraints. *IEEE Trans. Circuits and Systems 32* (1985), 647–663.

[40] Goldstein, A. A. *Constructive Real Analysis*. Harper & Row, Publishers, New York-London, 1967.

[41] Gonçalves, A., Facas Vicente, M. A., and Vitória, J. Optimal pair of two linear varieties. *Appl. Math. Sci. (Ruse) 9* (2015), 593–596.

[42] Gross, J., and Trenkler, G. On the least squares distance between affine subspaces. *Linear Algebra Appl. 237/238* (1996), 269–276. Special issue honoring Calyampudi Radhakrishna Rao.

[43] Gruber, P. M., and Lekkerkerker, C. G. *Geometry of Numbers*, second ed. North Holland, 1987.

[44] Gubin, L., Polyak, B., and Raik, E. The method of projections for finding the common point of convex sets. *USSR Computational Mathematics and Mathematical Physics 7* (1967), 1–24.

[45] Holmes, R. B. *Geometric Functional Analysis and its Applications*, vol. No. 24 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg, 1975.

[46] Jameson, G. J. O. *Topology and Normed Spaces*. Chapman and Hall, London; Halsted Press [John Wiley & Sons, Inc.], New York, 1974.

[47] Kirk, W. A., Reich, S., and Veeramani, P. Proximinal retracts and best proximity pair theorems. *Numer. Funct. Anal. Optim. 24* (2003), 851–862.

[48] Kopecká, E., and Reich, S. A note on the von Neumann alternating projections algorithm. *J. Nonlinear Convex Anal. 5* (2004), 379–386.

[49] Kopecká, E., and Reich, S. A note on alternating projections in Hilbert space. *J. Fixed Point Theory Appl. 12* (2012), 41–47.

[50] Lindenstrauss, J., and Tzafriri, L. *Classical Banach Spaces, II: Function Spaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]. Springer-Verlag, Berlin-New York, 1979.

[51] Luke, D. R. Finding best approximation pairs relative to a convex and prox-regular set in a Hilbert space. *SIAM J. Optim. 19* (2008), 714–739.

[52] Myers, S. B. Arcs and geodesics in metric spaces. *Trans. Amer. Math. Soc. 57* (1945), 217–227.

[53] Okabe, A., Boots, B., Sugihara, K., and Chiu, S. N. *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*, second ed. Wiley Series in Probability and Statistics. John Wiley & Sons Ltd., Chichester, 2000. With a foreword by D. G. Kendall.

[54] Pai, D. V. Proximal points of convex sets in normed linear spaces. *Yokohama Math. J. 22* (1974), 53–78.

[55] Patel, D. K., and Patel, B. Finding the best proximity point of generalized multivalued contractions with applications. *Numer. Funct. Anal. Optim. 44* (2023), 1602–1627.

[56] Prus, S. Geometrical background of metric fixed point theory. In *Handbook of Metric Fixed Point Theory*, W. A. Kirk and B. Sims, Eds. Kluwer Acad. Publ., Dordrecht, 2001, pp. 93–132.

[57] Raju Kosuru, G. S., and Veeramani, P. On existence of best proximity pair theorems for relatively nonexpansive mappings. *J. Nonlinear Convex Anal. 11* (2010), 71–77.

[58] Reem, D. An algorithm for computing Voronoi diagrams of general generators in general normed spaces. In *Proceedings of the sixth annual IEEE International Symposium on Voronoi Diagrams in Science and Engineering (ISVD 2009), Copenhagen, Denmark* (2009), pp. 144–152.

[59] Reem, D. On the computation of zone and double zone diagrams. *Discrete & Computational Geometry 59* (2018), 253–292. arXiv:1208.3124 [cs.CG] (2012) (current version: [v6], 31 Dec 2017).

[60] Reem, D. The projector algorithm: a simple parallel algorithm for computing Voronoi diagrams and Delaunay graphs. *Theoret. Comput. Sci. 970* (2023), Paper No. 114054, 38.
[61] Rockafellar, R. T. *Convex Analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, NJ, USA, 1970.
[62] Sadiq Basha, S., and Veeramani, P. Best proximity pair theorems for multifunctions with open fibres. *J. Approx. Theory 103* (2000), 119–129.
[63] Sadiq Basha, S., Veeramani, P., and Pai, D. V. Best proximity pair theorems. *Indian J. Pure Appl. Math. 32* (2001), 1237–1246.
[64] Sahney, B. N., and Singh, S. P. On best simultaneous approximation. In *Approximation theory, III (Proc. Conf., Univ. Texas, Austin, Tex., 1980)*, E. W. Cheney, Ed. Academic Press, New York-London, 1980, pp. 783–789.
[65] Singer, I. *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, vol. Band 171 of *Die Grundlehren der mathematischen Wissenschaften*. Publishing House of the Academy of the Socialist Republic of Romania, Bucharest; Springer-Verlag, New York-Berlin, 1970. Translated from the Romanian by Radu Georgescu.
[66] Stiles, W. J. Closest-point maps and their products. II. *Nieuw Arch. Wisk. (3) 13* (1965), 212–225.
[67] Suzuki, T., Kikkawa, M., and Vetro, C. The existence of best proximity points in metric spaces with the property UC. *Nonlinear Anal. 71* (2009), 2918–2926.
[68] van Tiel, J. *Convex Analysis: An Introductory Text*. John Wiley and Sons, Universities Press, Belfast, Northern Ireland, 1984.
[69] von Neumann, J. *Functional Operators. II. The Geometry of Orthogonal Spaces*, vol. No. 22 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1950.
[70] Willner, L. B. On the distance between polytopes. *Quart. Appl. Math. 26* (1968), 207–212.
[71] Xu, X. A result on best proximity pair of two sets. *J. Approx. Theory 54* (1988), 322–325.
[72] Youla, D. C., and Velasco, V. Extensions of a result on the synthesis of signals in the presence of inconsistent constraints. *IEEE Trans. Circuits and Systems 33* (1986), 465–468.

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