# A BILEVEL HIERARCHY OF STRENGTHENED COMPLEX MOMENT RELAXATIONS FOR COMPLEX POLYNOMIAL OPTIMIZATION*

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**Abstract.** This paper proposes a bilevel hierarchy of strengthened complex moment relaxations for complex polynomial optimization. The key trick entails considering a class of positive semidefinite conditions that arise naturally in characterizing the normality of the so-called *shift operators*. The relaxation problem in this new hierarchy is parameterized by the usual relaxation order as well as an extra normal order, thus providing more space of flexibility to balance the strength of relaxation and computational complexity. Extensive numerical experiments demonstrate the superior performance of the new hierarchy compared to the usual hierarchy.

**Key words.** complex polynomial optimization, complex moment relaxation, Lasserre's hierarchy, shift operator

**MSC codes.** Primary, 90C23; Secondary, 90C22,90C26

**1. Introduction.** Complex polynomial optimization is a generalization of the usual real polynomial optimization and arises from various fields, e.g., signal processing [2, 5, 7, 16, 23], power systems [4, 10], quantum information [6, 9, 15], combinatorial optimization [8, 19, 29]. In principle, complex polynomial optimization problems could be handled with Lasserre's hierarchy of moment relaxations [12] (referred to as the real moment hierarchy) after converting into real polynomial optimization problems at the price of doubling the number of variables. On the other hand, there is a complex variant of Lasserre's hierarchy (referred to as the complex moment hierarchy) to directly solve complex polynomial optimization problems [10]. The complex moment hierarchy has much lower computational complexity and some computational advantages over the real moment hierarchy were shown on the alternating-current optimal-power-flow problem in [10, 25]. However, the complex moment hierarchy typically converges more slowly [26] and its application is therefore limited.

In this paper, we explore the following question:

*Can we strengthen complex moment relaxations by "borrowing" cheap but useful constraints from the real moment hierarchy and accelerate the convergence of the complex moment hierarchy?*

**Contributions.** We propose a systematic approach to strengthen complex moment relaxations for complex polynomial optimization. The constraints that we employ for strengthening arise naturally in characterizing the normality of *shift operators* and are parameterized by a positive integer. As a consequence, we obtain a bilevel hierarchy of strengthened complex moment relaxations, offering one more level of flexibility to balance the strength of relaxation and computational complexity. Another feature of the strengthened complex moment hierarchy is that one could rely on the flat truncation condition to detect finite convergence. The strengthening approach could also be integrated with sparsity to deal with large-scale complex polynomial

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optimization. Numerical experiments are performed on a variety of complex polynomial optimization problems. It turns out that the strengthened hierarchy exhibits superior performance in practice. In particular, it allows one to improve the bounds without increasing relaxation orders and achieve global optimality at lower relaxation orders, thus (significantly) reducing computational costs.

**2. Notation and preliminaries.** Let $\mathbb{N}$ (resp. $\mathbb{N}^*$) be the set of nonnegative (resp. positive) integers. For $n \in \mathbb{N}^*$, let $[n] \coloneqq \{1, 2, \ldots, n\}$. For $\boldsymbol{\alpha} = (\alpha_i) \in \mathbb{N}^n$, let $|\boldsymbol{\alpha}| \coloneqq \sum_{i=1}^n \alpha_i$. For $r \in \mathbb{N}$, let $\mathbb{N}^n_r \coloneqq \{\boldsymbol{\alpha} \in \mathbb{N}^n \mid |\boldsymbol{\alpha}| \le r\}$. We use $A \succeq 0$ to indicate that the matrix $A$ is positive semidefinite (PSD). Let $\overline{a}$ denote the conjugate of a complex number $a$ and $\boldsymbol{v}^*$ (resp. $A^*$) denote the conjugate transpose of a complex vector $\boldsymbol{v}$ (resp. a complex matrix $A$). Let $\mathbf{z} = (z_1, \ldots, z_n)$ be a tuple of complex variables and $\overline{\mathbf{z}} = (\overline{z}_1, \ldots, \overline{z}_n)$ be the conjugate of $\mathbf{z}$. We denote by $\mathbb{C}[\mathbf{z}, \overline{\mathbf{z}}] \coloneqq \mathbb{C}[z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n]$ the complex polynomial ring in $\mathbf{z}, \overline{\mathbf{z}}$. A polynomial $f \in \mathbb{C}[\mathbf{z}, \overline{\mathbf{z}}]$ can be written as $f = \sum_{(\boldsymbol{\beta},\boldsymbol{\gamma}) \in \mathbb{N}^n \times \mathbb{N}^n} f_{\boldsymbol{\beta},\boldsymbol{\gamma}} \mathbf{z}^{\boldsymbol{\beta}} \overline{\mathbf{z}}^{\boldsymbol{\gamma}}$ with $f_{\boldsymbol{\beta},\boldsymbol{\gamma}} \in \mathbb{C}$. The *conjugate* of $f$ is defined as $\overline{f} = \sum_{(\boldsymbol{\beta},\boldsymbol{\gamma}) \in \mathbb{N}^n \times \mathbb{N}^n} \overline{f}_{\boldsymbol{\beta},\boldsymbol{\gamma}} \mathbf{z}^{\boldsymbol{\gamma}} \overline{\mathbf{z}}^{\boldsymbol{\beta}}$. The polynomial $f$ is *self-conjugate* if $\overline{f} = f$. It is clear that self-conjugate polynomials take only real values. For $d \in \mathbb{N}$, $[\mathbf{z}]_d$ (resp. $[\mathbf{z}, \overline{\mathbf{z}}]_d$) stands for the standard monomial basis in $\mathbf{z}$ (resp. $\mathbf{z}, \overline{\mathbf{z}}$) of degree up to $d$. Let $\delta_{\boldsymbol{v}}$ denote the Dirac measure centered at a point $\boldsymbol{v} \in \mathbb{C}^n$.

Consider the complex polynomial optimization problem:

$$\text{(CPOP)} \qquad f_{\min} \coloneqq \inf \{f(\mathbf{z}, \overline{\mathbf{z}}) : \mathbf{z} \in \mathbf{K}\},$$

where

$$\mathbf{K} \coloneqq \{\mathbf{z} \in \mathbb{C}^n \mid g_i(\mathbf{z}, \overline{\mathbf{z}}) \ge 0,\, i \in [m]\}\,, \tag{2.1}$$

and $f, g_1, \ldots, g_m \in \mathbb{C}[\mathbf{z}, \overline{\mathbf{z}}]$ are self-conjugate polynomials. By invoking Borel measures, (CPOP) admits the following equivalent reformulation:

$$\begin{cases} \inf\limits_{\mu \in \mathcal{M}_+(\mathbf{K})} & \int_{\mathbf{K}} f \, \mathrm{d}\mu \\ \quad \text{s.t.} & \int_{\mathbf{K}} \mathrm{d}\mu = 1, \end{cases} \tag{2.2}$$

where $\mathcal{M}_+(\mathbf{K})$ denotes the set of finite positive Borel measures on $\mathbf{K}$.

Suppose that $\mathbf{y} = (y_{\boldsymbol{\beta},\boldsymbol{\gamma}}) \in \mathbb{C}^{\mathbb{N}^n \times \mathbb{N}^n}$ is a complex sequence satisfying $y_{\boldsymbol{\beta},\boldsymbol{\gamma}} = \overline{y}_{\boldsymbol{\gamma},\boldsymbol{\beta}}$. We associate it with a linear functional $L_{\mathbf{y}} : \mathbb{C}[\mathbf{z}, \overline{\mathbf{z}}] \to \mathbb{C}$ by

$$p = \sum_{(\boldsymbol{\beta},\boldsymbol{\gamma})} p_{\boldsymbol{\beta},\boldsymbol{\gamma}} \mathbf{z}^{\boldsymbol{\beta}} \overline{\mathbf{z}}^{\boldsymbol{\gamma}} \longmapsto L_{\mathbf{y}}(p) = \sum_{(\boldsymbol{\beta},\boldsymbol{\gamma})} p_{\boldsymbol{\beta},\boldsymbol{\gamma}} y_{\boldsymbol{\beta},\boldsymbol{\gamma}}.$$

For any $r \in \mathbb{N}$, the $r$-th order *(pruned) complex moment matrix* $\mathbf{M}^{\mathbb{C}}_r(\mathbf{y})$ is the Hermitian matrix defined by

$$\mathbf{M}^{\mathbb{C}}_r(\mathbf{y}) = L_{\mathbf{y}} \left([\mathbf{z}]_r \cdot [\mathbf{z}]_r^*\right)$$

and the $r$-th order *full complex moment matrix* $\mathbf{M}^{2\mathbb{C}}_r(\mathbf{y})$ is the Hermitian matrix defined by

$$\mathbf{M}^{2\mathbb{C}}_r(\mathbf{y}) = L_{\mathbf{y}} \left([\mathbf{z}, \overline{\mathbf{z}}]_r \cdot [\mathbf{z}, \overline{\mathbf{z}}]_r^*\right).$$

Moreover, for a self-conjugate polynomial $g$, the $r$-th order *(pruned) complex localizing matrix* $\mathbf{M}_r^{\mathbb{C}}(g\mathbf{y})$ associated with $g$ is the Hermitian matrix defined by

$$\mathbf{M}_r^{\mathbb{C}}(g\mathbf{y}) = L_{\mathbf{y}}\left([\mathbf{z}]_r \cdot [\mathbf{z}]_r^* \cdot g\right)$$

and the $r$-th order *full complex localizing matrix* $\mathbf{M}_r^{2\mathbb{C}}(g\mathbf{y})$ associated with $g$ is the Hermitian matrix defined by

$$\mathbf{M}_r^{2\mathbb{C}}(g\mathbf{y}) = L_{\mathbf{y}}\left([\mathbf{z}, \overline{\mathbf{z}}]_r \cdot [\mathbf{z}, \overline{\mathbf{z}}]_r^* \cdot g\right).$$

Assume $f = \sum_{(\boldsymbol{\beta},\boldsymbol{\gamma})} f_{\boldsymbol{\beta},\boldsymbol{\gamma}} \mathbf{z}^{\boldsymbol{\beta}} \overline{\mathbf{z}}^{\boldsymbol{\gamma}}$ and $g_i = \sum_{(\boldsymbol{\beta},\boldsymbol{\gamma})} g^i_{\boldsymbol{\beta},\boldsymbol{\gamma}} \mathbf{z}^{\boldsymbol{\beta}} \overline{\mathbf{z}}^{\boldsymbol{\gamma}}$. Let $d_0 \coloneqq \max\{|\boldsymbol{\beta}|, |\boldsymbol{\gamma}| : f_{\boldsymbol{\beta},\boldsymbol{\gamma}} \neq 0\}$, $d_i \coloneqq \max\{|\boldsymbol{\beta}|, |\boldsymbol{\gamma}| : g^i_{\boldsymbol{\beta},\boldsymbol{\gamma}} \neq 0\}$ for $i \in [m]$, and $d_{\min} \coloneqq \max\{d_0, d_1, \ldots, d_m\}$. For a positive integer $r \geq d_{\min}$, the $r$-th order (pruned) complex moment relaxation [10] for (CPOP) is given by

$$\tau_r \coloneqq \begin{cases} \inf\limits_{\mathbf{y}} & L_{\mathbf{y}}(f) \\ \text{s.t.} & \mathbf{M}_r^{\mathbb{C}}(\mathbf{y}) \succeq 0, \quad y_{\mathbf{0},\mathbf{0}} = 1, \\ & \mathbf{M}_{r-d_i}^{\mathbb{C}}(g_i\mathbf{y}) \succeq 0, \quad i \in [m]. \end{cases} \tag{2.3}$$

Let $d'_i \coloneqq \max\{\lceil \frac{|\boldsymbol{\beta}|+|\boldsymbol{\gamma}|}{2} \rceil : g^i_{\boldsymbol{\beta},\boldsymbol{\gamma}} \neq 0\}$ for $i \in [m]$. The $r$-th order full complex moment relaxation for (CPOP) is given by

$$\rho_r \coloneqq \begin{cases} \inf\limits_{\mathbf{y}} & L_{\mathbf{y}}(f) \\ \text{s.t.} & \mathbf{M}_r^{2\mathbb{C}}(\mathbf{y}) \succeq 0, \quad y_{\mathbf{0},\mathbf{0}} = 1, \\ & \mathbf{M}_{r-d'_i}^{2\mathbb{C}}(g_i\mathbf{y}) \succeq 0, \quad i \in [m]. \end{cases} \tag{2.4}$$

*Remark* 2.1. By introducing real variables for the real and imaginary parts of each complex variable, respectively, (CPOP) could be converted into a real polynomial optimization problem. In fact, the $r$-th order full complex moment relaxation for (CPOP) is equivalent to the $r$-th order real moment relaxation for the equivalent real polynomial optimization problem; see [13].

We say that the sequence $\mathbf{y}$ admits a (finitely atomic) representing measure if it can be realized by a Borel (finitely atomic) measure $\mu$, i.e., $y_{\boldsymbol{\beta},\boldsymbol{\gamma}} = \int_{\mathbf{K}} \mathbf{z}^{\boldsymbol{\beta}} \overline{\mathbf{z}}^{\boldsymbol{\gamma}} \,\mathrm{d}\mu$ for any $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{N}^n$. Accordingly, $\mathbf{y}$ is called the moment sequence of $\mu$. It is clear that the relaxations (2.3) and (2.4) attain global optimality for (CPOP) if the sequence $\mathbf{y}$ admits a representing measure.

## 3. The main results.

**3.1. Strengthening the complex moment relaxation.** Let us begin with the following simple proposition.

PROPOSITION 3.1. *Suppose that* $\mathbf{y}$ *is a complex sequence that admits a finitely atomic representing measure. Then for* $i \in [n]$ *and* $s \in \mathbb{N}^*$, *it holds*

$$\begin{bmatrix} \mathbf{M}_s^{\mathbb{C}}(\mathbf{y}) & \mathbf{M}_s^{\mathbb{C}}(z_i\mathbf{y}) \\ \mathbf{M}_s^{\mathbb{C}}(\overline{z}_i\mathbf{y}) & \mathbf{M}_s^{\mathbb{C}}(|z_i|^2\mathbf{y}) \end{bmatrix} \succeq 0. \tag{3.1}$$

*Proof.* Assume that $\mathbf{y}$ admits a representing measure $\mu = \sum_{i=1}^t \lambda_i \delta_{\boldsymbol{v}_i}$, $\boldsymbol{v}_i \in \mathbb{C}^n, \lambda_i > 0, i \in [t]$. Let $b_s^i(\mathbf{z})$ be the column vector formed by concatenating $[\mathbf{z}]_s$ and $\overline{z}_i[\mathbf{z}]_s$. Then,

$$\begin{bmatrix} \mathbf{M}_s^{\mathbb{C}}(\mathbf{y}) & \mathbf{M}_s^{\mathbb{C}}(z_i\mathbf{y}) \\ \mathbf{M}_s^{\mathbb{C}}(\overline{z}_i\mathbf{y}) & \mathbf{M}_s^{\mathbb{C}}(|z_i|^2\mathbf{y}) \end{bmatrix} = L_{\mathbf{y}}\left(b_s^i(\mathbf{z}) b_s^i(\mathbf{z})^*\right) = \sum_{i=1}^t \lambda_i b_s^i(\boldsymbol{v}_i) b_s^i(\boldsymbol{v}_i)^* \succeq 0. \qquad \square \tag{3.2}$$

*Remark* 3.2. Note that $b_s^i(\mathbf{z})$ is a subvector of $[\mathbf{z},\overline{\mathbf{z}}]_{s+1}$. As a result, the matrices in (3.1) are principal submatrices of the $(s+1)$-th order full complex moment matrix.

We now propose to strengthen the complex moment relaxation (2.3) with the PSD conditions (3.1):

$$\tau'_{r,s} \coloneqq \begin{cases} \inf\limits_{\mathbf{y}} & L_{\mathbf{y}}(f) \\ \text{s.t.} & \mathbf{M}_r^{\mathbb{C}}(\mathbf{y}) \succeq 0, \quad y_{\mathbf{0},\mathbf{0}} = 1, \\ & \mathbf{M}_{r-d_i}^{\mathbb{C}}(g_i\mathbf{y}) \succeq 0, \quad i \in [m], \\ & \begin{bmatrix} \mathbf{M}_s^{\mathbb{C}}(\mathbf{y}) & \mathbf{M}_s^{\mathbb{C}}(z_i\mathbf{y}) \\ \mathbf{M}_s^{\mathbb{C}}(\overline{z}_i\mathbf{y}) & \mathbf{M}_s^{\mathbb{C}}(|z_i|^2\mathbf{y}) \end{bmatrix} \succeq 0, \quad i \in [n], \end{cases} \tag{3.3}$$

where $s \in \mathbb{N}^*$ is a tunable parameter. We refer the reader to [24] for reformulating the complex semidefinite program (SDP) (3.3) as a real SDP.

THEOREM 3.3. *With the above notation, the following are true for any $r \geq d_{\min}$ and $s \in \mathbb{N}^*$:*

(i) $\tau_r \leq \tau'_{r,s} \leq \tau'_{r,s+1} \leq f_{\min}$;
(ii) $\tau'_{r,s} \leq \tau'_{r+1,s}$;
(iii) $\tau'_{r,s} \leq \rho_{\max\{r,s+1\}}$.

*Proof.* As (3.3) involves more constraints than (2.3), it holds $\tau_r \leq \tau'_{r,s}$. If the infimum of (CPOP) is attained, let $\mathbf{w}$ be a minimizer of (CPOP) and $\mathbf{y}$ be the moment sequence of the Dirac measure $\delta_{\mathbf{w}}$. By Proposition 3.1, $\mathbf{y}$ is a feasible solution of (3.3) and $L_{\mathbf{y}}(f) = f_{\min}$. Thus, $\tau'_{r,s} \leq f_{\min}$. If the infimum of (CPOP) is not attained, let $\{\mathbf{w}^{(k)}\}_{k\geq 1}$ be a minimizing sequence of (CPOP) and $\mathbf{y}^{(k)}$ be the moment sequence of the Dirac measure $\delta_{\mathbf{w}^{(k)}}$ for $k \geq 1$. We have that every $\mathbf{y}^{(k)}$ is a feasible solution of (3.3) and $\lim_{k\to\infty} L_{\mathbf{y}^{(k)}}(f) = f_{\min}$. Thus, $\tau'_{r,s} \leq f_{\min}$. The validness of the inequalities $\tau'_{r,s} \leq \tau'_{r,s+1}$ and $\tau'_{r,s} \leq \tau'_{r+1,s}$ is straightforward from the constructions. Finally, the inequality $\tau'_{r,s} \leq \rho_{\max\{r,s+1\}}$ is implied by the fact that the matrices in (3.1) are principal submatrices of the $(s+1)$-th order full complex moment matrix. ◻

We shall call the bilevel hierarchy consisting of the complex SDP relaxations (3.3) indexed by $r$ and $s$ the *strengthened complex moment hierarchy* for (CPOP).

*Remark* 3.4. By Theorem 3.2 of [10], the complex moment hierarchy (2.3) has asymptotic convergence (i.e., $\lim_{r\to\infty} \tau_r = f_{\min}$) when a sphere constraint is present. As an immediate corollary, the strengthened complex moment hierarchy (3.3) also has asymptotic convergence (i.e., $\lim_{r\to\infty} \tau'_{r,s} = f_{\min}$) under the presence of a sphere constraint.

*Remark* 3.5. Similar but more complicated conditions (later simplified in [26, Corollary 4.1]) appear in [10, Proposition 4.1] for detecting global optimality of the relaxation (2.3).

**3.2. Link to the normality of shift operators.** As there are numerous ways to pick constraints from the full complex moment relaxations, one may wonder why we choose the particular conditions (3.1) to strengthen the complex moment relaxations. In the following, we show that those conditions arise naturally in ensuring the normality of shift operators and enable the strengthened complex moment hierarchy to share a similar global optimality condition with the real moment hierarchy.

Suppose that $\mathbf{y} \in \mathbb{C}^{\mathbb{N}^n\times\mathbb{N}^n}$ is a complex sequence that satisfies $\mathbf{M}_{s+1}^{\mathbb{C}}(\mathbf{y}) \succeq 0$ and $\operatorname{rank} \mathbf{M}_{s+1}^{\mathbb{C}}(\mathbf{y}) = \operatorname{rank} \mathbf{M}_s^{\mathbb{C}}(\mathbf{y})$. Let $t \coloneqq \operatorname{rank} \mathbf{M}_{s+1}^{\mathbb{C}}(\mathbf{y})$. Then we can factorize

$\mathbf{M}_{s+1}^{\mathbb{C}}(\mathbf{y})$ in Grammian form so that

$$[\mathbf{M}_{s+1}^{\mathbb{C}}(\mathbf{y})]_{\boldsymbol{\beta}\boldsymbol{\gamma}} = \mathbf{a}_{\boldsymbol{\beta}}^*\mathbf{a}_{\boldsymbol{\gamma}}, \quad \forall \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{N}_{s+1}^n, \tag{3.4}$$

for some $\{\mathbf{a}_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathbb{N}_{s+1}^n} \subseteq \mathbb{C}^t$. Because $\operatorname{rank}\mathbf{M}_{s+1}^{\mathbb{C}}(\mathbf{y}) = \operatorname{rank}\mathbf{M}_s^{\mathbb{C}}(\mathbf{y})$, we must have $\mathbb{C}^t = \operatorname{span}(\{\mathbf{a}_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathbb{N}_{s+1}^n}) = \operatorname{span}(\{\mathbf{a}_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathbb{N}_s^n})$. We thus define the shift operators $T_1, \ldots, T_n : \mathbb{C}^t \to \mathbb{C}^t$ by

$$T_i : \sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}}\mathbf{a}_{\boldsymbol{\alpha}} \longmapsto \sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}}\mathbf{a}_{\boldsymbol{\alpha}+\mathbf{e}_i}, \tag{3.5}$$

where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard vector basis of $\mathbb{R}^n$.

*Remark* 3.6. The shift operators are well-defined (i.e., each element in $\mathbb{C}^t$ has a unique image under $T_i$) when a sphere or ball constraint is present in (CPOP) or each variable has unit norm; see [10, 26].

We say that an operator $T$ is *normal* if $T^*T = TT^*$ where $T^*$ denotes the adjoint operator of $T$. In case that $T$ is of finite dimension, the normality of $T$ is equivalent to the condition $T^*T - TT^* \succeq 0$, which is further equivalent to

$$\begin{bmatrix} I & T^* \\ T & T^*T \end{bmatrix} \succeq 0. \tag{3.6}$$

THEOREM 3.7. *Assume that* (CPOP) *contains a sphere or ball constraint or each variable has unit norm. Suppose that* $\mathbf{y}$ *is a complex sequence satisfying* $\mathbf{M}_{s+1}^{\mathbb{C}}(\mathbf{y}) \succeq 0$ *and* $\operatorname{rank}\mathbf{M}_{s+1}^{\mathbb{C}}(\mathbf{y}) = \operatorname{rank}\mathbf{M}_s^{\mathbb{C}}(\mathbf{y})$. *Then for any* $i \in [n]$, *the shift operator* $T_i$ *defined in* (3.5) *is normal if and only if*

$$\begin{bmatrix} \mathbf{M}_s^{\mathbb{C}}(\mathbf{y}) & \mathbf{M}_s^{\mathbb{C}}(z_i\mathbf{y}) \\ \mathbf{M}_s^{\mathbb{C}}(\overline{z}_i\mathbf{y}) & \mathbf{M}_s^{\mathbb{C}}(|z_i|^2\mathbf{y}) \end{bmatrix} \succeq 0.$$

*Proof.* Let $\mathbf{M}_{s+1}^{\mathbb{C}}(\mathbf{y})$ be factorized as in (3.4) for some $\{\mathbf{a}_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathbb{N}_{s+1}^n} \subseteq \mathbb{C}^t$ with $t = \operatorname{rank}\mathbf{M}_{s+1}^{\mathbb{C}}(\mathbf{y})$. For any $\mathbf{u} = (u_{\boldsymbol{\beta}})_{|\boldsymbol{\beta}|\le s} \in \mathbb{C}^{\mathbb{N}_s^n}$ and $\boldsymbol{v} = (v_{\boldsymbol{\gamma}})_{|\boldsymbol{\gamma}|\le s} \in \mathbb{C}^{\mathbb{N}_s^n}$, let $u = \sum_{|\boldsymbol{\beta}|\le s} u_{\boldsymbol{\beta}} a_{\boldsymbol{\beta}} \in \mathbb{C}^t$ and $v = \sum_{|\boldsymbol{\gamma}|\le s} v_{\boldsymbol{\gamma}} a_{\boldsymbol{\gamma}} \in \mathbb{C}^t$. We have

$$\begin{aligned}
u^*u &= \sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} \overline{u}_{\boldsymbol{\beta}} u_{\boldsymbol{\gamma}} \overline{a}_{\boldsymbol{\beta}} a_{\boldsymbol{\gamma}} = \mathbf{u}^*\mathbf{M}_s^{\mathbb{C}}(\mathbf{y})\mathbf{u}, \\
u^*T_i^*v &= \sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} \overline{u}_{\boldsymbol{\beta}} v_{\boldsymbol{\gamma}} (T_i a_{\boldsymbol{\beta}})^* a_{\boldsymbol{\gamma}} = \sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} \overline{u}_{\boldsymbol{\beta}} v_{\boldsymbol{\gamma}} \overline{a}_{\boldsymbol{\beta}+\mathbf{e}_i} a_{\boldsymbol{\gamma}} = \mathbf{u}^*\mathbf{M}_s^{\mathbb{C}}(z_i\mathbf{y})\boldsymbol{v}, \\
v^*T_i u &= \sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} \overline{v}_{\boldsymbol{\beta}} u_{\boldsymbol{\gamma}} \overline{a}_{\boldsymbol{\beta}} (T_i a_{\boldsymbol{\gamma}}) = \sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} \overline{v}_{\boldsymbol{\beta}} u_{\boldsymbol{\gamma}} \overline{a}_{\boldsymbol{\beta}} a_{\boldsymbol{\gamma}+\mathbf{e}_i} = \boldsymbol{v}^*\mathbf{M}_s^{\mathbb{C}}(\overline{z}_i\mathbf{y})\mathbf{u},
\end{aligned}$$

and

$$v^*T_i^*T_i v = \sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} \overline{v}_{\boldsymbol{\beta}} v_{\boldsymbol{\gamma}} (T_i a_{\boldsymbol{\beta}})^*(T_i a_{\boldsymbol{\gamma}}) = \sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} \overline{v}_{\boldsymbol{\beta}} v_{\boldsymbol{\gamma}} \overline{a}_{\boldsymbol{\beta}+\mathbf{e}_i} a_{\boldsymbol{\gamma}+\mathbf{e}_i} = \boldsymbol{v}^*\mathbf{M}_s^{\mathbb{C}}(|z_i|^2\mathbf{y})\boldsymbol{v}.$$

It follows

$$\begin{bmatrix} u^* & v^* \end{bmatrix} \begin{bmatrix} I & T_i^* \\ T_i & T_i^*T_i \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \mathbf{u}^* & \boldsymbol{v}^* \end{bmatrix} \begin{bmatrix} \mathbf{M}_s^{\mathbb{C}}(\mathbf{y}) & \mathbf{M}_s^{\mathbb{C}}(z_i\mathbf{y}) \\ \mathbf{M}_s^{\mathbb{C}}(\overline{z}_i\mathbf{y}) & \mathbf{M}_s^{\mathbb{C}}(|z_i|^2\mathbf{y}) \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{v} \end{bmatrix}, \tag{3.7}$$

which gives the desired result. ∎

In view of Theorem 3.7, we call the index $s$ in (3.3) the *normal order*. Finite convergence of the strengthened complex moment hierarchy could be detected in the same way as for the real moment hierarchy [17], which is stated in the following theorem.

THEOREM 3.8 (Global optimality). *Assume that* (CPOP) *contains a sphere or ball constraint or each variable has unit norm. Let* $\mathbf{y}$ *be an optimal solution of* (3.3) *and* $d_{\mathbf{K}} \coloneqq \max\{2, d_1, \ldots, d_m\}$. *If there is an integer* $t$ *such that* $\max\{d_{\min}, d_{\mathbf{K}}\} \leq t \leq \min\{s + d_{\mathbf{K}}, r\}$ *and* $\operatorname{rank} \mathbf{M}_t^{\mathbb{C}}(\mathbf{y}) = \operatorname{rank} \mathbf{M}_{t-d_{\mathbf{K}}}^{\mathbb{C}}(\mathbf{y})$, *then the complex moment relaxation* (3.3) *for* (CPOP) *is tight, i.e.,* $\tau'_{r,s} = f_{\min}$.

*Proof.* It is directly deduced from Corollary 4.1[1] of [26]. ∎

*Remark* 3.9. The rank condition in Theorem 3.8 is called *flat truncation*. Once it is satisfied, one could extract $\operatorname{rank} \mathbf{M}_t^{\mathbb{C}}(\mathbf{y})$ minimizers for (CPOP) from the moment matrix [11].

**3.3. Integrating with correlative sparsity.** The strengthening approach can be further integrated with correlative sparsity to improve scalability. Suppose that the two index sets $[n]$ and $[m]$ can be decomposed into $\{I_1, \ldots, I_p\}$ and $\{J_1, \ldots, J_p\}$, respectively, such that 1) $f = f_1 + \cdots + f_p$ with $f_k \in \mathbb{C}[\mathbf{z}_{I_k}, \overline{\mathbf{z}}_{I_k}]$ for $k \in [p]$; 2) for all $k \in [p]$ and $i \in J_k$, $g_i \in \mathbb{C}[\mathbf{z}_{I_k}, \overline{\mathbf{z}}_{I_k}]$, where $\mathbb{C}[\mathbf{z}_{I_k}, \overline{\mathbf{z}}_{I_k}]$ denotes the polynomial ring in those variables indexed by $I_k$. Let $\mathbf{M}_r^{\mathbb{C}}(\mathbf{y}, I_k)$ (resp. $\mathbf{M}_r^{\mathbb{C}}(g\mathbf{y}, I_k)$) be the submatrix obtained from $\mathbf{M}_r^{\mathbb{C}}(\mathbf{y})$ (resp. $\mathbf{M}_r^{\mathbb{C}}(g\mathbf{y})$) by retaining only those rows and columns indexed by $\boldsymbol{\beta} = (\beta_i) \in \mathbb{N}_r^n$ of $\mathbf{M}_r^{\mathbb{C}}(\mathbf{y})$ (resp. $\mathbf{M}_r^{\mathbb{C}}(g\mathbf{y})$) with $\beta_i = 0$ if $i \notin I_k$. Then, we can strengthen the sparse complex moment relaxation for complex polynomial optimization as follows:

$$
\left\{
\begin{array}{ll}
\inf\limits_{\mathbf{y}} & L_{\mathbf{y}}(f) \\
\text{s.t.} & \mathbf{M}_r^{\mathbb{C}}(\mathbf{y}, I_k) \succeq 0, \quad k \in [p], \\
& \mathbf{M}_{r-d_i}^{\mathbb{C}}(g_i\mathbf{y}, I_k) \succeq 0, \quad i \in J_k, k \in [p], \\
& \begin{bmatrix} \mathbf{M}_s^{\mathbb{C}}(\mathbf{y}, I_k) & \mathbf{M}_s^{\mathbb{C}}(z_i\mathbf{y}, I_k) \\ \mathbf{M}_s^{\mathbb{C}}(\overline{z}_i\mathbf{y}, I_k) & \mathbf{M}_s^{\mathbb{C}}(|z_i|^2\mathbf{y}, I_k) \end{bmatrix} \succeq 0, \quad i \in I_k, k \in [p], \\
& y_{\mathbf{0},\mathbf{0}} = 1.
\end{array}
\right. \tag{3.8}
$$

*Remark* 3.10. One may also take term sparsity into account for the strengthened approach. We refer the reader to [27, 28] and omit the details for conciseness.

**4. Numerical experiments.** The strengthened complex moment hierarchy has been implemented in the Julia package `TSSOS`[2]. In this section, we evaluate its performance on diverse complex polynomial optimization problems using `TSSOS` where `Mosek` 10.2 [1] is employed as an SDP backend with default settings. When presenting the results, 'Mom' means the complex moment relaxation (2.3), 'FMom' means the full complex moment relaxation (2.4), and 'S-Mom' means the strengthened complex moment relaxation (3.3); the column labelled by 'opt' records optima of SDP relaxations and the column labelled by 'time' records computational time in seconds. Lower bounds are marked by '*' if global optimality is certified through the flatness condition. Moreover, the symbol '-' indicates that `Mosek` runs out of memory. Unless

[1] Corollary 4.1 in [26] is stated under the assumption of a sphere constraint. But it also holds true under the assumption of a ball constraint or each variable being of unit norm.

[2] `TSSOS` is freely available at https://github.com/wangjie212/TSSOS.

otherwise stated, the normal order $s$ is set to 1. All numerical experiments were performed on a desktop computer with an Intel(R) Core(TM) i9-10900 CPU@2.80GHz and 64G RAM.

**4.1. Minimizing a random complex quadratic polynomial with unit-norm variables.** Let us minimize a random complex quadratic polynomial with unit-norm variables:

$$\begin{cases} \inf\limits_{\mathbf{z}\in\mathbb{C}^n} & [\mathbf{z}]_1^* Q[\mathbf{z}]_1 \\ \text{s.t.} & |z_i|^2 = 1, \quad i = 1,\ldots,n, \end{cases} \tag{4.1}$$

where $Q \in \mathbb{C}^{(n+1)\times(n+1)}$ is a random Hermitian matrix whose entries (both real and imaginary parts) are selected with respect to the uniform probability distribution on $[0,1]$. For each $n \in \{10,20,30\}$, we solve three random instances and present the results in Table 1. We make the following observations on the results: 1) increasing the relaxation order, employing the strengthening approach, and utilizing the full complex moment relaxation all improve the bounds produced by the initial complex moment relaxation to global optimality (except for one instance); 2) the strengthening approach runs much faster (typically by orders of magnitude) than solving higher order relaxations.

TABLE 1
*Minimizing a random complex quadratic polynomial with unit-norm variables.*

| $n$ | trial | Mom ($r=1$) | | Mom ($r=2$) | | S-Mom ($r=1$) | | FMom ($r=2$) | |
|---|---|---|---|---|---|---|---|---|---|
| | | opt | time | opt | time | opt | time | opt | time |
| 10 | 1 | -12.925 | 0.008 | -12.904* | 1.10 | -12.904* | 0.11 | -12.904* | 8.07 |
| | 2 | -12.811 | 0.009 | -12.655* | 1.09 | -12.655* | 0.11 | -12.655* | 8.03 |
| | 3 | -15.142 | 0.008 | -14.842* | 1.05 | -14.842* | 0.13 | -14.842* | 8.90 |
| 20 | 1 | -42.367 | 0.03 | -42.011* | 1084 | -42.011* | 3.94 | - | - |
| | 2 | -38.653 | 0.03 | -36.930* | 1213 | -36.930* | 5.94 | - | - |
| | 3 | -42.946 | 0.03 | -41.618* | 1091 | -41.618* | 4.63 | - | - |
| 30 | 1 | -74.472 | 0.14 | - | - | -72.346 | 179 | - | - |
| | 2 | -75.051 | 0.12 | - | - | -73.384* | 176 | - | - |
| | 3 | -83.099 | 0.20 | - | - | -81.726* | 142 | - | - |

**4.2. Minimizing a random complex quartic polynomial on a sphere.** Let us minimize a random complex quartic polynomial on a unit sphere:

$$\begin{cases} \inf\limits_{\mathbf{z}\in\mathbb{C}^n} & [\mathbf{z}]_2^* Q[\mathbf{z}]_2 \\ \text{s.t.} & |z_1|^2 + \cdots + |z_n|^2 = 1, \end{cases} \tag{4.2}$$

where $Q \in \mathbb{C}^{|[\mathbf{z}]_2|\times|[\mathbf{z}]_2|}$ ($|[\mathbf{z}]_2|$ is the cardinality of $[\mathbf{z}]_2$) is a random Hermitian matrix whose entries (both real and imaginary parts) are selected with respect to the uniform probability distribution on $[0,1]$. For each $n \in \{5,10,15\}$, we solve three random instances and present the results in Table 2. We could make the following observations on the results: 1) increasing the relaxation order, employing the strengthening approach, and utilizing the full complex moment relaxation all significantly improve the bounds produced by the initial complex moment relaxation; 2) the strengthening approach runs much faster than the other two approaches.

Table 2
*Minimizing a random complex quartic polynomial on a unit sphere.*

| $n$ | trial | Mom ($r=2$) | | Mom ($r=3$) | | S-Mom ($r=2$) | | FMom ($r=2$) | |
|---|---|---|---|---|---|---|---|---|---|
| | | opt | time | opt | time | opt | time | opt | time |
| 5 | 1 | -3.8882 | 0.03 | -3.2848 | 1.16 | -3.0001 | 0.04 | -2.8994* | 0.10 |
| | 2 | -2.7110 | 0.03 | -2.2546 | 1.25 | -2.0609* | 0.05 | -2.0609* | 0.10 |
| | 3 | -3.5725 | 0.04 | -3.1123 | 1.27 | -2.8213* | 0.04 | -2.8213* | 0.09 |
| 10 | 1 | -6.1871 | 2.10 | - | - | -4.8130 | 2.99 | -4.4058* | 14.3 |
| | 2 | -5.6484 | 2.23 | - | - | -4.1882 | 2.36 | -3.4838* | 14.2 |
| | 3 | -5.6759 | 2.24 | - | - | -4.3203 | 2.54 | -3.8306* | 15.9 |
| 15 | 1 | -8.4667 | 57.8 | - | - | -6.5468 | 94.6 | - | - |
| | 2 | -7.7119 | 71.1 | - | - | -5.8359 | 94.1 | - | - |
| | 3 | -7.7191 | 57.9 | - | - | -5.8155 | 85.4 | - | - |

**4.3. Minimizing a random complex quartic polynomial with correlative sparsity on multi-spheres.** Let us minimize a random complex quartic polynomial with correlative sparsity on multi-spheres:

$$\begin{cases} \inf\limits_{\mathbf{z}\in\mathbb{C}^n} & \sum_{i=1}^{l}[\mathbf{z}_i]_2^*Q_i[\mathbf{z}_i]_2 \\ \text{s.t.} & \|\mathbf{z}_i\|^2=1, \quad i\in[l], \end{cases} \tag{4.3}$$

where $n=4l+2$, $\mathbf{z}_i \coloneqq \{z_{4i-3},\ldots,z_{4i+2}\}$, and $Q_i\in\mathbb{C}^{|[\mathbf{z}_i]_2|\times|[\mathbf{z}_i]_2|}$ is a random Hermitian matrix whose entries (both real and imaginary parts) are selected with respect to the uniform probability distribution on $[0,1]$. For each $l\in\{5,10,15,20\}$, we solve three random instances and present the results in Table 3. We again see from the table that: 1) increasing the relaxation order, employing the strengthening approach, and utilizing the full complex moment relaxation all significantly improve the bounds produced by the initial complex moment relaxation; 2) the strengthening approach is much faster than solving high order relaxations while providing comparable bounds, and is several times faster than solving the full complex moment relaxation while providing looser bounds.

**4.4. Smale's Mean Value conjecture.** The following complex polynomial optimization problem is borrowed from [26]:

$$\begin{cases} \sup\limits_{(\mathbf{z},u)\in\mathbb{C}^{n+1}} & |u| \\ \text{s.t.} & |H(z_i)|\geq|u|, \quad i=1,\ldots,n, \\ & z_1\cdots z_n=\frac{(-1)^n}{n+1}, \\ & |z_1|^2+|z_2|^2+\cdots+|z_n|^2=n\left(\frac{1}{n+1}\right)^{\frac{2}{n}}, \end{cases} \tag{4.4}$$

where $H(y)\coloneqq\frac{1}{y}\int_0^y p(z)\,\mathrm{d}z$ and $p(z)\coloneqq(n+1)(z-z_1)\cdots(z-z_n)$ with $p(0)=1$. This problem is used in [26] to verify Smale's Mean Value conjecture [21, 22] which is open for $n\geq4$ since 1981. The optimum of (4.4) is conjectured to be $\frac{n}{n+1}$. We refer the reader to [26] for more details. Here, we tackle (4.4) with $n=4$ using the complex moment relaxation and the strengthened complex moment relaxation. The computational results are presented in Table 4, from which we see that the strengthening

TABLE 3
*Minimizing a random complex quartic polynomial on multi-spheres.*

| $n$ | trial | Mom ($r=2$) | | Mom ($r=3$) | | S-Mom ($r=2$) | | FMom ($r=2$) | |
|---|---|---|---|---|---|---|---|---|---|
| | | opt | time | opt | time | opt | time | opt | time |
| 22 | 1 | -18.018 | 0.48 | -14.338 | 49.9 | -14.452 | 0.84 | -12.487* | 3.06 |
| | 2 | -15.922 | 0.47 | -12.635 | 48.3 | -12.851 | 0.63 | -11.390* | 2.33 |
| | 3 | -16.223 | 0.45 | -12.947 | 51.2 | -12.697 | 0.65 | -11.320* | 2.37 |
| 42 | 1 | -35.188 | 1.23 | -28.218 | 79.9 | -28.149 | 1.69 | -24.708* | 4.54 |
| | 2 | -32.871 | 1.34 | -26.513 | 87.7 | -26.405 | 1.62 | -23.541* | 4.50 |
| | 3 | -33.145 | 1.24 | -27.368 | 92.5 | -26.410 | 1.64 | -24.456* | 4.56 |
| 62 | 1 | -50.784 | 2.13 | -40.280 | 138 | -40.576 | 2.78 | -35.046* | 7.76 |
| | 2 | -48.489 | 2.19 | -38.418 | 118 | -38.590 | 2.77 | -33.886* | 6.61 |
| | 3 | -48.370 | 2.15 | -39.454 | 121 | -38.363 | 2.84 | -35.346* | 6.15 |
| 82 | 1 | -66.503 | 3.54 | -52.537 | 175 | -52.779 | 4.39 | -46.063* | 10.5 |
| | 2 | -64.849 | 3.33 | -51.223 | 197 | -51.518 | 4.67 | -45.242* | 9.06 |
| | 3 | -65.615 | 3.54 | -52.996 | 181 | -51.917 | 4.70 | -46.982* | 9.12 |

approach enables us to achieve global optimality at a much lower relaxation order so that the computational time is dramatically reduced.

TABLE 4
*The results for* (4.4) *with* $n = 4$.

| | $r=4$ | | $r=6$ | | $r=8$ | |
|---|---|---|---|---|---|---|
| Mom | opt | time | opt | time | opt | time |
| | 1.4218 | 0.12 | 0.8404 | 12.3 | 0.8003 | 1296 |
| | $r=4, s=1$ | | $r=4, s=2$ | | $r=4, s=3$ | |
| S-Mom | opt | time | opt | time | opt | time |
| | 1.4218 | 0.13 | 1.2727 | 0.15 | 0.8000 | 0.27 |

**4.5. The Mordell inequality conjecture.** Our next example concerns the Mordell inequality conjecture due to Birch in 1958: if the numbers $z_1, \ldots, z_n \in \mathbb{C}$ satisfies $|z_1|^2 + \cdots + |z_n|^2 = n$, then the maximum of $\prod_{1\le i<j\le n} |z_i - z_j|^2$ is $n^n$. This conjecture was proved for $n \le 4$ and disproved for $n \ge 6$, and so the only remaining open case is when $n = 5$. The reader is referred to [26] for more details. Without loss of generality, we may eliminate one variable and reformulate the conjecture as the following complex polynomial optimization problem:

$$\left\{\begin{array}{ll}\sup\limits_{\mathbf{z}\in\mathbb{C}^{n-1}} & \prod_{1\le i<j\le n-1} |z_i - z_j|^2 \prod_{i=1}^{n-1} |z_i + z_1 + \ldots + z_{n-1}|^2 \\ \text{s.t.} & |z_1|^2 + \cdots + |z_{n-1}|^2 + |z_1 + \ldots + z_{n-1}|^2 = n.\end{array}\right. \tag{4.5}$$

Here, we tackle (4.5) with $n = 3, 4$ using the complex moment relaxation and the strengthened complex moment relaxation. The computational results are presented in Tables 5 and 6, respectively. From the tables, we see that the strengthening approach enables us to achieve global optimality at much lower relaxation orders so that the computational time is dramatically reduced.

TABLE 5
*The results for* (4.5) *with* $n = 3$.

| | $r = 10$ | | $r = 14$ | | $r = 18$ | |
|---|---|---|---|---|---|---|
| Mom | opt | time | opt | time | opt | time |
| | 27.347 | 0.04 | 27.144 | 0.17 | 27.085 | 0.32 |
| | $r = 3, s = 0$ | | $r = 3, s = 1$ | | $r = 3, s = 2$ | |
| S-Mom | opt | time | opt | time | opt | time |
| | 54.000 | 0.004 | 54.000 | 0.005 | 27.000 | 0.005 |

TABLE 6
*The results for* (4.5) *with* $n = 4$.

| | $r = 10$ | | $r = 12$ | | $r = 14$ | | $r = 16$ | | $r = 18$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| Mom | opt | time | opt | time | opt | time | opt | time | opt | time |
| | 343.67 | 6.42 | 326.85 | 57.8 | 292.82 | 205 | 277.50 | 669 | - | - |
| | $r = 6, s = 1$ | | $r = 6, s = 2$ | | $r = 6, s = 3$ | | $r = 6, s = 4$ | | $r = 6, s = 5$ | |
| S-Mom | opt | time | opt | time | opt | time | opt | time | opt | time |
| | 1638.4 | 0.13 | 1337.6 | 0.19 | 932.20 | 0.14 | 582.86 | 0.15 | 256.00 | 0.16 |

**5. Discussions.** In this paper, we have presented a bilevel hierarchy of strengthened complex moment relaxations for complex polynomial optimization and have demonstrated its superior performance in practice. There still remain some theoretical problems for future research:

It is well known that the real moment hierarchy has finite convergence in the generic case under the Archimedean condition [18]. Moreover, Nie showed in [17] that the real moment hierarchy has finite convergence if and only if flat truncation holds for $r$ sufficiently large under some generic conditions. They together imply that flat truncation occurs in the generic case for the real moment hierarchy under the Archimedean condition. Therefore, it is natural to consider the following question:

*Does flat truncation also occur in the generic case for the strengthened complex moment hierarchy under the Archimedean condition?*

If this question has an affirmative answer, then Theorem 3.8 would yield an important corollary:

CONJECTURE 5.1. *The strengthened complex moment hierarchy has finite convergence in the generic case under the Archimedean condition.*

Another interesting question is determining the convergence rate of the strengthened complex moment hierarchy, following the line of recent research on the convergence rate of the real moment hierarchy [3, 6, 14, 20].

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