Sensitivity Analysis in Dantzig-Wolfe Decomposition

Kouhei Harada *

May 9, 2024

Abstract

Dantzig-Wolfe decomposition is a well-known classical method for solving huge linear optimization problems with a block-angular structure. The most computationally expensive process in the method is pricing: solving block subproblems for a dual variable to produce new columns. Therefore, when we want to solve a slightly perturbated problem in which the block-angular structure is preserved by warm-start Dantzig-Wolfe decomposition, the method's speed highly depends on whether we must generate new columns or not. Nevertheless, theoretical analysis from this point of view has yet to be investigated.

We consider two types of perturbations in this paper and give their sensitivity analysis. First, we propose the range of the right-hand side parameters where no new column generation is necessary. Second, we consider adding a new block to the original problem or removing an existing one. We demonstrate that we do not have to generate any new columns for existing blocks if our proposed condition, a small-sized linear equation has a positive solution, is satisfied under a mild assumption.

Key words. Dantzig-Wolfe decomposition, sensitivity analysis, Lagrangian relaxation, column generation

AMS subject classifications. 90C05, 90C46

1 Introduction

In this paper, we focus on linear optimization problems of the form,

$$\begin{array}{lll} (P) & \text{minimize} & \boldsymbol{c}^\top \boldsymbol{x} \\ & \text{subject to} & A\boldsymbol{x} = \boldsymbol{b}, \\ & & \boldsymbol{x} \in X, \end{array} \tag{1.1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is a vector of decision variables, $A \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{c} \in \mathbb{R}^n$ are parameters, and X is a polyhedron.

It is well known that any polyhedron can be represented as a convex combination of its extreme points and positive combination of its extreme rays [3, Theorem 16.2]. Therefore, we can convert problem (P) into another linear optimization problem in which variables are positive weights of extreme points and rays. The new form of optimization problems is called *Full Master Problem*. We abbriviate it as (FMP). It can be described in the following.

$$\begin{array}{lll} (FMP) & \text{minimize} & \boldsymbol{f}^\top \boldsymbol{\lambda} \\ & \text{subject to} & G\boldsymbol{\lambda} = \boldsymbol{b}, \\ & & \boldsymbol{\xi}^\top \boldsymbol{\lambda} = 1, \\ & & \boldsymbol{\lambda} \geq \boldsymbol{0}, \end{array}$$

*NTTDATA Mathematical Systems Inc, 1F, Shinanomachi Renga-Kan, 35, Shinanomachi, Shinjuku-ku, Tokyo, JAPAN, email:harada@msi.co.jp

where K and L are the number of extreme points and extreme rays, respectively, $\boldsymbol{\lambda} \in \mathbb{R}^{K+L}$ is a vector of variables corresponding to weight of extreme points and rays, $\boldsymbol{f} \in \mathbb{R}^{K+L}$, $G \in \mathbb{R}^{m \times (K+L)}$, and $\boldsymbol{\xi} = (\underbrace{1, \cdots, 1}_{K}, \underbrace{0, \cdots, 0}_{L}) \in \mathbb{R}^{K+L}$ are parameters. We will explain the precise relationship between (P) and (FMP) in section 2.

In general, (FMP) is not an attractive alternative to (P) because the number of extreme points and rays are usually much larger than that of variables, i.e., $K + L \gg n$. Moreover, obtaining all extreme points and rays is a formidable task [3]. Fortunately, not all extreme points and rays are necessary to solve (FMP). We can use this fact to construct an algorithm for solving (FMP). It recursively generates a new column and solves a master problem of which extreme points and extreme rays are restricted. The algorithm is known as *Dantzig-Wolfe decomposition* or *column generation* [7, 20][6, Chapter 23][3, Chapter 26][4, Part II, Section 2.4]. It is often called *Dantzig-Wolfe decomposition algorithm* or *Dantzig-Wolfe decomposition method* to stress that it is the name of an algorithm. From a dual point of view, the algorithm is equivalent to *Kelley's cutting plane method* [13][1, Section 6.3][2, Section 9.3.3][19, Section 3.3.2]. To confirm the relationships, see [1, Section 6.4][8, Chapter I]. Dantzig-Wolfe decomposition is also helpful for mixed integer linear optimization problems. In branch-and-bound procedures, the algorithm is applicable to solveing their relaxed subproblems. The combined framework is known as *branch-and-price* [5, Chapter 8.2.3][21, Section 11.4]. However, this is not the scope of this paper.

Let us go back to the original problem (P). When we want to solve a slightly perturbated (P), we can use some information about an optimal solution of (P). Such techniques are known as *sensitivity analysis*. When $X = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{x} \geq 0\}$, slightly perturbating $\boldsymbol{b}$, $\boldsymbol{c}$, and adding a new variable are typical themes of sensitivity analysis covered in lots of optimization textbooks. On the other hand, we are interested in the case that X is an arbitrary polyhedron and solving (FMP) instead of (P) by Dantzig-Wolfe decomposition. Although Dantzig-Wolfe decomposition has been broadly investigated [20, 16, 8, 15], its sensitivity analyses from theoretical points of view has yet to be reported to our knowledge. We will demonstrate two sensitivity analyses in the context of Dantzig-Wolfe decomposition. First, we will clarify the range of the right-hand side(RHS) parameters $\boldsymbol{b}$ that preserves the optimal basis in (FMP). At first glance, the result just directly applies a well-known classical fact to (FMP) with an elementary modification. However, the result can be improved when X has a block-angular structure such that $X = \prod_{i \in I} X^i$ and the number of blocks $|I|$ is large, which is an attractive and original situation for applying Dantzig-Wolfe decomposition [7]. Under the same condition, we consider another likely situation: adding a new block to (FMP) or removing an existing block from it. We demonstrate that it does not change the optimal basis of existing blocks if a small-sized linear equation has a positive solution.

The paper is organized as follows. Section 2 demonstrates some preliminary results for sensitivity analysis and Dantzig-Wolfe decomposition. In section 3, we consider the perturbation of the RHS parameters $\boldsymbol{b}$ and propose the range of perturbation that preserves the optimal basis. Section 4 is the main section of this paper. In this section, we consider the case of adding a new block or removing an existing block. We demonstrate that this manipulation preserves the optimal basis of existing blocks if an easily confirmed condition is satisfied. In section 5, we observe the proposed results from a dual point of view. In the final section, we conclude the paper and suggest for further research.

As described in this paper, we use the following notation. We use the bold style, such as $\boldsymbol{x}$, to stress that it is a vector. Let $A = (\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n) \in \mathbb{R}^{m \times n}$ be a matrix, where $\boldsymbol{a}_i \in \mathbb{R}^m$ are column vectors. We denote the convex hull of the column vectors $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$ by $\mathrm{conv}\,(A)$. Similarly, we denote the cone hull of $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$ by $\mathrm{cone}\,(A)$. We use the cursive style, such as $\mathcal{B}$, to stress that it is a set of indices. We denote the partial matrix of A of which columns are limited to an index set $\mathcal{B}$ by $A_{\mathcal{B}}$. For example $A_{\{1,3\}} = (\boldsymbol{a}_1, \boldsymbol{a}_3)$. We denote the optimal objective value of a problem (P) by obj(P). We denote n dimensional one vector $(1, \cdots, 1) \in \mathbb{R}^n$ as $\boldsymbol{1}_n$. We denote the direct sum of $\mathcal{B}$ and $\tilde{\mathcal{B}}$ as $\mathcal{B} \bigoplus \tilde{\mathcal{B}}$ in order to distinguish the same indices derived from different origins. We denote the open ball of radius r centered at point z as $B_r(z) = \{\, x \mid \|x - z\| < r \,\}$.

2 Preliminary Results

This section prepares known results regarding Dantzig-Wolfe decomposition and sensitivity analysis. We start by describing the precise relationships between (P) and (FMP).

2.1 Master Problems

It is well known that any polyhedron can be represented by its extreme points and rays [3, Theorem 16.2][18, Part I, Theorem 4.8].

Theorem 2.1. *(Minkowski's theorem) Let X be a nonempty polyhedron, then it can be represented by*

$$X = \left\{ \sum_{k=1}^{K} \lambda_k \boldsymbol{v}_k + \sum_{k=K+1}^{K+L} \lambda_k \boldsymbol{r}_k \;\middle|\; \sum_{k=1}^{K} \lambda_k = 1,\, \lambda_k \geq 0,\, \forall k \in \{1, \cdots, K+L\} \right\},$$

where $\{\boldsymbol{v}_k\}_{k=1}^{K}$ and $\{\boldsymbol{r}_k\}_{k=K+1}^{K+L}$ are extreme points and extreme rays of X, respectively.

Unlike its original description, we use same symbols, k and λ_k, for both extreme points and extreme rays to simplify the description in this paper. By theorem 2.1, we have

$$\boldsymbol{c}^\top \boldsymbol{x} = \boldsymbol{c}^\top \left(\sum_{k=1}^{K} \lambda_k \boldsymbol{v}_k + \sum_{k=K+1}^{K+L} \lambda_k \boldsymbol{r}_k \right) = \sum_{k=1}^{K+L} \lambda_k f_k = \boldsymbol{f}^\top \boldsymbol{\lambda},$$

where $\boldsymbol{f}$ is defined by $f_k = \boldsymbol{c}^\top \boldsymbol{v}_k$ for $k = 1, \cdots, K$ and $f_k = \boldsymbol{c}^\top \boldsymbol{r}_k$ for $k = K+1, \cdots, K+L$. We also have

$$Ax = A \left(\sum_{k=1}^{K} \lambda_k \boldsymbol{v}_k + \sum_{k=K+1}^{K+L} \lambda_k \boldsymbol{r}_k \right) = \sum_{k=1}^{K+L} \lambda_k \boldsymbol{g}_k = G\boldsymbol{\lambda},$$

where $\boldsymbol{g}_k$ is defined by $\boldsymbol{g}_k = A\boldsymbol{v}_k$ for $k = 1, \cdots, K$ and, $\boldsymbol{g}_k = A\boldsymbol{r}_k$ for $k = K+1, \cdots, K+L$, and G is defined by $G = (\boldsymbol{g}_1, \cdots, \boldsymbol{g}_{K+L})$. Using these parameters $\boldsymbol{f}$ and G, we can construct (FMP) as a linear optimization problem to obtain optimal weight variables $\boldsymbol{\lambda}$. The reformulation is said to be *Dantzig-Wolfe relaxation*, in the context of mixed integer linear optimization problems [5, Chapter 8.2].

For an arbitrary subset of indices $\mathcal{S} \subset \{1, \cdots, K+L\}$, we can construct its corresponding master problem of which extreme points and extreme rays are restricted. These restricted problems are called *Restricted Master Problems*. We denote it as (RMP).

$$\begin{array}{lll} (RMP) & \text{minimize} & (\boldsymbol{f}_{\mathcal{S}})^\top \boldsymbol{\lambda}_{\mathcal{S}} \\ & \text{subject to} & G_{\mathcal{S}} \boldsymbol{\lambda}_{\mathcal{S}} = \boldsymbol{b}, \\ & & (\boldsymbol{\xi}_{\mathcal{S}})^\top \boldsymbol{\lambda}_{\mathcal{S}} = 1, \\ & & \boldsymbol{\lambda}_{\mathcal{S}} \geq \boldsymbol{0}. \end{array}$$

2.2 Relationships between Dual Problems

This subsection considers the relationships between dual problems of (P) and (FMP). Let the dual problem of (P) be (D). Then, we can describe its formulation as follows.

$$(D) \quad \text{maximize} \quad \boldsymbol{b}^\top \boldsymbol{\pi} + d(\boldsymbol{\pi}),$$

where $\boldsymbol{\pi} \in \mathbb{R}^m$ are dual variables, and $d(\boldsymbol{\pi})$ is a piecewise-linear concave function defined as

$$d(\boldsymbol{\pi}) \equiv \min\{(\boldsymbol{c} - A^\top \boldsymbol{\pi})^\top \boldsymbol{x} \,|\, \boldsymbol{x} \in X\}.$$

The minimization problems obtaining $d(\boldsymbol{\pi})$ are said to be *Lagrangian relaxation problems* concerning constraint (1.1). We denote it by (LR). The solution of (LR) gives a lower bound of (P), i.e.

$$\text{obj(P)} \geq \text{obj(LR)} = d(\boldsymbol{\pi}).$$

Similarly, let the dual problem of (FMP) as (FMD). We give its formulation in the following.

$$
\begin{array}{lll}
(FMD) & \text{maximize} & \boldsymbol{b}^\top \boldsymbol{\pi} + \sigma \\
& \text{subject to} & G^\top \boldsymbol{\pi} + \boldsymbol{\xi}\sigma \leq \boldsymbol{f},
\end{array}
$$

where $\boldsymbol{\pi} \in \mathbb{R}^m$ and $\sigma \in \mathbb{R}$ are dual variables.

Some readers may feel confused that we use the same symbol $\boldsymbol{\pi}$ in both (D) and (FMD). However, the following proposition suggests a valid correspondence between them.

Proposition 2.2. *A dual variable $\boldsymbol{\pi} \in \mathbb{R}^m$ is feasible in (D), if and only if $(\boldsymbol{\pi}, d(\boldsymbol{\pi}))$ is feasible in (FMD).*

Proof. We first prove $\Rightarrow$. Since $\boldsymbol{\pi}$ is feasible, we have $d(\boldsymbol{\pi}) > -\infty$. It follows that $f_k - \boldsymbol{g}_k^\top \boldsymbol{\pi} \geq 0$ for all $k = K+1, \cdots, K+L$. Therefore, by definition, $d(\boldsymbol{\pi}) = \min_{k=1,\cdots,K}(f_k - \boldsymbol{g}_k^\top \boldsymbol{\pi})$. Combining the two properties, we have $G^\top \boldsymbol{\pi} + \boldsymbol{\xi} d(\boldsymbol{\pi}) \leq \boldsymbol{f}$. Hence, $(\boldsymbol{\pi}, d(\boldsymbol{\pi}))$ is feasible in (FMD). Conversely, if $(\boldsymbol{\pi}, d(\boldsymbol{\pi}))$ is feasible in (FMD), we have $G^\top \boldsymbol{\pi} + \boldsymbol{\xi} d(\boldsymbol{\pi}) \leq \boldsymbol{f}$. Thus, $f_k - \boldsymbol{g}_k^\top \boldsymbol{\pi} \geq 0$ holds for all $k = K+1, \cdots, K+L$, i.e., Then, by definition, $d(\boldsymbol{\pi}) = \min_{k=1,\cdots,K}(f_k - \boldsymbol{g}_k^\top \boldsymbol{\pi}) > -\infty$. $\boldsymbol{\pi}$ is feasible in (D). □

Corollary 2.3. *A dual variable $\boldsymbol{\pi}^*$ is optimal in (D) if and only if $(\boldsymbol{\pi}^*, d(\boldsymbol{\pi}^*))$ is optimal in (FMD).*

Proof. The claim directly follows from Proposition 2.2 and obj(D) = obj(FMD). □

Tebboth [20, Section 3] describes the relationships between the original and master problem in more detail. In fact, Proposition 2.2 corresponds to [20, Proposition 14, Section 3]. A simpler explanation is given in [5, Section 8.2]. However, their description uses an outer explicit representation of X, which is not provided in this paper to simplify the description.

2.3 Dantzig-Wolfe Decomposition

In this subsection, we briefly describe the process of Dantzig-Wolfe decomposition. Dantzig-Wolfe decomposition repeatedly solves (RMP) and (LR). The former provides a new dual variable, and the latter generates a new extreme point or extreme ray.

Algorithm 1 Dantzig-Wolfe decomposition

Inputs:
  Prepare initial feasible extreme points $\{\boldsymbol{v}_k\}_{k=1}^{k_0}$ and extreme rays $\{\boldsymbol{r}_k\}_{k=K+1}^{K+\ell_0}$.
Initialize:
  Set $\bar{k} = k_0$ and $\bar{\ell} = \ell_0$. Set $\mathcal{S} = \{1, \cdots, \bar{k}, K, \cdots, K+\bar{\ell}\}$.
  Calculate f_k and $\boldsymbol{g}_k$ for all $k \in \mathcal{S}$.
repeat
  Solve (RMP) for $\mathcal{S}$ to obtain its optimal dual solution $(\bar{\boldsymbol{\pi}}, \bar{\sigma})$. ▷ **Step 1: Dual update**
  Solve (LR) for $\boldsymbol{\pi} = \bar{\boldsymbol{\pi}}$. ▷ **Step 2: Pricing**
  if (LR) has an optimal solution **then**
    There exists an extreme point $\bar{\boldsymbol{v}}$ such that $\bar{\boldsymbol{v}} \in \arg\min_{x \in X}(\boldsymbol{c} - A^\top \bar{\boldsymbol{\pi}})^\top \boldsymbol{x}$.
    Calculate $f_{\bar{k}+1} = \boldsymbol{c}^\top \bar{\boldsymbol{v}}$ and $\boldsymbol{g}_{\bar{k}+1} = A\bar{\boldsymbol{v}}$.
    Insert $\bar{k}+1$ into $\mathcal{S}$, and update $\bar{k} = \bar{k}+1$.
  else
    There exists an extreme ray $\bar{\boldsymbol{r}}$ such that $(\boldsymbol{c} - A^\top \bar{\boldsymbol{\pi}})^\top \bar{\boldsymbol{r}} < 0$.
    Calculate $f_{K+\bar{\ell}+1} = \boldsymbol{c}^\top \bar{\boldsymbol{r}}$ and $\boldsymbol{g}_{K+\bar{\ell}+1} = A\bar{\boldsymbol{r}}$.
    Insert $K+\bar{\ell}+1$ into $\mathcal{S}$, and update $\bar{\ell} = \bar{\ell}+1$.
  end if
until obj(RMP) = $\boldsymbol{b}^\top \bar{\boldsymbol{\pi}}$ + obj(LR) ▷ **Step 3: Optimality Test**

Although explaining the algorithm's validity is outside the scope of the paper, we will provide several comments to help readers understand the procedure. We need initial sets of extreme points and rays to be feasible in inputs. If difficult, we can use artificial columns that act as slack variables [8, Chapter 12]. The optimality test implies that the solution of (RMP) is also optimal for (FMP). In practice, we finish the algorithm if its difference becomes sufficiently small. Since we have combined the index set of extreme points and that of extreme rays, there exists a side effect; we have to prepare K, which is generally not known beforehand, to describe Dantzig-Wolfe decomposition. In practice, K is not necessary. Its role is to distinguish the index set of extreme points and extreme rays.

We have not prepared any examples to trace the process of the algorithm. To confirm them, see [20],[3, Section 26], [4, Part II, Section 2.4].

2.4 Optimality Condition and Sensitivity Analysis

In this subsection, we assume that $X = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, \boldsymbol{x} \geq \boldsymbol{0}\}$ to review the optimality condition and sensitivity analysis for standard linear optimization problems. We first describe several definitions [18, Part I, Definition 3.1]. Let (P0) be linear optimization problems such that $X = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, \boldsymbol{x} \geq \boldsymbol{0}\}$ in (P). Let $\mathcal{B}$ be an arbitrary index set of (P0) such that $|\mathcal{B}| = m$, and let $\mathcal{N} = \{1, \cdots, n\} \setminus B$. Then, the $m \times m$ nonsingular matrix $A_{\mathcal{B}}$ is called a *basis*. The variables $\boldsymbol{x}_{\mathcal{B}}$ are called *basic variables* and the variables $\boldsymbol{x}_{\mathcal{N}}$ are called *nonbasic variables*. If $(\boldsymbol{x}_{\mathcal{B}}, \boldsymbol{x}_{\mathcal{N}}) = (A_{\mathcal{B}}^{-1}\boldsymbol{b}, \boldsymbol{0})$ is an optimal solution of (P0), then $A_{\mathcal{B}}$ is called an *optimal basis*.

In this paper, we often focus on the index set $\mathcal{B}$ of a basis $A_{\mathcal{B}}$. To simplify the description, we use the word *basis* not only for the nonsingular matrix $A_{\mathcal{B}}$, but also for its set of indices $\mathcal{B}$ as the same way in [10].

The following theorem, the optimality condition for standard linear optimization problems (P0), is elementary but fundamental. It is covered in many optimization textbooks. For example, see [3, Section 5][18, Part I.2, Proposition 3.1].

Theorem 2.4. *Let $\mathcal{B}$ be a basis of (P0). Then, the basis $\mathcal{B}$ becomes an optimal basis of (P) if and only if the following two conditions hold.*

(a) *(dual feasibility)* $\boldsymbol{c}_{\mathcal{N}}^\top - \boldsymbol{c}_{\mathcal{B}}^\top A_{\mathcal{B}}^{-1} A_{\mathcal{N}} \geq \boldsymbol{0}^\top$.

(b) *(primal feasibility)* $A_{\mathcal{B}}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$.

As described above, if $A_{\mathcal{B}}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$, we call that $A_{\mathcal{B}}$ is a *primal feasible basis*. Similarly, if $\boldsymbol{c}_{\mathcal{N}}^\top - \boldsymbol{c}_{\mathcal{B}}^\top A_{\mathcal{B}}^{-1} A_{\mathcal{N}} \geq \boldsymbol{0}^\top$, then $A_{\mathcal{B}}$ is called a *dual feasible basis*.

Corollary 2.5. *Let $\mathcal{B}$ be an optimal basis of (P0). Then, $A_{\mathcal{B}}^{-T}\boldsymbol{c}_{\mathcal{B}}$ is an optimal dual solution of (P0).*

Among many topics in sensitivity analysis, we focus on the perturbation of the RHS parameter. The following fact directly follows from theorem 2.4.

Corollary 2.6. *Let $\mathcal{B}$ be an optimal basis of (P0). Then, perturbating $\boldsymbol{b}$ to $\boldsymbol{b}+\boldsymbol{\Delta b}$ preserves the optimality of $\mathcal{B}$ as far as $A_{\mathcal{B}}^{-1}(\boldsymbol{b} + \boldsymbol{\Delta b}) \geq \boldsymbol{0}$ holds.*

2.5 Block-Angular Structure

In this subsection, we consider the case that X has a block structure such that $X = \prod_{i \in I} X^i$ and X^i is a polyhedron for each $i \in I$. Let us denote the problem as (PB).

$$
\begin{aligned}
(PB) \quad \text{minimize} \quad & \sum_{i \in I} (\boldsymbol{c}^i)^\top \boldsymbol{x}^i \\
\text{subject to} \quad & \sum_{i \in I} A^i \boldsymbol{x}^i = \boldsymbol{b}, \\
& \boldsymbol{x}^i \in X^i, \ \forall i \in I,
\end{aligned}
\tag{2.1}
$$

where $\boldsymbol{x}^i \in \mathbb{R}^{n_i}$ are decision variables , and $A^i \in \mathbb{R}^{m \times n_i}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{c}^i \in \mathbb{R}^n$ are parameters. In describing its master problems, we have two options: Applying Theorem 2.1 to whole X, or to X^i for each $i \in I$. The former and latter are said to be aggregated and disaggregated forms, respectively [9].

The superiority of the disaggregated form is described in [12, 9]. We denote the disaggregated form of the master problem as (FMPB). Its precise formulation is given as follows.

$$
\begin{aligned}
(FMPB) \quad \text{minimize} \quad & \sum_{i\in I} (\boldsymbol{f}^i)^\top \boldsymbol{\lambda}^i \\
\text{subject to} \quad & \sum_{i\in I} G^i \boldsymbol{\lambda}^i = \boldsymbol{b}, \\
& (\boldsymbol{\xi}^i)^\top \boldsymbol{\lambda}^i = 1, \, \forall i \in I, \\
& \boldsymbol{\lambda}^i \geq \boldsymbol{0}, \, \forall i \in I,
\end{aligned} \tag{2.2}
$$

where K_i and L_i are the numbers of extreme points and extreme rays for each $i \in I$, respectively, $\boldsymbol{\lambda}^i \in \mathbb{R}^{K_i+L_i}$ is a variable vector corresponding to the weight of extreme points and extreme rays for each $i \in I$, and $\boldsymbol{f}^i \in \mathbb{R}^{K_i+L_i}$, $G^i \in \mathbb{R}^{m\times(K_i+L_i)}$, and $\boldsymbol{\xi}^i = (\underbrace{1,\cdots,1}_{K_i},\underbrace{0,\cdots,0}_{L_i}) \in \mathbb{R}^{K_i+L_i}$ are parameters for each $i \in I$. Let $\{\boldsymbol{v}_k^i\}_{k=1}^{K_i}$ and $\{\boldsymbol{r}_k^i\}_{k=K_i+1}^{K_i+L_i}$ be extreme points and extreme rays of the polyhedron X_i for each $i \in I$. Then, theorem 2.1 implies that

$$
X_i = \left\{ \sum_{k=1}^{K_i} \lambda_k^i \boldsymbol{v}_k + \sum_{k=K_i+1}^{K_i+L_i} \lambda_k^i \boldsymbol{r}_k \;\middle|\; \sum_{k=1}^{K_i} \lambda_k^i = 1, \, \lambda_k^i \geq 0 \right\}.
$$

Using the representation, we have

$$
(\boldsymbol{c}^i)^\top \boldsymbol{x}^i = (\boldsymbol{c}^i)^\top \left(\sum_{k=1}^{K_i} \lambda_k^i \boldsymbol{v}_k^i + \sum_{k=K_i+1}^{K_i+L_i} \lambda_k^i \boldsymbol{r}_k^i \right) = \sum_{k=1}^{K_i+L_i} \lambda_k^i f_k^i = (\boldsymbol{f}^i)^\top \boldsymbol{\lambda}^i,
$$

where $\boldsymbol{f}^i$ is defined by $f_k^i = (\boldsymbol{c}^i)^\top \boldsymbol{v}_k^i$ for $k = 1, \cdots, K_i$ and $f_k^i = (\boldsymbol{c}^i)^\top \boldsymbol{r}_k^i$ for $k = K_i+1, \cdots, K_i+L_i$. We also have

$$
A^i \boldsymbol{x}^i = A^i \left(\sum_{k=1}^{K_i} \lambda_k^i \boldsymbol{v}_k^i + \sum_{k=K_i+1}^{K_i+L_i} \lambda_k^i \boldsymbol{r}_k^i \right) = \sum_{k=1}^{K_i+L_i} \lambda_k^i \boldsymbol{g}_k^i = G^i \boldsymbol{\lambda}^i,
$$

where $\boldsymbol{g}_k^i$ is defined by $\boldsymbol{g}_k^i = A^i \boldsymbol{v}_k^i$ for $k = 1, \cdots, K_i$ and, $\boldsymbol{g}_k^i = A^i \boldsymbol{r}_k^i$ for $k = K_i+1, \cdots, K_i+L_i$, and G^i is defined by $G^i = (\boldsymbol{g}_1^i, \cdots, \boldsymbol{g}_{K_i+L_i}^i)$.

Let the dual problem of (PB) be (DB). Then, its formulation is given as follows.

$$
(DB) \quad \text{maximize} \quad \boldsymbol{b}^\top \boldsymbol{\pi} + \sum_{i\in I} d_i(\boldsymbol{\pi}),
$$

where $\boldsymbol{\pi} \in \mathbb{R}^m$ are dual variables, and $d_i(\boldsymbol{\pi})$ is a piecewise-linear concave function defined as

$$
d_i(\boldsymbol{\pi}) \equiv \min\{(\boldsymbol{c}^i - (A^i)^\top \boldsymbol{\pi})^\top \boldsymbol{x}^i \,|\, \boldsymbol{x}^i \in X_i\},
$$

for each i. The minimization problems obtaining $d_i(\boldsymbol{\pi})$ are said to be *Lagrangian relaxation subproblems* with respect to constraint (2.1). We denote it by (LR(i)). The solutions of (LR(i)) give a lower bound of (PB), i.e.

$$
\text{obj(PB)} \geq \boldsymbol{b}^\top \boldsymbol{\pi} + \sum_{i\in I} \text{obj(LR}(i)) = \boldsymbol{b}^\top \boldsymbol{\pi} + \sum_{i\in I} d_i(\boldsymbol{\pi}).
$$

Let (FMDB) be a dual problem of (FMPB) defined as follows.

$$
(FMDB) \quad \text{maximize} \quad \boldsymbol{b}^\top \boldsymbol{\pi} + \sum_{i\in I} \sigma_i,
$$

where $\boldsymbol{\pi} \in \mathbb{R}^m$ and $\boldsymbol{\sigma} \in \mathbb{R}^{|I|}$ are dual variables. In the same way as in section 2.2, the following properties hold.

Proposition 2.7. *A dual variable $\boldsymbol{\pi} \in \mathbb{R}^m$ is feasible in (DB), if and only if $(\boldsymbol{\pi}, d_1(\boldsymbol{\pi}), \cdots, d_{|I|}(\boldsymbol{\pi})))$ is feasible in (FMDB).*

Corollary 2.8. *A dual variable $\boldsymbol{\pi}^*$ is optimal in (DB), if and only if $(\boldsymbol{\pi}^*, d_1(\boldsymbol{\pi}^*), \cdots, d_{|I|}(\boldsymbol{\pi}^*))$ is optimal in (FMDB).*

Applying Dantzig-Wolfe decomposition for (PB) or (FMPB), pricing is necessary for each $i \in I$. In addition, the optimality test becomes obj(RMPB) $= \boldsymbol{b}^\top \bar{\boldsymbol{\pi}} + \sum_{i \in I}$ obj(LR(i)), where (RMPB) is a restricted master problem of (PB).

3 Perturbing the Right-Hand Side Parameters

In this section, we focus on the perturbation of the RHS parameters. We start with the case that X is a general polyhedron. Then, we consider the case that X has a block-angular structure.

3.1 General Case

Let $\mathcal{B}$ be an optimal index set of (FMP). To describe the restricted index set within extreme points and extreme rays, we denote

$$\mathcal{K} \equiv \mathcal{B} \cap \{1, \cdots, K\}, \quad \mathcal{L} \equiv \mathcal{B} \cap \{K+1, \cdots, K+L\}.$$

By definition, we have $\mathcal{B} = \mathcal{K} \cup \mathcal{L}, \quad \mathcal{K} \cap \mathcal{L} = \emptyset$. We assume the following (A1) for (FMP) to ensure the existence of an optimal basis.

(A1) rank $\begin{pmatrix} G \\ \boldsymbol{\xi}^\top \end{pmatrix} = m + 1.$

Proposition 3.1. *Let $\mathcal{B}$ be an optimal basis of (FMP), then perturbating $\boldsymbol{b}$ to $\boldsymbol{b} + \Delta \boldsymbol{b}$ preserves its optimality if and only if the following condition holds.*

$$\boldsymbol{b} + \Delta \boldsymbol{b} \in \text{conv}\,(G_{\mathcal{K}}) + \text{cone}\,(G_{\mathcal{L}})$$

Proof. Applying Corollary 2.6 to (FMP) we have $\begin{pmatrix} G_{\mathcal{B}} \\ \boldsymbol{\xi}_{\mathcal{B}}^\top \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{b} \\ 1 \end{pmatrix} \geq \mathbf{0}$. By definition, it is equivalent to

$$\begin{aligned} \exists \boldsymbol{\lambda} \geq \mathbf{0}, \begin{pmatrix} G_{\mathcal{B}} \\ \boldsymbol{\xi}_{\mathcal{B}}^\top \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{b} + \Delta \boldsymbol{b} \\ 1 \end{pmatrix} &\Leftrightarrow \exists \boldsymbol{\lambda} \geq \mathbf{0}, \sum_{k \in \mathcal{K}} \lambda_k = 1, \, G_{\mathcal{B}} \boldsymbol{\lambda} = \boldsymbol{b} + \Delta \boldsymbol{b} \\ &\Leftrightarrow \boldsymbol{b} + \Delta \boldsymbol{b} \in \text{conv}\,(G_{\mathcal{K}}) + \text{cone}\,(G_{\mathcal{L}}) \end{aligned}$$

□

3.2 Block-Angulared Case

For a general polyhedron X, the result of proposition 3.1 is straightforward. In this subsection, we consider the case that X has a block-angular structure such that $X = \prod_{i \in I} X^i$ and X^i is a polyhedron for each $i \in I$.

Let $\mathcal{B}$ be an optimal basis of (FMPB). In (FMPB), we have to distinguish the same indices each of which belongs to a different block. Therefore, we describe the whole optimal basis as $\mathcal{B} = \bigoplus_{i \in I} \mathcal{B}(i)$, where $\mathcal{B}(i)$ be the optimal basis in block $i \in I$. To represent the restricted index set within extreme points and extreme rays for each block $i \in I$, we denote

$$\mathcal{K}(i) \equiv \mathcal{B}(i) \cap \{1, \cdots, K_i\}, \quad \mathcal{L}(i) \equiv \mathcal{B}(i) \cap \{K_i + 1, \cdots, K_i + L_i\}.$$

By definition, we have $\mathcal{B}(i) = \mathcal{K}(i) \cup \mathcal{L}(i)$ and $\mathcal{K}(i) \cap \mathcal{L}(i) = \emptyset$ for each $i \in I$. For (FMPB), we assume the following (A3) to ensure the existence of an optimal basis.

(A3) rank $\begin{pmatrix} G^1 & G^2 & \cdots & G^{|I|} \\ (\boldsymbol{\xi}^1)^\top & \mathbf{0}^\top & \cdots & \mathbf{0}^\top \\ \mathbf{0}^\top & (\boldsymbol{\xi}^2)^\top & \cdots & \mathbf{0}^\top \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}^\top & \mathbf{0}^\top & \cdots & (\boldsymbol{\xi}^{|I|})^\top \end{pmatrix} = m + |I|.$

Let $\mathcal{B} = \bigoplus_{i \in I} \mathcal{B}(i)$ an optimal basis of (FMPB). Then, due to corollary 2.5, its corresponding optimal dual solution $\begin{pmatrix} \boldsymbol{\pi}^* \\ \boldsymbol{\sigma}^* \end{pmatrix}$ is given by

$$\begin{pmatrix} \boldsymbol{\pi}^* \\ \boldsymbol{\sigma}^* \end{pmatrix}^\top = \begin{pmatrix} \boldsymbol{f}^1_{\mathcal{B}(1)} \\ \vdots \\ \boldsymbol{f}^{|I|}_{\mathcal{B}(|I|)} \end{pmatrix}^\top \begin{pmatrix} G^1_{\mathcal{B}(1)} & \cdots & G^{|I|}_{\mathcal{B}(|I|)} \\ (\boldsymbol{\xi}^1_{\mathcal{B}(1)})^\top & \cdots & \mathbf{0}^\top \\ \vdots & \ddots & \vdots \\ \mathbf{0}^\top & \cdots & (\boldsymbol{\xi}^{|I|}_{\mathcal{B}(|I|)})^\top \end{pmatrix}^{-1}. \tag{3.1}$$

For the problems with a block-angular structure, we can generalize proposition 3.1 as follows.

Proposition 3.2. *Let $\mathcal{B} = \bigoplus_{i \in I} \mathcal{B}(i)$ be an optimal basis of (FMPB), then perturbating $\boldsymbol{b}$ to $\boldsymbol{b} + \Delta \boldsymbol{b}$ preserves its optimality if and only if the following condition holds.*

$$\boldsymbol{b} + \Delta \boldsymbol{b} \in \sum_{i \in I} \left(\text{conv}\left(G^i_{\mathcal{K}(i)} \right) + \text{cone}\left(G^i_{\mathcal{L}(i)} \right) \right)$$

Proof. Applying Corollary 2.6 to (FMPB), we have

$$\begin{pmatrix} G^1_{\mathcal{B}(1)} & \cdots & G^{|I|}_{\mathcal{B}(|I|)} \\ (\boldsymbol{\xi}^1_{\mathcal{B}(1)})^\top & \cdots & \mathbf{0}^\top \\ \vdots & \ddots & \vdots \\ \mathbf{0}^\top & \cdots & (\boldsymbol{\xi}^{|I|}_{\mathcal{B}(|I|)})^\top \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{b} + \Delta \boldsymbol{b} \\ \mathbf{1}_{|I|} \end{pmatrix} \geq \mathbf{0}.$$

By definition, it is equivalent to

$$\exists \begin{pmatrix} \boldsymbol{\lambda}^1 \\ \vdots \\ \boldsymbol{\lambda}^{|I|} \end{pmatrix} \geq \mathbf{0}, \begin{pmatrix} G^1_{\mathcal{B}(1)} & \cdots & G^{|I|}_{\mathcal{B}(|I|)} \\ (\boldsymbol{\xi}^1_{\mathcal{B}(1)})^\top & \cdots & \mathbf{0}^\top \\ \vdots & \ddots & \vdots \\ \mathbf{0}^\top & \cdots & (\boldsymbol{\xi}^{|I|}_{\mathcal{B}(|I|)})^\top \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda}^1 \\ \vdots \\ \boldsymbol{\lambda}^{|I|} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b} + \Delta \boldsymbol{b} \\ \mathbf{1}_{|I|} \end{pmatrix} \tag{3.2}$$

$$\Leftrightarrow \forall i \in I, \exists \boldsymbol{\lambda}^i \geq \mathbf{0}, \sum_{k \in \mathcal{K}(i)} \lambda^i_k = 1, \sum_{i \in I} G^i_{\mathcal{B}(i)} \boldsymbol{\lambda}^i = \boldsymbol{b} + \Delta \boldsymbol{b}$$

$$\Leftrightarrow \boldsymbol{b} + \Delta \boldsymbol{b} \in \sum_{i \in I} \left(\text{conv}\left(G^i_{\mathcal{K}(i)} \right) + \text{cone}\left(G^i_{\mathcal{L}(i)} \right) \right).$$

□

To confirm the condition in Proposition 3.2, we have to solve $m + |I|$ dimensional linear equation (3.2). If the number of the blocks $|I|$ is too large, the confirmation appears to be a hard task at first glance. However, it is easier than it appears; we prove that the linear equation can be compressed into a $2m$-dimensional one under the following assumption.

(A3) The number of blocks is larger than the number of constraints, i.e., $|I| \geq m$.

Lemma 3.3. *Let $\bigoplus_{i \in I} \mathcal{B}(i)$ be an optimal basis of (FMPB) and assumption (A3) holds, then there exists at least $|I| - m$ blocks i such that $\mathcal{B}(i)$ becomes a singleton.*

Proof. Since (FMPB) has $m+|I|$ equality constraints, $|\bigoplus_{i\in I}\mathcal{B}(i)| = m+|I|$. Therefore, $\sum_{i\in I}|\mathcal{B}(i)| = m+|I|$. On the other hand, $\mathcal{B}(i)\neq\emptyset$, i.e, $|\mathcal{B}(i)|\geq 1$ for all i, due to constraint (2.2). Let $\bar{I}$ be a set of indices of singletons, namely, $\bar{I}=\{i\in I||\mathcal{B}(i)|=1\}$. Assume that $|\bar{I}|<|I|-m$ to achieve a contradiction. Then, we have

$$\sum_{i\in I}|\mathcal{B}(i)| = \sum_{i\in \bar{I}}|\mathcal{B}(i)| + \sum_{i\notin \bar{I}}|\mathcal{B}(i)| \geq |\bar{I}| + 2(|I|-|\bar{I}|) > |I|+m.$$

It contradicts to $\sum_{i\in I}|\mathcal{B}(i)| = m+|I|$. Hence, we have reached the desired result. □

Lemma 3.3 is inspired by a proof of Shapley-folkman theorem[22][1, Exercise 5.1.3].

Proposition 3.4. *Let $\mathcal{B}=\bigoplus_{i\in I}\mathcal{B}(i)$ be an optimal basis of (FMPB) and assumption (A3) holds, then perturbating $\boldsymbol{b}$ to $\boldsymbol{b}+\Delta\boldsymbol{b}$ preserves its optimality if and only if the following condition holds.*

$$\boldsymbol{b}+\Delta\boldsymbol{b} - \sum_{i\in\bar{I}}\boldsymbol{g}^i_{k(i)} \in \sum_{i\notin\bar{I}}\left(\operatorname{conv}\left(G^i_{\mathcal{K}(i)}\right) + \operatorname{cone}\left(G^i_{\mathcal{L}(i)}\right)\right), \tag{3.3}$$

where $\bar{I}=\{i\in I||\mathcal{B}(i)|=1\}$ be a set of indices each of which block is a singleton, $\mathcal{B}(i)=\{k(i)\}$ for each $i\in\bar{I}$. Furthermore, we have $|\bar{I}|\geq|I|-m$.

Proof. Due to proposition 3.2, we have

$$\boldsymbol{b}+\Delta\boldsymbol{b} \in \sum_{i\in I}\left(\operatorname{conv}\left(G^i_{\mathcal{K}(i)}\right) + \operatorname{cone}\left(G^i_{\mathcal{L}(i)}\right)\right). \tag{3.4}$$

On the other hand, we have $\mathcal{K}(i)=\{k(i)\}$ and $\mathcal{L}(i)=\emptyset$ for each $i\in\bar{I}$ due to constraint (2.2). It follows that $\boldsymbol{g}^i_{k(i)} = \operatorname{conv}\left(G^i_{\mathcal{K}(i)}\right)+\operatorname{cone}\left(G^i_{\mathcal{L}(i)}\right)$ for each $i\in\bar{I}$. Substituting it to (3.4), we conclude (3.3). Last inequality $|\bar{I}|\geq|I|-m$ follows from lemma 3.3. □

Example 3.5. *Consider an example of (PB) such that $I=\{1,\cdots,5\}$ in the following.*

$$\begin{aligned}
\text{minimize}\quad & \begin{pmatrix}6\\0\end{pmatrix}^\top \boldsymbol{x}^1 + \begin{pmatrix}0\\8\end{pmatrix}^\top \boldsymbol{x}^2 + \begin{pmatrix}4\\0\end{pmatrix}^\top \boldsymbol{x}^3 + \begin{pmatrix}0\\3\end{pmatrix}^\top \boldsymbol{x}^4 + \begin{pmatrix}0\\5\end{pmatrix}^\top \boldsymbol{x}^5\\
\text{subject to}\quad & \begin{pmatrix}6&0\\0&1\end{pmatrix}\boldsymbol{x}^1 + \begin{pmatrix}1&0\\0&8\end{pmatrix}\boldsymbol{x}^2 + \begin{pmatrix}2&0\\0&1\end{pmatrix}\boldsymbol{x}^3 + \begin{pmatrix}1&0\\0&2\end{pmatrix}\boldsymbol{x}^4 + \begin{pmatrix}5&0\\0&4\end{pmatrix}\boldsymbol{x}^5 = \begin{pmatrix}10\\6\end{pmatrix},\\
& \boldsymbol{x}^i\in X^i,\quad i\in I,
\end{aligned}$$

where $X^i=\{(x^i_1,x^i_2)\in\mathbb{R}^2\,|\,0\leq x^i_1\leq 1,\,0\leq x^i_2\leq 1,\,x^i_1+x^i_2\geq 1\}$.

The solution of the problem is $(\boldsymbol{x}^1,\boldsymbol{x}^2,\boldsymbol{x}^3,\boldsymbol{x}^4,\boldsymbol{x}^5) = \left(\begin{pmatrix}1/2\\1\end{pmatrix},\begin{pmatrix}1\\1/2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right)$. It is clear that extreme points of X^i are $\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}$, and $\begin{pmatrix}1\\1\end{pmatrix}$, and X^i has no extreme rays. Let us denote them $\boldsymbol{v}^i_1,\boldsymbol{v}^i_2$, and $\boldsymbol{v}^i_3$, respectively. Therefore, its full master problem (FMPB) becomes

$$\begin{aligned}
\text{minimize}\quad & \begin{pmatrix}6\\0\\6\end{pmatrix}^\top \boldsymbol{\lambda}^1 + \begin{pmatrix}0\\8\\8\end{pmatrix}^\top \boldsymbol{\lambda}^2 + \begin{pmatrix}4\\0\\4\end{pmatrix}^\top \boldsymbol{\lambda}^3 + \begin{pmatrix}0\\3\\3\end{pmatrix}^\top \boldsymbol{\lambda}^4 + \begin{pmatrix}0\\5\\5\end{pmatrix}^\top \boldsymbol{\lambda}^5\\
\text{subject to}\quad & \begin{pmatrix}6&0&6\\0&1&1\end{pmatrix}\boldsymbol{\lambda}^1 + \begin{pmatrix}1&0&1\\0&8&8\end{pmatrix}\boldsymbol{\lambda}^2 + \begin{pmatrix}2&0&2\\0&1&1\end{pmatrix}\boldsymbol{\lambda}^3 + \begin{pmatrix}1&0&1\\0&2&2\end{pmatrix}\boldsymbol{\lambda}^4 + \begin{pmatrix}5&0&5\\0&4&4\end{pmatrix}\boldsymbol{\lambda}^5 = \begin{pmatrix}10\\6\end{pmatrix},\\
& \boldsymbol{1}^\top\boldsymbol{\lambda}^i = 1,\quad i\in I,\\
& \boldsymbol{\lambda}^i\geq\boldsymbol{0},\quad i\in I,
\end{aligned}$$

where $\boldsymbol{\lambda}^i\in\mathbb{R}^3$. Furthermore, $\mathcal{B}=\bigoplus_{i=1}^5\mathcal{B}(i)$, where

$$\mathcal{B}(1)=\{2,3\},\ \mathcal{B}(2)=\{1,3\},\ \mathcal{B}(3)=\{2\},\ \mathcal{B}(4)=\{1\},\ \mathcal{B}(5)=\{1\}, \tag{3.5}$$

is an optimal basis of the (FMPB). Because of lemma 3.3, at least $5-2=3$ blocks become singletons. Indeed, $\mathcal{B}(3), \mathcal{B}(4)$, and $\mathcal{B}(5)$ are singletons. Since X^i has no extreme rays, we have $\mathcal{B}(i)=\mathcal{K}(i)$. Therefore, we have

$$G^1_{\mathcal{K}(1)} = \begin{pmatrix} 0 & 6 \\ 1 & 1 \end{pmatrix}, \quad G^2_{\mathcal{K}(2)} = \begin{pmatrix} 1 & 1 \\ 0 & 8 \end{pmatrix}, \quad \boldsymbol{g}^3_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{g}^4_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{g}^5_1 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}.$$

Due to proposition 3.4, $\begin{pmatrix} 10 \\ 6 \end{pmatrix} + \Delta \boldsymbol{b}$ preserves the optimality of $\mathcal{B}$ as far as

$$\begin{pmatrix} 10 \\ 6 \end{pmatrix} + \Delta \boldsymbol{b} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \end{pmatrix} \in \operatorname{conv} \begin{pmatrix} 0 & 6 \\ 1 & 1 \end{pmatrix} + \operatorname{conv} \begin{pmatrix} 1 & 1 \\ 0 & 8 \end{pmatrix}$$

holds. We can confirm the condition by solving a 4-dimensional linear equation in the following:

$$\begin{pmatrix} 0 & 6 & 1 & 1 \\ 1 & 1 & 0 & 8 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda^1_2 \\ \lambda^1_3 \\ \lambda^2_1 \\ \lambda^2_3 \end{pmatrix} = \begin{pmatrix} 4+\Delta b_1 \\ 5+\Delta b_2 \\ 1 \\ 1 \end{pmatrix}.$$

The linear equation is a compressed version of its original 7-dimensional linear equation

$$\begin{pmatrix} 0 & 6 & 1 & 1 & 0 & 1 & 5 \\ 1 & 1 & 0 & 8 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^1_2 \\ \lambda^1_3 \\ \lambda^2_1 \\ \lambda^2_3 \\ \lambda^3_2 \\ \lambda^4_1 \\ \lambda^5_1 \end{pmatrix} = \begin{pmatrix} 10+\Delta b_1 \\ 6+\Delta b_2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

which is obtained by proposition 3.2. The optimality of $\mathcal{B}$ preserves if the linear equation has a positive solution.

4 Adding or Removing a Block

In this section, we consider the case of adding a new block to or removing an existing block from the original problem (PB), which X has a block-angular structure. Under some mild assumptions, we consider the conditions that preserves the optimality of the basis when we add a new block or remove an existing one. We demonstrate that these conditions are perturbations of the RHS parameters that has been discussed in the previous chapter, under some mild assumptions.

4.1 Adding a New Block

We first consider the case of adding a new block to the original problem (PB), in which X has a block structure. Let us denote new block's index by τ and the new problem by (PB+).

$$\begin{aligned} (PB+) \quad \text{minimize} \quad & \sum_{i \in I \cup \{\tau\}} (\boldsymbol{c}^i)^\top \boldsymbol{x}^i \\ \text{subject to} \quad & \sum_{i \in I \cup \{\tau\}} A^i \boldsymbol{x}^i = \boldsymbol{b}, \\ & \boldsymbol{x}^i \in X^i, \forall i \in I \cup \{\tau\}, \end{aligned}$$

Then, we can describe its full master problem in the following. We denote it by (FMPB+).

$$\begin{aligned}(FMPB+)\quad \text{minimize}\quad & \sum_{i\in I\cup\{\tau\}} (\boldsymbol{f}^i)^\top \boldsymbol{\lambda}^i\\ \text{subject to}\quad & \sum_{i\in I\cup\{\tau\}} G^i\boldsymbol{\lambda}^i = \boldsymbol{b},\\ & (\boldsymbol{\xi}^i)^\top\boldsymbol{\lambda}^{(i)} = 1,\ \forall i\in I\cup\{\tau\},\\ & \boldsymbol{\lambda}^{(i)} \geq 0,\ \forall i\in I\cup\{\tau\}.\end{aligned}$$

where $K_\tau, L_\tau, \boldsymbol{f}^\tau, G^\tau, \boldsymbol{\xi}^\tau, \boldsymbol{\lambda}^\tau$ are defined as the same way for $i=\tau$ in section 2.5. For the index τ and a basis $\mathcal{B}$, we consider the following assumption (A4).

$(A4)$ Lagrangina relaxation subroblem (LR(τ)) has a unique solution for $\boldsymbol{\pi}=\boldsymbol{\pi}^*$ defined in (3.1).

When the assumpition (A4) holds, we denote its unique solution by $\boldsymbol{v}^\tau_{k(\tau)}$.

Proposition 4.1. *Let $\mathcal{B}=\bigoplus_{i\in I}\mathcal{B}(i)$ be an index set of (FMPB), and assumption (A4) holds. Then, two claims (i) and (ii) in the following are equivalent.*

(i) (a) The index set $\bigoplus_{i\in I}\mathcal{B}(i)$ is an optimal basis for (FMPB).

(b) Condition $\boldsymbol{b}-\boldsymbol{g}^\tau_{k(\tau)}\in\sum_{i\in I}\left(\operatorname{conv}\left(G_{\mathcal{K}(i)}\right)+\operatorname{cone}\left(G_{\mathcal{L}(i)}\right)\right)$ holds.

(ii) (a) The index set $\bigoplus_{i\in I\cup\{\tau\}}\mathcal{B}(i)$, where $\mathcal{B}(\tau)=\{k(\tau)\}$, is an optimal basis for (FMPB+).

(b) Condition $\boldsymbol{b}\in\sum_{i\in I}\left(\operatorname{conv}\left(G_{\mathcal{K}(i)}\right)+\operatorname{cone}\left(G_{\mathcal{L}(i)}\right)\right)$ holds.

Proof. We first prove (i) $\Rightarrow$ (ii). In view of theorem 2.4, we have (i)-(a) $\Rightarrow$ (ii)-(b). Therefore, we need to prove (ii)-(a). Due to assumption (A4), we have $G^\tau_{\mathcal{B}(\tau)}=\boldsymbol{g}^\tau_{k(\tau)}$ and $\boldsymbol{\xi}^\tau_{\mathcal{B}(\tau)}=1$. Then, Theorem 2.4 suggests that (ii)-(a) is equivalent to the following two conditions:

$$\begin{pmatrix}\boldsymbol{f}_{\mathcal{N}}\\ \boldsymbol{f}^\tau_{\mathcal{N}(\tau)}\end{pmatrix}^\top-\begin{pmatrix}\boldsymbol{f}_{\mathcal{B}}\\ f^\tau_{k(\tau)}\end{pmatrix}^\top\begin{pmatrix}B & B^\tau\\ \boldsymbol{0}^\top & 1\end{pmatrix}^{-1}\begin{pmatrix}N & N^\tau\\ \boldsymbol{0}^\top & (\boldsymbol{\xi}^\tau_{\mathcal{N}(\tau)})^\top\end{pmatrix}\geq\boldsymbol{0}^\top, \tag{4.1}$$

$$\begin{pmatrix}B & B^\tau\\ \boldsymbol{0}^\top & 1\end{pmatrix}^{-1}\begin{pmatrix}\boldsymbol{b}\\ \boldsymbol{1}_{|I|+1}\end{pmatrix}\geq\boldsymbol{0}, \tag{4.2}$$

where $\boldsymbol{f}_{\mathcal{B}}, B, B^\tau, \boldsymbol{f}_{\mathcal{N}}, N, N^\tau$ are defined as follows:

$$\boldsymbol{f}_{\mathcal{B}}=\begin{pmatrix}\boldsymbol{f}^1_{\mathcal{B}(1)}\\ \vdots\\ \boldsymbol{f}^{|I|}_{\mathcal{B}(|I|)}\end{pmatrix},\ B=\begin{pmatrix}G^1_{\mathcal{B}(1)} & \cdots & G^{|I|}_{\mathcal{B}(|I|)}\\ (\boldsymbol{\xi}^1_{\mathcal{B}(1)})^\top & \cdots & \boldsymbol{0}^\top\\ \vdots & \ddots & \vdots\\ \boldsymbol{0}^\top & \cdots & (\boldsymbol{\xi}^{|I|}_{\mathcal{B}(|I|)})^\top\end{pmatrix},\ B^\tau=\begin{pmatrix}\boldsymbol{g}^\tau_{k(\tau)}\\ 0\\ \vdots\\ 0\end{pmatrix},$$

$$\boldsymbol{f}_{\mathcal{N}}=\begin{pmatrix}\boldsymbol{f}^1_{\mathcal{N}(1)}\\ \vdots\\ \boldsymbol{f}^{|I|}_{\mathcal{N}(|I|)}\end{pmatrix},\ N=\begin{pmatrix}G^1_{\mathcal{N}(1)} & \cdots & G^{|I|}_{\mathcal{N}(|I|)}\\ (\boldsymbol{\xi}^1_{\mathcal{N}(1)})^\top & \cdots & \boldsymbol{0}^\top\\ \vdots & \ddots & \vdots\\ \boldsymbol{0}^\top & \cdots & (\boldsymbol{\xi}^{|I|}_{\mathcal{N}(|I|)})^\top\end{pmatrix},\ N^\tau=\begin{pmatrix}G^\tau_{\mathcal{N}(\tau)}\\ \boldsymbol{0}^\top\\ \vdots\\ \boldsymbol{0}^\top\end{pmatrix}.$$

Condition (4.2) is equivalent to claim (i)-(b). Condition (4.1) is equivalent to

$$\begin{aligned}
&\begin{pmatrix} \boldsymbol{f}_{\mathcal{N}} \\ \boldsymbol{f}^{\tau}_{\mathcal{N}(\tau)} \end{pmatrix}^{\top} - \begin{pmatrix} \boldsymbol{f}_{\mathcal{B}} \\ f^{\tau}_{k(\tau)} \end{pmatrix}^{\top} \begin{pmatrix} B^{-1} & -B^{-1}B^{\tau} \\ \mathbf{0}^{\top} & 1 \end{pmatrix} \begin{pmatrix} N & N^{\tau} \\ \mathbf{0}^{\top} & (\boldsymbol{\xi}^{\tau}_{\mathcal{N}(\tau)})^{\top} \end{pmatrix} \geq \mathbf{0}^{\top} \\
\Leftrightarrow &\begin{pmatrix} \boldsymbol{f}_{\mathcal{N}} \\ \boldsymbol{f}^{\tau}_{\mathcal{N}(\tau)} \end{pmatrix}^{\top} - \begin{pmatrix} \boldsymbol{f}_{\mathcal{B}} \\ f^{\tau}_{k(\tau)} \end{pmatrix}^{\top} \begin{pmatrix} B^{-1}N & B^{-1}N^{\tau} - B^{-1}B^{\tau}(\boldsymbol{\xi}^{\tau}_{\mathcal{N}(\tau)})^{\top} \\ \mathbf{0}^{\top} & (\boldsymbol{\xi}^{\tau}_{\mathcal{N}(\tau)})^{\top} \end{pmatrix} \geq \mathbf{0}^{\top} \\
\Leftrightarrow &\begin{cases} \boldsymbol{f}^{\top}_{\mathcal{N}} - \boldsymbol{f}^{\top}_{\mathcal{B}} B^{-1}N \geq \mathbf{0}^{\top}, \\ (\boldsymbol{f}^{\tau}_{\mathcal{N}(\tau)})^{\top} - \boldsymbol{f}^{\top}_{\mathcal{B}}(B^{-1}N^{\tau} - B^{-1}B^{\tau}(\boldsymbol{\xi}^{\tau}_{\mathcal{N}(\tau)})^{\top}) - f^{\tau}_{k(\tau)}(\boldsymbol{\xi}^{\tau}_{\mathcal{N}(\tau)})^{\top} \geq \mathbf{0}^{\top} \end{cases} \\
\Leftrightarrow &\begin{cases} \boldsymbol{f}^{\top}_{\mathcal{N}} - \begin{pmatrix} \boldsymbol{\pi}^* \\ \boldsymbol{\sigma}^* \end{pmatrix}^{\top} N \geq \mathbf{0}^{\top}, \\ (\boldsymbol{f}^{\tau}_{\mathcal{N}(\tau)})^{\top} - \begin{pmatrix} \boldsymbol{\pi}^* \\ \boldsymbol{\sigma}^* \end{pmatrix}^{\top} N^{\tau} \geq (f^{\tau}_{k(\tau)} - \begin{pmatrix} \boldsymbol{\pi}^* \\ \boldsymbol{\sigma}^* \end{pmatrix}^{\top} B^{\tau})(\boldsymbol{\xi}^{\tau}_{\mathcal{N}(\tau)})^{\top}. \end{cases}
\end{aligned} \tag{4.3}$$

First inequality follows from claim (i)-(a). Second inequality is equivalent to

$$\boldsymbol{f}^{\tau}_{k} - (\boldsymbol{\pi}^*)^{\top}\boldsymbol{g}^{\tau}_{k} \geq \begin{cases} f^{\tau}_{k(\tau)} - (\boldsymbol{\pi}^*)^{\top}\boldsymbol{g}^{\tau}_{k(\tau)}, \ \forall k \in \{1, \cdots, K_{\tau}\}, \\ 0, \ \forall k \in \{K_{\tau}+1, \cdots, K_{\tau}+L_{\tau}\}. \end{cases}$$

It is deduced from the assumption (A4). Therefore, we conclude (i) $\Rightarrow$ (ii).

Next, we will prove (ii) $\Rightarrow$ (i). In view of theorem 2.4, we have (ii)-(a) $\Rightarrow$ (i)-(b). The rest is to prove (i)-(a). Due to theorem 2.4, (i)-(a) is equivalent to

$$\boldsymbol{f}^{\top}_{\mathcal{N}} - \boldsymbol{f}^{\top}_{\mathcal{B}} B^{-1}N \geq \mathbf{0}^{\top}, \quad B^{-1}\begin{pmatrix} \boldsymbol{b} \\ \mathbf{1}_{|I|} \end{pmatrix} \geq \mathbf{0}.$$

We have already shown that (ii)-(a) $\Rightarrow$ $\boldsymbol{f}^{\top}_{\mathcal{N}} - \boldsymbol{f}^{\top}_{\mathcal{B}} B^{-1}N \geq \mathbf{0}^{\top}$ in (4.3). On the other hand, $B^{-1}\begin{pmatrix} \boldsymbol{b} \\ \mathbf{1}_{|I|} \end{pmatrix} \geq \mathbf{0}$ is equivalent to claim (ii)-(b). Therefore, we deduce (ii) $\Rightarrow$ (i). We complete the proof. □

The proposition claims that if the assumption (A4) and condition (i)-(b) hold, the optimality of the basis is preserved with an additional index for the new block τ. In other words, we do not have to generate new columns for existing blocks to obtain the optimal basis of (FMPB+). To confirm condition (i)-(b), we need to solve a $(m+|I|)$-dimensional linear equation. Fortunately, we can apply Proposition 3.4 to condition (i)-(b), where $\Delta\boldsymbol{b} = \boldsymbol{g}^{\tau}_{k(\tau)}$. Therefore, the $(m+|I|)$-dimensional linear equation is equivalent to at most $2m$-dimensional one because at least $|I|-m$ basic variables are fixed by nature. Therefore, the question is to what extent the assumption (A4) holds. We give a positive answer to the question.

Let us consider a parametric minimization problem (Q) as follows.

$$\begin{aligned} (Q) \quad &\text{minimize} \quad \boldsymbol{\theta}^{\top}\boldsymbol{y} \\ &\text{subject to} \quad \boldsymbol{y} \in Y, \end{aligned}$$

where $\boldsymbol{\theta} \in \mathbb{R}^N$ is a parameter, $\boldsymbol{y} \in \mathbb{R}^N$ is a variable, and Y is a polyhedron. By Theorem 2.1, Y can be represented as a convex combination of its extreme points $\{\boldsymbol{w}_t\}_{t\in T}$ plus positive combination of its extreme rays $\{\boldsymbol{s}_u\}_{u\in U}$. Positive combinations of its extreme rays form a cone. We denote the cone by S. For the problem (Q), the following lemma holds.

Lemma 4.2. *Problem (Q) has a unique optimal solution for almost every $\boldsymbol{\theta} \in \Theta \cap B_R(\mathbf{0})$, where Θ be a set of parameters such that (Q) is bounded, and $R > 0$ is a sufficiently large parameter.*

Proof. Let $S^* = \{\boldsymbol{\theta} \in \mathbb{R}^N | \langle \boldsymbol{\theta}, \boldsymbol{y} \rangle \geq 0, \forall \boldsymbol{y} \in S\}$ be a dual cone of S. We can confirm that $\Theta = S^*$ as follows.

$$\boldsymbol{\theta} \in \Theta \Leftrightarrow \langle \boldsymbol{\theta}, \boldsymbol{s}_u \rangle \geq 0, \forall u \in U \Leftrightarrow \boldsymbol{\theta} \in S^*.$$

By definition, Θ is a set of parameters such that (Q) has an optimal solution. Let us divide the set $\Theta = S^*$ into two disjoint subsets; Let Θ_1, Θ_M be sets of parameters such that (Q) has a unique optimal solution, and (Q) has multiple optimal solutions, respectively. By definition, we have

$$\Theta = \Theta_1 \cup \Theta_M,\ \Theta_1 \cap \Theta_M = \emptyset.$$

We have already confirmed that (Q) has an optimal solution for all $\boldsymbol{\theta} \in S^* \cap B_R(\mathbf{0})$. Therefore, to prove the lemma, we need to show that the measure of $\Theta_M \cap B_R(\mathbf{0})$ is negligible in $S^* \cap B_R(\mathbf{0})$. Let m_N be the Lubesgue measure in $\mathbb{R}^N$, it is sufficient that the following two statements hold.

$$m_N(\Theta_M \cap B_R(\mathbf{0})) = 0,\ m_N(S^* \cap B_R(\mathbf{0})) > 0. \tag{4.4}$$

We first show $m_N(\Theta_M \cap B_R(\mathbf{0})) = 0$. If $\boldsymbol{\theta} \in \Theta_M$, there exists a pair of extreme points $(\boldsymbol{w}_{t_1}, \boldsymbol{w}_{t_2})$ in Y such that $\boldsymbol{\theta}^\top \boldsymbol{w}_{t_1} = \boldsymbol{\theta}^\top \boldsymbol{w}_{t_2}$, where $(t_1, t_2) \in T^2$ and $t_1 \neq t_2$. It follows that $\boldsymbol{\theta} \in \ker(\boldsymbol{w}_{t_1} - \boldsymbol{w}_{t_2})$. Therefore, we have

$$\Theta_M \subseteq \bigcup_{t_1 \in T} \bigcup_{t_2 \in T \setminus \{t_1\}} \ker(\boldsymbol{w}_{t_1} - \boldsymbol{w}_{t_2})$$

Since $\dim(\ker(\boldsymbol{w}_{t_1} - \boldsymbol{w}_{t_2})) = N - 1$, we have $m_N(\ker(\boldsymbol{w}_{t_1} - \boldsymbol{w}_{t_2}) \cap B_R(\mathbf{0})) = 0$. It follows that

$$m_N(\Theta_M \cap B_R(\mathbf{0})) \leq \sum_{t_1 \in T} \sum_{t_2 \in T \setminus \{t_1\}} m_N(\ker(\boldsymbol{w}_{t_1} - \boldsymbol{w}_{t_2}) \cap B_R(\mathbf{0})) = 0$$

Next, we are to prove $m_N(S^* \cap B_R(\mathbf{0})) > 0$. If $\dim(\operatorname{Span} S^*) = N$, the statement obviously holds. Assume that $\dim(\operatorname{Span} S^*) \leq N - 1$ to achieve a contradiction. Since $\dim(\operatorname{Span} S^*) \leq N - 1$, we have $\dim((\operatorname{Span} S^*)^\perp) \geq 1$. Then, there exists a non-zero vector $\boldsymbol{d} \in (\operatorname{Span} S^*)^\perp$. By definition, we have $\langle \boldsymbol{d}, \boldsymbol{\theta} \rangle = 0$ for all $\boldsymbol{\theta} \in S^*$. If follows that $\pm\boldsymbol{d} \in (S^*)^*$. Since S is closed and convex, we have $(S^*)^* = S$. Therefore, we have $\pm\boldsymbol{d} \in S$. Since Y is a polyhedron, at least one extreme point $\boldsymbol{w} \in Y$ exists. On the other hand, we have $\boldsymbol{w} \pm \boldsymbol{d} \in Y$, because $\pm\boldsymbol{d} \in S$. It follows that $\boldsymbol{w} = (\boldsymbol{w} + \boldsymbol{d})/2 + (\boldsymbol{w} - \boldsymbol{d})/2$, which contradicts to the extremity of $\boldsymbol{w}$. Therefore, we have $\dim(\operatorname{Span} S^*) = N$. Hence, we conclude that $m_N(S^* \cap B_R(\mathbf{0})) > 0$. □

Corollary 4.3. *If Y is bounded, then problem (Q) has a unique solution for almost every $\boldsymbol{\theta} \in B_R(\mathbf{0})$ where $R > 0$ is a sufficiently large parameter.*

Let us review several types of researches related to lemma 4.2. There are necessary and sufficient conditions for a standard linear optimization problems to have a unique solution in [17]. However, they do not focus on to what extent the conditions are satisfied, which is a central theme of lemma 4.2. The theme has been broadly researched in the community of parametric optimization for more general optimization problems, such as [14][11].

Let us apply lemma 4.2 to block τ, i.e, $\boldsymbol{\theta} = \boldsymbol{c}^\tau - (A^\tau)^\top \boldsymbol{\pi}^*$ and $Y = X^\tau$ with $T = \{1, \cdots, K_\tau\}$ and $U = \{K_\tau + 1, \cdots K_\tau + L_\tau\}$. Then, if the Lagrangian relaxation problem (LR)(τ) is bounded, it has a unique solution in most cases. In case the problem has multiple solutions, lemma 4.2 also suggests that slight perturbation of $\boldsymbol{c}^\tau$ recovers the uniqueness. We can summarize the result in the following.

Theorem 4.4. *Let $\mathcal{B} = \bigoplus_{i \in I} \mathcal{B}(i)$ be an optimal basis of (FMPB), and $\begin{pmatrix} \boldsymbol{\pi}^* \\ \boldsymbol{\sigma}^* \end{pmatrix} \in \mathbb{R}^{m+|I|}$ be its corresponding optimal dual variables defined in (3.1). Then, the following two statements hold.*

(1) *Assumption (A4) holds for almost every $\boldsymbol{c}^\tau$ such that $\min\{(\boldsymbol{c}^\tau - (A^\tau)^\top \boldsymbol{\pi}^*)^\top \boldsymbol{x}^\tau \mid \boldsymbol{x}^\tau \in X^\tau\}$ is bounded.*

(2) *If assumption (A4) holds, following condition is a necessary and sufficient condition for $\bigoplus_{i \in I \cup \{\tau\}} \mathcal{B}(i)$, where $\mathcal{B}(\tau) = \{k(\tau)\}$, to be an optimal basis of (FMPB+).*

$$\boldsymbol{b} - \sum_{i \in \bar{I} \cup \{\tau\}} \boldsymbol{g}^i_{k(i)} \in \sum_{i \notin \bar{I}} \left(\operatorname{conv}\left(G_{\mathcal{K}(i)}\right) + \operatorname{cone}\left(G_{\mathcal{L}(i)}\right)\right),$$

where $\bar{I}$ is the set of indices $i \in I$ such that $\mathcal{B}(i)$ is a singleton.

Example 4.5. *Consider an example of (PB+) such that $I = \{1, \cdots, 5\}$ and $\tau = 6$ in the following.*

$$\begin{aligned} &\text{minimize} && \begin{pmatrix}6\\0\end{pmatrix}^\top \boldsymbol{x}^1 + \begin{pmatrix}0\\8\end{pmatrix}^\top \boldsymbol{x}^2 + \begin{pmatrix}4\\0\end{pmatrix}^\top \boldsymbol{x}^3 + \begin{pmatrix}0\\3\end{pmatrix}^\top \boldsymbol{x}^4 + \begin{pmatrix}0\\5\end{pmatrix}^\top \boldsymbol{x}^5 + \begin{pmatrix}2\\1\end{pmatrix}^\top \boldsymbol{x}^6 \\ &\text{subject to} && \begin{pmatrix}6 & 0\\0 & 1\end{pmatrix}\boldsymbol{x}^1 + \begin{pmatrix}1 & 0\\0 & 8\end{pmatrix}\boldsymbol{x}^2 + \begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}\boldsymbol{x}^3 + \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\boldsymbol{x}^4 + \begin{pmatrix}5 & 0\\0 & 4\end{pmatrix}\boldsymbol{x}^5 + \begin{pmatrix}2 & 0\\1 & 2\end{pmatrix}\boldsymbol{x}^6 = \begin{pmatrix}10\\6\end{pmatrix}, \\ &&& \boldsymbol{x}^i \in X^i, \quad i \in I, \end{aligned}$$

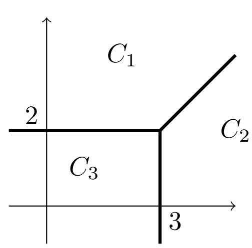
where $X^i = \{(x_1^i, x_2^i) \in \mathbb{R}^2 \,|\, 0 \le x_1^i \le 1,\, 0 \le x_2^i \le 1,\, x_1^i + x_2^i \ge 1\}$.

Its original problem (PB) is Example 3.5. We are to add a new block X^6 with $\boldsymbol{c}^6 = \begin{pmatrix}2\\1\end{pmatrix}$ and $A^6 = \begin{pmatrix}2 & 0\\1 & 2\end{pmatrix}$. The optimal dual solution of the original problem is $\boldsymbol{\pi}^* = \begin{pmatrix}1\\1\end{pmatrix}$. Then, we have $\boldsymbol{c}^6 - (A^6)^\top \boldsymbol{\pi}^* = \begin{pmatrix}-1\\-1\end{pmatrix}$. It follows that (LR(6)) has a unique solution $\boldsymbol{v}_3^6 = \begin{pmatrix}1\\1\end{pmatrix}$, which implies that assumption (A4) holds. We have $\boldsymbol{g}_3^6 = \begin{pmatrix}2\\3\end{pmatrix}$. Therefore, due to theorem 4.4 (2), the index set $\bigoplus_{i=1}^6 \mathcal{B}(i)$, where $\mathcal{B}(6) = \{3\}$, becomes an optimal basis of the new full master problem (FMPB+) because the following condition

$$\begin{pmatrix}10\\6\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}5\\0\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} \in \text{conv}\begin{pmatrix}0 & 6\\1 & 1\end{pmatrix} + \text{conv}\begin{pmatrix}1 & 1\\0 & 8\end{pmatrix}$$

is satisfied.

Let us confirm Theorem 4.4 (1). Let us regard that $\boldsymbol{c}^6$ be a parameter, then $\boldsymbol{c}^6 - (A^6)^\top \boldsymbol{\pi}^* = \begin{pmatrix}c_1^6 - 3\\c_2^6 - 2\end{pmatrix}$. Therefore, the space $\mathbb{R}^2$ can be divided into three regions C_1, C_2, and C_3 of which corresponding optimal solutions are $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}$, and $\begin{pmatrix}1\\1\end{pmatrix}$, respectively.



$$\begin{aligned} C_1 &= \left\{\, (c_1, c_2) \in \mathbb{R}^2 \,\middle|\, c_2 \ge 2,\, c_1 - c_2 \le 1 \,\right\} \\ C_2 &= \left\{\, (c_1, c_2) \in \mathbb{R}^2 \,\middle|\, c_1 \ge 3,\, c_1 - c_2 \ge 1 \,\right\} \\ C_3 &= \left\{\, (c_1, c_2) \in \mathbb{R}^2 \,\middle|\, c_1 \le 3,\, c_2 \le 2 \,\right\} \end{aligned}$$

Figure 1: Parameters and their corresponding solutions

The topology illustrated in Figure 1 clearly suggests that (LR(6)) has a unique solution almost everywhere. In case the parameter $\boldsymbol{c}^6$ is in the intersection of the regions, (LR(6)) has multiple solutions. Therefore, assumption (A4) does not hold. However, such a case seldom happens.

4.2 Removing an Existing Block

In this subsection, we consider the case of removing an existing block from the original problem (PB). Let us denote the index of the existing block by γ and the new problem by (PB-).

$$\begin{aligned} (PB-) \quad &\text{minimize} && \sum_{i \in I \setminus \{\gamma\}} (\boldsymbol{c}^i)^\top \boldsymbol{x}^i \\ &\text{subject to} && \sum_{i \in I \setminus \{\gamma\}} A^i \boldsymbol{x}^i = \boldsymbol{b}, \\ &&& \boldsymbol{x}^i \in X^i, \forall i \in I \setminus \{\gamma\}, \end{aligned}$$

Then, we can describe the full master problem without the block γ in the following. We denote it by (FMPB-).

$$
\begin{aligned}
(FMPB-) \quad \text{minimize} \quad & \sum_{i \in I \setminus \{\gamma\}} (\boldsymbol{f}^i)^\top \boldsymbol{\lambda}^i \\
\text{subject to} \quad & \sum_{i \in I \setminus \{\gamma\}} G^i \boldsymbol{\lambda}^i = \boldsymbol{b}, \\
& (\boldsymbol{\xi}^i)^\top \boldsymbol{\lambda}^{(i)} = 1, \, \forall i \in I \setminus \{\gamma\}, \\
& \boldsymbol{\lambda}^{(i)} \geq 0, \, \forall i \in I \setminus \{\gamma\}.
\end{aligned}
$$

Theorem 4.6. *Let $\mathcal{B} = \bigoplus_{i \in I} \mathcal{B}(i)$ be an optimal basis of (FMPB), and γ be an index such that $\mathcal{B}(\gamma)$ is a singleton. Then, the following condition is necessary and sufficient for $\bigoplus_{i \in I \setminus \{\gamma\}} \mathcal{B}(i)$ to be an optimal basis of (FMPB-)*

$$
\boldsymbol{b} - \sum_{i \in \bar{I} \setminus \{\gamma\}} \boldsymbol{g}^i_{k(i)} \in \sum_{i \notin \bar{I}} \left(\text{conv} \left(G_{\mathcal{K}(i)} \right) + \text{cone} \left(G_{\mathcal{L}(i)} \right) \right), \tag{4.5}
$$

where $\bar{I}$ is the set of indices $i \in I$ such that $\mathcal{B}(i) = \{k(i)\}$ is a singleton.

Proof. To apply proposition 4.1, let us regard (FMPB-), (FMPB), and γ be (FMPB), (FMPB+), and τ, respectively. Then, assumption (A4) holds, because $\mathcal{B}(\gamma)$ is a singleton. In our context, claims in proposition 4.1 become as follows.

(*i*) (a) The index set $\bigoplus_{i \in I \setminus \{\gamma\}} \mathcal{B}(i)$ is an optimal basis for (FMPB-).

(b) Condition $\boldsymbol{b} \in \sum_{i \in I} \left(\text{conv} \left(G_{\mathcal{K}(i)} \right) + \text{cone} \left(G_{\mathcal{L}(i)} \right) \right)$ holds.

(*ii*) (a) The index set $\bigoplus_{i \in I} \mathcal{B}(i)$, where $\mathcal{B}(\gamma) = \{k(\gamma)\}$, is an optimal basis for (FMPB).

(b) Condition $\boldsymbol{b} \in \sum_{i \in I \setminus \{\gamma\}} \left(\text{conv} \left(G_{\mathcal{K}(i)} \right) + \text{cone} \left(G_{\mathcal{L}(i)} \right) \right)$ holds.

Since (ii)-(a) is assumed, (i)-(b) is also assumed. Therefore, (ii)-(b) is a necessary and sufficient condition for (i)-(a). Furthermore, (ii)-(b) is equivalent to the proposed condition (4.5). We complete the proof. □

Due to proposition 3.4, we have $|\bar{I}| \geq |I| - m$. Therefore, if $|I|$ is large, assumption (A4) holds for many blocks.

Example 4.7. *Consider an example of (PB-) such that $I = \{1, \cdots, 5\}$ and $\gamma = 3$ in the following.*

$$
\begin{aligned}
\text{minimize} \quad & \begin{pmatrix} 6 \\ 0 \end{pmatrix}^\top \boldsymbol{x}^1 + \begin{pmatrix} 0 \\ 8 \end{pmatrix}^\top \boldsymbol{x}^2 + \begin{pmatrix} 0 \\ 3 \end{pmatrix}^\top \boldsymbol{x}^4 + \begin{pmatrix} 0 \\ 5 \end{pmatrix}^\top \boldsymbol{x}^5 \\
\text{subject to} \quad & \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{x}^1 + \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} \boldsymbol{x}^2 + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \boldsymbol{x}^4 + \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix} \boldsymbol{x}^5 = \begin{pmatrix} 10 \\ 6 \end{pmatrix}, \\
& \boldsymbol{x}^i \in X^i, \quad i \in I \setminus \{\gamma\},
\end{aligned}
$$

where $X^i = \{(x^i_1, x^i_2) \in \mathbb{R}^2 \,|\, 0 \leq x^i_1 \leq 1, \, 0 \leq x^i_2 \leq 1, \, x^i_1 + x^i_2 \geq 1\}$.

Its original problem (PB) is Example 3.5, again. We are to remove an existing block $\gamma = 3$. Since $\mathcal{B}(3) = \{2\}$ is a singleton, assumption (A4) holds. We also have

$$
\begin{pmatrix} 10 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \in \text{conv} \begin{pmatrix} 0 & 6 \\ 1 & 1 \end{pmatrix} + \text{conv} \begin{pmatrix} 1 & 1 \\ 0 & 8 \end{pmatrix}.
$$

Due to theorem 4.6, the index set $\bigoplus_{i \in \{1,2,4,5\}} \mathcal{B}(i)$ is an optimal basis of the new full master problem (FMPB-).

Let us consider another case; we will remove another block $\gamma = 5$ from the original problem. Assumption (A4) also holds, because $\mathcal{B}(5) = \{1\}$ is a singleton. On the other hand, we have

$$\begin{pmatrix}10\\6\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}9\\5\end{pmatrix} \notin \operatorname{conv}\begin{pmatrix}0 & 6\\1 & 1\end{pmatrix} + \operatorname{conv}\begin{pmatrix}1 & 1\\0 & 8\end{pmatrix}.$$

Therefore, in view of theorem 4.6, the index set $\bigoplus_{i=1}^{4} \mathcal{B}(i)$ is not an optimal basis of the new full master problem (FMPB-) where $\gamma = 5$.

5 Dual Interpretation

In this section, we observe the main results proved in the previous sections from a dual point of view. Let $\mathcal{B}$ be an optimal basis of (PB), and $\boldsymbol{\pi}^*$ be its corresponding optimal dual solution. By definition, the optimality of (DB) is given by $\boldsymbol{b} \in \partial(-\sum_{i\in I} d_i)(\boldsymbol{\pi^*})$. Let us visualize it. Note that $\sum_{i\in I} d_i(\boldsymbol{\pi})$ is a piecewise linear function. Figure 2 illustrates the optimality condition. It is clear that perturbating $\boldsymbol{b}$ to $\boldsymbol{b} + \Delta\boldsymbol{b}$ preserves the optimality as long as $-\sum_{i\in I} d_i(\boldsymbol{\pi})$ is located on the line of which slope is $\boldsymbol{b} + \Delta\boldsymbol{b}$.

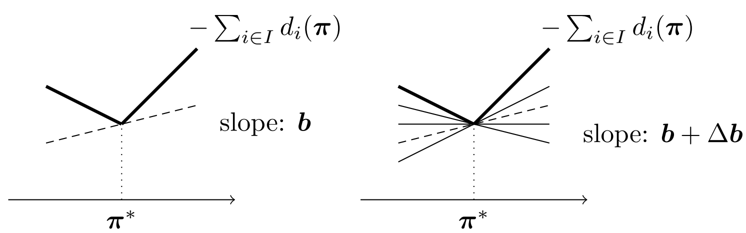


Figure 2: Dual optimality in perturbating $\boldsymbol{b}$ to $\boldsymbol{b} + \Delta\boldsymbol{b}$.

This visualization is also useful to understand Proposition 4.1. Function $d_\tau(\boldsymbol{\pi})$ is equal to a line of which slope is $-\boldsymbol{g}^{\tau}_{k(\tau)}$ in the neighborhood of $\boldsymbol{\pi}^*$ under assumption (A4). Therefore, whether $\boldsymbol{\pi}^*$ maximize $\sum_{i\in I\cup\{\tau\}} d_i(\boldsymbol{\pi}) + \boldsymbol{b}^\top\boldsymbol{\pi}$ or not is judged by whether $-\sum_{i\in I} d_i(\boldsymbol{\pi})$ is located on the line of which slope is $\boldsymbol{b} - \boldsymbol{g}^{\tau}_{k(\tau)}$ or not. See figure 3.

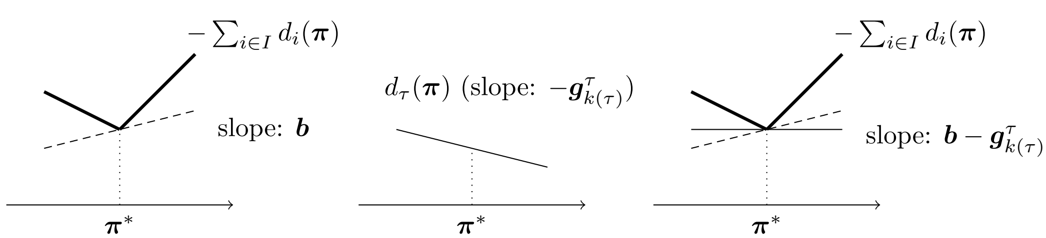


Figure 3: Dual optimality in adding a new block τ.

Similarly, whether $\boldsymbol{\pi}^*$ maximize $\sum_{i\in I\setminus\{\gamma\}} d_i(\boldsymbol{\pi}) + \boldsymbol{b}^\top\boldsymbol{\pi}$ or not is judged by whether $-\sum_{i\in I} d_i(\boldsymbol{\pi})$ is located on the line of which slope is $\boldsymbol{b} + \boldsymbol{g}^{\gamma}_{k(\gamma)}$ or not. See figure 4.

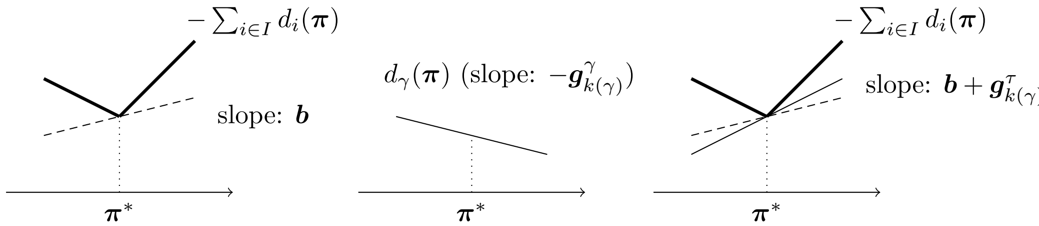


Figure 4: Dual optimality in removing an existing block γ.

In understanding these visualizations, we stress that $\partial(-\sum_{i\in I} d_i)(\boldsymbol{\pi^*})$ is not always equivalent to $\sum_{i\in I}\left(\text{conv}\left(G^i_{\mathcal{K}(i)}\right)+\text{cone}\left(G^i_{\mathcal{L}(i)}\right)\right)$. All we can prove is that $\sum_{i\in I}\left(\text{conv}\left(G^i_{\mathcal{K}(i)}\right)+\text{cone}\left(G^i_{\mathcal{L}(i)}\right)\right)\subseteq \partial(-\sum_{i\in I} d_i)(\boldsymbol{\pi^*})$.

6 Conclusion

In this paper, we have considered several perturbations in which no new column generation is necessary in the context of Dantzig-Wolfe decomposition. Our first focus is on the perturbation of the RHS parameter. When the RHS parameter is perturbed, solving a linear equation can judge whether new column generations are necessary. The claim itself is a direct application of well-known results. However, when the original problem has a block-angular structure, we have shown that we can compress the linear equation.

Then, we consider other perturbations: adding a new block and removing an existing block. We have demonstrated that they can be seen as special cases of the RHS parameter's perturbation under the condition that their corresponding Lagrangian relaxation subproblems have unique solutions. The condition is very mild. In adding a new block, it is satisfied for almost all objective parameters. In removing an existing block, it holds for at least $|I|-m$ blocks.

Although we have focused on linear optimization problems, the most attractive application of Dantzig-Wolfe decomposition is mixed integer linear optimization problems. We must combine our results into the branch-and-price framework or agrangian primal heuristics for further research to recover the feasibility problem.

References

[1] Dimitri P Bertsekas. *Nonlinear programming.* Athena Scientific Belmont, 1999.

[2] Joseph-Frédéric Bonnans, Jean Charles Gilbert, Claude Lemaréchal, and Claudia Sagastizábal. *Numerical Optimization: Theoretical and Practical Aspects.* Springer Science & Business Media, 2006.

[3] Vašek Chvátal. *Linear programming.* Macmillan, 1983.

[4] Antonio J Conejo, Enrique Castillo, Roberto Minguez, and Raquel Garcia-Bertrand. *Decomposition techniques in mathematical programming: engineering and science applications.* Springer Science & Business Media, 2006.

[5] Michele Conforti, Gérard Cornuéjols, Giacomo Zambelli, Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli. *Integer programming.* Springer, 2014.

[6] George B Dantzig. *Linear programming and extensions.* Princeton university press, 1963.

[7] George B Dantzig and Philip Wolfe. Decomposition principle for linear programs. *Operations research*, 8(1):101–111, 1960.

[8] Guy Desaulniers, Jacques Desrosiers, and Marius M Solomon. *Column generation*, volume 5. Springer Science & Business Media, 2006.

[9] J-L Goffin, Jacek Gondzio, Robert Sarkissian, and J-P Vial. Solving nonlinear multicommodity flow problems by the analytic center cutting plane method. *Mathematical programming*, 76(1):131–154, 1997.

[10] D Goldfarb and M.J. Todd. Chapter ii linear programming. *Handbooks in Operations Research and Management Science*, 1:141–170, 1989.

[11] A Ioffe and RE Lucchetti. Typical convex program is very well posed. *Mathematical programming*, 104(2):483–499, 2005.

[12] Kim L Jones, Irvin J Lustig, Judith M Farvolden, and Warren B Powell. Multicommodity network flows: The impact of formulation on decomposition. *Mathematical Programming*, 62:95–117, 1993.

[13] James E Kelley, Jr. The cutting-plane method for solving convex programs. *Journal of the society for Industrial and Applied Mathematics*, 8(4):703–712, 1960.

[14] Petar S Kenderov. Most of the optimization problems have unique solution. In *Parametric Optimization and Approximation: Conference Held at the Mathematisches Forschungsinstitut, Oberwolfach, October 16–22, 1983*, pages 203–216. Springer, 1985.

[15] Marco E Lübbecke. Column generation. *Wiley encyclopedia of operations research and management science*, 17:18–19, 2010.

[16] Marco E Lübbecke and Jacques Desrosiers. Selected topics in column generation. *Operations research*, 53(6):1007–1023, 2005.

[17] Olvi Mangasarian. Uniqueness of solution in linear programming. Technical report, University of Wisconsin-Madison Department of Computer Sciences, 1978.

[18] George L Nemhauser and Laurence A Wolsey. *Integer and combinatorial optimization*. John Wiley & Sons, 1988.

[19] Yurii Nesterov. Nonsmooth convex optimization. In *Lectures on Convex Optimization*, pages 139–240. Springer, 2018.

[20] James Richard Tebboth. A computational study of dantzig-wolfe decomposition. *University of Buckingham*, 2001.

[21] Laurence A Wolsey. *Integer programming*. John Wiley & Sons, 2020.

[22] Lin Zhou. A simple proof of the shapley-folkman theorem. *Economic Theory*, 3:371–372, 1993.