A Facial Reduction Algorithm for Standard Spectrahedra

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Abstract

Facial reduction is a pre-processing method aimed at reformulating a problem to ensure strict feasibility. The importance of constructing a robust model is widely recognized in the literature, and facial reduction has emerged an attractive approach for achieving robustness. In this note, we outline a facial reduction algorithm for a standard spectrahedra, the intersection of the cone of positive semidefinite matrices and a set of linear equalities. We address an optimization problem that serves as an intermediate step in the facial reduction process. To tackle this optimization problem, we employ an interior point method that uses the Gauss-Newton method.

1 Introduction

Facial reduction (**FR**) is a pre-processing method aimed at reformulating a problem to ensure strict feasibility [1,2]. The importance of constructing a robust model is widely recognized in the literature, and facial reduction has emerged an attractive approach for achieving robustness. In this note, we address an optimization problem that serves as an intermediate step in the facial reduction process. To tackle this optimization problem, we employ an interior point method that uses the Gauss-Newton method inspired by [8,12]. The Gauss-Newton method (e.g., see [5, Chapter 10]) offers a powerful framework for satisfying the first-order optimality conditions of the primal-dual pair. This approach interprets the optimality conditions of the problem as solving an overdetermined nonlinear system. Successful applications of this type can be found in [8,12,13].

Notation We let $\mathbb{R}^n, \mathbb{C}^n$ denote the set of vectors with n coordinates in real and complex vector spaces, respectively. We let $\mathbb{S}^n$ and $\mathbb{H}^n$ denote the space of n-be-n symmetric matrices and n-by-n Hermitian matrices, respectively. Given $x \in \mathbb{C}^n$, $X \in \mathbb{H}^n$, we use x^*, X^* to denote the conjugate transpose of x and X. We let $\mathbb{S}^n_+$ denote the set of positive semidefinite matrices in $\mathbb{S}^n$, i.e., $\mathbb{S}^n_+ = \{X \in \mathbb{S}^n : y^T X y \geq 0,\ \forall y \in \mathbb{R}^n\}$. Similarly, we use $\mathbb{H}^n_+$ to denote the set of positive semidefinite matrices in $\mathbb{H}^n$, i.e., $\mathbb{H}^n_+ = \{X \in \mathbb{H}^n : y^* X y \geq 0,\ \forall y \in \mathbb{C}^n\}$. The set of positive definite matrices are defined by $\mathbb{S}^n_{++} = \{X \in \mathbb{S}^n : y^T X y > 0,\ \forall y \in \mathbb{R}^n \setminus \{0\}\}$ and $\mathbb{H}^n_{++} = \{X \in \mathbb{H}^n : y^* X y > 0,\ \forall y \in \mathbb{C}^n \setminus \{0\}\}$. We often use the partial order notation $X \succeq 0$ ($X \succ 0$, respectively) to indicate that X is positive semidefinite (positive definite, respectively). We use $\langle \cdot, \cdot \rangle$ to denote the usual inner product in the spaces $\mathbb{R}^n, \mathbb{C}^n, \mathbb{S}^n$ and $\mathbb{H}^n$. I_n is used as the n-by-n identity matrix, and e_i is the i-th columns of the identity matrix. Given a matrix X, range(X) and null(X) denote the range of X and null-space of X, respectively. Given a linear map T, we let T^* denote the adjoint of T.

2 Facial Reduction Process

Facial reduction, first appeared in [1, 2], is a proprocessing method aimed at reformulating a problem to ensure strict feasibility. In this note, we focus on the standard feasible region in $\mathbb{H}^n$

(or $\mathbb{S}^n$) that arises in the class of semidefinite programs:

$$\mathcal{F} := \{X : \mathcal{A}(X) = b, X \succeq 0\}, \tag{2.1}$$

where $\mathcal{A}$ is a linear map from $\mathbb{H}^n$ to $\mathbb{R}^m$ (or $\mathbb{S}^n \to \mathbb{R}^m$) and $b \in \mathbb{R}^m$. The i-th equality of $\mathcal{A}(X) = b$ is often written using the usual inner product form $\langle A_i, X\rangle = b_i$.

The set $\mathcal{F}$ is said to hold *strict feasibility* if there exists a positive *definite* $\hat{X} \in \mathcal{F}$. If $\mathcal{F}$ fails strict feasibility, there exists a matrix V, with fewer columns than n, such that

$$\forall X \in \mathcal{F},\ X = VRV^*,\ \text{for some } R \succeq 0. \tag{2.2}$$

The role of V can be viewed as confining the range of feasible points in $\mathcal{F}$, and some researchers refer to such V a facial range vector. If V is identified in a way that $\bar{X} \in \mathcal{F}$ and $\bar{X} = VRV^*$ with $R \succ 0$, then V is called a *minimal* facial range vector. For detailed derivation of V, we lead the readers to [6,9]. Finding the matrix V can often be challenging, hence we typically rely on an auxiliary lemma. Lemma 2.1 below is fundamental to the construction of the **FR** process. There is a large number of literature that uses Lemma 2.1 analytically. For example, Lemma 2.1 is used for certifying absence of strict feasibility, e.g., [3, 18]. Moreover, Lemma 2.1 is often used for constructing an efficient algorithm for solving some classes of combinatorial problems, e.g., [4, 7, 14].

Lemma 2.1. *(Theorem of the alternative) [6, Theorem 3.1.3] For the feasible constraint system in (2.1), exactly one of the following statements holds:*

1. *There exists $X \succ 0$ such that $\mathcal{A}(X) = b$.*
2. *There exists y such that*

$$0 \neq \mathcal{A}^*(y) \succeq 0\ ,\ \langle b, y\rangle = 0. \tag{2.3}$$

The matrix $\mathcal{A}^*(y)$ in (2.3) is called an *exposing vector*. One can show that

$$\mathcal{A}^*(y)X = 0,\ \forall X \in \mathcal{F},$$

i.e., every feasible solution lies in the orthogonal complement of $\mathcal{A}^*(y)$. Consequently, a facial range vector V in (2.2) can be constructed by choosing a V such that $\mathcal{A}^*(y)V = 0$. Our goal is to find y that satisfies (2.3) so that we can find a facial range vector.

An explicit solution y to (2.3) is often unavailable for an arbitrary problem. In this case, finding a solution y to (2.3) can be passed to a numerical solver in the form of

$$p^* = \min_y\{\langle b, y\rangle : \mathcal{A}^*(y) \succeq 0,\ \text{trace}(\mathcal{A}^*(y)) = 1\}. \tag{2.4}$$

The constraint $\text{trace}(\mathcal{A}^*(y)) = 1$ in (2.4) is imposed to avoid having the trivial solution $y^* = 0$. Here, we use a different equality to allow some flexibilities. Let $P \succ 0$ and $\alpha > 0$. Consider the following problem motivated by (2.4):

$$(\mathcal{P}_{\mathbf{FR}})\quad p^*_{\mathbf{FR}} := \min_y \{\langle b, y\rangle\ :\ \mathcal{A}^*y \succeq 0,\ \langle P, \mathcal{A}^*y\rangle = \alpha\}. \tag{2.5}$$

If $P = I$ and $\alpha = 1$, (2.5) is identical to (2.4). We note that the dual $(\mathcal{D}_{\mathbf{FR}})$ of $(\mathcal{P}_{\mathbf{FR}})$ is

$$(\mathcal{D}_{\mathbf{FR}})\quad d^*_{\mathbf{FR}} := \max_{\lambda, W} \{\alpha\lambda\ :\ \mathcal{A}(W) + \lambda\mathcal{A}(P) = b,\ W \succeq 0\}. \tag{2.6}$$

Note that if $\mathcal{F}$ has a feasible point $\bar{X}$, the dual (2.6) is also feasible with $(\bar{\lambda}, \bar{W}) = (0, \bar{X})$. Let y^* be an optimal solution to (2.6), i.e., $p^*_{\mathbf{FR}} = \langle b, y^*\rangle$. Given a feasible $\mathcal{F}$, the optimal value $p^*_{\mathbf{FR}}$ is nonnegative. If $p^*_{\mathbf{FR}} = 0$, then solution y^* is a solution to (2.3). If $p^*_{\mathbf{FR}} < 0$, then $\mathcal{F}$ is infeasible; and the conditions $\mathcal{A}^*y^* \succeq 0$ and $\langle b, y^*\rangle < 0$ serve as certificate if infeasibility.

We now list some properties of the primal-dual pair $(\mathcal{P}_{\mathbf{FR}})$ and $(\mathcal{D}_{\mathbf{FR}})$.

Lemma 2.2. *Let $\mathcal{F}$ be a feasible system. Then the following holds.*

1. *There exists a positive definite P such that the problem (2.6) possesses a Slater point.*
2. *If the problem data $\{A_i\}_{i=1}^m$ contains a positive definite or negative definite matrix, the Slater condition holds for $(\mathcal{P}_{\boldsymbol{FR}})$.*
3. *Let (λ^*, W^*) be an optimal solution to (2.6) with $d^*_{\boldsymbol{FR}} > 0$. Then $W^* + \lambda^* P$ is a Slater point of $\mathcal{F}$.*

Proof. 1. Let $\bar{W} \succeq 0$ be any feasible point to $\mathcal{F}$. Let

$$\bar{W} = \begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} D_V & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V & U \end{bmatrix}^T$$

be the spectral decomposition of $\bar{W}$. We let $\bar{T} = U D_U U^T \succeq 0$, where $D_U \succ 0$. Consider $\hat{W} = 2\bar{W} + 2\bar{T}$ and $P = 2\bar{T} + \bar{W}$. Clearly, both $\hat{W}$ and P are positive definite. We note that

$$\mathcal{A}(\hat{W}) - 1 \cdot \mathcal{A}(P) = \mathcal{A}(\hat{W} - P) = \mathcal{A}(\bar{W}) = b.$$

Hence, (2.6) contains a strictly feasible point.

2. Without loss of generality, let A_1 be positive definite. Then choosing $\bar{y} = (\alpha/\langle A_1, P\rangle, 0, \ldots, 0)^T$ gives the desired property. If A_1 is negative definite, we may replace $\langle A_1, X\rangle = b_1$ with $\langle -A_1, X\rangle = -b_1$ and apply the above.

3. Let (λ^*, W^*) be an optimal solution to (2.6) with $d^*_{\mathbf{FR}} > 0$. Then

$$\mathcal{A}(W^*) + \lambda^* \mathcal{A}(P) = \mathcal{A}(W^*) + \lambda^* \mathcal{A}(P) = b \text{ and } W^* + \lambda^* P \succ 0.$$

□

Item 1 of Lemma 2.2 is particularly useful since the purpose of **FR** is to obtain an exposing vector for $\mathcal{F}$. When the dual $(\mathcal{D}_{\mathbf{FR}})$ has a Slater point, the primal $(\mathcal{P}_{\mathbf{FR}})$ must attain its optimal value. Hence, by strong duality, Lemma 2.2 guarantees the attainability of the exposing vector, which is the primary objective of the **FR** process. It is noteworthy that for a structured problem, a feasible solution to $\mathcal{F}$ is often available. Hence, we may choose P in (2.5) as constructed in the proof of Lemma 2.2.

Regarding item 2 of Lemma 2.2, some classes of problems permit positive definite data matrices. For example, the feasible solution that arises in the quantum information theory frequently involve the unit trace constraint, e.g., see [8]. Moreover, it is shown that **SDP** relaxations of quadratic assignment problem and protein side-chain positioning problem incorporate a type of trace constraint, see [4, 7]. We also note that if there exists $\bar{y}$ satisfying $\mathcal{A}^*(\bar{y}) \succ 0$, (2.5) holds strict feasibility and item 2 of Lemma 2.2 reduces to a special case. Items 1 and 2 of Lemma 2.2 together endow strict feasibility of both primal-dual pair (2.5) and (2.6), which is crucial for the performance of interior point methods.

3 Algorithm

3.1 Optimality Conditions

In this section we consider the optimality conditions of the primal-dual pair (2.5) and (2.6) to be used for construction an interior point method. Currently, there is a **FR** algorithm in the

literature that uses the self-dual embedding [17]. In this note, we propose an alternative approach. We propose an algorithm motivated by the approach in [8] that arises from the interior point method for solving an application in the quantum information theory. In nutshell, [8] solves a convex optimization problem with a nonlinear objective over a standard spectraheron via a tailored interior point method. Here, we construct a tailored interior point method for (2.5).

We employ a slack variable $S \in \mathbb{S}^n_+$ to (2.5) and write the perturbed optimality conditions for the primal-dual pair ($\mathcal{P}_{\mathbf{FR}}$) and ($\mathcal{D}_{\mathbf{FR}}$). Let $W, S \succ 0, \mu > 0$ and consider

$$\begin{array}{lll} \text{dual feasibility} & \mathcal{A}(W) + \lambda \mathcal{A}(P) - b & = 0 \\ \text{primal feasibility 1} & \mathcal{A}^* y - S & = 0 \\ \text{primal feasibility 2} & \text{trace}(\mathcal{A}^* y) - \alpha & = 0 \\ \text{perturbed complementarity} & WS - \mu I & = 0. \end{array} \tag{3.1}$$

One may substitute primal feasibility 1 ($\mathcal{A}^*(y) - S = 0$) in (3.1) to the complementary slackness condition.

$$0 = \begin{bmatrix} \mathcal{A}(W) + \lambda \mathcal{A}(P) - b \\ \text{trace}(\mathcal{A}^* y) - \alpha \\ W(\mathcal{A}^* y) - \mu I \end{bmatrix}$$

Depending on the linear map $\mathcal{A}$, we may not be able to find y satisfying $\mathcal{A}^* y \succ 0$. In this case we would be impossible to maintain $\mathcal{A}^* y \succ 0$ in the complementary block $W(\mathcal{A}^* y) - \mu I$ during the execution of the interior point method. Thus we work with (3.1) directly without substituting primal feasibility 1 ($\mathcal{A}^*(y) - S = 0$) in (3.1) to the complementarity condition. Consequently, we are interested solving the following overdetermined system of equations:

$$0 = \begin{bmatrix} F_\mu^{\text{df}}(W, \lambda) \\ F_\mu^{\text{pf}_1}(y) \\ F_\mu^{\text{pf}_2}(y, S) \\ F_\mu^{\text{cs}}(W, S) \end{bmatrix} = \begin{array}{ll} \mathcal{A}(W) + \lambda \mathcal{A}(P) - b & \text{(dual feasibility)} \\ \text{trace}(\mathcal{A}^* y) - \alpha & \text{(primal feasibility 1)} \\ \mathcal{A}^* y - S & \text{(primal feasibility 2)} \\ WS - \mu I & \text{(pertuberd complementarity)} \end{array}$$

3.2 Projected Gauss-Newton Direction

Let $\hat{W}, \hat{\lambda}$ be a pair of particular solutions to the dual feasibility equation, i.e., $\mathcal{A}(\hat{W}) + \hat{\lambda}\mathcal{A}(P) = b$. We replace the dual feasibility $F_\mu^{\text{df}}(W, \lambda) = \mathcal{A}(W) + \lambda \mathcal{A}(P) - b = 0$ using the null-space representation

$$\begin{bmatrix} \mathcal{N}_W \\ \mathcal{N}_\lambda \end{bmatrix} v + \begin{bmatrix} \hat{W} \\ \hat{\lambda} \end{bmatrix} - \begin{bmatrix} W \\ \lambda \end{bmatrix} = \begin{bmatrix} F_\mu^{d,1} \\ F_\mu^{d,2} \end{bmatrix} \begin{array}{l} \text{(dual feasibility 1)} \\ \text{(dual feasibility 2).} \end{array}$$

Here, $\mathcal{N}_W(v) = \sum_i N_W^i v_i$ and $\mathcal{N}_\lambda(v) = N_\lambda^1 v$ for some $\{N_W^i\}$ and N_λ^1. Similarly, let $\hat{y}, \hat{S}$ be a pair of particular solutions to the primal feasibility equations $F_\mu^{\text{pf}_1}(y) = 0$ and $F_\mu^{\text{pf}_2}(y, S) = 0$, i.e., $\text{trace}(\mathcal{A}^* \hat{y}) = \alpha$ and $\mathcal{A}^* \hat{y} - \hat{S} = 0$. We replace the primal feasibilities using the null-space representation

$$\begin{bmatrix} F_\mu^{p,1} \\ F_\mu^{p,2} \end{bmatrix} = \begin{bmatrix} \text{trace}(\mathcal{A}^* y) - \alpha \\ \mathcal{A}^* y - S \end{bmatrix} = \begin{bmatrix} \mathcal{N}_y \\ \mathcal{N}_S \end{bmatrix} u + \begin{bmatrix} \hat{y} \\ \hat{S} \end{bmatrix} - \begin{bmatrix} y \\ S \end{bmatrix} \begin{array}{l} \text{(primal feasibility 1)} \\ \text{(primal feasibility 2).} \end{array}$$

Finally, the perturbed optimality conditions then become

$$0 = F_\mu(v, W, \lambda, u, y, S) = \begin{bmatrix} F_\mu^{d,1}(v, W) \\ F_\mu^{d,2}(v, \lambda) \\ F_\mu^{p,1}(u, y) \\ F_\mu^{p,2}(u, S) \\ F_\mu^{c}(W, S) \end{bmatrix} = \begin{bmatrix} \mathcal{N}_W v + \hat{W} - W \\ \mathcal{N}_\lambda v + \hat{\lambda} - \lambda \\ \mathcal{N}_y u + \hat{y} - y \\ \mathcal{N}_S u + \hat{S} - S \\ WS - \mu I \end{bmatrix}. \tag{3.2}$$

We solve (3.2) by viewing the optimality conditions as an over-determined system equalities. Note that the over-determinedness originates from the complementarity equation. Hence we solve (3.2) via Gauss-Newton method while driving $\mu \downarrow 0$. For detailed discussion of the usage of the Gauss-Newton method, we lead the readers to [9, Section 2.4.2].

The Gauss-Newton method requires linearization step. The linearization of (3.2) yields

$$F'_\mu \Delta d = \begin{bmatrix} \mathcal{N}_W \Delta v - \Delta W \\ \mathcal{N}_\lambda \Delta v - \Delta \lambda \\ \mathcal{N}_y \Delta u - \Delta y \\ \mathcal{N}_S \Delta u - \Delta S \\ (\Delta W)S + W(\Delta S) \end{bmatrix} = - \begin{bmatrix} F_\mu^{d,1} \\ F_\mu^{d,2} \\ F_\mu^{p,1} \\ F_\mu^{p,2} \\ F_\mu^c \end{bmatrix}, \tag{3.3}$$

where $\Delta d = (\Delta v, \Delta W, \Delta \lambda, \Delta u, \Delta y, \Delta S)$. We note that the first four blocks of (3.3) produce

$$\Delta W = \mathcal{N}_W \Delta v + F_\mu^{d,1}, \ \Delta \lambda = \mathcal{N}_\lambda \Delta v + F_\mu^{d,2}, \ \Delta y = \mathcal{N}_y \Delta u + F_\mu^{p,1}, \ \Delta S = \mathcal{N}_S \Delta u + F_\mu^{p,2}. \tag{3.4}$$

We use $(\Delta W, \Delta S)$ to make substitutions into the last block of (3.3)

$$\begin{aligned} -F_\mu^c &= (\Delta W)S + W(\Delta S) \\ &= (\mathcal{N}_W \Delta v + F_\mu^{d,1})S + W(\mathcal{N}_S \Delta u + F_\mu^{p,2}) \\ &= (\mathcal{N}_W \Delta v)S + (F_\mu^{d,1})S + W(\mathcal{N}_S \Delta u) + W F_\mu^{p,2}. \end{aligned}$$

Rearranging the terms, we solve the following system

$$(\mathcal{N}_W \Delta v)S + W(\mathcal{N}_S \Delta u) = -F_\mu^c - (F_\mu^{d,1})S - W(F_\mu^{p,2}). \tag{3.5}$$

We then use $(\Delta v, \Delta u)$, the solution from (3.5), to backsolve and obtain the remaining components of the search direction $(\Delta W, \Delta \lambda, \Delta y, \Delta S)$ via (3.4):

$$\begin{pmatrix} \Delta W \\ \Delta \lambda \\ \Delta y \\ \Delta S \end{pmatrix} = \begin{pmatrix} \mathcal{N}_W \Delta v \\ \mathcal{N}_\lambda \Delta v \\ \mathcal{N}_y \Delta u \\ \mathcal{N}_S \Delta u \end{pmatrix} + \begin{pmatrix} F_\mu^{d,1} \\ F_\mu^{d,2} \\ F_\mu^{p,1} \\ F_\mu^{p,2} \end{pmatrix}.$$

Finally, we update the iterates along these directions with appropriate positive step sizes α_y, α_λ, α_W, α_S:

$$\begin{pmatrix} W \\ \lambda \\ y \\ S \end{pmatrix} \leftarrow \begin{pmatrix} W \\ \lambda \\ y \\ S \end{pmatrix} + \begin{pmatrix} \alpha_W \Delta W \\ \alpha_\lambda \Delta \lambda \\ \alpha_y \Delta y \\ \alpha_S \Delta S \end{pmatrix}. \tag{3.6}$$

Note that the stepsizes are taken so that the positive definiteness of W, S is maintained.

Matrix Representation We provide the matrix representation of $(\mathcal{N}_W \Delta v)S + W(\mathcal{N}_S \Delta u)$ in the system (3.5) to be used in the implementation. We first define some related notations. We use $\mathrm{vec}(X)$ to denote the usual vectorization map that stacks the columns of a real matrix X into a single vector. Given a complex matrix X, let $\Re(X)$ and $\Im(X)$ be the real and imaginary parts of X, respectively. We define the mapping $\mathrm{Cvec} : \mathbb{C}^{m \times n} \to \mathbb{R}^{2mn}$ by

$$\mathrm{Cvec}(X) = \begin{pmatrix} \mathrm{vec}(\Re(X)) \\ \mathrm{vec}(\Im(X)) \end{pmatrix}.$$

We now consider $\mathcal{F}$ in $\mathbb{H}^n_+$. Let N^i_W be the i-th data matrix in $\mathcal{N}_W$. In other words, $N^i_W = \mathcal{N}_W(e_i)$. Then

$$\operatorname{Cvec}\left((\mathcal{N}_W\Delta v)S\right) = \begin{bmatrix}\operatorname{Cvec}(N^1_W S) & \cdots & \operatorname{Cvec}(N^{n^2-m}_W S)\end{bmatrix}\begin{bmatrix}\Delta v_1\\ \vdots \\ \Delta v_{n^2-m}\end{bmatrix}.$$

Let N^i_S be i-th data matrix in $\mathcal{N}_S$, i.e., $N^i_S = \mathcal{N}_S(e_i)$. Then we obtain

$$\operatorname{Cvec}\left(W\mathcal{N}_S\Delta u\right) = \begin{bmatrix}\operatorname{Cvec}(WN^1_S) & \cdots & \operatorname{Cvec}(WN^{m-1}_S)\end{bmatrix}\begin{bmatrix}\Delta u_1\\ \vdots \\ \Delta u_{m-1}\end{bmatrix}.$$

Putting all together, the solution $(\Delta v, \Delta u)$ of

$$(\mathcal{N}_W\Delta v)S + W(\mathcal{N}_S\Delta u) = -F^c_\mu - (F^{d,1}_\mu)S - W(F^{p,2}_\mu)$$

is obtained by solving the following linear equation in the usual matrix-vector form:

$$\begin{bmatrix}\left[\operatorname{Cvec}(N^i_W S)\right]_{i=1,\ldots,n^2-m}, \left[\operatorname{Cvec}(W\mathcal{N}^j_S)\right]_{j=1,\ldots,m-1}\end{bmatrix}\begin{bmatrix}\Delta v\\ \Delta u\end{bmatrix} = -\operatorname{Cvec}\left(F^c_\mu + (F^{d,1}_\mu)S + W(F^{p,2}_\mu)\right).$$

The analogous representation of (3.5) for $\mathcal{F}$ in $\mathbb{S}^n_+$ is obtained by replacing Cvec with vec, and with appropriate dimensions.

Predictor-Corrector Steps The traditional search direction of interior point method is determined by solving (3.5). Many researchers adopt predictor-corrector steps to enhance computational efficiency, e.g., see [10, 11, 15, 16]. Typically, a set of predictor-corrector steps involves solving two linear systems, differing only in their right-hand-side data. That is, the data matrix $(\mathcal{N}_W\Delta v)S + W(\mathcal{N}_S\Delta u)$ in (3.5) remains unchanged while the right-hand-side data are altered in these two steps. Below, we outline how these right-hand-side data are computed.

The predictor step is performed by solving

$$(\mathcal{N}_W\Delta v)S + W(\mathcal{N}_S\Delta u) = \mathrm{RHS}_p, \tag{3.7}$$

where

$$\mathrm{RHS}_p = -\left(WS + F^{p,2}_\mu + F^{d,1}_\mu S\right).$$

The remaining components of the direction are obtained using the backsolve step (3.4). In particular, we use $\Delta W^{\mathrm{aff}}, \Delta S^{\mathrm{aff}}$ to denote the W, S related search directions. The direction obtained by the predictor step is often called the affine scaling direction.

For the corrector step, choose step-sizes $\alpha^{\mathrm{aff}}_W, \alpha^{\mathrm{aff}}_S \in (0, 1]$ so that $W + \alpha^{\mathrm{aff}}_W\Delta W^{\mathrm{aff}} \succ 0$ and $S + \alpha^{\mathrm{aff}}_S\Delta S^{\mathrm{aff}} \succ 0$ are satisfied. Compute

$$\mu^{\mathrm{aff}} = \frac{1}{n}\operatorname{trace}\left((W + \alpha^{\mathrm{aff}}_W\Delta W^{\mathrm{aff}})\cdot(S + \alpha^{\mathrm{aff}}_S\Delta S^{\mathrm{aff}})\right) \text{ and } \sigma = (\mu^{\mathrm{aff}}/\mu)^3, \tag{3.8}$$

where μ is the duality gap measure $\operatorname{trace}(WS)$ evaluated by the previous iterates W, S. We then solve

$$(\mathcal{N}_W\Delta v)S + W(\mathcal{N}_S\Delta u) = \mathrm{RHS}_c, \tag{3.9}$$

where

$$\mathrm{RHS}_c = -\left(WS + \Delta W^{\mathrm{aff}}\Delta S^{\mathrm{aff}} - \sigma\mu I + WF^{p,2}_\mu + F^{d,1}_\mu S\right).$$

Finally, using the solutions in (3.9) to obtain the remaining components of the search direction via (3.4) completes the corrector step.

Stopping Conditions The algorithm terminates if one of the following conditions holds:

1. (Stalling) The minimum eigenvalues of W or S is near 0 for a certain number of consecutive iterations.
2. (Max iteration) The algorithm reached the pre-defined maximum number of iterations.
3. (Optimality measure) For a pre-defined tolerance $\epsilon > 0$,
   - $\min\{\text{trace}(WS), \text{best primal objective value-best dual objective value}\} < \epsilon$
   - $\|F_\mu(v, W, \lambda, y, u, S)\| < \epsilon$
4. (Objective value)
   - The dual feasibility is achieved with a positive dual objective value. (This means that the primal optimal value is never 0 hence we obtain a Slater point by item 3 of Lemma 2.2.)
   - Similarly, the primal feasibility is achieved with a negative primal objective value. (This means that $\mathcal{F} \neq \emptyset$. In this case, the vector $\mathcal{A}^*(y)$ provides a certificate of infeasibility.)

Algorithm 1 below summarizes a step of the **FR** process.

Algorithm 1 Pseudo-Code for Checking Strict Feasibility of $\mathcal{F}$

Require: data $(\mathcal{A}, b)$ for spectrahedon $\mathcal{F} = \{X \succeq 0 : \mathcal{A}(X) = b\}$

while stopping criteria are not satisfied **do**
  Solve (3.7) (predictor step)
  Compute the parameters μ^{aff} and σ via (3.8)
  Solve (3.9) (corrector step)
  Choose step lengths and update iterates using (3.6)
end while
if $p^*_{\mathbf{FR}} = 0$ **then**
  Compute a full column rank matrix V such that $\text{range}(V) = \text{null}(\mathcal{A}^*(y))$
else if $p^*_{\mathbf{FR}} > 0$ **then**
  strict feasibility holds for $\mathcal{F}$
else if $p^*_{\mathbf{FR}} < 0$ **then**
  $\mathcal{F}$ is infeasible
end if

4 Conclusions

We have addressed an optimization problem that serves as an intermediate step on the facial reduction process. Specifically, we outlined a primal-dual interior point method that uses Gauss-Newton method for finding a point that satisfies the first optimality conditions. However, future improvements are needed. In cases where the dual feasible region is unbounded, the algorithm inflates the variable W, causing S to become relatively small due to the complementarity condition $WS = \mu I$. This premature termination occurs because S cannot make significant progress due to its small magnitude. We aim to devise a method to mitigate this issue. Additionally, the termination of Algorithm 1 provides the information about whether $\mathcal{F}$ satisfies strict feasibility.

However, this termination does not necessarily produce an exposing vector that fully identifies the minimal facial range vector, especially when singularity degree of $\mathcal{F}$ exceeds 1. We intend to incorporate functionality to address these scenarios. Lastly, we wish to include computational results obtained by pre-processing the instances from SDPLIB.

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