STRICT EFFICIENCY IN SET OPTIMIZATION STUDIED WITH THE SET APPROACH

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Abstract. This paper is devoted to strict efficiency in set optimization studied with the set approach. Strict efficient solutions are defined with respect to the l-type less order relation and the possibly less order relation. Scalar characterization and necessary and/or sufficient conditions for such solutions are obtained. In particular, we establish some conditions expressed in terms of a high-order directional derivative and a subdifferential of the set-valued objective map. Various illustrating examples are presented.

Key words: Set optimization, strict efficient solution, optimality condition, directional derivative, subdifferential, set-valued map

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1. Introduction

A set-valued optimization problem, in brief (SOP), of the form

$$\text{Min}_{x\in\Omega} F(x),$$

where $F : \Omega \subseteq X \rightrightarrows Y$ is a set-valued map and X and Y are normed spaces, has been recently attracted more attention due to its extensive real-world applications, see [20, 26] and the references therein.

There are several approaches to defining an efficient solution $\bar{x} \in \Omega$ for (SOP), among which we would like to mention the classical vector approach and the set approach introduced by Kuroiwa [28]. Roughly speaking, one compares some vector $\bar{y} \in F(\bar{x})$ with other vectors in the set $\cup_{x\in\Omega} F(x)$ with respect to a partial order on Y in the first approach and one compares the whole set $F(\bar{x})$ with other sets $F(x)$ with respect to some set order relations on 2^Y in the second approach. In this paper, we restrict ourselves to (SOP) studied with the set approach.

Various types of solutions were considered in scalar optimization, among which strict solutions are of particular importance for several reasons including that one that strict solutions are, in contrast to regular minimum points, the most likely to be found by numerical algorithms. Recall that this kind of solution (the name of which varies in different papers) was introduced by Auslender [4] as follows. Let $f : \Omega \to \mathbb{R}$ be a function. A point $\bar{x} \in \Omega$ is said to be an isolated local minimum of order m ($m \in \{1, 2\}$) if there exist a scalar $\alpha > 0$ and a neighborhood U of $\bar{x}$ such that for all $x \in U \cap \Omega$, $x \neq \bar{x}$, one has

$$f(x) > f(\bar{x}) + \alpha\|x - \bar{x}\|^m.$$

The concept of strict minimizer has been extended and developed successfully in vector optimization by Bednarczuk [6], Ginchev et al [13], Jiménez [23, 24] and in (SOP) studied with the vector approach by Crespi et al. [8], Flores-Bazán et al. [11]. Note that in [11], the authors introduced the notion of strict minimizer, called ϕ-strict minimizer, for problem (SOP) with respect to an admissible function ϕ, in such a way that this concept generalizes, in a unified manner, to problems with set-valued maps all the definitions of strict minimizer of order m and ϕ-strict minimizer for a vector-valued function given in the literature. Recently, the concept of strict minimizer of order m with respect to the possibly less set order relation for (SOP) studied with the set approach has been introduced by Michalak and Studniarski [32]. Optimality conditions for strict minimizers have been obtained through different kinds of derivative such as directional derivative, contingent derivative, Studniarski derivative, radial derivative, among others. Nowadays, there exists an important literature dedicated to strict efficiency. In addition to the papers mentioned above, we refer an interested reader to [9, 13, 14, 30] for historical comments and different approaches to strict efficiency in optimization.

In this paper, we introduce for (SOP) studied with the set approach the concepts of $(\preceq_r, \phi)$-strict solutions ($r \in \{l, p\}$) with respect to the l-type less order relation $\preceq_l$ or the possibly less set order relation $\preceq_p$ and an admissible function ϕ. Our concept in case $r = p$ is a slightly extended version of the strict minimizer introduced in [32]. We obtain scalar characterization in terms of the signed Hausdorff-type half-distance [15] and an abstract scalarizing functional, and some necessary and/or sufficient conditions for these solutions. Appropriate set-valued versions of high-order directional derivative and subdifferential are presented in order to handle optimality conditions for strict efficiency of (SOP) under the set approach paradigm. We also provide illustrative examples.

The paper is organized as follows. Next section contains auxiliary results. In Section 3, we study the high order directional derivative and a subdifferential of set-valued maps. Section 4 is devoted to concepts of $(\preceq_r, \phi)$-strict efficient solutions of (SOP). Scalar characterization and optimality conditions for these solutions are obtained in Sections 5 and 6, respectively. Some conclusions are given in the last section.

2. Preliminaries

2.1. **Notations and some facts from vector optimization.** Throughout the paper, let X and Y be normed spaces. Given a normed space, say X, we denote its dual by X^*, the pairing between X and X^* by $\langle \cdot, \cdot \rangle_X$, its closed unit ball by $\mathbb{B}_X$, its norm by $\|.\|_X$ and the distance from a point u to a nonempty set U of X by $d_U(u)$. When no confusion occurs, we omit the subscript X in these notations. For $a, b \in Y$, denote $[a, b] := \{\lambda a + (1-\lambda)b \colon 0 \leq \lambda \leq 1\}$. For nonempty subsets A and B of X and a scalar t, $A+B := \{a+b \colon a \in A,\ b \in B\}$, $A - B := \{a - b \colon a \in A,\ b \in B\}$ and $tA := \{ta \colon a \in A\}$. By $\mathbb{R}^q$ and $\mathbb{R}^q_+$ we mean the q-dimensional euclidean space and its nonnegative orthant.

We recall some concepts from vector optimization [31]. Let $K \subset Y$ be a pointed closed convex cone. Here, K is a *cone* if $k \in K$ implies $tk \in K$ for all $t \geq 0$ and K is *pointed* if $K \cap (-K) = \{0\}$. We say that K is *solid* if $\mathrm{int}K \neq \emptyset$, where $\mathrm{int}K$ denotes the interior of K. The nonnegative dual cone of K is the set $K^* := \{y^* \in Y^* \colon \langle y^*, k\rangle \geq 0,\ \forall\ k \in K\}$. A partial order induced by K in Y is defined by: for any $y_1, y_2 \in Y$,

$$y_1 \leq_K y_2 :\Longleftrightarrow\ y_2 - y_1 \in K$$

(we also write $y_2 \geq_K y_1$). For simplicity, we omit the subscript K in these notations.

Let $A \subset Y$ be a nonempty set. $\bar{a} \in A$ is said to be an *efficient point* or a *Pareto minimal point* of A with respect to K (denoted by $\bar{a} \in \mathrm{Min}(A)$) if there is no $a \in A \setminus \{\bar{a}\}$ such that $a \leq \bar{a}$. We say that A is *K-bounded* if there exists a bounded nonempty set $M \subset Y$ such that $A \subset M + K$, A is *K-compact* if any cover of the form $\{U_\alpha + K \colon \alpha \in I,\ U_\alpha \text{ are open}\}$ of A admits a finite subcover and A has the *domination property* if $\mathrm{Min}(A)$ is nonempty and $A \subseteq \mathrm{Min}(A) + K$. It is known that if A is K-compact then it is K-bounded, if A has the domination property then $A + K = \mathrm{Min}(A) + K$ and if A is K-compact then $\mathrm{Min}(A) \neq \emptyset$ and A has the domination property [31, Theorem 3 and Lemma 3.5]. One can check that if $A+K = B+K$ then $\mathrm{Min}(A) = \mathrm{Min}(B)$ and if A is K-bounded or K-compact, then so is B.

2.2. **Set order relations.**

In the set approach to (SOP), various set order relations are used to define optimal solutions, see [22, 28, 29]. In this paper, we are working with the following ones.

Definition 2.1. Let A and B be nonempty subsets of Y.

(i) The l–type less order relation $\preceq_l$ is defined by

$$A \preceq_l B\ :\Longleftrightarrow\ B \subseteq A + K \Longleftrightarrow\ \big(\forall b \in B\ \exists\ a \in A : a \leq b\big).$$

(ii) The possibly less order relation $\preceq_p$ is defined by

$$A \preceq_p B\ :\Longleftrightarrow (A - B) \cap (-K) \neq \emptyset\ \Longleftrightarrow\ \big(\exists a \in A\ \exists\ b \in B : a \leq b\big).$$

It is easy to see that $A \preceq_l B$ implies $A \preceq_p B$.

2.3. **Abstract scalarizing functions.**

Let $\theta : Y \to \mathbb{R}$ be a function, which is nontrivial in the sense that $\theta(y) \neq 0$ for some $y \in Y$. We will consider the following properties.

(P1) *Global Lipschizity*: $\exists L > 0$ such that $|\theta(y_1) - \theta(y_2)| \leq L||y_1 - y_2||$ for all $y_1, y_2 \in Y$.

(P2) *Sublinearity*: $\theta(ty) = t\theta(y)$ for any $y \in Y$ and scalar $t \geq 0$ (positive homogeneity) and $\theta(y_1 + y_2) \leq \theta(y_1) + \theta(y_2)$ for all $y_1, y_2 \in Y$ (subadditivity).

(P3) *K-monotonicity*: $\theta(y_1) \leq \theta(y_2)$ for all $y_1, y_2 \in Y$ satisfying $y_1 \leq y_2$.

(P4) *Cone representation property*: $-K = \{y \in Y : \theta(y) \leq 0\}$.

(P5) *Cone interior representation property* (if K is solid): $-\text{int}K = \{y \in Y : \theta(y) < 0\}$.

(P6) *Uniform positivity*: For any $\alpha > 0$, there exists a scalar $\beta > 0$ such that $y + \alpha\mathbb{B} \subset Y \setminus (-K)$ implies $\theta(y) \geq \beta$.

We say that θ is an *abstract scalarizing function* if it has the properties (P1)-(P4) [16].

Let us recall some functions that are frequently used for scalarization in vector and set optimization.

(a) *The function proposed by Gerstewizt (Tammer)* [12]. Assume that K is solid and $k_0 \in \text{int}K$. Define a function $\varphi_{k_0} : Y \to \mathbb{R}$ by

$$\varphi_{k_0}(y) := \inf\{t \in \mathbb{R} : y \in tk_0 - K\} = \inf\{t \in \mathbb{R} : y \leq tk_0\}.$$

(b) *The Hiriart-Urruty signed distance* Δ_{-K} associated to the cone K defined by

$$\Delta_{-K}(y) := d_{-K}(y) - d_{Y\setminus(-K)}(y) = \begin{cases} -d_{Y\setminus(-K)}(y) & \text{if } y \in -K \\ d_{-K}(y) & \text{otherwise,} \end{cases}$$

see [19] for the definition in the general case. This function is popularly known in the literature also as the oriented distance function.

(c) *The function introduced by Kasimbeyli* [25]. Given a continuous linear functional $\ell \in Y^*$ with $\|\ell\| \geq 1$, a Bishop-Phelps cone $C(\ell)$ is defined as follows

$$C(\ell) := \{y \in Y : \ell(y) \geq \|y\|\},$$

see [5]. When $\|\ell\| = 1$, $C(\ell)$ has the form $C(\ell) = \{y \in Y : \ell(y) = \|y\|\}$ and it is called a Bishop-Phelps cone given by an equation in [17].

Let $K = C(\ell)$. The function $\xi_\ell : Y \to \mathbb{R}$ used for scalarization by Kasimbeyli is defined as follows: for every $y \in Y$

$$\xi_\ell(y) := \ell(y) + \|y\|.$$

It has been proved that these functions satisfy Properties (P1)-(P6) (the function introduced by Kasimbeily satisfies (P5)-(P6) under an additional condition that $\|\ell\| > 1$) [16, Propositions 3.1-3.3].

2.4. **The signed Hausdorff-type half-distances and the Hausdorff-type distance.** We recall some quantities associated to a pair (A, B) of nonempty subsets of Y that have been successfully used for scalarization in set optimization and will play an important role in characterizing our strict minimizers. From now on, let θ be an abstract scalarizing function.

The signed Hausdorff-type half-distances $h^l_\theta(A,B)$, $h^p_\theta(A,B)$ and the Hausdorff-type distance $d_\theta(A,B)$ are defined as follows [15, 16]:

$$h^l_\theta(A,B) := \sup_{b\in B}\inf_{a\in A}\theta(a-b),$$

$$h^p_\theta(A,B) := \inf_{a\in A}\inf_{b\in B}\theta(a-b),$$

and

$$d_\theta(A,B) := \max\{h^l_\theta(A,B), h^l_\theta(B,A)\}.$$

Below, we collect some properties of $h^r_\theta(A,B)$ ($r\in\{l,p\}$) and $d_\theta(A,B)$ that will be used later, see [15, Proposition 3.2], [16, Propositions 4.1,4.2, 4.5, 4.6, 6.2, 6.3 and Corollary 4.1]. Note that the Hausdorff-type distance satisfies some properties of a metric such as nonnegativity, symmetry and triangle inequality, among others.

Lemma 2.1. *The following assertions are true.*

(a) $h^r_\theta(A,B)$ *is finite, provided* A, B *are* K*-bounded in case* $r=l$ *and* A, $-B$ *are* K*-bounded in case* $r=p$.

(b) *The "sup" and "inf" in the definition of* $h^r_\theta(A,B)$ *can be replaced by "max" and "min", provided* A, B *are* K*-compact in case* $r=l$ *and* A, $-B$ *are* K*-compact in case* $r=p$.

(c) $h^r_\theta(A,A)\le 0$.

(d) $h^r_\theta(A,A)=0$, *provided* A *is* K*-bounded in case* $r=l$ *and* $A=\mathrm{Min}(A)$ *in case* $r=p$.

(e) $d_\theta(A,B)=d_\theta(A,\mathrm{Min}(B))=d_\theta(\mathrm{Min}(A),\mathrm{Min}(B))$ *and there exist* $a\in\mathrm{Min}(A)$ *and* $b\in\mathrm{Min}(B)$ *such that* $d_\theta(A,B)=\theta(a-b)$ *or* $d_\theta(A,B)=\theta(b-a)$, *provided that* A *and* B *are* K*-compact.*

3. Directional derivative and subdifferential of a set-valued map

In this section, we present a high order version of the directional derivative introduced in [15] and define subdifferential in terms of coderivative for a set-valued map.

From now on, let $\Omega\subseteq X$ be a nonempty set and $F:\Omega\subseteq X\rightrightarrows Y$ be a set-valued map with nonempty values. The graph and epigraph of F are the sets $\mathrm{gr}F := \{(x,y)\colon x\in\Omega, y\in F(x)\}$ and $\mathrm{epi}F := \{(x,y)\colon x\in\Omega, y\in F(x)+K\}$, respectively.

Let $\bar{x}\in\Omega$. We say that $d\in Y$ is an *admissible direction* of F at $\bar{x}$ (or simply an admissible direction) if there exists $\delta>0$ such that $\bar{x}+td\in\Omega$ for all $t\in]0,\delta[$. For such d and t, denote

$$A_t := \frac{F(\bar{x}+td)-F(\bar{x})}{t^k}$$

and

$$W^k_\theta(\bar{x},d) := \{A\subset Y\colon\ A \text{ is } K\text{−compact and } \lim_{t\downarrow 0^+} d_\theta(A_t,A)=0\}.$$

Definition 3.1. The directional derivative $D^k_\theta F(\bar{x}, d)$ of order k of F at $\bar{x}$ in an admissible direction d is defined as follows:

$$D^k_\theta F(\bar{x}, d) := \begin{cases} \mathrm{Min}(A) \text{ for some } A \in W^k_\theta(\bar{x}, d) & \text{if } W^k_\theta(\bar{x}, d) \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

Note that when $k = 1$ and $\theta = \Delta_{-K}$, Definition 3.1 collapses into the definition of directional derivative for a set-valued map introduced in [15, Definition 4.1]. Observe that if the set-valued map F has bounded values, then the distance $d_\theta(A_t, A)$ considered in the definition of the set $W^k_\theta(\bar{x}, d)$ is finite and if $A \in W^k_\theta(\bar{x}, d)$, then A being K-compact has the domination property and one can prove that the set $\mathrm{Min}(A)$ also is K-compact.

It turns out that the directional derivative of the order k is well-defined. Namely, we have the following.

Proposition 3.1. *Assume that* $W^k_\theta(\bar{x}, d) \neq \emptyset$.

(i) *The directional derivative* $D^k_\theta F(\bar{x}, d)$ *is well-defined in the sense that it does not depend on the choice of* $A \in W^k_\theta(\bar{x}, d)$ *and it satisfies*

$$\lim_{t \downarrow 0^+} d_\theta(A_t, D^k_\theta F(x, d)) = 0.$$

(ii) *If* $B \subset Y$ *is a nonempty* K*-compact set, then* $D^k_\theta F(\bar{x}, d) = B$ *if and only if*

$$\lim_{t \downarrow 0^+} d_\theta(A_t, B) = 0 \text{ and } \mathrm{Min}(B) = B.$$

(iii) *If* $D^k_\theta F(\bar{x}, d)$ *is nonempty, then so is* $D^k_\theta F(\bar{x}, \lambda d)$ *for any* $\lambda > 0$ *and*

$$D^k_\theta F(\bar{x}, \lambda d) = \lambda^k D^k_\theta F(\bar{x}, d).$$

We omit the proof, which is similar to the one for the case $k = 1$ and $\theta = \Delta_{-K}$ [15, Proposition 4.1].

Example 3.1. Let $\Omega = X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. In this example, $\theta = \Delta_{-\mathbb{R}^2_+}$ and $\bar{x} = 0$. Due to Proposition 3.1 (iii), we can restrict ourselves to $d = 1$ and $d = -1$. We provide some detailed proofs to demonstrate how to calculate the directional derivative in these concrete cases.

(i) Let

$$F(x) := \begin{cases} \{(2x^2, \mathrm{e}^{x^2}), (x^2, 2\mathrm{e}^{x^2} - 1)\} & \text{if } x \neq 0 \\ \{(0, 0), (0, 1)\} & \text{if } x = 0 \end{cases}$$

For $d \in \{-1, 1\}$ and $t > 0$, we have

$$F(\bar{x} + td) = \{(2t^2, \mathrm{e}^{t^2}), (t^2, 2\mathrm{e}^{t^2} - 1)\},$$

$$F(\bar{x} + td) - F(\bar{x}) = \{(2t^2, \mathrm{e}^{t^2}), (t^2, 2\mathrm{e}^{t^2} - 1), (2t^2, \mathrm{e}^{t^2} - 1), (t^2, 2\mathrm{e}^{t^2} - 2)\}$$

and

$$A_t = \{(2, \frac{e^{t^2}}{t^2}), (1, \frac{2e^{t^2}-1}{t^2}), (2, \frac{e^{t^2}-1}{t^2}), (1, \frac{2e^{t^2}-2}{t^2})\}.$$

Let $A := \{(1,2),(2,1)\}$. One can check that $\mathrm{Min}(A) = A$ and $\mathrm{Min}(A_t) = \{(1, \frac{2e^{t^2}-2}{t^2}), (2, \frac{e^{t^2}-1}{t^2})\}$. A simple calculation gives

$$\begin{aligned} & h^l_{\Delta_{-K}}(\mathrm{Min}(A_t), \mathrm{Min}(A)) = \\ = \ & \max\{\min\{\Delta_{-K}((0, \tfrac{2e^{t^2}-2}{t^2} - 2)), \Delta_{-K}((1, \tfrac{e^{t^2}-1}{t^2} - 2))\}, \\ & \quad \min\{\Delta_{-K}((-1, \tfrac{2e^{t^2}-2}{t^2} - 1)), \Delta_{-K}((0, \tfrac{e^{t^2}-1}{t^2} - 1))\}\} \\ = \ & \max\{\tfrac{2e^{t^2}-2}{t^2} - 2, \tfrac{e^{t^2}-1}{t^2} - 1\} = 2(\tfrac{e^{t^2}-1}{t^2} - 1) \end{aligned}$$

and

$$\begin{aligned} & h^l_{\Delta_{-K}}(\mathrm{Min}(A), \mathrm{Min}(A_t)) = \\ = \ & \max\{\min\{\Delta_{-K}((0, 2 - \tfrac{2e^{t^2}-2}{t^2})), \Delta_{-K}((1, 1 - \tfrac{2e^{t^2}-2}{t^2}))\}, \\ & \quad \min\{\Delta_{-K}((-1, 2 - \tfrac{e^{t^2}-1}{t^2})), \Delta_{-K}((0, 1 - \tfrac{e^{t^2}-1}{t^2}))\}\} \\ = \ & \max\{0, 0\} = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{t\downarrow 0^+} d_{\Delta_{-K}}(A_t, A) &= \lim_{t\downarrow 0^+} d_{\Delta_{-K}}(\mathrm{Min}(A_t), \mathrm{Min}(A)) \\ &= \lim_{t\downarrow 0^+} \max\{2(\tfrac{e^{t^2}-1}{t^2} - 1), 0\} = 0, \end{aligned}$$

which means that

$$D^2_\theta F(0,1) = D^2 F(0,-1) = \{(1,2),(2,1)\}.$$

(ii) Let

$$F(x) := \begin{cases} \{(x,1),(x,2)\} & \text{if } x \neq 0 \\ \{(0,0),(0,1)\} & \text{if } x = 0. \end{cases}$$

Then

$$D^1_\theta F(0,d) = \begin{cases} \{(1,0)\} & \text{if } d = 1 \\ \{(-1,0)\} & \text{if } d = -1 \end{cases}$$

(see Example 4.1 in [15]).

(iii) Let

$$F(x) := \begin{cases} \{(x,0),(x,1)\} & \text{if } x \neq 0 \\ \{(0,0),(0,1)\} & \text{if } x = 0 \end{cases}$$

Then

$$D^1_\theta F(0,d) = \begin{cases} \{(1,0)\} & \text{if } d = 1 \\ \emptyset & \text{if } d = -1 \end{cases}$$

We omit the proof of the equality $D^1_\theta F(0,1) = \{(1,0)\}$ and show that $D^1_\theta F(0,-1) = \emptyset$. For $d = -1$ and $t > 0$, we have

$$F(\bar{x} + td) = \{(-t,0),(-t,1)\},$$

and

$$A_t = \{((-1,0),(-1,\frac{1}{t}),(-1,-\frac{1}{t})\}.$$

One can check that $\mathrm{Min}(A_t) = \{a_t\}$, where $a_t = (-1, -\frac{1}{t})$. Suppose to the contrary that there exists a nonempty $\mathbb{R}^2_+$-compact subset $A \subset \mathbb{R}^2$ such that

$$\lim_{t\downarrow 0^+} d_{\Delta_{-\mathbb{R}^2_+}}(\mathrm{Min}(A_t), \mathrm{Min}(A)) = 0.$$

Note that since A is $\mathbb{R}^2_+$-compact, we have

$$t^* := \inf\{a\colon \exists a' \text{ such that } (a,a') \in A \text{ or } (a',a) \in A\} > -\infty.$$

By Lemma 2.1, for each $t > 0$ (sufficiently small), there exists $\bar{a}_t = (\bar{u}_t, \bar{v}_t) \in A$ such that either

$$d_{\Delta_{-\mathbb{R}^2_+}}(\mathrm{Min}(A), \mathrm{Min}(A_t)) = \Delta_{-\mathbb{R}^2_+}(\bar{a}_t - a_t) = \Delta_{-\mathbb{R}^2_+}((\bar{u}_t + 1, \bar{v}_t + \frac{1}{t}))$$

or

$$d_{\Delta_{-\mathbb{R}^2_+}}(\mathrm{Min}(A), \mathrm{Min}(A_t)) = \Delta_{-\mathbb{R}^2_+}(a_t - \bar{a}_t) = \Delta_{-\mathbb{R}^2_+}(-(\bar{u}_t + 1, \bar{v}_t + \frac{1}{t})).$$

Let consider the first case. Since $\bar{u}_t \geq t^*$, $\bar{v}_t \geq t^*$ and $\frac{1}{t} \to +\infty$, we get (for t sufficiently small such that $\bar{v}_t + \frac{1}{t} > 0$)

$$\Delta_{-\mathbb{R}^2_+}((\bar{u}_t + 1, \bar{v}_t + \frac{1}{t})) = \begin{cases} \sqrt{(\bar{u}_t+1)^2 + (\bar{v}_t + \frac{1}{t})^2} & \text{if } \bar{u}_t + 1 \geq 0 \\ \bar{v}_t + \frac{1}{t} & \text{if } \bar{u}_t + 1 < 0. \end{cases}$$

Then it follows that

$$\lim_{t\downarrow 0^+} d_{\Delta_{-\mathbb{R}^2_+}}(\mathrm{Min}(A_t), \mathrm{Min}(A)) = +\infty,$$

a contradiction. The second case can be considered in a similar way. Thus, $W^1_\theta(0,-1) = \emptyset$ and $D^1_\theta F(0,-1) = \emptyset$.

(iv) Let

$$F(x) := \begin{cases} \{(x^k, 1), (0, x^k \mathrm{e}^x + 1)\} & \text{if } x \neq 0 \\ \{(0,0),(0,1)\} & \text{if } x = 0. \end{cases}$$

If k is even, then $D^k_\theta F(0,1) = D^k_\theta F(0,-1) = \{(1,0),(0,1)\}$.

Next, we define subdifferential of the map F. In what follows, Ω is assumed to be convex whenever some convexity assumption is imposed on the map F.

Definition 3.2. We say that

(i) F is *convex* [2] if for all $x_1, x_2 \in \Omega$ and $\lambda \in [0,1]$

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(\lambda x_1 + (1-\lambda)x_2).$$

(ii) F is *K-convex* [31] if for all $x_1, x_2 \in \Omega$ and $\lambda \in [0,1]$

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(\lambda x_1 + (1-\lambda)x_2) + K.$$

We associate to F a set-valued map $F_+ : \Omega \rightrightarrows Y$ defined by

$$F_+(x) = F(x) + K.$$

It is known that F is convex if and only if its graph is convex and F is K-convex if and only if its epigraph is convex, for the second fact see [27, Proposition 2.2]. Since $\text{epi}F = \text{graph}F_+$, the map F is K-convex if and only if the map F_+ is convex.

Let us recall some concepts from convex analysis and nonsmooth analysis. Given a convex set A in a normed space X, a *normal cone* to A at $\bar{x} \in A$ is the set

$$N(A, \bar{x}) := \{x^* \in X^* \colon \langle x^*, a - \bar{a}\rangle \leq 0, \forall a \in A\}.$$

For a convex set-valued map $G : \Omega \rightrightarrows Y$, its *coderivative* $D^*G(\bar{x}, \bar{y})$ at $(\bar{x}, \bar{y}) \in \text{graph}G$ is a set-valued map from Y^* to X^* defined as follows [2]: for $y^* \in Y^*$

$$D^*G(\bar{x}, \bar{y})(y^*) = \{x^* \in X^* \colon (x^*, -y^*) \in N(\text{graph}G, (\bar{x}, \bar{y}))\}.$$

Definition 3.3. Assume that F is K-convex and $\bar{x} \in \Omega$. The subdifferential of F at $\bar{x}$ is the set

$$\partial F(\bar{x}) := \cup_{\bar{y} \in F(\bar{x})} \cup_{y^* \in K^*, \|y^*\|=1} D^*F_+(\bar{x}, \bar{y})(y^*).$$

It follows from the definition that

$$\partial F(\bar{x}) = \{ \begin{array}{l} x^* \in X^* \colon \exists \bar{y} \in F(\bar{x})\ \exists y^* \in K^*, \|y^*\| = 1 \\ \text{such that } (x^*, -y^*) \in N(\text{epi}F, (\bar{x}, \bar{y})\}. \end{array}$$

One can check that if $(x^*, -y^*) \in N(\text{epi}F, (\bar{x}, \bar{y}))$, then $y^* \in K^*$ and if F is a convex function from Ω to $\mathbb{R}$, then the subdifferential given by Definition 3.3 reduces to the subdifferential of convex analysis.

Example 3.2. Let $\Omega = X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $\theta = \Delta_{-K}$ and F be the map defined by

$$F(x) := [(|x|, \mathrm{e}^{|x|}), (|x|, 2\mathrm{e}^{|x|} - 1)].$$

Since the set $\text{epi}F = \{(x, y, z) \colon x \in \mathbb{R}, y \geq |x|, z \geq \mathrm{e}^{|x|}\}$ is convex, the map F is K-convex. One can check that $F(0) = \{(0,1)\}$, $D^1_\theta F(0,1) = D^1_\theta F(0,-1) = \{(1,1)\}$ and $\partial F(0) = [-1,1]$.

We conclude this section with some comments.

Remark 3.1. Various concepts of directional derivative and subdifferential for a set-valued map F have been introduced in literature and each of them has shown to be useful for specific optimization problems [7, 10, 15, 18, 21, 30, 32]. Here, we follow the lines of [7, 10, 15, 21] in taking into consideration a point $\bar{x}$ in the domain of F and the whole set $F(\bar{x})$, and not a pair $(\bar{x}, \bar{y})$ from the graph of F like in [18, 30, 32], in order to be closer in spirit to the set approach in set-valued optimization and appropriately formulate optimality conditions for strict minimizers by means of these constructions. We also note that our subdifferential is defined through normal cone and coderivative, which are known concepts in convex analysis and variational analysis, in contrast to [10, 18, 21], where subdifferential is defined with the help of new constructions.

4. Concepts of strict solutions of a set-valued optimization problem studied with the set approach

In this section, we introduce concepts of strict solutions for (SOP) studied with the set approach. From now on, let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an *admissible function*, i.e. ϕ is nondecreasing, $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$. Let us recall main concepts of strict efficiency in scalar optimization and set optimization.

(a) *Scalar optimization problem*

Let $g : \Omega \to \mathbb{R}$ be a function. Consider a scalar optimization problem (OP)

$$\mathrm{Min}_{x\in\Omega} g(x).$$

Definition 4.1. We say that $\bar{x} \in \Omega$ is a ϕ-strict local solution of (OP) or $\bar{x}$ is a ϕ-strict local minimizer of g over Ω (with constant $\alpha > 0$) if there exists a neighborhood U of $\bar{x}$ such that for all $x \in U \cap \Omega$, $x \neq \bar{x}$, one has

$$g(x) > g(\bar{x}) + \alpha\phi(\|x - \bar{x}\|).$$

Remark that strict minimizers for a scalar function have been introduced with $\phi(t) = t^m$, $m = 1, 2$ [4] and considered also for $m \geq 2$ [34]. Definition 4.1 involving an admissible function ϕ is motivated by the one presented for a vector-valued function in [6].

(b) *Set optimization problem studied with the vector approach.*

Definition 4.2. [11, Definition 3.2] We say that a pair $(\bar{x}, \bar{y}) \in \mathrm{graph} F$ is a ϕ-strict local efficient solution of (SOP) or $(\bar{x}, \bar{y})$ is a ϕ-strict local minimizer of F over Ω if $\bar{y} \in \mathrm{Min}(F(\bar{x}))$ and there exists a neighborhood U of $\bar{x}$ such that for all $x \in U \cap \Omega$, $x \neq \bar{x}$, one has

$$(F(x) + K) \cap (\bar{y} + \alpha\phi(\|x - \bar{x}\|)\mathbb{B}) = \emptyset. \tag{1}$$

Condition (1) can be expressed in the following equivalent way: for all $x \in U \cap \Omega$, $x \neq \bar{x}$, $y \in F(x)$, $b \in \mathbb{B}$, one has

$$y + \alpha\phi(\|x - \bar{x})b \not\leq \bar{y}. \tag{2}$$

(c) *Set optimization problem studied with the set approach*

Let us recall some concepts of solutions to (SOP) studied with the set approach. From now on, **we assume that** $r \in \{l, p\}$.

Definition 4.3. [15, Definition 5.1] We say that $\bar{x}$ is a strict $\preceq_r$-efficient local solution of (SOP) if there exists a neighborhood U of $\bar{x}$ such that for all $x \in U \cap \Omega$, $x \neq \bar{x}$, one has

$$F(x) \not\preceq_r F(\bar{x}).$$

Definition 4.4. [32, Definition 11] We say that a point $\bar{x}$ is a set-based strict local minimizer of order m for F over Ω with respect to the order set relation $\preceq_p$ if there exists a neighborhood U of $\bar{x}$ such that for all $x \in U \cap \Omega$, $x \neq \bar{x}$, one has

$$F(x) \not\preceq_p \mathbb{B}(F(\bar{x}), \alpha\|x - \bar{x}\|^m), \tag{3}$$

where $\mathbb{B}(A, t) := \cup_{a \in A}(a + t\mathbb{B})$ for any nonempty subset A of Y and nonnegative scalar t.

Remark that the order relation $\preceq_p$ has been referred to as "possibly less domination elation" in [32].

Now, let us introduce concepts of strict efficiency for (SOP) studied with the set approach.

Definition 4.5. We say that $\bar{x}$ is a $(\preceq_r, \phi)$-strict local efficient solution of (SOP) or $\bar{x}$ is a $(\preceq_r, \phi)$-strict local minimizer of F over Ω (with constant $\alpha > 0$) if there exists a neighborhood U of $\bar{x}$ such that for all $x \in U \cap \Omega$, $x \neq \bar{x}$, one has

$$F(x) + \alpha\phi(\|x - \bar{x}\|)\mathbb{B} \not\preceq_r F(\bar{x}). \tag{4}$$

When $U = \Omega$ in Definitions 4.1-4.5, we have corresponding global concepts.

Let us state some facts about relationships among concepts of efficiency given in Definitions 4.2-4.5. In what follows, when no confusion occurs, we omit "local/global" or " over Ω" while speaking about solutions of (SOP)/minimizers of F.

Lemma 4.1. (a) *If $\bar{x}$ is a $(\preceq_p, \phi)$-strict solution of (SOP), then it is a $(\preceq_l, \phi)$-strict solution of (SOP).*

(b) *If $\bar{x}$ is a $(\preceq_r, \phi)$-strict solution of (SOP), then then it is a strict $\preceq_r$-efficient solution of (SOP).*

We omit an easy proof of this lemma.

Lemma 4.2. (a) *If $\bar{x}$ is a set-based strict local minimizer of order m of F with respect to the order set relation $\preceq_p$, then it is a $(\preceq_p, \phi)$-strict minimizer of F with $\phi(t) = t^m$.*

(b) *If $\bar{x}$ is a $(\preceq_p, \phi)$-strict minimizer of F and there exists $\bar{y} \in \operatorname{Min}(F(\bar{x}))$, then $(\bar{x}, \bar{y})$ is a ϕ-strict minimizer of F.*

Proof. The first assertion is true because (3) is equivalent to (4) when $r = p$ and $\phi(t) = t^m$ and the second one follows from the fact that if (3) is satisfied and $\bar{y} \in \operatorname{Min}(F(\bar{x}))$, then (2) holds for all $x \in U \cap \Omega$, $x \neq \bar{x}$, $y \in F(x)$ and $b \in \mathbb{B}$. □

Remark 4.1. The second assertion of Lemma 4.2 is motivated by [32, Theorem 4], which has been proved for a strict minimizer in the sense of Definition 4.4.

We provide some examples to illustrate the above concepts of strict efficiency as well as to show that the converses to the assertions of Lemma 4.1 are not true.

Example 4.1. Let $\Omega = X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$ and $\bar{x} = 0$.

(i) *$\bar{x}$ is a $(\preceq_p, \phi)$-strict global minimizer.* Let F be the map in Example 3.1 (i) and $\phi(t) = t^2$. Then $\bar{x}$ is a $(\preceq_p, \phi)$-strict global minimizer of F and (4) is globally satisfied in case $r = p$ with any constant $\alpha \in]1, 0[$. Indeed, observe that

$$F(x) + \alpha|x|^2\mathbb{B} = \quad \{(2x^2 + \alpha x^2 u, \mathrm{e}^{x^2} + \alpha x^2 v) \colon (u, v) \in \mathbb{B}\} \\ \cup \quad \{(x^2 + \alpha x^2 u, 2\mathrm{e}^{x^2} - 1 + \alpha x^2 v) \colon (u, v) \in \mathbb{B}\}.$$

Since $\alpha \in]0, 1[$ and $u \in [-1, 1]$, we have $2x^2 + \alpha x^2 u = x^2(2 + \alpha u) > 0$ and $x^2 + \alpha x^2 u = x^2(1 + \alpha u) > 0$. Hence, if $y \in F(x) + \alpha x^2\mathbb{B}$ and $\bar{y} \in F(\bar{x})$, then $y \not\leq \bar{y}$. This means that (4) holds in case $r = p$ for all $x \neq \bar{x}$.

(ii) *$\bar{x}$ is not a $(\preceq_p, \phi)$–strict local minimizer.* Let F be the map considered in Example 3.1 (ii) and $\phi(t) = t$. Then $\bar{x}$ is not a $(\preceq_p, \phi)$-strict local minimizer of F. Indeed, led $\alpha > 0$ be an arbitrary scalar. For any $x < 0$, take $y = (x, 1)$. Then $y \in F(x) + \alpha|x|\mathbb{B}$ and $y \leq (0, 1) \in F(0)$. Thus, for any neighborhood U of zero, there exists $x \in U$, $x \neq 0$ such that (4) does not hold in case $r = p$.

(iii) *$\bar{x}$ is a $(\preceq_l, \phi)$-strict local minimizer but it is not a $(\preceq_p, \phi)$-strict local minimizer.* Let F be the map

$$F(x) := \begin{cases} \{(-3, \frac{1}{2}\mathrm{e}^x), (-2, \frac{1}{3}\mathrm{e}^x)\} & \text{if } x \neq 0 \\ \{(0, 0), (0, 1)\} & \text{if } x = 0 \end{cases}$$

and $\phi(t) = t$. Let $\alpha > 0$ be an arbitrary scalar. We show that for x sufficiently close to zero, (4) is satisfied in case $r = l$ but is not satisfied in case $r = p$. Indeed, observe that

$$F(x) + \alpha|x|\mathbb{B} = \quad \{(-3 + \alpha|x|u, \tfrac{1}{2}\mathrm{e}^x + \alpha|x|v) \colon (u, v) \in \mathbb{B}\} \\ \cup \quad \{(-2 + \alpha|x|u, \tfrac{1}{3}\mathrm{e}^x + \alpha|x|v) \colon (u, v) \in \mathbb{B}\}.$$

For x sufficiently close to zero and $|u| \leq 1, |v| \leq 1$, we have

$$0 < \min\{\frac{1}{2}\mathrm{e}^x + \alpha|x|v, \frac{1}{3}\mathrm{e}^x + \alpha|x|u\},$$

which means that for any $y \in F(x) + \alpha|x|\mathbb{B}$, one has $y \nleq (0,0)$. Thus, (4) is satisfied in case $r = l$. For x sufficiently close to zero and $y = (-3, \frac{1}{2}\mathrm{e}^x) \in F(x) + \alpha|x|\mathbb{B}$, we have $y \leq (0,1)$, which means that (4) is not satisfied in case $r = p$.

(iv) $\bar{x}$ *is a strict* $\preceq_r$*-efficient local solution of (SOP) but it is not a* $(\preceq_r, \phi)$*-strict local efficient solution.* This happens when F is the map defined by $F(x) = \{(x^2, 0), (x^2, 1)\}$ and $\phi(t) = t$.

(v) $\bar{x}$ *is a* $(\preceq_p, \phi)$*-strict local minimizer but* $\mathrm{Min}(F(\bar{x})) = \emptyset$. Let $\phi(t) = t^2$ and F be the map defined by

$$F(x) := \begin{cases} \{(2x^2, \mathrm{e}^{x^2}), (x^2, 2\mathrm{e}^{x^2} - 1)\} & \text{if } x \neq 0 \\ \{(0,t)\colon\ 0 < t \leq 1\} & \text{if } x = 0. \end{cases}$$

Similar to the case (i), we can show that $\bar{x}$ is a $(\preceq_p, \phi)$-strict local minimizer. The fact that $\mathrm{Min}(F(\bar{x})) = \emptyset$ is obvious.

Next, we show that the concepts of $(\preceq_r, \phi)$-strict local/global minimizer may coincide under the K-convexity condition.

Proposition 4.1. *Assume that F is K-convex and $\phi(t) = t$. If $\bar{x}$ is a $(\preceq_r, \phi)$-strict local minimizer of F, then it is a $(\preceq_r, \phi)$-strict global minimizer of F with the same constant α.*

Proof. Assume that $\bar{x}$ is a $(\preceq_r, \phi)$-strict local minimizer of F. Then there exists a (convex) neighborhood U of $\bar{x}$ such that for all $x \in U \cap \Omega$, $x \neq \bar{x}$, one has

$$F(x) + \alpha\|x - \bar{x}\|\mathbb{B} \not\preceq_r F(\bar{x}). \tag{5}$$

Suppose to the contrary that $\bar{x}$ is not a $(\preceq_r, \phi)$-strict global minimizer of F with the constant α. Then there exists $x \in \Omega$ such that

$$F(x) + \alpha\|x - \bar{x}\|\mathbb{B} \preceq_r F(\bar{x}). \tag{6}$$

Let $x_\lambda := \lambda x + (1-\lambda)\bar{x}$, where $\lambda \in]0,1]$. Observe that $x_\lambda \in U \cap \Omega$, $x_\lambda \neq \bar{x}$ for λ sufficiently close to zero. Therefore, (5) implies

$$F(x_\lambda) + \alpha\|x_\lambda - \bar{x}\|\mathbb{B} \not\preceq_r F(\bar{x}). \tag{7}$$

On the other hand, the K-convexity and (6) imply

$$\begin{aligned}
& F(x)+\alpha\|x-\bar{x}\|\mathbb{B} \preceq_l F(\bar{x}) \\
\Longrightarrow\ & F(\bar{x}) \subseteq F(x)+\alpha\|x-\bar{x}\|\mathbb{B}+K \\
\Longrightarrow\ & \lambda F(\bar{x}) \subseteq \lambda F(x)+\alpha\lambda\|x-\bar{x}\|\mathbb{B}+K = \lambda F(x)+\alpha\|x_\lambda-\bar{x}\|\mathbb{B}+K \\
\Longrightarrow\ & F(\bar{x}) \subseteq \lambda F(\bar{x})+(1-\lambda)F(\bar{x}) \subseteq \lambda F(x)+(1-\lambda)F(\bar{x})+\alpha\|x_\lambda-\bar{x}\|\mathbb{B}+K \\
\Longrightarrow\ & F(\bar{x}) \subseteq F(x_\lambda)+\alpha\|x_\lambda-\bar{x}\|\mathbb{B}+K \\
\Longrightarrow\ & F(x_\lambda)+\alpha\|x_\lambda-\bar{x}\|\mathbb{B} \preceq_l F(\bar{x})
\end{aligned}$$

in case $r=l$ and

$$\begin{aligned}
& F(x)+\alpha\|x-\bar{x}\|\mathbb{B} \preceq_p F(\bar{x}) \\
\Longrightarrow\ & \exists \bar{y}\in F(\bar{x}):\ \bar{y}\in F(x)+\alpha\|x-\bar{x}\|\mathbb{B}+K \\
\Longrightarrow\ & \exists \bar{y}\in F(\bar{x}): \lambda\bar{y}\in \lambda F(x)+\alpha\lambda\|x-\bar{x}\|\mathbb{B}+K = \lambda F(x)+\alpha\|x_\lambda-\bar{x}\|\mathbb{B}+K \\
\Longrightarrow\ & \exists \bar{y}\in F(\bar{x}): \bar{y}=\lambda\bar{y}+(1-\lambda)\bar{y}\in \lambda F(x)+(1-\lambda)F(\bar{x})+\alpha\|x_\lambda-\bar{x}\|\mathbb{B}+K \\
\Longrightarrow\ & \exists \bar{y}\in F(\bar{x}): \bar{y}\in F(x_\lambda)+\alpha\|x_\lambda-\bar{x}\|\mathbb{B}+K \\
\Longrightarrow\ & F(x_\lambda)+\alpha\|x_\lambda-\bar{x}\|\mathbb{B} \preceq_p F(\bar{x})
\end{aligned}$$

in case $r=p$. This is a contradiction to (7). □

5. Scalarization for strict minimizers

Scalarization is one of the most important techniques in vector and set optimization. In this section, we show that $(\preceq_r,\phi)$-strict local minimizers of F are in fact ϕ-strict local minimizers of some scalar functions. Afterward, we study the convexity and Lipschizity properties of these scalarizing functions.

Let $\theta : Y \to \mathbb{R}$ be an abstract scalarizing function and $\bar{x}\in\Omega$. We define a function $g^r_\theta : \Omega \to \mathbb{R}$ as follows:

$$g^r_\theta(x) = h^r_\theta(F(x),F(\bar{x}))$$

(some assumptions will be made to ensure that g^r_θ has finite values). In what follows, we say that F is N-valued if for all $x\in\Omega$, $F(x)$ is N, where N is K-compact, compact or K-bounded.

Proposition 5.1. *Assume that K is solid, θ satisfies Property (P5) and the following conditions are satisfied.*

(a) *(Case $r=l$) F is K-compact-valued.*
(b) *(Case $r=p$) F is compact-valued and $F(\bar{x})=\mathrm{Min}(F(\bar{x}))$.*

Then $\bar{x}$ is a $(\preceq_r,\phi)$-strict local minimizer of F if and only if it is a ϕ-strict local minimizer of g^r_θ (in case $r=p$, the "only if part" is true without the assumption that $F(\bar{x})=\mathrm{Min}(F(\bar{x}))$).

Proof. Observe that since K is solid and θ satisfies Property (P5), one has

$$\beta_1 := \sup_{b\in B} -\theta(b) > 0 \text{ and } \beta_2 := \sup_{b\in B} \theta(b) > 0. \tag{8}$$

This follows from the fact that for $b \in \text{int}K \cap \mathbb{B}$, we have $\theta(-b) < 0$ and, since $\theta(b)+\theta(-b) \geq \theta(0) = 0$, we also have $\theta(b) > 0$. Observe further that under the assumptions (a)-(b), the function g^r_θ has finite values and $g^r_\theta(\bar{x}) = 0$ due to Lemma 2.1.

The "only if" part: We show that if (4) holds then we have

$$g^r_\theta(x) > g^r_\theta(\bar{x}) + \alpha\beta_1\phi(\|x - \bar{x}\|).$$

Indeed, (4) implies that the following situations occur:

- Case $r = l$: $\exists \bar{y} \in F(\bar{x})$ such that $\forall y \in F(x)$, $\forall b \in \mathbb{B}$, one has $y + \alpha\phi(\|x - \bar{x}\|)b \not\leq \bar{y}$.
- Case $r = p$: $\forall \bar{y} \in F(\bar{x})$, $\forall y \in F(x)$, $\forall b \in \mathbb{B}$, one has $y + \alpha\phi(\|x - \bar{x}\|)b \not\leq \bar{y}$.

In both cases we have $\theta(y - \bar{y} + \alpha\phi(\|x - \bar{x}\|)b) > 0$ and

$$\theta(y - \bar{y}) \geq \theta(y - \bar{y} + \alpha\phi(\|x - \bar{x}\|)k_0) - \theta(\alpha\phi(\|x - \bar{x}\|)b) > \alpha(-\theta(b))\phi(\|x - \bar{x}\|).$$

As $b \in \mathbb{B}$ is arbitrary, we get

$$\theta(y - \bar{y}) > \alpha(\sup_{b\in\mathbb{B}} -\theta(b))\phi(\|x - \bar{x}\|) = \alpha\beta_1\phi(\|x - \bar{x}\|). \tag{9}$$

Recall that by Lemma 2.1 and the conditions (a)-(c), the "sup" and "inf" in the definition of h^r_θ can be replaced by "max" and "min". Therefore, (9) implies that

$$h^r_\theta(F(x), F(\bar{x})) > \alpha\beta_1\phi(\|x - \bar{x}\|).$$

Thus, we have $g^r_\theta(x) > g^r_\theta(\bar{x}) + \alpha\beta_1\phi(\|x - \bar{x}\|)$, as it was to be shown.

The "if" part: We show that if (4) does not hold then

$$g^r_\theta(x) \leq g^r_\theta(\bar{x}) + \alpha\beta_2\phi(\|x - \bar{x}\|).$$

Indeed, if (4) does not hold then the following situations occur:

- Case $r = l$: $\forall \bar{y} \in F(\bar{x})$, $\exists y \in F(x)$, $\exists b \in \mathbb{B}$ such that $y - \alpha\phi(\|x - \bar{x}\|)b \leq \bar{y}$.
- Case $r = p$: $\exists \bar{y} \in F(\bar{x})$, $\exists y \in F(x)$, $\exists b \in \mathbb{B}$ such that $y - \alpha\phi(\|x - \bar{x}\|)b \leq \bar{y}$.

In both cases we have $\theta(y - \bar{y} - \alpha\phi(\|x - \bar{x}\|)b) \leq 0$ and

$$\theta(y - \bar{y}) \leq \theta(y - \bar{y} - \alpha\phi(\|x - \bar{x}\|)b) + \theta(\alpha\phi(\|x - \bar{x}\|)b) \leq \alpha(\theta(b))\phi(\|x - \bar{x}\|).$$

Hence,

$$\theta(y - \bar{y}) \leq \alpha \sup_{b\in\mathbb{B}}(\theta(b))\phi(\|x - \bar{x}\|) = \alpha\beta_2\phi(\|x - \bar{x}\|).$$

It follows that

$$h^r_\theta(F(x), F(\bar{x})) \leq \alpha\beta_2\phi(\|x - \bar{x}\|).$$

Thus, we have $g^r_\theta(x) \leq \alpha\beta_2\phi(\|x - \bar{x}\|)$, as it was to be shown. ∎

Remark 5.1. Since $-\theta(-b) \leq \theta(b)$ for all $b \in Y$, we have $\beta_1 \geq \beta_2$. Let $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. One can check that $\beta_1 = \sqrt{2}/2$, $\beta_2 = 1$ when θ is the Hiriart-Urruty signed distance function $\Delta_{-\mathbb{R}^2_+}$ and $\beta_1 = 1$, $\beta_2 = \sqrt{2}$ when θ is the Gerstewizt scalarizing function φ_{k_0} and $k_0 = (1, 1)$.

Next, we consider the convexity and Lipschitzity properties of the functions g^r_θ.

Definition 5.1. We say that

(i) F is locally Lipschitz around $\bar{x}$ [3] if there is a scalar $L > 0$ and a neighborhood U of $\bar{x}$ such that for any $x_1, x_2 \in U \cap \Omega$

$$F(x_1) \subseteq F(x_2) + L\|x_1 - x_2\|\mathbb{B}.$$

(ii) F is locally K-Lipschitz around $\bar{x}$ [1] if there is a scalar $L > 0$ and a neighborhood U of $\bar{x}$ such that for any $x_1, x_2 \in U \cap \Omega$

$$F(x_1) \subseteq F(x_2) + L\|x_1 - x_2\|\mathbb{B} + K.$$

Proposition 5.2. *Assume that the functions g^r_θ have finite values.*

(a) *If F is K-convex, then g^r_θ are convex.*

(b) *Assume that K is solid and θ also has Property (P5). If F is locally K-Lipschitz around $\bar{x} \in \Omega$, then g^r_θ are locally Lipschitz around $\bar{x}$.*

Proof. (a) Let $x_1, x_2 \in \Omega$ and $\lambda \in [0, 1]$. Denote $x_\lambda := \lambda x_1 + (1 - \lambda)x_2$. Let $\hat{y}_1 \in F(x_1), \hat{y}_2 \in F(x_2)$. Since F is K-convex, there exists $\hat{y}_\lambda \in F(x_\lambda)$ such that $\lambda\hat{y}_1 + (1 - \lambda)\hat{y}_2 \geq \hat{y}_\lambda$. Then

$$\lambda(\hat{y}_1 - \bar{y}) + (1 - \lambda)(\hat{y}_2 - \bar{y}) \geq \hat{y}_\lambda - \bar{y}, \quad \forall \bar{y} \in F(\bar{x})$$

and

$$\lambda\theta(\hat{y}_1 - \bar{y}) + (1 - \lambda)\theta(\hat{y}_2 - \bar{y}) \geq \theta(\hat{y}_\lambda - \bar{y}) \geq \inf_{y_\lambda \in F(x_\lambda)} \theta(y_\lambda - \bar{y}), \quad \forall \bar{y} \in F(\bar{x}).$$

Since $\hat{y}_1 \in F(x_1)$ and $\hat{y}_2 \in F(x_2)$ can be arbitrarily chosen, we get

$$\lambda \inf_{y_1 \in F(x_1)} \theta(y_1 - \bar{y}) + (1 - \lambda) \inf_{y_2 \in F(x_2)} \theta(y_2 - \bar{y}) \geq \inf_{y_\lambda \in F(x_\lambda)} \theta(y_\lambda - \bar{y}), \quad \forall \bar{y} \in F(\bar{x}).$$

Again, since $\bar{y} \in F(\bar{x})$ can be arbitrarily chosen, we get

$$\lambda g^l_\theta(x_1) + (1 - \lambda)g^l_\theta(x_2) \geq g^l_\theta(x_\lambda)$$

and

$$\lambda g^p_\theta(x_1) + (1 - \lambda)g^p_\theta(x_2) \geq g^p_\theta(x_\lambda).$$

This means that the functions g^r_θ are convex.

(b) Let L, U, x_1 and x_2 as in the definition of the K-Lipschizity. For any $\hat{y}_1 \in F(x_1)$ there exists $\hat{y}_2 \in F(x_2)$ and $\hat{b} \in \mathbb{B}$ such that $\hat{y}_2 - \hat{y}_1 \leq L\|x_1 - x_2\|\hat{b}$. Then for all $\bar{y} \in F(\bar{x})$, we have

$$\theta(\hat{y}_2 - \bar{y}_1) \leq \theta(L\|x_1 - x_2\|\bar{b}) \leq L \sup_{b \in B} \theta(b)\|x_1 - x_2\| = L\beta_2\|x_1 - x_2\|$$

and

$$\theta(\hat{y}_2 - \bar{y}) \leq \theta(\hat{y}_1 - \bar{y}) + \theta(\hat{y}_2 - \bar{y}_1) \leq \theta(\hat{y}_1 - \bar{y}) + L\beta_2\|x_1 - x_2\|,\ \forall \bar{y} \in F(\bar{x}),$$

where β_2 is defined by (8). Therefore,

$$\inf_{y_2 \in F(x_2)} \theta(y_2 - \bar{y}) \leq \theta(\hat{y}_2 - \bar{y}) \leq \theta(\hat{y}_1 - \bar{y}) + L\beta_2\|x_1 - x_2\|, \quad \forall \bar{y} \in F(\bar{x})$$

and since $\hat{y}_1 \in F(x_1)$ is arbitrary, we obtain

$$\inf_{y_2 \in F(x_2)} \theta(y_2 - \bar{y}) \leq \inf_{y_1 \in F(x_1)} \theta(y_1 - \bar{y}) + L\beta_2\|x_1 - x_2\|, \quad \forall \bar{y} \in F(\bar{x}).$$

Again, since $\bar{y} \in F(\bar{x})$ can be arbitrarily chosen, we get

$$g^l_\theta(x_2) \leq g^l_\theta(x_1) + L\beta_2\|x_1 - x_2\|$$

and

$$g^p_\theta(x_2) \leq g^p_\theta(x_1) + L\beta_2\|x_1 - x_2\|.$$

Thus, the functions g^r_θ are locally Lipschitz around $\bar{x}$. □

Example 5.1. Let $\Omega = X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $\theta = \Delta_{-K}$, $\bar{x} = 0$ and F be the map defined by $F(x) := \{(0, \mathrm{e}^{|x|}), (|x|, 1)\}$. It is easy to see that F is locally Lipschitz. Observe that $F(\bar{x}) = \{(0, 1)\}$ and hence, $g^l_\theta = g^p_\theta$. One can check that

$$g^l_\theta(x) = \inf_{y \in F(x)} \Delta_{-\mathbb{R}^2_+}(y - (0,1)) = \begin{cases} x & \text{if } x \geq 0 \\ \mathrm{e}^{-x} - 1 & \text{if } x < 0 \end{cases}$$

and the functions g^l_θ and g^p_θ are locally Lipschitz.

6. Necessary and sufficient conditions for $(\preceq_r, \phi)$-strict minimizers

The aim of this section is to establish optimality conditions for $\bar{x} \in \Omega$ to be a $(\preceq_r, \phi)$-strict local/global minimizer of F. These conditions are either imposed on the set difference $F(x) - F(\bar{x})$, on $h^r_\theta(F(x), F(\bar{x}))$ or expressed in terms of the directional derivative $D^k_\theta F(\bar{x}, d)$ and the subdifferential $\partial F(\bar{x})$. In what follows, the notation "$x \to_\Omega \bar{x}$" means $x \to \bar{x}$ and $x \in \Omega$.

Let us start with optimality conditions for a $(\preceq_p, \phi)$-strict minimizer.

Theorem 6.1. *The following assertions are equivalent.*

(a) *$\bar{x}$ is a $(\preceq_p, \phi)$-strict local minimizer of F.*

(b) *There exist a neighbrhood U of $\bar{x}$ and a scalar $\alpha > 0$ such that the following equality holds*

$$(F(x) + K) \cap (F(\bar{x}) + \alpha\phi(\|x - \bar{x}\|)\mathbb{B}) = \emptyset, \quad \forall x \in (U \cap \Omega) \setminus \{\bar{x}\}. \tag{10}$$

(c) *The following relation holds*

$$0 \notin \text{Limsup}_{x \to_\Omega \bar{x},\ x \neq \bar{x}} \frac{F(x) - F(\bar{x}) + K}{\phi(\|x - \bar{x}\|)} \tag{11}$$

(the set in the right hand side of (11) is the set of all cluster points).

Proof. (a) $\Longleftrightarrow$ (b) It is immediate from the fact that if A and B are nonempty subsets of Y and t is a nonegative scalar, then

$$A \not\preceq_p B \Longleftrightarrow (A + K) \cap B = \emptyset$$

and

$$(A + t\mathbb{B}) \cap B = \emptyset \Longleftrightarrow A \cap (B + t\mathbb{B}) = \emptyset.$$

(b) $\Longrightarrow$ (c) Suppose to contrary that (10) is true but (11) is not. Then there exist a sequence $x_n \to_\Omega \bar{x}$, $x_n \neq \bar{x}$, $y_n \in F(x_n)$, $\bar{y}_n \in F(\bar{x})$ and $k_n \in K$ such that

$$\frac{y_n - \bar{y}_n + k_n}{\phi(\|x_n - \bar{x}\|)} \to 0.$$

For any $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that for $n \geq n_0$ one has

$$\|y_n + k_n - \bar{y}_n\| \leq \epsilon\phi(\|x_n - \bar{x}\|),$$

which means that $y_n + k_n \in \bar{y}_n + \epsilon\phi(\|x_n - \bar{x}\|)\mathbb{B}$. Since we can take $\epsilon < \alpha/2$, it follows that $(F(x_n) + K) \cap (F(\bar{x}) + \alpha\phi(\|x_n - \bar{x}\|)\mathbb{B}) \neq \emptyset$, a contradiction to (10).

(c) $\Longrightarrow$ (b) Suppose to contrary that (11) is true but (10) is not, i.e., for any positive integer n, there exists x_n such that $x_n \to_\Omega \bar{x}$, $x_n \neq \bar{x}$ and

$$(F(x_n) + K) \cap (F(\bar{x}) + \frac{1}{n}\phi(\|x_n - \bar{x}\|)\mathbb{B}) \neq \emptyset, \quad \forall n.$$

Then one can find $y_n \in F(x_n)$, $k_n \in K$, $\bar{y}_n \in F(\bar{x})$ and $b_n \in \mathbb{B}$ such that $y_n + k_n = \bar{y}_n + \frac{1}{n}\phi(\|x_n - \bar{x}\|)b_n$. It follows that

$$\frac{y_n - \bar{y}_n + k_n}{\phi(\|x_n - \bar{x}\|)} = \frac{1}{n}b_n,$$

a contradiction to (11). □

Corollary 6.1. *If $\bar{x} \in \Omega$ is a $(\preceq_p, \phi)$-strict local minimizer of F, then*

$$\text{Limsup}_{x \to_\Omega \bar{x},\ x \neq \bar{x}} \frac{F(x) - F(\bar{x})}{\phi(\|x - \bar{x}\|)} \cap (-K) = \emptyset. \tag{12}$$

In the case $\Omega \subset \mathbb{R}^m$ and $K = \mathbb{R}^m_+$, $\bar{x}$ is a $(\preceq_p, \phi)$-strict local minimizer of F if

$$\text{Limsup}_{x \to_\Omega \bar{x},\ x \neq \bar{x}} \frac{F(x) - F(\bar{x})}{\phi(\|x - \bar{x}\|)} \cap [-\infty, 0]^m = \emptyset. \tag{13}$$

Proof. First, suppose to the contrary that (12) does not hold. Then there exist $k \in K$, a sequence $x_n \to_\Omega \bar{x}$, $x_n \neq \bar{x}$, $y_n \in F(x_n)$ and $\bar{y}_n \in F(\bar{x})$ such that

$$\frac{y_n - \bar{y}_n}{\phi(\|x_n - \bar{x}\|)} \to -k$$

and hence,

$$\frac{y_n - \bar{y}_n + k\phi(\|x_n - \bar{x}\|)}{\phi(\|x_n - \bar{x}\|)} \to 0,$$

which is a contradiction to (11).

Next, suppose to the contrary that $\bar{x} \in \Omega$ is not a $(\preceq_p, \phi)$-strict local minimizer of F. Proposition 6.1 implies that (11) does not hold. Then there exist a sequence $x_n \to_\Omega \bar{x}$, $x_n \neq \bar{x}$, $y_n \in F(x_n)$, $\bar{y}_n \in F(\bar{x})$ and $k_n \in \mathbb{R}^m_+$ such that

$$\frac{y_n - \bar{y}_n + k_n}{\phi(\|x_n - \bar{x}\|)} \to 0.$$

Applying the arguments used in the proof of Proposition 3.5 (b) in [23], we can find subsequences $x_{n_j} \to_\Omega \bar{x}$, $y_{n_j} \in F(x_{n_j})$, $\bar{y}_{n_j} \in F(\bar{x})$ and $k \in [0, \infty]^m$ such that

$$\frac{y_{n_j} - \bar{y}_{n_j}}{\phi(\|x_{n_j} - \bar{x}\|)} \to -k.$$

This is a contradiction to (13). □

Let us illustrate Theorem 6.1 and Corollary 6.1.

Example 6.1. (i) Let F be the map in Example 3.1 (i) and $\phi(t) = t^2$. Condition (11) is satisfied, namely,

$$(0,0) \notin \text{Limsup}_{x \to_\Omega \bar{x},\ x \neq \bar{x}} \frac{F(x) - F(\bar{x}) + K}{\phi(\|x - \bar{x}\|)} = \{(u, v)\colon u \geq 1, v \geq 0\}.$$

Hence, $\bar{x} = 0$ is a $(\preceq_p, \phi)$-strict global minimizer (with $\alpha \in]0, 1[$).

(ii) Let F be the map in Example 3.1 (ii) and $\phi(t) = t$. Condition (12) is not satisfied, namely,

$$\text{Limsup}_{x \to_\Omega \bar{x},\ x \neq \bar{x}} \frac{F(x) - F(\bar{x})}{\phi(\|x - \bar{x}\|)} = \{(-1, 0)\} \in \mathbb{R}^2_+.$$

Hence, $\bar{x} = 0$ is not a $(\preceq_p, \phi)$-strict global minimizer.

(iii) Let F be the map considered in Example 3.2 and $\phi(t) = t$. Condition (13) is satisfied, namely,

$$\text{Limsup}_{x \to_\Omega \bar{x},\ x \neq \bar{x}} \frac{F(x) - F(\bar{x})}{\phi(\|x - \bar{x}\|)} = [(1, 1), (1, 2)] \cap [-\infty, 0]^2 = \emptyset.$$

Hence, $\bar{x} = 0$ is a $(\preceq_p, \phi)$-strict local minimizer and since the map F is K-convex, $\bar{x}$ is a $(\preceq_p, \phi)$-strict global minimizer.

Theorem 6.2. *Assume that K is solid, θ satisfies (P5), $\phi(t) = t^k$ and the directional derivative $D^k_\theta F(\bar{x}, d)$ exists for all admissible direction d.*

(a) *If $\bar{x}$ is a $(\preceq_p, \phi)$-strict local minimizer of F, then for any admissible direction d we have*

$$D^k_\theta F(\bar{x}, d) \cap (-K) = \emptyset \tag{14}$$

and

$$\inf\{\Delta_{-K}(u)\colon u \in D^k_\theta F(\bar{x}, d)\} \geq 0. \tag{15}$$

(b) *Assume that F is K-convex, $\theta = \Delta_{-K}$ and $k = 1$. Then $\bar{x}$ is a $(\preceq_p, \phi)$-strict global minimizer of F if*

$$\inf\{\Delta_{-K}(u)\colon u \in D^1_\theta F(\bar{x}, d), d \text{ is an admissible direction, } \|d\| = 1\} > 0. \tag{16}$$

Proof. (a) Let α and U be as in the definition of a $(\preceq_p, \phi)$-strict local minimizer of F. Suppose to the contrary that there exist an admissible direction d and a vector $v \in D^k_\theta(\bar{x}, d) \cap (-K)$. Without lost of generality, we may assume that $\|d\| = 1$. Let $\delta > 0$ be a scalar such that $\bar{x} + td \in (U \cap \Omega) \setminus \{\bar{x}\}$ for all $t \in]0, \delta[$. Fix $\hat{t} \in]0, \delta[$ and denote $\hat{x} := \bar{x} + \hat{t}d$. Observe that $\hat{t} = \|\hat{x} - \bar{x}\|$. Then (4) (in case $r = p$) implies that for all $\hat{y} \in F(\hat{x})$, $\bar{y} \in F(\bar{x})$ and $b \in \mathbb{B}$ we have $\hat{y} + \alpha\|\hat{x} - \bar{x}\|^k b \not\leq \bar{y}$. Since $v \in -K$, we get

$$\frac{\hat{y} - \bar{y}}{\|\hat{x} - \bar{x}\|^k} - v + \alpha b \notin -K$$

and therefore,

$$\theta(\frac{\hat{y} - \bar{y}}{\|\hat{x} - \bar{x}\|^k} - v + \alpha b) > 0.$$

We have

$$\theta(\frac{\hat{y} - \bar{y}}{\|\hat{x} - \bar{x}\|^k} - v) \geq \theta(\frac{\hat{y} - \bar{y}}{\|\hat{x} - \bar{x}\|^k} - v + \alpha b) - \theta(\alpha b) > \alpha(-\theta(b)$$

and since $b \in \mathbb{B}$ is arbitrary, we get

$$\theta(\frac{\hat{y} - \bar{y}}{\|\hat{x} - \bar{x}\|^k} - v) \geq \alpha \sup_{b \in B}(-\theta(b)) = \alpha\beta_1,$$

where β_1 is defined by (8). Hence,

$$h^l_\theta(\frac{F(\hat{x}) - F(\bar{x})}{\|\hat{x} - \bar{x}\|^k}, D^k_\theta F(\bar{x}, d)) = \sup_{v \in D^k F(\bar{x}, d)} \inf_{\hat{y} \in F(\hat{x}), \bar{y} \in F(\bar{x})} \theta(\frac{\hat{y} - \bar{y}}{\|\hat{x} - \bar{x}\|^k} - v) \geq \alpha\beta_1.$$

Since $d_\theta(A, B) \geq h^l_\theta(A, B)$ by the definition, we get

$$d_\theta(\frac{F(\bar{x} + \hat{t}d) - F(\bar{x})}{\hat{t}^k}, D^k_\theta F(\bar{x}, d)) \geq \alpha\beta_1.$$

The just obtained inequality holds for all $\hat{t} \in]0, \delta[$, and this is a contradiction to the definition of the directional derivative. The fact that (14) implies (15) is obvious.

(b) Suppose to the contrary that (16) holds but $\bar{x}$ is not a $(\preceq_p, \phi)$-strict global minimizer of F. Since F is K-convex, Proposition 4.1 implies that $\bar{x}$ is not a $(\preceq_p, \phi)$-strict local minimizer of F. Proposition 6.1 implies

$$0 \in \mathrm{Limsup}_{x \to_\Omega \bar{x},\ x \neq \bar{x}} \frac{F(x) - F(\bar{x}) + K}{\|x - \bar{x}\|}.$$

Recall that for any $x \in \Omega$, $x - \bar{x}$ is an admissible direction because Ω is a convex set. Let $\theta = \Delta_{-K}$, which is known to be an abstract scalarizing function satisfying Properties (P5). Due to [15, Proposition 4.4], we have $F(x) - F(\bar{x}) \subseteq D^1_\theta F(\bar{x}, x - \bar{x}) + K$. Hence,

$$\frac{F(x) - F(\bar{x}) + K}{\|x - \bar{x}\|} \subseteq D^1_\theta F(\bar{x}, \frac{x - \bar{x}}{\|x - \bar{x}\|}) + K$$

and

$$\mathrm{Limsup}_{x \to_\Omega \bar{x}, x \neq \bar{x}} \frac{F(x) - F(\bar{x}) + K}{\|x - \bar{x}\|} \subseteq \mathrm{Limsup}_{x \to_\Omega \bar{x}, x \neq \bar{x}} (D^1_\theta F(\bar{x}, \frac{x - \bar{x}}{\|x - \bar{x}\|}) + K).$$

It follows that

$$0 \in \mathrm{Limsup}_{x \to \bar{x}, x \neq \bar{x}} (D^1_\theta F(\bar{x}, \frac{x - \bar{x}}{\|x - \bar{x}\|}) + K).$$

One can find sequences $x_i \to_\Omega \bar{x}$, $x_i \neq \bar{x}$, $u_i \in D^1_\theta F(\bar{x}, \frac{x_i - \bar{x}}{\|x_i - \bar{x}\|})$, $k_i \in K$ such that $u_i + k_i \to 0$. Since $\Delta_{-K}(u_i) \leq \Delta_{-K}(u_i + k_i)$, we get

$$\limsup_{i \to \infty} \Delta_{-K}(u_i) \leq \lim_{i \to \infty} \Delta_{-K}(u_i + k_i) = 0,$$

a contradiction to (16). □

Example 6.2. Conditions (14)-(15) are satisfied for the map considered in Example 3.1 (i) with $\phi(t) = t^2$ and are not satisfied for the map considered in Example 3.1 (ii) with $\phi(t) = t$. Conditions (16) is satisfied for the map considered in Example 3.2.

The following sufficient condition is a set-valued version of [9, Proposition 3.4].

Theorem 6.3. *Assume that X is a Banch space, F is K-convex and there exists $\mu > 0$ such that*

$$\mu \mathbb{B}_{X^*} \subset \partial F(\bar{x}) + N(\Omega, \bar{x}). \tag{17}$$

Let $\phi(t) = t$. Then $\bar{x}$ is a $(\preceq_p, \phi)$-strict global minimizer of F over Ω with (any) $\alpha \in]0, \mu[$.

Proof. Suppose to the contrary that there exists $\tilde{x} \in \Omega$, $\tilde{x} \neq \bar{x}$ such that

$$F(\tilde{x}) + \alpha \|\tilde{x} - \bar{x}\| \mathbb{B} \preceq_p F(\bar{x})$$

and therefore, one can find $\tilde{y} \in F(\tilde{x})$, $\bar{y} \in F(\bar{x}), b \in \mathbb{B}$ and $k \in K$ such that

$$\tilde{y} + \alpha \|\tilde{x} - \bar{x}\| b + k = \bar{y}$$

It follows from a consequence of the Hahn-Banach theorem that there exists $u^* \in X^*$ such that $\|u^*\| = 1$ and

$$\langle u^*, \tilde{x} - \bar{x} \rangle = \|\tilde{x} - \bar{x}\|.$$

By the assumption, there exists $x^* \in \partial F(\bar{x})$ such that $\mu u^* - x^* \in N(\Omega, \bar{x})$. By the definition of the normal cone, we have $\langle \mu u^* - x^*, \tilde{x} - \bar{x} \rangle \leq 0$. Hence, we get

$$\mu\|\tilde{x} - \bar{x}\| = \langle \mu u^*, \tilde{x} - \bar{x} \rangle \leq \langle x^*, \tilde{x} - \bar{x} \rangle. \tag{18}$$

On the other hand, since $x^* \in \partial F(\bar{x})$, there exist $\bar{y} \in F(\bar{y})$, $k^* \in K^*$, $\|k^*\| = 1$ such that $(x^*, -k^*) \in N(\text{epi}F(\bar{x}, \bar{y}))$. Then $\langle (x^*, -k^*), (\tilde{x}, \tilde{y}) - (\bar{x}, \bar{y}) \rangle \leq 0$ and hence,

$$\begin{aligned} \langle x^*, \tilde{x} - \bar{x} \rangle &\leq \langle k^*, \tilde{y} - \bar{y} \rangle = \langle k^*, -\alpha\|\tilde{x} - \bar{x}\|b - k \rangle \leq \langle k^*, -\alpha\|\tilde{x} - \bar{x}\|b \rangle \\ &\leq \alpha\|\tilde{x} - \bar{x}\|\|k^*\|\|b\| < \mu\|\tilde{x} - \bar{x}\|, \end{aligned}$$

which is a contradiction to (18). □

Example 6.3. Condition (17) is satisfied with $\mu = 1$ for the map considered in Example 3.2 and hence, $\bar{x}$ is a $(\preceq_p, \phi)$-strict global minimizer of F.

Remark 6.1. If there exists $\bar{y} \in \text{Min}(F(\bar{x}))$, then due to Lemma 4.2 (ii), one can get necessary conditions for $\bar{x}$ to be a $(\preceq_p, \phi)$-strict local/global minimizer from the known ones for a ϕ-strict minimizer $(\bar{x}, \bar{y})$ in the sense of Definition 4.2. This argument has been used in the proof of [32, Theorem 5].

Next, we consider optimality conditions for a $(\preceq_r, \phi)$-strict minimizer of F in both the cases $r = l$ and $r = p$. Our argument is based on using the scalar characterization of these minimizers and optimality conditions for strict local minimizers of a scalar-valued function obtained by Studniarski [34] and Durea [9].

Theorem 6.4. *Assume that $\Omega = X$ and the conditions of Proposition 5.1 are satisfied. Then $\bar{x}$ is a $(\preceq_r, \phi)$-strict local minimizer of F with $\phi(t) = t$ if and only if the following condition is satisfied:*

(i) *Case F is locally K-Lipschitz: For all $v \in X \setminus \{0\}$*

$$\liminf_{t \downarrow 0^+} \frac{h^r_\theta(F(\bar{x} + tv), F(\bar{x}))}{t} > 0.$$

(ii) *Case F is K-convex: For all $v \in X \setminus \{0\}$*

$$\lim_{t \downarrow 0^+} \frac{h^r_\theta(F(\bar{x} + tv), F(\bar{x}))}{t} > 0.$$

Proof. Due to Proposition 5.1, $\bar{x}$ is a $(\preceq_r, \phi)$-strict local minimizer of F with $\phi(t) = t$ if and only if $\bar{x}$ is a strict local minimizer of the function g^r_θ, where $g^r_\theta(x) = h^r_\theta(F(x), F(\bar{x}))$.

Proposition 5.2 implies that g^r_θ is locally Lipschitz or convex if F is locally K-Lipschitz or K-convex, respectively. Theorem 2.1 in [34] applied to g^r_θ states that $\bar{x}$ is a strict local minimizer of the function g^r_θ if and only if the following condition is satisfied: For all $v \in X \setminus \{0\}$

$$\liminf_{t\downarrow 0^+} \frac{g^r_\theta(\bar{x}+tv) - g^r_\theta(\bar{x})}{t} > 0$$

in case g^r_θ is locally Lipschitz and

$$\lim_{t\downarrow 0^+} \frac{g^r_\theta(\bar{x}+tv) - g^r_\theta(\bar{x})}{t} > 0$$

in case g^r_θ is convex. Since $g^r_\theta(\bar{x}) = 0$ due to Lemma 2.1, the assertion follows. □

Example 6.4. The locally K-Lipschitz map F considered in Example 5.1 satisfies conditions of Theorem 6.4. Indeed, for $r \in \{l, p\}$, $\theta = \Delta_{-\mathbb{R}^2_+}$ and $v \in \mathbb{R}$, $v \neq 0$, we have

$$h^r_\theta(F(\bar{x}+tv), F(\bar{x})) = g^r_\theta(tv) = \begin{cases} tv & \text{if } v \geq 0 \\ \mathrm{e}^{-tv} - 1 & \text{if } v < 0 \end{cases}$$

and

$$\liminf_{t\downarrow 0^+} \frac{h^r_\theta(F(\bar{x}+tv), F(\bar{x}))}{t} = \begin{cases} v & \text{if } v \geq 0 \\ -v & \text{if } v < 0. \end{cases}$$

Remark 6.2. Assume in the case (i) of Theorem 6.4 that X is an Asplund space. Theorem 3.7 in [9] states that $\bar{x}$ is a ϕ-strict local minimizer of g^r_θ with $\alpha > 0$ and $\phi(t) = t$ (so $\bar{x}$ is a $(\preceq_r, \phi)$-strict local minimizer of F) only if

$$\alpha\mathbb{B} \subset \tilde{\partial} g^r_\theta(\bar{x}) + \tilde{N}(\Omega, \bar{x}),$$

where $\tilde{\partial} g^r_\theta(\bar{x})$ and $\tilde{N}(\Omega, \bar{x})$ are the limiting (or Mordukhovich) subdifferential of the function g^r_θ at $\bar{x}$ and the limiting (or the Mordukhovich) normal cone to the set Ω at $\bar{x}$. We refer an interested reader for the concepts of Asplund space, limiting subdifferential and limiting cone to the book [33].

7. Conclusions

We introduced the concepts of $(\preceq_r, \phi)$-strict efficient solutions ($r \in \{l, p\}$) for a set optimization problem studied with the set approach with respect to the l-less set order relation $\preceq_l$ and the possibly less set order relations $\preceq_p$ and an admissible function ϕ. We obtained scalar characterization and some necessary/sufficient conditions for these solutions and provided various illustrative examples. In particular, some conditions for $(\preceq_p, \phi)$-strict efficient solutions have been expressed in terms of high-order directional derivative and subdifferential of a set-valued map. It is of interest to study further optimality conditions for $(\preceq_l, \phi)$-strict efficient solutions.

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