An exponential cone representation of the general power cone

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Abstract

Chandrasekaran and Shah (2017) used the exponential cone to model the second-order cone in demonstration of its modeling capabilities. We simplify and extend this result to general power cones.

1 Main result

Considering the identity $x^a = \exp(a\log(x))$ on $x > 0$, it is not hard to imaging that at least some subset of powers should be representable in terms of the exponential cone. Indeed, under the definition

$$K_{\mathrm{exp}} := \mathbf{cl}\left\{(t,s,x) \in \mathbb{R}^3 \mid s > 0,\ t \geq s\exp\left(\frac{x}{s}\right)\right\}, \tag{1}$$

in which the closure operator adds the face $\left\{(t,s,x) \in \mathbb{R}^3 \mid s = 0,\ t \geq 0,\ x \leq 0\right\}$, we find that

$$\begin{array}{rl} & t \geq x^a, \\ \Leftrightarrow & t \geq \exp(a\log(x)), \\ \Leftrightarrow & t \geq \exp(au),\ u \leq \log(x), \\ \Leftrightarrow & (t,1,au) \in K_{\mathrm{exp}},\ (x,1,u) \in K_{\mathrm{exp}}, \end{array} \tag{2}$$

holds on $x > 0$ for any power $a \leq 0$, as verified by standard convex composition rules noting that $\exp(au)$ is convex nonincreasing and $\log(x)$ is concave. These steps can be repeated for the general identity $\prod_{i=1}^k x_i^{\alpha_i} = \exp(\sum_{i=1}^k \alpha_i \log(x_i))$ on $x \in \mathbb{R}^k_{++}$, leading straight to an important intermediate result.

Lemma 1.1 *The monomial inequality, $t \geq x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ on $x \in \mathbb{R}^k_{++}$, can be represented for any set of negative powers, $\alpha_i \leq 0$, by*

$$(t, 1, \textstyle\sum_{i=1}^k \alpha_i u_i) \in K_{\mathrm{exp}}, \quad (x_i, 1, u_i) \in K_{\mathrm{exp}}\ \forall i = 1,\ldots,k.$$

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With a bit of rearrangement and rescaling, the result above provides us with a representation of a general power cone inequality.

Lemma 1.2 *The power cone inequality,* $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} \geq r \geq 0$ *on* $x \in \mathbb{R}_+^k$*, can be represented on the subset of powers,* $\sum_{i=1}^k \alpha_i = 1$ *and* $\alpha_i \geq 0$*, by*

$$\textstyle\sum_{i=1}^k \alpha_i u_i = 0, \quad (x_i, r, u_i) \in K_{\exp}\ \forall i = 1, \ldots, k. \tag{3}$$

Proof *Assume* $r = 1$ *and* $x \in \mathbb{R}_{++}^k$ *to rearrange the inequality as*

$$1 \geq x_1^{-\alpha_1} x_2^{-\alpha_2} \cdots x_k^{-\alpha_k},$$

and apply Lemma 1.1. The constraint $(1, 1, \sum_{i=1}^k -\alpha_i u_i) \in K_{\exp}$ *simplifies to* $\sum_{i=1}^k \alpha_i u_i \geq 0$ *and is enforced with equality in* (3) *by exploiting the opportunity for slack in* $(x_i, 1, u_i) \in K_{\exp} \Leftrightarrow u_i \leq \log(x_i)$*. Finally, by homogenization, substituting* x_i *by* $\frac{x_i}{r}$*, we obtain the claimed result on* $x \in \mathbb{R}_{++}^k$ *and* $r > 0$*. The lack of singularities on the boundary allows us to relax the domain.*

2 Examples and variants

The *quadratic power cone*, $x_1 x_2 \geq x_3^2$ on $x \in \mathbb{R}_+^2 \times \mathbb{R}$, can be rewritten as $x_1^{0.5} x_2^{0.5} \geq |x_3|$ and represented by

$$(x_1, r, u) \in K_{\exp}, \quad (x_2, r, -u) \in K_{\exp}, \quad r \geq x_3 \geq -r. \tag{4}$$

In turn, we can rewrite the *second-order cone*, $t \geq \|x\|_2$, as

$$t \geq \textstyle\sum_{i=1}^k w_i, \quad t w_i \geq x_i^2\ \forall i = 1, \ldots, k, \tag{5}$$

and use (4) on the quadratic power cones to complete the reformulation.

Finally, the power cone represented by Lemma 1.2, namely

$$K_{\text{pow}}^{\alpha} := \{(x, r) \in \mathbb{R}_+^{k+1} : x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} \geq r\}, \tag{6}$$

can be used to construct more common variants of itself. The *hypograph power cone*, found by replacing $r \in \mathbb{R}_+$ with $y \in \mathbb{R}$, is simply $(x, r) \in K_{\text{pow}}^{\alpha},\ r \geq y$, and the *radial power cone*, found by replacing r with $\|y\|_2$, is just $(x, r) \in K_{\text{pow}}^{\alpha}$ and $r \geq \|y\|_2$, leveraging the second-order cone representation in (5). These representations all offer the luxury of not requiring parameterized cones, simplifying software interfaces and algorithms. Whether they improve performance under any scenario is unknown, however, and in many cases unlikely.

References

[1] Venkat Chandrasekaran and Parikshit Shah. "Relative entropy optimization and its applications". In: *Mathematical Programming* 161 (2017), pp. 1–32.