Stabilizing GNEP-Based Model Predictive Control: Quasi-GNEPs and End Constraints

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Abstract. We present a feedback scheme for non-cooperative dynamic games and investigate its stabilizing properties. The dynamic games are modeled as generalized Nash equilibrium problems (GNEP), in which the shared constraint consists of linear time-discrete dynamic equations (e.g., sampled from a partial or ordinary differential equation), which are jointly controlled by the players' actions. Further, the individual objectives of the players are interdependent and defined over a fixed time horizon. The feedback law is synthesized by moving-horizon model predictive control (MPC). We investigate the asymptotic stability of the resulting closed-loop dynamics. To this end, we introduce α-quasi GNEPs, a family of auxiliary problems based on a modification of the Nikaido–Isoda function, which approximate the original games. Basing the MPC scheme on these auxiliary problems, we derive conditions on the players' objectives, which guarantee asymptotic stability of the closed-loop if stabilizing end constraints are enforced. This analysis is based on showing that the associated optimal-value function is a Lyapunov function. Additionally, we identify a suitable Lyapunov function for the MPC scheme based on the original GNEP, whose solution fulfills the stabilizing end constraints. The theoretical results are complemented by numerical experiments.

1. Introduction

Non-cooperative systems is a field of research of growing importance in control and optimization. They can be found in the context of economics [16, 4], traffic modeling [7], or robotics [12]. Due to disturbances, uncertainties, or model errors, feedback schemes are essential, especially for stabilization tasks. Model predictive control (MPC) is a widely used and flexible feedback mechanism, for which stability guarantees can be derived for a wide range of centralized control systems. This work deals with the question of whether these stability guarantees can be transferred to non-cooperative distributed MPC schemes.

The basic idea of MPC is to solve the underlying (open-loop) optimal control problem for a possibly short look-ahead horizon repetitively and to implement only the first part of the solution to the system during a given sampling time. Due to this very simple construction, this scheme can be applied to a variety of control problems. It has been studied for both linear [28] and nonlinear systems [18] with regard to its stabilizing properties. Although technically it only provides a semi-feedback, considering the scheme for time-discrete systems, usually derived from a discretization of the underlying continuous dynamics, yields a structure for which MPC can be considered as a "real" feedback mechanism. Grüne et al. studied such discrete-time MPC loops extensively in [18] and provided a vast canon of stability analysis based on classic Lyapunov theory and Bellmann's dynamic programming principle. This also includes MPC based on multi-objective [19] and economic dynamic optimization problems [17]. MPC schemes have also been analyzed for continuous dynamics directly [10, 23, 24]. Otherwise, in order to transfer the stability result from the inherent time-discrete closed-loop to its continuous counterpart, energy estimates may be required; see, e.g., [1]. The incorporation of stabilizing end constraints provides a general framework,

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which guarantees asymptotic stability; see [23] for infinite-dimensional systems and [18] for time-discrete ones.

Rather recently, MPC schemes for distributive non-cooperative control schemes have been studied in the context of engineering applications such as robotics [8, 12, 27] or water irrigation systems [33]. These systems are controlled by multiple agents, where private and shared constraints are imposed on the controllers. Each agent aims to minimize a private cost function, which is not only dependent on their own control component but also on the other players' decisions. Hence, a Nash equilibrium problem (NEP) arises. If, furthermore, the agents' sets of admissible control decisions are interdependent, the problem is classified as a generalized Nash equilibrium problem (GNEP). Both problem classes are fundamental pillars of game theory. A comprehensive survey article tackling the existence, uniqueness, and characterization of such problems in finite dimensions can be found in [14]. GNEPs arising in control settings are usually governed by dynamics that enter the problem as a constraint. Noticeable work on dynamic games includes [3, 6] as well as [13, 21, 22, 25] for PDE-constrained problems. It is also well-known that for dynamical games, dynamic programming principles similar to the Bellman principle apply. Regarding MPC schemes for GNEPs, only a limited number of studies treat the question of asymptotic stability. In [12], different scenarios for systems of self-driving cars have been considered. Additionally, the authors of [31] derived conditions guaranteeing stabilizing performance for MPC based on affine-quadratic NEPs by finding an explicit characterization of the optimal trajectories.

Similarly to [31], in this paper, we derive conditions guaranteeing asymptotic stability for MPC schemes based on the class of jointly convex GNEPs governed by jointly controlled dynamics. We do so by finding suitable Lyapunov functions for the associated closed-loop dynamics. In each MPC iteration we solve the GNEP by finding a variational Nash equilibrium. Using a reformulation based on the Nikaido–Isoda function, these open-loop dynamic games can be expressed as a joint minimization over the set of admissible controls. Unfortunately, unlike for MPC schemes based on optimal control problems with stage-additive cost functionals, the associated optimal-value function does not provide a suitable Lyapunov function for the MPC-GNEP closed loop since it is constantly equal to zero for every initial value.

Hence, we introduce a class of auxiliary problems based on a modification of the Nikaido–Isoda function and some auxiliary parameter $\alpha \in [0, 1]$, which we call α-quasi GNEPs, for which the optimal-value function does provide a good candidate for a Lyapunov function. For MPC loops based on these problems instead, we obtain stabilizing behavior if additional stabilizing terminal constraints are enforced, as is often done in MPC schemes. We utilize the associated Lyapunov function in combination with perturbation analysis regarding the continuous dependence of minimizers on the parameter and the initial value to transfer the stability result to the original problem. In the process, we identify a suitable Lyapunov function for the MPC-GNEP loop under assumptions consistent with the terminal constraints.

The paper is structured as follows. In Section 2 we provide an overview of GNEPs as well as stability analysis for MPC. In Section 3 we introduce the general feedback scheme of basing MPC on generalized Nash equilibria. In Section 4, we discuss our auxiliary problems and derive conditions for asymptotic stability based on equilibrium end constraints. In Section 5 we provide the sensitivity analysis on the parameters used in Section 6 to transfer the stability guarantee to MPC schemes based on GNEPs consistent with equilibrium end constraints. Lastly, in Section 7 we provide some numerical examples.

2. Preliminaries

2.1. **Generalized Nash Equilibria.**

We consider a GNEP with shared constraints consisting of K players. Each player ν controls the variable $u^\nu \in \mathbb{R}^{d_\nu}$. These decision variables form the vector $u = (u^1, \dots, u^K) \in \mathbb{R}^d$ with $d = \sum_\nu d_\nu$. As typical in game theory, we use the notation $u = (u^\nu, u^{-\nu})$, where $u^{-\nu}$ refers to an block of components u^μ with $\mu \neq \nu$. Each player solves the problem

$$\min_{u^\nu} \quad \theta^\nu(u^\nu, u^{-\nu}) \quad \text{s.t.} \quad u^\nu \in \mathbb{U}_\nu(u^{-\nu}) := \left\{u^\nu \in \mathbb{R}^{d_\nu} : (u^\nu, u^{-\nu}) \in \mathbb{U}\right\}$$

with continuous payoff function θ^ν, which is quasi-convex in the player's variable u^ν and a strategy set $\mathbb{U}_\nu(u^{-\nu})$ that depends on the decisions $u^{-\nu}$ of the other players. Moreover, we denote the shared constraint set by $\mathbb{U} \subset \mathbb{R}^d$. For such a setup, the GNEP refers to the problem of finding a *Nash equilibrium (NE)*, which is a vector $u^* = (u^{*,1}, \ldots, u^{*,K}) \in \mathbb{U}$, so that for every player ν we have

$$\theta^\nu(u^{*,\nu}, u^{*,-\nu}) \leq \theta^\nu(u^\nu, u^{*,-\nu}) \quad \forall u^\nu \in \mathbb{U}_\nu(u^{*,-\nu}).$$

In what follows, we will assume the shared constraint set $\mathbb{U}$ is convex, compact, and non-empty and consider so-called variational Nash equilibria (VNE). To this end, we use the Nikaido–Isoda function

$$\Psi(u, v) := \sum_{\nu=1}^{K} \left(\theta^\nu(u^\nu, u^{-\nu}) - \theta^\nu(v^\nu, u^{-\nu})\right).$$

A variational equilibrium of the GNEP is then defined as a point $u \in \mathbb{R}^d$ with

$$\sup_{v \in \mathbb{U}} \ \Psi(u^*, v) = 0.$$

Each VNE is an NE; see, e.g., [14]. If, furthermore, the strategy sets are independent, i.e., $\mathbb{U}_\nu(u^{-\nu}) = \mathbb{U}_\nu$ for all $\nu = 1, \ldots, K$ and every admissible $u^{-\nu}$, then the two notions coincide. For the setup defined so far, existence of at least one VNE is guaranteed; see Section 4.1 in [14].

We define the u-parameterized optimal-value function

$$V(u) := \sup_{v \in \mathbb{U}} \ \Psi(u, v), \quad u \in \mathbb{U}.$$

Due to the compactness and continuity assumptions, we can directly deduce the continuity of the Nikaido–Isoda function and ensure that for every $u \in \mathbb{U}$ there is a point $v_u \in \mathbb{U}$ with $V(u) = \Psi(u, v_u)$. Furthermore, the function $V : \mathbb{U} \to \mathbb{R}$ is non-negative [14, Theorem 7.1] and, hence, we can compute a VNE by globally minimizing the min-max problem

$$\min_u \quad V(u) \quad \text{s.t.} \quad u \in \mathbb{U}.$$

2.2. **Model Predictive Control.** We briefly recap a general MPC scheme for time-discrete dynamics and discuss strategies for showing its stabilizing properties.

Let U, X be finite-dimensional vector spaces and let $N \in \mathbb{N}$. Furthermore, let $f : X \times U \to X$ be a transition map and let $J_N : X^N \times U^N \to \mathbb{R}$ be a cost functional. For every $x \in X$, let there be a non-empty set of admissible controls $\mathbb{U}^N(x) \subset U^N$. Then, an MPC scheme for the discrete-time dynamics $x^+ = f(x, u)$ is given by the following method; see also Algorithm 3.1 in [18].

Algorithm 2.1 (Basic MPC with horizon length N)**.**
At each sampling time $n = 0, 1, 2, \ldots$

(1) *Evaluate the state $x(n) \in X$ and set $x_0 := x(n)$.*
(2) *Solve the optimization problem*

$$\begin{aligned} \inf_{u \in \mathbb{U}^N(x_0)} \quad & J_N(x, u) \\ \text{s.t.} \quad & x(n) = f(x(n), u(n)) \quad \text{for } n = 0, \ldots, N-1, \\ & x(0) = x_0 \end{aligned}$$

and let $\bar{u} \in \mathbb{U}^N(x_0)$ denote the solution.

(3) *Define the MPC feedback by* $\mu_N(x_0) = \mu_N(x(n)) = \bar{u}(0)$ *and use this control value in the next sampling period.*

This defines a closed-loop system via

$$x^+ = g(x) \coloneqq f(x, \mu_N(x)). \tag{1}$$

Our goal, for now, is to investigate whether the MPC feedback law stabilizes the dynamics in the sense that the closed-loop system given by (1) is asymptotically stable at a reference point $x^* \in X$. A prerequisite for doing so is the existence of a reference control u^* with $x^* = f(x^*, u^*)$. The stability of the closed-loop system governed by a given feedback law can be derived from the existence of a Lyapunov function. For this purpose, we use the following classes of comparison functions:

$$\begin{aligned}
\mathscr{K} &\coloneqq \{\alpha \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \colon \alpha \text{ is continuous and strictly increasing with } \alpha(0) = 0\}, \\
\mathscr{K}_\infty &\coloneqq \{\alpha \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \colon \alpha \in \mathscr{K} \text{ and } \alpha \text{ is unbounded}\}, \\
\mathscr{L} &\coloneqq \left\{\delta \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \colon \delta \text{ is continuous, strictly decreasing with } \lim_{t \to 0} \beta(t) = 0\right\}, \\
\mathscr{K}\mathscr{L} &\coloneqq \{\beta \colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \colon \beta \text{ is continuous, } \beta(\cdot, t) \in \mathscr{K},\ \beta(r, \cdot) \in \mathscr{L}\}.
\end{aligned}$$

Definition 2.2. Let X be a metric space and consider the function $g \colon X \to X$, which defines a dynamic system $x^+ = g(x)$. A function $V \colon S \to \mathbb{R}_{\geq 0}$ is called a Lyapunov function if there exists a subset $S \subset X$ and a point $x^* \in X$, such that the following conditions hold.

(1) There exist functions $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$ so that

$$\alpha_1(\|x - x^*\|) \leq V(x) \leq \alpha_2(\|x - x^*\|) \tag{2}$$

holds for all $x \in S$.

(2) There exists a function $\alpha_V \in \mathscr{K}$ so that for each $x \in S$ with $g(x) \in S$ we have

$$V(g(x)) - V(x) \leq -\alpha_V(\|x - x^*\|). \tag{3}$$

Theorem 2.3 ([18, Theorem 2.19])**.** *Let x^* be an equilibrium point of the closed loop system $x^+ = g(x)$, i.e., $g(x^*) = x^*$. Assume further that there exists a Lyapunov function V on a subset $S \subset X$ containing x^*. If S contains a ball $B_\eta(x^*)$ with radius $\eta > 0$ such that $g(x) \in S$ for all $x \in B_\eta(x^*)$, then x^* is locally asymptotically stable, i.e., there exists $\eta' > 0$ and a function $\beta \in \mathscr{K}\mathscr{L}$ such that the inequality*

$$\|x(n, x_0) - x^*\| \leq \beta(\|x_0 - x^*\|, n)$$

holds for all $x_0 \in B_{\eta'}(x^*)$.

For the closed MPC loop, a canonical candidate for the Lyapunov function is given by the optimal-value function.

Definition 2.4. For any $x_0 \in X$ and $N \in \mathbb{N}$, define the optimal-value function by

$$V_N(x_0) = \min_{u \in \mathbb{U}^N(x)} J_N(x, u)$$

where $x(0) = x_0$ and $x(n) = f(x(n), u(n))$ for $n = 0, \ldots, N-1$. To emphasize the influence of the initial value we will occasionally denote the cost functional as $J_N(x_0, u)$.

In [18], the stabilizing behavior of MPC feedback for time discrete dynamics has been thoroughly studied for stage-additive cost functionals of the form

$$J(x, u) \coloneqq \sum_{n=0}^{N-1} \ell(x(n), u(n))$$

with stage costs $\ell \colon X \times U \to \mathbb{R}$. The analysis exploits the dynamic programming principle in discrete time to derive a relaxed dynamic programming inequality, which is equivalent to Condition (3) for suitable stage costs. The other property of a Lyapunov function must be ensured separately.

3. MPC for GNEPs

3.1. **A Discrete-Time Dynamic GNEP.** Let us now consider a time-depending GNEP over a finite time horizon $n = 0, 1, \ldots, N-1$. Let X and U_ν, $\nu = 1, \ldots, K$, be finite-dimensional vector spaces. We denote the control space by $U \coloneqq U_1 \times \cdots \times U_K$. Furthermore, let $f\colon X \times U \to X$ be a continuous function. We consider the discrete-time dynamics

$$x(n+1) = f(x(n), u^1(n), \ldots, u^K(n)),$$

where for each player ν, their respective control variable is denoted by $u^\nu(n) \in U_\nu$ and $u^\nu \in U_\nu^N$, respectively. For a finite control sequence $u \in U^N$ with $N \in \mathbb{N}$ and an initial value $x_0 \in X$, we denote the corresponding trajectory via

$$x_u(0) = x_0, \quad x_u(n+1) = f(x(n), u(n)) \quad \text{for } n = 0, \ldots, N-1. \tag{4}$$

We consider a set of admissible controls independent of the state variable. More precisely, let $\mathbb{U} \subset U$ denote the set of admissible controls. Then, the set of admissible control sequences of length N is given by $\mathbb{U}^N \in U^N$. Since we do not impose state constraints, this directly yields viability of the admissible controls if $\mathbb{U}$ is non-empty. Each player ν is equipped with an optimization problem

$$\begin{aligned} \min_{u^\nu} \quad & \sum_{n=0}^{N-1} \theta^\nu(u^\nu(n), u^{-\nu}(n), x(n)) \\ \text{s.t.} \quad & u^\nu \in \mathbb{U}_\nu^N(u^{-\nu}), \\ & x(n+1) = f(x(n), u^\nu(n), u^{-\nu}(n)) \quad \text{for all } n = 1, \ldots, N-1, \\ & x(0) = x_0 \end{aligned}$$

with the constraint set $\mathbb{U}_\nu^N(u^{-\nu}) \coloneqq \{u^\nu \colon (u^\nu, u^{-\nu}) \in \mathbb{U}^N\}$. Throughout this paper, we will impose the following assumptions.

Assumption 1. *Let the following hold:*

(1) *The payoff functions $\theta^\nu\colon U \times X \to \mathbb{R}$ are continuous for all $\nu = 1, \ldots, K$;*
(2) *The payoff functions $\theta^\nu(\cdot, u^{-\nu}, \cdot)$ are convex in the state variable as well as in u^ν, respectively, for all $u^{-\nu} \in U_{-\nu}$;*
(3) *The set $\mathbb{U} \subset U$ is compact, convex, and non-empty;*
(4) *The transition map $f\colon X \times U \to X$ is affine-linear;*
(5) *There exists a point $x^* \in X$ and an admissible control $u^* \in \mathbb{U}$, such that $x^* = f(x^*, u^*)$;*
(6) *For each player ν the pay-off function $\theta^\nu\colon U \times X \to \mathbb{R}$ is non-negative.*

The first three assumptions are the conditions of Theorem 5.9 in [14], which ensure the existence of a VNE for every initial value $x_0 \in X$. This can easily be seen by introducing a *control-to-state operator* $S\colon U^N \to X^N$, $u \mapsto x_u$, as well as the reduced pay-off functions defined by $\tilde{\theta}^\nu(u) \coloneqq \theta^\nu(u, S(u))$ for $u \in U^N$. For each $x_0 \in X$, the corresponding (reduced) Nikaido–Isoda function is given by

$$\Psi^N(x_0; u, v) = \sum_\nu \sum_{n=0}^{N-1} \left(\theta^\nu(u^\nu(n), u^{-\nu}(n), x_u(n)) - \theta^\nu(v^\nu(n), u^{-\nu}(n), y_{u,v}^\nu(n))\right),$$

where for each player ν, the inner dynamics is given by

$$\begin{aligned} y_{u,v}^\nu(0) &= x_0, \\ y_{u,v}^\nu(n+1) &= f(y_{u,v}^\nu(n), v^\nu(n), u^{-\nu}(n)) \quad \text{for all } n = 0, \ldots, N-1, \end{aligned} \tag{5}$$

which we combine in $y_{u,v} = (y_{u,v}^\nu)_\nu$. A VNE for the dynamic GNEP can be found by solving the problem

$$\min_{u \in \mathbb{U}^N} \max_{v \in \mathbb{U}^N} \; \Psi^N(x_0; u, v) \tag{6}$$

with optimal value equal to zero.

MPC Scheme. In the following, we study MPC schemes, where in each step a VNE for the presented GNEP is computed. In each loop, this can be expressed as an optimization problem via the min-max problem (6). Hence, the cost functional is given by

$$J_N(x_0, u) \coloneqq \max_{v \in \mathbb{U}^N} \Psi^N(x_0; u, v)$$

and the resulting scheme reads as follows.

Algorithm 3.1 (MPC for GNEPs with horizon length N)**.**
At each sampling time $n = 0, 1, 2, \dots$

(1) *Evaluate the state $x(n) \in X$ and set $x_0 \coloneqq x(n)$.*
(2) *Solve the optimization problem*

$$\begin{aligned}
&\min_{u \in \mathbb{U}^N} \left(\max_{v \in \mathbb{U}^N} \Psi^N(x_0; u, v) \right) \\
&\text{s.t.} \quad x(n+1) = f(x(n), u(n)) \quad \text{for } n = 0, \dots, N-1, \\
&\qquad y^\nu(n+1) = f(y^\nu(n), v^\nu(n), u^{-\nu}(n)) \quad \text{for } n = 0, \dots, N-1, \ \nu = 1, \dots, K, \\
&\qquad x(0) = y^\nu(0) = x_0 \quad \text{for } \nu = 1, \dots, K,
\end{aligned}$$

and let $\bar{u} \in \mathbb{U}^N$ denote the solution.

(3) *Define the MPC feedback by $\mu_N(x_0) = \bar{u}(0)$ and use this control value in the next sampling period.*

Unfortunately, the resulting optimal-value function is not a suitable candidate for a Lyapunov function because

$$\min_{u \in \mathbb{U}^N} \max_{v \in \mathbb{U}^N} \ \Psi^N(x_0; u, v) = 0$$

holds for all initial data $x_0 \in X$. Hence, we cannot directly adapt the stability analysis from [18].

4. α-Quasi GNEPs

For classic MPC schemes, a widely applicable strategy for guaranteeing asymptotic stability is showing that the optimal-value function is a Lyapunov function for the closed loop. Although the reformulation of a GNEP based on the Nikaido–Isoda function allows us to express the GNEP as a joint optimization problem, we still face a major difficulty: By construction, the associated optimal-value function is equal to zero. Hence, further modification of the problem might be necessary. There are several modifications of GNEPs based on regularizations of the Nikaido–Isoda function [20, 29, 32]. They all conserve the VNE property of the solutions while introducing smoothness to the problem. Further, the involved optimal values are still equal to zero.

In order to exploit the relation between optimal-value functions and Lyapunov functions, we propose a modification of the Nikaido–Isoda function, which instead of preserving the VNEs only approximates them in favor of a non-constant optimal-value function. We call this the aggregated cost perturbation and the associated problems are called α-quasi GNEPs.

In this chapter, we will discuss the aggregated cost perturbations as well as stabilizing MPC schemes based on α-quasi GNEPs. Similar to MPC algorithms with stage-additive cost, we will impose equilibrium end constraints to guarantee stability. Passing our approximation to a limit, based on the associated Lyapunov functions, we will be able to identify a suitable Lyapunov function for the original GNEP-based closed MPC loop, which will be discussed in Section 6.

4.1. **Aggregated Cost Perturbation.** We now consider the modification of the Nikaido–Isoda function

$$\Psi_\alpha^N(x_0;u,v) := \sum_{n=0}^{N-1}\sum_{\nu}\Big(\theta^\nu(u^\nu(n),u^{-\nu}(n),x_u(n)) \\ -(1-\alpha)\theta^\nu(v^\nu(n),u^{-\nu}(n),y^\nu_{u,v}(n))\Big)$$

with $\alpha\in(0,1)$, where x_u and $y_{u,v}$ are given by (4) and (5), respectively, with initial value $x_0 = x_u(0) = y^\nu_{u,v}(0)$. In this chapter, we will study an alteration of the MPC scheme given by Algorithm 3.1 by replacing the Ψ^N for Ψ^N_α. In this case, the optimal-value functions encode more information. Since each player provides a non-negative pay-off function, we have

$$\min_u\max_v \Psi_\alpha^N(u,v) \geq \min_u \Psi_\alpha^N(u,u) = \alpha\min_u \sum_{n=0}^{N-1}\sum_\nu \theta^\nu(u^\nu(n),u^{-\nu}(n),x_u(n)) \geq 0$$

for any initial value $x_0\in X$. In general, the optimal-value function of the associated min-max problem is not constant. However, the solution is not necessarily given by a VNE. We call the min-max problems over the modified Nikaido–Isoda function α-quasi GNEPs. For every configuration of controls $u\in\mathbb{U}$, the vector $v_u := \arg\min_v \sum_\nu \theta^\nu(u,v)$ can be be understood component-wise as the best deviation from u^ν that each player can make if the other players stick to $u^{-\nu}$. By construction, a VNE satisfies $u - v_u = 0$. The parameter α counteracts the idea of minimizing the discrepancy between u and v_u in favor of some aggregated cost aspect. These problems connect the non-cooperative case ($\alpha = 0$) to the aggregated-cost case ($\alpha = 1$). For aggregated cost, we find a minimization problem over $\mathbb{U}$, where the cost functional is given by the sum of all players' costs. The continuity of the corresponding solutions, in particular for $\alpha\to 0$, is discussed in Section 5. We now shortly discuss a dynamic programming principle (DPP) for a time-discrete min-max problem controlled by two entities.

Theorem 4.1. *Let X, U, V be Banach spaces. Let $f\colon \mathbb{N}\times X\times U\times V\to X$ and $\ell\colon \mathbb{N}\times U\times V\times X\to\mathbb{R}$ be continuous functions. For any $n_0\in\mathbb{N}$, $x_0,x_1\in X$, denote the sets of admissible control sequences of length N by $U^N(n_0,x_0)\subset U^N$ and $V^N(n_0,x_0)\subset V^N$, respectively. Let the admissible control sets fulfill the following non-anticipativity properties.*

- *Consider $u\in U^N(n_0,x_0)$, $1<N\in\mathbb{N}$. If there exists a control $v\in V^N(n_0,x_0)$ such that $x_1 = x(n_0,u(n_0),v(n_0))$ holds, then $u|_{n=n_0+1}^{n_0+N}\in U^{N-1}(n_0+1,x_1)$ follows.*
- *Consider $v^{N-1}\in V^{N-1}(n_0+1,x_1)$, $N\in\mathbb{N}$. If there exists an admissible $u\in U^1(n_0,x_0)$ and an admissible $v\in V^1(n_0,x_0)$ such that $x_1 = x(n_0,x_0,u(n_0),v(n_0))$, then the control sequence given by*

  $$v^N(n_0) = v,\quad v^N(n) = v^{N-1}(n)\quad \text{for all } n = n_0+1,\dots,N-1,$$

  satisfies $v^N\in V^N(n_0,x_0)$.

Moreover, suppose that both statements still hold if the roles of v and u are reversed.

Consider now the problem

$$\min_{u\in U^N(n_0,x_0)}\ \max_{v\in V^N(n_0,x_0)}\ J_N(n_0,x_0,u,v),$$

with

$$J_N(n_0,x_0,u,v) := \sum_{n=n_0}^{n_0+N-1}\ell(n,u(n),v(n),x(n)),$$

where the dynamics is given by

$$x(n_0) = x_0,\quad (n+1) = f(n,u(n),v(n)),\quad n = n_0,n_0+1,\dots$$

for some initial data $(n_0, x_0) \in \mathbb{N} \times X$*. For any choice* $(n_0, x_0) \in \mathbb{N} \times X$*, assume that the sets* $U^N(n_0, x_0), V^N(n_0, x_0)$ *are compact. Next, define the optimal-value function by*

$$V_N(n_0, x_0) = \min_{u \in U^N(n_0,x_0)} \max_{v \in V^N(n_0,x_0)} J_N(n_0, x_0, u, v).$$

Then, the dynamic programming principle

$$V_N(n_0, x_0) = \min_{u \in U^N} \max_{v \in V^N} \{\ell(n_0, u(n_0), v(n_0), x_0) + V_{N-1}(n_0 + 1, x_1)\} \tag{7}$$

holds with $x_1 = f(n_0, x_0, u(n_0), v(n_0))$*.*

Proof. The proof can be found in Appendix 9.1. □

The non-anticipativity property ensures that the tails of admissible controls are admissible. These results are directly applicable to the min-max problems over the (modified) Nikaido–Isoda function. This enables us to derive a relaxed dynamic programming principle for the modified problem.

Lemma 4.2 (Relaxed dynamic programming principle for outer minimization)**.** *Let* $N \in \mathbb{N}$ *be fixed and let the family of compact sets* $(U^{N'}(x))_x \subset \mathbb{U}^{N'}$ *with* $x \in X, N' \in \mathbb{N}, N' \leq N$*, fulfill the non-anticipativity property. Consider the function* $V_N^\alpha \colon X \to \mathbb{R}$ *defined by*

$$x_0 \mapsto \min_{u \in U^N(x_0)} J_N^\alpha(x_0; u) \quad \textit{with} \quad J_N^\alpha \coloneqq \max_{v \in \mathbb{U}^N} \Psi_\alpha^N(x_0; u, v).$$

Let $x_0 \in X$ *be fixed and denote the minimizer by* $\bar{u}_\star \in U^N(x_0)$*. Then,* $\bar{u} \coloneqq \bar{u}_\star(0)$ *satisfies*

$$V_N^\alpha(x_0) \geq \alpha \sum_\nu \theta^\nu(\bar{u}^\nu, \bar{u}^{-\nu}, x_0) + V_{N-1}^\alpha(f(x_0, \bar{u})).$$

Proof. First note that Theorem 4.1 is applicable. The set of admissible (outer) control sets is non-anticipative by definition and the inner sets $\mathbb{U}$ are so, too, since the admissible control set is constant in time and independent of the state. Here, we can identify the time-independent state trajectory by $\tilde{f}(z, u, v) = (f(x, u), (f(x, v^\nu, u^{-\nu}))_\nu)$. The time-independent stage cost is given by

$$\ell(u, v, z) = \sum_\nu \theta^\nu(u^\nu, u^{-\nu}, x) - (1 - \alpha)\theta^\nu(v^\nu, u^{-\nu}, y^\nu) \quad \text{with} \quad z = (x, (y^\nu)_\nu.$$

Hence, we can apply the DPP for min-max problems. Let $\tilde{V}_{N'} \colon X \times X^K \to \mathbb{R}$, denote optimal-value function for the min-max problem with a horizon of length $N' \leq N$ and let $z_0 = (x_0)^{K+1}$. Thus, we can conclude

$$\begin{aligned}
V_N^\alpha(x_0) = \tilde{V}(z_0) &= \min_u \max_v \left\{\ell(u(0), v(0), z_0) + \tilde{V}_{N-1}\left(\tilde{f}(z_0, u(0), v(0))\right)\right\} \\
&= \max_v \left\{\ell(\bar{u}_\star(0), v(0), z_0)) + \tilde{V}_{N-1}\left(\tilde{f}(z_0, \bar{u}_\star(0), v(0))\right)\right\} \\
&\geq \ell(\bar{u}, \bar{u}, x_0) + \tilde{V}_{N-1}\left(\tilde{f}(z_0, \bar{u}, \bar{u})\right) \\
&= \alpha \sum_\nu \theta^\nu(\bar{u}^\nu, \bar{u}^{-\nu}, x_0) + V_{N-1}^\alpha(f(x_0, \bar{u})).
\end{aligned}$$

□

Remark 4.3. First note that the previous lemma covers the case $U^{N'}(x) = \mathbb{U}^{N'}$ for $N' \in \mathbb{N}$. Second, the DPP is also naturally applicable to the (unmodified) GNEP case. Since tails of VNEs are VNEs themselves (see Lemma 4.1 in [30]), it reduces to a trivial equation, where all min-max terms are equal to zero.

4.2. **Auxiliary MPC with Stabilizing End Constraints.** A common strategy to ensure stability of the MPC closed-loop system with stage-additive costs is to use terminal conditions such as equilibrium end constraints or Lyapunov terminal cost [18]. We derive conditions on the pay-off function, which allow us to generalize the stability analysis to cost functions given by the modified Nikaido–Isoda function. First, let us focus on equilibrium end constraints. To be specific, we modify Algorithm 3.1 by exchanging the optimization problem for

$$\begin{aligned} \min_{u\in\mathbb{U}^N_{\mathbb{X}_0}(x_0)} \quad & J_N(x_0,u) = \max_{v\in\mathbb{U}^N} \Psi^N_\alpha(x_0;u,v) \\ \text{s.t.} \quad & x(0)=x_0, \quad x(n+1)=f(x(n),u(n)), \quad n=0,\dots,N-1, \end{aligned} \tag{8}$$

where $\mathbb{X}_0 = \{x^*\}$ and $\mathbb{U}^N_{\mathbb{X}_0}(x_0) = \{u \in \mathbb{U}^N(x_0)\colon x_u(N) \in \mathbb{X}_0\}$ with $x_u(0) = x_0$ holds. Denote the solution by $\bar{u} \in \mathbb{U}^N(x_0)$. We denote the associated feedback by $\mu^\alpha_N\colon X \to U$ and the corresponding optimal-value function by $V^\alpha_N\colon X \to \mathbb{R}$. We can generalize the notation of $\mathbb{X}_0$ by considering the set of initial values, which can be brought to the stationary point $x^* \in X$ in k steps by considering

$$\mathbb{X}_k \coloneqq \left\{x \in X\colon \text{there exists } u \in \mathbb{U}^k \text{ with } x_u(k) = x^* \text{ if } x_u(0) = 0\right\}.$$

In this auxiliary scheme, we consider an MPC feedback, where in each iteration a solution for an α-quasi GNEP is computed. Starting in the stationary point, meaning $x_0 = x^*$, for a large class of pay-off functions, these solutions will coincide with the corresponding VNE; see Lemma 4.6 below. Moreover, these solutions can be seen as approximations of the VNE (for any initial value), which we discuss in section 5.

Theorem 4.4. *Let the following assumptions hold.*

(1) *The point $x^* \in X$ is an equilibrium point for an admissible control, i.e., there exists a control $u^* \in \mathbb{U}$ such that $x^* = f(x^*, u^*)$.*
(2) *For any choice $y_\nu \in X$, $\nu = 1,\dots,K$, we have*

$$\sum_\nu \theta^\nu(u^{*,\nu}, u^{*,-\nu}, x^*) \le (1-\alpha)\sum_\nu \theta^\nu(v^\nu, u^{*,-\nu}, y_\nu)$$

for any $v \in \mathbb{U}$ and $n = 0,\dots,N-1$.

Then, the optimal-value function from Algorithm 3.1 based on the minimization problem (8) satisfies

$$V^\alpha_N(x_0) \le V^\alpha_{N-1}(x_0) \tag{9}$$

for every $x_0 \in \mathbb{X}_{N-1}$ and $N \ge 2$. For $x_0 \in \mathbb{X}_{N-1}$, this implies

$$V^\alpha_N(x_0) \ge \alpha \sum_\nu \theta^\nu(\mu^\alpha_N(x_0)^\nu, \mu^\alpha_N(x_0)^{-\nu}, x_0) + V^\alpha_N(f(x_0, \mu^\alpha_N(x_0))). \tag{10}$$

Proof. Let $x_0 \in X$ be fixed and note that

$$\ell(u^*, v, x^*, y) = \sum_\nu \theta^\nu(u^{*,\nu}, u^{*,-\nu}, x^*) - (1-\alpha)\sum_\nu \theta^\nu(\bar{v}^\nu, u^{*,-\nu}, y_\nu) \le 0$$

holds for every $v \in \mathbb{U}$ and $y = (y_\nu) \in X^K$. Let $u^{N-1} \in \mathbb{U}^{N-1}_{\mathbb{X}_0}(x_0)$, then by Condition (1), the control sequence given by

$$u^N(n) = u^{N-1}(n) \text{ for } n = 0,\dots,N-2, \quad u^N(N-1) = u^*$$

satisfies $u^N \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)$. Furthermore, by employing the dynamic programming principle for the (inner) maximization problem, we have

$$\begin{aligned} J_N(x_0, u^N) &= \max_{v \in \mathbb{U}^N} \Psi^N_\alpha(u^N, v, x_{u^N}, y_{u^N,v}) \\ &= \max_{v \in \mathbb{U}^N} \left\{ \Psi^{N-1}_\alpha(u^N, v, x_{u^N}, y_{u,v}) + \max_{\bar{v} \in \mathbb{U}} \ell(u^*, v, x^*, y_{u^N,v}(N-1)) \right\} \\ &\le \max_{v \in \mathbb{U}^N} \left\{ \Psi^{N-1}_\alpha(u^N, v, x_{u^N}, y_{u^N,v}) \right\} = \max_{v \in \mathbb{U}^{N-1}} \left\{ \Psi^{N-1}_\alpha(u^N, v, x_{u^N}, y_{u^N,v}) \right\} \\ &= J_{N-1}(u^{N-1}, x_0). \end{aligned}$$

Considering the set $\tilde{\mathbb{U}}^N_{\mathbb{X}_0}(x_0) \coloneqq \mathbb{U}^{N-1}_{\mathbb{X}_0}(x_0) \times \{u^*\}$, this leads to

$$\inf_{u \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)} J_N(u, x_0) \le \inf_{u^N \in \tilde{\mathbb{U}}^N_{\mathbb{X}_0}(x_0)} J_N(u^N, x_0) \le \inf_{u^{N-1} \in \mathbb{U}^{N-1}_{\mathbb{X}_0}(x_0)} J_{N-1}(u^{N-1}, x_0),$$

which translates to $V^\alpha_N(x_0) \le V^\alpha_{N-1}(x_0)$. Now fix $x_0 \in \mathbb{X}_N$. Note that $\mathbb{U}^N_{\mathbb{X}_0}(x_0)$ fulfills the non-anticipativity assumption from Theorem 4.1. Hence, by Lemma 4.2 we get the relaxed DPP inequality

$$V^\alpha_N(x_0) \ge \alpha \sum_\nu \theta^\nu(\mu^\alpha_N(x_0)^\nu, \mu^\alpha_N(x_0)^{-\nu}, x_0) + V^\alpha_{N-1}(f(x_0, \mu^\alpha_N(x_0))).$$

Since $\mu^\alpha_N(x_0) = \bar{u}(0)$ holds for an $\bar{u} \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)$, we have $f(\mu^\alpha_N(x_0)) \in \mathbb{X}_{N-1}$. Combining the relaxed DPP inequality with Equation (9) evaluated in $f(\mu^\alpha_N(x_0))$, we directly can obtain Inequality (10). □

Remark 4.5. Note that Condition (2) of Theorem 4.4 is satisfied if $\theta^\nu(x^*, u^*) = 0$ holds for every player ν.

Lemma 4.6. *If the conditions from Theorem 4.4 are fulfilled, then u^* is a VNE for the initial value x^* and any horizon length $k \in \mathbb{N}$. Furthermore, we have $V^\alpha_k(x^*) = 0$ for every $k \in \mathbb{N}$.*

Proof. Since $1 - \alpha > 0$, Condition 2 directly implies $\sum_\nu \theta^\nu(u^{*,\nu}, u^{*,-\nu}, x^*) = 0$. Using Condition 2 iteratively leads to

$$\sum_{n=0}^{k} \sum_\nu \theta^\nu(u^{*,\nu}, u^{*,-\nu}, x^*) \le \sum_{n=0}^{k} \sum_\nu \theta^\nu(v^\nu(n), u^{*,-\nu}, y_{u^*,v}(x^*; n))$$

for any $v \in \mathbb{U}^k$, which means that the constant control sequence u^* is a VNE of the GNEP with initial point x^* and any horizon length $k \in \mathbb{N}$.

Considering the optimal-value function, we deduce

$$0 \le \min \Psi^k_\alpha(u, u) \le V^\alpha_k(x^*) \le \max_v \Psi^k_\alpha(u^*, v) = \Psi^k_\alpha(u^*, u^*) = 0,$$

for any $k \in \mathbb{N}$. □

Lemma 4.7. *Under the assumption of Theorem 4.4, the set $\mathbb{X}_N$ is forward invariant under the feedback law $\mu^\alpha_N \colon X \to U$ for every $N \in \mathbb{N}$, i.e., $f(x, \mu^\alpha_N(x)) \in \mathbb{X}_N$ for every $x \in \mathbb{X}_N$.*

Proof. Let $x \in \mathbb{X}_N$. Then, we have $f(x, \mu^\alpha_N(x)) \in \mathbb{X}_{N-1}$. Hence, there exists $u^{N-1} \in \mathbb{U}^{N-1}_{\mathbb{X}_0}(f(x, \mu^\alpha_N(x)))$. Now, by Assumption 1 of Theorem 4.4, we can construct the control sequence

$$u^N(n) = u^{N-1}(n), \quad n = 0, \dots, N-2, \quad u^N(N-1) = u^*,$$

which produces the state

$$x_{u^N}(N) = f(x_{u^{N-1}}(N-1), x^*) = f(x^*, u^*) = x^*.$$

Hence, we have $\mathbb{X}_{N-1} \subset \mathbb{X}_N$ and, thus, $f(x, \mu^\alpha_N(x)) \in \mathbb{X}_N$. □

Theorem 4.8 (Asymptotic stability using endpoint constraints)**.** *Consider the MPC scheme with stabilizing endpoint constraints given by Algorithm 3.1 based on the modified minimization problem (8) and prediction horizon $N \in \mathbb{N}$. Furthermore, let the assumptions of Theorem 4.4 hold as well as the following conditions.*

(1) *There exists a function $\alpha_1 \in \mathscr{K}$ such that*
$$\alpha \sum_{\nu} \theta^\nu(\mu_N^\alpha(x_0)^\nu, \mu_N^\alpha(x_0)^{-\nu}, x_0) \geq \alpha_1(\|x_0 - x^*\|)$$
holds for all $x_0 \in \mathbb{X}_N.v$

(2) *There exists functions $\alpha_2, \alpha_3 \in \mathscr{K}_\infty$ such that*
$$\alpha_2(\|x_0 - x^*\|) \leq V_N^\alpha(x_0) \leq \alpha_3(\|x_0 - x^*\|)$$
holds for all $x_0 \in \mathbb{X}_N$.

Then, the closed-loop system is asymptotically stable.

Proof. By Theorem 4.4, we have
$$V_N^\alpha(x_0) \geq \alpha \sum_{\nu} \theta^\nu(\mu_N^\alpha(x_0)^\nu, \mu_N^\alpha(x_0)^{-\nu}, x_0) + V_N^\alpha(f(x_0, \mu_N^\alpha(x_0))).$$
Combing this with the first condition we obtain Equation (3) for $V = V_N^\alpha$, $g = \mu_N^\alpha$, and $S = \mathbb{X}_N$, $\alpha_V = \alpha_1$. Condition (2) leads to Equation (2) for these entities directly. Hence, by Theorem 2.3 the closed-loop system is asymptotically stable on $\mathbb{X}_N$. □

We see that by imposing equilibrium end constraints we can easily enforce the optimal-value function to fulfill one criterion of a Lyapunov function, namely the decay along the trajectories. The second property is more dependent on the design of the cost functional. However, there is a close connection between the optimal-value function V_N^α and the optimal-value function of a problem with stage-additive cost, which can be exploited to find suitable $\mathcal{K}_\infty$-bounds.

Definition 4.9. Consider non-negative pay-off functions. We define the aggregated cost functional as $J_{\text{agg}}^N : U^N \times X \to \mathbb{R}$ via
$$(x, u) \mapsto \sum_{n=0}^{N-1} \sum_{\nu} \theta^\nu(u^\nu(n), u^{-\nu}(n), x_u(n)),$$
where $x_u(0) = x$ and $x_u(n+1) = f(x(n), u(n))$ for all $n = 0, \dots, N-1$. Similarly, we define the aggregated optimal-value function by
$$V_N^{\text{agg}}(x_0) := \min_{u \in \mathbb{U}_{\mathbb{X}_0}^N(x_0)} J_{\text{agg}}^N(x_0, u)$$
for every $x_0 \in \mathbb{X}_N$.

Based on this optimal-value function, we can find $\mathscr{K}_\infty$-bounds for V_N^α with $\alpha \in (0, 1)$.

Lemma 4.10. *Consider non-negative pay-off functions and let V_N^α denote the optimal-value function from Theorem 4.4 for $\alpha \in (0, 1)$. If there exist functions $\gamma_1, \gamma_2 \in \mathscr{K}_\infty$ such that*
$$\gamma_1(\|x_0 - x^*\|) \leq V_N^{agg}(x_0) \leq \gamma_2(\|x_0 - x^*\|)$$
holds for every $x_0 \in \mathbb{X}_N$, then for every $\alpha \in (0, 1)$ we have
$$\alpha\gamma_1(\|x_0 - x^*\|) \leq V_N^\alpha(x_0) \leq \gamma_2(\|x_0 - x^*\|).$$

Proof. Concerning the lower bound, we can easily derive
$$\begin{aligned} V_N^\alpha(x_0) &= \min_u \max_v \theta(u, u) - (1 - \alpha)\theta(v, u) \\ &\geq \min_u \alpha\theta(u, u) = \alpha V_N^{\text{agg}}(x_0) \geq \alpha\gamma_1(\|x - x^*\|). \end{aligned}$$

For the upper bound, due to $\theta \geq 0$, we have

$$V_N^\alpha(x_0) = \min_u \max_v \theta(u,u) - (1-\alpha)\theta(v,u) \leq \min_u \theta(u,u) \leq \gamma_2(\|x - x^*\|). \qquad \square$$

Note that the existence of such $\mathscr{K}_\infty$-bounds can be guaranteed under the assumptions of the next lemma.

Lemma 4.11. (1) *Assume that there exists a function $\alpha_1 \in \mathscr{K}_\infty$ such that*

$$\sum_\nu \theta^\nu(u^\nu, u^{-\nu}, x) \geq \alpha_1(\|x - x^*\|) \quad \text{for every } x \in \mathbb{X}_N \text{ and } u \in \mathbb{U}.$$

Then, we have $V_N^\alpha(x_0) \geq \alpha_1(\|x_0 - x^*\|)$ for all $x_0 \in \mathbb{X}_N$.

(2) *Assume that f and all pay-off functions are continuous on $X \times \mathbb{U}$. Let there exist a $\eta > 0$ such that for every $x \in B_\eta(x^*) \subset X$, $\eta > 0$, there exists a $u_x \in \mathbb{U}$ such that $f(x, u_x) = x^*$ holds, as well as a function $\alpha_2 \in \mathscr{K}_\infty$ that satisfies*

$$\sum_\nu \theta^\nu(u_x^\nu, u_x^{-\nu}, x) \leq \alpha_2(\|x - x^*\|).$$

Then, there exists $\alpha_3 \in \mathscr{K}_\infty$ such that

$$V_N^\alpha(x_0) \leq \alpha_3(\|x_0 - x^*\|) \quad \text{for all } x_0 \in \mathbb{X}_N.$$

Proof. Choosing $\ell(n,x,u) = \sum_\nu \theta^\nu(x,u)$, we fit to the setting of Proposition 5.7 in [18], which yields $\mathcal{K}_\infty$-bounds for V_N^{agg}. Combined with Lemma 4.10, the statements follow. $\square$

5. Continuous dependence of the minimizer on the initial value and parameters

In this section, we show stability of the minimizers with respect to perturbations of the initial value and the auxiliary parameter α. We are especially interested in limits including $\alpha \to 0$. The first result is concerned with the variation of the auxiliary parameter around the point $\alpha = 0$. It shows that the min-max problems for the modified Nikaido–Isoda function approximate the problem of finding a VNE within the subset $\mathbb{U}_{\mathbb{X}_0}^N(x_0)$.

Theorem 5.1. *Let the initial value $x_0 \in X$ be given and fixed and let, with a slight abuse of notation, $\Psi^N, \Psi_\alpha^N \colon U^N \times U^N \to \mathbb{R}_0$ denote the reduced objective functionals with eliminated dynamics. Consider the functionals*

$$F \colon \mathbb{U}_{\mathbb{X}_0}^N(x_0) \to \mathbb{R}_{\geq 0}, \quad u \mapsto \max_{v \in \mathbb{U}^N} \Psi^N(u,v),$$

$$F_\alpha \colon \mathbb{U}_{\mathbb{X}_0}^N(x_0) \to \mathbb{R}_{\geq 0}, \quad u \mapsto \max_{v \in \mathbb{U}^N} \Psi_\alpha^N(u,v).$$

Any accumulation point of a sequence of minimizers $(u_\alpha)_\alpha$, corresponding to F_α each, minimizes F.

Proof. In what follows, we abbreviate

$$\theta(u,v) := \sum_\nu \sum_{n=0}^{N-1} \theta^\nu(v^\nu(n), u^{-\nu}(n), y_{u,v}^\nu(n)).$$

Due to the compactness of $\mathbb{U}^N$ and the continuity of the pay-off functions and the transition map, Theorem 2.1 in [11] yields that the functions F, F_α are well-defined and continuous. The function $\tilde{f} \colon U^N \to X$, $u \mapsto x_u(x_0; N)$, is continuous. Thus, the set

$$\mathbb{U}_{\mathbb{X}_0}^N(x_0) = \mathbb{U}^N \cap \tilde{f}^{-1}(\{x^*\})$$

is compact as a closed subset of $\mathbb{U}^N$. Hence, both functions attain a minimum. For any $u \in \mathbb{U}_{\mathbb{X}_0}^N(x_0)$, denote

$$v_u := \arg\min_{v \in \mathbb{U}^N} \theta(u,v).$$

First note that F_α converges point-wisely to F as $\alpha \to 0$. For $\alpha \in (0,1)$, let u_α be a minimizer of F_α and let u_0 be the minimizer of F. Let $u_\alpha \to u$ for some $u \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)$ as $\alpha \to 0$. Then, we have

$$F_\alpha(u_0) \geq F_\alpha(u_\alpha) = \theta(u_\alpha, u_\alpha) - (1-\alpha)\min_v \theta(u_\alpha, v) \geq \theta(u_\alpha, u_\alpha) - (1-\alpha)\theta(u_\alpha, v_u)$$

Taking the limit, we obtain

$$F(u_0) = \lim_{\alpha\to 0} F_\alpha(u_0) \geq \lim_{\alpha \to 0} \theta(u_\alpha, u_\alpha) - (1-\alpha)\theta(u_\alpha, v_u) = F(u). \qquad \square$$

Remark 5.2. The proof can be carried over to the case $\alpha \to \alpha'$ with $\alpha' \in (0,1]$.

Deriving a stability result for an additional variation of the initial value is more challenging because the initial value influences the set of admissible controls, whereas the parameter α does not. To succeed, we will make use of the following assumptions.

Assumption 2 (Completely controllable dynamics)**.** *Let the following hold.*

(1) *The transition map is given by $f\colon X \times U \to X$, $(x,u) \mapsto Ax + Bu$, with a matrix pair (A,B) so that the system $x^+ = f(x,u)$ is completely controllable in every step.*
(2) *Assume that u^* is in the interior $\mathring{\mathbb{U}}$ of $\mathbb{U}$.*

Lemma 5.3. *Let Assumption 2.1 hold. For any $(x,u) \in X \times U$ with $f(x,u) = x^*$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $y \in B_\delta(x)$, there exists $v \in B_\varepsilon(u)$ with $f(y,v) = x^*$.*

Proof. Let $d \in X$ with $\|d\| = 1$ be given arbitrarily. Due to the complete controllability, there exists a $w \in U$ with $0 = Ad + Bw$. One solution to this equation is given by $v_d = B^\dagger Ad$, where $B^\dagger$ is the Moore–Penrose pseudo-inverse of B. Now, let $\varepsilon > 0$ and consider $\delta < \varepsilon/\|B^\dagger A\|$. Consider $y \in B_\delta(x)$. Then, we can find the representation $y = x + \delta d$ for some $d \in X$. Choosing the control $v = u + \delta v_d$, we obtain $Ay + Bv = Ax + Bu + \delta(Ad + Bv_d) = x^*$ with $\|u - v\| = \delta\|v_d\| < \varepsilon$. $\square$

Theorem 5.4. *Let Assumption 2 hold. Consider the multi-function $S\colon \mathbb{R} \times X \to 2^U$ defined by*

$$(\alpha, x_0) \mapsto \operatorname*{arg\,min}_{u \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)} J_N^\alpha(x_0, u) \quad \textit{with} \quad J_N^\alpha(x_0,u) = \max_{v \in \mathbb{U}^N} \Psi_N^\alpha(x_0; u, v).$$

Then, S is upper semi-continuous at any point $(\alpha, x_0) \in [0,1] \times \mathbb{X}_{N-1}$. Furthermore, the associated optimal-value function $V\colon \mathbb{R} \times X \to \mathbb{R}$ is continuous at every point $(\alpha, x_0) \in [0,1] \times \mathbb{X}_{N-1}$.

Proof. We show this result by applying Proposition 4.4 of [5]. In the following, we check the conditions of this theorem. Let $\alpha \in [0,1]$ be given and let $x_0 \in \mathbb{X}_{N-1}$. Here, the parameter space is given by $\mathbb{R} \times X$ and the objective is given by $J_N^\alpha\colon \mathbb{R} \times X \times U^N \to \mathbb{R}$. Firstly, we have to ensure that the objective is continuous. Note that Ψ is continuous in x_0, u, v, and α as a composition of continuous functions. The continuity of the function J_N^α follows from Theorem 2.1 in [11].

Secondly, we have to ensure that the multi-function $\Phi\colon \mathbb{R} \times X \to 2^{U^N}$, $(\alpha, x_0) \mapsto \mathbb{U}^N_{\mathbb{X}_0}(x_0)$ is closed. Note that this mapping is independent of α. For every $x_0 \in X$, the function $\tilde{f}_{x_0}\colon U^N \to X$, $u \mapsto x_u(x_0, N)$, is continuous since f is continuous. Furthermore, define $\bar{F}\colon U^N \times X \to X$, $(u, x_0) \mapsto x_u(x_0, N)$, which is also a continuous function. Note that $\Phi(x_0, \alpha) = \tilde{f}_{x_0}^{-1}(x^*) \cap \mathbb{U}^N$ is a closed set. Hence, Φ is closed-valued. Furthermore, the graph of Φ is $\mathbb{R} \times \big(\bar{F}^{-1}(x^*) \cap (X \times \mathbb{U}^N)\big)$ and thus closed.

Thirdly, we show that there exists a constant $M \in \mathbb{R}$, a compact set $C \subset U^N$, and a neighborhood U of (α, x_0) such that for any $(\beta, y_0) \in U$ the level set

$$\operatorname{lev}_M J_N^\beta(y_0, \cdot) := \Big\{ x \in \mathbb{U}^N_{\mathbb{X}_0}(x_0) \colon J_N^\beta(y_0, u) \leq M \Big\}$$

is non-empty and contained in C. Since $\mathbb{U}$ is compact, we can choose $C = \mathbb{U}^N$. Then, there exists a $u \in \mathbb{U}^{N-1}_{\mathbb{X}_0}(x_0) \subset \mathbb{U}^N_{\mathbb{X}_0}(x_0) = \Phi(\alpha, x_0)$. Lemma 5.3 implies that there exists $\delta > 0$ such that for every $y \in B_\delta(x^*)$ there exists an $u_y \in \mathring{\mathbb{U}}$ with $f(y, u_y) = x^*$. Since f is continuous, there exists a $\eta > 0$ such that for every $y_0 \in B_\eta(x_0)$, we have

$$\|x_u(y_0; N-1) - x_u(x_0; N-1)\| = \|x_u(y_0; N-1) - x^*\| < \delta.$$

Hence, the sequence given by

$$v(n) = u(n) \text{ for } n = 0, \dots, N-2, \quad v(N-1) = u_z \text{ with } z = x_u(y_0; N-1)$$

is contained in $\mathbb{U}^N_{\mathbb{X}_0}(y_0)$. Thus, for every $(\beta, y_0) \in (\alpha - \eta, \alpha + \eta) \times B_\eta(x_0)$, we have $\Phi(\beta, y_0) \neq \emptyset$. Now, consider

$$M \coloneqq \max \left\{ J^\beta_N(y_0, u) \colon \beta \in [\alpha - \eta, \alpha + \eta], y_0 \in \bar{B}_\eta(x_0), u \in \mathbb{U}^N \right\}.$$

Then, for any $u \in \Phi(\beta, y_0)$ with $(\beta, y_0) \in (\alpha - \eta, \alpha + \eta) \times B_\eta(x_0)$ we have $J^\beta_N(y_0, u) \leq M$ and, consequently, the level set $\operatorname{lev}_M J^\alpha_N(y_0, \cdot)$ is non-empty.

Lastly, we need to show that for any open neighborhood V_U of $S(\alpha, x_0)$, there exists a neighborhood V_P around (α, x_0) such that $V_U \cap \Psi(\beta, y_0) \neq \emptyset$ if $(\beta, y_0) \in V_P$. Consider an open neighborhood V_U of $S(\alpha, x_0)$. Then, for each $u \in S(\alpha, x_0)$ there exists an $\varepsilon > 0$ with $B_\varepsilon(x_0) \subset V_U$. We will show that for any choice of $\varepsilon > 0$ and $u \in S(\alpha, x_0)$, there exists a $\delta > 0$ such that for any $y_0 \in B_\delta(x_0)$, there exists a $v \in \mathbb{U}^N_{\mathbb{X}_0}(y_0)$ with $v \in B_\varepsilon(u)$.

Since the set of admissible controls is independent of α, the desired property follows by choosing the neighborhood $V_X = (\alpha - \delta, \alpha + \delta) \times B_\delta(x_0)$ around the parameter (α, x_0). Now, fix $u \in S(\alpha, x_0)$ and $\varepsilon < 1$. Since $x_0 \in \mathbb{X}_{N-1}$, there exists a $\tilde{u} \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)$ with $\tilde{u}(N-1) \in \mathring{\mathbb{U}}$. Let $m = \max\{\|u\| \colon u \in \mathbb{U}^N\}$. Consider the control sequence $u_\varepsilon = u + \frac{\varepsilon}{4m}(\tilde{u} - u)$. Since the admissible control set $\mathbb{U}$ is convex, we have $u_\varepsilon(n) \in \mathbb{U}$ for every $n = 0, \dots, N-1$. Additionally, we have $u_\varepsilon(N-1) \in \mathring{\mathbb{U}}$ due to Lemma 9.1 in the appendix. Since the dynamics is linear, we can deduce $x_{u_\varepsilon}(N, x_0) = x^*$ as well.

There exists a ball $B_{\varepsilon_1}(u_\varepsilon(N-1)) \subset \mathring{\mathbb{U}}$ with $\varepsilon_1 \leq \varepsilon/2$. Furthermore, due to Lemma 5.3, there exists a $\delta_1 > 0$ such that for any $z \in B_{\delta_1}(x_{u_\varepsilon}(x_0; N-1))$ there exists a $v_z \in B_{\varepsilon_1}(u_\varepsilon(N-1)) \subset \mathring{\mathbb{U}}$ with $f(z, v) = x^*$. Furthermore, since the map $x \mapsto x_{u_\varepsilon}(x; N-1)$ is continuous, we can find $\delta > 0$ such that

$$\|x_{u_\varepsilon}(x_0; N-1) - x_{u_\varepsilon}(y_0; N-1)\| < \delta_1 \quad \text{for any} \quad y_0 \in B_\delta(x_0).$$

Now, for any $y_0 \in B_\delta(x_0)$ consider the control sequence given by

$$\begin{aligned} v(n) &= u_\varepsilon(n) \quad \text{for} \quad n = 0, \dots, N-2, \\ v(N-1) &= v_z \quad \text{with} \quad z = x_{u_\varepsilon}(y_0; N-1). \end{aligned}$$

Then, we have $v \in \mathbb{U}^N_{\mathbb{X}_0}(y_0)$ by construction and

$$\|u - v\| \leq \|u - u_\varepsilon\| + \|u_\varepsilon - v\| \leq \frac{\varepsilon}{4m} 2m + \|u_\varepsilon(N-1) - v(N-1)\| \leq \frac{\varepsilon}{2} + \varepsilon_1 < \varepsilon.$$

Hence, we can apply Proposition 4.4 of [5] and obtain the continuous dependence on the parameters at any $\alpha \in [0, 1]$ and $x_0 \in \mathbb{X}_{N-1}$. □

6. Stability of the Limiting Scheme

In this section, we extend the stability analysis from Section 4.2 to the case $\alpha = 0$. This corresponds to the MPC scheme with equilibrium end constraints, where the objective is given by

$$J^0_N(x_0, u) = \max_{v \in \mathbb{U}^N} \Psi^N(x_0, u, v),$$

meaning that in each iteration a true VNE is computed. As before, we define

$$\theta(v,u) := \sum_{n=0}^{N-1} \sum_{\nu} \theta^{\nu}(v^{\nu}(n), u^{-\nu}(n), y_{u,v}^{\nu}(n))).$$

In the following, we presume the following set of assumptions.

Assumption 3. *Let the following hold:*

(1) *The pair $(x^*, u^*) \in X \times U$ fulfills $f(x^*, u^*) = x^*$ with $u^* \in \mathring{\mathbb{U}}$.*
(2) *For any choice $\alpha \in (0,1)$, $y_\nu \in X$, $\nu = 1, \dots, K$, we have*

$$\sum_{\nu} \theta^{\nu}(u^{*,\nu}, u^{*,-\nu}, x^*) \le (1-\alpha) \sum_{\nu} \theta^{\nu}(v^{\nu}, u^{*,-\nu}, y_{\nu})$$

for any $v \in \mathbb{U}$.
(3) *There exists $\gamma_1, \gamma_2, \gamma_V \in \mathscr{K}_\infty$ such that*

$$\gamma_1(\|x_0 - x^*\|) \le V_N^{agg}(x_0) \le \gamma_2(\|x_0 - x^*\|),$$
$$\sum_{\nu} \theta^{\nu}(u^{\nu}, u^{-\nu}, x_0) \ge \gamma_V(\|x_0 - x^*\|)$$

holds for any $x_0 \in \mathbb{X}_N$ and any $u \in \mathbb{U}$.
(4) *The dynamics $f\colon X \times U \to X$ is completely controllable in one step.*
(5) *For any $x_0 \in \mathbb{X}_N$, we have*

$$\min_{u \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)} \max_{v \in \mathbb{U}^N} \Psi^N(x_0; u, v) = 0.$$

Furthermore, the minimizer is unique.

By the results of Section 4, the first three assumptions ensure asymptotic stability for the closed MPC loop for any $\alpha \in (0,1)$. Note that for this result alone, the function γ_V only needs to be non-negative as well as strictly increasing and to satisfy $\gamma_V(0) = 0$. Furthermore, Assumption 3.5 is fulfilled if for any $x_0 \in \mathbb{X}_N$ there exists a unique VNE to the associated GNEP with $x_u(N) = x^*$. Altogether, this gives the following desired result.

Theorem 6.1. *Consider the MPC scheme feedback scheme with equilibrium end constraints $\mu_N\colon X \to X$, where the objective is given by*

$$J_N^0(x_0, u) = \max_{v \in \mathbb{U}^N} \Psi^N(x_0; u, v).$$

Let Assumption 3 hold. For $x_0 \in \mathbb{X}_N$, define

$$V_N^0(x_0) := \sum_{n=0}^{N-1} \sum_{\nu} \theta^{\nu}(u^{\nu}(n), u^{-\nu}(n), x_u(n)),$$

where $u \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)$ is the optimal solution to the problem

$$\min_{u \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)} J_N^0(x_0, u).$$

Then, the function $V_N^0\colon \mathbb{X}_N \to \mathbb{R}$ fulfills

$$\gamma_1(\|x_0 - x^*\|) \le V_N^0(x_0) \le \gamma_2(\|x_0 - x^*\|),$$
$$V_N^0(x_0) - V_N^0(f(x_0, \mu_N(x_0))) \ge \gamma_V(\|x - x^*\|)$$

for any $x_0 \in \mathbb{X}_N$, where $\gamma_1, \gamma_2, \gamma_V$ are given by Assumption 3.

Proof. Let $x_0 \in \mathbb{X}_N$ be given and fixed. For any $\alpha \in (0,1)$, consider the auxiliary MPC scheme based on the objective

$$J_N^{\alpha}(x_0, u) = \max_{v \in \mathbb{U}^N} \Psi_{\alpha}^N(x_0; u, v).$$

Due to Assumptions 3.1–3.3, Theorem 4.4, Theorem 2.3, and Lemma 4.10, the associated MPC feedback μ_N^α and the optimal-value function V_N^α fulfill

$$\alpha\gamma_1(\|x_0 - x^*\|) \le V_N^\alpha(x_0) \le \gamma_2(\|x_0 - x^*\|), \tag{11}$$

$$V_N^\alpha(x_0) - V_N^\alpha(f(x_0, \mu_N^\alpha(x_0))) \ge \alpha\gamma_V(\|x_0 - x^*\|) \tag{12}$$

for some $\gamma_V, \gamma_1, \gamma_2 \in \mathscr{K}_\infty$.

We now show the decay along the trajectories. Since γ_V is a $\mathscr{K}_\infty$-function, it is invertible on $[0, \infty)$ and Equation (12) can be re-written as

$$\|x_0 - x^*\| \le \gamma_V^{-1}\left(\frac{1}{\alpha}\left(V_N^\alpha(x_0) - V_N^\alpha(f(x_0, \mu_N^\alpha(x_0)))\right)\right)$$

with $\gamma_V^{-1} \in \mathscr{K}_\infty$. Furthermore, the solution

$$u^0(x_0) \coloneqq \operatorname*{arg\,min}_{u \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)} J_N^0(x_0, u)$$

is unique due to Assumption 3.5. This assumption also gives $J_N^0(x_0, u^0(x_0)) = 0$. Hence, we can derive

$$\begin{aligned}
\frac{1}{\alpha}V_N^\alpha(x_0) &= \frac{1}{\alpha}\min_{u \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)} J_N^\alpha(x_0, u) \le \frac{1}{\alpha}J_N^\alpha(x_0, u^0(x_0)) \\
&= \frac{1}{\alpha}(J_N^0(x_0, u^0(x_0)) + \alpha \min_v \theta(v, u^0(x_0))) = \min_v \theta(v, u^0(x_0)).
\end{aligned}$$

For any $x_0 \in \mathbb{X}_N$, we choose $u^\alpha(x) \in \operatorname{arg\,min}_{u \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)} J_N^\alpha(x_0, u)$ arbitrarily. Furthermore, we deduce

$$\begin{aligned}
\frac{1}{\alpha}V_N^\alpha(f(x_0, \mu_N^\alpha(x_0))) &= \frac{1}{\alpha}J_N^\alpha(f(x_0, \mu_N^\alpha(x_0)), u^\alpha(f(x_0, \mu_N^\alpha(x_0)))) \\
&= \frac{1}{\alpha}\max_v \Psi_\alpha^N(u^\alpha(f(x_0, \mu_N^\alpha(x_0))), v) \\
&\ge \frac{1}{\alpha}\Psi_\alpha^N(u^\alpha(f(x_0, \mu_N^\alpha(x_0))), u^\alpha(f(x_0, \mu_N^\alpha(x_0)))) \\
&= \theta(u^\alpha(f(x_0, \mu_N^\alpha(x_0))), u^\alpha(f(x_0, \mu_N^\alpha(x_0)))).
\end{aligned}$$

Since Assumption 3.5 holds, $u^0(x_0)$ is a VNE and satisfies

$$\min_v \theta(v, u^0(x_0)) = \theta(u^0(x_0), u^0(x_0)).$$

Hence, by the monotonicity of γ_V, we have

$$\|x_0 - x^*\| \le \gamma_V^{-1}\left(\theta(u^0(x_0), u^0(x_0)) - \theta(u^\alpha(f(x_0, \mu_N^\alpha(x_0))), u^\alpha(f(x_0, \mu_N^\alpha(x_0))))\right). \tag{13}$$

Next, we show that

$$\lim_{\alpha \to 0} u^\alpha(f(x_0, \mu_N^\alpha(x_0)) = u^0(f(x_0, \mu_N(x_0))) \tag{14}$$

holds independently of the choice u^α. To this end, let $\varepsilon > 0$. Consider the function $S\colon \mathbb{R} \times X \to 2^{U^N}$ given by

$$(\alpha, x_0) \mapsto \operatorname*{arg\,min}_{u \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)} J_N^\alpha(x_0, u).$$

Due to Assumption 3.4, we can employ Theorem 5.4 and deduce that S is upper semi-continuous at any $(\alpha, x_0) \in [0, 1] \times \mathbb{X}_{N-1}$. Furthermore, note that by Assumption 3.5, for any $x \in \mathbb{X}_{N-1}$ the set $S(0, x) \subset U^N$ is single-valued. In particular, for $x_1 \coloneqq f(x_0, \mu_N(x_0)) \in \mathbb{X}_{N-1}$ this means that there exists $\delta > 0$ such that for $(\beta, y_1) \in (-\delta, \delta) \times B_\delta(x_1)$, we have $\|u^0(x_1) - u^\beta(y_1)\| \le \varepsilon$ for any choice $u^\beta(y_1) \in S(\beta, y_1)$. By Theorem 5.1, we obtain that for any $x \in \mathbb{X}_N$, for $\alpha \to 0$, any convergent sequence $(u^\alpha(x))_\alpha$ with $u^\alpha(x) \in S(\alpha, x)$ converges to the unique minimizer $u^0(x) \in \mathbb{U}^N_{\mathbb{X}_0}(x)$. Since $\mathbb{U}^N_{\mathbb{X}_0}(x)$ is bounded, by using the subsequence principle, we can deduce

$$\lim_{\alpha\to 0} u^\alpha(x) = u^0(x)$$

for arbitrary $u^\alpha(x) \in S(\alpha, x)$ and $x \in \mathbb{X}_N$. Since the transition map f is continuous, this implies

$$\lim_{\alpha\to 0} f(x, (u^\alpha(x))_0) = f(x, (u^0(x))_0).$$

By this, for $x_0 \in \mathbb{X}_N$ specifically, we can find $\delta_1 > 0$ such that for $\alpha \in (-\delta_1, \delta_1)$, we have

$$\|f(x, \mu^\alpha_N(x_0)) - f(x, \mu_N(x_0))\| = \|f(x, (u^\alpha(x_0))_0) - f(x, (u^0(x_0))_0)\| < \delta$$

for any $u^\alpha(x_0) \in S(\alpha, x_0)$. Now, by choosing $\alpha \le \min(\delta, \delta_1)$, we can deduce (14). Now, taking Equation (13) to the limit $\alpha \to 0$, we obtain

$$\begin{aligned}\|x_0 - x^*\| &\le \gamma_V^{-1}\left(\theta(u^0(x_0), u^0(x_0)) - \theta(u^0(f(x_0, \mu_N(x_0))), u^0(f(x_0, \mu_N(x_0))))\right).\\ &= \gamma_V^{-1}\left(V^0_N(x_0) - V^0_N(f(x_0, \mu_N(x_0)))\right).\end{aligned}$$

Consequently, applying γ_V to the equation conserves the order and we are left with

$$\gamma_V(\|x_0 - x^*\|) \le V^0_N(x_0) - V^0_N(f(x_0, \mu_N(x_0)).$$

Next, we show that γ_1 is a lower bound. In the same fashion as above, for $x_0 \in \mathbb{X}_N$, we can deduce

$$\begin{aligned}\|x_0 - x^*\| \le \gamma_1^{-1}(\frac{1}{\alpha}V^\alpha_N(x_0)) &\le \gamma_1^{-1}(\min_v \theta(v, u^0(x_0)))\\ &= \gamma_1^{-1}(\theta(u^0(x_0), u^0(x_0))) = \gamma_1^{-1}(V^0_N(x_0)).\end{aligned}$$

Finally, we construct the upper bound directly. Note that

$$J_{\text{agg}}(x_0, u^0(x_0)) = \theta(u^0(x_0), u^0(x_0)), \quad J_{\text{agg}}(x^*, (u^*)^N) = J^0_N(x^*, (u^*)^N) = 0.$$

Furthermore, due to Lemma 4.6, we have $u^0(x^*) = (u^*)^N$. Since the dynamics is completely controllable there exists a $\delta > 0$ with $B_\delta(x^*) \subset \mathbb{X}_1 \subset \mathbb{X}_{N-1}$.

Since the pay-off functions are convex, the function $(x, u) \mapsto J^0_N(x, u)$ is Lipschitz continuous on $\bar{B}_{\delta'}(x^*) \times \mathbb{U}^N$ for some $\delta' < \delta$. Furthermore, due to Assumption 3.3, it is also non-constant. Hence, there exists $L > 0$ such that for any $x_0 \in \bar{B}_{\delta'}(x^*)$, we have

$$V^0_N(x_0) = J^0_N(x_0, u^0(x_0)) - J^0_N(x^*, (u^*)^N)) \le L\left(\|x_0 - x^*\| + \|u^0(x_0) - (u^*)^N\|\right).$$

Define

$$\tilde{\alpha}(r) \coloneqq \max_{x\in\bar{B}_r(x^*)} L\left(\|x - x^*\| + \|u^0(x) - u^*\|\right).$$

By Theorem 5.4, the function $x \mapsto S(0, x_0) = \arg\min_u J^0_N(x, u)$ is upper semi-continuous. Since for any $x \in \mathbb{X}_N$ the set of minimizers $S(0, x)$ is single-valued, the function $x \mapsto u^0(x)$ is continuous at every $x \in B_\delta(x^*) \subset \mathbb{X}_{N-1}$. Thus, the function $\tilde{\alpha}$ is continuous on $[0, \delta')$. Furthermore, $\tilde{\alpha}$ is monotonically increasing on $[0, \delta')$ and we have $\tilde{\alpha}(0) = J^0_N(x^*, u^*) = 0$. Outside of $B_{\delta'}(x^*)$, we can consider

$$\hat{\alpha}(r) \coloneqq \max\left\{J^0_N(x, u) \colon x \in \bar{B}_r(x^*), u \in \mathbb{U}^N\right\}.$$

Since J^0_N is continuous on $\mathbb{X} \times \mathbb{U}^N$ and for each $r > 0$, the set $\bar{B}_r(x^*) \times \in \mathbb{U}^N$ is compact, the function $\hat{\alpha}$ is continuous. Furthermore, $\hat{\alpha}$ is monotonically increasing and we have

$$V^0_N(x_0) \le \hat{\alpha}(\|x_0 - x^*\|).$$

Now, consider the function

$$\alpha_2(r) = r + \begin{cases}\tilde{\alpha}(\delta') + \hat{\alpha}(r) & \text{for } r \ge \delta',\\ \tilde{\alpha}(r) + \frac{r}{\delta'}\hat{\alpha}(r) & \text{for } r \in [0, \delta').\end{cases}$$

This function is continuous, strictly increasing, unbounded (due to the addition of the first term), and fulfills $\alpha_2(0) = 0$. Hence, $\alpha_2 \in \mathscr{K}_\infty$. Furthermore, we have $\alpha_2(r) \geq \tilde{\alpha}(r)$ for $0 \leq r \leq \delta'$ as well as $\alpha_2(r) \geq \hat{\alpha}(r)$ for $r \geq \delta'$. Thus, α_2 gives an upper bound for V_N^0. □

Lemma 6.2. *Under Assumption 3, the set $\mathbb{X}_N$ is forward invariant under the feedback law $\mu_N \colon X \to X$ presented in Theorem 6.1.*

Proof. The proof follows the same construction as used for the proof of Lemma 4.7. □

Corollary 6.3. *Under Assumption 3, the closed loop $x^+ = f(x, \mu_N(x))$ with $\mu_N \colon X \to X$ presented in Theorem 6.1 is asymptotically stable at $x^* \in X$ on the domain $\mathbb{X}_N$.*

Proof. Theorem 6.1 gives us the criteria for asymptotic stability from Theorem 2.3 with $V = V_N^0$, $S = \mathbb{X}_N$, and $g = f(\cdot, \mu_N)$. □

Even if Assumption 3.5 is not satisfied for every initial value, we can still make the following observation.

Lemma 6.4. *Let Assumption 3.1–3.4 hold. Consider the set*

$$\mathbb{Y} \coloneqq \left\{ x_0 \in X \colon \text{the GNEP (6) has a VNE } u \in \mathbb{U}^N_{\mathbb{X}_0}(x_0) \right\}.$$

Then, the feedback law $\mu_N \colon X \to X$ presented in Theorem 6.1 renders the set $\mathbb{Y}$ invariant.

Proof. Let $x_0 \in \mathbb{Y}$ be given. Then, there exists $u \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)$, which is a VNE of Problem (6). Hence, we have

$$\sum_{n=0}^{N-1} \sum_{\nu} \theta^\nu(u^\nu(n), u^{-\nu}(n), x_u(n)) \leq \sum_{n=0}^{N-1} \sum_{\nu} \theta^\nu(v^\nu(n), u^{-\nu}(n), y^\nu_{u,v}(n))$$

for every $v \in \mathbb{U}^N$. Choosing $v(0) = u(0)$, this yields

$$\sum_{n=1}^{N-1} \sum_{\nu} \theta^\nu(u^\nu(n), u^{-\nu}(n), x_u(n)) \leq \sum_{n=1}^{N-1} \sum_{\nu} \theta^\nu(v^\nu(n), u^{-\nu}(n), y^\nu_{u,v}(n))$$

for every $v \in \mathbb{U}^{N-1}$. Now consider $x_1 = f(x_0, u(0)) = f(x_0, \mu_N(x_0))$ and define

$$u_1(n) = u(n+1), \text{ for } n = 0, \dots, N-2, \quad u_1(N-1) = u^*.$$

Then, we have $u_1 \in \mathbb{U}^N_{\mathbb{X}_0}(x_1)$. Due to Assumption 3.2, we have $\theta^\nu(x^*, u^*) = 0$ and we can deduce

$$\begin{aligned}
&\sum_{n=0}^{N-1} \sum_{\nu} \theta^\nu(u_1^\nu(n), u^{-\nu}(n), x_{u_1}(n)) \\
&= \sum_{n=1}^{N-1} \sum_{\nu} \theta^\nu(u^\nu(n), u^{-\nu}(n), x_u(n)) + \sum_{\nu} \theta^\nu(u^*, x_u(n)) \\
&\leq \sum_{n=1}^{N-1} \sum_{\nu} \theta^\nu(v^\nu(n), u^{-\nu}(n), y^\nu_{u,v}(n)) + \sum_{\nu} \theta^\nu(u^{\nu,*}, v_1^{-\nu}, y^\nu_{u,v}(N)))
\end{aligned}$$

for every $(v, v_1) \in \mathbb{U}^{N-1} \times \mathbb{U}$. Employing an index shift, we see that u_1 is a VNE of Problem 6 with initial value $x_1 \in X$ and, hence, $x_1 \in \mathbb{Y}$. □

We give an example for which the VNE of a GNEP satisfies the equilibrium end constraint at least on a subdomain of the state space.

Example 6.5. Let the dynamics be given by $x^+ = f(x, u) = Ax + Bu$, where $A \colon X \to X$ and $B \colon U \to X$ are linear operators. Let the pair $(x^*, u^*) \subset X \times U$ satisfy $f(x^*, u^*) = x^*$. Furthermore, the operator B has a diagonal block matrix structure $B = \operatorname{diag}\{B_1, \dots, B_K\}$, where each B_ν acts on the νth player's control with $\nu = 1, \dots, K$. We will also write

$B = B_\nu + B_{-\nu}$. For each ν and any fixed choice $u^{-\nu}$, the dynamics' νth component is completely controllable in u^ν in one step. Furthermore, let the pay-off functions be given by

$$\theta^\nu(v^\nu, u^{-\nu}, x) = \|x^{*,\nu} - x^\nu\|^2 + \|x^{*,\nu} - f(x, v^\nu, u^{-\nu})_\nu)\|^2.$$

Moreover, let $\mathbb{U} \subset U$ be a closed, compact set such that $u^* \in \mathring{\mathbb{U}}$. Then, there exists a domain $\Omega \subset X$ such that for any $x \in \Omega$, there exists a VNE of

$$\min_{u\in\mathbb{U}} \max_{v\in\mathbb{U}} \sum_\nu \sum_{k=0}^{N-1} \left(\theta^\nu(u^\nu(k), u^{-\nu}(k), x_u(k)) - \theta^\nu(v^\nu(k), u^{-\nu}(k), y^\nu_{u,v}(k))\right),$$

which satisfies $x_u(N) = x^*$ for any $x_0 \in \mathbb{X}_N$.

Proof. Let $\eta > 0$ be given. Then, the set

$$Y = \{x_u(N-1) \colon u \in \mathbb{U}^N,\ x_u(0) \in B_\eta(x^*)\}$$

is bounded. Due to the controllability properties of the system and Lemma 5.3, there exists a ball $B_\delta(x^*)$ with $\delta > 0$, such that for any $y \in B_\delta(x^*)$, there exists a $v_y \in \mathbb{U}$ with $x^* = f(y, v)$. Choosing η small enough, we ensure $Y \subset B_\delta(x^*)$ due to the continuity of the transition map. Note that due to the diagonal block structure of B, for any $u \in \mathbb{U}$, the point $z = f(y, v^\nu_y, u^{-\nu})$ fulfills $z^\nu = x^{*,\nu}$ for any component $\nu = 1, \dots, K$ and $y \in B_\delta(x^*)$. Let $u = (u(0), \dots, u(N-1))$ be a solution to the GNEP.

Then, we have $\sum_\nu \theta^\nu(u, u) \le \sum_\nu \theta^\nu(v, u)$ for any $v \in \mathbb{U}^N$. Consider $v = (u(0), \dots, u(N-1), v_z)$, where v_z is chosen as described above for $z = x_u(N-1)$. Using this choice of v, we can deduce

$$\sum_\nu \theta^\nu(u^\nu(N-1), u^{-\nu}(N-1), x_u(N-1)) \le \sum_\nu \theta^\nu(v^\nu_x, u^{-\nu}(N-1), x_u(N-1)),$$
$$\|x^{*,\nu} - x^\nu_u(N-1)\|^2 + \|x^{*,\nu} - x^\nu_u(N)\|^2 \le \|x^{*,\nu} - x^\nu_u(N-1)\|^2 + \|x^{*,\nu} - x^{*,\nu}\|^2,$$

which gives $x^\nu_u(N) = x^{*,\nu}$ for every $\nu = 1, \dots, K$. □

Remark 6.6. Equilibrium end constraints are not applicable if the system is not controllable to $x^\star$. In classic MPC, the terminal constraint can be relaxed by further imposing Lyapunov end costs $F\colon X \to \mathbb{R}$. This construction can also be used to design a stabilizing feedback scheme for MPC based on α-quasi GNEPs by generalizing the conditions found in [18, Section 5.3]. When it comes to transferring the result to $\alpha = 0$, one could aim for a similar construction as in Section 6. Note that this would require finding an upper bound for

$$\frac{1}{\alpha} V^\alpha_N(x_0) = \frac{1}{\alpha}\left(\min_{u\in\mathbb{U}^N_{\mathbb{X}_0}(x_0)} \max_{v\in\mathbb{U}^N} \Psi^N_\alpha(x_0, u, v) + F(x_u(N))\right)$$

and, subsequently, passing to the limit $\alpha \to 0$. Note that the presence of terms of the form $\frac{1}{\alpha}F(x)$ might hinder convergence. The analysis for this case is an interesting direction for future work.

7. Numerical Examples

In order to shed some light on how the proposed MPC schemes perform in practice, we present two dynamics for which we implemented the GNEP based MPC feedback given by Algorithm 3.1 (without equilibrium end constraints) and its alteration based on Problem (8) (with equilibrium end constraints and $\alpha \neq 0$). The first dynamics is given by the time-discrete linear system

$$x^+ = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u, \tag{15}$$

which fits to the conditions of Theorem 6.5. Here, each component of the state corresponds to a player $\nu = 1, 2$. The pay-off functions are given by the tracking term pay-offs given in Example 6.5. For the second example, we consider the dynamics given by two inverted pendulums coupled with a spring. The dynamics is modeled by the system

$$\begin{aligned}\ddot{x}_\nu &= u_\nu + k(x_{-\nu} - x_\nu \pm d),\\ \ddot{\vartheta}_\nu &= \frac{g}{l}\sin\vartheta_\nu - \frac{1}{l}\cos\vartheta_\nu(u_\nu + k(x_{-\nu} - x_\nu \pm d))\end{aligned}$$

for $\nu = 1, 2$, where g is the gravitational constant, $l \in [0, \infty)$ is the length of the rods, $k \in \mathbb{R}$ refers to the spring constant, and d is the distance the pendulums keep, when the spring is rested. This can be rewritten into a dynamical system given by the variables $(\vartheta_1, \dot{\vartheta}_1, x_1, \dot{x}_1, \vartheta_2, \dot{\vartheta}_2, x_2, \dot{x}_2)$. Furthermore, with a slight abuse of notation, we write $x^\nu = (\vartheta_\nu, \dot{\vartheta}_\nu, x_\nu, \dot{x}_\nu)$ for each player ν. A stationary point of this system is given by

$$x^* = (0, 0, 0, 0, 0, 0, d, 0), \quad u^* = (0, 0).$$

To ensure the existence of a unique solution to the modified equilibrium problems we encounter in each iteration, we linearize the dynamic at (x^*, u^*). We use an explicit RK4 scheme to translate the resulting system into a time-discrete form, which has the form $x^+ = (\mathrm{Id} + A_{\mathrm{RK}})x + B_{\mathrm{RK}}u - A_{\mathrm{RK}}x^* \coloneqq Ax + Bu + C$. In order to fulfill the complete controllability condition for the GNEP case, we concatenate four discrete time steps to arrive at an affine-linear time-discrete system, that is completely controllable in one step. This results in the control space $U = \mathbb{R}^8$ and the dynamics

$$x^+ = A^4 x + [B \ AB \ A^2B \ A^3B]u + \sum_{k=0}^{3} A^k C.$$

Note that this system does not have the block structure required to guarantee the existence of a VNE that satisfies the equilibrium end constraints. For the MPC loop based on Problem (8), we forgo the concatenation since it is not required. We furthermore equip the dynamics with pay-off functions of the form

$$\theta^{\nu,k}(x, u) = \|x^\nu(k) - x^{*,\nu}\|^2 + \|u^\nu(k)\|^2.$$

Regarding the optimization,when working with the modified Nikaido–Isoda function and equilibrium end constraints, we solve the outer minimization with a sequential least-squares programming (SLSQP) algorithm [26]. The derivative is computed based on Danskin's theorem (see, e.g., [15]) for which the inner maximization needs to be solved, which we accomplish by using an L-BFGS-B algorithm [9, 34].

In each iteration of Algorithm 3.1, we compute the VNE based on a reformulation of the GNEP via the regularized Nikaido–Isoda function [14, 20] resulting in the optimality condition from Theorem 3.6 of [20], which we again solve via a L-BFGS-B algorithm. Here, we do not enforce the equilibrium end constraints, since our stability region is only given by points, where the GNEP solution satisfies them anyway.

For the first example, based on the dynamics (15), the closed MPC-GNEP trajectory can be found in Figure 1. Due to the structure of Example 6.5, there exists a stability region for the GNEP-MPC scheme. Outside, we cannot guarantee that the computed VNE satisfies the terminal constraint. This is mirrored in the corresponding Lyapunov function V^0, which is not monotone outside this region, but starts decaying when the state enters the stability region. Subsequently, we see stabilization of the state trajectory.

For the dynamics given by the linearized coupled inverted pendulum, the results computed via the auxiliary MPC scheme (with terminal end constraints and modified Nikaido–Isoda function) and the scheme given by Algorithm 3.1 can be found in Figures 2 and 3, respectively. For the auxiliary scheme, we see stabilizing behavior, which is in accordance to our theoretical results from Section 4.

Regarding the MPC-GNEP scheme, note that for this example, Assumption 3.5 is not fulfilled. Hence, we cannot provide a theoretical justification for a converging closed-loop trajectory. Nonetheless, asymptotically stabilizing behavior can be observed. As

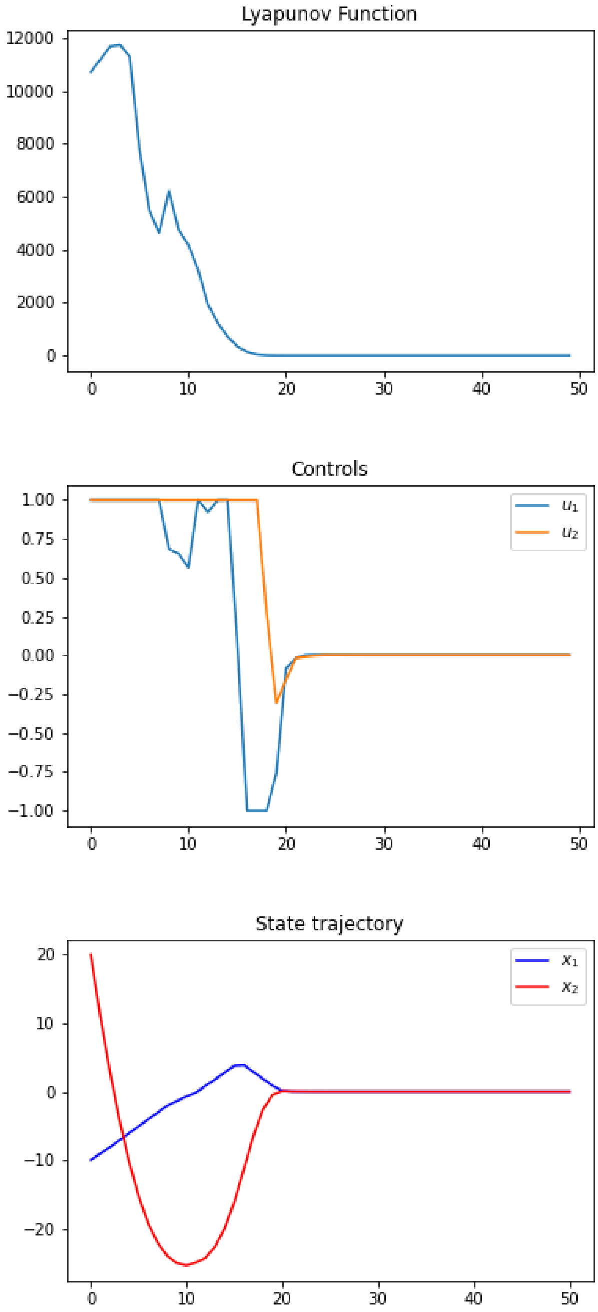

FIGURE 1. MPC based on Algorithm 3.1 for the dynamics (15). We use the horizon length $N = 10$ and bounds $\mathbb{U}_N = [-1, 1]$. Furthermore, $1\,\%$ uniformly distributed noise is added to the dynamics. Note that the Lyapunov function is non-monotone outside of the stability region.

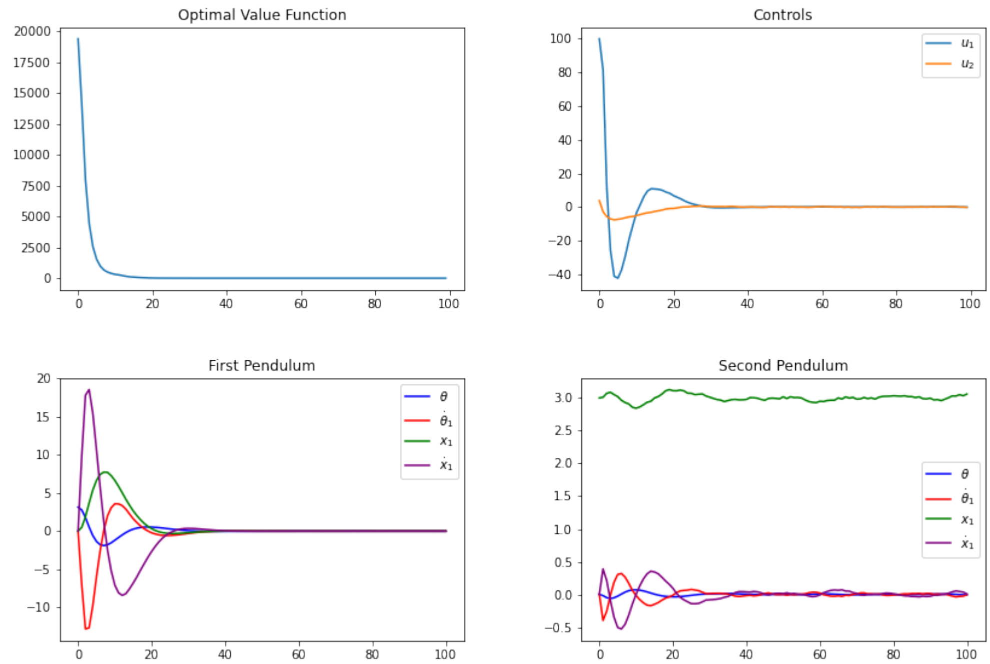

FIGURE 2. MPC based on problem (8) with equilibrium end constraints and modified Nikaido–Isoda function for the coupled inverted pendulum. We use the horizon length $N = 20$, $\gamma = 1$, $\alpha = 0.1$, $\mathbb{U} = [-100, 100]$. Furthermore, 1 % uniformly distributed noise is added to the dynamics.

time progresses, the trajectory seems to oscillate around a stable point with a decaying amplitude. Simultaneously, the Lyapunov function V^0 seems to decay in an oscillating fashion as well. This suggests that the stability analysis can be extended to cases in which the VNE computed in each iteration do not fulfill the equilibrium end constraints.[1]

8. Conclusion and Outlook

We presented a stability analysis for MPC schemes based on dynamic GNEPs and an approximation thereof: α-quasi GNEPs. For both types of problems, we were able to derive conditions guaranteeing asymptotic stability by employing terminal constraints.

For the α-quasi GNEP, both equilibrium end constraints and Lyapunov end costs can be utilized. Although α-quasi GNEPs do not exactly correspond to game-theoretical problems, they can be used for feedback synthesis nonetheless. Furthermore, they play an important role in transferring the analysis to the MPC scheme for GNEPs. In this field, stabilizing behavior has barely been studied analytically. We provide a stability result for an abstract class of jointly convex GNEPs based on classic Lyapunov theory. A crucial assumption is that the corresponding variational Nash equilibria fulfill the equilibrium end constraint. Although this condition is not met generally, our numerical examples suggest that MPC for GNEPs can be stabilizing despite its violation. This raises the question of whether the end constraint can be omitted and, consequently, whether a stability result can again be derived on the basis of α-quasi GNEPs. Similarly to classic MPC, exponentially stabilizing dynamics [18, 1] paired with a suitably large look-ahead horizon could be key in this analysis.

[1] The jupyter notebooks containing these computations can be found at `https://github.com/anttop/MPC_GNEP`.

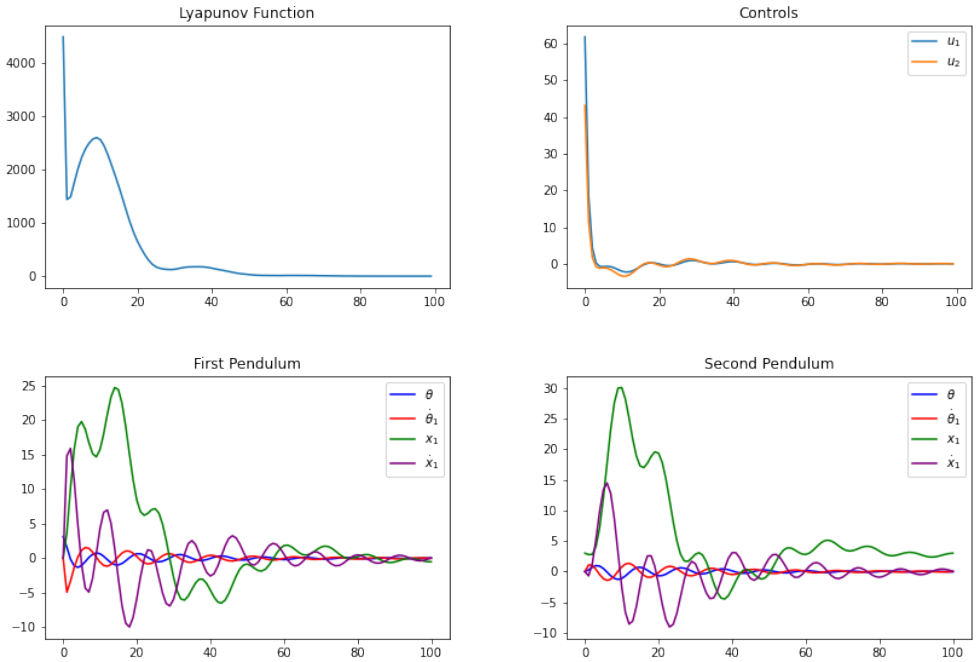

FIGURE 3. MPC based on Algorithm 3.1 for the coupled inverted pendulum. We use the horizon length $N = 5$, $\gamma = 1$, $\mathbb{U} = [-100, 100]$. Furthermore, 1. % uniformly distributed noise is added to the dynamics. Note that for this setup, Condition 3.5 is not fulfilled in general.

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9. Appendix

9.1. Proof of Theorem 4.1.

Proof. The proof follows the arguments used in Theorem VIII.1.9 of [2] for showing a DPP for dynamic games with respect to the lower value. In the following, we will denote the right-hand side of Equation (7) by $W_N(n_0+1, x_0)$.

For $u \in U^N(n_0, x_0)$, the ith component of the control will be denoted by $u(n_0+i)$ or $v(n_0+i)$, respectively. First note that due to the compactness of the sets of admissible controls and the continuity of the functions f and ℓ, each minimum and maximum is well-defined.

Let $(n_0, x_0) \in \mathbb{N} \times X$ be fixed. We begin with showing $V_N \leq W_N$. Let $\varepsilon > 0$ and choose $z \in X$, $n \in \mathbb{N}$ arbitrarily. Then, there exists a $u_z \in U^{N-1}(n, z)$ with

$$V_{N-1}(n, z) \geq \max_{v \in V^{N-1}(n,z)} J_{N-1}(n, z, u_z, v) + \frac{\varepsilon}{2}.$$

Furthermore, there exists $\bar{u} \in U^N(n_0, x_0)$ such that

$$W(n_0, x_0) \geq \max_{v \in V^N(n_0, z_0)} \Big\{ \ell(n_0, \bar{u}(n_0), v(n_0), x_0) + V_{N-1}(n_0+1, x_{\bar{u},v}(n_0+1)) \Big\} + \frac{\varepsilon}{2}.$$

Choosing $z = x_{\bar{u},v}(n_0+1)$ as well as $n = n_0+1$ (for every $v \in V^N(n_0, x_0)$) and exploiting the first non-anticipativity property for the control v, we can conclude

$$\begin{aligned}
&W(n_0, x_0) \\
&\geq \max_{v \in V^N} \Big\{ \ell(n_0, \bar{u}(n_0), v(n_0), x_0) + \\
&\geq \max_{\tilde{v} \in V^{N-1}} J_{N-1}(n_0, x_{\bar{u},v}(n_0+1), u_z, \tilde{v}) + \frac{\varepsilon}{2} \Big\} + \frac{\varepsilon}{2} \\
&\geq \max_{v \in V^N} \{ \ell(n_0, \bar{u}(n_0), v(n_0), x_0) + J_{N-1}(n_0, x_{\bar{u},v}(n_0+1), u_z, v\big|_{n=n_0+1}^{n_0+N} \} + \varepsilon \\
&\geq \min_{u \in U^N} \max_{v \in V^N} J_N(n_0, x_0, u, v) + \varepsilon,
\end{aligned}$$

which yields the desired inequality.

Lastly, we show $V_N \geq W_N$ in a similar way. Again, let $\varepsilon > 0$. There exists a $u \in U^N(n_0, x_0)$ with

$$V_N(n_0, x_0) \geq \max_{v \in V^N(n_0,x_0)} J_N(n_0, x_0, u, v) + \varepsilon. \tag{16}$$

Furthermore, we have

$$W(n_0, x_0) \leq \max_{v \in V^N(n_0,x_0)} \Big\{ \ell(n_0, x_0, u(n_0), v(n_0)) + V_{N-1}(n_0 + 1, x_{u,v}(n_0 + 1)) \Big\}. \tag{17}$$

There exists $v \in V^N(n_0, x_0)$ such that

$$\begin{aligned} &\max_{\tilde{v} \in V^N} \Big\{ \ell(n_0, x_0, u(n_0), \tilde{v}(n_0)) + V_{N-1}(n_0 + 1, x_{u,\tilde{v}}(n_0 + 1)) \Big\} \\ &\leq \ell(n_0, x_0, u(n_0), v(n_0)) + V_{N-1}(n_0 + 1, x_{u,v}(n_0 + 1)) + \frac{\varepsilon}{2}. \end{aligned} \tag{18}$$

Note that $u \in U^N(n_0, x_0)$ implies $u(\cdot + 1) \in U^{N-1}(n_0 + 1, x_{u,\tilde{v}}(n_0 + 1))$ for every $\tilde{v} \in V^N(n_0, x_0)$. Hence, for $z = x_{u,v}(n_0 + 1)$, we find $\bar{v} \in V^{N-1}(n_0 + 1, x_{u,v}(n_0 + 1))$ with

$$\begin{aligned} V_{N-1}(n_0 + 1, z) &= \min_{u \in U^{N-1}} \max_{v \in V^{N-1}} J_{N-1}(n_0 + 1, z, u, v) \\ &\leq \max_{v \in V^{N-1}} J_{N-1}(n_0 + 1, z, u(\cdot + 1), v) \\ &\leq J_{N-1}(n_0 + 1, z, u(\cdot + 1), \bar{v}) + \frac{\varepsilon}{2}. \end{aligned} \tag{19}$$

Combining (17)–(19) and finally (16), we can conclude

$$\begin{aligned} W_N(n_0, x_0) &\leq \ell(n_0, x_0, u(n_0), v(n_0)) + V_{N-1}(n_0 + 1, x_{u,v}(n_0 + 1)) + \frac{\varepsilon}{2} \\ &\leq \ell(n_0, x_0, u(n_0), v(n_0)) + J_{N-1}(n_0 + 1, z, u(\cdot + 1), \bar{v}) + \varepsilon \\ &\leq \max_{v \in V^N} J_N(n_0, x_0, u, v) + \varepsilon \\ &\leq V_N(n_0, x_0). \end{aligned}$$

□

9.2. **Miscellaneous.**

Lemma 9.1. *Let $C \subset \mathbb{R}^n$ be a convex set with $\mathring{C} \neq \emptyset$. Let $x \in \partial C$ and $y \in \mathring{C}$. Then, the inclusion*

$$\{x + \varepsilon(y - x) \,:\, \varepsilon \in (0, 1]\} \subset \mathring{C}$$

holds.

Proof. Let $\varepsilon > 0$ and consider $m = x + \varepsilon(y - x)$. Then, $m \in C$. Furthermore, the function $f \colon \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$z \mapsto x + \frac{1}{\varepsilon}(z - x)$$

is continuous with $f^{-1}(C) \subset C$, since for $w \in C$ we have

$$w = x + \frac{1}{\varepsilon}(z - w) \iff z = x + \varepsilon(w - x).$$

Due to continuity, $f^{-1}(\mathring{C}) \subset C$ is open. Furthermore, we have $m \in f^{-1}(\mathring{C})$ since $f(m) = y$ and, thus, $m \in \mathring{C}$. □

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