A four-operator splitting algorithm for nonconvex and nonsmooth optimization

Jan Harold Alcantara* Ching-pei Lee† Akiko Takeda‡

June 26, 2024

Abstract

In this work, we address a class of nonconvex nonsmooth optimization problems where the objective function is the sum of two smooth functions (one of which is proximable) and two nonsmooth functions (one weakly convex and proximable and the other continuous and weakly concave). We introduce a new splitting algorithm that extends the Davis-Yin splitting (DYS) algorithm to handle such four-term nonconvex nonsmooth problems. We prove that with appropriately chosen step sizes, our algorithm exhibits global subsequential convergence to stationary points with a stationarity measure converging at a rate of $1/k$. When specialized to the setting of the DYS algorithm, our results allow for larger stepsizes compared to existing bounds in the literature. Experimental results demonstrate the practical applicability and effectiveness of our proposed algorithm.

Keywords. operator splitting; Davis-Yin splitting; nonconvex optimization; nonsmooth optimization

1 Introduction

We consider the nonsmooth and nonconvex problem

$$\min_{x\in\mathbb{R}^n} \ \Psi(x) \coloneqq f(x) + g(x) + h(x) + p(x) \tag{1.1}$$

under the following assumptions.

Assumption 1.1. *The functions, f, g, h and p satisfy the following:*

(a) $f : \mathbb{R}^n \to (-\infty, \infty)$ *is an* L_{f}*-smooth function and is "proximable", in the sense that its proximal mapping either has a closed form or is easily computable;*

(b) $g : \mathbb{R}^n \to (-\infty, \infty]$ *is a proper, closed, and* ρ_{g}*-weakly convex function; that is,* $g + \frac{\rho_{\mathrm{g}}}{2}\|\cdot\|^2$ *is convex where* $\rho_{\mathrm{g}} \geq 0$*;*

(c) $h : \mathbb{R}^n \to (-\infty, \infty)$ *is an* L_{h}*-smooth function; and*

*janharold.alcantara@riken.jp. Center for Advanced Intelligence Project, RIKEN, Tokyo, Japan.

†chingpei@ism.ac.jp. Department of Advanced Data Science, Institute of Statistical Mathematics, Tokyo, Japan.

‡takeda@mist.i.u-tokyo.ac.jp. Department of Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo, Tokyo, Japan, and Center for Advanced Intelligence Project, RIKEN, Tokyo, Japan

(d) $p : \mathbb{R}^n \to \mathbb{R}$ *is continuous (possibly nonsmooth) on an open set containing the domain of* g *such that* $-p$ *is* L_{p}*-weakly convex with* $L_{\mathrm{p}} \geq 0$*; that is,* $\frac{L_{\mathrm{p}}}{2}\|\cdot\|^2 - p$ *is convex.*

A notable special case of (1.1) is the three-term optimization problem, where $p \equiv 0$. The Davis-Yin splitting (DYS) algorithm, introduced in Davis and Yin (2017), tackles this problem for convex functions f, g, and h, assuming h additionally meets Assumption 1.1(c). To extend the algorithm's applicability to nonconvex problems, Bian and Zhang (2021) leveraged techniques from Li and Pong (2016) to show that with a sufficiently small stepsize, the DYS algorithm achieves global subsequential convergence. This requires f and h to satisfy conditions (a) and (c) of Assumption 1.1, respectively, and g to be any proper closed function. However, despite not requiring the functions to be convex, this extension is still limited as it can only admit at most one nonsmooth function.

In scenarios involving objective functions comprised of multiple nonsmooth, nonconvex terms − such as those encountered in nonconvex fused Lasso Parekh and Selesnick (2015), ℓ_{1-2} regularized optimization problems Yin et al. (2015), simultaneous sparse and low-rank optimization problems Richard et al. (2012), among others − the DYS algorithm becomes inapplicable. Indeed, the problem setting of the DYS framework is generally not suited for several classes of nonsmooth nonconvex optimization problems such as the following ones.

DC optimization. The DYS algorithm is not applicable to difference-of-convex (DC) regularized optimization problems, which take the form

$$\min_{x\in\mathbb{R}^n} \; f(x) + g(x) - \hat{g}(x), \tag{1.2}$$

where g and $\hat{g}$ are convex. The limitation arises from the potential nonsmoothness of the concave component $-\hat{g}$. Meanwhile, by setting $h \equiv 0$ and $p \coloneqq -\hat{g}$, we satisfy Assumption 1.1 (c) and (d) with $L_{\mathrm{h}} = L_{\mathrm{p}} = 0$.

General nonsmooth nonconvex regularized optimization. Another example is the general class of regularized problems considered in Liu et al. (2019a), given by

$$\min_{x\in\mathbb{R}^n} \; f(x) + g(x) + \sum_{i=1}^{r} P_i(A_i x),$$

where g and P_i are nonsmooth, nonconvex and nonnegative regularizers, and $A_i \in \mathbb{R}^{n_i\times n}$. This difficult class of optimization problems was addressed by Liu et al. (2019a) through the use of the Moreau envelope $M_{\lambda_i P_i}$ of each P_i to obtain an approximate problem:

$$\min_{x\in\mathbb{R}^n} \; f(x) + g(x) + \sum_{i=1}^{r} M_{\lambda_i P_i}(A_i x). \tag{1.3}$$

However, note that even after this reformulation, the DYS algorithm is still not applicable to (1.3), as the third term is in general a nonsmooth function due to the nonconvexity of the P_i's. On the other hand, we can utilize the fact (see (Lucet, 2006, Proposition 3)) that for any function P,

$$\frac{1}{2\lambda}\|x\|^2 - M_{\lambda P}(x) = \sup_{y\in\mathrm{dom}(P)} \left\{ \frac{1}{\lambda} x^\top y - \frac{1}{2\lambda}\|y\|^2 - P(y) \right\}. \tag{1.4}$$

By noting the convexity of the function on the right-hand side of (1.4) (being the supremum of affine (convex) functions), we may then fit (1.3) in the form (1.1) by setting $h \equiv 0$ and $p \coloneqq \sum_{i=1}^{r} M_{\lambda_i P_i} \circ A_i$ so that Assumption 1.1(d) holds with $L_{\mathrm{p}} = \sum_{i=1}^{r} \lambda_i^{-1}$.

Nonconvex feasibility problems. We can also draw examples from feasibility problems reformulated as optimization problems: Let A, B, C and D_i for $i \in \{1, 2, \ldots, r\}$ be closed nonempty sets where A, B, C are convex, and the D_i's are not necessarily convex. The *feasibility problem* is given by:

$$\text{Find } x \in A \cap B \cap C \cap \bigcap_{i=1}^{r} D_i.$$

By defining $d_S(x) \coloneqq \inf_{w \in S} \|w - x\|$ for any set S, this problem can be reformulated as the following problem that conforms the form of (1.1):

$$\min_{x \in \mathbb{R}^n} \underbrace{\frac{1}{2} d_A^2(x)}_{f(x)} + \underbrace{\frac{1}{2} d_B^2(x)}_{g(x)} + \underbrace{\frac{1}{2} d_C^2(x)}_{h(x)} + \underbrace{\sum_{i=1}^{r} \frac{1}{2} d_{D_i}^2(x)}_{p(x)}. \tag{1.5}$$

The original setting of the DYS algorithm in Davis and Yin (2017) can handle this problem when $r = 0$. On the other hand, the extended setting considered in Bian and Zhang (2021) can handle the case $B = \mathbb{R}^n$ and $r = 1$ (*i.e.*, the feasibility problem with two convex sets and one nonconvex set), but it requires a very small stepsize. Meanwhile, by the convexity of A, B and C, we see that Assumption 1.1(a), (b) and (c) hold for (1.5). The function p, on the other hand, is a nonsmooth function due to the nonconvexity of each D_i. Nevertheless, noting that the half-squared distance function is the Moreau envelope of the indicator function, we have from (1.4) that p satisfies Assumption 1.1(d) with $L_{\mathrm{p}} = r$.

In this paper, we propose a splitting algorithm that generalizes the DYS algorithm to the four-term optimization problem (1.1), which encompasses a broader range of nonconvex and nonsmooth problem classes, including those described above. We prove that our algorithm is globally subsequentially convergent to stationary points for appropriately chosen stepsizes, with a first-order optimality measure converging at a rate of $1/k$. Another major contribution of this work is the derivation of upper bounds for stepsizes for such convergence results. When specialized to $p \equiv 0$, our results yield stepsize estimates for the DYS algorithm that are significantly larger than the existing bounds in Bian and Zhang (2021). Up to our knowledge, for this special case of $p \equiv 0$, our iteration complexity result is also new for the DYS algorithm. During the final stages of preparing this manuscript, we came across a recent preprint by Dao et al. (2024) that examines a specific case of our problem in which $L_{\mathrm{p}} = 0$ in Assumption 1.1(d). Notably, our analysis yields upper bounds for stepsizes that are considerably larger than those reported in their work.

This paper is organized as follows. In Section 2, we summarize important definitions, notations and known results that we will use in this paper. We propose our algorithm and establish its global subsequential convergence and convergence rates in Section 3. Experiments to demonstrate the applicability and efficiency of our method are presented in Section 4, and concluding remarks are given in Section 5.

2 Preliminaries

Throughout the paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space endowed with the inner product $\langle \cdot, \cdot \rangle$, and we denote its induced norm by $\|\cdot\|$. The set $(-\infty, \infty]$ is the extended real-line, and we adopt the conventions that $\frac{a}{\infty} = 0$, $\frac{a}{0} = \infty$ for any $a \neq 0$, and $\frac{\infty}{\infty} = 1$.

Let $\phi : \mathbb{R}^n \to (-\infty, \infty]$ be an extended-valued function. The *domain* of ϕ is given by the set $\mathrm{dom}(\phi) = \{x \in \mathbb{R}^n : \phi(x) < \infty\}$. We say that ϕ is a *proper* function if $\mathrm{dom}(\phi) \neq \emptyset$, and that ϕ is

closed if it is lower semicontinuous. ϕ is said to be a σ_ϕ-*convex* function if $\phi - \frac{\sigma_\phi}{2}\|\cdot\|^2$ is convex for some $\sigma_\phi \in \mathbb{R}$. If $\sigma_\phi > 0$, then ϕ is σ_ϕ-*strongly convex*. If $\sigma_\phi \leq 0$, we denote $\rho_\phi := -\sigma_\phi$ and call ϕ a ρ_ϕ-*weakly convex function*. In other words, ϕ is ρ_ϕ-weakly convex for $\rho_\phi \geq 0$ if $\phi + \frac{\rho_\phi}{2}\|\cdot\|^2$ is convex.

The *subdifferential* of ϕ at a point $x \in \mathrm{dom}(\phi)$ is defined as

$$\partial\phi(x) := \left\{\xi \in \mathbb{R}^n : \exists\{(x^k, \xi^k)\} \text{ such that } x^k \xrightarrow{\phi} x,\ \xi^k \in \hat{\partial}\phi(x^k),\ \text{and } \xi^k \to \xi\right\}, \tag{2.1}$$

where $x^k \xrightarrow{\phi} x$ means $x^k \to x$ and $\phi(x^k) \to \phi(x)$, and

$$\hat{\partial}\phi(x) := \left\{\xi \in \mathbb{R}^n : \liminf_{\bar{x}\to x, \bar{x}\neq x} \frac{\phi(\bar{x}) - \phi(x) - \langle \xi, \bar{x} - x\rangle}{\|\bar{x} - x\|} \geq 0\right\}.$$

When ϕ is convex, (2.1) coincides with the classical subdifferential in convex analysis:

$$\partial\phi(x) = \{\xi \in \mathbb{R}^n : \phi(y) \geq \phi(x) + \langle \xi, y - x\rangle,\ \forall y \in \mathbb{R}^n\}.$$

If ϕ is continuously differentiable, the subdifferential reduces to a singleton containing the gradient of ϕ. We also note that the definition of the subdifferential gives the following property:

$$\left\{\xi \in \mathbb{R}^n : \exists\{(x^k, \xi^k) \text{ such that } x^k \xrightarrow{\phi} x,\ \xi^k \in \partial\phi(x^k),\ \text{and } \xi^k \to \xi\right\} \subseteq \partial\phi(x). \tag{2.2}$$

For a proper and closed function ϕ, we define the *proximal mapping* of ϕ as

$$\mathrm{prox}_{\gamma\phi}(x) := \operatorname*{arg\,min}_{w\in\mathbb{R}^n} \phi(w) + \frac{1}{2\gamma}\|w - x\|^2, \quad \gamma > 0. \tag{2.3}$$

For a set $S \subseteq \mathbb{R}^n$, we define $\mathrm{prox}_{\gamma\phi}(S) := \bigcup_{x\in S} \mathrm{prox}_{\gamma\phi}(x)$. From the optimality condition of (2.3), we have

$$y \in \mathrm{prox}_{\gamma\phi}(x) \quad \Longrightarrow \quad x - y \in \gamma\partial\phi(y), \tag{2.4}$$

and the converse holds if $\phi + \frac{1}{2\gamma}\|\cdot\|^2$ is convex. We also use the notation $T_{\gamma\phi} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ to denote

$$T_{\gamma\phi} := Id - \gamma\partial\phi,$$

where Id is the identity map on $\mathbb{R}^n$. Given a point x and a subgradient $\xi \in \partial\phi(x)$, for any $\gamma > 0$, we define

$$Q_{\gamma\phi}(w; x, \xi) := \phi(x) + \langle \xi, w - x\rangle + \frac{1}{2\gamma}\|w - x\|^2, \quad \forall w \in \mathbb{R}^n.$$

When ϕ is smooth, we write $Q_{\gamma\phi}(w; x)$, and ξ is understood to be equal to $\nabla\phi(x)$.

Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function $\phi : \mathbb{R}^n \to \mathbb{R}$. We say that ϕ is L_ϕ-*smooth* if its gradient satisfies

$$\|\nabla\phi(x) - \nabla\phi(y)\| \leq L_\phi\|x - y\|, \quad \forall x, y \in \mathbb{R}^n.$$

We collect some properties of L_ϕ-smooth functions.

Lemma 2.1. *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be an L_ϕ-smooth function. Then for any $x, y \in \mathbb{R}^n$,*

(a) $|\phi(y) - \phi(x) - \langle \nabla\phi(x), y - x\rangle| \leq \frac{L_\phi}{2}\|y - x\|^2$. *(Descent Lemma)*

(b) If in addition, ϕ is ρ_ϕ-weakly convex, then for any $L \geq L_\phi$ such that $L > \rho_\phi$, we have

$$\phi(y) - \phi(x) - \langle \nabla\phi(x), y - x\rangle \geq \frac{1}{2(L-\rho_\phi)}\|\nabla\phi(y) - \nabla\phi(x)\|^2 - \frac{\rho_\phi L}{2(L-\rho_\phi)}\|y - x\|^2.$$

Proof. Parts (a) and (b) follow from (Bertsekas, 2016, Proposition A.24) and (Themelis and Patrinos, 2020, Theorem 2.2), respectively. □

Remark 2.2. By Lemma 2.1(a), ϕ is ρ_ϕ-weakly convex for some $\rho_\phi \in [0, L_\phi]$ if ϕ is L_ϕ-smooth. If ϕ is convex and L_ϕ-smooth, we obtain from Lemma 2.1(b) that

$$\phi(y) - \phi(x) - \langle \nabla\phi(x), y - x\rangle \geq \frac{1}{2L_\phi}\|\nabla\phi(y) - \nabla\phi(x)\|^2, \quad \forall x, y \in \mathbb{R}^n. \tag{2.5}$$

3 Proposed algorithm and its convergence

In this section, we propose our algorithm and provide its convergence analysis under various parameter settings.

3.1 Proposed algorithm

We present our proposed algorithm for solving (1.1) in Algorithm 1.

Algorithm 1: A four-operator splitting algorithm for nonsmooth nonconvex optimization

Step 0. Choose an initial point $(y^0, z^0) \in \mathbb{R}^n \times \mathbb{R}^n$ and stepsize parameters $\tau > 0$, $\gamma \in (0, \infty)$, and $\alpha, \beta \in (0, \infty]$ such that $\frac{1}{\gamma} = \frac{1}{\alpha} + \frac{1}{\beta}$.

Step 1. Set

$$x^k \in \mathrm{prox}_{\alpha f}(z^k), \tag{3.1}$$

$$y^{k+1} \in \mathrm{prox}_{\gamma g}\left(\frac{\gamma}{\alpha}(2x^k - z^k - \alpha\nabla h(x^k)) + \frac{\gamma}{\beta}T_{\beta p}(y^k)\right), \tag{3.2}$$

$$z^{k+1} = z^k + \tau(y^{k+1} - x^k). \tag{3.3}$$

Step 2. If a termination criterion is not met, go to Step 1.

For convenience in later discussions, we define $P_\Lambda : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ as

$$P_\Lambda(y, z) \coloneqq \bigcup_{x \in \mathrm{prox}_{\alpha f}(z)} \mathrm{prox}_{\gamma g}\left(\frac{\gamma}{\alpha}(2x - z - \alpha\nabla h(x)) + \frac{\gamma}{\beta}T_{\beta p}(y)\right), \tag{3.4}$$

where $\Lambda \coloneqq (\alpha, \beta, \gamma)$, $\alpha, \beta \in (0, \infty]$ and $\gamma \in (0, \infty)$. Hence, the y-step in (3.2) can be written as $y^{k+1} \in P_\Lambda(y^k, z^k)$. Observe that since f is differentiable, $x \in \mathrm{prox}_{\alpha f}(z)$ implies that $z - x = \alpha\nabla f(x)$ by (2.4), so we may also write P_Λ as

$$P_\Lambda(y, z) = \bigcup_{x \in \mathrm{prox}_{\alpha f}(z)} \mathrm{prox}_{\gamma g}\left(\frac{\gamma}{\alpha}T_{\alpha(f+h)}(x) + \frac{\gamma}{\beta}T_{\beta p}(y)\right). \tag{3.5}$$

When $L_{\mathrm{p}} = 0$, we in particular set $\beta = \infty$, so that $\gamma = \alpha$ by Step 0 of Algorithm 1. On the other hand, when $L_{\mathrm{f}} = L_{\mathrm{h}} = 0$, we set $\alpha = \infty$ and therefore $\gamma = \beta$.

Remark 3.1. The above algorithm covers several algorithms in the literature.

1. (Davis-Yin splitting algorithm). When $p \equiv 0$, $\tau = 1$, and $\gamma = \alpha$, the algorithm reduces to the Davis-Yin splitting (DYS) algorithm Davis and Yin (2017), which itself covers the gradient descent algorithm (when $f \equiv g \equiv 0$), the proximal gradient algorithm for the sum of a smooth and a nonsmooth function (when $f \equiv 0$), and the Douglas-Rachford algorithm Douglas and Rachford (1956) (when $h \equiv 0$).

2. (Proximal subgradient method for sum of two nonsmooth functions). When $f \equiv h \equiv 0$ so that the stepsizes satisfy $\gamma = \beta$ and $\alpha = \infty$, the algorithm reduces to the proximal subgradient algorithm:
$$y^{k+1} = \mathrm{prox}_{\gamma g}(y^k - \gamma \partial p(y^k),$$
but covers a wider range of problems compared to DYS with $f \equiv 0$ since the function p could be nonsmooth.

3. (Proximal DC algorithm). When $f \equiv 0$, the algorithm simplifies to
$$y^{k+1} \in \mathrm{prox}_{\gamma g}\left(\frac{\gamma}{\alpha}(z^k - \alpha \nabla h(z^k)) + \frac{\gamma}{\beta} T_{\beta p}(y^k)\right),\ z^{k+1} = (1-\tau)z^k + \tau y^{k+1}.$$
Thus, when $\tau = 1$ and $L_{\mathrm{p}} = 0$ so that $\beta = \infty$ and $\gamma = \alpha$, we further obtain
$$y^{k+1} \in \mathrm{prox}_{\gamma g}(y^k - \gamma \nabla h(y^k) - \gamma \partial \phi(y^k)), \tag{3.6}$$
which is the proximal DC algorithm Wen et al. (2018) for solving (1.2).

3.2 Stepsize bounds

We now derive appropriate stepsizes for Algorithm 1 that will guarantee sufficient descent of some merit function. First, we introduce some notations. Given $\xi \in \partial p(y)$, we define $V_{\Lambda,\xi} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as
$$V_{\Lambda,\xi}(y,z,x) \coloneqq \min_{w \in \mathbb{R}^n} \Phi_{\Lambda,\xi}(w;y,z,x), \tag{3.7}$$
where
$$\Phi_{\Lambda,\xi}(w;y,z,x) \coloneqq Q_{\alpha(f+h)}(w;x) + Q_{\beta p}(w;y,\xi) + g(w). \tag{3.8}$$

We will use (3.7) as a merit function for our proposed algorithm. It is not difficult to verify that when $p \equiv 0$ and $\alpha = \gamma$ as in the setting described in the first item of Remark 3.1, the function given by (3.7) with $\alpha < L_{\mathrm{f}}^{-1}$ and $x = \mathrm{prox}_{\alpha f}(z)$ simplifies to the Davis-Yin envelope function introduced in Liu and Yin (2019), which covers the forward-backward envelope Themelis et al. (2018) and the Douglas-Rachford envelope Patrinos et al. (2014).

Lemma 3.2. *Suppose that f and h are continuously differentiable functions, g satisfies Assumption 1.1 (b), and $\gamma > 0$ satisfies $\frac{1}{\gamma} = \frac{1}{\alpha} + \frac{1}{\beta}$ and $\gamma \leq \frac{1}{\rho_{\mathrm{g}}}$, with $\alpha, \beta > 0$. Then $P_\Lambda(y,z) \neq \emptyset$ and*
$$P_\Lambda(y,z) = \bigcup_{\substack{x \in \mathrm{prox}_{\alpha f}(z) \\ \xi \in \partial p(y)}} \operatorname*{arg\,min}_{w \in \mathbb{R}^n} \Phi_{\Lambda,\xi}(w;y,z,x) \tag{3.9}$$
for all $(y,z) \in \mathrm{dom}(g) \times \mathbb{R}^n$, where P_Λ is given by (3.4).

Proof. Given $y^+ \in \arg\min_{w\in\mathbb{R}^n} \Phi_{\Lambda,\xi}(w;y,z,x)$ for some $x \in \mathrm{prox}_{\alpha f}(z)$ and $\xi \in \partial p(y)$, we have from the optimality condition of (3.7) that

$$0 \in \nabla f(x) + \nabla h(x) + \frac{1}{\alpha}(y^+ - x) + \left(\xi + \frac{1}{\beta}(y^+ - y)\right) + \partial g(y^+). \tag{3.10}$$

Rearranging the terms, we see that

$$0 \in \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) y^+ - \frac{1}{\alpha}\left(x - \alpha\nabla f(x) - \alpha\nabla h(x)\right) - \frac{1}{\beta}\left(y - \beta\xi\right) + \partial g(y^+) \tag{3.11}$$

Multiplying the above by γ and noting that $\frac{1}{\gamma} = \frac{1}{\alpha} + \frac{1}{\beta}$, we obtain

$$0 \in y^+ - \frac{\gamma}{\alpha} T_{\alpha(f+h)}(x) - \frac{\gamma}{\beta} T_{\beta p}(y) + \gamma\partial g(y^+). \tag{3.12}$$

Meanwhile, we note that

$$\mathrm{prox}_{\gamma g}\left(\frac{\gamma}{\alpha} T_{\alpha(f+h)}(x) + \frac{\gamma}{\beta} T_{\beta p}(y)\right) \overset{(3.5)}{\subseteq} P_\Lambda(y,z). \tag{3.13}$$

Since g is ρ_{g}-weakly convex, $\gamma \le \frac{1}{\rho_{\mathrm{g}}}$, and (3.12) holds, we conclude that y^+ is an element of the set on the left-hand side of (3.13) by looking at its optimality condition. Consequently, $y^+ \in P_\Lambda(y,z)$ by (3.13) and therefore "$\supseteq$" in (3.9) holds. The other inclusion can be proved by reversing the above arguments and noting that our setting for γ, α, β ensures that $\Phi_{\Lambda,\xi}(w;y,z,x)$ is convex with respect to w, so (3.10) is also a sufficient condition for optimality. □

The following lemma provides an upper bound for the merit function $V_{\Lambda,\xi}$.

Lemma 3.3. *Under the assumptions of Lemma* 3.2*, we have*

$$V_{\Lambda,\xi}(y,z,x) \le Q_{\alpha(f+h)}(y;x) + p(y) - \left(\frac{1-\gamma\rho_{\mathrm{g}}}{2\gamma}\right)\|y - y^+\|^2 + g(y)$$

for all $(y,z) \in \mathrm{dom}(g) \times \mathbb{R}^n$*,* $x \in \mathrm{prox}_{\alpha f}(z)$*,* $\xi \in \partial p(y)$*,* $y^+ \in \arg\min_{w\in\mathbb{R}^n} \Phi_{\Lambda,\xi}(w;y,z,x)$*.*

Proof. Given $\xi \in \partial p(y)$, note that

$$Q_{\beta p}(y^+;y,\xi) = p(y) + \langle \xi, y^+ - y\rangle + \frac{1}{2\beta}\|y^+ - y\|^2. \tag{3.14}$$

Meanwhile,

$$\begin{aligned}
Q_{\alpha f}(y^+;x) &= f(x) + \langle \nabla f(x), y^+ - x\rangle + \frac{1}{2\alpha}\|y^+ - x\|^2 \\
&= \left(f(x) + \langle \nabla f(x), y - x\rangle + \frac{1}{2\alpha}\|y - x\|^2\right) + \langle \nabla f(x), y^+ - y\rangle \\
&\quad + \frac{1}{2\alpha}\|y^+ - y\|^2 + \frac{1}{\alpha}\langle y - x, y^+ - y\rangle \\
&= Q_{\alpha f}(y;x) + \frac{1}{\alpha}\langle y - T_{\alpha f}(x), y^+ - y\rangle + \frac{1}{2\alpha}\|y^+ - y\|^2.
\end{aligned}$$

It follows that

$$Q_{\alpha(f+h)}(y^+;x) = Q_{\alpha(f+h)}(y;x) + \frac{1}{\alpha}\left\langle y - T_{\alpha(f+h)}(x), y^+ - y\right\rangle + \frac{1}{2\alpha}\|y^+ - y\|^2. \tag{3.15}$$

Using (3.14) and (3.15) and noting (3.8), we have

$$\begin{aligned}\Phi_{\Lambda,\xi}(y^+;y,z,x) = Q_{\alpha(f+h)}(y;x) + p(y) + \frac{1}{2\gamma}\|y^+ - y\|^2 \\ + \left\langle \frac{1}{\alpha}\left(y - T_{\alpha(f+h)}(x)\right) + \xi, y^+ - y\right\rangle + g(y^+)\end{aligned} \tag{3.16}$$

On the other hand, since $y^+ \in \arg\min_{w\in\mathbb{R}^n} \Phi_{\Lambda,\xi}(w;y,z,x)$, we have from (3.11) and the ρ_{g}-weak convexity of g that

$$\begin{aligned} g(y) &\geq g(y^+) + \left\langle \frac{1}{\alpha}T_{\alpha(f+h)}(x) + \frac{1}{\beta}(y - \beta\xi) - \frac{1}{\gamma}y^+, y - y^+\right\rangle - \frac{\rho_{\mathrm{g}}}{2}\|y - y^+\|^2 \\ &= g(y^+) + \left\langle \frac{1}{\alpha}T_{\alpha(f+h)}(x) + \frac{1}{\beta}(y - \beta\xi) - \frac{1}{\gamma}y, y - y^+\right\rangle - \left(\frac{\rho_{\mathrm{g}}}{2} - \frac{1}{\gamma}\right)\|y - y^+\|^2. \end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17), we obtain

$$\Phi_{\Lambda,\xi}(y^+;y,z,x) \leq Q_{\alpha(f+h)}(y;x) + p(y) + g(y) + \left(\frac{1}{2\gamma} - \frac{1}{\gamma} + \frac{\rho_{\mathrm{g}}}{2}\right)\|y - y^+\|^2.$$

Simplifying the last expression and using (3.7), we obtain the claim of the lemma. □

We now compute a lower bound for $V_{\Lambda,\xi}$. We make use of the estimate provided in Lemma 2.1(b) in the following lemma, inspired by Themelis and Patrinos (2020) when they studied the Douglas-Rachford algorithm.

Lemma 3.4. *Suppose that Assumption* 1.1 *holds. Let $\rho_{\mathrm{f}} \geq 0$ be such that f is ρ-weakly convex and $L \geq L_{\mathrm{f}}$ be such that $L - \rho_{\mathrm{f}} > 0$. Then*

$$\begin{aligned} &V_{\Lambda,\xi}(y,z,x) \\ \geq &Q_{\alpha(f+h)}(y^+;\hat{x}) + p(y^+) + g(y^+) + \left\langle \nabla f(\hat{x}) - \nabla f(x) - \frac{1}{\alpha}(\hat{x} - x), x - y^+\right\rangle \\ &+ \langle \nabla h(x) - \nabla h(\hat{x}), y^+ - \hat{x}\rangle + \frac{1}{2(L-\rho_{\mathrm{f}})}\|\nabla f(x) - \nabla f(\hat{x})\|^2 \\ &- \left(\frac{L_{\mathrm{h}}}{2} + \frac{1}{2\alpha} + \frac{\rho_{\mathrm{f}}L}{2(L-\rho_{\mathrm{f}})}\right)\|\hat{x} - x\|^2 + \frac{1 - \beta L_{\mathrm{p}}}{2\beta}\|y^+ - y\|^2 \end{aligned} \tag{3.18}$$

for all $(y,z) \in \mathrm{dom}(g)\times\mathbb{R}^n$, $x \in \mathrm{prox}_{\alpha f}(z)$, $\hat{x} \in \mathbb{R}^n$, $\xi \in \partial p(y)$, and $y^+ \in \arg\min_{w\in\mathbb{R}^n} \Phi_{\Lambda,\xi}(w;y,z,x)$.

Proof. We have from Lemma 2.1(b) that for all $\hat{x} \in \mathbb{R}^n$,

$$\begin{aligned} Q_{\alpha f}(y^+;x) &\geq \left(f(\hat{x}) + \langle \nabla f(\hat{x}), x - \hat{x}\rangle + \frac{1}{2(L-\rho_{\mathrm{f}})}\|\nabla f(x) - \nabla f(\hat{x})\|^2 \right. \\ &\qquad \left. - \frac{\rho_{\mathrm{f}}L}{2(L-\rho_{\mathrm{f}})}\|x - \hat{x}\|^2\right) + \langle \nabla f(x), y^+ - x\rangle + \frac{1}{2\alpha}\|y^+ - x\|^2 \\ &= f(\hat{x}) + \langle \nabla f(\hat{x}) - \nabla f(x), x - y^+\rangle + \langle \nabla f(\hat{x}), y^+ - \hat{x}\rangle \\ &\quad + \frac{1}{2(L-\rho_{\mathrm{f}})}\|\nabla f(x) - \nabla f(\hat{x})\|^2 - \frac{\rho_{\mathrm{f}}L}{2(L-\rho_{\mathrm{f}})}\|x - \hat{x}\|^2 + \frac{1}{2\alpha}\|y^+ - x\|^2, \end{aligned}$$

for any $L \geq L_{\mathrm{f}}$ such that $L > \rho_{\mathrm{f}}$. Using the identity $\|a-b\|^2 - \|a-c\|^2 = -\|b-c\|^2 + 2\langle b-c, b-a\rangle$ and after some routine calculations, we further obtain

$$\begin{aligned} Q_{\alpha f}(y^+; x) \geq Q_{\alpha f}(y^+; \hat{x}) + \left\langle \nabla f(\hat{x}) - \nabla f(x) - \frac{1}{\alpha}(\hat{x} - x), x - y^+ \right\rangle \\ - \left(\frac{1}{2\alpha} + \frac{\rho_{\mathrm{f}} L}{2(L - \rho_{\mathrm{f}})} \right) \|x - \hat{x}\|^2 + \frac{1}{2(L - \rho_{\mathrm{f}})} \|\nabla f(x) - \nabla f(\hat{x})\|^2. \end{aligned} \tag{3.19}$$

On the other hand, we have from Lemma 2.1(a) that

$$\begin{aligned} & h(x) + \langle \nabla h(x), y^+ - x \rangle \\ \geq & \left(h(\hat{x}) - \langle \nabla h(x), \hat{x} - x \rangle - \frac{L_{\mathrm{h}}}{2} \|\hat{x} - x\|^2 \right) + \langle \nabla h(x), y^+ - x \rangle \\ = & h(\hat{x}) + \langle \nabla h(\hat{x}), y^+ - \hat{x} \rangle + \langle \nabla h(x) - \nabla h(\hat{x}), y^+ - \hat{x} \rangle - \frac{L_{\mathrm{h}}}{2} \|\hat{x} - x\|^2 \end{aligned} \tag{3.20}$$

for all $\hat{x} \in \mathbb{R}^n$. By Assumption 1.1(d) and the fact that

$$\partial \left(\frac{L_{\mathrm{p}}}{2} \|\cdot\|^2 - p \right)(y) = L_{\mathrm{p}} y - \partial p(y), \tag{3.21}$$

which holds by (Rockafellar and Wets, 1998, Exercise 8.8(c)), we have

$$\frac{L_{\mathrm{p}}}{2} \|y^+\|^2 - p(y^+) \geq \frac{L_{\mathrm{p}}}{2} \|y\|^2 - p(y) + \left\langle L_{\mathrm{p}} y - \xi, y^+ - y \right\rangle,$$

for any $\xi \in \partial p(y)$. This implies that

$$p(y) \geq p(y^+) - \langle \xi, y^+ - y \rangle - \frac{L_{\mathrm{p}}}{2} \|y^+ - y\|^2 \tag{3.22}$$

and hence

$$Q_{\beta p}(y^+; y, \xi) \geq \left(p(y^+) + \frac{1 - \beta L_{\mathrm{p}}}{2\beta} \|y^+ - y\|^2 \right). \tag{3.23}$$

Using the fact that $V_{\Lambda,\xi}(y, z, x) = \Phi_{\Lambda,\xi}(y^+; y, z, x)$ together with the bounds (3.19), (3.20) and (3.23), we obtain the desired conclusion. □

Regarding the requirement of ρ_{f} in Lemma 3.4, by Assumption 1.1(a) and Remark 2.2, we see that there indeed exists $\rho_{\mathrm{f}} \geq 0$ such that $f + \frac{\rho_{\mathrm{f}}}{2} \|\cdot\|^2$ is convex.

We now show that $\{V_{\Lambda,\xi^k}(y^k, z^k, x^k)\}$ is a nonincreasing sequence for appropriately chosen stepsizes. To simplify the notations, we denote

$$V_k \coloneqq V_{\Lambda,\xi^k}(y^k, z^k, x^k) = \Phi_{\Lambda,\xi^k}(y^{k+1}; y^k, z^k, x^k).$$

In what follows, we discuss the cases $\tau \in (0, 1]$, $\tau \in (1, 2)$ and $\tau \in [2, \infty)$ separately. For each case, we will show that when $L_{\mathrm{f}} + L_{\mathrm{h}} > 0$, there exist a function $c(\alpha)$ and a finite interval $I \subseteq (0, \infty)$ such that (i) $c(\alpha) \leq 0$ on I, (ii) $c(\alpha) < 0$ on the interior of I, and (iii) the inequality

$$V_{k-1} - V_k \geq -\frac{c(\alpha)}{2\tau\alpha} \left\| x^k - x^{k-1} \right\|^2 + \frac{1 - \beta L_{\mathrm{p}}}{2\beta} \left\| y^k - y^{k-1} \right\|^2 + \frac{1 - \gamma \rho_{\mathrm{g}}}{2\gamma} \left\| y^k - y^{k+1} \right\|^2 \tag{3.24}$$

holds for all k. For the case $L_{\mathrm{f}} + L_{\mathrm{h}} = 0$, the above inequality also holds but with $\alpha = \infty$ and $c(\alpha)$ a negative constant, so that the first term on the right-hand side vanishes; see Remark 3.8.

Theorem 3.5 (Stepsize for $\tau \in (0,1]$)**.** *Suppose Assumption* 1.1 *holds and* $L_{\rm f} + L_{\rm h} > 0$. *If* $\{(x^k, y^k, z^k)\}$ *is generated by Algorithm* 1 *with* $\tau \in (0,1]$ *and* $\alpha \in (0, \bar{\alpha}]$,

$$\bar{\alpha} := \begin{cases} \frac{1}{L_{\rm f}+L_{\rm h}} & \text{if } (2-\tau)L_{\rm f} - 2\rho_{\rm f} \geq \tau L_{\rm h}, \\ \frac{\tau}{2\eta^*} & \text{otherwise}, \end{cases} \tag{3.25}$$

and η^* *is the positive root of*

$$q(\eta) := 2(2-\tau)\eta^2 - \tau((2-\tau)L_{\rm h} + \rho_{\rm f}\tau)\eta - \tau(\rho_{\rm f}^2 + L_{\rm f}L_{\rm h}), \tag{3.26}$$

then (3.24) *holds with*

$$c(\alpha) := \begin{cases} 2L_{\rm f}(L_{\rm f}+L_{\rm h})\alpha^2 + ((2-\tau)L_{\rm h} - \tau L_{\rm f})\,\alpha - (2-\tau) & \text{if } (2-\tau)L_{\rm f} - 2\rho_{\rm f} \geq \tau L_{\rm h}, \\ 2L_{\rm f}L_{\rm h}\alpha^2 + \left((2-\tau)L_{\rm h} + \frac{\rho_{\rm f}\tau(\eta^*+\rho_{\rm f})}{\eta^*}\right)\alpha - (2-\tau) & \text{otherwise.} \end{cases} \tag{3.27}$$

In particular, $\{V_k\}$ *is nonincreasing if* $\alpha \leq \bar{\alpha}$, $\beta \leq L_{\rm p}^{-1}$ *and* $\gamma \leq \rho_{\rm g}^{-1}$, *and strictly decreasing if at least one holds with strict inequality.*

Proof. From Lemma 3.2, we know that for each k, there exists $\xi^k \in \partial p(y^k)$ such that $y^{k+1} \in \arg\min_{w\in\mathbb{R}^n} \Phi_{\Lambda,\xi^k}(w; y^k, z^k, x^k)$. By Lemma 3.3,

$$V_k \leq Q_{\alpha(f+h)}(y^k; x^k) + p(y^k) - \left(\frac{1-\gamma\rho_{\rm g}}{2\gamma}\right)\left\|y^k - y^{k+1}\right\|^2 + g(y^k). \tag{3.28}$$

On the other hand, setting $(y,z,x) = (y^{k-1}, z^{k-1}, x^{k-1})$ and $\hat{x} = x^k$ in Lemma 3.4, we have

$$\begin{aligned} V_{k-1} \geq\ & Q_{\alpha(f+h)}(y^k; x^k) + p(y^k) + g(y^k) \\ & + \left\langle \nabla f(x^k) - \nabla f(x^{k-1}) - \frac{1}{\alpha}(x^k - x^{k-1}), x^{k-1} - y^k \right\rangle \\ & + \left\langle \nabla h(x^{k-1}) - \nabla h(x^k), y^k - x^k \right\rangle + \frac{1}{2(L-\rho_{\rm f})}\left\|\nabla f(x^{k-1}) - \nabla f(x^k)\right\|^2 \\ & - \left(\frac{L_{\rm h}}{2} + \frac{1}{2\alpha} + \frac{\rho_{\rm f}L}{2(L-\rho_{\rm f})}\right)\left\|x^k - x^{k-1}\right\|^2 + \frac{1-\beta L_{\rm p}}{2\beta}\left\|y^k - y^{k-1}\right\|^2. \end{aligned} \tag{3.29}$$

Subtracting (3.28) from (3.29), we get

$$\begin{aligned} V_{k-1} - V_k \geq\ & \left\langle \nabla f(x^k) - \nabla f(x^{k-1}) - \frac{1}{\alpha}(x^k - x^{k-1}), x^{k-1} - y^k \right\rangle \\ & + \left\langle \nabla h(x^{k-1}) - \nabla h(x^k), y^k - x^k \right\rangle + \frac{1}{2(L-\rho_{\rm f})}\left\|\nabla f(x^{k-1}) - \nabla f(x^k)\right\|^2 \\ & - \left(\frac{L_{\rm h}}{2} + \frac{1}{2\alpha} + \frac{\rho_{\rm f}L}{2(L-\rho_{\rm f})}\right)\left\|x^k - x^{k-1}\right\|^2 + \frac{1-\beta L_{\rm p}}{2\beta}\left\|y^k - y^{k-1}\right\|^2 \\ & + \left(\frac{1-\gamma\rho_{\rm g}}{2\gamma}\right)\left\|y^k - y^{k+1}\right\|^2. \end{aligned} \tag{3.30}$$

Meanwhile, we have from (3.3) that $y^k = \frac{1}{\tau}(z^k - z^{k-1}) + x^{k-1}$ and from (3.1) and (2.4) that

$$z^k - z^{k-1} = (x^k + \alpha\nabla f(x^k)) - (x^{k-1} + \alpha\nabla f(x^{k-1})). \tag{3.31}$$

Thus,

$$y^k - x^{k-1} = \frac{1}{\tau}(x^k - x^{k-1}) + \frac{\alpha}{\tau}(\nabla f(x^k) - \nabla f(x^{k-1})), \tag{3.32}$$

which can also be rewritten as

$$y^k - x^k = \left(\frac{1}{\tau} - 1\right)(x^k - x^{k-1}) + \frac{\alpha}{\tau}(\nabla f(x^k) - \nabla f(x^{k-1})). \tag{3.33}$$

With these, algebraic calculations lead to

$$\begin{aligned} &\left\langle \nabla f(x^k) - \nabla f(x^{k-1}) - \frac{1}{\alpha}(x^k - x^{k-1}), x^{k-1} - y^k \right\rangle \\ \overset{(3.32)}{=}\ & -\frac{\alpha}{\tau}\left\|\nabla f(x^k) - \nabla f(x^{k-1})\right\|^2 + \frac{1}{\tau\alpha}\left\|x^k - x^{k-1}\right\|^2 \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} \left\langle \nabla h(x^{k-1}) - \nabla h(x^k), y^k - x^k \right\rangle &\overset{(3.33)}{=} \left(1 - \frac{1}{\tau}\right)\left\langle \nabla h(x^k) - \nabla h(x^{k-1}), x^k - x^{k-1} \right\rangle \\ &\quad - \frac{\alpha}{\tau}\left\langle \nabla h(x^k) - \nabla h(x^{k-1}), \nabla f(x^k) - \nabla f(x^{k-1}) \right\rangle. \qquad (3.35) \\ &\geq \frac{\tau - 1}{\tau}\left\|\nabla h(x^k) - \nabla h(x^{k-1})\right\| \cdot \left\|x^k - x^{k-1}\right\| \\ &\quad - \frac{\alpha}{\tau}\left\|\nabla h(x^k) - \nabla h(x^{k-1})\right\| \cdot \left\|\nabla f(x^k) - \nabla f(x^{k-1})\right\| \\ &\geq \frac{\tau - 1}{\tau}L_{\mathrm{h}}\left\|x^k - x^{k-1}\right\|^2 - \frac{\alpha}{\tau}L_{\mathrm{f}}L_{\mathrm{h}}\left\|x^k - x^{k-1}\right\|^2, \qquad (3.36) \end{aligned}$$

where the first inequality is by the Cauchy-Schwarz inequality and noting that $\tau \in (0, 1]$, while the last inequality holds by the Lipschitz continuity of the gradients of f and h. Combining (3.30), (3.34) and (3.36), we obtain

$$\begin{aligned} V_{k-1} - V_k \geq\ & \left(-\frac{\alpha}{\tau} + \frac{1}{2(L - \rho_{\mathrm{f}})}\right)\left\|\nabla f(x^{k-1}) - \nabla f(x^k)\right\|^2 \\ & + \left[\left(\frac{1}{\tau} - \frac{1}{2}\right)\frac{1}{\alpha} - \frac{L_{\mathrm{h}}}{2} - \frac{\rho_{\mathrm{f}}L}{2(L - \rho_{\mathrm{f}})} + \frac{\tau - 1}{\tau}L_{\mathrm{h}} - \frac{\alpha}{\tau}L_{\mathrm{h}}L_{\mathrm{f}}\right]\left\|x^k - x^{k-1}\right\|^2 \\ & + \frac{1 - \beta L_{\mathrm{p}}}{2\beta}\left\|y^k - y^{k-1}\right\|^2 + \left(\frac{1 - \gamma\rho_{\mathrm{g}}}{2\gamma}\right)\left\|y^k - y^{k+1}\right\|^2. \end{aligned} \tag{3.37}$$

Now, we discuss two disjoint cases.

Case 1. Suppose that $(2 - \tau)L_{\mathrm{f}} - 2\rho_{\mathrm{f}} \geq \tau L_{\mathrm{h}}$. Then $L_{\mathrm{f}} - \rho_{\mathrm{f}} \geq \frac{\tau}{2}(L_{\mathrm{f}} + L_{\mathrm{h}}) > 0$ and we may take $L = L_{\mathrm{f}}$ in (3.37). Let $\hat{\alpha} > 0$ be such that $\hat{\alpha} \geq \frac{\tau}{2(L_{\mathrm{f}} - \rho_{\mathrm{f}})}$, and suppose that $0 < \alpha \leq \hat{\alpha}$. Then

$$-\frac{\alpha}{\tau} + \frac{1}{2(L_{\mathrm{f}} - \rho_{\mathrm{f}})} \geq -\frac{\hat{\alpha}}{\tau} + \frac{1}{2(L_{\mathrm{f}} - \rho_{\mathrm{f}})},$$

where the quantity on the right-hand side is at most zero. Together with the Lipschitz conti-

nuity of ∇f and (3.37) with $L = L_{\mathrm{f}}$, for any $\alpha \in (0, \hat{\alpha}]$ we have

$$\begin{aligned} V_{k-1} - V_k &\geq \left(\frac{2-\tau}{2\tau\hat{\alpha}} - \hat{\alpha}\frac{L_{\mathrm{f}}(L_{\mathrm{f}}+L_{\mathrm{h}})}{\tau} + \frac{L_{\mathrm{f}}}{2} - \frac{L_{\mathrm{h}}}{2} + \frac{\tau-1}{\tau}L_{\mathrm{h}} \right) \left\| x^k - x^{k-1} \right\|^2 \\ &\quad + \frac{1-\beta L_{\mathrm{p}}}{2\beta} \left\| y^k - y^{k-1} \right\|^2 + \left(\frac{1-\gamma\rho_{\mathrm{g}}}{2\gamma} \right) \left\| y^k - y^{k+1} \right\|^2 \\ &= -\frac{c(\hat{\alpha})}{2\tau\hat{\alpha}} \left\| x^k - x^{k-1} \right\|^2 + \frac{1-\beta L_{\mathrm{p}}}{2\beta} \left\| y^k - y^{k-1} \right\|^2 \\ &\quad + \left(\frac{1-\gamma\rho_{\mathrm{g}}}{2\gamma} \right) \left\| y^k - y^{k+1} \right\|^2, \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} c(\hat{\alpha}) &\coloneqq 2L_{\mathrm{f}}(L_{\mathrm{f}}+L_{\mathrm{h}})\hat{\alpha}^2 + ((2-\tau)L_{\mathrm{h}} - \tau L_{\mathrm{f}})\,\hat{\alpha} - (2-\tau) \\ &= \left(\hat{\alpha} - \frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}} \right) (2L_{\mathrm{f}}(L_{\mathrm{f}}+L_{\mathrm{h}})\hat{\alpha} + (2-\tau)(L_{\mathrm{f}}+L_{\mathrm{h}}))\,, \end{aligned} \tag{3.39}$$

which is nonpositive when $\hat{\alpha} \leq \frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}}$. Hence, (3.38) holds with nonpositive $c(\hat{\alpha})$ when $\hat{\alpha} \in \left[\frac{\tau}{2(L_{\mathrm{f}}-\rho_{\mathrm{f}})}, \frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}} \right]$, which is a nonempty interval due to our hypothesis that $(2-\tau)L_{\mathrm{f}} - 2\rho_{\mathrm{f}} \geq \tau L_{\mathrm{h}}$. It is clear that $c(\alpha) < 0$ when $\alpha < \bar{\alpha}$.

Case 2. Suppose now that

$$(2-\tau)L_{\mathrm{f}} - 2\rho_{\mathrm{f}} < \tau L_{\mathrm{h}}. \tag{3.40}$$

Given $L \geq L_{\mathrm{f}}$ with $L - \rho_{\mathrm{f}} > 0$, we define $\hat{\alpha}(L) \coloneqq \frac{\tau}{2(L-\rho_{\mathrm{f}})}$ and select $\alpha \in (0, \hat{\alpha}(L)]$. Then

$$-\frac{\alpha}{\tau} + \frac{1}{2(L-\rho_{\mathrm{f}})} \geq -\frac{\hat{\alpha}(L)}{\tau} + \frac{1}{2(L-\rho_{\mathrm{f}})} = 0.$$

Hence, we have from (3.37) that

$$\begin{aligned} V_{k-1} - V_k &\geq -\frac{c(\hat{\alpha}(L))}{2\tau\hat{\alpha}(L)} \left\| x^k - x^{k-1} \right\|^2 + \frac{1-\beta L_{\mathrm{p}}}{2\beta} \left\| y^k - y^{k-1} \right\|^2 \\ &\quad + \left(\frac{1-\gamma\rho_{\mathrm{g}}}{2\gamma} \right) \left\| y^k - y^{k+1} \right\|^2, \end{aligned} \tag{3.41}$$

where

$$c(\hat{\alpha}(L)) \coloneqq 2L_{\mathrm{f}}L_{\mathrm{h}}\hat{\alpha}(L)^2 + \left((2-\tau)L_{\mathrm{h}} + \frac{\rho_{\mathrm{f}}\tau L}{L-\rho_{\mathrm{f}}} \right) \hat{\alpha}(L) - (2-\tau). \tag{3.42}$$

To determine the largest allowable stepsize $\hat{\alpha}(L)$ so that $c(\hat{\alpha}) \leq 0$, we calculate

$$L^* \coloneqq \min\{L : c(\hat{\alpha}(L)) \leq 0\ , L \geq L_{\mathrm{f}}, L > \rho_{\mathrm{f}}\}, \tag{3.43}$$

so that $\hat{\alpha}(L^*)$ is the desired stepsize. By some routine calculations, it can be shown that $c(\hat{\alpha}(L)) = -\frac{1}{2(L-\rho_{\mathrm{f}})^2} q(L-\rho_{\mathrm{f}}) = -\frac{1}{2\eta^2} q(\eta)$, where q is given by the polynomial (3.26) and $\eta \coloneqq L - \rho_{\mathrm{f}}$. Hence, if η^* is the (strictly) positive root of q, then $L^* \coloneqq \max\{\eta^* + \rho_{\mathrm{f}}, L_{\mathrm{f}}\}$. We now claim that $L^* = \eta^* + \rho_{\mathrm{f}}$. If $L_{\mathrm{f}} = \rho_{\mathrm{f}}$, this immediately holds since $\eta^* > 0$. Suppose now that $L_{\mathrm{f}} > \rho_{\mathrm{f}}$. By the definition of $\hat{\alpha}(L)$, note that we may write c as

$$c(\hat{\alpha}(L)) = c(\hat{\alpha}(L)) + 2LL_{\mathrm{f}}\hat{\alpha}(L)^2 - \frac{\tau L L_{\mathrm{f}}}{L-\rho_{\mathrm{f}}}\hat{\alpha}(L)$$

for any $L > \rho_{\mathrm{f}}$. Simplifying this expression, we obtain

$$c(\hat{\alpha}(L)) = 2L_{\mathrm{f}}(L_{\mathrm{h}} + L)\hat{\alpha}(L)^2 + \left((2-\tau)L_{\mathrm{h}} - \frac{\tau L(L_{\mathrm{f}} - \rho_{\mathrm{f}})}{L - \rho_{\mathrm{f}}}\right)\hat{\alpha}(L) - (2-\tau). \tag{3.44}$$

Since $L_{\mathrm{f}} > \rho_{\mathrm{f}}$, $c(\hat{\alpha}(L_{\mathrm{f}}))$ is well-defined and can be calculated as

$$\begin{aligned} c(\hat{\alpha}(L_{\mathrm{f}})) &= 2L_{\mathrm{f}}(L_{\mathrm{f}} + L_{\mathrm{h}})\hat{\alpha}(L_{\mathrm{f}})^2 + ((2-\tau)L_{\mathrm{h}} - \tau L_{\mathrm{f}})\,\hat{\alpha}(L_{\mathrm{f}}) - (2-\tau) \\ &= \left(\hat{\alpha}(L_{\mathrm{f}}) - \frac{1}{L_{\mathrm{f}} + L_{\mathrm{h}}}\right)\left(2L_{\mathrm{f}}(L_{\mathrm{f}} + L_{\mathrm{h}})\hat{\alpha}(L_{\mathrm{f}}) + (2-\tau)(L_{\mathrm{f}} + L_{\mathrm{h}})\right). \end{aligned}$$

Since (3.40) implies that $\hat{\alpha}(L_{\mathrm{f}}) > \frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}}$, it follows that $c(\hat{\alpha}(L_{\mathrm{f}})) > 0$. Hence, $L^* \neq L_{\mathrm{f}}$ by the definition of L^* in (3.43). The claim that $L^* = \eta^* + \rho_{\mathrm{f}}$ now follows. We also note that since $c(\hat{\alpha}(L^*)) \leq 0$, it follows from (3.42) that $c(\alpha) < 0$ for any $\alpha < \hat{\alpha}(L^*)$, where $c(\alpha)$ is as defined in (3.27). □

Remark 3.6. To gain insight on the magnitude of the stepsize upper bound $\bar{\alpha} = \frac{\tau}{2\eta^*}$ in the second case of the above proof, consider $L := \frac{\tau L_{\mathrm{h}} + 2\rho_{\mathrm{f}}}{2-\tau}$. By (3.40), $L > L_{\mathrm{f}}$ and $L > \rho_{\mathrm{f}}$. In addition, for this choice of L, $c(\hat{\alpha}(L)) \leq 0$. By (3.43), it holds that $L \geq L^*$, ensuring $\frac{\tau}{2\eta^*} \geq \frac{\tau}{2(L-\rho_{\mathrm{f}})} = \frac{2-\tau}{2(L_{\mathrm{h}}+\rho_{\mathrm{f}})}$. To obtain an upper bound, we note that from (3.44), it can be verified that an alternative way to express $c(\hat{\alpha}(L))$ is

$$\begin{aligned} c(\hat{\alpha}(L)) &= \left[2L_{\mathrm{f}}(L_{\mathrm{f}} + L_{\mathrm{h}})\hat{\alpha}(L)^2 + ((2-\tau)L_{\mathrm{h}} - \tau L_{\mathrm{f}})\hat{\alpha}(L) - (2-\tau)\right] \\ &\quad + 2L_{\mathrm{f}}(L - L_{\mathrm{f}})\hat{\alpha}(L)^2 + \left(\tau L_{\mathrm{f}} - \frac{\tau L(L_{\mathrm{f}} - \rho_{\mathrm{f}})}{L_{\mathrm{f}} - \rho_{\mathrm{f}}}\right)\hat{\alpha}(L) \\ &= \left(\hat{\alpha}(L) - \frac{1}{L_{\mathrm{f}} + L_{\mathrm{h}}}\right)\left(2L_{\mathrm{f}}(L_{\mathrm{f}} + L_{\mathrm{h}})\hat{\alpha}(L) + (2-\tau)(L_{\mathrm{f}} + L_{\mathrm{h}})\right) \\ &\quad + 2L_{\mathrm{f}}(L - L_{\mathrm{f}})\hat{\alpha}(L)^2 + \frac{\tau\rho_{\mathrm{f}}(L - L_{\mathrm{f}})}{L - \rho_{\mathrm{f}}}\hat{\alpha}(L). \end{aligned}$$

Note that the last two terms are nonnegative when $L = L^*$ since $L^* \geq L_{\mathrm{f}}$ and $L^* > \rho_{\mathrm{f}}$. Since $c(\hat{\alpha}(L^*)) = 0$, it follows that $\hat{\alpha}(L^*) - \frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}} \leq 0$. That is, $\frac{\tau}{2\eta^*} \leq \frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}}$. In summary, when $\tau \in (0, 1]$, we have $\frac{2-\tau}{2(L_{\mathrm{h}}+\rho_{\mathrm{f}})} \leq \frac{\tau}{2\eta^*} \leq \frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}}$ and therefore

$$\bar{\alpha} \leq \frac{1}{L_f + L_h}. \tag{3.45}$$

Remark 3.7 (Stepsize comparison with Bian and Zhang (2021))**.** For the case $\tau = 1$, the above theorem implies that strict monotonicity of $\{V_k\}$ holds when $\alpha < \bar{\alpha}$, where

$$\bar{\alpha} = \begin{cases} \frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}} & \text{if } L_{\mathrm{f}} - 2\rho_{\mathrm{f}} \geq L_{\mathrm{h}}, \\ \frac{2}{L_{\mathrm{h}}+\rho_{\mathrm{f}}+\sqrt{(L_{\mathrm{h}}+\rho_{\mathrm{f}})^2+8(\rho_{\mathrm{f}}^2+L_{\mathrm{f}}L_{\mathrm{h}})}} & \text{otherwise.} \end{cases}$$

On the other hand, the bound derived in (Bian and Zhang, 2021, Lemma 3.3) for the DYS algorithm (*i.e.*, $p \equiv 0$) indicates that the stepsize $\alpha > 0$ should satisfy

$$\frac{1}{2}\left(\frac{1}{\alpha} - \rho_{\mathrm{f}}\right) - L_{\mathrm{h}} - \left(\frac{1}{\alpha} + \frac{L_{\mathrm{h}}}{2}\right)(2\alpha\rho_{\mathrm{f}} + 2\alpha L_{\mathrm{f}} + \alpha^2 L_{\mathrm{f}}^2) > 0,$$

or equivalently, under the constraint $\alpha > 0$,

$$d(\alpha) \coloneqq L_{\mathrm{f}}^2 L_{\mathrm{h}} \alpha^3 + 2(L_{\mathrm{f}}^2 + L_{\mathrm{h}} L_{\mathrm{f}} + \rho_{\mathrm{f}} L_{\mathrm{h}})\alpha^2 + (5\rho_{\mathrm{f}} + 2L_{\mathrm{h}} + 4L_{\mathrm{f}})\alpha - 1 < 0.$$

Hence, the upper bound for α is the unique positive root $\hat{\alpha}$ of the polynomial d given above. Consider $c(\alpha)$ given in (3.27). In the first case, that is, when $L_{\mathrm{f}} - 2\rho_{\mathrm{f}} \geq L_{\mathrm{h}}$, as long as $L_{\mathrm{f}} + L_{\mathrm{h}} > 0$ and $\alpha > 0$, we have

$$c(\alpha) - d(\alpha) = -L_{\mathrm{f}}^2 L_{\mathrm{h}} \alpha^3 - 2\rho_{\mathrm{f}} L_{\mathrm{h}} \alpha^2 - (5\rho_{\mathrm{f}} + L_{\mathrm{h}} + 5L_{\mathrm{f}}) < 0.$$

Hence, $-d(\bar{\alpha}) = c(\bar{\alpha}) - d(\bar{\alpha}) < 0$, and since $\hat{\alpha}$ is the unique positive root of d, it follows that $\hat{\alpha} < \bar{\alpha}$. In the second case, recall that $\bar{\alpha} = \hat{\alpha}(L^*) = \frac{1}{2\eta^*} = \frac{1}{2(L^* - \rho_{\mathrm{f}})}$, so that

$$\begin{aligned} c(\bar{\alpha}) &= c(\bar{\alpha}) + 2\rho_{\mathrm{f}} L_{\mathrm{f}} \bar{\alpha}^2 - \frac{\rho_{\mathrm{f}} L_{\mathrm{f}}}{L^* - \rho_{\mathrm{f}}} \bar{\alpha} \\ &= 2(L_{\mathrm{f}} L_{\mathrm{h}} + \rho_{\mathrm{f}} L_{\mathrm{f}})\bar{\alpha}^2 + \left(L_{\mathrm{h}} + \frac{\rho_{\mathrm{f}}(L^* - L_{\mathrm{f}})}{L^* - \rho_{\mathrm{f}}} \right) \bar{\alpha} - 1 \\ &\leq 2(L_{\mathrm{f}} L_{\mathrm{h}} + L_{\mathrm{f}}^2)\bar{\alpha}^2 + (L_{\mathrm{h}} + \rho_{\mathrm{f}})\, \bar{\alpha} - 1, \end{aligned}$$

where the last inequality holds since $\rho_{\mathrm{f}} \leq L_{\mathrm{f}}$ and $L^* > \rho_{\mathrm{f}}$. Then, provided that $L_{\mathrm{f}} + L_{\mathrm{h}} > 0$, we have

$$c(\bar{\alpha}) - d(\bar{\alpha}) \leq -L_{\mathrm{f}}^2 L_{\mathrm{h}} \bar{\alpha}^3 - 2\rho_{\mathrm{f}} L_{\mathrm{h}} \bar{\alpha}^2 - (4\rho_{\mathrm{f}} + L_{\mathrm{h}} + 4L_{\mathrm{f}}) < 0,$$

and similar to the previous case, we get that $d(\bar{\alpha}) > 0$ and therefore $\hat{\alpha} < \bar{\alpha}$. This shows that our stepsize upper bound is always larger than that in Bian and Zhang (2021). The significant gap between the computed stepsizes is evident in Fig. 1.

Remark 3.8 (The case $L_{\mathrm{f}} + L_{\mathrm{h}} = 0$)**.** Suppose that $L_{\mathrm{f}} + L_{\mathrm{h}} = 0$, in which case $\alpha = \infty$ and $\gamma = \beta$. From (3.30), we immediately obtain

$$V_{k-1} - V_k \geq \frac{1 - \beta L_{\mathrm{p}}}{2\beta} \left\| y^k - y^{k-1} \right\|^2 + \frac{1 - \gamma \rho_{\mathrm{g}}}{2\gamma} \left\| y^k - y^{k+1} \right\|^2,$$

for any $\tau > 0$ (Note that in this case, the x and z sequences generated by Algorithm 1 are irrelevant). Thus, we still obtain the desired inequality (3.24) by setting $c(\alpha)$ to be any negative number.

The following provides a result similar to the previous theorem but for parameters τ in $(1, 2)$.

Theorem 3.9 (Stepsize for $\tau \in (1, 2)$)**.** *Suppose that Assumption* 1.1 *holds and* $L_{\mathrm{f}} + L_{\mathrm{h}} > 0$. *Let* $\sigma_{\mathrm{h}} \in \mathbb{R}$ *be such that* $h - \frac{\sigma_{\mathrm{h}}}{2}\|\cdot\|^2$ *is convex. If* $\{(x^k, y^k, z^k)\}$ *is generated by Algorithm* 1 *with* $\tau \in (1, 2)$ *and* $\alpha \in (0, \bar{\alpha}]$*, where*

$$\bar{\alpha} \coloneqq \begin{cases} \bar{\alpha}_1 & \text{if } \tau \leq 2\bar{\alpha}_1(L_{\mathrm{f}} - \rho_{\mathrm{f}}) \\ \frac{\tau}{2\eta^*} & \text{otherwise} \end{cases},$$

$\bar{\alpha}_1$ *is the positive root of*

$$c(\alpha) \coloneqq 2L_{\mathrm{f}}(L_{\mathrm{f}} + L_{\mathrm{h}})\alpha^2 + (\tau L_{\mathrm{h}} - 2(\tau - 1)\sigma_{\mathrm{h}} - \tau L_{\mathrm{f}})\, \alpha - (2 - \tau), \tag{3.46}$$

and η^* *is the positive root of*

$$q(\eta) \coloneqq 2(2 - \tau)\eta^2 - \tau(\tau L_{\mathrm{h}} - 2(\tau - 1)\sigma_{\mathrm{h}} + \rho_{\mathrm{f}} \tau)\eta - \tau^2(\rho_{\mathrm{f}}^2 + L_{\mathrm{f}} L_{\mathrm{h}}),$$

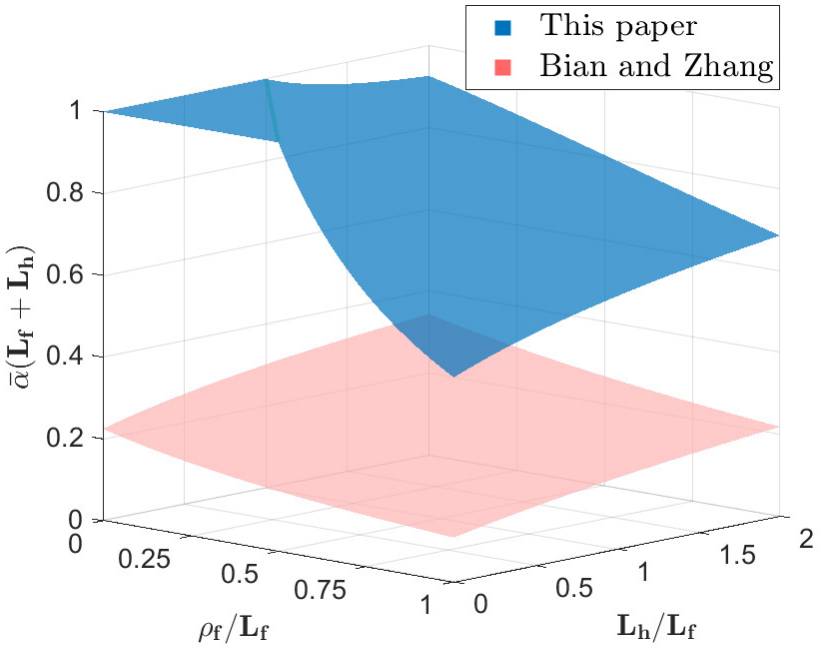


Figure 1: Comparison of stepsize upper bounds (denoted by $\bar{\alpha}$) for the DYS algorithm in this paper (with $\tau = 1$) and in Bian and Zhang (2021).

then (3.24) *holds with* $c(\alpha)$ *given by* (3.46) *if* $\tau \leq 2\bar{\alpha}_1(L_{\rm f} - \rho_{\rm f})$, *and*

$$c(\alpha) = 2L_{\rm f}L_{\rm h}\alpha^2 + \left(\tau L_{\rm h} - 2(\tau-1)\sigma_{\rm h} + \frac{\rho_{\rm f}\tau(\eta^* + \rho_{\rm f})}{\eta^*}\right)\alpha - (2-\tau))$$

otherwise. In particular, $\{V_k\}$ *is nonincreasing if* $\alpha \leq \bar{\alpha}$, $\beta \leq L_{\rm p}^{-1}$ *and* $\gamma \leq \rho_{\rm g}^{-1}$, *and strictly decreasing if at least one holds with strict inequality.*

Proof. Given $\tau \geq 1$, we have from (3.35) and the monotonicity of $\nabla h - \sigma_{\rm h} Id$ that

$$\left\langle \nabla h(x^{k-1}) - \nabla h(x^k), y^k - x^k \right\rangle \geq \left(\frac{\tau-1}{\tau}\right)\sigma_{\rm h}\left\|x^k - x^{k-1}\right\|^2 - \frac{\alpha}{\tau}L_{\rm f}L_{\rm h}\left\|x^k - x^{k-1}\right\|^2. \tag{3.47}$$

The rest of the proof follows arguments similar to those in the proof of Theorem 3.5. □

We note that there indeed exists $\sigma_{\rm h} \in \mathbb{R}$ such that $h - \frac{\sigma_{\rm h}}{2}\|\cdot\|^2$ is convex; for instance, we can take $\sigma_{\rm h} = -L_{\rm h}$ by Lemma 2.1(a).

Remark 3.10. From Theorem 3.9, it can be shown that the stepsize bounds for $\tau \in (1,2)$ are also given by (3.25) if $\sigma_{\rm h} = L_{\rm h}$. In particular, when $h = 0$, we obtain that $\bar{\alpha} = \min\left\{\frac{1}{L_{\rm f}}, \frac{2-\tau}{2\rho_{\rm f}}\right\}$ for any $\tau \in (0,2)$, which is the same as the one obtained in (Themelis and Patrinos, 2020, Theorem 4.1), where the case $h = p = 0$ is considered. Another scenario for $\sigma_{\rm h} = L_{\rm h}$ is when h is a quadratic function with its Hessian being a positive multiple of the identity matrix.

Remark 3.11. Similar to Remark 3.6, the bound (3.45) also holds for $\tau \in (1,2)$. To see this, note that $c(\alpha)$ in (3.46) can also be written as

$$c(\alpha) = \hat{c}(\alpha) + 2(\tau-1)(L_{\rm h} - \sigma_{\rm h})\alpha,$$

where $\hat{c}(\alpha) \coloneqq 2L_{\mathrm{f}}(L_{\mathrm{f}}+L_{\mathrm{h}})\alpha^2 + ((2-\tau)L_{\mathrm{h}} - \tau L_{\mathrm{f}})\,\alpha - (2-\tau)$. Since $\frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}}$ is the positive root of $\hat{c}(\alpha)$ (see (3.39)) and $L_{\mathrm{h}} - \sigma_{\mathrm{h}} \geq 0$ by Remark 2.2, it holds that $c\left(\frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}}\right) = \frac{2(\tau-1)(L_{\mathrm{h}}-\sigma_{\mathrm{h}})}{L_{\mathrm{f}}+L_{\mathrm{h}}} \geq 0$. Since $c(0) = \tau - 2 < 0$ and $\bar{\alpha}_1$ is the positive root of $c(\alpha)$, it follows that $\bar{\alpha}_1 \leq \frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}}$. On the other hand, following the same arguments in Remark 3.6, it can be shown that $\tau/(2\eta^*) \leq \bar{\alpha}_1$, so (3.45) holds as claimed.

Lastly, we derive stepsize upper bounds for $\tau \geq 2$.

Theorem 3.12 (Stepsize for $\tau \in [2,\infty)$)**.** *Suppose that Assumption* 1.1 *holds, f is σ_{f}-strongly convex for some $\sigma_f > 0$, and $\rho_{\mathrm{h}} \in [0, L_{\mathrm{h}}]$ such that $h + \frac{\rho_{\mathrm{h}}}{2}\|\cdot\|^2$ is convex. Let $\tau \geq 2$ be such that*

$$\delta(\tau) \coloneqq (\tau\nu - \tau\theta_1 - 2(\tau-1)\theta_2)^2 - 8(\theta_0+\nu)(\tau-2) > 0, \tag{3.48}$$

and

$$\tau\nu - \tau\theta_1 - 2(\tau-1)\theta_2 > 0, \tag{3.49}$$

where $\nu \coloneqq \frac{\sigma_{\mathrm{f}}}{L_{\mathrm{f}}+L_{\mathrm{h}}}$, $\theta_0 \coloneqq \frac{L_{\mathrm{h}}(L_{\mathrm{f}}^2-\sigma_{\mathrm{f}}^2)}{L_{\mathrm{f}}(L_{\mathrm{f}}+L_{\mathrm{h}})^2}$, $\theta_1 \coloneqq \frac{L_{\mathrm{h}}}{L_{\mathrm{f}}+L_{\mathrm{h}}}$ and $\theta_2 \coloneqq \frac{\rho_{\mathrm{h}}}{L_{\mathrm{f}}+L_{\mathrm{h}}}$, so that

$$r(\mu) \coloneqq \tau^2(\theta_0+\nu)\mu^2 - \tau^2\left(\nu - \theta_1 - \frac{2(\tau-1)}{\tau}\theta_2\right)\mu + 2(\tau-2) \tag{3.50}$$

has two distinct roots μ_* and μ^* with $0 \leq \mu_* < \mu^* \leq 1$. If $\{(x^k, y^k, z^k)\}$ is generated by Algorithm 1 *with stepsize $\alpha = \frac{\tau\mu}{2(L_{\mathrm{f}}+L_{\mathrm{h}})}$, where $\mu > 0$ satisfies $\mu \in [\mu_*, \mu^*]$, then* (3.24) *holds with $c(\alpha)$ given by*

$$c(\alpha) = 2L_{\mathrm{h}}L_{\mathrm{f}}\alpha^2 + \left(\tau\mu\sigma_{\mathrm{f}} + \tau L_{\mathrm{h}} + 2(\tau-1)\rho_{\mathrm{h}} - \tau\sigma_{\mathrm{f}} - \frac{-\tau L_{\mathrm{h}}\sigma_{\mathrm{f}}^2\mu}{L_{\mathrm{f}}(L_{\mathrm{f}}+L_{\mathrm{h}})}\right)\alpha - (2-\tau). \tag{3.51}$$

In particular, $\{V_k\}$ is nonincreasing if $\frac{\tau\mu_*}{2(L_{\mathrm{f}}+L_{\mathrm{h}})} \leq \alpha \leq \frac{\tau\mu^*}{2(L_{\mathrm{f}}+L_{\mathrm{h}})}$, $\beta \leq L_{\mathrm{p}}^{-1}$ and $\gamma \leq \rho_{\mathrm{g}}^{-1}$, and strictly decreasing if at least one holds with strict inequality.

Proof. σ_{f}-strong convexity implies that

$$f(x) \geq f(\hat{x}) + \langle \nabla f(\hat{x}), x - \hat{x}\rangle + \frac{\sigma_{\mathrm{f}}}{2}\|x-\hat{x}\|^2, \quad \forall x, \hat{x} \in \mathbb{R}^n.$$

By the above inequality together with the L_{f}-smoothness of f and (2.5), we have

$$f(x) \geq f(\hat{x}) + \langle \nabla f(\hat{x}), x - \hat{x}\rangle + \frac{\mu}{2L_{\mathrm{f}}}\|\nabla f(x) - \nabla f(\hat{x})\|^2 + \frac{(1-\mu)\sigma_{\mathrm{f}}}{2}\|x-\hat{x}\|^2,$$

for any $x, \hat{x} \in \mathbb{R}^n$ and $\mu \in [0,1]$. Following the arguments in Lemma 3.4, we obtain

$$\begin{aligned} &V_{\Lambda,\xi}(y,z,x) \geq \\ &\quad Q_{\alpha(f+h)}(y^+;\hat{x}) + p(y^+) + g(y^+) + \left\langle \nabla f(\hat{x}) - \nabla f(x) - \frac{1}{\alpha}(\hat{x}-x), x - y^+ \right\rangle \\ &\quad + \langle \nabla h(x) - \nabla h(\hat{x}), y^+ - \hat{x}\rangle + \frac{\mu}{2L_{\mathrm{f}}}\|\nabla f(x) - \nabla f(\hat{x})\|^2 \\ &\quad - \left(\frac{L_{\mathrm{h}}}{2} + \frac{1}{2\alpha} - \frac{(1-\mu)\sigma_{\mathrm{f}}}{2}\right)\|\hat{x}-x\|^2 + \frac{1-\beta L_{\mathrm{p}}}{2\beta}\|y^+ - y\|^2. \end{aligned}$$

By using this bound and inequality (3.47) with $\sigma_{\mathrm{h}} = -\rho_{\mathrm{h}}$, we obtain by following the same arguments and calculations in the proof of Theorem 3.5 that

$$\begin{aligned} V_{k-1} - V_k \geq & \left(-\frac{\alpha}{\tau} + \frac{\mu}{2L_{\mathrm{f}}}\right) \left\|\nabla f(x^{k-1}) - \nabla f(x^k)\right\|^2 \\ & + \left[\frac{2-\tau}{2\tau\alpha} - \frac{L_{\mathrm{h}}}{2} + \frac{(1-\mu)\sigma_{\mathrm{f}}}{2} - \frac{\tau-1}{\tau}\rho_{\mathrm{h}} - \frac{\alpha}{\tau}L_{\mathrm{h}}L_{\mathrm{f}}\right] \left\|x^k - x^{k-1}\right\|^2 \\ & + \frac{1-\beta L_{\mathrm{p}}}{2\beta}\left\|y^k - y^{k-1}\right\|^2 + \left(\frac{1-\gamma\rho_{\mathrm{g}}}{2\gamma}\right)\left\|y^k - y^{k+1}\right\|^2. \end{aligned}$$

By setting $\alpha = \frac{\tau\mu}{2(L_{\mathrm{f}}+L_{\mathrm{h}})}$ with $\mu \in (0,1]$, we have

$$\left(-\frac{\alpha}{\tau} + \frac{\mu}{2L_{\mathrm{f}}}\right) \left\|\nabla f(x^{k-1}) - \nabla f(x^k)\right\|^2 \geq \frac{\mu L_{\mathrm{h}}\sigma_{\mathrm{f}}^2}{2L_{\mathrm{f}}(L_{\mathrm{f}}+L_{\mathrm{h}})}\left\|x^{k-1} - x^k\right\|^2,$$

where the inequality follows from the strong convexity of f. Therefore, (3.24) holds with $c(\alpha)$ given by (3.51). To obtain a nonincreasing sequence $\{V_k\}$, we need to find the range of μ that makes $c(\alpha) \leq 0$. Plugging in $\alpha = \frac{\tau\mu}{2(L_{\mathrm{f}}+L_{\mathrm{h}})}$ in (3.51), some algebraic calculations lead to

$$c(\alpha) = c\left(\frac{\tau\mu}{2(L_{\mathrm{f}}+L_{\mathrm{h}})}\right) = \frac{1}{2}r(\mu),$$

with $r(\mu)$ given by (3.50). We note that $r(\mu)$ has two distinct roots μ_* and μ^* with $\mu_* < \mu^*$ if and only if its discriminant is positive. This condition is equivalent to (3.48). Meanwhile, since (3.49) holds and $\rho_{\mathrm{h}} \geq 0$, we see that

$$0 < \frac{\tau\nu - \tau\theta_1 - 2(\tau-1)\theta_2}{2\tau(\theta_0+\nu)} \leq \frac{\tau\nu}{2\tau\nu} = \frac{1}{2}.$$

That is to say, the first coordinate of the vertex of the parabola defined by $r(\mu)$ lies in $(0, \frac{1}{2}]$. Since $\tau \geq 2$, it follows that $0 \leq \mu_* < \mu^* \leq 1$. Thus, $c(\alpha) \leq 0$ for $\alpha = \frac{\tau\mu}{2(L_{\mathrm{f}}+L_{\mathrm{h}})}$ with $\mu \in [\mu_*, \mu^*]$, and $c(\alpha) < 0$ if $\mu \in (\mu_*, \mu^*)$. This completes the proof. □

Remark 3.13 (Stepsize for $\tau = 2$)**.** Suppose that $\tau = 2$. Then (3.48) and (3.49) are equivalent to having $\sigma_{\mathrm{f}} > L_{\mathrm{h}} + \rho_{\mathrm{h}}$, and the roots of (3.50) are $\mu_* = 0$ and $\mu^* = \frac{L_{\mathrm{f}}(L_{\mathrm{f}}+L_{\mathrm{h}})(\sigma_{\mathrm{f}}-L_{\mathrm{h}}-\rho_{\mathrm{h}})}{L_{\mathrm{h}}(L_{\mathrm{f}}^2-\sigma_{\mathrm{f}}^2)+\sigma_{\mathrm{f}}L_{\mathrm{f}}(L_{\mathrm{f}}+L_{\mathrm{h}})}$. Hence, $c(\alpha)$ is strictly negative for all α such that

$$0 < \alpha < \frac{L_{\mathrm{f}}(\sigma_{\mathrm{f}} - L_{\mathrm{h}} - \rho_{\mathrm{h}})}{L_{\mathrm{h}}(L_{\mathrm{f}}^2 - \sigma_{\mathrm{f}}^2) + \sigma_{\mathrm{f}}L_{\mathrm{f}}(L_{\mathrm{f}} + L_{\mathrm{h}})}.$$

Remark 3.14. If $h \equiv 0$, the condition (3.49) automatically holds since $\sigma_{\mathrm{f}} > 0$. Moreover, the condition (3.48) with the constraint $\tau \geq 2$ is equivalent to having

$$2 \leq \tau < \frac{4}{1+\sqrt{1-\nu}} \quad \text{or} \quad \tau > \frac{4}{1-\sqrt{1-\nu}}. \tag{3.52}$$

On the other hand, the roots of (3.50) are given by

$$\mu_* = \frac{1}{2} - \frac{\sqrt{\nu(\nu\tau^2 - 8\tau + 16)}}{2\tau\nu} \quad \text{and} \quad \mu^* = \frac{1}{2} + \frac{\sqrt{\nu(\nu\tau^2 - 8\tau + 16)}}{2\tau\nu}.$$

Theorem 3.12 asserts that if we choose τ that satisfies (3.52), then $\{V_k\}$ is strictly decreasing provided that the stepsize α satisfies

$$\frac{\tau\nu - \sqrt{\nu(\nu\tau^2 - 8\tau + 16)}}{4\sigma_{\mathrm{f}}} < \alpha < \frac{\tau\nu + \sqrt{\nu(\nu\tau^2 - 8\tau + 16)}}{4\sigma_{\mathrm{f}}}, \tag{3.53}$$

which are the bounds obtained in (Themelis and Patrinos, 2020, Theorem 4.1), where the case $h \equiv p \equiv 0$ is considered. However, in the said result, the analysis restricts τ to satisfy only the first condition in (3.52), due to their imposed constraint that αL_{f} must be at most 1. Meanwhile, the analysis we provide in the proof of Theorem 3.12 does not require this condition to establish the nonincreasing property, and therefore we have shown that the range of τ can be widened to include those that satisfy $\tau \geq \frac{4}{1-\sqrt{1-\nu}}$. For instance, if we are given $\nu = 3/4$, following (3.52) we can choose $\tau = 12$. By (3.53), for this value we may allow any stepsize α that satisfies $3 - \frac{\sqrt{21}}{3} < \alpha L_{\mathrm{f}} < 3 + \frac{\sqrt{21}}{3}$, and the lower bound is strictly greater than 1. Nevertheless, we point out that this wider range of τ provided above is only sufficient to guarantee monotonicity of $\{V_k\}$. If we want to establish boundedness of a sequence generated by Algorithm 1, the restriction that $\alpha(L_{\mathrm{f}} + L_{\mathrm{h}}) < 1$ will inevitably be required (see Proposition 3.15).

3.3 Subsequential convergence

After obtaining the stepsize upper bounds, our next goal is to show subsequential convergence and convergence rates of Algorithm 1. We first establish the boundedness of its iterate sequence.

Proposition 3.15. *Suppose that Assumption* 1.1 *holds,* Ψ *has bounded level sets,* $\beta \in (0, L_{\mathrm{p}}^{-1}]$*, and* $\gamma \in (0, \rho_{\mathrm{g}}^{-1}]$*. In addition, suppose that* $L_{\mathrm{f}} + L_{\mathrm{h}} > 0$*, and* $\alpha, \tau > 0$ *are chosen such that*

(a) $\tau \in (0, 1]$ *and* $\alpha \in (0, \bar{\alpha})$*, where* $\bar{\alpha}$ *is given in Theorem* 3.5*;*

(b) $\tau \in (1, 2)$ *and* $\alpha \in (0, \bar{\alpha})$*, where* $\bar{\alpha}$ *is given in Theorem* 3.9*; or*

(c) $\tau \geq 2$ *satisfies* (3.48) *and* (3.49)*, and* $\alpha \in \left(\frac{\tau\mu_*}{2(L_{\mathrm{f}}+L_{\mathrm{h}})}, \frac{\tau\mu^*}{2(L_{\mathrm{f}}+L_{\mathrm{h}})}\right) \cap \left(0, \frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}}\right)$.

If $\{(x^k, y^k, z^k)\}$ *is generated by Algorithm* 1*, then*

(i) $\{(x^k, y^k, z^k)\}$ *is bounded; and*

(ii) $\left\|(x^k, y^k, z^k) - (x^{k-1}, y^{k-1}, z^{k-1})\right\| \to 0$.

Proof. We recall from (3.7) that $V_{\Lambda,\xi}(y, z, x) = \Phi_{\Lambda,\xi}(y^+; y, z, x)$ where $y^+ \in \arg\min_{w\in\mathbb{R}^n} \Phi_{\Lambda,\xi}(w; y, z, x)$. We then have from (3.8) that

$$\begin{aligned} V_{\Lambda,\xi}(y, z, x) &= Q_{\alpha(f+h)}(y^+; x) + Q_{\beta p}(y^+; y, \xi) + g(y^+) \\ &\geq f(y^+) + h(y^+) + \frac{1 - \alpha(L_{\mathrm{f}} + L_{\mathrm{h}})}{2\alpha}\|y^+ - x\|^2 + Q_{\beta p}(y^+; y, \xi) + g(y^+) \\ &\overset{(3.23)}{\geq} \Psi(y^+) + \frac{1 - \alpha(L_{\mathrm{f}} + L_{\mathrm{h}})}{2\alpha}\|y^+ - x\|^2 + \frac{1 - \beta L_{\mathrm{p}}}{2\beta}\|y^+ - y\|^2, \end{aligned} \tag{3.54}$$

where the first inequality holds by $(L_{\mathrm{f}} + L_{\mathrm{h}})$-smoothness of $f + h$ and Lemma 2.1(a). Setting $(y, z, x) = (y^{k-1}, z^{k-1}, x^{k-1})$, we have from (3.54) that

$$V_{k-1} \geq \Psi(y^k) + \frac{1 - \alpha(L_{\mathrm{f}} + L_{\mathrm{h}})}{2\alpha}\left\|y^k - x^{k-1}\right\|^2 + \frac{1 - \beta L_{\mathrm{p}}}{2\beta}\left\|y^k - y^{k-1}\right\|^2. \tag{3.55}$$

By our choice of (α, β, γ), it follows from Theorems 3.5, 3.9 and 3.12 that $\{V_k\}$ is strictly decreasing. Meanwhile, we have from Remark 3.6 and Remark 3.11 that under conditions (a) and (b), respectively, it holds that $\alpha < \frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}}$. In condition (c), it is explicitly assumed that $\alpha < \frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}}$. Hence, the second term in (3.55) is always bounded below by zero. By our choice of β, the last term is also nonnegative. With these, we conclude that $\{\Psi(y^k)\}$ is bounded above, and therefore $\{y^k\}$ is a bounded sequence by the level-boundedness of Ψ. On the other hand, since Ψ is closed, Ψ is also bounded below. It thus follows from (3.55) that $\{\|y^k - x^{k-1}\|\}$ is bounded above. Since $\{y^k\}$ is bounded, it then follows that $\{x^{k-1}\}$ is also bounded. Finally, since $z^k = x^k + \alpha\nabla f(x^k)$ by (3.1) and ∇f is continuous, we obtain the boundedness of $\{z^k\}$. This completes the proof of part (i).

To prove part (ii), note that since $\{V_k\}$ is a bounded decreasing sequence, it follows that $V_{k-1} - V_k \to 0$. Meanwhile, under our hypotheses, the inequality (3.24) holds and the coefficient of $\|x^k - x^{k-1}\|^2$ is strictly positive since $\alpha < \frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}}$ as mentioned above. Using these facts, we have that $\|x^k - x^{k-1}\| \to 0$. From (3.31) and the L_{f}-smoothness of f, we have

$$\left\|z^k - z^{k-1}\right\| = \left\|x^k - x^{k-1} + \alpha\left(\nabla f(x^k) - \nabla f(x^{k-1})\right)\right\| \leq (1 + \alpha L_{\mathrm{f}})\left\|x^k - x^{k-1}\right\|, \tag{3.56}$$

and so $\|z^k - z^{k-1}\| \to 0$. From (3.3), we have $\|y^k - x^{k-1}\| \to 0$, and therefore $\|y^k - y^{k-1}\| \to 0$ by the triangle inequality. □

Remark 3.16. When $L_{\mathrm{f}} + L_{\mathrm{h}} = 0$, x and z play no role in the algorithm. We thus obtain the following inequality similar to (3.54):

$$V_{\Lambda,\xi}(y, z, x) \geq \Psi(y^+) + \frac{1 - \beta L_{\mathrm{p}}}{2\beta}\|y^+ - y\|^2.$$

Hence, if Ψ has bounded level sets and $\beta < \frac{1}{L_{\mathrm{p}}}$, we obtain the boundedness of $\{y^k\}$ and that $\|y^k - y^{k-1}\| \to 0$.

We will next show that accumulation points of the x and y sequence generated from Algorithm 1 are *stationary points* of Ψ. We say that a point w^* is a stationary point of Ψ (see (Wen et al., 2018, Definition 4.1)) if

$$0 \in \nabla f(w^*) + \partial g(w^*) + \nabla h(w^*) + \partial p(w^*). \tag{3.57}$$

Note that any local minimizer of Ψ is a stationary point of Ψ (see (Pham and Thi, 1997, Theorem 2(i))). To establish stationarity of accumulation points, we will prove that these points are fixed points of the defining operator of Algorithm 1. Observe that Algorithm 1 can be concisely written as fixed-point iterations of a certain map; in particular,

$$(y^{k+1}, z^{k+1}) \in T_\Lambda(y^k, z^k), \quad k = 0, 1, \ldots,$$

where $T_\Lambda : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is given by

$$T_\Lambda(y, z) \coloneqq \left\{ \begin{pmatrix} y^+ \\ z + \tau(y^+ - \mathrm{prox}_{\alpha f}(z)) \end{pmatrix} : y^+ \in P_\Lambda(y, z) \right\}. \tag{3.58}$$

We say that (y, z) is a *fixed point* of T_Λ if $(y^*, z^*) \in T_\Lambda(y^*, z^*)$. The set of fixed points of T_Λ is denoted by $\mathrm{Fix}(T_\Lambda)$. The following proposition shows that fixed points of T_Λ are stationary points of Φ.

Proposition 3.17. *Suppose that f is L_{f}-smooth, g is proper and closed, h is continuously differentiable, and p is continuous on an open set containing* $\mathrm{dom}(g)$. *In addition, let $\alpha, \beta, \gamma > 0$ satisfy $\frac{1}{\gamma} = \frac{1}{\alpha} + \frac{1}{\beta}$ with $\alpha < L_{\mathrm{f}}^{-1}$. If $(y^*, z^*) \in \mathrm{Fix}(T_\Lambda)$, then y^* is a stationary point of Ψ.*

Proof. If $(y^*, z^*) \in \mathrm{Fix}(T_\Lambda)$, then $y^* \in P_\Lambda(y^*, z^*)$, $z^* \in z^* + \tau(y^* - \mathrm{prox}_{\alpha f}(z^*))$. Hence, by noting that $\mathrm{prox}_{\alpha f}$ is single-valued since $\alpha < L_{\mathrm{f}}^{-1}$, we have

$$y^* = \mathrm{prox}_{\alpha f}(z^*) \tag{3.59}$$

and thus

$$\frac{1}{\alpha}(z^* - y^*) = \nabla f(y^*). \tag{3.60}$$

From (3.59) and the optimality condition of (3.4), we also have that there exists $\xi^* \in \partial p(y^*)$ such that

$$\frac{1}{\alpha}(2y^* - z^* - \alpha\nabla h(y^*)) + \frac{1}{\beta}(y^* - \beta\xi^*) - \frac{1}{\gamma}y^* \in \partial g(y^*). \tag{3.61}$$

Adding (3.60) and (3.61) then leads to (3.57). □

Thus, we define the *residual function* for Algorithm 1 as

$$R(y, z) \coloneqq \mathrm{dist}(0, (y, z) - T_\Lambda(y, z)), \tag{3.62}$$

and use it as a measure of stationarity from the view that $R(y, z) = 0$ if and only if $(y, z) \in \mathrm{Fix}(T_\Lambda)$. This residual function reduces to that in Lee and Wright (2019) when $f \equiv 0$ and $p \equiv 0$, and to that in Liu and Takeda (2022) when $f \equiv 0$. A direct consequence of inequality (3.24) and Proposition 3.15 is the iteration complexity given in the following result.

Theorem 3.18. *Suppose that the hypotheses of Proposition* 3.15 *hold, and let (x^*, y^*, z^*) be an accumulation point of $\{(x^k, y^k, z^k)\}$. Then $x^* = y^*$ and $(y^*, z^*) \in \mathrm{Fix}(T_\Lambda)$. Moreover, if either $\beta < L_{\mathrm{p}}^{-1}$ or $\gamma < \rho_{\mathrm{g}}^{-1}$, then*

$$\min_{k=1,2,\ldots,N} R(y^k, z^k)^2 \leq \frac{\omega}{N},$$

where $\omega > 0$ is given by

$$\omega \coloneqq \begin{cases} \frac{2\beta(V_0 - \Psi^*)}{\min\{2\beta\zeta, 1-\beta L_{\mathrm{p}}\}} & \text{if } \beta < L_{\mathrm{p}}^{-1}, \\ \frac{4\gamma(V_0 - \Psi^*)}{\min\{2\gamma\zeta, 1-\gamma\rho_{\mathrm{g}}\}} & \text{if } \gamma < \rho_{\mathrm{g}}^{-1}, \end{cases} \quad \text{and} \quad \zeta \coloneqq \frac{-c(\alpha)}{2\tau\alpha(1+\alpha L_{\mathrm{f}})^2} > 0,$$

and $\Psi^* \coloneqq \min_{w \in \mathbb{R}^n} \Psi(w)$.

Proof. Let $\{(x^{k_j}, y^{k_j}, z^{k_j})\}$ be such that $(x^{k_j}, y^{k_j}, z^{k_j}) \to (x^*, y^*, z^*)$. From the proof of Proposition 3.15, we have that $\|y^k - x^{k-1}\| \to 0$ and $\|y^k - y^{k-1}\| \to 0$. It then follows that $y^{k_j} \to x^*$ by the triangle inequality, and therefore $x^* = y^*$. To show that $(y^*, z^*) \in \mathrm{Fix}(T_\Lambda)$, we first note that since $x^* = y^*$ and $z^k = x^k + \alpha\nabla f(x^k)$ by (2.4), it follows that $z^* = y^* + \alpha\nabla f(y^*)$. Since $\alpha < \frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}} \leq \frac{1}{L_{\mathrm{f}}}$, then $f + \frac{1}{2\alpha}\|\cdot\|^2$ is strongly convex by Remark 2.2 and therefore $y^* \in \mathrm{prox}_{\alpha f}(z^*)$. Thus, $z^* = z^* + \tau(y^* - \mathrm{prox}_{\alpha f}(z^*))$. Hence, to show that $(y^*, z^*) \in \mathrm{Fix}(T_\Lambda)$, it remains to prove that $y^* \in P_\Lambda(y^*, z^*)$. To this end, note that by the convexity and continuity of $\frac{L_{\mathrm{p}}}{2}\|\cdot\|^2 - p$ on $\mathrm{dom}(g)$ (by Assumption 1.1(d)) and the formula (3.21), we have from (Beck, 2017, Theorem 3.16) and (3.21) that $\{\xi^{k_j}\}$ is bounded. Using again the continuity of p on $\mathrm{dom}(g)$ together with (2.2) and the fact that $\{y^k\} \subseteq \mathrm{dom}(g)$, we see that accumulation points of $\{\xi^{k_j}\}$ belong to $\partial p(y^*)$. Without loss of generality, we may assume that $\xi^{k_j} \to \xi^*$ where $\xi^* \in \partial p(y^*)$.

Meanwhile, we have from (3.2) that

$$y^{k_j+1} \in \mathrm{prox}_{\gamma g}(u^{k_j}) \quad \text{where } u^{k_j} := \frac{\gamma}{\alpha}(2x^{k_j} - z^{k_j} - \alpha\nabla h(x^{k_j}) + \frac{\gamma}{\beta}(y^{k_j} - \beta\xi^{k_j}). \tag{3.63}$$

Note that $y^{k_j+1} \to y^*$ and $u^{k_j} \to u^* := \frac{\gamma}{\alpha}(2y^* - z^* - \alpha\nabla h(y^*)) + \frac{\gamma}{\beta}(y^* - \beta\xi^*)$, where we have used the fact that $\xi^{k_j} \to \xi^*$ and $x^* = y^*$. To prove the claim that $y^* \in P_\Lambda(y^*, z^*)$, we only need to show that $y^* \in \mathrm{prox}_{\gamma g}(u^*)$, noting that $y^* \in \mathrm{prox}_{\alpha f}(z^*)$.

From Lemma 2.1(a) and (3.22), we have that for all $x \in \mathbb{R}^n$,

$$\begin{aligned}\Psi^* &\le f(x) + h(x) + p(x) + g(x)\\ &\le (f+h+p)(\bar{x}) + \left\langle \nabla(f+h)(\bar{x}) + \bar{\xi}, x - \bar{x}\right\rangle + \frac{L_{\mathrm{f}} + L_{\mathrm{h}} + L_{\mathrm{p}}}{2}\|x - \bar{x}\|^2 + g(x),\end{aligned}$$

where $\bar{x}$ is a fixed but arbitrary element of $\mathrm{dom}(g)$ and $\bar{\xi} \in \partial p(\bar{x})$. We may write the above inequality as

$$0 \le a + \langle v, x\rangle + \frac{L_{\mathrm{f}} + L_{\mathrm{h}} + L_{\mathrm{p}}}{2}\|x\|^2 + g(x) \tag{3.64}$$

for some $a \in \mathbb{R}$ and $v \in \mathbb{R}^n$. It then follows that $\liminf_{\|x\|\to\infty} \frac{g(x)}{\|x\|^2} \ge -\frac{L_{\mathrm{f}}+L_{\mathrm{h}}+L_{\mathrm{p}}}{2}$ by dividing both sides of (3.64) by $\|x\|^2$ and using Cauchy-Schwarz inequality. By (Rockafellar and Wets, 1998, Exercise 1.24), the fact that $\frac{1}{\gamma} > L_{\mathrm{f}} + L_{\mathrm{h}} + \frac{1}{\beta} \ge L_{\mathrm{f}} + L_{\mathrm{h}} + L_{\mathrm{p}}$ (that is, $\gamma < \frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}+L_{\mathrm{p}}}$), and (Rockafellar and Wets, 1998, Theorem 1.25), we know that $\{y^{k_j+1}\}$ given in (3.63) is bounded, with accumulation points lying in $\mathrm{prox}_{\gamma g}(u^*)$. Since $y^{k_j+1} \to y^*$, it follows that $y^* \in \mathrm{prox}_{\gamma g}(u^*)$, as desired.

To prove the last claim, we note first from (3.55) that $V_k \ge \Psi^*$. Summing (3.24) from 2 to $N+1$, we get

$$\sum_{k=2}^{N+1} -\frac{c(\alpha)}{2\tau\alpha}\left\|x^k - x^{k-1}\right\|^2 + \sum_{k=2}^{N+1}\frac{1-\beta L_{\mathrm{p}}}{2\beta}\left\|y^k - y^{k-1}\right\|^2 \le V_1 - \Psi^*. \tag{3.65}$$

Meanwhile, we have $R(y^k, z^k)^2 \le \left\|(y^k, z^k) - (y^{k+1}, z^{k+1})\right\|^2$ since $(y^{k+1}, z^{k+1}) \in T_\Lambda(y^k, z^k)$. These together with (3.24) and (3.56) give the desired result for the case $\beta < L_{\mathrm{p}}^{-1}$. For the other case, the summation of (3.24) from 1 to N gives

$$\sum_{k=1}^{N}\frac{1-\gamma\rho_{\mathrm{g}}}{2\gamma}\left\|y^k - y^{k+1}\right\|^2 \le V_0 - \Psi^*.$$

Using (3.65), we obtain

$$\sum_{k=1}^{N} -\frac{c(\alpha)}{2\tau\alpha}\left\|x^{k+1} - x^k\right\|^2 + \sum_{k=1}^{N}\frac{1-\gamma\rho_{\mathrm{g}}}{2\gamma}\left\|y^k - y^{k+1}\right\|^2 \le V_0 + V_1 - 2\Psi^*,$$

from where we can easily infer the result. □

3.4 Comments on the weak convexity assumption on g

We provide some remarks about the weak convexity assumption on g. First, this assumption is needed for guaranteeing the formula

$$P_\Lambda(y, z) = \bigcup_{x\in\mathrm{prox}_{\alpha f}(z), \xi\in\partial p(y)} \operatorname*{arg\,min}_{w\in\mathbb{R}^n} \Phi_{\Lambda,\xi}(w; y, z, x). \tag{3.66}$$

in Lemma 3.2, but we can actually still ensure (3.66) without this weak convexity assumption on g in some special cases, such as when either (I) $f = h = 0$, or (II) $L_{\mathrm{p}} = 0$. In the former case, $\mathrm{prox}_{\alpha f}(z) \equiv z$ and $\gamma = \beta$, and consequently $P_\Lambda = \mathrm{prox}_{\gamma g}(T_{\gamma p}(y))$, which is precisely the right-hand side of (3.66). On the other hand, when $L_{\mathrm{p}} = 0$, we have $\gamma = \alpha$ and hence

$$\begin{aligned} P_\Lambda(y,z) &= \bigcup_{\substack{x\in\mathrm{prox}_{\alpha f}(z)\\ \xi\in\partial p(y)}} \mathrm{prox}_{\gamma g}\left(x-\gamma\nabla f(x)-\gamma\nabla h(x)-\gamma\partial p(y)\right), \\ &= \bigcup_{\substack{x\in\mathrm{prox}_{\alpha f}(z)\\ \xi\in\partial p(y)}} \operatorname*{arg\,min}_{w\in\mathbb{R}^n} g(w)+\frac{1}{2\gamma}\|w-x+(\gamma\nabla f(x)+\gamma\nabla h(x)+\gamma\xi)\|^2, \end{aligned}$$

which is clearly equal to the right-hand side of (3.66).

We also note that when g is not weakly convex, we can get a weaker version of Lemma 3.3. Indeed, from (3.7), we can obtain by setting $w = y$ that

$$V_{\Lambda,\xi}(y,z,x) \le Q_{\alpha(f+h)}(y;x) + p(y) + g(y) \tag{3.67}$$

for all $(y,z) \in \mathrm{dom}(g)\times\mathbb{R}^n$, $\xi\in\partial p(y)$ and $x\in\mathrm{prox}_{\alpha f}(z)$. The only difference is that we now lose the term $-\frac{1-\gamma\rho_{\mathrm{g}}}{2\gamma}\|y-y^+\|^2$ in Lemma 3.3. Nevertheless, using this, we can obtain results parallel to Proposition 3.15 and Theorem 3.18 without requiring weak convexity, as long as the formula (3.66) holds true such as in cases (I) and (II) described above.

Proposition 3.19. *Suppose that Assumption* 1.1*(a), (b) and (d) are fulfilled, g is a proper closed function, the formula* (3.66) *holds true, Ψ has bounded level sets, and $\beta\in(0,L_{\mathrm{p}}^{-1})$ if $L_{\mathrm{p}}>0$. In addition, suppose that $L_{\mathrm{f}}+L_{\mathrm{h}}>0$, and $\alpha,\tau>0$ are chosen such that they satisfy (a), (b) or (c) in Proposition* 3.15*. If $\{(x^k,y^k,z^k)\}$ is generated by Algorithm* 1*, then*

(i) $\{(x^k,y^k,z^k)\}$ *is bounded;*

(ii) $\left\|(x^k,y^k,z^k)-(x^{k-1},y^{k-1},z^{k-1})\right\|\to 0$*;*

(iii) If (x^*,y^*,z^*) is an accumulation point, then $x^*=y^*$ and $(y^*,z^*)\in\mathrm{Fix}(T_\Lambda)$.

(iv) There exists $\omega\in(0,\infty)$ (dependent on the stepsizes and the problem) such that the residual converges to 0 with rate

$$\min_{k=1,2,\ldots,N} R(y^k,z^k)^2 \le \frac{\omega}{N}.$$

Proof. Using (3.67), Lemma 3.4, and (3.66), we can derive identical stepsize upper bounds for α as in Theorems 3.5, 3.9 and 3.12 to guarantee that the inequality

$$V_{k-1}-V_k \ge \begin{cases} -\frac{c(\alpha)}{2\tau\alpha}\left\|x^k-x^{k-1}\right\|^2+\frac{1-\beta L_{\mathrm{p}}}{2\beta}\left\|y^k-y^{k-1}\right\|^2 & \text{if } L_{\mathrm{p}}>0,\\ -\frac{c(\alpha)}{2\tau\alpha}\left\|x^k-x^{k-1}\right\|^2 & \text{if } L_{\mathrm{p}}=0, \end{cases} \tag{3.68}$$

holds for all k, where $c(\alpha)$ is as defined in these theorems. With this, the proofs of parts (i) and (ii) are identical to that of Proposition 3.15. The proof of part (iii) is identical to the first part of the proof of Theorem 3.18, noting that weak convexity is not employed in the proof. For part (iv), when $L_{\mathrm{p}}>0$, the result follows from the exact same arguments in Theorem 3.18 for this particular

case, given the hypothesis that $\beta < L_{\mathrm{p}}^{-1}$. Suppose now that $L_{\mathrm{p}} = 0$. From (3.68), we have for any $N \geq 0$ that

$$\sum_{k=1}^{N+1} -\frac{c(\alpha)}{2\tau\alpha}\left\|x^k - x^{k-1}\right\|^2 \leq V_0 - \Psi^*. \tag{3.69}$$

On the other hand, from (3.3), we get $y^{k+1} - y^k = x^k - x^{k-1} + \tau^{-1}(z^{k+1} - z^k) - \tau^{-1}(z^k - z^{k-1})$ It follows from the triangle inequality and the estimate $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ (for $a, b, c \in \mathbb{R}$) that

$$\left\|y^{k+1} - y^k\right\|^2 \leq 3\left\|x^k - x^{k-1}\right\|^2 + 3\tau^{-2}\left\|z^{k+1} - z^k\right\|^2 + 3\tau^{-2}\left\|z^k - z^{k-1}\right\|^2.$$

Then

$$\begin{aligned}
\sum_{k=1}^{N}\left\|y^{k+1} - y^k\right\|^2 &\leq 3\sum_{k=1}^{N}\left\|x^k - x^{k-1}\right\|^2 + 3\tau^{-2}\sum_{k=1}^{N}\left\|z^{k+1} - z^k\right\|^2 + 3\tau^{-2}\sum_{k=1}^{N}\left\|z^k - z^{k-1}\right\|^2 \\
&\overset{(3.56)}{\leq} 3\sum_{k=1}^{N}\left\|x^k - x^{k-1}\right\|^2 + 3\tau^{-2}(1+\alpha L_{\mathrm{f}})^2\sum_{k=1}^{N}\left\|x^{k+1} - x^k\right\|^2 \\
&\quad + 3\tau^{-2}(1+\alpha L_{\mathrm{f}})^2\sum_{k=1}^{N}\left\|x^k - x^{k-1}\right\|^2 \\
&= \left(3 + 3\tau^{-2}(1+\alpha L_{\mathrm{f}})^2\right)\sum_{k=1}^{N}\left\|x^k - x^{k-1}\right\|^2 + 3\tau^{-2}(1+\alpha L_{\mathrm{f}})^2\sum_{k=1}^{N}\left\|x^{k+1} - x^k\right\|^2 \\
&\overset{(3.69)}{\leq} \left(3 + 6\tau^{-2}(1+\alpha L_{\mathrm{f}})^2\right)\left(-\frac{2\tau\alpha(V_0 - \Psi^*)}{c(\alpha)}\right).
\end{aligned}$$

Combining this with (3.69) yields

$$\sum_{k=1}^{N}\left\|x^{k+1} - x^k\right\|^2 + \sum_{k=1}^{N}\left\|y^{k+1} - y^k\right\|^2 \leq \left(4 + 6\tau^{-2}(1+\alpha L_{\mathrm{f}})^2\right)\left(-\frac{2\tau\alpha(V_0 - \Psi^*)}{c(\alpha)}\right),$$

from where the result follows. □

We remark that in particular, our results apply to the exact same nonconvex setting of the DYS algorithm studied in Bian and Zhang (2021) (that is, without weak convexity assumption on g). Together with Remark 3.7 and Fig. 1, we have significantly improved the stepsize bounds derived by Bian and Zhang (2021) for the DYS algorithm.

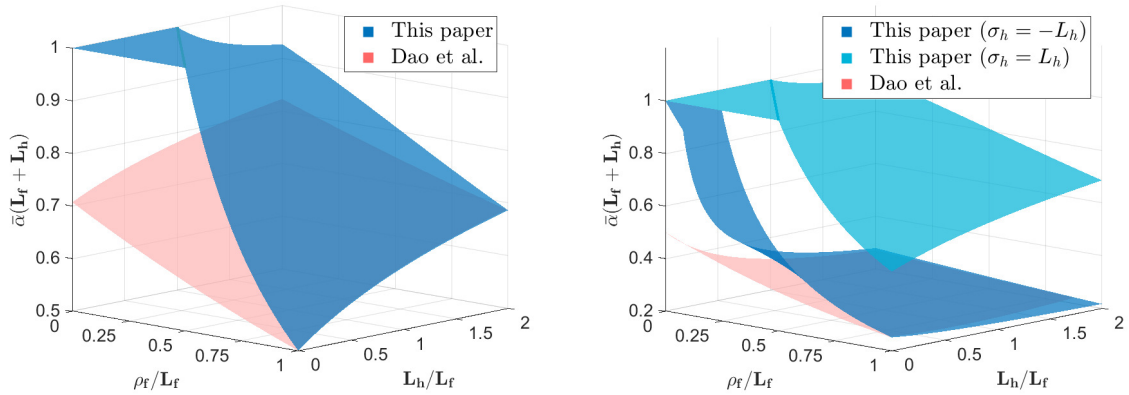
Generalizing the above discussion, provided that the minimization of the function $\Phi_{\Lambda,\xi^k}(\cdot, y^k, z^k, x^k)$ is not difficult or expensive to compute, we can simply replace the y-update rule (3.2) in Algorithm 1 with the formula

$$y^{k+1} \in \bigcup_{x^k \in \mathrm{prox}_{\alpha f}(z^k),\, \xi^k \in \partial p(y^k)} \operatorname*{arg\,min}_{w \in \mathbb{R}^n} \Phi_{\Lambda,\xi^k}(w; y^k, z^k, x^k). \tag{3.70}$$

In this case, we can also apply the results to situations outside conditions (I) and (II) above. Similarly, we can obtain identical estimates for the stepsizes and derive results analogous to Proposition 3.19, where T_Λ is as defined in (3.58) with P_Λ given by the right-hand side of (3.66).

While finalizing this paper, we became aware of a recent preprint by Dao et al. (2024) that proposed to solve (1.1) with $L_{\mathrm{p}} = 0$ using an algorithm similar to ours.[1] Subsequential convergence

[1] Specifically, setting $\theta = 1$ in (Dao et al., 2024, Algorithm 1) coincides with our algorithm for the case $L_{\mathrm{p}} = 0$.



(a) $\tau = 1$ (b) $\tau = 1.5$ and $\sigma_{\mathrm{h}} = \pm L_{\mathrm{h}}$

Figure 2: Comparison of stepsize upper bounds (denoted by $\bar{\alpha}$) computed in the present work and in Dao et al. (2024) for $\tau = 1$ and $\tau = 1.5$. We can see that the stepsize upper bounds in this work are always larger than those in Dao et al. (2024). Figure (a) demonstrates that the upper bound we have derived for the stepsize of the Davis-Yin splitting algorithm is $\frac{1}{L_{\mathrm{f}}+L_{\mathrm{h}}}$ provided that $\frac{L_{\mathrm{h}}}{L_{\mathrm{f}}} + \frac{2\rho_{\mathrm{f}}}{L_{\mathrm{f}}} \leq 1$ (see Theorem 3.5). Figure (b) demonstrates Theorem 3.9 with $\tau = 1.5$ using a rough estimate $-\sigma_{\mathrm{h}} = L_{\mathrm{h}}$, but this can be further improved when $-\sigma_{\mathrm{h}} < L_{\mathrm{h}}$. For instance, the case of $h(x) = \lambda\|x\|_2^2$ ($\sigma_{\mathrm{h}} = L_{\mathrm{h}}$) is also shown in figure (b) (see Remark 3.10).

was proved in Dao et al. (2024) without the weak convexity assumption on g, which is also covered by our framework, in view of Proposition 3.19, as it corresponds to case (II) above. We highlight that another advantage of our analysis for this special case in which the settings coincide is that the upper bounds for stepsizes we computed are significantly larger than those in Dao et al. (2024); see Figs. 2 and 3 for geometric illustrations. Meanwhile, Dao et al. (2024) also proved global convergence of the full sequence under a Kurdyka-Łojasiewicz hypothesis, using the techniques by Liu et al. (2019b), and a similar proof strategy can be employed to prove the same global convergence property for our algorithm.

4 Numerical experiments

In this section, we present experiments to corroborate the theoretical findings in the previous section. All experiments were conducted in MATLAB. For our method, we experiment with various values of τ. For each τ, we use the corresponding $\bar{\alpha}$ in Theorems 3.5, 3.9 and 3.12 times 9/10 to ensure global subsequential convergence as suggested by Theorem 3.18. For comparing different methods that can fit in the framework of Algorithm 1, we report their required number of iterations and running time to reach $R(y, z) \leq 10^{-6}$.

4.1 Nonnegative and low-rank matrix completion

The first experiment we consider is nonnegative matrix completion/factorization Lee and Seung (1999), whose goal is to recover missing entries of a partially observed matrix using nonnegative entries, and we impose the additional structure requirement that the recovered matrix should be low-rank. Given a data matrix $M \in \mathbb{R}^{m\times n}$, we let Ω denote the entries (i, j) oberserved and P_Ω denote the associated projection operation such that $P_\Omega(X)_{i,j}$ outputs $X_{i,j}$ if $(i, j) \in \Omega$ and 0

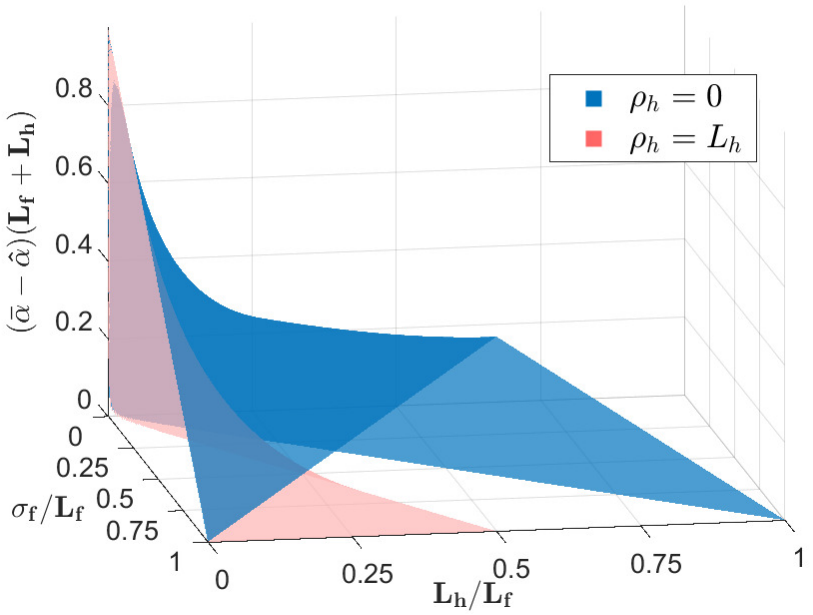


Figure 3: Gaps between the stepsize upper bounds in the present work and in Dao et al. (2024), denoted respectively by $\bar{\alpha}$ and $\hat{\alpha}$, when $\tau = 2$. In this case, Dao et al. (2024) always requires that $\sigma_{\mathrm{f}} > 2L_{\mathrm{h}}$. Using the rough estimate $\rho_{\mathrm{h}} = L_{\mathrm{h}}$ in Theorem 3.9, the red graph shows that the difference $\bar{\alpha} - \hat{\alpha}$ between our and their stepsizes is always nonnegative. For a convex h (namely when $\rho_{\mathrm{h}} = 0$), the upper bound $\hat{\alpha}$ provided in Dao et al. (2024) does not change. As depicted in the blue graph, ours, on the other hand, provides a larger stepsize upper bound for the convex case, resulting in a bigger gap between $\bar{\alpha}$ and $\hat{\alpha}$. Ours also has a wider region in which $\tau = 2$ is applicable, since our condition on the parameters (namely $\sigma_{\mathrm{f}} > L_{\mathrm{h}} + \rho_{\mathrm{h}}$) becomes weaker than requiring $\sigma_{\mathrm{f}} > 2L_{\mathrm{h}}$.

otherwise. We consider (1.1) with $p(X) \equiv 0$ and

$$f(X) = \frac{\lambda_1}{2} \min_{Y \in \mathbb{R}^{m\times n}_+} \|X - Y\|_F^2, \quad g(X) = \lambda_2 \|X\|_*, \quad h(X) = \frac{1}{2}\|P_\Omega(X - M)\|_F^2,$$

where $\lambda_1, \lambda_2 \geq 0$ are weights for the respective terms, $\|\cdot\|_*$ is the nuclear norm, $\|\cdot\|_F$ is the Frobenius norm, and $\mathbb{R}^{m\times n}_+$ is the nonnegative orthant of $\mathbb{R}^{m\times n}$.

We follow Toh and Yun (2010) to generate $M \in \mathbb{R}^{m\times n}$ with a specific rank r as the product of an $m \times r$ and an $r \times n$ matrix whose entries are all identically and independently distributed as standard Gaussian, and then randomly select s entries in uniform random to form Ω. We in particular use $m = n$ and consider $(n, s) \in \{(100, 1000), (500, 10000)\}$, $r \in \{10, 30\}$, $\lambda_1 = 5$ and $\lambda_2 = 10$. We set $\beta = \infty$ for our method due to the absence of the p term. Since f is not strongly convex, our theoretical results indicate that we should use $\tau < 2$. We thus run our method with $\tau \in \{1, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9\}$,[2] and compare with the proximal gradient method (PG) by treating $f + h$ together as the smooth term. In addition, for this problem, our method with $\tau = 1$ recovers the Davis-Yin splitting (DYS). All algorithms are run with a cap of $30,000$ iterations. We can clearly see from the results shown in Table 1 that our method constantly outperforms DYS and PG.

[2] We have also experimented with $\tau \in (0, 1)$, but those values tend to give worse performance. We therefore omit these results in our experiments.

Table 1: Comparison between methods for nonnegative and low-rank matrix completion. $*$ denotes that the maximum iteration of 30000 is reached. $\tau = t$ denotes our method with the specified τ.

Method	$n = 100, r = 10$		$n = 100, r = 30$		$n = 500, r = 10$		$n = 500, r = 30$	
	iter	time	iter	time	iter	time	iter	time
PG	6269	36.57	12044	70.16	17221	2299.74	*30000	*4108.18
DYS	6892	49.13	13217	94.71	18937	2350.59	*30000	*5117.43
$\tau = 1.1$	6364	45.05	12223	87.18	17483	2175.75	*30000	*5081.06
$\tau = 1.2$	5922	42.04	11391	81.11	16269	2024.97	*30000	*5092.04
$\tau = 1.3$	5549	40.00	10686	76.25	15242	1897.01	*30000	*5077.33
$\tau = 1.4$	5231	37.35	10083	72.04	14365	1780.56	*30000	*5075.39
$\tau = 1.5$	4956	34.99	9564	68.71	13611	1683.91	28883	4894.85
$\tau = 1.6$	4720	33.73	9115	64.84	12959	1609.62	27500	4666.92
$\tau = 1.7$	**4514**	**31.78**	**8725**	**62.30**	**12395**	**1542.70**	**26303**	**4472.93**
$\tau = 1.8$	5516	39.46	10624	75.42	15152	1879.76	*30000	*5106.69
$\tau = 1.9$	8537	61.16	16306	116.10	23471	2908.76	*30000	*5089.89

4.2 Cardinality-constrained problems

Our next experiment considers a least-square problem with a penalized form of the cardinality constraint $\|x\|_0 \leq k$ for some given positive integer k. As argued in Gotoh et al. (2018), this constraint is equivalent to $\|x\|_1 - \|x\|_{(k)} = 0$, where $\|x\|_{(k)}$ is the Ky Fan k-norm that computes the sum of the top k largest elements of the element-wise absolute value of x, and we follow Gotoh et al. (2018) to take $\|x\|_1 - \|x\|_{(k)}$ as a penalty in the objective function instead of enforcing the hard constraint. We also follow a common convention in machine learning to add in a small Tikhonov regularization of the form $\|x\|_2^2$ to improve the problem condition. This leads to the following setting for (1.1).

$$f(x) = \frac{\lambda_1}{2}\|x\|_2^2, \quad g(x) = \lambda_2\|x\|_1, \quad p(x) = -\lambda_2\|x\|_{(k)}, \quad h(x) = \frac{1}{2}\|Ax - b\|_2^2. \tag{4.1}$$

Given the existence of the p term in this experiment, PG is not applicable, but by again treating $(f + h)$ as the smooth term, we can use the proximal DC algorithm described in (3.6). Our experiment uses publicly available real-world data[3] listed in Table 2. In this experiment, we set $\lambda_1 = 0.01$, $\lambda_2 = 0.005$, and $k = \lfloor n/10 \rfloor$. As (3.49) does not hold for any $\tau \geq 2$, for our method we use the same range of τ as the previous experiment. All algorithms are run with a cap of $100,000$ iterations. The results are shown in Table 3. We can see that with appropriately selected τ, our method performs similarly to proximal DC on ijcnn1, and is much more efficient than it on all the other datasets.

5 Conclusion

In this work, we presented a new splitting algorithm for the four-term optimization problem (1.1). Our algorithm generalizes the Davis-Yin splitting algorithm for three-term optimization to allow for an additional nonsmooth term. We derived stepsize estimates for the proposed algorithm that ensure global subsequential convergence to stationary points of the objective function. A notable

[3] Downloaded from `https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/`.

Data	m	n
colon-cancer	62	2000
duke	38	7129
ijcnn1	49900	22
phishing	11055	68
heart	270	13

Table 2: Datasets used in the experiment of least-square problems.

Table 3: Comparison between proximal DC (denoted by PDC) and our method for (4.1) $*$ denotes that the maximum iteration of 100000 is reached. $\tau = t$ denotes our method with the specified τ.

Method	colon		duke		ijcnn1		phishing		heart	
	iter	time	iter	time	iter	time	iter	time	iter	time
PDC	59796	63.88	19108	33.09	**1367**	**2.97**	*100000	*104.20	*100000	*27.56
$\tau = 1$	49934	53.09	13600	26.41	1502	3.33	*100000	*104.31	*100000	*24.96
$\tau = 1.1$	37384	39.76	8447	17.08	1650	3.69	*100000	*104.94	*100000	*25.31
$\tau = 1.2$	20352	21.71	5538	10.30	1834	4.06	*100000	*103.98	*100000	*25.70
$\tau = 1.3$	4772	5.07	**5500**	**10.06**	2066	4.60	*100000	*104.64	*100000	*26.02
$\tau = 1.4$	**4672**	**4.97**	5847	11.64	2370	5.26	*100000	*104.08	*100000	*31.02
$\tau = 1.5$	5148	5.48	6450	13.61	2787	6.19	*100000	*114.73	*100000	*34.09
$\tau = 1.6$	5968	6.39	7365	13.56	3396	7.55	**28347**	**29.39**	*100000	*30.16
$\tau = 1.7$	7335	7.84	8790	17.05	4380	9.71	32633	33.78	*100000	*31.07
$\tau = 1.8$	9881	10.52	11220	21.21	6257	13.92	42635	44.08	*100000	*33.36
$\tau = 1.9$	16308	17.54	16169	31.22	11457	25.25	68361	70.27	**52222**	**14.46**

implication of our results is the significant improvement in the upper bound estimates for the stepsize of the DYS algorithm that guarantee subsequential convergence. Our experiments demonstrated the applicability and effectiveness of our proposed algorithm.

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