Inverse of the Gomory Corner Relaxation of Integer Programs

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Abstract

We analyze the inverse of integer programs (IPs) using the Gomory corner relaxation (GCR). We prove the inverse GCR is equivalent to the inverse of a shortest path problem, yielding a polyhedral representation of the GCR inverse-feasible region. We propose a linear program formulation for the inverse GCR under L_1 and L_∞ norms. The inverse GCR bounds the inverse IP optimal value as tightly as the inverse linear relaxation under mild conditions. We relate the inverse feasible region of IP and the inverse feasible regions of GCRs.

Keywords: Inverse optimization; Integer programming; Gomory corner relaxation.

1 Introduction

Given a (forward) optimization problem and a feasible solution, the *inverse-feasible region* is the set of objective vectors under which the given feasible solution is optimal to the forward problem. The *inverse optimization problem* finds an inverse-feasible vector that is closest (by some given metric) to a given target vector. Inverse optimization has many applications. Tarantola [33] applied inverse optimization in geophysical sciences, such as estimating the epicenter of a seismic event. He also showed applications in statistics, e.g., for linear regression. Inverse optimization is also useful in healthcare: Estimating liver-transplant patients' preferences over healthcare outcomes [16], medical imaging [4], designing cancer treatment plans [3, 11], estimating the physical properties of solid materials [9], and traffic equilibrium models [5].

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The inverse of integer programs (IPs) and the inverse of mixed integer programs (MIPs) are particularly interesting because of their wide applicability. Schaefer [30] and Lamperski and Schaefer [27] established polyhedral representations of the inverse-feasible regions of IPs and MIPs using the superadditive duality of the forward problems. This characterization enabled linear programming (LP) formulations for inverse IPs and inverse MIPs. However, the number of variables and constraints in these LP formulations grow super-exponentially (in the size of the forward problem) and are thus intractable for most instances. Huang [23] reformulated the inverse IP as the inverse of a shortest path problem; the number of vertices and arcs in the graph of this shortest path problem grow super-exponentially (on the number of constraints in the forward IP).

Cutting plane algorithms have been proposed as an alternative to LP formulations for solving inverse IPs and MIPs. Wang [35] provided a cutting plane algorithm for solving inverse MIPs by repeatedly generating optimality cuts from the extreme points of the convex hull of the feasible region of the forward problem. His empirical analysis demonstrated the algorithm's tractability for small inverse MIPs. The algorithm was improved upon by Duan and Wang [14], who introduced a heuristic algorithm for computing the extreme points and bounds for Wang's algorithm [35]. Bodur et al. [8] introduced another cutting plane algorithm for solving inverse MIPs, which generates optimality cuts from interior points of the convex hull of the feasible region of the forward problem. Their empirical analysis showed runtime improvements over Wang's algorithm [35] because the interior points are often easier to compute than the extreme points. These cutting plane algorithms are far more tractable than the LP formulations proposed by Schaefer [30] and Lamperski and Schaefer [27], but the cutting plane algorithms do not characterize the polyhedral structure of the inverse-feasible regions of IPs and MIPs. Inverse IP and inverse MIPs remain theoretically and computationally challenging.

IPs and MIPs are often studied by relaxing the integrality constraints, obtaining the LP relaxation. Therefore, a natural approach to studying inverse IP and inverse MIPs is to solve inverse LP problems, which typically exhibit more structure. Zhang and Liu [37] proposed a solution for general inverse LP problems under the L_1 norm, from which they obtained strongly polynomial algorithms for solving the inverse minimum cost flow problem and the inverse assignment problem. Zhang and Liu [38] proposed a solution for inverse LP problems when both the given feasible solution and an optimal solution under the original objective vector are composed of only zeros and ones, which is common in network flow problems. Ahuja and Orlin [2] showed that if a problem with a linear objective function is polynomially solvable, as is the case for LP problems, then the inverse of that problem under the L_1 or L_∞ norm is also polynomially solvable. Tavaslıoğlu et al. [34] studied the polyhedral structure of the inverse-feasible region of LP problems, while Chan et al. [12] introduced a goodness-of-fit framework for evaluating inverse LP problems where the provided feasible solution for the forward LP problem cannot be made optimal outside of the trivial zero-objective case.

The Gomory corner relaxation (GCR) is an alternative method for relaxing IPs, obtained by relaxing the nonnegativity constraint of each variable in a basis of the LP relaxation while preserving variable integrality [19]. Gomory [20] noted that the forward GCR reveals the underlying structure of the original IP; for example, the facets of the convex hull of the feasible region of the GCR provide cutting planes for the original IP. Gomory [19], Hoşten and Thomas [22], and Richard and Dey [29] enumerated several classes of IP instances where the optimal solutions for the GCR are also optimal solutions for the original IP. Fischetti and Monaci [17] demonstrated that for many problem instances, the gap between the IP and GCR optimal values is much tighter the gap between the IP and LP relaxation optimal values. Köppe et al. [26] characterized the geometry of several reformulations of the GCR. The GCR can be further relaxed to obtain the master group relaxation, which can be applied to broader classes of problems because of its more general structure [29]. The GCR is NP-hard [28], and the most efficient known algorithms for solving the GCR exhibit polynomial runtime complexity with respect to the size of the determinant of the basis matrix of the LP relaxation, which can be very large [28, 29]. Several algorithms for solving the forward GCR reduce the GCR to an instance of the shortest path problem [13, 26, 29], a technique first developed by Shapiro [31].

We show that the inverse GCR can be solved as the inverse of a shortest path problem, which manipulates a graph's arc weights such that a given path becomes shortest from among all paths that connect the associated origin and destination vertices. The inverse shortest path problem has been extensively studied. The forward shortest path problem can be reduced to a minimum cost flow problem, so the inverse of the shortest path problem under the L_1 norm can be solved using a strongly polynomial algorithm provided by Zhang and Liu [37]. Ahuja and Orlin [2] showed that the inverse shortest path problem under the L_1 norm can be reduced to a forward shortest path problem. Zhang et al. [39] proposed a column generation framework for solving a variant of the inverse shortest path problem where several given paths each need to become shortest from among paths that connect their respective origin and destination vertices. Burton and Toint [10] proposed a quadratic programming formulation for solving the inverse shortest path problem under the L_2 norm. Xu and Zhang [36] characterized the feasible region of the inverse shortest path problem as a polyhedral cone.

We represent the inverse-feasible region of the GCR as a nonempty polyhedral cone and propose an LP formulation for the inverse GCR under the L_1 and L_∞ norms. We show that the inverse GCR bounds the inverse IP optimal value as tightly as the bounds provided by the inverse LP, assuming nondegeneracy. Our formulation of the inverse GCR is much smaller than the exact inverse IP formulation proposed by Schaefer [30].

We study the structure of inverse-feasible regions of IP and GCRs. We demonstrate that solving the inverse of a set of GCR problems, each defined by a different basis of the LP relaxation, provides more information about the inverse of IP than solving only one inverse GCR problem. We also show that the conic hull of the inverse-feasible regions of this set of GCR problems is a subset of the inverse-feasible region of IP. We provide the conditions under which the union of inverse-feasible regions of GCRs is the same as the inverse-feasible region of IP. Additionally, we identify the conditions under which the union of the inverse-feasible regions of GCR is a superset of the inverse-feasible region of the LP relaxation. In the absence of degeneracy, we show that the inverse-feasible region of GCR for some basis always performs as well as the inverse-feasible region of LP relaxation in terms of covering the inverse-feasible region of IP.

2 Preliminaries

2.1 Gomory Corner Relaxation

Given $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$, and $c \in \mathbb{R}^n$, let $\mathcal{IP}$ denote the following IP problem, which we assume has nonempty feasible region. Let $\mathcal{LP}$ denote the LP relaxation of $\mathcal{IP}$:

$$\min\{c^\intercal x \mid Ax = b, x \geq \mathbf{0}, x \in \mathbb{Z}^n\}, \tag{$\mathcal{IP}$}$$

$$\min\{c^\intercal x \mid Ax = b, x \geq \mathbf{0}, x \in \mathbb{R}^n\}. \tag{$\mathcal{LP}$}$$

Let $B, N \subseteq \{1, ..., n\}$ respectively denote the indices of the basic and nonbasic variables of a basic solution for $\mathcal{LP}$. Assume A is full row rank and let $m \leq n$, so $|B| = m, |N| = n - m$ and $B \cap N = \varnothing$. Let c_B, x_B (c_N, x_N) denote the vectors comprised of the B-indexed (N-indexed) components of c, x, respectively. Let A_B (A_N) be the matrix comprised of the B-indexed (N-indexed) columns of A. Observe that A_B is nonsingular. Then, the GCR of $\mathcal{IP}$ with respect to B, denoted by $\mathcal{G}^B$, is obtained by relaxing the nonnegativity constraints of the decision variables in the selected basis B [29]:

$$\min\{c_B^\intercal x_B + c_N^\intercal x_N \mid A_B x_B + A_N x_N = b, x_N \geq \mathbf{0}, x \in \mathbb{Z}^n\}. \qquad (\mathcal{G}^B)$$

For a given $d \in \mathbb{R}^n$, let $\mathcal{IP}^d$, $\mathcal{LP}^d$, $\mathcal{G}^{B,d}$ denote the problems $\mathcal{IP}, \mathcal{LP}, \mathcal{G}^B$, where the original objective vector c has been replaced by d. The feasible regions of $\mathcal{IP}^d$, $\mathcal{LP}^d$, $\mathcal{G}^{B,d}$ remain the same as the feasible regions of the original problems $\mathcal{IP}$, $\mathcal{LP}$, $\mathcal{G}^B$, respectively.

For a given optimization problem $\mathcal{P}$, let $z(\mathcal{P})$ denote the optimal objective value, and let $Opt(\mathcal{P})$ denote the set of optimal solutions.

Remark 1. We allow B to be an infeasible basis of $\mathcal{LP}$. Though Gomory [19] also allows B to be an infeasible basis, he assumes B is an optimal basis to find conditions where solving $\mathcal{G}^B$ also solves $\mathcal{IP}$. Richard and Dey [29] and Fischetti and Monaci [17] assume B is an optimal basis. Allowing B to be an infeasible basis, as done by Köppe et al. [26], permits a more general representation of the inverse GCR. Our results hold for both feasible and infeasible bases of $\mathcal{LP}$. In Section 4.2, we will show if d is in the inverse-feasible region of $\mathcal{G}^B$, where B is a feasible basis for $\mathcal{LP}$, then B must be an optimal basis for $\mathcal{LP}^d$.

2.2 Gomory Corner Relaxation as a Shortest Path Problem

We summarize how the GCR is reformulated as an instance of the shortest path problem as described by Richard and Dey [29], based on a reformulation first proposed by Shapiro [31].

Lemma 1. **[29]** *There exist unimodular matrices $S, T \in \mathbb{Z}^{m \times m}$ and a vector $w \in \mathbb{Z}_+^m$ such that $SA_BT = \text{diag}(w)$, where $\text{diag}(w)$ is the $m \times m$ matrix whose diagonal is given by w and whose off-diagonal entries are all zero.*

The formulation in Lemma 1 is the Smith Normal Form of A_B [32]. There are several efficient algorithms for computing S, T, and w [7, 15, 24, 25]. S, T, and w (as well as several objects we will define later) all depend on the selected basis B, but we decline to notate this dependence on B for clarity.

For a given vector $u \in \mathbb{Z}^m$, we define the modulo operator $u(\text{mod } w)$ to denote an m-dimensional vector whose ith component is given by $(u(\text{mod } w))_i = u_i \text{ mod } w_i$ for each $i \in [m] = \{1, ..., m\}$. For example, if $u = (3, 5, -2)$ and $w = (2, 3, 3)$, then $u(\text{mod } w) = (1, 2, 1)$.

We define linear function $r : \mathbb{R}^n \to \mathbb{R}^{n-m}$ by $r(d) = d_N - (A_B^{-1} A_N)^\intercal d_B$ to denote the reduced costs of the N-indexed variables in the basic solution $x_B = A_B^{-1} b$, $x_N = \mathbf{0}$ for $\mathcal{LP}^d$. Observe that r depends on the selected basis B. We use this notation to define a directed graph G with the vertex set V and the arc set E as follows:

$$V := \prod_{i \in [m]} \{0, 1, ..., w_i - 1\}, \quad E := \bigcup_{j \in [n-m]} E_j,$$

where, for each $j \in [n - m]$, $(SA_N)_j$ is the jth column vector of SA_N, and

$$E_j := \Big\{ \Big(u, \big(u + (SA_N)_j\big)(\text{mod } w)\Big) \mid u \in V \Big\}.$$

Since S and T are unimodular, $|\det A_B| = \prod_{i \in [m]} w_i$, and therefore, $|V| = |\det A_B|$ and $|E| = (n - m)|\det A_B|$.

Let $\mathcal{S}^B$ denote the problem of finding a shortest path from source vertex $\mathbf{0}$ to destination vertex $Sb(\text{mod } w)$ in graph G, where each arc in E_j is weighted by $r(c)_j$. For a given $d \in \mathbb{R}^n$, let $\mathcal{S}^{B,d}$ denote the same problem of finding a shortest $\mathbf{0}$-to-$Sb(\text{mod } w)$ path in G, except each arc in E_j is weighted by $r(d)_j$ instead of $r(c)_j$.

Consider any vector $y \in \mathbb{Z}_+^{n-m}$. For problem $\mathcal{S}^B$, consider all paths that start from source vertex $\mathbf{0}$ and are composed of some permutation of exactly y_j arcs from E_j for each $j \in [n - m]$. (E.g., if $y = (1, 2)$, consider the path that traverses one E_1 arc then two E_2 arcs, the path that traverses one E_2 arc then one E_1 arc then one E_2 arc, and the path that traverses two E_2 arcs then one E_1 arc.) Such a path always exists because each vertex is the tail of an E_j arc for each $j \in [n-m]$. Each E_j arc has the same weight $r(c)_j$, so all of these paths have the same weight $r(c)^\intercal y$. All of these paths also have the same destination vertex $(SA_N)y \pmod{w}$. Thus, if we consider all of these paths to be (possibly infeasible) solutions for $\mathcal{S}^B$, then y provides their objective value (path weight $r(c)^\intercal y$) and feasibility (if the destination vertex $(SA_N)y \pmod{w}$ is equal to $Sb (\bmod\ w)$). We therefore represent potential solutions for $\mathcal{S}^B$ as vectors from $\mathbb{Z}_+^{n-m}$, where the vector $y \in \mathbb{Z}_+^{n-m}$ corresponds to a path starting at vertex $\mathbf{0}$ that is composed of some permutation of exactly y_j arcs from E_j.

Lemma 2 formalizes the relationship between $\mathcal{G}^{B,d}$ and $\mathcal{S}^{B,d}$ for a given $d \in \mathbb{R}^n$. The lemma is given by Richard and Dey [29] for $\mathcal{G}^B$ and $\mathcal{S}^B$, and their results hold more generally for $\mathcal{G}^{B,d}$ and $\mathcal{S}^{B,d}$ because their proof does not depend on if B is an optimal/feasible basis of the linear relaxation. Their proof offers the following intuition: x is a feasible solution for $\mathcal{G}^{B,d}$ if and only if x_N is a $\mathbf{0}$-to-$Sb (\bmod\ w)$ path for $\mathcal{S}^{B,d}$ and $x_B = A_B^{-1}b - A_B^{-1}A_N x_N$. The objective value of a solution x for $\mathcal{G}^{B,d}$ differs from the weight of the path x_N for $\mathcal{S}^{B,d}$ by exactly a fixed value: $d^\intercal x = r(d)^\intercal x_N + c_B^\intercal A_B^{-1} d$.

Lemma 2. **[29]** *For a given $d \in \mathbb{R}^n$, we have $x \in Opt(\mathcal{G}^{B,d})$ if and only if $x_N \in Opt(\mathcal{S}^{B,d})$ and $x_B = A_B^{-1}b - A_B^{-1}A_N x_N$.*

This shortest path reformulation $\mathcal{S}^B$ of $\mathcal{G}^B$ will be used to represent the inverse GCR as the inverse of a shortest path problem.

2.3 Inverse Optimization

Let $\mathcal{P}$ be an optimization problem from among $\mathcal{IP}, \mathcal{LP}, \mathcal{G}^B, \mathcal{S}^B$. Let x° be a feasible solution for $\mathcal{P}$. The inverse-feasible region of $\mathcal{P}$ with respect to x°, denoted by $IFR(\mathcal{P}, x^\circ)$, is the set of vectors $d \in \mathbb{R}^n$ for which x° is an optimal solution for $\mathcal{P}^d$:

$$IFR(\mathcal{P}, x^\circ) = \{d \in \mathbb{R}^n \mid x^\circ \in Opt(\mathcal{P}^d)\}.$$

The inverse problem of $\mathcal{P}$ with respect to x°, denoted by $Inv(\mathcal{P}, x^\circ)$, is the problem of finding a vector $d \in IFR(\mathcal{P}, x^\circ)$ that minimizes the (possibly weighted) L_p norm of $d - c$:

$$Inv(\mathcal{P}, x^\circ) : \min\{\|d - c\|_p \mid d \in IFR(\mathcal{P}, x^\circ)\}.$$

We now give a motivating example where the inverse IP is exactly solved by the inverse GCR but is *not* exactly solved by the inverse LP relaxation. We later show that generally, the inverse GCR may be easier to compute than the inverse IP while providing a better approximation of the inverse IP than that of the inverse LP relaxation.

Example 1. Suppose the feasible region of $\mathcal{IP}$ is given by $\{(x_1, x_2) \in \mathbb{Z}^2 \mid x_1+2x_2 = 3,\ x_1, x_2 \geq 0\}$. Let $x^\circ = (1, 1)$, see Figure 1. Since $(3, 0)$ is the only other feasible solution for $\mathcal{IP}$, $IFR(\mathcal{IP}, x^\circ) = \{d \in \mathbb{R}^2 \mid -2d_1 + d_2 \leq 0\}$. The convex hull of the feasible region of $\mathcal{G}^B$, $B = \{2\}$ is the ray with origin x° and direction $(2, -1)$, so $IFR(\mathcal{G}^B, x^\circ) = IFR(\mathcal{IP}, x^\circ)$. $IFR(\mathcal{LP}, x^\circ) = \text{span}\{(1, 2)\}$ by inspection of Figure 1 or direct computation [1, 35]. Thus, $IFR(\mathcal{LP}, x^\circ) \subsetneq IFR(\mathcal{G}^B, x^\circ) = IFR(\mathcal{IP}, x^\circ)$.

Finally, $Inv(\mathcal{LP}, x^\circ) \geq Inv(\mathcal{G}^B, x^\circ) = Inv(\mathcal{IP}, x^\circ)$ because each inverse problem has the same objective function. $Inv(\mathcal{LP}, x^\circ) > Inv(\mathcal{IP}, x^\circ)$ for many given target objective vectors c; for example, if $c \in IFR(\mathcal{IP}, x^\circ) \setminus IFR(\mathcal{LP}, x^\circ)$, then $Inv(\mathcal{LP}, x^\circ) > 0 = Inv(\mathcal{G}^B, x^\circ) = Inv(\mathcal{IP}, x^\circ)$. See, Figure 2.

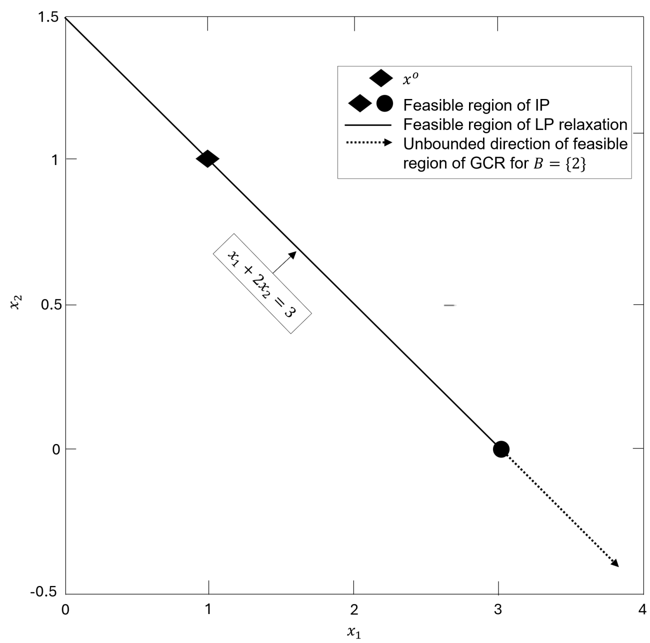

Figure 1: The feasible region of $\mathcal{IP}$ in Example 1 is shown by the filled diamond at $x^\circ = (1, 1)$ and the filled circle at $(3, 0)$. The feasible region of $\mathcal{G}^B$, $B = \{2\}$ is unbounded; two points that are feasible for $\mathcal{G}^B$ but infeasible for $\mathcal{IP}$ are shown by the crosses. The feasible region of $\mathcal{LP}$ is shown by the line segment between $(0, 1.5)$ and $(3, 0)$.

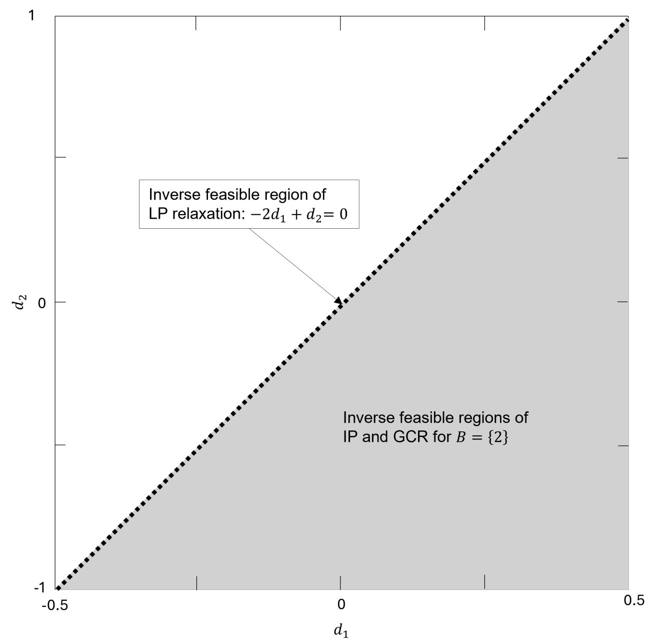

Figure 2: $IFR(\mathcal{G}^B, x^\circ) = IFR(\mathcal{IP}, x^\circ)$ are shown as the shaded region, and $IFR(\mathcal{LP}, x^\circ)$ is shown as the solid line.

3 Characterizing the Inverse of Integer Programs

Let $\mathcal{B}$ be the set of all bases of $\mathcal{LP}$ (both feasible and infeasible), and consider a feasible solution x° for $\mathcal{IP}$. For any $B \in \mathcal{B}$, the intersection of the feasible regions of $\mathcal{LP}$ and $\mathcal{G}^B$ is exactly the feasible region of $\mathcal{IP}$ because the constraints of $\mathcal{LP}$ enforce nonnegativity and the constraints of $\mathcal{G}^B$ enforce integrality. Furthermore, if $\bigcap_{B \in \mathcal{B}} B = \varnothing$, then the intersection of the feasible regions of $\mathcal{G}^B, B \in \mathcal{B}$ is exactly the feasible region of $\mathcal{IP}$ because the constraints of $\mathcal{G}^B, B \in \mathcal{B}$ collectively enforce nonnegativity for all decision variables. Thus, it may be possible to obtain large portions of $IFR(\mathcal{IP}, x^\circ)$ using $IFR(\mathcal{LP}, x^\circ)$ and all $IFR(\mathcal{G}^B, x^\circ), B \in \mathcal{B}$.

Though we focus on the inverse of the GCR and its relationship with the inverse LP relaxation, our findings in this section apply to any IP relaxation, such as the master group relaxation [19, 29] or the Lagrangean relaxation [6].

Lemma 3. *Let x° be a feasible solution for $\mathcal{IP}$, and let $\mathcal{R}$ be a relaxation of $\mathcal{IP}$. Then, $IFR(\mathcal{R}, x^\circ) \subseteq IFR(\mathcal{IP}, x^\circ)$, and $z(Inv(\mathcal{R}, x^\circ)) \geq z(Inv(\mathcal{IP}, x^\circ))$.*

Proof. Consider any $d \in IFR(\mathcal{R}, x^\circ)$, so $x^\circ \in Opt(\mathcal{R}^d)$. Since $\mathcal{R}^d$ is a relaxation of $\mathcal{IP}^d$, $x^\circ \in Opt(\mathcal{IP}^d)$. Thus, $d \in IFR(\mathcal{IP}^d, x^\circ)$ and $IFR(\mathcal{R}, x^\circ) \subseteq IFR(\mathcal{IP}, x^\circ)$. $Inv(\mathcal{R}, x^\circ)$ and $Inv(\mathcal{IP}, x^\circ)$

have the same objective function, so $z(Inv(\mathcal{R}, x^\circ)) \geq z(Inv(\mathcal{IP}, x^\circ))$. $\square$

From Proposition 1, in cases where $\bigcup_{B \in \mathcal{B}} IFR(\mathcal{G}^B, x^\circ) \neq IFR(\mathcal{IP}, x^\circ)$, we may be able to contain more of $IFR(\mathcal{IP}, x^\circ)$ using $\text{cone}(\{IFR(\mathcal{G}^B, x^\circ), B \in \mathcal{B}\})$.

Proposition 1. *Let x° be a feasible solution for $\mathcal{IP}$, and let $\mathcal{R}_1, ..., \mathcal{R}_r$ be relaxations of $\mathcal{IP}$. Then, $\bigcup_{i \in [r]} IFR(\mathcal{R}_i, x^\circ) \subseteq \text{cone}(\{IFR(\mathcal{R}_i, x^\circ), i \in [r]\}) \subseteq IFR(\mathcal{IP}, x^\circ)$.*

Proof. We first show $IFR(\mathcal{IP}, x^\circ)$ is a polyhedral cone. Schaefer [30] proved that $IFR(\mathcal{IP}, x^\circ)$ is a polyhedron. $IFR(\mathcal{IP}, x^\circ)$ is a cone because if $d \in IFR(\mathcal{IP}, x^\circ)$, then $td \in IFR(\mathcal{IP}, x^\circ)$ for any $t \in \mathbb{R}_+$ [6].

Clearly, $\bigcup_{i \in [r]} IFR(\mathcal{R}_i, x^\circ) \subseteq \text{cone}(\{IFR(\mathcal{R}_i, x^\circ), i \in [r]\})$. Next, $IFR(\mathcal{IP}, x^\circ)$ is a polyhedral cone and $IFR(\mathcal{R}_i, x^\circ) \subseteq IFR(\mathcal{IP}, x^\circ)$ for each $i \in [r]$, so $\text{cone}(\{IFR(\mathcal{R}_i, x^\circ), i \in [r]\}) \subseteq \text{cone}(IFR(\mathcal{IP}, x^\circ)) = IFR(\mathcal{IP}, x^\circ)$. $\square$

$IFR(\mathcal{IP}, x^\circ)$ can be fully contained by the inverse-feasible regions of relaxations of $\mathcal{IP}$.

Theorem 1. *Let x° be a feasible solution for $\mathcal{IP}$, and let $\mathcal{R}_1, ..., \mathcal{R}_r$ be relaxations of $\mathcal{IP}$. Suppose for any selection of one feasible solution for each of $\mathcal{R}_i$, $i \in [r]$, there exists a convex combination of those feasible solutions that lies within* $\text{conv}(\{Ax = b, x \geq \mathbf{0}, x \in \mathbb{Z}^n\})$. *Then, $\bigcup_{i \in [r]} IFR(\mathcal{R}_i, x^\circ) = IFR(\mathcal{IP}, x^\circ)$.*

Proof. Lemma 3 implies $\bigcup_{i \in [r]} IFR(\mathcal{R}_i, x^\circ) \subseteq IFR(\mathcal{IP}, x^\circ)$. We show $IFR(\mathcal{IP}, x^\circ) \subseteq \bigcup_{i \in [r]} IFR(\mathcal{R}_i, x^\circ)$. By contradiction, suppose there exists $d \in IFR(\mathcal{IP}, x^\circ) \setminus \bigcup_{i \in [r]} IFR(\mathcal{R}_i, x^\circ)$. Then, for each $i \in [r]$, $d \in IFR(\mathcal{IP}, x^\circ) \setminus IFR(\mathcal{R}_i, x^\circ)$, so there exists a feasible solution y^i for $\mathcal{R}_i$ such that $d^\intercal y^i < d^\intercal x^\circ$. Let w be the convex combination of $y^i, i \in [r]$ where $w \in \text{conv}(\{Ax = b, x \geq \mathbf{0}, x \in \mathbb{Z}^n\})$. Since w is a convex combination of $y^i, i \in [r]$, we have $d^\intercal w < d^\intercal x^\circ$. Also, since $w \in \text{conv}(\{Ax = b, x \geq \mathbf{0}, x \in \mathbb{Z}^n\})$ and $d \in IFR(\mathcal{IP}, x^\circ)$, $d^\intercal w \geq d^\intercal x^\circ$. This contradiction indicates that $IFR(\mathcal{IP}, x^\circ) \subseteq \bigcup_{i \in [r]} IFR(\mathcal{R}_i, x^\circ)$. $\square$

4 Inverse Gomory Corner Relaxation

4.1 Inverse-Feasible Region of Shortest Path Reformulation

This subsection provides a polyhedral representation of $IFR(\mathcal{S}^B, x_N^\circ)$, where x_N° encodes some given $\mathbf{0}$-to-$Sb(\text{mod } w)$ path. Ahuja et al. [1] derived conditions for a given path to be a shortest path, which we apply to x_N° for problem $\mathcal{S}^{B,d}$ to obtain Lemma 4.

Lemma 4. *For a given $d \in \mathbb{R}^n$, x_N° is a shortest $\mathbf{0}$-to-$Sb(\text{mod } w)$ path for problem $\mathcal{S}^{B,d}$ if and only if for each vertex $u \in V$, there exists an associated $y_u \in \mathbb{R}$ such that*

$$y_{\mathbf{0}} = 0, \tag{1a}$$

$$y_{Sb(\text{mod } w)} = r(d)^\intercal x_N^\circ, \tag{1b}$$

$$y_v - y_u \leq r(d)_j, \qquad \forall (u,v) \in E_j,\ \forall j \in [n-m]. \tag{1c}$$

$IFR(\mathcal{S}^B, x_N^\circ)$ is the set of all $d \in \mathbb{R}^n$ such that x_N° is a shortest $\mathbf{0}$-to-$Sb(\text{mod } w)$ path for problem $\mathcal{S}^{B,d}$, so we formulate $IFR(\mathcal{S}^B, x_N^\circ)$ by defining the set of all $d \in \mathbb{R}^n$ that satisfy the conditions in Lemma 4 given by (1a), (1b), (1c).

Proposition 2.

$$\begin{aligned} IFR(\mathcal{S}^B, x_N^\circ) = &\ \{d \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^{|V|} \text{ such that (1a), (1b), (1c)}\} \\ = &\ \text{proj}_{\mathbb{R}^n}\{d \in \mathbb{R}^n, y \in \mathbb{R}^{|V|} \mid \text{(1a), (1b), (1c)}\}, \end{aligned}$$

which is a polyhedral cone that contains $\mathbf{0}$.

Note that $Inv(\mathcal{S}^B, x_N^\circ)$ is a special case of the general inverse shortest path problem (e.g., studied by Zhang and Liu [37] and Ahuja and Orlin [2]) because each arc in E_j has the same weight.

4.2 Feasible Region and Linear Programming Formulation of the Inverse Gomory Corner Relaxation

Theorem 2. *For a given feasible solution x° for $\mathcal{G}^B$, $IFR(\mathcal{S}^B, x_N^\circ) = IFR(\mathcal{G}^B, x^\circ)$.*

Proof. Since x° is a feasible solution for $\mathcal{G}^B$, we have $x^\circ_B = A_B^{-1}b - A_B^{-1}A_N x^\circ_N$. Then, by Lemma 2,

$$\begin{aligned} d \in IFR(\mathcal{G}^B, x^\circ) &\iff x^\circ \in Opt(\mathcal{G}^{B,d}) \\ &\iff x^\circ_N \in Opt(\mathcal{S}^{B,d}) \\ &\iff d \in IFR(\mathcal{S}^B, x^\circ_N). \end{aligned}$$

□

Theorem 2 implies that the inverse GCR is equivalent to the inverse of a shortest path problem, which also implies the GCR inverse-feasible region is a nonempty polyhedral cone by Proposition 2.

Most research on the GCR assumes B is an optimal basis of $\mathcal{LP}$ (e.g., [17, 19, 29]). Proposition 3 addresses this condition.

Proposition 3. *Consider a feasible basis B for $\mathcal{LP}$ and a feasible solution x° for $\mathcal{G}^B$. Then, B is an optimal basis of $\mathcal{LP}^d$ for all $d \in IFR(\mathcal{G}^B, x^\circ)$.*

Proof. By contradiction, suppose there exists $d \in IFR(\mathcal{G}^B, x^\circ)$ such that B is a non-optimal feasible basis of $\mathcal{LP}^d$, and let $N = [n] \setminus B$. Non-optimality implies that at least one of the N-indexed reduced costs of $\mathcal{LP}^d$ must be negative [6]. The arc weights of $\mathcal{S}^B$ are defined by the N-indexed reduced costs, so there exists $j \in [n-m]$ such that the arcs in E_j have negative weight. Since there are finitely many vertices, and each vertex is the tail of an arc from E_j, we can then construct a negative-weight cycle by repeatedly augmenting a path with arcs from E_j until a cycle is formed [29]. The existence of a negative-weight cycle implies x°_N is not a shortest path, so $d \notin IFR(\mathcal{S}^B, x^\circ_N) = IFR(\mathcal{G}^B, x^\circ)$. □

Corollary 1. *Let x° be a feasible solution for $\mathcal{IP}$. Consider a feasible basis B for $\mathcal{LP}$ and the associated basic feasible solution $x^*_B = A_B^{-1}b$, $x^*_N = \mathbf{0}$. Then, $IFR(\mathcal{G}^B, x^\circ) \subseteq IFR(\mathcal{LP}, x^*)$.*

We obtain the following LP formulation for $Inv(\mathcal{G}^B, x^\circ)$ under the L_1 norm. The constraints are derived from Proposition 2, and we linearize the objective function $\min\|d-c\|_1$ by substituting $d := c - e + f$ for $e, f \in \mathbb{R}^n_+$.

Proposition 4. *For a given feasible solution x° for $\mathcal{G}^B$, an optimal solution for $Inv(\mathcal{G}^B, x^\circ)$ under the L_1 norm weighted by a given $w \in \mathbb{R}^n_+$ is equal to $c - e^* + f^*$, where e^*, f^*, y^* is an optimal* *solution for the following LP problem:*

$$\min \sum_{k=1}^{n} w_k(e_k + f_k) \tag{2a}$$

$$\text{s.t.} \quad y_{\mathbf{0}} = 0, \tag{2b}$$

$$y_{Sb(\text{mod } w)} = (r(c) - r(e) + r(f))^\intercal x^\circ_N, \tag{2c}$$

$$y_v - y_u \leq r(c)_j - r(e)_j + r(f)_j, \forall (u,v) \in E_j, \ \forall j \in [n-m], \tag{2d}$$

$$e, f \in \mathbb{R}^n_+, \ y \in \mathbb{R}^{|V|}. \tag{2e}$$

The LP in (2) can be modified to solve $Inv(\mathcal{G}^B, x^\circ)$ under the L_∞ norm [2]. The LP formulation for $Inv(\mathcal{G}^B, x^\circ)$ can be quite large depending on $\det A_B$.

5 Comparing Inverse Formulations

5.1 Comparison with Inverse Linear Programming Relaxation

Theorems 3 and 4 show how the GCR inverse-feasible regions may contain as much of the IP inverse-feasible region as the LP relaxation inverse-feasible region. We compare the optimal values of the inverse IP, inverse GCR, and inverse LP relaxation.

Theorem 3. *Let x° be a feasible solution for $\mathcal{IP}$ that is a basic feasible solution for $\mathcal{LP}$. Let $\bar{\mathcal{B}} \subseteq \mathcal{B}$ be the set of feasible bases B of $\mathcal{LP}$ that satisfy $x^\circ_B = A_B^{-1} b, x^\circ_N = \mathbf{0}$. Then, $IFR(\mathcal{LP}, x^\circ) = \bigcup_{B \in \bar{\mathcal{B}}} IFR(\mathcal{G}^B, x^\circ)$.*

Proof. By Corollary 1, $\bigcup_{B \in \bar{\mathcal{B}}} IFR(\mathcal{G}^B, x^\circ) \subseteq IFR(\mathcal{LP}, x^\circ)$. To prove $IFR(\mathcal{LP}, x^\circ) \subseteq \bigcup_{B \in \bar{\mathcal{B}}} IFR(\mathcal{G}^B, x^\circ)$, consider any $d \in IFR(\mathcal{LP}, x^\circ)$. Then, x° is an optimal solution for $\mathcal{LP}^d$, so there exists $B \in \bar{\mathcal{B}}$ such that the reduced costs of the N-indexed variables are nonnegative for $\mathcal{LP}^d$ [29]. Since x° is feasible for $\mathcal{IP}^d$, x° is also feasible for $\mathcal{G}^{B,d}$, and based on Theorem 2, $x^\circ_N = \mathbf{0}$ is feasible for $\mathcal{S}^{B,d}$. Thus, the source and destination vertices are the same in $\mathcal{S}^{B,d}$, and since the arc weights are defined by the reduced costs of the N-indexed variables for $\mathcal{LP}^d$, the arc weights are then nonnegative. Therefore, $x^\circ_N = \mathbf{0}$ is an optimal solution for $\mathcal{S}^{B,d}$. Hence, x° is an optimal solution for $\mathcal{G}^{B,d}$, and thus $d \in IFR(\mathcal{G}^B, x^\circ)$. □

Theorem 4. *Let x° be a feasible solution for $\mathcal{IP}$ that is not a basic feasible solution for $\mathcal{LP}$. Let $K := \{k \in [n] \mid x^\circ_k > 0\}$.*

(a) For any basis B of $\mathcal{LP}$ where $B \subseteq K$, we have $IFR(\mathcal{LP}, x^\circ) \subseteq IFR(\mathcal{G}^B, x^\circ)$.

(b) In the absence of degeneracy, there always exists a feasible basis B such that $B \subseteq K$.

Proof. Let B be a basis of $\mathcal{LP}$ where $B \subseteq K$. Consider any $d \in \mathbb{R}^n$ such that $d \notin IFR(\mathcal{G}^B, x^\circ)$. To prove *(a)*, we will show $IFR(\mathcal{LP}, x^\circ) \subseteq IFR(\mathcal{G}^B, x^\circ)$ by proving that $d \notin IFR(\mathcal{LP}, x^\circ)$, or equivalently, that x° is not an optimal solution for $\mathcal{LP}^d$.

x° is not an optimal solution for $\mathcal{G}^{B,d}$, so there exists a feasible solution y for $\mathcal{G}^{B,d}$ such that $d^\intercal y < d^\intercal x^\circ$. We consider two cases.

Case 1. Suppose $y \geq \mathbf{0}$. y is a feasible solution for $\mathcal{G}^B$, so $Ay = b$, and therefore y is a feasible solution for $\mathcal{LP}^d$. x° is not optimal for $\mathcal{LP}^d$.

Case 2. Suppose there exist some $k \in B$ such that $y_k < 0$. Let $C := \{k \in B \mid y_k < 0\}$ denote the indices of the negative components of y. We construct w that is a convex combination of x° and y. Let $\lambda := \max\left\{\frac{-y_k}{x^\circ_k - y_k} \mid k \in C\right\}$. For each $k \in C$, $x^\circ_k > 0 > y_k$, so $0 < \frac{-y_k}{x^\circ_k - y_k} < 1$, which implies $\lambda \in (0,1)$. Let $w := \lambda x^\circ + (1-\lambda)y$. Then, for each $k \in C$,

$$w_k = y_k + \lambda(x^\circ_k - y_k) \geq y_k + \frac{-y_k}{x^\circ_k - y_k}(x^\circ_k - y_k) = 0.$$

Also, for each $k \in [n] \setminus C$, we have $x^\circ_k, y_k \geq 0$, so $w_k \geq 0$. Therefore, $w \geq \mathbf{0}$. Furthermore,

$$d^\intercal w = \lambda d^\intercal x^\circ + (1-\lambda) d^\intercal y < \lambda d^\intercal x^\circ + (1-\lambda) d^\intercal x^\circ = d^\intercal x^\circ, \tag{3}$$

and $Aw = A(\lambda x^\circ + (1-\lambda)y) = \lambda A x^\circ + (1-\lambda)Ay = b$, where the last equality holds because both x° and y are feasible solutions for $\mathcal{G}^{B,d}$. Thus, w is a feasible solution for $\mathcal{LP}$, and, by (3), x° is not an optimal solution for $\mathcal{LP}^d$.

To prove *(b)*, define $\delta \in \mathbb{R}^n$ by $\delta_k = 0$ for $k \in K$, and $\delta_k = 1$ for $k \in [n] \setminus K$. Then, $0 \leq z(\mathcal{LP}^\delta) \leq \delta^\intercal x^\circ = 0$. Thus, $\mathcal{LP}^\delta$ has an optimal basis B^* associated with the optimal basic solution x^* given by $x^*_{B^*} = A^{-1}_{B^*} b$, $x^*_{N^*} = \mathbf{0}$, where $N^* = [n] \setminus B^*$. Clearly, $\delta^\intercal x^* = 0$. By contradiction, suppose $B^* \not\subseteq K$. Then, there exists $k \in B^* \setminus K$. $\delta_k = 1$ because $k \notin K$, and $x^*_k > 0$ because $k \in B^*$, assuming nondegeneracy. We reach the contradiction $0 < \delta_k x^*_k \leq \delta^\intercal x^* = 0$. □

Corollary 2. *Let x° be a feasible solution for $\mathcal{IP}$ that is an interior point of $\mathcal{LP}$. Then, $IFR(\mathcal{LP}, x^\circ) \subseteq IFR(\mathcal{G}^B, x^\circ)$ for all bases B of $\mathcal{LP}$.*

Theorem 5. *Let x° be a feasible solution for $\mathcal{IP}$. In the absence of degeneracy, there exists a feasible basis B of $\mathcal{LP}$ such that*

(a) $IFR(\mathcal{LP}, x^\circ) \subseteq IFR(\mathcal{G}^B, x^\circ) \subseteq IFR(\mathcal{IP}, x^\circ)$.

(b) $z(Inv(\mathcal{LP}, x^\circ)) \geq z(Inv(\mathcal{G}^B, x^\circ)) \geq z(Inv(\mathcal{IP}, x^\circ))$.

Proof. If x° is a basic feasible solution for $\mathcal{LP}$, then there is exactly one feasible basis B such that $x^\circ_B = A_B^{-1}b$, $x^\circ_N = \mathbf{0}$. Theorem 3 implies $IFR(\mathcal{LP}, x^\circ) \subseteq IFR(\mathcal{G}^B, x^\circ)$.

If x° is not a basic feasible solution for $\mathcal{LP}$, then Theorem 4 implies that there exists a feasible basis B such that $IFR(\mathcal{LP}, x^\circ) \subseteq IFR(\mathcal{G}^B, x^\circ)$.

Lemma 3 implies $IFR(\mathcal{G}^B, x^\circ) \subseteq IFR(\mathcal{IP}, x^\circ)$, proving *(a)*.

Since all of three inverse problems have the same objective function, part *(a)* implies the bounds on the objective values of the inverse problems in part *(b)*. □

5.2 Comparison with Exact Inverse Integer Programming Formulation

Schaefer [30] obtained an exact LP formulation for inverse IPs using superadditive duality, albeit of enormous size. This introduces the question of whether our LP formulation for the inverse GCR in (2) is smaller than solving the inverse GCR as an inverse IP.

We compared the number of variables and constraints in our LP formulation for the inverse GCR in (2) against the number of variables and constraints in Schaefer's [30] LP formulation for the inverse IP interpretation of the inverse GCR under the L_1 norm. Table 1 summarizes this comparison for each of five pure IP instances obtained from MIPLIB 2017 [18]. For each instance, B was set to an optimal basis of the LP relaxation, computed using Gurobi 10.0.2 [21]. Our LP formulation has $2n + |\det A_B|$ variables and $2 + (n - m)|\det A_B|$ constraints. Schaefer's [30] LP formulation has $2n + \prod_{i \in [m]} (|b_i| + 1)$ variables and $3 + n + 2\left(\prod_{i \in [m]} \frac{(|b_i|+1)(|b_i|+2)}{2} - \prod_{i \in [m]} (|b_i| + 1)\right)$ constraints.

Our formulation has many magnitudes fewer variables and constraints when compared to Schaefer's [30] formulation. We conclude that our formulation, which exploits specific GCR properties,

Table 1: Comparison of the number of variables and constraints in our LP formulation (2) with Schaefer's [30] LP formulation for the inverse IP interpretation of the inverse GCR under the L_1 norm. IP instances were obtained from MIPLIB 2017 [18]. **Size of IP Instance** lists the number of variables and constraints in the IP instance after converting it to $Ax = b, x \in \mathbb{Z}^n_+$ form. **Inv GCR** and **Inv IP** list the $\log_{10}$ of the number of variables and constraints in our formulation and Schaefer's [30], respectively.

Size of IP Instance			**Inv GCR**		**Inv IP**	
Name	var	con	$\log_{10}$ var	$\log_{10}$ con	$\log_{10}$ var	$\log_{10}$ con
gen-ip016	52	24	2.9	4.3	105.8	197.6
gen-ip054	57	27	11.6	13.0	77.6	141.0
gen-ip002	65	24	20.1	21.7	103.1	192.2
gen-ip021	63	28	10.1	11.7	104.6	193.0
ns1952667	13264	41	32.8	36.9	244.5	464.7

yields smaller LP formulations than can be found by solving the inverse GCR as an inverse IP. However, Schaefer's [30] formulation exactly solves inverse IPs, where our approach only solves the inverse of a relaxation.

6 Conclusion

The inverse GCR can be formulated as the inverse of a shortest path problem. We obtained a polyhedral representation of the inverse-feasible region of the GCR, and we proposed an LP formulation for the inverse GCR under the L_1 and L_∞ norms. A GCR inverse-feasible region contains as much of the IP inverse-feasible region as is contained by the LP relaxation inverse-feasible region, in the absence of LP degeneracy. Our formulation of the inverse GCR is much smaller than the exact inverse IP formulation proposed by Schaefer [30].

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