Toll Setting with Robust Wardrop Equilibrium Conditions Under Budgeted Uncertainty

Yasmine Beck, Martine Labbé, Martin Schmidt

Abstract. We consider two variants of the toll-setting problem in which a traffic authority uses tolls either to maximize revenue or to alleviate bottlenecks in the traffic network. The users of the network are assumed to act according to Wardrop's user equilibrium so that the overall toll-setting problems are modeled as mathematical problems with equilibrium constraints. We present nonconvex mixed-integer nonlinear reformulations that exploit binary variables and big-M constants for these problems, derive valid big-Ms, prove the existence of optimal solutions, and provide valid inequalities. Moreover, we consider the setting in which the network users hedge against uncertainties in the travel costs. We model this setting using robust Wardrop equilibria under budgeted uncertainty and prove existence of robust solutions. Finally, we present a computational case study to illustrate the effects of considering robustified travel decisions.

1. Introduction

In traffic networks, collecting tolls is a powerful tool for network management and influencing travel behavior. For instance, revenues generated by imposing tolls may support the maintenance of existing infrastructure or fund the construction of new roads. In addition, tolls may be used to manage traffic flow by alleviating congestion and encouraging a more efficient use of road capacity. Hence, determining optimal tolls in a traffic network is an important aspect of transportation science. In this context, the traffic authority has to decide on the tolls while anticipating the reaction of the users of the traffic network, who typically try to minimize costs and time. The overall toll-setting problem can thus be seen as a single-leader multi-follower game in which the traffic authority acts as the leader and the users of the traffic network act as the followers. Influential works in this context include, e.g., Brotcorne et al. (2001), Dempe and Zemkoho (2012), Dewez et al. (2008), Kalashnikov et al. (2020), and Labbé et al. (1998, 2000).

In this paper, we consider a multi-commodity traffic network in which a traffic authority decides on the tolls for (some of) the commodities on (some of) the arcs of the network. We explicitly account for different travel costs and tolls for each commodity, which allows us to represent various vehicle types—such as heavy-duty vehicles, private cars, or motorcycles—that may incur different cost structures due to technical properties, driver behavior, or regulatory reasons. As collecting tolls can serve multiple purposes, we consider two exemplary variants of the toll-setting problem that reflect different objectives of the traffic authority. First, we consider the setting in which tolls are used to maximize the revenue generated by imposing tolls. Second, we study the setting in which the traffic authority uses tolls to encourage a more efficient use of road capacity by alleviating bottlenecks in the network. Regarding the users of the traffic network, we assume that they act

Date: February 10, 2026.

2020 *Mathematics Subject Classification.* 90C26, 90C33, 90C35, 90C70.

Key words and phrases. Transportation networks, Pricing, Wardrop equilibrium, Mixed-integer nonlinear optimization, Robust optimization.

according to Wardrop's user equilibrium (Wardrop 1952; Wardrop and Whitehead 1952), minimizing their individual travel costs that are parameterized by the imposed tolls. The overall toll-setting problems are thus modeled as mathematical problems with equilibrium constraints (MPECs); see, e.g., Luo et al. (1996) for a general overview.

To the best of our knowledge, Dempe and Zemkoho (2012) is the closest related work to ours. Therein, similar toll-setting problems with separable travel cost functions are studied from a theoretical point of view. In contrast, we do not assume separability and approach the toll-setting problems rather from a computational perspective. To this end, we reformulate them as nonconvex mixed-integer nonlinear problems (MINLPs) using binary variables and big-M constants, which can be tackled by state-of-the-art general-purpose solvers. In addition, we provide results on the existence of optimal solutions to these problems, present valid inequalities to enhance their formulations, and derive valid big-M constants. Beyond Dempe and Zemkoho (2012), we are not aware of any other work that addresses toll-setting problems in which the travelers' behavior is modeled using Wardrop user equilibria. Hence, this paper provides a first step towards solving such problems to global optimality and motivates future research on tailored solution methods.

Building on the deterministic framework, we further extend our analysis to account for uncertainties in the travel costs, which may arise, e.g., from accidents, maintenance work, or changing weather conditions. To address these uncertainties, we use techniques from robust optimization (Ben-Tal et al. 2009; Bertsimas et al. 2011; Soyster 1973). To avoid overly conservative solutions, which is a common criticism of classic robust approaches, we adopt a budgeted uncertainty (or Γ-robust) modeling (Bertsimas and Sim 2003, 2004; Sim 2004), which has also been applied in other transportation contexts such as network design (Mattia 2019; Mattia and Poss 2018) and traffic assignment (Ito 2011; Ordóñez and Stier-Moses 2007, 2010). In particular, we pursue similar ideas compared to those in Ito (2011) and Ordóñez and Stier-Moses (2007, 2010), who also study robust Wardrop equilibria. In Ito (2011), a strictly robust setup with ellipsoidal uncertainty sets is considered to hedge against uncertainties in the travel costs. The author makes the necessary continuity assumptions on the robustified travel cost functions to ensure that robust Wardrop equilibria exist. Ordóñez and Stier-Moses (2007) propose a Γ-robust approach to hedge against uncertain travel costs, provide existence results for robust Wardrop equilibria, and present a column-generation algorithm to compute them. In a follow-up paper, Ordóñez and Stier-Moses (2010) extend their work with additional equilibrium concepts—the percentile equilibrium and the added-variability equilibrium—along with more theoretical and computational details. We point out that Ito (2011) and Ordóñez and Stier-Moses (2007, 2010) focus on the robust traffic assignment problem in which a path-based formulation is used to model the travelers' behavior. Our framework differs from these approaches in two aspects. First, we study the problem of determining optimal tolls in a traffic network that incorporates robust Wardrop equilibria in the constraints. Second, we study robust Wardrop equilibria under budgeted uncertainty that are based on a node-arc formulation, which is the more natural model for toll-setting problems. To the best of our knowledge, no other works in the literature consider such a robust network pricing model. To illustrate the effects of considering robust travel decisions, we conduct a case study based on the well-known Sioux Falls network (LeBlanc et al. 1975). Our computational results show that accounting for uncertainties in the travel costs can significantly affect travel behavior, network congestion, and revenues from tolls. In particular, we observe that robustifying travel costs can lead to substantially higher toll revenues.

The remainder of this paper is organized as follows. In Section 2, we introduce two variants of the toll-setting problem, which we model as MPECs. In Section 3, we present MINLP reformulations of these problems, derive valid big-Ms, prove existence of optimal solutions, and provide valid inequalities. In Section 4, we consider robustified variants of the toll-setting problems under budgeted uncertainty. We also present MINLP reformulations for the robustified toll-setting MPECs, derive valid big-Ms, prove existence of optimal solutions, and provide valid inequalities. In Section 5, we conduct a computational study to illustrate the effects of considering robust travel decisions. Finally, we conclude in Section 6.

2. Problem Statement

We consider a multi-commodity traffic network, which we model using a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ with node set $\mathcal{N}$ and arc set $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$. Node subsets $\mathcal{O} \subseteq \mathcal{N}$ and $\mathcal{D} \subseteq \mathcal{N}$ denote the sets of origin and destination nodes, respectively, and we refer to the set of all origin-destination (OD) pairs of the network as $\mathcal{K} \subseteq \mathcal{O} \times \mathcal{D}$. For the ease of presentation, we consider a single commodity for each OD pair $k \in \mathcal{K}$. Moreover, we make the following assumption about the network's connectivity. Possible relaxations of this assumption are discussed later in this paper.

Assumption 1. *For every node $n \in \mathcal{N}$, there is at least one path that connects node n to each destination node $\omega \in \mathcal{D}$.*

In this paper, we study the problem in which a traffic authority decides on the tolls $\tau = (\tau^k)_{k\in\mathcal{K}}$ with $\tau^k = (\tau_a^k)_{a\in\mathcal{A}} \in \mathbb{R}^{|\mathcal{A}|}$ that are imposed for (some of) the commodities on (some of) the arcs of the traffic network. Users of the network are assumed to act according to Wardrop's user equilibrium (Wardrop 1952; Wardrop and Whitehead 1952), minimizing their individual travel costs that are parameterized by the imposed tolls. In Section 2.1, we formally state the Wardrop equilibrium conditions to model the travelers' behavior. Afterward, in Section 2.2, we elaborate on feasible tolls and present the overall toll-setting problems.

2.1. Wardrop Equilibrium Conditions.

Let $x^k = (x_a^k)_{a\in\mathcal{A}} \in \mathbb{R}^{|\mathcal{A}|}$ denote the flow vector of commodity $k \in \mathcal{K}$. The vector of overall arc flows is then given by

$$f = \sum_{k\in\mathcal{K}} x^k \in \mathbb{R}^{|\mathcal{A}|}. \tag{1}$$

Throughout this paper, we make the following assumption.

Assumption 2. *For every $k \in \mathcal{K}$, the travel demand $d_k \in \mathbb{R}$ is positive and fixed.*

Assumption 2 is w.l.o.g. because any elastic-demand problem can equivalently be reformulated as a fixed-demand problem; see, e.g., Dantzig et al. (1976) and Gartner (1980). For each commodity $k = (\alpha_k, \omega_k) \in \mathcal{K}$, flow conservation is modeled via

$$\sum_{a\in\delta^{\text{in}}(n)} x_a^k - \sum_{a\in\delta^{\text{out}}(n)} x_a^k = d_n^k, \quad n \in \mathcal{N}, \tag{2}$$

with

$$d_n^k = \begin{cases} +d_k, & n = \omega_k, \\ 0, & n \in \mathcal{N} \setminus \{\alpha_k, \omega_k\}, \\ -d_k, & n = \alpha_k. \end{cases}$$

Here, $\delta^{\text{in}}(n)$ and $\delta^{\text{out}}(n)$ denote the sets of in- and outgoing arcs of node $n \in \mathcal{N}$, respectively. Let us now elaborate on Wardrop's second principle to model user-optimized behavior. Note that the user optimum is commonly referred to as Wardrop's second principle in the literature, despite it being introduced first in Wardrop and Whitehead (1952); see, e.g., the respective discussion in Ferris and

Pang (1997). Following Wardrop's second principle, we assume that the users of the traffic network aim to minimize their individual travel costs so that no commodity can reduce costs by unilaterally changing routes. This behavior can be modeled as

$$0 \le c_a^k(f; \tau_a^k) + t_j^k - t_i^k \perp x_a^k \ge 0, \quad a = (i, j) \in \mathcal{A},\ k \in \mathcal{K}. \tag{3}$$

In (3), the cost for commodity $k \in \mathcal{K}$ to travel along an arc $a \in \mathcal{A}$ is given by the function $c_a^k(f; \tau_a^k)$ that depends on the overall arc flows f and that is parameterized by the toll τ_a^k imposed for this commodity on that arc. Moreover, we use t_n^k to denote the minimum cost for commodity $k \in \mathcal{K}$ to reach its destination from node $n \in \mathcal{N}$. In what follows, we abbreviate $t = (t^k)_{k \in \mathcal{K}}$ and $t^k = (t_n^k)_{n \in \mathcal{N}} \in \mathbb{R}^{|\mathcal{N}|}$. Overall, the set of τ-parameterized Wardrop equilibria is given by

$$S(\tau) := \{(f, x) \colon \exists t \text{ such that } (f, x, t) \text{ solves (1)–(3)}\}.$$

For the remainder of this paper, we make the following assumptions about the properties of the travel cost functions.

Assumption 3. *For every commodity $k \in \mathcal{K}$ and every arc $a \in \mathcal{A}$, the travel cost function $c_a^k : \mathbb{R}^{|\mathcal{A}|} \times \mathbb{R}^{|\mathcal{A}| \cdot |\mathcal{K}|} \to \mathbb{R}_{>0}$, $(f, \tau) \mapsto c_a^k(f; \tau_a^k)$ is positive and continuous in f and τ.*

The latter is a standard assumption and satisfied by many travel cost functions studied in the literature such as, e.g., the BPR function (U.S. Bureau of Public Roads 1964); see also Section 1.5 in Patriksson (2015) for further discussions. Moreover, note that we do not make any assumptions about the separability of the travel cost functions. In traffic assignment problems, it is often interesting to consider travel costs that are non-separable. This means that the costs $c_a^k(f; \tau_a^k)$ for commodity $k \in \mathcal{K}$ may not only depend on the flow f_a on arc $a \in \mathcal{A}$ itself, but also on the flows $f_{a'}$, $a' \neq a \in \mathcal{A}$, on other arcs. For further discussions of non-separable travel costs, we refer to Dafermos (1971). Finally, we mention that Assumption 3 explicitly accounts for different travel costs and tolls for each commodity. This feature of our model allows to represent different vehicle types, which may incur different cost structures due to technical properties, driver behavior, or regulatory reasons. Nevertheless, our model naturally covers the setting in which all commodities face the same travel costs and tolls. A similar setting is, e.g., considered in Section 3.6.2 in Ferris and Pang (1997).

2.2. **The Toll-Setting MPEC.** We now elaborate on feasible toll-setting policies and the overall toll-setting problem. In practice, tolls are often subject to regulatory constraints. For instance, the European Commission sets guidelines on toll caps and toll exemptions to ensure fairness and accessibility (European Commission 2025). To capture such restrictions, we make the following assumption.

Assumption 4. *For each commodity $k \in \mathcal{K}$, the tolls τ^k are subject to constraints described by a polytope $\mathcal{T}^k = \{\tau^k \in \mathbb{R}^{|\mathcal{A}|} \colon B^k \tau^k \le b^k\} \neq \emptyset$ for some matrix B^k and a vector b^k of appropriate dimension.*

The sets $\mathcal{T}^k$, $k \in \mathcal{K}$, model, e.g., lower and upper bounds on the tolls or toll-free arcs. Here and in what follows, we assume that the set $\mathcal{T}^k$ imposes a lower bound of zero and a finite upper bound of $\bar{\tau}_a^k$ on the toll τ_a^k for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$. Within these bounds, collecting tolls can serve multiple purposes. For instance, revenues from tolls may support the maintenance of existing infrastructure. In addition, tolls may be used to encourage a more efficient use of road capacity. Reflecting these two objectives, we consider two variants of the toll-setting problem.

Revenue Maximization. First, we study the situation in which the traffic authority aims to maximize the revenue
$$\sum_{a \in \mathcal{A}} \sum_{k \in \mathcal{K}} \tau_a^k x_a^k$$
that is realized by charging tolls on some arcs of the network. The overall toll-setting problem under revenue maximization can thus be stated as
$$\max_{\tau, f, x} \quad \sum_{a \in \mathcal{A}} \sum_{k \in \mathcal{K}} \tau_a^k x_a^k \tag{4a}$$
$$\text{s.t.} \quad \tau^k \in \mathcal{T}^k, \, k \in \mathcal{K}, \tag{4b}$$
$$(f, x) \in S(\tau). \tag{4c}$$
Note that Problem (4) is a nonlinear and nonconvex optimization problem due to the bilinear terms in the revenue maximization objective and the complementarity constraints in the set $S(\tau)$.

Bottleneck Minimization. Second, we consider the setting in which the traffic authority uses tolls to encourage a more efficient use of road capacity by alleviating bottlenecks in the network. For this purpose, we introduce the so-called practical capacity $0 < u_a < \infty$ of an arc $a \in \mathcal{A}$, which represents the maximum flow that the arc can accommodate under ideal operating conditions without causing congestion. Practical capacities commonly reflect physical or operational limitations such as road geometry, signal timing, and speed regulations; see, e.g., Section 1.5 in Patriksson (2015). To capture capacity utilization, we consider congestion as the ratio of arc flow to practical capacity. In particular, we focus on the setting in which the traffic authority aims to minimize the worst-case congestion across the network modeled as
$$\max_{a \in \mathcal{A}} \frac{f_a}{u_a}.$$
The overall toll-setting problem under bottleneck minimization thus reads
$$\min_{\tau, f, x} \quad \max_{a \in \mathcal{A}} \frac{f_a}{u_a} \tag{5a}$$
$$\text{s.t.} \quad \tau^k \in \mathcal{T}^k, \quad k \in \mathcal{K}, \tag{5b}$$
$$(f, x) \in S(\tau). \tag{5c}$$
Although this objective also captures a relevant traffic-management goal, we primarily adopt it for tractability reasons. Specifically, by using an epigraph formulation, the bottleneck objective can be linearized, which makes the overall toll-setting problem more amenable to general-purpose solvers.

Both toll-setting problems (4) and (5) are MPECs, which can be interpreted as single-leader multi-follower games in which the traffic authority acts as the leader and the users of the traffic network act as the followers. By optimizing over the tolls τ and the flow variables f and x, we adopt the so-called optimistic approach as it is known in this context; see, e.g., Dempe (2002). This means that, whenever multiple optimal route choices exist, users select those that favor the leader the most w.r.t. the objective function value. The latter is a common assumption in the literature; see, e.g., Brotcorne et al. (2011, 2001), Bui et al. (2022), Dempe and Zemkoho (2012), Didi-Biha et al. (2006), and Labbé et al. (1998).

3. MINLP Reformulations

The complementarity constraints (3) used to model Wardrop's user equilibrium render (4) and (5) nonlinear and nonconvex optimization problems, which makes them inherently challenging to solve. However, because of the disjunctive nature of these constraints, we can exploit techniques from mixed-integer optimization to obtain reformulations that may be easier to handle in practice. To this end, we introduce additional binary variables $z_a^k \in \{0,1\}$ and sufficiently large big-M constants M_a^k for $a \in \mathcal{A}$ and all $k \in \mathcal{K}$ so that we can re-write (3) as

$$0 \le x_a^k \le M_a^k z_a^k, \qquad a \in \mathcal{A},\ k \in \mathcal{K}, \tag{6a}$$

$$0 \le c_a^k(f;\tau_a^k) + t_j^k - t_i^k \le M_a^k(1 - z_a^k), \qquad a = (i,j) \in \mathcal{A},\ k \in \mathcal{K}. \tag{6b}$$

Using the constraints in (6), we can now re-state the toll-setting problems (4) and (5). For the case in which the traffic authority aims to maximize revenues, we obtain the problem

$$\max_{\tau,f,x,t,z} \quad \sum_{a\in\mathcal{A}} \sum_{k\in\mathcal{K}} \tau_a^k x_a^k \tag{7a}$$

$$\text{s.t.} \quad (1), (2), (6), \tag{7b}$$

$$\tau^k \in \mathcal{T}^k, \quad k \in \mathcal{K}, \tag{7c}$$

$$z_a^k \in \{0,1\}, \quad a \in \mathcal{A},\ k \in \mathcal{K}, \tag{7d}$$

whereas, for the bottleneck minimization case replacing (3) with (6), yields

$$\min_{\tau,f,x,t,z} \quad \max_{a\in\mathcal{A}} \frac{f_a}{u_a} \tag{8a}$$

$$\text{s.t.} \quad (1), (2), (6), \tag{8b}$$

$$\tau^k \in \mathcal{T}^k, \quad k \in \mathcal{K}, \tag{8c}$$

$$z_a^k \in \{0,1\}, \quad a \in \mathcal{A},\ k \in \mathcal{K}. \tag{8d}$$

Problems (7) and (8) are nonconvex mixed-integer nonlinear problems due to their nonlinear and nonconvex objectives and possible nonlinearities in the travel cost functions $c_a^k(f;\tau_a^k)$, $a \in \mathcal{A}$, $k \in \mathcal{K}$. By construction, both problems are equivalent to their respective original MPEC formulations if the big-M constants M_a^k, $a \in \mathcal{A}$, $k \in \mathcal{K}$, are chosen sufficiently large. Although the resulting MINLPs remain nonconvex, we emphasize that the bilinear terms in Problem (7) can still be tackled using state-of-the-art general-purpose solvers. In contrast, however, the bottleneck minimization objective is not directly supported by such solvers. To address this, we further reformulate the problem using an epigraph formulation, which reads

$$\min_{\tau,f,x,t,z,\eta} \quad \eta \tag{9a}$$

$$\text{s.t.} \quad \eta \ge \frac{f_a}{u_a}, \quad a \in \mathcal{A}, \tag{9b}$$

$$(1), (2), (6), \tag{9c}$$

$$\tau^k \in \mathcal{T}^k, \quad k \in \mathcal{K}, \tag{9d}$$

$$\eta \in \mathbb{R},\ z_a^k \in \{0,1\}, \quad a \in \mathcal{A},\ k \in \mathcal{K}. \tag{9e}$$

Note that Problem (9) is an MINLP in which the only nonlinearities stem from the potentially nonlinear travel cost functions.

The remainder of this section is now organized as follows. In Section 3.1, we elaborate on how to obtain sufficiently large big-M constants that can be used in (6). Afterward, in Section 3.2, we prove the existence of optimal solutions to the two considered toll-setting problems under revenue maximization and bottleneck minimization. Finally, in Section 3.3, we derive valid inequalities to obtain stronger formulations for these problems.

3.1. **Computing Big-Ms.** In what follows, we provide bounds for the flow variables f and x, as well as for the minimum travel costs t. Having such bounds at hand is essential for deriving sufficiently large big-M constants that can be used in (6). For this purpose, we first establish the existence of a Wardrop equilibrium for any given tolls $\tau = (\tau^k)_{k \in \mathcal{K}}$ with $\tau^k \in \mathcal{T}^k$ for all $k \in \mathcal{K}$.

Lemma 1. *Suppose that Assumptions 1–3 hold and let $\tau = (\tau^k)_{k \in \mathcal{K}}$ with $\tau^k \in \mathcal{T}^k$ for all $k \in \mathcal{K}$ be given arbitrarily. Then, there exists a Wardrop equilibrium for the given tolls τ, i.e., $S(\tau) \neq \emptyset$ holds.*

Proof. By Assumption 1, there exists at least one path that connects α_k and ω_k for each commodity $k = (\alpha_k, \omega_k) \in \mathcal{K}$. Moreover, by Assumption 2, the travel demand is fixed, positive, and bounded from above. In addition, the travel cost functions $c_a^k(f; \tau_a^k)$ are positive and continuous for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$ by Assumption 3. Under Assumptions 1–3, we can thus apply Theorem 5.5 in Aashtiani and Magnanti (1981), which yields the existence of a Wardrop equilibrium in the path formulation. As a consequence, there also exists a Wardrop equilibrium in the node-arc formulation; see, e.g., the discussion in Section 2.2.2 in Patriksson (2015) for further details. $\square$

Next, we provide bounds for the commodity flow variables x.

Proposition 1. *Suppose that Assumptions 1–3 hold and let $\tau = (\tau^k)_{k \in \mathcal{K}}$ with $\tau^k \in \mathcal{T}^k$ for all $k \in \mathcal{K}$ be given arbitrarily. Then, for every $(f, x) \in S(\tau)$, it holds*

$$0 \leq x_a^k \leq d_k, \quad a \in \mathcal{A}, \; k \in \mathcal{K}.$$

Moreover, $x_a^k = 0$ holds for all $a \in \delta^{in}(\alpha_k) \cup \delta^{out}(\omega_k)$ with $k = (\alpha_k, \omega_k) \in \mathcal{K}$.

Proof. By Lemma 1, there exists a Wardrop equilibrium $(f, x) \in S(\tau)$ for the given tolls τ. Hence, the non-negativity of x immediately follows from (3). We now prove the upper bound for the commodity flow variables. To this end, let $k = (\alpha_k, \omega_k) \in \mathcal{K}$ be given arbitrarily. The classic flow decomposition theorem yields

$$x_a^k = \sum_{\{p \in \mathcal{P}^k : \, a \in p\}} h_p^k + \sum_{\{\ell \in \mathcal{C} : \, a \in \ell\}} g_\ell^k, \quad a \in \mathcal{A};$$

see, e.g., Theorem 3.5 in Ahuja et al. (1993). Here, $\mathcal{P}^k$ denotes the set of all simple paths between the origin and the destination of commodity k and $\mathcal{C}$ denotes the set of all cycles in the traffic network. The vectors $h^k = (h_p^k)_{p \in \mathcal{P}^k}$ and $g^k = (g_\ell^k)_{\ell \in \mathcal{C}}$ are used for the path and cycle flows, respectively. We now show that, under Assumption 3, there cannot be a cycle with positive flow. We prove this by contradiction. To this end, suppose that there is a cycle $\ell \in \mathcal{C}$ with positive flow, i.e., $g_\ell^k > 0$ holds. This implies that $x_{a'}^k > 0$ holds for all $a' \in \ell$. Summing over all arcs in the cycle, i.e., summing over all $a' \in \ell$, Wardrop's second principle (3) yields

$$\sum_{a' \in \ell} c_{a'}^k(f; \tau_{a'}^k) = 0,$$

which is a contradiction to the positivity of the travel costs due to Assumption 3. Hence, there cannot be a cycle with positive flow. In particular, this implies that $x_a^k = 0$ has to hold for all $a \in \delta^{\text{in}}(\alpha_k) \cup \delta^{\text{out}}(\omega_k)$ with $k = (\alpha_k, \omega_k) \in \mathcal{K}$. From flow conservation (2), we thus obtain

$$\sum_{a \in \delta^{\text{in}}(\omega_k)} x_a^k = \sum_{a \in \delta^{\text{out}}(\alpha_k)} x_a^k = d_k. \tag{10}$$

as well as

$$d_k = \sum_{a \in \delta^{\text{in}}(\omega_k)} x_a^k \geq \sum_{a \in \delta^{\text{out}}(i)} x_a^k = \sum_{a \in \delta^{\text{in}}(i)} x_a^k, \quad i \in \mathcal{N} \setminus \{\alpha_k, \omega_k\}. \tag{11}$$

Finally, the non-negativity of the commodity flows together with (10) and (11) yield $x_a^k \leq d_k$ for all $a \in \mathcal{A}$, which concludes the proof. $\square$

Because the overall arc flows f and the commodity flows x^k, $k \in \mathcal{K}$, are linearly coupled, Proposition 1 also yields valid bounds for the arc flows. We formally state these bounds in the following.

Proposition 2. *Suppose that Assumptions 1–3 hold and let* $\tau = (\tau^k)_{k\in\mathcal{K}}$ *with* $\tau^k \in \mathcal{T}^k$ *for all* $k \in \mathcal{K}$ *be given arbitrarily. Then, for every* $(f, x) \in S(\tau)$*, it holds*

$$0 \leq f_a \leq \sum_{k\in\mathcal{K}} d_k, \quad a \in \mathcal{A}.$$

Proof. The existence of a Wardrop equilibrium $(f, x) \in S(\tau)$ for the given tolls τ follows from Lemma 1. For all $a \in \mathcal{A}$, we then have

$$0 \leq f_a = \sum_{k\in\mathcal{K}} x_a^k \leq \sum_{k\in\mathcal{K}} d_k.$$

Here, the first inequality follows from (1) and the non-negativity of the commodity flows given by (3). The equality again follows from (1). Finally, the last inequality is due to Proposition 1. $\square$

Next, we provide bounds for the minimum travel costs t. The key idea for obtaining these bounds is that, whenever a point (f, x, t) solves (1)–(3), we can shift all values of t by the same amount while still satisfying these conditions.

Proposition 3. *Suppose that Assumptions 1–4 hold and let* $\tau = (\tau^k)_{k\in\mathcal{K}}$ *with* $\tau^k \in \mathcal{T}^k$ *for all* $k \in \mathcal{K}$ *be given arbitrarily. Then, for every* $(f, x) \in S(\tau)$*, there exists* t *such that* (f, x, t) *solves* (1)–(3) *and* t *has the following properties:*

(i) *For all* $k = (\alpha_k, \omega_k) \in \mathcal{K}$*, it holds* $t_{\omega_k}^k = 0$.
(ii) *For all* $k \in \mathcal{K}$ *and* $n \in \mathcal{N}$*, there exists* $0 \leq \bar{t}_n^k < \infty$ *such that* $0 \leq t_n^k \leq \bar{t}_n^k$.

Proof. By Lemma 1, there exists a Wardrop equilibrium $(f, x) \in S(\tau)$ for the given tolls τ, i.e., there exists t such that (f, x, t) solves (1)–(3). For all $k \in \mathcal{K}$, we now set $\Delta t^k = t_{\omega_k}^k$ and consider $t_i^k - \Delta t^k$ instead of t_i^k for all $i \in \mathcal{N}$. By construction, this yields $t_{\omega_k}^k = 0$ for all $k \in \mathcal{K}$. Moreover, we have

$$0 \leq c_a^k(f; \tau_a^k) + (t_j^k - \Delta t^k) - (t_i^k - \Delta t^k) = c_a^k(f; \tau_a^k) + t_j^k - t_i^k \perp x_a^k \geq 0,$$

for all $a = (i, j) \in \mathcal{A}$ and all $k \in \mathcal{K}$, i.e., Wardrop's second principle (3) remains satisfied. Hence, and because Conditions (1) and (2) do not depend on t, there exists (f, x, t) with $t_{\omega_k}^k = 0$ for all $k \in \mathcal{K}$ that solves (1)–(3) for the given tolls τ. This proves (i). Moreover, we have $0 \leq t_{\omega_k}^k \leq \bar{t}_{\omega_k}^k := 0$ for all $k \in \mathcal{K}$.

For what follows, let $k \in \mathcal{K}$ be given arbitrarily. Summing over Conditions (3) and applying (i) then yields

$$t_n^k \leq \sum_{a\in p} c_a^k(f; \tau_a^k) + t_{\omega_k}^k = \sum_{a\in p} c_a^k(f; \tau_a^k), \quad n \in \mathcal{N},\ p \in \mathcal{P}_n^k. \tag{12}$$

Here, we use $\mathcal{P}_n^k$ to denote the set of all simple paths between nodes n and ω_k. The conditions in (12) are equivalent to

$$t_n^k \leq \min_{p\in\mathcal{P}_n^k} \left\{ \sum_{a\in p} c_a^k(f; \tau_a^k) \right\}, \quad n \in \mathcal{N}. \tag{13}$$

To prove the lower and upper bounds for the variables t^k, we now show that the inequality in (13) is satisfied with equality for every node $n \in \mathcal{N}$ that is traversed in a path with positive commodity flow. To this end, let a path $p \in \mathcal{P}_n^k$, $n \in \mathcal{N}$, be given arbitrarily and suppose that $x_a^k > 0$ holds for all $a \in p$. Because of the complementarity constraint in (3), we obtain $t_i^k = c_a^k(f;\tau_a^k) + t_j^k$ for all $a = (i,j) \in p$. The latter together with (i) yields

$$t_n^k = \sum_{a \in p} c_a^k(f;\tau_a^k).$$

In particular, this means that p is a minimum-cost path from node n to ω_k. We now show that at least one equilibrium is preserved by setting

$$t_n^k = \min_{p' \in \mathcal{P}_n^k} \left\{ \sum_{a \in p'} c_a^k(f;\tau_a^k) \right\} \tag{14}$$

for all nodes $n \in \mathcal{N} \setminus \{\omega_k\}$. By our previous considerations, it suffices to consider arcs with zero flow, i.e., arcs $a = (i,j) \in \mathcal{A}$ with $x_a^k = 0$. In this case, the complementarity constraint in (3) is trivially satisfied. Hence, we only need to show that $t_i^k \le c_a^k(f;\tau_a^k) + t_j^k$ holds for t_i^k and t_j^k chosen according to (14). To this end, let

$$p_j = \operatorname*{arg\,min}_{p' \in \mathcal{P}_j^k} \left\{ \sum_{a' \in p'} c_{a'}^k(f;\tau_{a'}^k) \right\}$$

be a minimum-cost path from node j to the destination ω_k of commodity k. If we augment path p by arc a, we obtain a path from node i to ω_k, i.e., $p_j \cup \{a\} \in \mathcal{P}_i^k$ holds. This yields

$$t_i^k = \min_{p' \in \mathcal{P}_i^k} \left\{ \sum_{a' \in p'} c_{a'}^k(f;\tau_{a'}^k) \right\} \le c_a^k(f;\tau_a^k) + \sum_{a' \in p_j} c_{a'}^k(f;\tau_{a'}^k) = c_a^k(f;\tau_a^k) + t_j^k.$$

Here, both equalities follow from (14). Because $k \in \mathcal{K}$ has been chosen arbitrarily, we can thus conclude that (f,x,t) with $t = (t^k)_{k\in\mathcal{K}}$, $t_{\omega_k}^k = 0$, and t_n^k for $n \in \mathcal{N} \setminus \{\omega_k\}$ defined as in (14) solves (1)–(3) for the given tolls τ.

The non-negativity of the variables t now follows from (14) and the non-negativity of the travel cost functions, which is due to Assumption 3. Moreover, again by Assumption 3, the travel cost functions are continuous in the flows f and the tolls τ. Hence, and because f and τ are finitely bounded due to Proposition 2 and Assumption 4, respectively, there exist finite upper bounds $\bar{t}_n^k$ with

$$t_n^k = \min_{p' \in \mathcal{P}_n^k} \left\{ \sum_{a \in p'} c_a^k(f;\tau_a^k) \right\} \le \bar{t}_n^k$$

for all $n \in \mathcal{N} \setminus \{\omega_k\}$ and $k \in \mathcal{K}$. Taking all previous considerations into account, this proves (ii). □

In Proposition 3, we have shown the existence of finite upper bounds for the variables t. Determining explicit bounds, however, strongly depends on the specific structure of the travel cost functions. In the next proposition, we derive such bounds for the exemplary case of affine-linear travel cost functions, which are also used in our computational study in Section 5.

Proposition 4. *Suppose that the requirements of Proposition 3 are satisfied and that, for every $k \in \mathcal{K}$, the travel cost functions $c^k(f;\tau) = (c_a^k(f;\tau_a^k))_{a\in\mathcal{A}}$ are affine-linear in the flows, i.e., there exists a matrix $C^k \in \mathbb{R}_{\ge 0}^{|\mathcal{A}|\times|\mathcal{A}|}$ and a vector $c^{\mathrm{fix},k} \in \mathbb{R}_{>0}^{|\mathcal{A}|}$ with $c^k(f;\tau) = C^k f + c^{\mathrm{fix},k} + \tau^k$. Then, for every $(f,x) \in S(\tau)$, there exists $t = (t^k)_{k\in\mathcal{K}}$ such that (f,x,t) solves (1)–(3) and, for every $k \in \mathcal{K}$, t^k satisfies $t_{\omega_k}^k = 0$*

as well as

$$0 \le t_n^k \le \min_{p \in \mathcal{P}_n^k} \left\{ \sum_{a \in p} \left(\sum_{a' \in \mathcal{A}} C_{aa'}^k \sum_{q \in \mathcal{K}} d_q + c_a^{\mathit{fix},k} + \bar{\tau}_a \right) \right\}, \quad n \in \mathcal{N} \setminus \{\omega_k\},$$

where $\mathcal{P}_n^k$ denotes the set of all simple paths between nodes n and ω_k.

Proof. The claim follows from the last displayed formula in the proof of Proposition 3, Assumption 4, and Proposition 2. □

Finally, sufficiently large big-M constants to be used in (6) can be obtained by exploiting Assumptions 3 and 4 as well as Propositions 1, 2, and 3.

Remark 1. *In Assumption 1, we assume that there exists at least one path that connects each node $n \in \mathcal{N}$ to each destination node $\omega \in \mathcal{D}$. Hence, the set of all simple paths $\mathcal{P}_n^k$ between node $n \in \mathcal{N}$ and the destination $\omega_k \in \mathcal{D}$ of commodity $k \in \mathcal{K}$ is non-empty. The latter is used in Proposition 3 to prove finite bounds for the variables t. However, let us mention that Assumption 1 can be relaxed. In fact, it is sufficient to impose the following:*

> *For each commodity $k = (\alpha_k, \omega_k) \in \mathcal{K}$, there exists at least one path that connects α_k to ω_k.*

Relaxing Assumption 1 in this way neither affects the results in Lemma 1 nor the ones in Propositions 1 and 2. However, it requires minor adjustments in the derivation of valid upper bounds for the variables t. This is because the construction in (14) yields $\bar{t}_n^k = \infty$ for a commodity $k \in \mathcal{K}$ and some node $n \in \mathcal{N} \setminus \{\omega_k\}$ with $\mathcal{P}_n^k = \emptyset$. Nevertheless, to obtain finite upper bounds, we can proceed as follows. Because the travel costs are positive under Assumption 3, the are no cycles with positive flow in a Wardrop equilibrium. Consequently, if there is no path from node $n \in \mathcal{N}$ to the destination ω_k of some commodity k, there is also no commodity flow on arcs leading to or originating from that node. In such cases, the respective complementarity constraints in (3) are trivially satisfied. Hence, it suffices to set the variables t so that the remaining inequality constraints in (3) hold and the set of Wardrop equilibria is preserved. Exploiting Proposition 3, this can be done by setting

$$0 \le t_n^k \le \begin{cases} 0, & n = \omega_k, \\ \min\left\{\bar{t}_n^k, \, \bar{t}_{\alpha_k}^k + 1\right\}, & n \in \mathcal{N} \setminus \{\omega_k\}, \end{cases} \tag{15}$$

for all $k = (\alpha_k, \omega_k) \in \mathcal{K}$. Note that, because there exists a path from α_k to ω_k for each commodity $k \in \mathcal{K}$, we have $\bar{t}_{\alpha_k}^k < \infty$ and, thus, the upper bounds in (15) are finite as well.

3.2. **Existence of Solutions.** We now show the existence of optimal solutions to to Problems (4) and (5). To this end, we start with the following auxiliary result.

Lemma 2. *Suppose that Assumptions 1–4 hold. Then, the set*

$$\mathcal{F} = \left\{(\tau, f, x) \colon \tau = (\tau^k)_{k \in \mathcal{K}}, \, \tau^k \in \mathcal{T}^k, \, k \in \mathcal{K}, \, (f, x) \in S(\tau)\right\}$$

is non-empty and compact.

Proof. Due to Assumption 4, $\mathcal{T}^k$ is non-empty and compact for all $k \in \mathcal{K}$. From Propositions 1–3, we further obtain that $S(\tau)$ is bounded for all $\tau = (\tau^k)_{k \in \mathcal{K}}$ with $\tau^k \in \mathcal{T}^k$ for all $k \in \mathcal{K}$. By Lemma 1, the set $S(\tau)$ is non-empty for all feasible tolls τ. Finally, the set-valued map S also has a closed graph because it is described by finitely many continuous equality and inequality constraints in (1)–(3). This implies that all sets are non-empty and compact, which concludes the proof. □

Using Lemma 2 and the Weierstraß theorem, we can now prove the existence of optimal solutions to the toll-setting problems under revenue maximization and bottleneck minimization.

Theorem 1. *Under Assumptions 1–4, the toll-setting problems* (4) *and* (5) *each admit an optimal solution* (τ, f, x).

Proof. By Lemma 2, the feasible set of Problems (4) and (5) is non-empty and compact. Moreover, the functions $(\tau, f) \mapsto \sum_{a\in\mathcal{A}} \tau_a f_a$ and $f \mapsto \max_{a\in\mathcal{A}} f_a/u_a$ are both continuous. Overall, the Weierstraß theorem thus ensures that Problem (4) and (5) each admit an optimal solution. □

3.3. **Valid Inequalities.** To conclude this section, we provide valid inequalities for the feasible set of the toll-setting problems as well as valid inequalities for optimal solutions to these problems. These inequalities build on the auxiliary binary variables introduced to reformulate the complementarity constraints in Wardrop's second principle and can be used to strengthen the problem formulation. In particular, this may be seen as a structural advantage that arises from the MINLP reformulations (7) and (8) of the toll-setting problems.

Proposition 5. *Suppose that Assumptions 3 and 4 hold and let* $\tau = (\tau^k)_{k\in\mathcal{K}}$ *with* $\tau^k \in \mathcal{T}^k$ *for all* $k \in \mathcal{K}$ *be given arbitrarily. Further, let* $i, j \in \mathcal{N}$ *be such that* $(i, j), (j, i) \in \mathcal{A}$ *holds. Then, the inequalities*

$$z^k_{(i,j)} + z^k_{(j,i)} \le 1, \quad k \in \mathcal{K},$$

are valid for the feasible sets of the toll-setting MINLPs (7) *and* (8).

Proof. We prove the claim by contradiction. To this end, let (τ, f, x, t, z) be feasible for Problem (7) and let $k \in \mathcal{K}$ be given arbitrarily. Suppose that $z^k_{(i,j)} = 1 = z^k_{(j,i)}$ holds. Then, the constraints in (6b) yield

$$c^k_{(i,j)}(f; \tau^k_{(i,j)}) + t^k_j = t^k_i \quad \text{and} \quad c^k_{(j,i)}(f; \tau^k_{(j,i)}) + t^k_i = t^k_j.$$

From the latter, we obtain

$$\begin{aligned} t^k_i &= c^k_{(i,j)}(f; \tau^k_{(i,j)}) + \left(c^k_{(j,i)}(f; \tau^k_{(j,i)}) + t^k_i\right) \\ \iff 0 &= c^k_{(i,j)}(f; \tau^k_{(i,j)}) + c^k_{(j,i)}(f; \tau^k_{(j,i)}), \end{aligned}$$

which is a contradiction to the positivity of the travel cost functions due to Assumption 3. Hence, a feasible point for Problem (7) satisfies $z^k_{(i,j)} + z^k_{(j,i)} \le 1$. The result for Problem (8) can be shown by applying the same arguments. □

From Proposition 5, we particularly obtain $0 \le x^k_{(i,j)} \perp x^k_{(j,i)} \ge 0$ for all $i, j \in \mathcal{N}$ with $(i, j), (j, i) \in \mathcal{A}$ and all $k \in \mathcal{K}$. This means that, under the assumption of positive travel costs, there cannot be positive commodity flow on both an arc and its reversed arc. Finally, we provide valid inequalities for optimal tolls τ.

Proposition 6. *Under Assumptions 1–4, the toll-setting MINLPs* (7) *and* (8) *each admit an optimal solution* (τ, f, x, t, z) *that satisfies*

$$\tau^k_a \ge \bar{\tau}^k_a (1 - z^k_a), \quad a \in \mathcal{A},\ k \in \mathcal{K}. \tag{16}$$

Proof. An optimal solution (τ, f, x) to Problems (4) and (5) exists due to Theorem 1. Hence, there also exist t and z so that (τ, f, x, t, z) is an optimal solution to Problem (7) or (8), respectively. Let now $k \in \mathcal{K}$ be given arbitrarily. By Assumption 4, an optimal solution satisfies $\tau^k_a \ge 0$ for all $a \in \mathcal{A}$. Hence, if the set $\mathcal{T}^k$ imposes the upper bound $\bar{\tau}^k_a = 0$ on an arc $a \in \mathcal{A}$, the associated inequality in (16) is trivially satisfied. Therefore, we only need to consider arcs $a \in \mathcal{A}$ with $\bar{\tau}^k_a > 0$ in the following. Suppose that there is an arc $a \in \mathcal{A}$ with $\bar{\tau}^k_a > 0$ for which the inequality in (16) is violated. Because $\tau_a^k \geq 0$ holds by Assumption 4, this implies that $\tau_a^k < \bar{\tau}_a^k$ as well as $z_a^k = 0$ has to hold. In particular, the latter implies $x_a^k = 0$. We now define an alternative toll-setting policy for commodity k, which is given by

$$\hat{\tau}_{a'}^k = \begin{cases} \tau_{a'}^k, & a' \in \mathcal{A} \setminus \{a\}, \\ \bar{\tau}_{a'}^k, & a' = a. \end{cases}$$

By Assumption 4, we have $\hat{\tau}^k \in \mathcal{T}^k$. For all $k' \in \mathcal{K} \setminus \{k\}$, we further set $\hat{\tau}^{k'} = \tau^{k'}$. Because the constraints in (1) and (2) do not depend on the tolls, they are satisfied by the alternative point $(\hat{\tau}, f, x, t, z)$. Moreover, for sufficiently large big-M constants, $(\hat{\tau}, f, x, t, z)$ also satisfies the constraints in (6). Hence, we have $(f, x) \in S(\hat{\tau})$ and the point $(\hat{\tau}, f, x, t, z)$ is feasible for Problems (7) and (8). Because the bottleneck minimization objective does not depend on the tolls, $(\hat{\tau}, f, x, t, z)$ is thus optimal for Problem (8). Moreover, no revenue is generated by imposing tolls for commodity k on arc a, i.e., we further have $\tau_{a'}^k x_{a'}^k = \hat{\tau}_{a'}^k x_{a'}^k$ for all $a' \in \mathcal{A}$ by construction. As a result, the point $(\hat{\tau}, f, x, t, z)$ solves Problem (7) as well. Finally note that $\hat{\tau}_a^k \geq \bar{\tau}_a^k(1 - z_a^k)$ holds by construction. Hence, repeating the previous procedure until there are no arcs left that violate one of the inequalities in (16) concludes the proof. □

Proposition 6 is used to fix a toll τ_a^k to its upper bound $\bar{\tau}_a^k$ whenever $z_a^k = 0$ holds for $k \in \mathcal{K}$ and $a \in \mathcal{A}$. The latter may be beneficial to avoid exploring multiple practically equivalent solutions to the toll-setting problems, thereby reducing computational effort.

4. Robustification

Up to now, we have considered the setting in which the users of the traffic network perfectly know the travel costs. In practice, however, travelers often face uncertainty when making their decisions, e.g., due to accidents, maintenance work, or changing weather conditions. In this section, we study the toll-setting problems (4) and (5) under uncertain travel costs. We address these uncertainties using techniques from robust optimization and pursue similar ideas compared to those in Ito (2011) and Ordóñez and Stier-Moses (2007, 2010), who study the robust traffic assignment problem. In Section 4.1, we present the robustified variants of the toll-setting problems in which the network users hedge against uncertain travel costs within predefined and user-specific uncertainty sets. We model these situations as mathematical problems with robustified Wardrop equilibrium conditions, for which we present MINLP reformulations that exploit binary variables and big-M constants in Section 4.2. We derive valid big-Ms in Section 4.3 and we prove the existence of robust solutions in Section 4.4. Finally, we present valid inequalities for the robustified toll-setting problems in Section 4.5.

4.1. **Robust Toll-Setting Problems.** We start from the nominal Wardrop equilibrium model given by Conditions (1)–(3), for which we now assume that the travel costs of each arc $a \in \mathcal{A}$ and each commodity $k \in \mathcal{K}$ are not known exactly. More formally, we impose the following.

Assumption 5. *For all $a \in \mathcal{A}$ and $k \in \mathcal{K}$, the travel costs $c_a^k(f; \tau_a^k)$ are subject to additive deviations $Y_a^k \Delta c_a^k$ with Y_a^k being a random variable with support in $[0, 1]$ and $0 \leq \Delta c_a^k < \infty$.*

We use Δc_a^k to denote the maximum possible delay that a commodity $k \in \mathcal{K}$ may face when traveling along an arc $a \in \mathcal{A}$. Because it seems unlikely that travel costs realize in a worst-case sense on every arc of the network and, thus, to avoid being overly conservative, we consider a budgeted uncertainty modeling. This means that each commodity $k \in \mathcal{K}$ hedges against the situation in which at most $\Gamma^k \in \{0, \dots, |\mathcal{A}|\}$ arcs experience their worst-case delay. The robustified version of Wardrop's second principle (3) then reads

$$0 \leq c_a^k(f; \tau_a^k) + y_a^k \Delta c_a^k + t_j^k - t_i^k \perp x_a^k \geq 0, \quad a = (i,j) \in \mathcal{A},\, k \in \mathcal{K}. \tag{17}$$

Here, for a commodity $k \in \mathcal{K}$ and a given flow vector x^k, the vector y^k solves

$$\begin{aligned} \max_{y^k} \quad & \sum_{a \in \mathcal{A}} (\Delta c_a^k x_a^k) y_a^k && \text{(18a)}\\ \text{s.t.} \quad & \sum_{a \in \mathcal{A}} y_a^k \leq \Gamma^k, && \text{(18b)}\\ & 0 \leq y_a^k \leq 1, \quad a \in \mathcal{A}. && \text{(18c)} \end{aligned}$$

Problem (18) models the setting in which commodity $k \in \mathcal{K}$ hedges against delays that adversely affect its travel time. Here, we multiply the maximum deviation Δc_a^k with the commodity flow x_a^k in the objective function because delays on an arc $a \in \mathcal{A}$ that is not used by the commodity, i.e., $x_a^k = 0$, are irrelevant for its travel decision. Hence, a commodity only hedges against delays on arcs that are actually used.

Note that Problem (18) is a linear problem for fixed x^k, $k \in \mathcal{K}$. Hence, the Karush–Kuhn–Tucker (KKT) conditions are necessary and sufficient optimality conditions, i.e., replacing Problem (18) with its KKT conditions yields an equivalent reformulation of (17) and (18) that is given by

$$\begin{aligned} 0 \leq c_a^k(f; \tau_a^k) + y_a^k \Delta c_a^k + t_j^k - t_i^k \perp x_a^k \geq 0, &\qquad a = (i,j) \in \mathcal{A},\, k \in \mathcal{K}, && \text{(19a)}\\ 0 \leq \xi^k + \zeta_a^k - \Delta c_a^k x_a^k \perp y_a^k \geq 0, &\qquad a \in \mathcal{A},\, k \in \mathcal{K}, && \text{(19b)}\\ 0 \leq 1 - y_a^k \perp \zeta_a^k \geq 0, &\qquad a \in \mathcal{A},\, k \in \mathcal{K}, && \text{(19c)}\\ 0 \leq \Gamma^k - \sum_{a \in \mathcal{A}} y_a^k \perp \xi^k \geq 0, &\qquad k \in \mathcal{K}. && \text{(19d)} \end{aligned}$$

For notational convenience, we use $\tilde{c}_a^k(f; \tau_a^k) := c_a^k(f; \tau_a^k) + y_a^k \Delta c_a^k$ to denote the robustified travel costs for commodity $k \in \mathcal{K}$ on arc $a \in \mathcal{A}$ in the following. We further emphasize that the equilibrium conditions (1) and (2) do not explicitly depend on the travel costs. Hence, the set of robust Wardrop equilibria for given tolls τ and fixed $\Gamma = (\Gamma^k)_{k \in \mathcal{K}}$ can be stated as

$$S_{\text{rob}}(\tau) = \{(f, x) \colon \exists (t, y, \xi, \zeta) \text{ such that } (f, x, t, y, \xi, \zeta) \text{ solves } (1), (2), \text{and } (19)\}.$$

The overall robustified toll-setting problem under revenue maximization is now given by

$$\begin{aligned} \max_{\tau, f, x} \quad & \sum_{a \in \mathcal{A}} \sum_{k \in \mathcal{K}} \tau_a^k x_a^k && \text{(20a)}\\ \text{s.t.} \quad & \tau^k \in \mathcal{T}^k, \quad k \in \mathcal{K}, && \text{(20b)}\\ & (f, x) \in S_{\text{rob}}(\tau), && \text{(20c)} \end{aligned}$$

whereas the toll-setting problem under bottleneck minimization reads

$$\begin{aligned} \min_{\tau, f, x} \quad & \max_{a \in \mathcal{A}} \frac{f_a}{u_a} && \text{(21a)}\\ \text{s.t.} \quad & \tau^k \in \mathcal{T}^k, \quad k \in \mathcal{K}, && \text{(21b)}\\ & (f, x) \in S_{\text{rob}}(\tau). && \text{(21c)} \end{aligned}$$

4.2. **MINLP Reformulations.** Similar as it is done in Section 3, we exploit sufficiently large big-M constants and additional binary variables to reformulate the complementarity constraints in (19). An MINLP reformulation of the robustified version of Wardrop's second principle (19) thus reads

$$
\begin{aligned}
&0 \le x_a^k \le M_a^k z_a^k, && a \in \mathcal{A},\, k \in \mathcal{K}, && \text{(22a)}\\
&0 \le \tilde{c}_a^k(f;\tau_a^k) + t_j^k - t_i^k \le M_a^k(1 - z_a^k), && a = (i,j) \in \mathcal{A},\, k \in \mathcal{K}, && \text{(22b)}\\
&0 \le \xi^k + \zeta_a^k - \Delta c_a^k x_a^k \le N_a^k w_a^k, && a \in \mathcal{A},\, k \in \mathcal{K}, && \text{(22c)}\\
&0 \le y_a^k \le 1 - w_a^k, && a \in \mathcal{A},\, k \in \mathcal{K}, && \text{(22d)}\\
&v_a^k \le y_a^k \le 1, && a \in \mathcal{A},\, k \in \mathcal{K}, && \text{(22e)}\\
&0 \le \zeta_a^k \le L_a^k v_a^k, && a \in \mathcal{A},\, k \in \mathcal{K}, && \text{(22f)}\\
&0 \le \xi^k \le R^k q_k, && k \in \mathcal{K}, && \text{(22g)}\\
&0 \le \Gamma^k - \sum_{a \in \mathcal{A}} y_a^k \le R^k(1 - q_k), && k \in \mathcal{K}, && \text{(22h)}\\
&q_k,\, v_a^k,\, w_a^k,\, z_a^k \in \{0,1\}, && a \in \mathcal{A},\, k \in \mathcal{K}. && \text{(22i)}
\end{aligned}
$$

By construction, (22) is equivalent to (19) for sufficiently large constants L_a^k, M_a^k, N_a^k, and R^k for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$. Before we elaborate on how to obtain such constants in Section 4.3, we provide enhanced formulations for (22) in the remainder of this section. For this purpose, we need the following auxiliary lemma.

Lemma 3. *Let $k \in \mathcal{K}$, $x^k \in \mathbb{R}_{\ge 0}^{|\mathcal{A}|}$, and $\Gamma^k \in \{0, \dots, |\mathcal{A}|\}$ be given arbitrarily. Then, Problem (18) has an optimal solution y^k that satisfies*

$$\sum_{a \in \mathcal{A}} y_a^k = \Gamma^k.$$

Proof. Problem (18) is always feasible and an optimal solution y^k that satisfies (18b) with equality can be constructed in a greedy way. To this end, we sort all arcs such that their coefficients $\Delta_a^k x_a^k$ are given in non-increasing order. Then, setting $y^k = 1$ for the first Γ^k arcs and $y^k = 0$ for the remaining ones yields an optimal solution to Problem (18); see, e.g., Section 2.1 in Pisinger and Toth (1998). □

By Lemma 3, we can eliminate the auxiliary binary variables q used for the linearization of the complementarity in Constraint (19d). To this end, we replace the constraints in (22g) and (22h) with

$$\xi^k \ge 0, \quad \sum_{a \in \mathcal{A}} y_a^k = \Gamma^k, \quad k \in \mathcal{K}.$$

Although the complementarity constraint in (19d) is satisfied for any $\xi \ge 0$, we emphasize that finite upper bounds for the variables ξ are required to obtain sufficiently large big-M constants N_a^k, $a \in \mathcal{A}$, $k \in \mathcal{K}$, for the constraints in (22c). Therefore, we additionally impose $\xi^k \le R^k$ for all $k \in \mathcal{K}$.

Taking all previous considerations into account, an equivalent MINLP reformulation of the robustified toll-setting problem under revenue maximization (20) is now given by

$$
\begin{aligned}
\max_{\tau,f,x,t,r} \quad & \sum_{a\in\mathcal{A}}\sum_{k\in\mathcal{K}} \tau_a^k x_a^k && \text{(23a)}\\
\text{s.t.} \quad & (1), (2), (22a)–(22f), && \text{(23b)}\\
& \sum_{a\in\mathcal{A}} y_a^k = \Gamma^k, \quad k\in\mathcal{K}, && \text{(23c)}\\
& 0 \le \xi^k \le R^k, \quad k\in\mathcal{K}, && \text{(23d)}\\
& \tau^k \in \mathcal{T}^k, \quad k\in\mathcal{K}, && \text{(23e)}\\
& v_a^k,\, w_a^k,\, z_a^k \in \{0,1\}, \quad a\in\mathcal{A},\, k\in\mathcal{K}. && \text{(23f)}
\end{aligned}
$$

Here, r contains all variables that are used for the robustification of the travel costs as well as the variables that are introduced for the reformulation of the complementarity constraints, i.e., $r := (y,\xi,\zeta,v,w,z)$. By construction, Problem (23) is equivalent to the robustified toll-setting problem (20) for sufficiently large constants L_a^k, M_a^k, N_a^k, and R^k for all $a\in\mathcal{A}$ and $k\in\mathcal{K}$. Similarly, an equivalent MINLP reformulation of the robustified toll-setting problem under bottleneck minimization (21) reads

$$
\min_{\tau,f,x,t,r} \quad \max_{a\in\mathcal{A}} \frac{f_a}{u_a} \quad \text{s.t.} \quad (23b)–(23f). \tag{24}
$$

4.3. **Computing Big-Ms.** We now derive bounds for the variables of the robustified toll-setting problems, which we exploit to obtain sufficiently large big-M constants L_a^k, M_a^k, N_a^k, and R^k for all $a\in\mathcal{A}$ and $k\in\mathcal{K}$.

Corollary 1. *Suppose that Assumptions 1, 2, 3, and 5 hold, and let $\tau = (\tau^k)_{k\in\mathcal{K}}$ with $\tau^k \in \mathcal{T}^k$ as well as $\Gamma = (\Gamma^k)_{k\in\mathcal{K}}$ with $\Gamma^k \in \{0,\dots,|\mathcal{A}|\}$ for all $k\in\mathcal{K}$ be given arbitrarily. Then, for every $(f,x)\in S_{rob}(\tau)$, the following properties are satisfied:*

(i) *For all $a\in\mathcal{A}$ and $k\in\mathcal{K}$, it holds $0\le x_a^k \le d_k$.*
(ii) *For all $k=(\alpha_k,\omega_k)\in\mathcal{K}$ and $a\in\delta^{in}(\alpha_k)\cup\delta^{out}(\omega_k)$, it holds $x_a^k = 0$.*
(iii) *For all $a\in\mathcal{A}$, it holds $0\le f_a \le \sum_{k\in\mathcal{K}} d_k$.*

Corollary 2. *Suppose that Assumptions 1, 2, 3, and 5 hold, and let $\tau = (\tau^k)_{k\in\mathcal{K}}$ with $\tau^k \in \mathcal{T}^k$ as well as $\Gamma = (\Gamma^k)_{k\in\mathcal{K}}$ with $\Gamma^k \in \{0,\dots,|\mathcal{A}|\}$ for all $k\in\mathcal{K}$ be given arbitrarily. Then, for every $(f,x)\in S_{rob}(\tau)$, there exists (t,y,ξ,ζ) such that (f,x,t,y,ξ,ζ) solves (1), (2) and (19), and t has the following properties:*

(i) *For all $k=(\alpha_k,\omega_k)\in\mathcal{K}$, it holds $t_{\omega_k}^k = 0$.*
(ii) *For all $k\in\mathcal{K}$ and $n\in\mathcal{N}$, there exists $0\le \bar{t}_n^k < \infty$ such that $0 \le t_n^k \le \bar{t}_n^k$.*

The last two corollaries state bounds for f and x if they exist, i.e., if $S_{\text{rob}}(\tau)\neq\emptyset$, and can be shown in analogy to the proofs of Propositions 1, 2, and 3 by replacing the nominal travel cost functions $c_a^k(f;\tau_a^k)$ with the robustified travel cost functions $\tilde{c}_a^k(f;\tau_a^k)$. We now prove finite bounds for the variables ξ and ζ.

Proposition 7. *Suppose that Assumptions 1, 2, 3, and 5 hold, and let $\tau = (\tau^k)_{k\in\mathcal{K}}$ with $\tau^k \in \mathcal{T}^k$ as well as $\Gamma = (\Gamma^k)_{k\in\mathcal{K}}$ with $\Gamma^k \in \{0,\dots,|\mathcal{A}|\}$ for all $k\in\mathcal{K}$ be given arbitrarily. Then, for every $(f,x)\in S_{rob}(\tau)$, there exists (t,y,ξ,ζ) such that (f,x,t,y,ξ,ζ) solves (1), (2) and (19) with*

$$
0 \le \xi^k \le \max_{a\in\mathcal{A}} \left\{\Delta c_a^k d_k\right\}, \quad k\in\mathcal{K}, \tag{25}
$$

and

$$
0 \le \zeta_a^k \le \Delta c_a^k d_k, \quad a\in\mathcal{A},\, k\in\mathcal{K}. \tag{26}
$$

In particular, it holds

$$S_{rob}(\tau) = \{(f, x) : \exists (t, y, \xi, \zeta) \text{ such that } (f, x, t, y, \xi, \zeta) \text{ solves } (1), (2), (19), (25), \text{ and } (26)\}.$$

Proof. The non-negativity of ξ and ζ immediately follows from the feasibility w.r.t. Conditions (19). Hence, we only need to prove the upper bounds. To this end, let $k \in \mathcal{K}$ be given arbitrarily. If $\Gamma^k = 0$ holds, commodity k does not hedge against any uncertainties regarding the travel costs, i.e., no additional variables ξ^k and ζ^k are introduced for the robustification of the travel costs of commodity k. Consequently, it suffices to consider the case in which $\Gamma^k \geq 1$ holds. Due to Lemma 3, we can assume w.l.o.g. that $\sum_{a \in \mathcal{A}} y_a^k = \Gamma^k$ holds. In particular, this implies that there exists at least one arc $a \in \mathcal{A}$ with $y_a^k > 0$. The complementarity constraint in (19b) then yields $\xi^k + \zeta_a^k = \Delta c_a^k x_a^k$. From the non-negativity of ξ^k and ζ_a^k, we thus obtain

$$0 \leq \zeta_a^k \leq \Delta c_a^k x_a^k \quad \text{and} \quad 0 \leq \xi^k \leq \Delta c_a^k x_a^k$$

for all $a \in \mathcal{A}$ with $y_a^k > 0$. Moreover, for all $a \in \mathcal{A}$ with $y_a^k = 0$, we obtain $\zeta_a^k = 0$ from (19c). Exploiting Corollary 1 as well as the non-negativity of Δc_a^k and d_k due to Assumptions 2 and 5, we thus obtain the bounds

$$\begin{aligned} &0 \leq \xi^k \leq \max_{a \in \mathcal{A}} \left\{\Delta c_a^k x_a^k\right\} \leq \max_{a \in \mathcal{A}} \left\{\Delta c_a^k d_k\right\}, && k \in \mathcal{K}, \\ &0 \leq \zeta_a^k \leq \Delta c_a^k d_k, && a \in \mathcal{A},\ k \in \mathcal{K}. \end{aligned}$$

Finally, we note that imposing these bounds does not affect the flows (f, x) in a robust Wardrop equilibrium. This concludes the proof. □

Next, we prove the existence of a robust Wardrop equilibrium for any given tolls $\tau = (\tau^k)_{k \in \mathcal{K}}$ with $\tau^k \in \mathcal{T}^k$ for all $k \in \mathcal{K}$.

Theorem 2. *Suppose that Assumptions 1, 2, 3, and 5 hold, and let $\tau = (\tau^k)_{k \in \mathcal{K}}$ with $\tau^k \in \mathcal{T}^k$ as well as $\Gamma = (\Gamma^k)_{k \in \mathcal{K}}$ with $\Gamma^k \in \{0, \dots, |\mathcal{A}|\}$ for all $k \in \mathcal{K}$ be given arbitrarily. Then, there exists a robust Wardrop equilibrium for the given tolls τ, i.e., $S_{rob}(\tau) \neq \emptyset$ holds.*

Proof. To show that $S_{\text{rob}}(\tau) \neq \emptyset$ holds, we show that for given f and x that satisfy (1) and (2), there always exists a solution to the complementarity problem (19). By abbreviating the variables of this problem with $\varphi = (f, x, t, y, \zeta, \xi)$ and the respective function of the system with F, we can write (19) as $0 \leq F(\varphi) \perp \varphi \geq 0$. It is well-known that any complementarity problem can be re-stated as finding the roots of the function $G(\varphi) = \min\{\varphi, F(\varphi)\}$, which is again equivalent to finding a fixed point of $H(\varphi) = \max\{0, \varphi - F(\varphi)\}$. From Corollaries 1 and 2 we know that we can restrict the search for fixed points to a bounded set for the variables f, x, and t. Moreover, all solutions y to the complementarity problem are bounded as well due to the bound constraints in Problem (18). Finally, all ζ and ξ values are bounded by Proposition 7. Hence, the complementarity problem is equivalent to the fixed point problem for H on a non-empty, convex, and compact set that is given by the above mentioned upper bounds. Because H is continuous under Assumption 3, the existence follows from the fixed-point theorem by Brouwer. □

Sufficiently large big-M constants to be used in (22) can now be obtained by exploiting Assumptions 3, 4, and 5, Corollaries 1 and 2, as well as Proposition 7.

4.4. **Existence of Solutions.** We now show that optimal solutions to the robustified toll-setting problems (20) and (21) exist. To this end, we again start with an auxiliary lemma and then prove the existence of solutions using the Weierstraß theorem.

Lemma 4. *Suppose that Assumptions 1–5 hold. Then, the set*

$$\mathcal{F}_{rob} = \left\{(\tau, f, x) \colon \tau = (\tau^k)_{k \in \mathcal{K}},\, \tau^k \in \mathcal{T}^k,\, k \in \mathcal{K},\, (f, x) \in S_{rob}(\tau)\right\}$$

is non-empty and compact.

Proof. Due to Assumption 4, the sets $\mathcal{T}^k$ are non-empty and compact for all $k \in \mathcal{K}$. By Theorem 2, the set $S_{\text{rob}}(\tau)$ is non-empty for all feasible tolls τ. From Corollary 1, we further obtain the boundedness of $S_{\text{rob}}(\tau)$ for all τ. Finally, S_{rob} is described my finitely many continuous equality and inequality constraints in (1), (2), and (19). Hence, the set-valued map S_{rob} has a closed graph, which completes the proof. □

Theorem 3. *Under Assumptions 1–5, the robustified toll-setting problems* (20) *and* (21) *each admit an optimal solution* (τ, f, x)*.*

Proof. By Lemma 4, the feasible set of Problems (20) and (21) is non-empty and compact. Moreover, the functions $(\tau, f) \mapsto \sum_{a \in \mathcal{A}} \tau_a f_a$ and $f \mapsto \max_{a \in \mathcal{A}} f_a / u_a$ are both continuous. Overall, the Weierstraß theorem thus ensures that Problem (20) and (21) each admit an optimal solution. □

4.5. **Valid Inequalities.** Under Assumptions 3, 4, and 5, the robustified travel cost functions $\tilde{c}_a^k(f; \tau_a^k)$ are positive for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$. Thus, the valid inequalities derived in Proposition 5 are also valid for Problems (23) and (24). Moreover, because we do not use any information about the travel costs to prove the validity of the inequalities in (16), the latter are valid for optimal solutions to the robustified toll-setting problems as well. To conclude this section, we now provide additional valid inequalities for the feasible sets of Problems (23) and (24).

Proposition 8. *The inequalities*

$$v_a^k + w_a^k \leq 1, \quad a \in \mathcal{A},\, k \in \mathcal{K},$$

are valid for the feasible sets of Problems (23) *and* (24)*.*

Proof. We prove the claim by contradiction. To this end, let $(\tau, f, x, t, y, \xi, \zeta, v, w, z)$ be feasible for Problem (23) and suppose that there exists an arc $a \in \mathcal{A}$ and a commodity $k \in \mathcal{K}$ for which the inequality $v_a^k + w_a^k \leq 1$ is violated. This means that $v_a^k = 1 = w_a^k$ has to hold. As a consequence, Constraint (22e) yields $y_a^k = 1$. However, this is a contradiction to $y_a^k = 0$, which is obtained from Constraint (22d). Hence, the inequality $v_a^k + w_a^k \leq 1$ is valid for the feasible set of Problem (23). The result for Problem (24) can be shown by applying the same arguments. □

5. Case Study

In this section, we present a case study to compare the toll-setting problems under revenue maximization and bottleneck minimization. Moreover, we illustrate how the consideration of travelers, who hedge against travel cost uncertainty in a robust way, may affect toll-setting policies and travel behavior. In Sections 5.1 and 5.2, we briefly discuss the considered test instances and the computational setup. In Section 5.3, we then discuss the computational results of our case study.

5.1. **Test Instances.** We consider the well-known Sioux Falls network (LeBlanc et al. 1975), whose data is publicly available at `https://github.com/bstabler/TransportationNetworks`. In addition to the full Sioux Falls network, we also study a subnetwork, which we refer to as “Sioux Falls East”. An illustration of both the entire Sioux Falls network and the “Sioux Falls East” subnetwork is given in Figure 1. In our computational study, we vary the number of commodities and toll arcs in the two networks. In the following, we discuss the selection of commodities and the generation of toll arcs, as well as the considered travel cost functions and the specification of the uncertainty parameters.

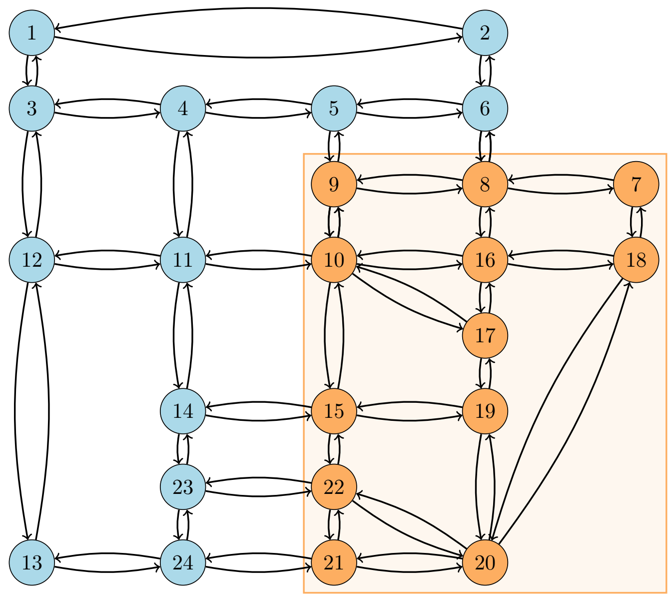

FIGURE 1. The entire Sioux Falls network (blue and orange nodes) consisting of 24 nodes and 76 arcs and the "Sioux Falls East" network (orange nodes) consisting of 12 nodes and 36 arcs.

Travel Cost Functions. The travel costs provided for the Sioux Falls network are defined by the BPR function (U.S. Bureau of Public Roads 1964), which is given by

$$c_a(f_a) = c_a^{\text{fix}} \left(1 + 0.15 \left(\frac{f_a}{u_a}\right)^4\right), \quad a \in \mathcal{A}.$$

Here, $c_a^{\text{fix}} > 0$ denotes the fixed costs ("free-flow time") and $u_a > 0$ denotes the practical capacity of an arc $a \in \mathcal{A}$. Note that the BPR function is separable and that every commodity faces the same travel costs. Although our modeling framework also allows for more general, non-separable travel cost functions, we build our case study on the costs provided with the instance data for tractability reasons. Moreover, we note that the BPR function does not include tolls. In our case study, we thus adapt the BPR function to incorporate tolls as well. We further restrict ourselves to the case of affine-linear travel cost functions. More specifically, we consider

$$c_a(f_a; \tau_a) = c_a^{\text{fix}} \left(1 + 1.5 \frac{f_a}{u_a}\right) + \tau_a, \quad a \in \mathcal{A}.$$

Here, the factor 1.5 is chosen so that, after linearizing the original BPR function (replacing the power 4 with 1), congestion effects remain visible and are not obscured by too small coefficients. Although we acknowledge that considering affine-linear travel costs is a strong assumption, we make this modeling choice as a first step towards solving the highly challenging toll-setting problems to global optimality, rendering them more computationally tractable and amenable to state-of-the-art general-purpose solvers such as, e.g., Gurobi. Note that affine-linear travel cost functions lead to toll-setting problems that are optimization problems with only mixed-integer and linear constraints. As a result, we obtain a toll-setting MILP in the case of bottleneck minimization and a nonconvex toll-setting MINLP in the case of revenue maximization. The bilinearities in the latter can still be tackled using solvers such as Gurobi. Several works in the related literature also rely on the assumption of affine-linear travel cost functions for similar reasons; see, e.g., Brotcorne et al.

(2011, 2001), Bui et al. (2022), and Didi-Biha et al. (2006). Nevertheless, as we will see in Section 5.3, solving the toll-setting problems remains a highly challenging task even under this simplifying assumption.

Selection of Commodities. We consider instances with a varying number of commodities. To obtain challenging network instances, we select the origin-destination (OD) pairs using the following procedure. We randomly select an OD pair from the original network, compute its shortest path, and include the OD pair in the reduced network if the shortest path traverses at least three arcs. We stop the procedure as soon as the desired number of OD pairs is selected. In this way, we aim to avoid trivial instances in which OD pairs are too close or in which commodity flows can be routed along disjoint paths. Overall, we expect that this selection procedure yields network instances that are rather challenging to solve. In our case study, we consider instances with 4 to 10 commodities.

Generation of Toll Arcs. To ensure consistency with the available data for the Sioux Falls network, which does not differentiate travel costs by commodity, we also consider unified tolls for all commodities in our case study. This means that we consider $\tau_a^k = \tau_a$ for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$. Here and in what follows, arcs $a \in \mathcal{A}$ with an upper bound of $\bar{\tau}_a = 0$ for the tolls are called toll-free arcs. All remaining arcs are called toll arcs. Let us point out that the used data for the Sioux Falls network does not include toll arcs. Hence, we have generated toll arcs using a procedure similar to the one considered in Brotcorne et al. (2000) and Bui et al. (2022). In our case study, we vary the number of toll arcs in the considered Sioux Falls (sub-)network. The procedure for generating toll arcs works as follows. For a given set of OD pairs, we determine the shortest path for each commodity. For each arc of the network, we then determine the number of shortest paths that go through that arc. Afterward, we sort the arcs of the network in decreasing order w.r.t. the number of shortest paths traversing it. Following this order, we convert each arc and its reversed arc into a toll arc until $2/3$ of the desired number of toll arcs is reached. The remaining $1/3$ of the desired number of toll arcs is chosen randomly among the remaining arcs. Again, if an arc is converted into a toll arc, we also convert its reversed arc. We impose a lower bound of 0 for all tolls. The upper bound $\bar{\tau}_a$ on the tolls is set to the fixed travel costs c_a^{fix} on all toll arcs. As usual in the literature, we half the costs c_a^{fix} for toll arcs after their conversion. Note that the toll-setting problems are bounded in our setting. In contrast to Brotcorne et al. (2000) and Bui et al. (2022), we thus do not need to ensure that at least one toll-free path is preserved for each commodity when converting arcs into toll arcs.

Generation of Uncertainty Parameters. As the used data for the Sioux Falls network does not include uncertainties, we have randomly generated the uncertainty parameters Δc_a^k and Γ^k, $a \in \mathcal{A}$, $k \in \mathcal{K}$. For all commodities, we consider the same travel cost uncertainties, i.e., we consider $\Delta c_a^k = \Delta c_a$ for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$. Here, Δc_a is a uniformly distributed random integer value in the interval $[0.5c_a^{\text{fix}}, 2c_a^{\text{fix}}]$. Moreover, for each commodity $k \in \mathcal{K}$, the parameter Γ^k takes a uniformly distributed integer value in the interval $[0, 0.5|\mathcal{A}|]$. We emphasize that the value Γ^k, which is used to control the level of conservatism of a solution, may differ across commodities.

5.2. **Computational Setup.** All tests were realized on an Intel XEON SP 6126 at 2.6 GHz (4 cores) with 32 GB RAM, which is part of the high performance cluster "Elwetritsch" at TU Kaiserslautern within the "Alliance of High Performance Computing Rheinland-Pfalz" (AHRP). The toll-setting problems (7), (8), (23), and (24) are implemented in Python 3.7.11 and we use Gurobi 12.0.0 to solve them. Because the bottleneck minimization objective is not directly supported by Gurobi, we reformulate the toll-setting problems under bottleneck minimization using an epigraph formulation. This reformulation is shown in Problem (9) for the nominal case, but the same technique can be applied to the robust setting as well. Moreover, our implementations include the valid inequalities presented in Propositions 5, 6, and 8. Preliminary computational tests revealed that including these inequalities significantly enhances the solution process. As the toll-setting problems under revenue maximization (7) and (20) are nonconvex MINLPs, we further need to set the Gurobi parameter NonConvex to 2 for these models. All other parameters were left at their default settings. For each test run, we set a time limit (TL) of 1 h.

5.3. **Computational Results.** We now compare the nominal and robustified toll-setting problems under revenue maximization and bottleneck minimization. We first report the results for the full Sioux Falls network in Section 5.3.1, and then turn to the "Sioux Falls East" subnetwork in Section 5.3.2. In what follows, the objective values of the revenue maximization and bottleneck minimization problems are reported as revenues and maximum (or worst-case) congestion, respectively. In addition, we evaluate the alternative metric ex post for each solution, allowing a comparison of revenue and congestion across different toll configurations.

5.3.1. *Full Sioux Falls Network.* We start by considering the deterministic setting. Reflected by the runtimes and the number of investigated branch-and-bound nodes shown in Table 1, we observe that the resources required to solve Problems (7) and (8) increase with the number of OD pairs. This is to be expected as the number of OD pairs directly influences the size of the respective toll-setting problems. For each additional OD pair, we introduce $2|\mathcal{A}| + |\mathcal{N}| = 176$ variables and $4|\mathcal{A}| + |\mathcal{N}| = 328$ constraints in the models. Thus, it is evident that increasing the number of OD pairs increases the amount of resources required to solve the respective toll-setting problems. In particular, we observe that only the smallest instance, i.e., the one with 5 commodities and 4 toll arcs, can be solved to global optimality in the case of revenue maximization. Recall that the toll-setting problem under revenue maximization (7) is a nonconvex MINLP, whereas the one under bottleneck minimization is an MILP. Hence, solving Problem (7) requires special algorithmic treatment. In this context, the results in Table 1 show that increasing the number of toll arcs in the network further increases the computational burden for the toll-setting problem under revenue maximization. This is due to the fact that more toll arcs lead to more nonconvex terms in the objective function of Problem (7), which Gurobi tackles using spatial branching based on convex envelopes. In contrast, the number of toll arcs affects the computational effort for the bottleneck minimization problem only slightly. Overall, the large discrepancies in the number of branch-and-bound nodes between the two settings may thus be explained by the additional nodes that are explored during spatial branching to handle the nonconvexities in the revenue maximization problem.

Compared to their nominal counterparts, the robustified toll-setting problems (23) and (24) are significantly larger w.r.t. the number of variables and constraints. For the Sioux Falls network, we introduce $229|\mathcal{K}|$ additional variables and $611|\mathcal{K}|$ additional constraints for the robustification of the travel costs and the reformulation of the complementarity constraints. The computational challenges resulting from larger models is thus even more pronounced in the robust setting. As shown in Table 2, none of the instances of the robust toll-setting problem under revenue maximization can be solved within the time limit of 1 h. For the instances with 10 commodities, not even a feasible point is found. In the case of bottleneck minimization, we also observe that the size of the models becomes a limiting factor. Here, only the instances with 5 commodities can be solved to global optimality.

Table 1. The revenues realized by imposing tolls, the runtimes (in s), and the number of investigated branch-and-bound nodes required to solve the nominal toll-setting problems under revenue maximization and bottleneck minimization for the full Sioux Falls network with varying numbers of OD pairs ("$|\mathcal{K}|$") and toll arcs ("$\left|\mathcal{A}^{\text{toll}}\right|$"). Additionally, the maximum congestion and the optimality gaps (in %) are shown.

objective	$\|\mathcal{K}\|$	$\left\|\mathcal{A}^{\text{toll}}\right\|$	congestion	revenues	runtime	B&B nodes	gap
revenue	5	4	0.16	4468.47	138.19	32 745	0.01
		8	0.16	5779.11	TL	1 387 900	11.52
	10	4	0.18	3502.29	TL	89 747	110.43
		8	0.18	5948.52	TL	236 242	302.10
bottleneck	5	4	0.16	2555.75	6.60	4540	0.00
		8	0.16	2069.74	8.24	4911	0.00
	10	4	0.18	1328.40	241.87	22 105	0.00
		8	0.18	5602.30	140.91	25 421	0.00

Table 2. The revenues realized by imposing tolls, the runtimes (in s), and the number of investigated branch-and-bound nodes required to solve the robustified toll-setting problems under revenue maximization and bottleneck minimization for the full Sioux Falls network with varying numbers of OD pairs ("$|\mathcal{K}|$") and toll arcs ("$\left|\mathcal{A}^{\text{toll}}\right|$"). Additionally, the maximum congestion and the optimality gaps (in %) are shown.

objective	$\|\mathcal{K}\|$	$\left\|\mathcal{A}^{\text{toll}}\right\|$	congestion	revenues	runtime	B&B nodes	gap
revenue	5	4	0.16	5769.44	TL	128 024	59.48
		8	0.16	5983.53	TL	163 943	227.31
	10	4	–	–	TL	28 521	–
		8	–	–	TL	26 583	–
bottleneck	5	4	0.16	3941.72	196.41	10 552	0.00
		8	0.16	4317.60	202.24	11 522	0.00
	10	4	0.20	3519.42	TL	74 431	17.72
		5	0.20	7780.37	TL	173 110	11.42

Overall, even under the simplifying assumption of affine-linear travel costs (see Section 5.1), solving the toll-setting problems is a highly challenging task and the size of the instances becomes a limiting factor. In particular, none of the robust toll-setting problems under revenue maximization can be solved within 1 h. To enable meaningful comparisons between the nominal and the robust case, and to gain insight into the travel behavior and toll-setting policies across different settings, we thus focus on the "Sioux Falls East" subnetwork in the following.

5.3.2. *Sioux Falls East Subnetwork.* For the "Sioux Falls East" network, we make the same observations as before regarding the computationally tractability and hardness of the considered toll-setting problems. As the number of commodities and toll arcs increases, solving the problems requires more computational resources. Similarly,

TABLE 3. The number of solved instances (out of the 28 instances considered in the nominal and the robust setting, respectively) for the toll-setting problems under revenue maximization and bottleneck minimization.

	revenue	bottleneck
nominal	24	28
robust	8	24

the robust toll-setting problems remain significantly more challenging because of the number of additional variables and constraints introduced for the robustification of the travel costs and the reformulation of the complementarity constraints. As before, the computational challenges are particularly pronounced for the case of revenue maximization, which is supported by the number of solved instances shown in Table 3. Detailed results for the runtimes and the number of branch-and-bound nodes required to solve both the nominal and the robust toll-setting problems under revenue maximization and bottleneck minimization can be found in Appendix A.1. We now focus on the qualitative analysis of the toll-setting problems and study the travel behavior as well as the toll-setting policies in the nominal and robust settings.

In Tables 4 and 5, we show the revenues generated by imposing tolls and the maximum congestion obtained using the toll-setting problems under revenue maximization and bottleneck minimization, respectively. For the revenue maximization problem, we observe that the revenues realized by imposing tolls are significantly higher in the robust compared to the nominal setting. In the case of bottleneck minimization, this is no longer the case. Here, revenues can be larger (see, e.g., the instance with 4 commodities and 4 toll arcs) or smaller (see, e.g., the instance with 4 commodities and 6 toll arcs) in the robust setting. In particular, we observe that the robustification of uncertain travel costs always leads to changes in the travel behavior and the tolls, which is reflected by the different revenues obtained in the nominal and robust settings. Similarly, the maximum congestion can also be larger or smaller in the robust compared to the nominal setting. Note that hedging against uncertain travel costs in a robust way leads to more congested networks for the majority of the considered instances. In this context, we emphasize that it is not the traffic authority that hedges against uncertain travel costs, but the users of the traffic network. In particular, the users of the traffic network decide on their route choices in a "here-and-now" fashion, i.e., before the uncertainty realizes. Viewing the overall toll-setting problem as a single-leader multi-follower game, this means that we consider multiple "here-and-now" followers. Because this problem is considered from the leader's perspective, having higher revenues or less congestion in the robust setting is thus not in contrast to classic robust optimization theory. To further illustrate this, we report the nominal and the robustified travel costs faced by each commodity in the "Sioux Falls East" network with 4 commodities in Table 6. It can be seen that, when hedging against uncertain travel costs in a robust way, users of the traffic network always face increased travel costs to reach their destination. Moreover, the travel costs are higher in the revenue maximization setting than in the bottleneck minimization case for the majority of the considered instances. Overall, the previous observations indicate that, although the set of feasible flows do not change in the robust compared to the nominal setting, the set of Wardrop equilibria may change. Hence, for given tolls $\tau \in \mathcal{T}$, neither $S_{\mathrm{rob}}(\tau) \subseteq S(\tau)$ nor $S_{\mathrm{rob}}(\tau) \supseteq S(\tau)$ holds in general.

TABLE 4. The revenues realized by imposing tolls for both the nominal and robust toll-setting problems under revenue maximization for the "Sioux Falls East" network with varying numbers of OD pairs ("$|\mathcal{K}|$") and toll arcs ("$|\mathcal{A}^{\text{toll}}|$"). Additionally, the maximum congestion and the optimality gaps (in %) are shown.

		nominal			robust		
$\lvert\mathcal{K}\rvert$	$\lvert\mathcal{A}^{\text{toll}}\rvert$	congestion	revenues	gap	congestion	revenues	gap
4	4	0.35	6000.00	0.00	0.43	8757.14	0.01
	6	0.35	7176.11	0.01	0.41	10 086.90	0.01
	8	0.35	7176.12	0.01	0.43	11 794.84	0.01
	10	0.35	13 500.00	0.00	0.28	16 897.52	0.74
5	4	0.35	6647.06	0.00	0.42	7934.73	0.01
	6	0.35	7176.11	0.01	0.40	11 638.56	0.02
	8	0.35	7835.83	0.01	0.42	12 450.18	1.86
	10	0.35	15 497.98	0.01	0.39	19 208.61	3.43
6	4	0.35	8258.36	0.00	0.43	9719.71	0.01
	6	0.35	7176.11	0.00	0.41	9955.25	0.11
	8	0.35	9447.12	0.01	0.43	12 093.90	0.02
	10	0.35	15 497.98	0.01	0.35	18 745.50	8.96
7	4	0.35	8335.27	0.00	0.43	9703.87	4.94
	6	0.35	7194.21	0.00	0.41	12 130.98	0.09
	8	0.35	9576.70	0.01	0.43	14 378.96	0.85
	10	0.35	17 067.11	0.01	0.39	21 685.01	9.56
8	4	0.35	8935.27	0.00	0.42	11 693.38	0.01
	6	0.35	16 614.47	0.00	0.43	22 090.28	0.01
	8	0.35	18 543.04	0.01	0.48	25 047.73	3.33
	10	0.35	22 281.52	0.08	0.52	26 588.36	1.05
9	4	0.45	9311.43	0.01	0.64	9651.38	5.79
	6	0.35	12 449.58	0.01	0.64	18 502.59	0.00
	8	0.54	18 251.39	2.20	0.66	23 405.38	14.34
	10	0.40	27 735.72	0.13	0.74	32 531.35	3.02
10	4	0.50	10 238.44	0.01	0.65	7601.56	60.71
	6	0.50	15 147.29	0.00	0.61	17 285.09	22.76
	8	0.63	20 269.78	0.01	0.85	24 193.01	3.82
	10	0.51	30 396.42	0.37	0.85	37 082.86	7.87

For the ease of presentation, we now focus on the "Sioux Falls East" instance with 4 OD pairs and 4 toll arcs to illustrate the impact of robustified travel decisions on the route choices and the imposed tolls. In Figure 2, we show the flows in a nominal Wardrop equilibrium and the tolls imposed by the traffic authority, which are both obtained from solving the toll-setting problem under revenue maximization (7). It can be seen that revenues are only generated by imposing tolls on arc $(15, 22)$. The remaining toll arcs are not used by any commodity. Nevertheless, we emphasize that imposing tolls on arcs with zero flow may still be beneficial for the traffic authority, even if no revenues are generated. In this way, the traffic authority can influence the travelers' decisions such as to encourage or discourage the use of specific arcs. This may lead to overall higher revenues. Moreover, we emphasize that decreasing the

Table 5. The revenues realized by imposing tolls for both the nominal and robust toll-setting problems under bottleneck minimization for the "Sioux Falls East" network with varying numbers of OD pairs ("$|\mathcal{K}|$") and toll arcs ("$\left|\mathcal{A}^{\text{toll}}\right|$"). Additionally, the maximum congestion and the optimality gaps (in %) are shown.

		nominal			robust		
$\lvert\mathcal{K}\rvert$	$\left\lvert\mathcal{A}^{\text{toll}}\right\rvert$	congestion	revenues	gap	congestion	revenues	gap
4	4	0.35	1731.25	0.00	0.37	3372.58	0.00
	6	0.35	5903.33	0.00	0.28	4024.41	0.00
	8	0.35	3903.33	0.00	0.26	6439.12	0.00
	10	0.29	3715.01	0.00	0.25	10 004.21	0.01
5	4	0.35	1959.91	0.00	0.41	7866.02	0.01
	6	0.35	2731.99	0.00	0.39	9359.26	0.00
	8	0.35	600.00	0.00	0.39	8072.96	0.00
	10	0.32	3431.34	0.00	0.37	13 378.71	0.00
6	4	0.35	6524.49	0.00	0.43	8050.91	0.01
	6	0.35	6089.05	0.00	0.35	6000.00	0.00
	8	0.35	6004.26	0.00	0.39	8532.51	0.01
	10	0.32	4931.34	0.00	0.35	8829.59	0.00
7	4	0.35	8335.27	0.00	0.42	9654.19	0.00
	6	0.35	3605.79	0.00	0.38	8370.54	0.01
	8	0.35	7850.26	0.00	0.41	7190.78	0.00
	10	0.32	4931.34	0.00	0.35	6278.22	0.00
8	4	0.35	2681.17	0.00	0.42	10 537.77	0.01
	6	0.35	8551.40	0.00	0.43	20 749.37	0.00
	8	0.35	12 399.29	0.00	0.44	21 619.02	0.00
	10	0.27	14 428.12	0.01	0.41	22 150.00	0.01
9	4	0.35	9164.85	0.00	0.60	5862.63	0.01
	6	0.35	7172.36	0.00	0.64	6620.13	1.07
	8	0.35	8624.44	0.01	0.64	22 269.26	0.01
	10	0.28	16 093.29	0.00	0.59	23 599.56	0.01
10	4	0.50	5353.02	0.00	0.54	7084.17	46.79
	6	0.50	7709.01	0.00	0.56	13 248.28	38.80
	8	0.50	15 923.03	0.00	0.55	14 682.19	7.74
	10	0.40	18 069.02	0.00	0.52	25 048.34	0.00

imposed tolls on arcs with zero flow may affect the flows in a Wardrop equilibrium. Let us now consider the specific routes taken by each commodity. In Figure 2, we observe that the green and the blue commodities, i.e., OD pairs $(9, 21)$ and $(17, 22)$, take the most direct route to reach their destination. In doing so, they accept to pay tolls along the way. The orange and the purple commodities, i.e., OD pairs $(8, 20)$ and $(16, 21)$, do not take the most direct route and prefer to take a detour to avoid being charged tolls. However, the situation may change significantly if users of the traffic network hedge against uncertain travel costs in a robust way. In Figure 3, we show the imposed tolls and the flows in a robust Wardrop equilibrium, which are both obtained from solving Problem (23). There are five aspects that we find remarkable. First, revenues are now additionally generated by imposing tolls on

TABLE 6. Nominal vs. robust travel costs for each OD pair in the "Sioux Falls East" network with 4 OD pairs and a varying number of toll arcs ("$|\mathcal{A}^{\text{toll}}|$").

		revenue maximization		bottleneck minimization	
$\|\mathcal{A}^{\text{toll}}\|$	OD pair	nominal	robust	nominal	robust
4	(8, 20)	10.02	15.92	10.02	15.92
	(9, 21)	16.44	24.87	14.30	23.49
	(16, 21)	14.59	27.66	14.59	23.82
	(17, 22)	11.55	18.56	9.42	16.58
6	(8, 20)	10.02	15.92	10.02	16.10
	(9, 21)	15.93	24.64	15.30	22.37
	(16, 21)	13.68	28.40	14.59	24.37
	(17, 22)	11.55	18.46	11.50	16.99
8	(8, 20)	10.02	15.92	10.02	16.07
	(9, 21)	15.93	24.58	14.30	24.57
	(16, 21)	13.68	27.34	14.59	26.20
	(17, 22)	11.55	18.73	10.50	16.66
10	(8, 20)	10.02	16.10	10.09	16.14
	(9, 21)	17.35	24.43	17.04	22.36
	(16, 21)	16.02	26.05	13.00	20.41
	(17, 22)	12.79	19.14	12.06	18.07

the arcs that connect nodes 16 and 17. This is in contrast to the nominal setting. Second, we note that robust travel decisions do not affect the actual tolls charged on the arcs of the network for this instance. Although the traffic authority imposes higher tolls on arc $(16, 17)$ in the robust setting (0 vs. 2), imposing tolls of 2 would be optimal in the nominal setting as well; cf. Proposition 6 in which we show that tolls dominating optimal toll-setting policies are also optimal. Third, the green and the orange commodities do not change their travel decisions when hedging against uncertain travel cost in a robust way. Also in the robust setting, the green commodity takes the most direct route, accepting the toll charges, whereas the orange commodity takes a detour to avoid toll arcs. Fourth, the travel decision of the purple commodity changes completely in the robust setting. Instead of taking a toll-free detour, it now takes the most direct route, which includes a toll arc. Finally, we point out that the flow of the blue commodity is split between the most direct tolled route and the toll-free detour. Moreover, the flows of the blue commodity are split between $(20, 21, 22)$ and $(20, 22)$ on the toll-free path. For the case of bottleneck minimization, we observe the same qualitative behavior as in the case of revenue maximization. The corresponding figures and additional discussion for this setting are provided in Appendix A.2.

To sum up, our case study illustrates that making robust travel decisions due to travel cost uncertainties can significantly impact the travel behavior, the revenues realized by imposing tolls, and congestion. We have seen that users of the traffic network, who hedge against uncertain travel costs within their user-specific uncertainty set, may be indifferent to uncertainties, change their travel decisions completely, or decide on something in between.

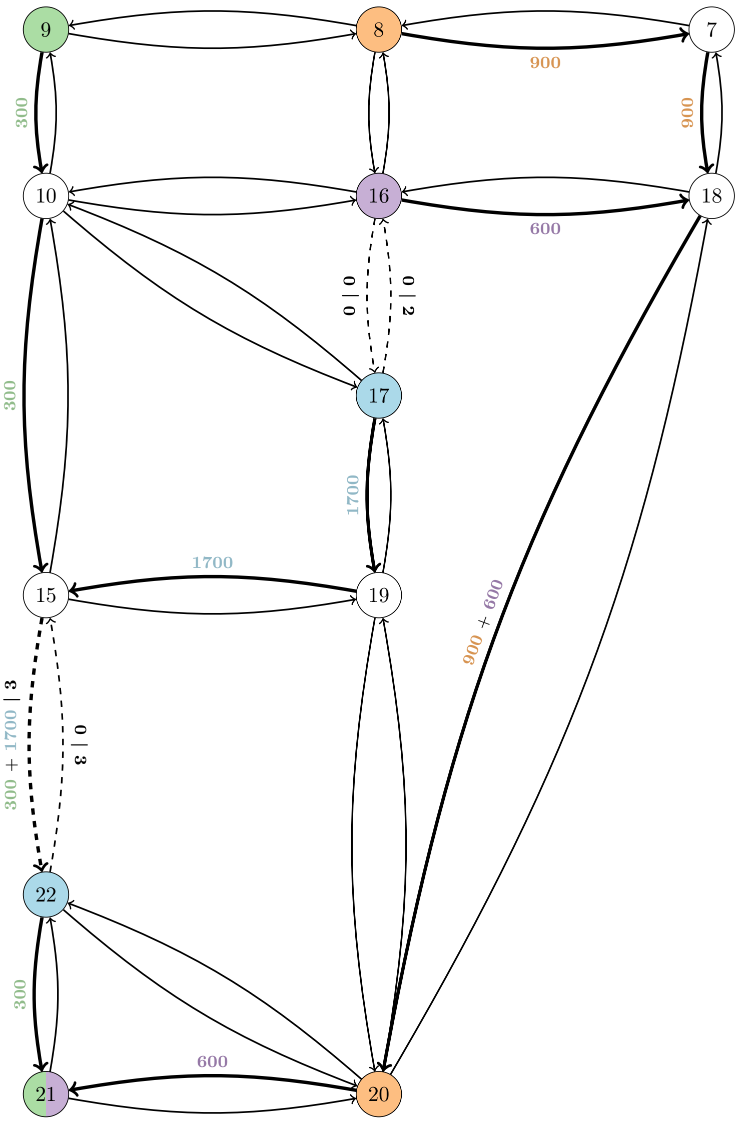

Figure 2. The "Sioux Falls East" network with 4 OD pairs and 4 toll arcs under revenue maximization. Each OD pair is color-coded (orange, green, blue, purple). Dashed arcs represent toll arcs and solid arcs represent toll-free arcs. Edge labels correspond to commodity flows. For toll arcs, edge labels are given in the format "flow | toll". If no label is shown, there is no flow on that edge.

6. Conclusion

In this paper, we consider a multi-commodity traffic network in which a traffic authority imposes tolls on some arcs of the network. As collecting tolls can serve multiple purposes, we consider two exemplary variants of the toll-setting problem, which reflect different aims of the traffic authority. First, we consider the setting in which tolls are used to maximize revenues. Second, we study the setting in which the traffic authority uses tolls to encourage a more efficient use of road capacity by alleviating bottlenecks in the network. Users of the traffic network are assumed to act in the sense of Wardrop's user equilibrium, minimizing their individual travel costs.

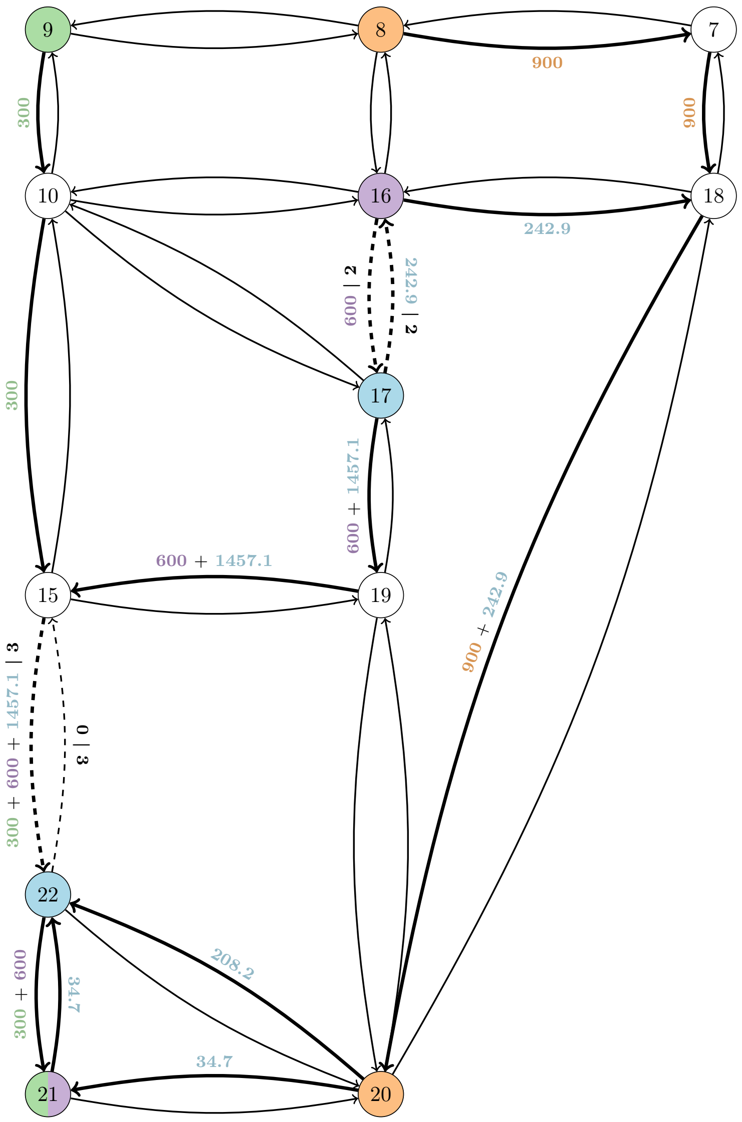

FIGURE 3. The robust "Sioux Falls East" network with 4 OD pairs and 4 toll arcs under revenue maximization. Each OD pair is color-coded (orange, green, blue, purple). Dashed arcs represent toll arcs and solid arcs represent toll-free arcs. Edge labels correspond to commodity flows. For toll arcs, edge labels are given in the format "flow | toll". If no label is shown, there is no flow on that edge.

We model the toll-setting problems as mathematical problems with equilibrium constraints, for which we present nonconvex mixed-integer nonlinear reformulations that exploit binary variables and big-M constants. We prove the existence of optimal solutions to these problems, derive correct big-Ms, and provide valid inequalities. Moreover, we consider the setting in which the network users pursue a robust approach to hedge against uncertainties in their travel costs. We reformulate the robustified toll-setting problems as nonconvex mixed-integer nonlinear problems and prove the existence of robust solutions. To illustrate the effects of considering robust travel decisions, we conduct a case study based on the well-known Sioux

Falls network. We observe that addressing uncertainties in the travel costs can significantly impact the travel behavior, the revenues realized by imposing tolls, and congestion. In particular, we observe that the robustification of the travel costs may lead to increased revenues realized by imposing tolls.

Given that solving the robustified toll-setting problems is significantly more challenging than solving their nominal counterparts, a potential future research question could be to identify situations in which quality guarantees for nominal toll-setting policies in the robust setting are available. Finally, as this paper serves as a first step towards solving toll-setting problems under Wardrop equilibria to global optimality, the development of tailored solution methods seems essential to address these challenging models effectively.

Acknowledgment

We kindly acknowledge the support of RHRK (https://rz.rptu.de/en/). Moreover, Martine Labbé thanks the DFG for their support within the Research Training Group 2126: "Algorithmic Optimization". This research was funded in whole or in part by the French National Research Agency (ANR) under project ANR-24-CE48-3565-03.

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Appendix A. Supplementary Material for the Case Study

A.1. **Additional Tables.** In this section, we present extended versions of Tables 4 and 5, which additionally include the runtimes and the number of branch-and-bound nodes required to solve the nominal and robust toll-setting problems under revenue maximization and bottleneck minimization. The detailed results are provided in Tables 7–10.

Table 7. The revenues realized by imposing tolls, the runtimes (in s), and the number of investigated branch-and-bound nodes required to solve the nominal toll-setting problem under revenue maximization for the "Sioux Falls East" network with varying numbers of OD pairs ("$|\mathcal{K}|$") and toll arcs ("$|\mathcal{A}^{\text{toll}}|$"). Additionally, the maximum congestion and the optimality gaps (in %) are shown.

$\lvert\mathcal{K}\rvert$	$\lvert\mathcal{A}^{\text{toll}}\rvert$	congestion	revenues	runtime	B&B nodes	gap
4	4	0.35	6000.00	1.60	3700	0.00
	6	0.35	7176.11	3.36	11 496	0.01
	8	0.35	7176.12	7.37	26 498	0.01
	10	0.35	13 500.00	3.46	5272	0.00
5	4	0.35	6647.06	0.82	128	0.00
	6	0.35	7176.11	11.93	33 395	0.01
	8	0.35	7835.83	172.71	543 050	0.01
	10	0.35	15 497.98	210.97	709 912	0.01
6	4	0.35	8258.36	1.34	11	0.00
	6	0.35	7176.11	4.70	4081	0.00
	8	0.35	9447.12	105.48	251 172	0.01
	10	0.35	15 497.98	885.38	1 660 983	0.01
7	4	0.35	8335.27	2.05	1854	0.00
	6	0.35	7194.21	9.78	9346	0.00
	8	0.35	9576.70	712.78	1 617 701	0.01
	10	0.35	17 067.11	2235.69	4 641 879	0.01
8	4	0.35	8935.27	5.30	4476	0.00
	6	0.35	16 614.47	13.79	13 501	0.00
	8	0.35	18 543.04	2111.70	7 638 570	0.01
	10	0.35	22 281.52	TL	6 178 299	0.08
9	4	0.45	9311.43	31.17	27 341	0.01
	6	0.35	12 449.58	1007.71	1 661 633	0.01
	8	0.54	18 251.39	TL	4 160 954	2.20
	10	0.40	27 735.72	TL	8 183 421	0.13
10	4	0.50	10 238.44	14.19	9096	0.01
	6	0.50	15 147.29	78.18	60 742	0.00
	8	0.63	20 269.78	88.86	97 800	0.01
	10	0.51	30 396.42	TL	6 579 466	0.37

A.2. **Nominal vs. Robust Travel Behavior Under Bottleneck Minimization.** We now illustrate the impact of robustified travel decisions on the route choices and the imposed tolls for the toll-setting problem under bottleneck minimization. As in Section 5.3, we focus on the "Sioux Falls East" network with 4 OD pairs and 4 toll arcs for the ease of presentation.

Table 8. The revenues realized by imposing tolls, the runtimes (in s), and the number of investigated branch-and-bound nodes required to solve the robust toll-setting problem under revenue maximization for the "Sioux Falls East" network with varying numbers of OD pairs ("$|\mathcal{K}|$") and toll arcs ("$\left|\mathcal{A}^{\text{toll}}\right|$"). Additionally, the maximum congestion and the optimality gaps (in %) are shown.

$\|\mathcal{K}\|$	$\left\|\mathcal{A}^{\text{toll}}\right\|$	congestion	revenues	runtime	B&B nodes	gap
4	4	0.43	8757.14	83.31	63 943	0.01
	6	0.41	10 086.90	118.40	110 862	0.01
	8	0.43	11 794.84	1489.22	1 378 122	0.01
	10	0.28	16 897.52	TL	1 964 263	0.74
5	4	0.42	7934.73	734.22	861 032	0.01
	6	0.40	11 638.56	TL	6 390 494	0.02
	8	0.42	12 450.18	TL	3 369 419	1.86
	10	0.39	19 208.61	TL	1 379 948	3.43
6	4	0.43	9719.71	636.39	668 277	0.01
	6	0.41	9955.25	TL	4 908 624	0.11
	8	0.43	12 093.90	TL	3 777 374	0.02
	10	0.35	18 745.50	TL	629 620	8.96
7	4	0.43	9703.87	TL	3 496 359	4.94
	6	0.41	12 130.98	TL	866 147	0.09
	8	0.43	14 378.96	TL	1 995 523	0.85
	10	0.39	21 685.01	TL	582 480	9.56
8	4	0.42	11 693.38	916.32	320 812	0.01
	6	0.43	22 090.28	3003.24	2 636 792	0.01
	8	0.48	25 047.73	TL	448 738	3.33
	10	0.52	26 588.36	TL	1 216 606	1.05
9	4	0.64	9651.38	TL	329 696	5.79
	6	0.64	18 502.59	2676.47	271 821	0.00
	8	0.66	23 405.38	TL	334 875	14.34
	10	0.74	32 531.35	TL	692 869	3.02
10	4	0.65	7601.56	TL	655 427	60.71
	6	0.61	17 285.09	TL	369 817	22.76
	8	0.85	24 193.01	TL	454 167	3.82
	10	0.85	37 082.86	TL	356 413	7.87

In Figure 4, we show the flows in a nominal Wardrop equilibrium and the tolls imposed by the traffic authority, which are both obtained from solving the toll-setting problem under bottleneck minimization (8). We observe that the equilibrium flows in the nominal Wardrop user equilibrium are identical to those in the revenue maximization case. As before, revenues are only generated by imposing tolls on arc $(15, 22)$, whereas the remaining toll arcs are not used by any commodity. However, the toll imposed on arc $(15, 22)$ is now lower than in the revenue maximization setting (0.9 vs. 3). This indicates that smaller tolls are sufficient to maintain the current equilibrium flows at the worst-case congestion of 0.35, which is observed on arc $(19, 15)$. This arc is part of the downtown area of "Sioux Falls East", which consists of nodes 16, 17, and 19. Overall, the additional revenues gained by

TABLE 9. The revenues realized by imposing tolls, the runtimes (in s), and the number of investigated branch-and-bound nodes required to solve the nominal toll-setting problem under bottleneck minimization for the "Sioux Falls East" network with varying numbers of OD pairs ("$|\mathcal{K}|$") and toll arcs ("$|\mathcal{A}^{\text{toll}}|$"). Additionally, the maximum congestion and the optimality gaps (in %) are shown.

$\lvert\mathcal{K}\rvert$	$\lvert\mathcal{A}^{\text{toll}}\rvert$	congestion	revenues	runtime	B&B nodes	gap
4	4	0.35	1731.25	0.37	159	0.00
	6	0.35	5903.33	0.26	1	0.00
	8	0.35	3903.33	0.59	629	0.00
	10	0.29	3715.01	0.38	1	0.00
5	4	0.35	1959.91	0.62	160	0.00
	6	0.35	2731.99	0.49	126	0.00
	8	0.35	600.00	0.52	627	0.00
	10	0.32	3431.34	1.09	2681	0.00
6	4	0.35	6524.49	0.49	129	0.00
	6	0.35	6089.05	0.55	178	0.00
	8	0.35	6004.26	0.58	558	0.00
	10	0.32	4931.34	1.47	2249	0.00
7	4	0.35	8335.27	0.89	566	0.00
	6	0.35	3605.79	1.78	1249	0.00
	8	0.35	7850.26	2.55	4295	0.00
	10	0.32	4931.34	1.99	3657	0.00
8	4	0.35	2681.17	0.49	76	0.00
	6	0.35	8551.40	2.47	3493	0.00
	8	0.35	12 399.29	2.47	4741	0.00
	10	0.27	14 428.12	2.90	1699	0.01
9	4	0.35	9164.85	0.81	67	0.00
	6	0.35	7172.36	0.95	45	0.00
	8	0.35	8624.44	1.72	1176	0.01
	10	0.28	16 093.29	2.85	1535	0.00
10	4	0.50	5353.02	4.60	2327	0.00
	6	0.50	7709.01	5.53	3490	0.00
	8	0.50	15 923.03	4.02	1487	0.00
	10	0.40	18 069.02	11.28	22 616	0.00

considering the alternative metric of revenue maximization instead of bottleneck minimization thus amount to $(3 - 0.9)2000 = 4200$ monetary units.

As in the revenue maximization case, the situation may change significantly if users of the traffic network hedge against uncertain travel costs in a robust way. In Figure 5, we show the imposed tolls and the flows in a robust Wardrop equilibrium, which are both obtained from solving Problem (24). In contrast to the revenue maximization case, no additional revenues are generated by imposing tolls on the arcs connecting nodes 16 and 17. However, compared to the nominal bottleneck minimization setting, the toll charged on arc $(15, 22)$ is now slightly higher (1.4 vs. 0.9). Moreover, we again observe that travelers, who hedge against uncertain travel costs in a robust way, may be indifferent to uncertainties, change their decisions completely, or decide on something in between. In Figure 5, the

TABLE 10. The revenues realized by imposing tolls, the runtimes (in s), and the number of investigated branch-and-bound nodes required to solve the robust toll-setting problem under bottleneck minimization for the "Sioux Falls East" network with varying numbers of OD pairs ("$|\mathcal{K}|$") and toll arcs ("$|\mathcal{A}^{\text{toll}}|$"). Additionally, the maximum congestion and the optimality gaps (in %) are shown.

$\lvert\mathcal{K}\rvert$	$\lvert\mathcal{A}^{\text{toll}}\rvert$	congestion	revenues	runtime	B&B nodes	gap
4	4	0.37	3372.58	14.10	11 426	0.00
	6	0.28	4024.41	8.80	8055	0.00
	8	0.26	6439.12	10.17	15 505	0.00
	10	0.25	10 004.21	18.84	41 336	0.01
5	4	0.41	7866.02	188.88	364 141	0.01
	6	0.39	9359.26	997.05	2 646 274	0.00
	8	0.39	8072.96	35.49	34 122	0.00
	10	0.37	13 378.71	222.73	140 962	0.00
6	4	0.43	8050.91	435.62	379 474	0.01
	6	0.35	6000.00	15.71	1936	0.00
	8	0.39	8532.51	69.84	47 618	0.01
	10	0.35	8829.59	14.23	5339	0.00
7	4	0.42	9654.19	113.67	42 588	0.00
	6	0.38	8370.54	1792.48	764 745	0.01
	8	0.41	7190.78	2113.66	2 427 423	0.00
	10	0.35	6278.22	25.88	5307	0.00
8	4	0.42	10 537.77	1101.95	430 373	0.01
	6	0.43	20 749.37	114.29	21 251	0.00
	8	0.44	21 619.02	134.19	20 265	0.00
	10	0.41	22 150.00	187.99	168 549	0.01
9	4	0.60	5862.63	1430.63	614 133	0.01
	6	0.64	6620.13	TL	1 522 597	1.07
	8	0.64	22 269.26	200.75	62 177	0.01
	10	0.59	23 599.56	685.35	858 456	0.01
10	4	0.54	7084.17	TL	755 023	46.79
	6	0.56	13 248.28	TL	460 062	38.80
	8	0.55	14 682.19	TL	903 806	7.74
	10	0.52	25 048.34	1469.98	1 479 879	0.00

green and the orange commodities, i.e., OD pairs $(9, 21)$ and $(8, 20)$, do not change their travel decisions when hedging against travel cost uncertainties. In the robust setting, the green commodity still takes the most direct route, accepting the toll charges, whereas the orange commodity takes a detour to avoid toll arcs. The purple commodity, i.e., OD pair $(16, 21)$, changes its route completely in the robust setting. Instead of taking a toll-free detour, it now follows the most direct route, which includes a toll arc. Finally, the flow of the blue commodity, i.e., OD pair $(17, 22)$, is split between several tolled and toll-free paths. This results in more arcs in the network being used and, consequently, commodity flows being more evenly spread across the network; see also Table 11. As a result, the downtown area of "Sioux Falls East" is less congested when considering the bottleneck minimization compared to the revenue maximization objective. For robust bottleneck minimization, the

Table 11. The number of arcs (out of the 36 arcs considered in the "Sioux Falls East" network with 4 OD pairs and 4 toll arcs) that carry positive flow in a Wardrop equilibrium obtained from the toll-setting problems under revenue maximization and bottleneck minimization.

	revenue	bottleneck
nominal	11	11
robust	14	16

maximum congestion is 0.37, which occurs on arc $(17, 19)$, compared to 0.43 in the robust revenue maximization case.

To sum up, considering the bottleneck minimization objective for the toll-setting problem can help alleviate congestion in the busier areas of the "Sioux Falls East" network. For the robust setting, it can be seen that this leads to commodity flows being more evenly spread across the network. Moreover, as in the revenue maximization case, we observe that making robust travel decisions because of travel cost uncertainties can significantly affect the travelers' behavior.

(Y. Beck) Eindhoven University of Technology, Department of Industrial Engineering and Innovation Sciences, PO Box 513, 5600 MB Eindhoven, the Netherlands
Email address: y.beck@tue.nl

(M. Labbé) (A) Université Libre de Bruxelles, Department of Computer Science, Boulevard du Triomphe, CP212, 1050 Brussels, Belgium; (B) Inria Lille - Nord Europe, Parc scientifique de la Haute Borne, 40 Avenue Halley - Bâtiment A - Park Plaza, 59650 Villeneuve d'Ascq, France
Email address: martine.labbe@ulb.be

(M. Schmidt) Trier University, Department of Mathematics, Universitätsring 15, 54296 Trier, Germany
Email address: martin.schmidt@uni-trier.de

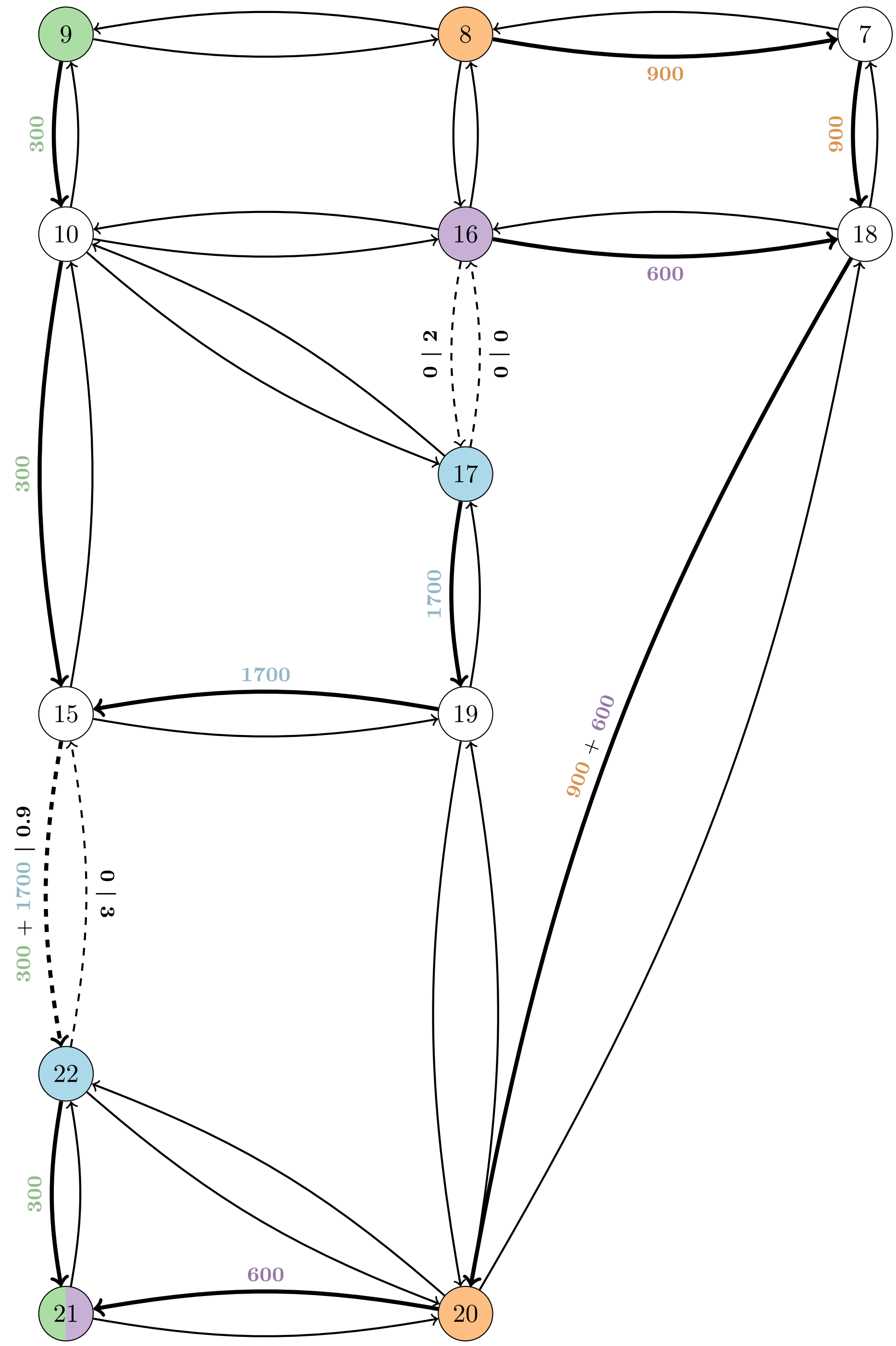

Figure 4. The "Sioux Falls East" network with 4 OD pairs and 4 toll arcs under bottleneck minimization. Each OD pair is color-coded (orange, green, blue, purple). Dashed arcs represent toll arcs and solid arcs represent toll-free arcs. Edge labels correspond to commodity flows. For toll arcs, edge labels are given in the format "flow | toll". If no label is shown, there is no flow on that edge.

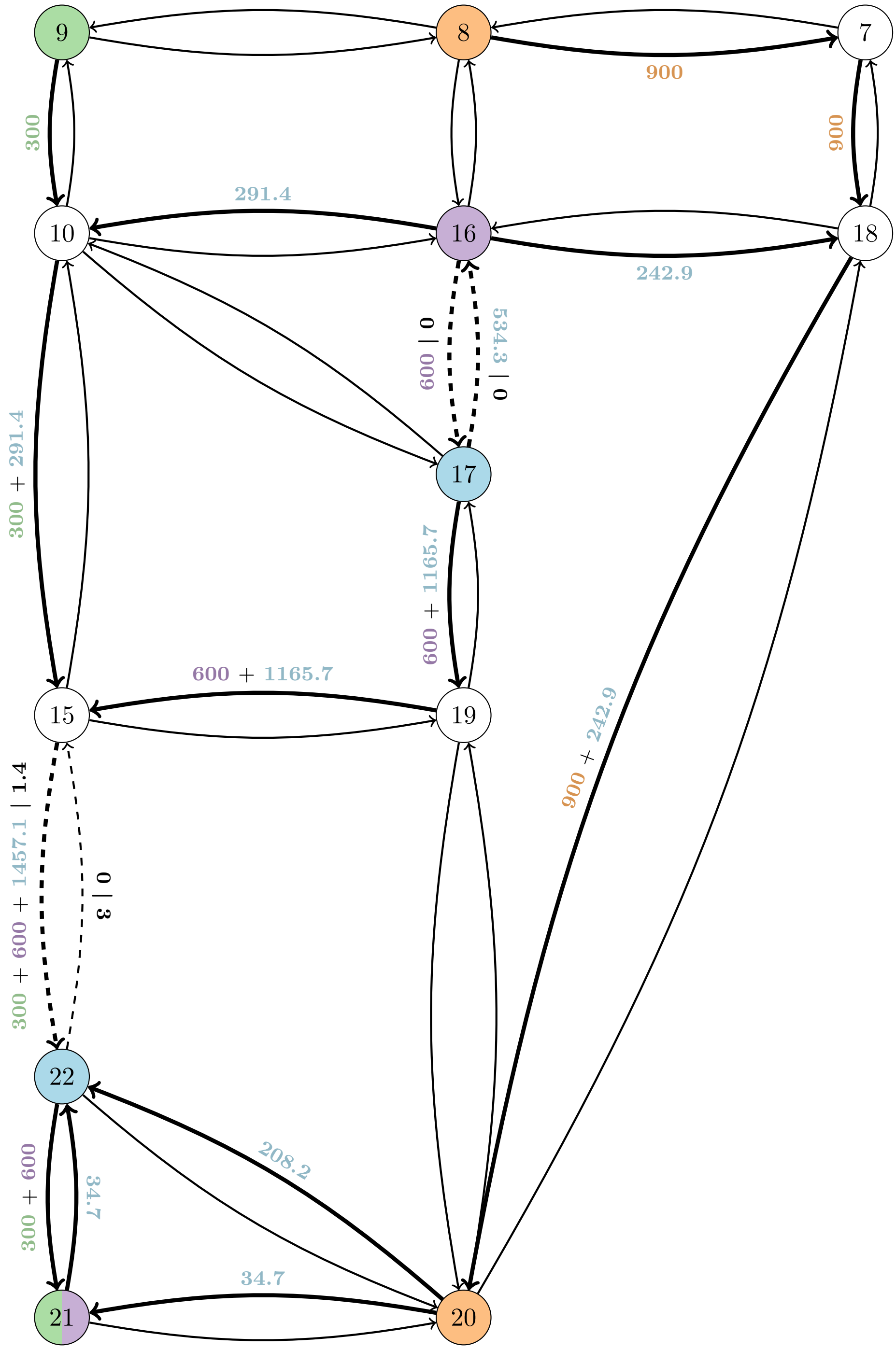

Figure 5. The robust "Sioux Falls East" network with 4 OD pairs and 4 toll arcs under bottleneck minimization. Each OD pair is color-coded (orange, green, blue, purple). Dashed arcs represent toll arcs and solid arcs represent toll-free arcs. Edge labels correspond to commodity flows. For toll arcs, edge labels are given in the format "flow | toll". If no label is shown, there is no flow on that edge.