A Toll-Setting Problem with Robust Wardrop Equilibrium Conditions Under Budgeted Uncertainty

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Abstract. We consider the problem of determining optimal tolls in a traffic network in which a toll-setting authority aims to maximize revenues and the users of the network act in the sense of Wardrop's user equilibrium. The setting is modeled as a mathematical problem with equilibrium constraints and a mixed-integer, nonlinear, and nonconvex reformulation is presented that exploits binary variables and big-M constants. We prove existence of optimal solutions to this problem, derive correct big-Ms, and provide valid inequalities. Moreover, we consider the setting in which the network users hedge against uncertainties regarding their travel costs. We model this setting using robust Wardrop equilibria under budgeted uncertainty and prove existence of robust solutions. Finally, we present preliminary computational results to illustrate the impact of considering robust travel decisions on the revenues realized by the toll-setting authority.

1. Introduction

In traffic networks, collecting tolls is a powerful tool for network management and for influencing travel behavior. For instance, revenues generated by imposing tolls may support the maintenance of existing infrastructure or fund the construction of new roads. In addition, tolls may be used to manage traffic flow by alleviating congestion and encouraging the more efficient use of road capacity. Thus, it is evident that determining optimal tolls in a traffic network is an important aspect of transportation science. In this context, the toll-setting authority has to decide on the tolls while anticipating the reaction of the users of the traffic network, who usually try to minimize costs and time spent on travel. The overall toll-setting problem can thus be seen as a single-leader multi-follower game in which the toll-setting authority acts as the leader and the users of the traffic network act as the followers. Influential works in this context include, e.g., Brotcorne et al. (2001), Dempe and Zemkoho (2012), Dewez et al. (2008), Kalashnikov et al. (2020), and Labbé et al. (1998, 2000).

In this paper, we consider a multi-commodity traffic network in which a toll-setting authority decides on the tolls of (some of) the arcs of the network. While we consider the setting in which the toll-setting authority aims to maximize revenues by imposing tolls, we emphasize that other objective functions may be possible as well. Regarding the users of the traffic network, we assume that they act according to Wardrop's user equilibrium (Wardrop 1952; Wardrop and Whitehead 1952), minimizing their individual travel costs that are parameterized by the imposed tolls. In this paper, we do not make any assumptions about the separability of the travel costs but they are assumed to be affine-linear in the traffic flows. We model the overall toll-setting problem as a mathematical problem with equilibrium

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constraints (MPEC); see, e.g., Luo et al. (1996) for a general overview. In contrast to Dempe and Zemkoho (2012), who study a similar setting for separable cost functions from a theoretical point of view, we consider the toll-setting problem more from a computational perspective. To this end, we reformulate the problem as a mixed-integer, nonlinear, and nonconvex problem (nonconvex MINLP) that exploits binary variables and big-M constants. The latter can be tackled using state-of-the-art general-purpose solvers. We provide results on the existence of optimal solutions to this problem as well as valid inequalities to enhance the problem formulation. Moreover, we derive valid big-M constants.

In addition, we study the toll-setting problem in which the users of the traffic network face uncertainties regarding their travel costs, which we tackle using techniques from robust optimization; see, e.g., Ben-Tal et al. (2009), Bertsimas et al. (2011), and Soyster (1973). To this end, we pursue similar ideas compared to those in Ito (2011) and Ordóñez and Stier-Moses (2007, 2010), who also consider so-called robust Wardrop equilibria. In Ito (2011), a strictly robust setup is considered to hedge against uncertainties regarding the travel costs. The author makes the necessary continuity assumptions on the robustified travel cost functions to ensure that robust Wardrop equilibria exist. In particular, the author considers ellipsoidal uncertainty sets. In Ordóñez and Stier-Moses (2007), the authors pursue a Γ-robust approach (Bertsimas and Sim 2003; Sim 2004) to hedge against uncertain travel costs. The authors provide existence results for robust Wardrop equilibria and present a column-generation algorithm to compute them. In a follow-up paper, Ordóñez and Stier-Moses (2010) provide more extensive theoretical and computational details on this approach, along with further equilibrium concepts to hedge against uncertain travel costs. All aforementioned works have in common that the authors focus on the robust traffic assignment problem in which a path-based formulation is used to model the travelers' behavior. The modeling framework considered in this paper differs from those in Ito (2011) and Ordóñez and Stier-Moses (2007, 2010) in the following two aspects. First, we consider the problem of determining optimal tolls in a traffic network that incorporates robust Wardrop equilibria in the constraints of the problem. Second, we study robust Wardrop equilibria under budgeted uncertainty, which necessitates the use of a node-arc formulation to model the travelers' behavior. To the best of our knowledge, there are no other works in the literature that consider such a network pricing under robust Wardrop equilibria. We illustrate the impact of considering robust travel decisions on the revenues realized by the toll-setting authority through a case study on a subnetwork of the well-known Sioux Falls network (LeBlanc et al. 1975). Here, we observe that addressing uncertainties in the travel costs may significantly impact the travel behavior and, in particular, lead to increased revenues realized by imposing tolls.

The remainder of this paper is organized as follows. In Section 2, we present the overall toll-setting problem, which we model as an MPEC. In Section 3, we present an MINLP reformulation of the toll-setting problem, prove existence of optimal solutions, derive valid big-Ms, and provide valid inequalities. In Section 4, we present a robustified variant of the toll-setting problem under budgeted uncertainty. We provide an MINLP reformulation of this problem and prove existence of robust solutions. In Section 5, we present preliminary computational results to illustrate the impact of considering robust travel decisions on the revenues realized by the toll-setting authority. Finally, we conclude in Section 6.

2. The MPEC Model

We consider a traffic network that is modeled using a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ with node set $\mathcal{N}$ and arc set $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$. We elaborate on the graph's connectivity later in this section. In what follows, we denote $f = (f_a)_{a\in\mathcal{A}}$ as the vector of all arc flows and $\tau = (\tau_a)_{a\in\mathcal{A}}$ as the tolls imposed on the arcs of the network. The aim of the toll-setting authority is to maximize the revenues

$$\sum_{a\in\mathcal{A}} \tau_a f_a$$

that are realized by charging tolls on certain arcs of the network. Throughout this paper, we make the following assumption.

Assumption 1. *The tolls τ are subject to constraints described by a polytope $\mathcal{T} = \{\tau \in \mathbb{R}^{|\mathcal{A}|} \colon B\tau \leq b\} \neq \emptyset$ for some matrix B and a vector b of appropriate dimension.*

The constraints $B\tau \leq b$ are used to model, e.g., lower and upper bounds on the tolls or toll-free arcs. Here and in what follows, we assume that the set $\mathcal{T}$ induces a finite upper bound τ_a^+ as well as a lower bound of zero for the toll τ_a on each arc $a \in \mathcal{A}$. Arcs $a \in \mathcal{A}$ for which the set $\mathcal{T}$ imposes the upper bound $\tau_a^+ = 0$ are called toll-free arcs. All remaining arcs are called toll arcs. The overall toll-setting problem can now be stated as

$$\max_{\tau,f,x} \quad \sum_{a\in\mathcal{A}} \tau_a f_a \quad \text{s.t.} \quad \tau \in \mathcal{T}, \ (f,x) \in S(\tau). \tag{1}$$

Here, the set $S(\tau)$ is used to denote the Wardrop equilibria that are parameterized by the imposed tolls τ, which we discuss in detail in the following section. In particular, Problem (1) can be interpreted as a single-leader multi-follower problem in which the toll-setting authority acts as the leader and the users of the traffic network act as the followers. By optimizing over the tolls τ and the variables f and x, we consider the so-called optimistic approach as it is known in bilevel optimization; see, e.g., Dempe (2002). This means that, whenever there are multiple optimal route choices for the users of the network, they choose the ones that favor the leader the most w.r.t. the associated revenues. The latter is a common assumption in the literature; see, e.g., Brotcorne et al. (2001) and Labbé et al. (1998).

2.1. **Wardrop Equilibrium Conditions.** For node subsets $\mathcal{O}, \mathcal{D} \subseteq \mathcal{N}$, we denote the set of all origin-destination (OD) pairs of the network as $\mathcal{K} \subseteq \mathcal{O} \times \mathcal{D}$. For the ease of presentation, we consider a single commodity for each OD pair $k \in \mathcal{K}$. Let $x^k = (x_a^k)_{a\in\mathcal{A}} \in \mathbb{R}^{|\mathcal{A}|}$ denote the flow vector of commodity $k \in \mathcal{K}$. The vector of arc flows is then given by

$$f = \sum_{k\in\mathcal{K}} x^k \in \mathbb{R}^{|\mathcal{A}|}. \tag{2}$$

Throughout this paper, we make the following assumptions.

Assumption 2. *For every node $i \in \mathcal{N}$, there is at least one path that connects node i to each destination node $j \in \mathcal{D}$.*

Assumption 3. *For every commodity $k \in \mathcal{K}$, the travel demand $d_k \in \mathbb{R}$ is nonnegative and fixed.*

We emphasize that Assumption 2 is a standard assumption in the literature; cf., e.g., Assumption 2.A in Patriksson (2015). Let us further mention that Assumption 3 is w.l.o.g. since any elastic-demand problem can equivalently be reformulated as a fixed-demand problem; see, e.g., Dantzig et al. (1976) and Gartner (1980). For each commodity $k = (\alpha_k, \omega_k) \in \mathcal{K}$, flow conservation can now be modeled via

$$\sum_{a\in\delta^{\text{in}}(i)} x_a^k - \sum_{a\in\delta^{\text{out}}(i)} x_a^k = d_i^k, \quad i \in \mathcal{N}, \tag{3}$$

with

$$d_i^k = \begin{cases} +d_k, & i = \omega_k, \\ 0, & i \in \mathcal{N} \setminus \{\alpha_k, \omega_k\}, \\ -d_k, & i = \alpha_k. \end{cases}$$

Here, $\delta^{\text{in}}(i)$ and $\delta^{\text{out}}(i)$ denote the sets of in- and outgoing arcs of node $i \in \mathcal{N}$, respectively. Next, we elaborate on Wardrop's second[1] principle to model user-optimized behavior. It is assumed that the users of the traffic network seek to minimize their individual travel costs such that no user can reduce costs by unilaterally changing routes. This behavior can be modeled as

$$0 \leq c_a^k(f; \tau_a) + t_j^k - t_i^k \perp x_a^k \geq 0, \quad a = (i,j) \in \mathcal{A},\ k \in \mathcal{K}. \tag{4}$$

A similar setting is, e.g., considered in Section 3.6.2 in Ferris and Pang (1997). In (4), the cost for commodity $k \in \mathcal{K}$ to travel along an arc $a \in \mathcal{A}$ is given by the function $c_a^k(f; \tau_a)$ that depends on the overall flows f and that is parameterized by the imposed toll τ_a. Moreover, t_i^k denotes the minimum cost to reach the destination of commodity $k \in \mathcal{K}$ from node $i \in \mathcal{N}$. We abbreviate $t = (t^k)_{k \in \mathcal{K}}$ with $t^k = (t_i^k)_{i \in \mathcal{N}} \in \mathbb{R}^{|\mathcal{N}|}$. To sum up, the τ-parameterized set of Wardrop equilibria is given by

$$S(\tau) := \{(f, x) \colon \exists t \text{ such that } (f, x, t) \text{ solves (2)–(4)}\}.$$

3. An MINLP Reformulation

We introduce additional binary variables $z \in \{0,1\}^{|\mathcal{A}| \cdot |\mathcal{K}|}$ to obtain a reformulation of Problem (1) that is given by

$$\max_{\tau, f, x, t, z} \quad \sum_{a \in \mathcal{A}} \tau_a f_a \tag{5a}$$

$$\text{s.t.} \quad \tau \in \mathcal{T}, \quad f = \sum_{k \in \mathcal{K}} x^k, \tag{5b}$$

$$\sum_{a \in \delta^{\text{in}}(i)} x_a^k - \sum_{a \in \delta^{\text{out}}(i)} x_a^k = d_i^k, \qquad i \in \mathcal{N},\ k \in \mathcal{K}, \tag{5c}$$

$$x_a^k \geq 0, \quad c_a^k(f; \tau_a) + t_j^k - t_i^k \geq 0, \qquad a = (i,j) \in \mathcal{A},\ k \in \mathcal{K}, \tag{5d}$$

$$c_a^k(f; \tau_a) + t_j^k - t_i^k \leq M_a^k (1 - z_a^k), \qquad a = (i,j) \in \mathcal{A},\ k \in \mathcal{K}, \tag{5e}$$

$$x_a^k \leq M_a^k z_a^k, \qquad a \in \mathcal{A},\ k \in \mathcal{K}, \tag{5f}$$

$$z_a^k \in \{0,1\}, \qquad a \in \mathcal{A},\ k \in \mathcal{K}. \tag{5g}$$

Problem (5) is as a mixed-integer nonlinear problem (MINLP) due to bilinearities in the objective function as well as possible nonlinearities in the travel cost functions $c_a^k(f; \tau_a)$, $a \in \mathcal{A}$, $k \in \mathcal{K}$. By construction, Problem (5) is equivalent to the toll-setting problem (1) if the big-M constants M_a^k, $a \in \mathcal{A}$, $k \in \mathcal{K}$, are chosen sufficiently large. To obtain such constants, however, we need further knowledge about the travel cost functions $c_a^k(f; \tau_a)$. In this paper, we assume that the travel cost functions are affine-linear in the flows f so that, in Problem (5), we consider a bilinear objective that is optimized over mixed-integer and linear constraints. We acknowledge that this is a strong assumption. However, we illustrate in Section 5 that even under this simplifying assumption, solving the toll-setting problem is a highly challenging task.

[1]In the literature, the user optimum is commonly referred to as Wardrop's second principle, despite it being introduced first in Wardrop and Whitehead (1952); see, e.g., the respective discussion by Ferris and Pang (1997).

Assumption 4. *For every commodity $k \in \mathcal{K}$, the travel cost functions $c^k(f;\tau) = \left(c_a^k(f;\tau_a)\right)_{a\in\mathcal{A}}$ are affine-linear in the flows, i.e., there exists a matrix $C^k \in \mathbb{R}_{\geq 0}^{|\mathcal{A}|\times|\mathcal{A}|}$ and a vector $c^{fix,k} \in \mathbb{R}_{>0}^{|\mathcal{A}|}$ with $c^k(f;\tau) = C^k f + c^{fix,k} + \tau$.*

We emphasize that we do not make any assumptions about the separability of the travel cost functions in Assumption 4. In traffic assignment problems, it is often interesting to consider travel costs that are non-separable. This means that, for $k \in \mathcal{K}$, the costs $c_a^k(f;\tau_a)$ may not only depend on the flow f_a on arc $a \in \mathcal{A}$ itself but also on the flows $f_{a'}$, $a' \neq a \in \mathcal{A}$, on the other arcs. For motivating examples as well as further discussions on non-separable travel costs, we refer to Dafermos (1971). In Assumption 4, non-separability implies that the travel cost matrix C^k is not a diagonal matrix.

The remainder of this section is organized as follows. In Section 3.1, we elaborate on how to obtain sufficiently large big-M constants that can be used in Problem (5). Afterward, in Section 3.2, we prove the existence of an optimal solution to the toll-setting problem (5). In Section 3.3, we provide valid inequalities to strengthen the formulation in (5).

3.1. **Computing Big-Ms.** In what follows, we provide bounds for the flow variables f and x as well as for the minimum travel costs t, which are essential for obtaining sufficiently large big-Ms that can be used in the MINLP reformulation (5) of the toll-setting problem. For this purpose, we first establish the existence of a Wardrop equilibrium for any given toll-setting policy $\tau \in \mathcal{T}$.

Lemma 1. *Let $\tau \in \mathcal{T}$ be given arbitrarily. Then, under Assumptions 1–4, $S(\tau) \neq \emptyset$ holds, i.e., there exists a Wardrop equilibrium for the given tolls τ.*

Proof. For every commodity $k = (\alpha_k, \omega_k) \in \mathcal{K}$, there exists at least one path that connects α_k and ω_k due to Assumption 2. Moreover, Assumption 1 implies $\tau_a \geq 0$ for all $a \in \mathcal{A}$. Hence, by Assumptions 1 and 4, the travel cost functions $c_a^k(f;\tau_a)$ are positive and continuous for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$. Moreover, by Assumption 3, the travel demand is fixed, positive, and bounded from above. Under Assumptions 1–4, we can thus apply Theorem 5.5 in Aashtiani and Magnanti (1981), which yields the existence of a Wardrop equilibrium in the path formulation. As a consequence, there also exists a Wardrop equilibrium in the node-arc formulation; see, e.g., the discussion in Section 2.2.2 in Patriksson (2015) for further details. □

Next, we provide bounds for the commodity flow variables x in a Wardrop equilibrium.

Proposition 1. *Let $\tau \in \mathcal{T}$ be given arbitrarily. Then, under Assumptions 1–4, there exists $(f, x) \in S(\tau)$ that satisfies*

$$0 \leq x_a^k \leq d_k, \quad a \in \mathcal{A}, \ k \in \mathcal{K}.$$

In particular, $x_a^k = 0$ holds for all $a \in \delta^{in}(\alpha_k) \cup \delta^{out}(\omega_k)$ with $k = (\alpha_k, \omega_k) \in \mathcal{K}$.

Proof. By Assumptions 1–4, we can apply Lemma 1, i.e., there exists $(f, x) \in S(\tau)$. The non-negativity of the commodity flows x immediately follows from (4). We now prove the upper bound. To this end, let $k = (\alpha_k, \omega_k) \in \mathcal{K}$ be given arbitrarily. By the flow decomposition theorem, we obtain

$$x_a^k = \sum_{\{p\in\mathcal{P}^k:\, a\in p\}} h_p + \sum_{\{\ell\in\mathcal{C}:\, a\in\ell\}} g_\ell^k, \quad a \in \mathcal{A};$$

see, e.g., Theorem 3.5 in Ahuja et al. (1993). Here, $\mathcal{P}^k$ denotes the set of all simple paths between the origin α_k and the destination ω_k of commodity k and $\mathcal{C}$ denotes the set of all cycles in the traffic network. The vectors $h = (h_p)_{p\in\mathcal{P}^k}$

and $g^k = (g^k_\ell)_{\ell \in \mathcal{C}}$ are used for the path and cycle flows, respectively. Suppose that there is a cycle $\ell \in \mathcal{C}$ with positive flow, i.e., $g^k_\ell > 0$ holds. This implies that $x^k_{a'} > 0$ holds for all $a' \in \ell$. Wardrop's second principle (4) thus yields

$$\sum_{a' \in \ell} c^k_{a'}(f; \tau_{a'}) = 0,$$

which is a contradiction to Assumptions 1 and 4. Hence, there cannot be a cycle with positive flow. Consequently, $x^k_a = 0$ holds for all $a \in \delta^{\text{in}}(\alpha_k) \cup \delta^{\text{out}}(\omega_k)$ with $k = (\alpha_k, \omega_k) \in \mathcal{K}$. From flow conservation (3), we thus obtain

$$\sum_{a \in \delta^{\text{in}}(\omega_k)} x^k_a = \sum_{a \in \delta^{\text{out}}(\alpha_k)} x^k_a = d_k. \tag{6}$$

Moreover, again due to flow conservation (3), we have

$$d_k = \sum_{a \in \delta^{\text{in}}(\omega_k)} x^k_a \geq \sum_{a \in \delta^{\text{out}}(i)} x^k_a = \sum_{a \in \delta^{\text{in}}(i)} x^k_a, \quad i \in \mathcal{N} \setminus \{\alpha_k, \omega_k\}. \tag{7}$$

The non-negativity of the commodity flows as well as (6) and (7) finally yield $x^k_a \leq d_k$ for all $a \in \mathcal{A}$, which concludes the proof. □

Since the overall arc flows f and the commodity flows x^k, $k \in \mathcal{K}$, are linearly coupled, Proposition 1 also yields valid bounds for the arc flows.

Proposition 2. *Let $\tau \in \mathcal{T}$ be given arbitrarily. Then, under Assumptions 1–4, there exists $(f, x) \in S(\tau)$ that satisfies*

$$0 \leq f_a \leq \sum_{k \in \mathcal{K}} d_k, \quad a \in \mathcal{A}.$$

Proof. By Assumptions 1–4, we can apply Lemma 1, i.e., there exists $(f, x) \in S(\tau)$. For all $a \in \mathcal{A}$, we have

$$0 \leq f_a = \sum_{k \in \mathcal{K}} x^k_a \leq \sum_{k \in \mathcal{K}} d_k.$$

Here, the first inequality follows from (2) and the non-negativity of the commodity flows given by (4), the equality follows from (2), and the last inequality is due to Proposition 1. □

In the following proposition, we provide bounds for the minimum travel costs t. The key idea is that, whenever a vector (f, x, t) solves (2)–(4), we can shift all values of t by the same amount while still satisfying the conditions.

Proposition 3. *Let $\tau \in \mathcal{T}$ be given arbitrarily and suppose that Assumptions 1–4 hold. Then, for all $(f, x) \in S(\tau)$, there exists t such that (f, x, t) solves (2)–(4) and t has the following properties:*

(i) *For all $k = (\alpha_k, \omega_k) \in \mathcal{K}$, it holds $t^k_{\omega_k} = 0$.*
(ii) *For all $k = (\alpha_k, \omega_k) \in \mathcal{K}$ and $i \in \mathcal{N} \setminus \{\omega_k\}$, it holds*

$$0 \leq t^k_i \leq \min_{p \in \mathcal{P}^k_i} \left\{ \sum_{a \in p} \left(\sum_{a' \in \mathcal{A}} C^k_{aa'} \sum_{q \in \mathcal{K}} d_q + c^{\mathit{fix},k}_a + \tau^+_a \right) \right\}$$

with $\mathcal{P}^k_i$ being the set of all simple paths between nodes i and ω_k.

Proof. Under Assumptions 1–4, there exists t such that (f, x, t) solves (2)–(4) due to Lemma 1. For all $k \in \mathcal{K}$, let now $\Delta t^k = t^k_{\omega_k}$ and consider $t^k_i - \Delta t^k$ instead of t^k_i for all $i \in \mathcal{N}$. Then, by construction, $t^k_{\omega_k} = 0$ holds for all $k \in \mathcal{K}$. Moreover, we have

$$0 \leq c^k_a(f; \tau_a) + (t^k_j - \Delta t^k) - (t^k_i - \Delta t^k) = c^k_a(f; \tau_a) + t^k_j - t^k_i \perp x^k_a \geq 0,$$

for all $a = (i,j) \in \mathcal{A}$ and all $k \in \mathcal{K}$, i.e., Wardrop's second principle (4) remains satisfied. Hence, and since Conditions (2) and (3) do not depend on t, there exists (f, x, t) with $t^k_{\omega_k} = 0$ for all $k \in \mathcal{K}$ that solves (2)–(4) for the given tolls τ. This proves (i). Let now $k \in \mathcal{K}$ be given arbitrarily. Summing over Conditions (4) and applying (i) yields

$$t^k_i \leq \sum_{a \in p} c^k_a(f;\tau_a) + t^k_{\omega_k} = \sum_{a \in p} c^k_a(f;\tau_a), \quad i \in \mathcal{N},\ p \in \mathcal{P}^k_i,$$

which is equivalent to

$$t^k_i \leq \min_{p' \in \mathcal{P}^k_i} \left\{ \sum_{a \in p'} c^k_a(f;\tau_a) \right\}, \quad i \in \mathcal{N}. \tag{8}$$

We now show that for every node $i \in \mathcal{N}$ that is traversed in a path with positive commodity flow, the corresponding inequality in (8) is satisfied with equality. To this end, let $p \in \mathcal{P}^k_i$, $i \in \mathcal{N}$, be a path with positive commodity flow, i.e., $x^k_a > 0$ for all $a \in p$. Due to the complementarity in (4), we thus have $t^k_n = c^k_a(f;\tau_a) + t^k_m$ for all $a = (n,m) \in p$, which yields

$$t^k_i = \sum_{a \in p} c^k_a(f;\tau_a).$$

In particular, this means that p is a minimum-cost path from node i to ω_k. Next, we show that at least one equilibrium is preserved by setting

$$t^k_i = \min_{p' \in \mathcal{P}^k_i} \left\{ \sum_{a \in p'} c^k_a(f;\tau_a) \right\} \tag{9}$$

for all nodes $i \in \mathcal{N} \setminus \{\omega_k\}$. By our previous considerations, it suffices to consider arcs with zero flow, i.e., $a = (n,m) \in \mathcal{A}$ with $x^k_a = 0$. In this case, the complementarity in (4) is trivially satisfied. Hence, we only need to show that $t^k_n \leq c^k_a(f;\tau_a) + t^k_m$ holds for t^k_n and t^k_m as defined in (9). To this end, let

$$p_m = \operatorname*{arg\,min}_{p' \in \mathcal{P}^k_m} \left\{ \sum_{a' \in p'} c^k_{a'}(f;\tau_{a'}) \right\}.$$

Since $p_m \cup \{a\} \in \mathcal{P}^k_n$, we have

$$t^k_n = \min_{p' \in \mathcal{P}^k_n} \left\{ \sum_{a' \in p'} c^k_{a'}(f;\tau_{a'}) \right\} \leq c^k_a(f;\tau_a) + \sum_{a' \in p_m} c^k_{a'}(f;\tau_{a'}) = c^k_a(f;\tau_a) + t^k_m,$$

where both equalities follow from (9).

Since $k \in \mathcal{K}$ was chosen arbitrarily, we conclude that (f,x,t) with $t = (t^k)_{k \in \mathcal{K}}$, $t^k = (t^k_i)_{i \in \mathcal{N}}$ as in (9), and $t^k_{\omega_k} = 0$ solves (2), (3) and (4) for the given tolls τ. The non-negativity of t now follows from Assumptions 1 and 4. For all $i \in \mathcal{N} \setminus \{\omega_k\}$ and $k \in \mathcal{K}$, applying Assumptions 1 and 4 as well as Propositions 1 and 2 to (9) finally yields

$$\begin{aligned} t^k_i &\leq \min_{p' \in \mathcal{P}^k_i} \left\{ \sum_{a \in p'} \left(\sum_{a' \in \mathcal{A}} C^k_{aa'} f_{a'} + c^{\text{fix},k}_a + \tau_a \right) \right\} \\ &\leq \min_{p' \in \mathcal{P}^k_i} \left\{ \sum_{a \in p'} \left(\sum_{a' \in \mathcal{A}} C^k_{aa'} \sum_{q \in \mathcal{K}} d_q + c^{\text{fix},k}_a + \tau^+_a \right) \right\}. \end{aligned}$$

□

Finally, we note that sufficiently large big-M constants for Problem (5) can be obtained by exploiting Assumptions 1 and 4 as well as Propositions 2 and 3.

3.2. **Existence of Solutions.** We now show the existence of an optimal solution to the overall toll-setting problem (1). To this end, we start with the following result.

Corollary 1. *Let $\tau \in \mathcal{T}$ be given arbitrarily. Then, under Assumptions 1–4, it holds*

$$S(\tau) = \left\{(f, x) \colon \exists t \text{ so that } (f, x, t) \text{ solves } (2)\text{–}(4) \text{ with } 0 \le t_i^k \le u_i^k,\ i \in \mathcal{N},\ k \in \mathcal{K}\right\},$$

where, for all $k = (\alpha_k, \omega_k) \in \mathcal{K}$, we have $u_{\omega_k}^k = 0$ and

$$u_i^k := \min_{p \in \mathcal{P}_i^k} \left\{ \sum_{a \in p} \left(\sum_{a' \in \mathcal{A}} C_{aa'}^k \sum_{q \in \mathcal{K}} d_q + c_a^{\mathrm{fix},k} + \tau_a^+ \right) \right\}, \quad i \in \mathcal{N} \setminus \{\omega_k\}.$$

Corollary 1 immediately follows from Proposition 3 in which we state that imposing $t_i^k \le u_i^k$, $i \in \mathcal{N}$, $k \in \mathcal{K}$, does not affect the flows (f, x) in a Wardrop equilibrium.

Theorem 1. *Under Assumptions 1–4, the toll-setting problem (1) has an optimal solution (τ, f, x).*

Proof. By Assumption 1 and Lemma 1, the toll-setting problem (1) is feasible, i.e.,

$$\mathcal{F} := \{(\tau, f, x) \colon \tau \in \mathcal{T},\ (f, x) \in S(\tau)\} \neq \emptyset.$$

From Corollary 1 and Propositions 1–3, we further obtain that the feasible set $\mathcal{F}$ of Problem (1) is bounded. Moreover, the set $\mathcal{F}$ is described by a finite number of continuous functions, which implies that $\mathcal{F}$ is closed. Since the function $(\tau, f) \mapsto \sum_{a \in \mathcal{A}} \tau_a f_a$ is continuous as well, the Weierstraß theorem thus ensures that the toll-setting problem (1) has an optimal solution. □

3.3. **Valid Inequalities.** We now conclude this section by providing valid inequalities for the feasible set of Problem (5) as well as valid inequalities for optimal solutions to the problem.

Proposition 4. *Let $\tau \in \mathcal{T}$ be given arbitrarily. Further, let $i, j \in \mathcal{N}$ be such that $(i, j), (j, i) \in \mathcal{A}$ holds. Then, under Assumptions 1 and 4, the inequalities*

$$z_{(i,j)}^k + z_{(j,i)}^k \le 1, \quad k \in \mathcal{K},$$

are valid for the feasible set of Problem (5).

Proof. We prove the claim by contradiction. To this end, let (τ, f, x, t, z) be feasible for Problem (5) and let $k \in \mathcal{K}$ be given arbitrarily. Suppose now that $z_{(i,j)}^k = 1 = z_{(j,i)}^k$ holds. Then, Constraints (5e) yield

$$c_{(i,j)}^k(f; \tau_{(i,j)}) + t_j^k = t_i^k \quad \text{and} \quad c_{(j,i)}^k(f; \tau_{(j,i)}) + t_i^k = t_j^k.$$

From the latter, we obtain

$$t_i^k = c_{(i,j)}^k(f; \tau_{(i,j)}) + \left(c_{(j,i)}^k(f; \tau_{(j,i)}) + t_i^k\right) \iff 0 = c_{(i,j)}^k(f; \tau_{(i,j)}) + c_{(j,i)}^k(f; \tau_{(j,i)}),$$

which is a contradiction to Assumptions 1 and 4. Hence, a feasible point for Problem (5) satisfies $z_{(i,j)}^k + z_{(j,i)}^k \le 1$. □

From Proposition 4, we particularly obtain $0 \le x_{(i,j)}^k \perp x_{(j,i)}^k \ge 0$ for all nodes $i, j \in \mathcal{N}$ with $(i, j), (j, i) \in \mathcal{A}$ and all commodities $k \in \mathcal{K}$. This means that, under the assumption of positive travel costs, there cannot be positive commodity flow on both an arc and its reversed arc. Finally, we provide valid inequalities for the tolls τ in an optimal solution to Problem (5).

Proposition 5. *Under Assumptions 1–4, there exists an optimal solution (τ, f, x, t, z) to Problem (5) that satisfies*

$$\tau_a \geq \tau_a^+ \left(1 - \sum_{k \in \mathcal{K}} z_a^k\right), \quad a \in \mathcal{A}. \tag{10}$$

Proof. Under Assumptions 1–4, there exists an optimal solution (τ, f, x, t, z) to Problem (5) due to Theorem 1. By Assumption 1, an optimal solution particularly satisfies $\tau_a \geq 0$ for all $a \in \mathcal{A}$. For all arcs $a \in \mathcal{A}$ for which the set $\mathcal{T}$ imposes the upper bound $\tau_a^+ = 0$, Inequality (10) is trivially satisfied. Hence, we only need to consider arcs $a \in \mathcal{A}$ with $\tau_a^+ > 0$ in the following. Suppose that there is an arc $a \in \mathcal{A}$ with $\tau_a^+ > 0$ for which Inequality (10) is violated. If there exists $k \in \mathcal{K}$ with $z_a^k = 1$, this implies $\tau_a < 0$, which is a contradiction to the feasibility of τ. Hence, if a feasible point violates Inequality (10), $\tau_a < \tau_a^+$ and $z_a^k = 0$ for all $k \in \mathcal{K}$ needs to hold. In particular, the latter implies $x_a^k = 0$ for all $k \in \mathcal{K}$ and, thus, $f_a = 0$. We now set

$$\hat{\tau}_{a'} = \begin{cases} \tau_{a'}, & a' \in \mathcal{A} \setminus \{a\}, \\ \tau_{a'}^+, & a' = a. \end{cases}$$

By construction and due to Assumption 1, $(\hat{\tau}, f, x, t, z)$ satisfies Constraints (5b), (5c), (5d), (5f), and (5g). Moreover, (5e) is satisfied for a sufficiently large big-M constant M_a^k for all $k \in \mathcal{K}$. Hence, $(\hat{\tau}, f, x, t, z)$ is feasible for Problem (5). By construction, we further have $\tau_{a'} f_{a'} = \hat{\tau}_{a'} f_{a'}$ for all $a' \in \mathcal{A}$, i.e., $(\hat{\tau}, f, x, t, z)$ solves Problem (5) as well. In particular, $\hat{\tau}_a \geq \tau_a^+(1 - \sum_{k \in \mathcal{K}} z_a^k)$ holds. Repeating the previous procedure until there are no arcs left that violate Inequality (10) concludes the proof. □

4. Robustification

Up to now, we have considered the setting in which the users of the traffic network act under perfect information. In real-world applications, however, travelers often face uncertainties when making their decisions. For instance, the travel costs may be subject to uncertainty due to unforeseen events such as accidents, maintenance work, or changing weather conditions. Hence, the assumption of perfect information seems to be rather strong. In this section, we consider the toll-setting problem (1) under uncertainties regarding the travel costs, which we tackle using techniques from robust optimization. In Section 4.1, we present a robustified variant of the toll-setting problem in which the network users hedge against uncertain travel costs within a predefined and user-specific uncertainty set. We model this setting as a mathematical problem with robustified Wardrop equilibrium conditions, for which we present an MINLP reformulation that exploits binary variables and big-M constants in Section 4.2. Section 4.3 is devoted to deriving valid big-Ms. We conclude by proving the existence of robust solutions in Section 4.4.

4.1. A Robust Toll-Setting Problem.

We start from the nominal Wardrop equilibrium model given by Conditions (2)–(4), for which we now assume that the travel costs of each arc $a \in \mathcal{A}$ and each commodity $k \in \mathcal{K}$ are not known exactly. More formally, we impose the following.

Assumption 5. *For all $a \in \mathcal{A}$ and $k \in \mathcal{K}$, the travel costs $c_a^k(f; \tau_a)$ are subject to additive deviations $Y_a^k \Delta c_a^k$ with Y_a^k being a random variable with support in $[0, 1]$ and $\Delta c_a^k \geq 0$.*

The parameters $\Delta c_a^k \geq 0$ denote upper bounds on the possible deviation from the nominal travel costs. Since it is unlikely that the costs realize in a worst-case sense on every arc of the network and, hence, to avoid being overly conservative, we assume that each commodity $k \in \mathcal{K}$ hedges against deviations of up to $\Gamma^k \in \{0, \dots, |\mathcal{A}|\}$. The robustified version of Wardrop's second principle (4) then reads

$$0 \le c_a^k(f; \tau_a) + y_a^k \Delta c_a^k + t_j^k - t_i^k \perp x_a^k \ge 0, \quad a = (i,j) \in \mathcal{A},\ k \in \mathcal{K}. \tag{11}$$

Here, for a commodity $k \in \mathcal{K}$ and a given flow vector x^k, the vector y^k solves

$$\begin{aligned} \max_{y^k} \quad & \sum_{a \in \mathcal{A}} (\Delta c_a^k x_a^k) y_a^k && \text{(12a)} \\ \text{s.t.} \quad & \sum_{a \in \mathcal{A}} y_a^k \le \Gamma^k, && \text{(12b)} \\ & 0 \le y_a^k \le 1, \quad a \in \mathcal{A}. && \text{(12c)} \end{aligned}$$

Problem (12) is a linear problem for fixed x^k, $k \in \mathcal{K}$. Hence, the KKT conditions are necessary and sufficient optimality conditions, i.e., replacing Problem (12) by its KKT conditions yields an equivalent reformulation of (11) and (12) that is given by

$$\begin{aligned} 0 \le c_a^k(f; \tau_a) + y_a^k \Delta c_a^k + t_j^k - t_i^k \perp x_a^k \ge 0, && a = (i,j) \in \mathcal{A},\ k \in \mathcal{K}, && \text{(13a)} \\ 0 \le \xi^k + \zeta_a^k - \Delta c_a^k x_a^k \perp y_a^k \ge 0, && a \in \mathcal{A},\ k \in \mathcal{K}, && \text{(13b)} \\ 0 \le 1 - y_a^k \perp \zeta_a^k \ge 0, && a \in \mathcal{A},\ k \in \mathcal{K}, && \text{(13c)} \\ 0 \le \Gamma^k - \sum_{a \in \mathcal{A}} y_a^k \perp \xi^k \ge 0, && k \in \mathcal{K}. && \text{(13d)} \end{aligned}$$

For notational convenience, we use $\tilde{c}_a^k(f; \tau_a) := c_a^k(f; \tau_a) + y_a^k \Delta c_a^k$ to denote the robustified travel costs for commodity $k \in \mathcal{K}$ on arc $a \in \mathcal{A}$ in the following. We further emphasize that the equilibrium conditions (2) and (3) do not explicitly depend on the travel costs. Hence, the set of robust Wardrop equilibria for given tolls τ and fixed $\Gamma = (\Gamma^k)_{k \in \mathcal{K}}$ can be stated as

$$S_{\text{rob}}(\tau) = \{(f, x) \colon \exists (t, y, \xi, \zeta) \text{ such that } (f, x, t, y, \xi, \zeta) \text{ solves } (2), (3), \text{and } (13)\}.$$

The overall robustified toll-setting problem is then given by

$$\max_{\tau, f, x} \quad \sum_{a \in \mathcal{A}} \tau_a f_a \quad \text{s.t.} \quad \tau \in \mathcal{T},\ (f, x) \in S_{\text{rob}}(\tau). \tag{14}$$

4.2. **An MINLP Reformulation.** Similar as it is done in Section 3, we exploit sufficiently large big-M constants and additional binary variables to linearize the complementarity constraints in (13). An MINLP reformulation of the robustified toll-setting problem (14) then reads

$$\begin{aligned} \max_{\tau, f, x, t, r} \quad & \sum_{a \in \mathcal{A}} \tau_a f_a && && \text{(15a)} \\ \text{s.t.} \quad & \tau \in \mathcal{T}, \quad f = \sum_{k \in \mathcal{K}} x^k, && && \text{(15b)} \\ & \sum_{a \in \delta^{\text{in}}(i)} x_a^k - \sum_{a \in \delta^{\text{out}}(i)} x_a^k = d_i^k, && i \in \mathcal{N},\ k \in \mathcal{K}, && \text{(15c)} \\ & \tilde{c}_a^k(f; \tau_a) + t_j^k - t_i^k \ge 0, && a = (i,j) \in \mathcal{A},\ k \in \mathcal{K}, && \text{(15d)} \\ & \tilde{c}_a^k(f; \tau_a) + t_j^k - t_i^k \le M_a^k(1 - z_a^k), && a = (i,j) \in \mathcal{A},\ k \in \mathcal{K}, && \text{(15e)} \\ & x_a^k \ge 0, \quad x_a^k \le M_a^k z_a^k, && a \in \mathcal{A},\ k \in \mathcal{K}, && \text{(15f)} \\ & \xi^k + \zeta_a^k - \Delta c_a^k x_a^k \ge 0, && a \in \mathcal{A},\ k \in \mathcal{K}, && \text{(15g)} \\ & \xi^k + \zeta_a^k - \Delta c_a^k x_a^k \le N_a^k w_a^k, && a \in \mathcal{A},\ k \in \mathcal{K}, && \text{(15h)} \\ & y_a^k \ge 0, \quad y_a^k \le 1 - w_a^k, && a \in \mathcal{A},\ k \in \mathcal{K}, && \text{(15i)} \end{aligned}$$

$$y_a^k \le 1, \quad y_a^k \ge v_a^k, \qquad a \in \mathcal{A},\ k \in \mathcal{K}, \tag{15j}$$
$$\zeta_a^k \ge 0, \quad \zeta_a^k \le L_a^k v_a^k, \qquad a \in \mathcal{A},\ k \in \mathcal{K}, \tag{15k}$$
$$\xi^k \ge 0, \quad \xi^k \le R^k q_k, \qquad k \in \mathcal{K}, \tag{15l}$$
$$\sum_{a \in \mathcal{A}} y_a^k \le \Gamma^k, \qquad k \in \mathcal{K}, \tag{15m}$$
$$\Gamma^k - \sum_{a \in \mathcal{A}} y_a^k \le R^k(1 - q_k), \qquad k \in \mathcal{K}, \tag{15n}$$
$$q_k,\ v_a^k,\ w_a^k,\ z_a^k \in \{0, 1\}, \qquad a \in \mathcal{A},\ k \in \mathcal{K}. \tag{15o}$$

Here, r contains all variables that are used for the robustification of the travel costs as well as the variables that are introduced for the linearization of the complementarity constraints, i.e., $r := (y, \xi, \zeta, q, v, w, z)$. By construction, Problem (15) is equivalent to the robustified toll-setting problem (14) for sufficiently large constants L_a^k, M_a^k, N_a^k, and R^k for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$. Before we elaborate on how to obtain such constants in Section 4.3, we provide enhanced formulations for Problem (15) in the remainder of this section. For this purpose, we need the following auxiliary lemma.

Lemma 2. *Let $k \in \mathcal{K}$, $x^k \in \mathbb{R}^{|\mathcal{A}|}_{\ge 0}$, and $\Gamma^k \in \{0, \dots, |\mathcal{A}|\}$ be given arbitrarily. Then, Problem (12) has an optimal solution y^k that satisfies*

$$\sum_{a \in \mathcal{A}} y_a^k = \Gamma^k.$$

Proof. The objective function of Problem (12) is linear for fixed x^k, the zero vector is feasible for Problem (12), and the feasible set of Problem (12) is compact. By the Weierstraß theorem, Problem (12) thus has an optimal solution y^k. Moreover, the system $My^k \le v$ that describes the feasible set of Problem (12) consists of a totally unimodular matrix $M \in \{0,1\}^{(|\mathcal{A}|+1)\times|\mathcal{A}|}$ and the right-hand side vector $v = \left(\Gamma^k, 1, \dots, 1\right)^\top \in \mathbb{Z}^{|\mathcal{A}|+1}$. Hence, by Proposition 3.3 in Wolsey (2020), Problem (12) has an integer solution. For $\Gamma^k = 0$, Constraint (12b) is satisfied with equality since $y^k = 0$ is the only feasible point for Problem (12). If $\sum_{a \in \mathcal{A}} y_a^k < \Gamma^k$ holds for some $\Gamma^k \ge 1$, the integrality of y^k implies that there exists at least one arc $a \in \mathcal{A}$ with $y_a^k = 0$. If $\Delta c_a^k x_a^k > 0$ holds, we have a contradiction to the optimality of y^k. Hence, there exists $a \in \mathcal{A}$ with $y_a^k = 0$ and $\Delta c_a^k x_a^k = 0$. We set

$$\hat{y}_{a'}^k = \begin{cases} y_{a'}^k, & a' \in \mathcal{A} \setminus \{a\}, \\ 1, & a' = a. \end{cases}$$

By construction, $\hat{y}$ solves Problem (12) as well. Repeating the previous procedure until Constraint (12b) is satisfied with equality concludes the proof. □

By Lemma 2, we can reduce the size of Problem (15) by eliminating the auxiliary binary variables q used for the linearization of the complementarity in (13d). To this end, we can replace Constraints (15l), (15m), and (15n) by

$$\xi^k \ge 0, \quad \sum_{a \in \mathcal{A}} y_a^k = \Gamma^k, \quad k \in \mathcal{K}.$$

While the complementarity in (13d) is satisfied for any $\xi \ge 0$, we emphasize that finite upper bounds on the variables ξ are required to obtain valid big-M constants N_a^k, $a \in \mathcal{A}$, $k \in \mathcal{K}$, for Constraint (15h). Hence, we additionally impose $\xi^k \le R^k$ for all $k \in \mathcal{K}$.

Proposition 6. *The inequalities*

$$v_a^k + w_a^k \le 1, \quad a \in \mathcal{A},\ k \in \mathcal{K},$$

are valid for the feasible set of Problem (15).

Proof. We prove the claim by contradiction. To this end, let $(\tau, f, x, t, y, \xi, \zeta, q, v, w, z)$ be feasible for Problem (15) and suppose that there exists an arc $a \in \mathcal{A}$ and a commodity $k \in \mathcal{K}$ for which the inequality $v_a^k + w_a^k \leq 1$ is violated. Hence, we have $v_a^k = 1 = w_a^k$. Then, we obtain $y_a^k = 1$ by Constraint (15j), which contradicts $y_a^k = 0$ that is obtained from Constraint (15i). □

Finally, we note that the robustified travel costs $\tilde{c}_a^k(f; \tau_a)$ are positive for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$ under Assumptions 1, 4, and 5. Thus, the valid inequalities derived in Proposition 4 are also valid for Problem (15). Moreover, since we do not use any information about the travel costs to prove the validity of Inequalities (10), the latter are valid for optimal solutions to the robustified toll-setting problem as well.

4.3. **Computing Big-Ms.** We now derive bounds for the variables of Problem (15), which we exploit to obtain sufficiently large big-M constants L_a^k, M_a^k, N_a^k, and R^k for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$. For this purpose, we first prove the existence of a robust Wardrop equilibrium for given tolls $\tau \in \mathcal{T}$.

Theorem 2. *Let $\tau \in \mathcal{T}$ as well as $\Gamma = (\Gamma^k)_{k \in \mathcal{K}}$ with $\Gamma^k \in \{0, \dots, |\mathcal{A}|\}$ for all $k \in \mathcal{K}$ be given arbitrarily. Then, under Assumptions 1–5, there exists $(f, x) \in S_{rob}(\tau)$.*

Proof. By Lemma 2, Problem (12) has an optimal solution y^k for arbitrarily given $x^k \in \mathbb{R}_{\geq 0}^{|\mathcal{A}|}$, $k \in \mathcal{K}$. Since the KKT conditions are necessary and sufficient for Problem (12), there exist ξ^k and $\zeta^k = (\zeta_a^k)_{a \in \mathcal{A}}$ such that (y^k, ξ^k, ζ^k) solves (13b)–(13d) for all $k \in \mathcal{K}$. Hence, Conditions (13b)–(13d) cannot induce any infeasibility. Similar as it is done in the proof of Lemma 1, we now apply Theorem 5.5 in Aashtiani and Magnanti (1981) to prove the existence of a robust Wardrop equilibrium. For the application of this theorem, it only remains to show that the robustified travel cost functions $\tilde{c}_a^k(f; \tau_a)$ are positive and continuous for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$. The positivity of the travel cost functions immediately follows from Assumptions 1, 4, and 5. Moreover, Problem (12) is a linear problem for given commodity flows x^k so that we can use classic sensitivity results as, e.g., Proposition 4.3.3 in Bertsekas (2016). As a consequence, the function $x^k \mapsto y_a^k \Delta c_a^k$ with y^k being an optimal solution to the x^k-parameterized linear problem (12), is continuous for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$. Thus, under Assumption 4, the robustified travel cost functions $\tilde{c}_a^k(f; \tau_a) := c_a^k(f; \tau_a) + y_a^k \Delta c_a^k$ are continuous. This concludes the proof. □

Remark 1. *The budgeted uncertainty modeling used in Problem (12) is closely related to the so-called Γ-robust approach presented in Bertsimas and Sim (2003) and Sim (2004). Pursuing a Γ-robust approach in our setting, however, would imply that the users of the traffic network hedge against uncertain travel costs on at most Γ^k many arcs of the network. This requires imposing integrality on the variables y^k in Problem (12), which leads to robustified travel cost functions $\tilde{c}_a^k(f; \tau_a) := c_a^k(f; \tau_a) + y_a^k \Delta c_a^k$ that are no longer continuous. Continuity is needed to prove the existence of robust Wardrop equilibria using Theorem 5.5 in Aashtiani and Magnanti (1981). Hence, proving existence of Γ-robust Wardrop equilibria in the sense of Bertsimas and Sim (2003) and Sim (2004) most likely requires different techniques compared to those used in the last proof.*

In the remainder of this section, we provide bounds for the flow variables f and x as well as for the variables t, ξ, and ζ in a robust Wardrop equilibrium.

Corollary 2. *Let $\tau \in \mathcal{T}$ as well as $\Gamma = (\Gamma^k)_{k \in \mathcal{K}}$ with $\Gamma^k \in \{0, \dots, |\mathcal{A}|\}$ for all $k \in \mathcal{K}$ be given arbitrarily. Then, under Assumptions 1–5, there exists $(f, x) \in S_{rob}(\tau)$ that satisfies*

$$0 \leq x_a^k \leq d_k, \quad a \in \mathcal{A}, \ k \in \mathcal{K},$$

as well as

$$0 \le f_a \le \sum_{k \in \mathcal{K}} d_k, \quad a \in \mathcal{A}.$$

Under Assumptions 1, 4, and 5, the robustified travel cost functions $\tilde{c}_a^k(f;\tau_a)$ are positive for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$. Hence, Corollary 2 can be shown in analogy to the proofs of Propositions 1 and 2 by replacing the nominal travel cost functions $c_a^k(f;\tau_a)$ with the robustified travel cost functions $\tilde{c}_a^k(f;\tau_a)$ for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$.

Corollary 3. *Let $\tau \in \mathcal{T}$ as well as $\Gamma = (\Gamma^k)_{k \in \mathcal{K}}$ with $\Gamma^k \in \{0, \dots, |\mathcal{A}|\}$ for all $k \in \mathcal{K}$ be given arbitrarily and suppose that Assumptions 1–5 hold. Then, for all $(f, x) \in S_{rob}(\tau)$, there exists (t, y, ξ, ζ) such that (f, x, t, y, ξ, ζ) solves (2), (3) and (13), and t has the following properties:*

(i) *For all $k = (\alpha_k, \omega_k) \in \mathcal{K}$, it holds $t^k_{\omega_k} = 0$.*

(ii) *For all $k = (\alpha_k, \omega_k) \in \mathcal{K}$ and $i \in \mathcal{N} \setminus \{\omega_k\}$, it holds*

$$0 \le t_i^k \le \min_{p \in \mathcal{P}_i^k} \left\{ \sum_{a \in p} \left(\sum_{a' \in \mathcal{A}} C_{aa'}^k \sum_{q \in \mathcal{K}} d_q + c_a^{fix,k} + \tau_a^+ + \Delta c_a^k \right) \right\} =: u_i^k$$

with $\mathcal{P}_i^k$ being the set of all simple paths between nodes i and ω_k. In particular, it holds

$$\begin{aligned} S_{rob}(\tau) = \{(f, x) : \exists (t, y, \xi, \zeta) \text{ such that } (f, x, t, y, \xi, \zeta) \text{ solves } (2), (3), \text{and } (13) \\ \text{with } 0 \le t_i^k \le u_i^k,\ i \in \mathcal{N},\ k \in \mathcal{K}\}. \end{aligned}$$

Corollary 3 can be shown in analogy to the proof of Proposition 3 by replacing the nominal travel cost functions $c_a^k(f;\tau_a)$ with the robustified travel cost functions $\tilde{c}_a^k(f;\tau_a)$ for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$.

Proposition 7. *Let $\tau \in \mathcal{T}$ as well as $\Gamma = (\Gamma^k)_{k \in \mathcal{K}}$ with $\Gamma^k \in \{0, \dots, |\mathcal{A}|\}$ for all $k \in \mathcal{K}$ be given arbitrarily. Then, under Assumptions 1–5, there exists (f, x, t, y, ξ, ζ) that solves (2), (3), and (13) with*

$$0 \le \xi^k \le \max_{a \in \mathcal{A}} \left\{ \Delta c_a^k d_k \right\}, \quad k \in \mathcal{K}, \tag{16}$$

and

$$0 \le \zeta_a^k \le \Delta c_a^k d_k, \quad a \in \mathcal{A},\ k \in \mathcal{K}. \tag{17}$$

In particular, it holds

$$\begin{aligned} S_{rob}(\tau) = \{(f, x) : \exists (t, y, \xi, \zeta) \text{ such that } (f, x, t, y, \xi, \zeta) \\ \text{solves } (2), (3), (13), (16) \text{ and } (17)\}. \end{aligned}$$

Proof. By Theorem 2, there exists (f, x, t, y, ξ, ζ) that solves (2), (3), and (13). Since the non-negativity of ξ and ζ immediately follows from the feasibility w.r.t. Conditions (13), we only need to prove the upper bounds. To this end, let $k \in \mathcal{K}$ be given arbitrarily. If $\Gamma^k = 0$ holds, commodity k does not hedge against any uncertainties regarding the travel costs, i.e., no additional variables ξ^k and ζ^k are introduced for the robustification of commodity k. Consequently, it suffices to consider the case $\Gamma^k \ge 1$. Due to Lemma 2, we can assume w.l.o.g. that $\sum_{a \in \mathcal{A}} y_a^k = \Gamma^k$ holds in a robust Wardrop equilibrium. In particular, this implies that there exists at least one arc $a \in \mathcal{A}$ with $y_a^k > 0$. Condition (13b) then yields $\xi^k + \zeta_a^k = \Delta c_a^k x_a^k$. From the non-negativity of ξ^k and ζ_a^k, we thus obtain

$$0 \le \zeta_a^k \le \Delta c_a^k x_a^k \quad \text{and} \quad 0 \le \xi^k \le \Delta c_a^k x_a^k$$

for all $a \in \mathcal{A}$ with $y_a^k > 0$. Moreover, for all $a \in \mathcal{A}$ with $y_a^k = 0$, we obtain $\zeta_a^k = 0$ from (13c). Taking all previous considerations into account, we obtain valid bounds

$$\begin{aligned} &0 \le \xi^k \le \max_{a \in \mathcal{A}} \left\{ \Delta c_a^k x_a^k \right\} \le \max_{a \in \mathcal{A}} \left\{ \Delta c_a^k d_k \right\}, && k \in \mathcal{K}, \\ &0 \le \zeta_a^k \le \Delta c_a^k d_k, && a \in \mathcal{A},\ k \in \mathcal{K}, \end{aligned}$$

by exploiting Corollary 2 as well as Assumptions 3 and 5. We further note that imposing these bounds does not affect the flows (f, x) in a robust Wardrop equilibrium for the given tolls τ. This concludes the proof. □

Finally, sufficiently large big-M constants for Problem (15) can be obtained by exploiting Assumptions 1, 4, and 5, Corollaries 2 and 3, as well as Proposition 7.

4.4. **Existence of Solutions.** We conclude this section by showing the existence of solutions to the robustified toll-setting problem (14).

Theorem 3. *Under Assumptions 1–5, the robustified toll-setting problem (14) has an optimal solution (τ, f, x).*

Proof. We consider the problem

$$\begin{aligned} \max_{\tau, f, x, t, y, \xi, \zeta} \quad & \sum_{a \in \mathcal{A}} \tau_a f_a && \text{(18a)} \\ \text{s.t.} \quad & \tau \in \mathcal{T},\ (f, x, t, y, \xi, \zeta) \text{ solves } (2), (3), \text{and } (13), && \text{(18b)} \\ & 0 \le t_i^k \le u_i^k, \qquad i \in \mathcal{N},\ k \in \mathcal{K}, && \text{(18c)} \\ & \zeta_a^k \le \Delta c_a^k d_k, \qquad a \in \mathcal{A},\ k \in \mathcal{K}, && \text{(18d)} \\ & \xi^k \le \max_{a \in \mathcal{A}} \left\{ \Delta c_a^k d_k \right\}, \qquad k \in \mathcal{K}, && \text{(18e)} \end{aligned}$$

where, for all $k \in \mathcal{K}$, we have $u_{\omega_k}^k = 0$ and

$$u_i^k := \min_{p \in \mathcal{P}_i^k} \left\{ \sum_{a \in p} \left(\sum_{a' \in \mathcal{A}} C_{aa'}^k \sum_{q \in \mathcal{K}} d_q + c_a^{\text{fix},k} + \tau_a^+ + \Delta c_a^k \right) \right\}, \quad i \in \mathcal{N} \setminus \{\omega_k\}.$$

By Assumption 1 and Theorem 2, the feasible set of Problem (18) is non-empty. Moreover, the feasible set of Problem (18) is bounded due to Assumption 1, Corollaries 2 and 3, as well as Proposition 7. In particular, the feasible set of Problem (18) is described by a finite number of continuous functions, which implies its closedness. Since the objective function of Problem (18) is continuous, the Weierstraß theorem thus ensures that Problem (18) has an optimal solution. From Part (iii) of Corollary 3 and from Proposition 7, we further obtain

$$\begin{aligned} S_{\text{rob}}(\tau) &= \{(f, x) \colon \exists t \text{ s.t. } (f, x, t, y, \xi, \zeta) \text{ solves } (2), (3), \text{and } (13)\} \\ &= \{(f, x) \colon \exists t \text{ s.t. } (f, x, t, y, \xi, \zeta) \text{ solves } (18b)\text{–}(18e)\} \end{aligned}$$

for arbitrarily given $\tau \in \mathcal{T}$. Since we consider the same objective functions in Problems (14) and (18), an optimal solution to Problem (18) solves the robust toll-setting problem (14) as well. □

5. Case Study

In this section, we present a case study to illustrate how the consideration of travelers, who hedge against travel cost uncertainty in a robust way, may impact toll-setting policies. In Sections 5.1 and 5.2, we briefly discuss the computational setup and the considered test instances. In Section 5.3, we discuss the computational results of our case study.

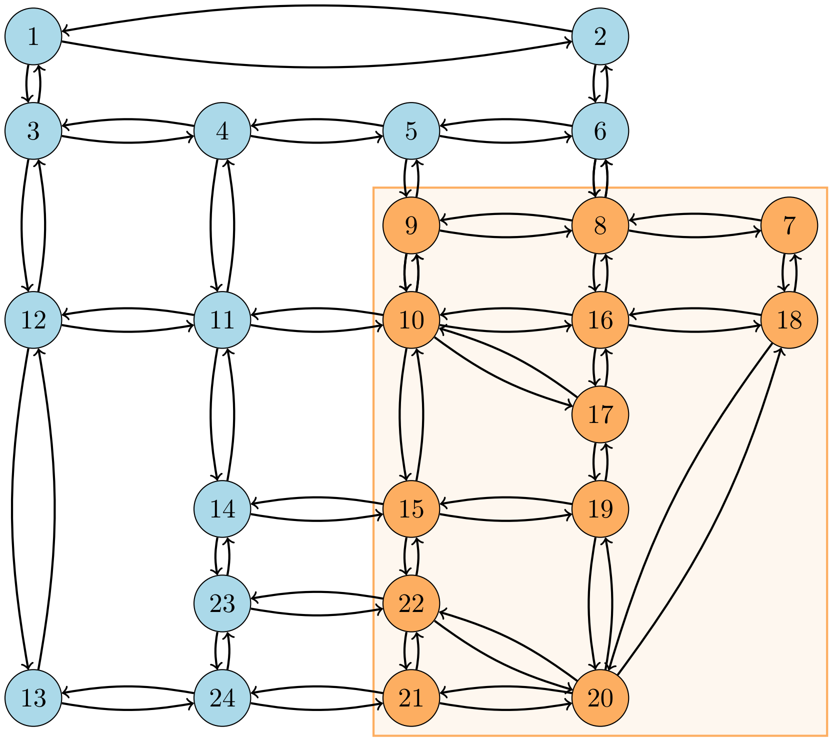

FIGURE 1. The entire Sioux Falls network (blue and orange nodes) consisting of 24 nodes and 76 arcs and the "Sioux Falls East" network (orange nodes) consisting of 12 nodes and 36 arcs.

5.1. **Computational Setup.** All tests have been realized on an Intel XEON SP 6126 at 2.6 GHz (4 cores) with 32 GB RAM, which is part of the high performance cluster "Elwetritsch" at TU Kaiserslautern within the "Alliance of High Performance Computing Rheinland-Pfalz" (AHRP).[2] The toll-setting problems (5) and (15) are implemented in Python 3.7.11 and we use Gurobi 10.0.3 to solve them. In particular, the implementation of our models includes the valid inequalities presented in Propositions 4, 5, and 6. Preliminary computational tests revealed that including these inequalities significantly enhances the solution process. Since the toll-setting models are nonconvex MINLPs, we need to set the Gurobi parameter NonConvex to 2. All other parameters have been left at their default settings. For each test run, we set a time limit (TL) of 1 h.

5.2. **Test Instances.** We consider a subnetwork of the Sioux Falls network (LeBlanc et al. 1975), which is publicly available at https://github.com/bstabler/TransportationNetworks. The subnetwork, which we refer to as "Sioux Falls East", consists of 12 nodes and 36 arcs. In this case study, we consider a varying number of origin-destination (OD) pairs ranging from 4 to 8. An illustration of both the entire Sioux Falls network and the "Sioux Falls East" subnetwork is given in Figure 1. The travel costs that have been provided for the Sioux Falls network are defined by the BPR function (U.S. Bureau of Public Roads 1964), which is given by

$$c_a(f_a) = c_a^{\text{fix}} \left(1 + 0.15 \left(\frac{f_a}{u_a} \right)^4 \right), \quad a \in \mathcal{A}.$$

[2]We kindly acknowledge the support of RHRK (https://rz.rptu.de/en/).

Here, $c_a^{\text{fix}} > 0$ denotes the fixed costs ("free-flow time") and $u_a > 0$ denotes the capacity of an arc $a \in \mathcal{A}$. In this paper, we address the problem of determining optimal tolls in a network with travel cost functions that are affine-linear in the flows f; cf. Assumption 4. Hence, we adapt the BPR functions to account for our setting. In this case study, we consider the travel cost functions

$$c_a(f_a; \tau_a) = c_a^{\text{fix}} \left(1 + 1.5\, \frac{f_a}{u_a}\right) + \tau_a, \quad a \in \mathcal{A}.$$

We emphasize that these travel costs are separable and that each commodity faces the same costs. Since the used data for the Sioux Falls network does not contain toll arcs, we have generated toll arcs using a procedure similar to the one considered in Brotcorne et al. (2000) and Minh Bui et al. (2022). The method works as follows. For a given set of OD pairs, we determine the shortest path for each commodity. For each arc of the network, we then determine the number of shortest paths that go through that arc. Afterward, we sort the arcs of the network in decreasing order w.r.t. the number of shortest paths traversing it. Following this order, we convert each arc and its reversed arc into a toll arc until $2/3$ of the desired number of toll arcs is reached. The remaining $1/3$ of the desired number of toll arcs is chosen randomly among the remaining arcs. Again, if an arc is converted to a toll arc, we also convert its reversed arc. For all arcs of the network, we impose a lower bound of 0 on the tolls and we set the upper bound $\tau_a^+ = 0$ for toll-free arcs. For toll arcs, the upper bounds τ_a^+ on the tolls are set to the fixed travel costs c_a^{fix}. As usual in the literature, we half the costs c_a^{fix} for toll arcs after their conversion. Since we define finite upper bounds on the tolls, the revenues of the toll-setting authority are bounded from above; cf. Proposition 2. In contrast to Brotcorne et al. (2000) and Minh Bui et al. (2022), we thus do not need to ensure that at least one toll-free path is preserved for each commodity when converting arcs into toll arcs. Moreover, since the used data for the Sioux Falls network does not include uncertainties, we have randomly generated the uncertainty parameters Δc_a^k and Γ^k, $a \in \mathcal{A}$, $k \in \mathcal{K}$. For all commodities, we consider the same travel cost uncertainties, i.e., we consider $\Delta c_a^k = \Delta c_a$ for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$. Here, Δc_a is a uniformly distributed random integer value in the interval $[0.5 c_a^{\text{fix}}, 2 c_a^{\text{fix}}]$. Moreover, for each commodity $k \in \mathcal{K}$, the parameter Γ^k takes a uniformly distributed integer value in the interval $[0, 0.5|\mathcal{A}|]$. We emphasize that Γ^k may differ among commodities.

5.3. **Computational Results.** We start by considering the nominal setting, i.e., the setting without any uncertainties regarding the travel costs. Reflected by the running times and the number of investigated branch-and-bound nodes shown in Table 1, we observe that the resources required to solve Problem (5) increase with the number of OD pairs. This is to be expected as the number of OD pairs directly influences the size of the toll-setting problem. For each additional OD pair, we introduce $2|\mathcal{A}| + |\mathcal{N}| = 84$ additional variables and $3|\mathcal{A}| + |\mathcal{N}| = 120$ additional constraints in the model. Moreover, since every arc in the "Sioux Falls East" network has one reversed arc, we further add $0.5|\mathcal{A}| = 18$ valid inequalities for each OD pair. Thus, it is evident that increasing the number of OD pairs increases the amount of resources required to solve the respective toll-setting problems. In addition, the results in Table 1 indicate that increasing the number of toll arcs in the network further increases the computational burden. This is due to the fact that more toll arcs lead to more nonconvex terms in the objective function of the toll-setting problem (5), which require a special algorithmic treatment. Gurobi tackles these nonconvexities using spatial branching based on convex envelopes. Overall, even under the assumption of affine-linear travel cost functions (see Assumption 4), solving Problem (5) is a highly challenging task. The latter is particularly reflected

Table 1. The revenues realized through imposing tolls as well as the runtimes (in s) and the number of investigated branch-and-bound nodes required to solve the respective nominal toll-setting optimization problem for the "Sioux Falls East" network with varying numbers of OD pairs ("$|\mathcal{K}|$") and toll arcs ("$|\mathcal{A}^{\text{toll}}|$"). Additionally, the optimality gap is shown (in %).

$\|\mathcal{K}\|$	$\|\mathcal{A}^{\text{toll}}\|$	revenues	runtime	nodes	gap
4	4	6000.00	1.90	3544	0.00
	6	7176.11	2.76	9765	0.01
	8	7176.11	11.58	58 979	0.01
5	4	6000.00	1.28	1100	0.00
	6	7415.86	25.57	155 363	0.01
	8	7415.86	27.18	153 352	0.01
6	4	6659.88	2.33	3707	0.01
	6	7415.86	40.75	168 903	0.01
	8	8086.66	3529.97	13 370 122	0.01
7	4	8271.18	2.49	4416	0.00
	6	7415.86	162.27	845 334	0.01
	8	9697.96	TL	11 804 881	0.40
8	4	8348.09	4.04	2280	0.00
	6	7445.88	710.55	2 306 930	0.01
	8	9835.07	TL	9 180 328	11.94

Table 2. The revenues realized through imposing tolls as well as the runtimes (in s) and the number of investigated branch-and-bound nodes required to solve the respective robustified toll-setting optimization problem for the "Sioux Falls East" network with varying numbers of OD pairs ("$|\mathcal{K}|$") and toll arcs ("$|\mathcal{A}^{\text{toll}}|$"). Additionally, the optimality gap is shown (in %).

$\|\mathcal{K}\|$	$\|\mathcal{A}^{\text{toll}}\|$	revenues	runtime	nodes	gap
4	4	8757.14	286.03	250 597	0.01
	6	10 086.90	TL	3 973 530	3.49
	8	11 794.84	TL	3 491 529	4.88
5	4	7934.72	TL	2 774 570	4.24
	6	9807.33	TL	2 840 066	3.81
	8	10 624.36	TL	1 578 972	15.90

by the two instances of the "Sioux Falls East" network that cannot be solved within the time limit of 1 h.

Compared to the nominal toll-setting problem (5), the robustified toll-setting problem (15) is significantly larger w.r.t. the number of variables and constraints. For the "Sioux Falls East" network, we introduce $145|\mathcal{K}|$ additional variables and $363|\mathcal{K}|$ additional constraints for the robustification and the linearization of robustified constraints. Moreover, we add $36|\mathcal{K}|$ additional valid inequalities; cf. Proposition 6. Since, even for the nominal setting, the problem size is a limiting factor for solving the respective toll-setting problem, the computational challenges resulting from

TABLE 3. Nominal vs. robust travel costs for each OD pair in the "Sioux Falls East" network with 4 (left) and 5 OD pairs (right) with a varying number of toll arcs ("$|\mathcal{A}^{\text{toll}}|$").

$\lvert\mathcal{A}^{\text{toll}}\rvert$	OD pair	nominal	robust
4	(8, 20)	10.02	15.92
	(9, 21)	16.44	24.87
	(16, 21)	14.59	27.66
	(17, 22)	11.55	18.56
6	(8, 20)	10.02	15.92
	(9, 21)	15.93	24.64
	(16, 21)	13.68	28.40
	(17, 22)	11.55	18.46
8	(8, 20)	10.02	15.92
	(9, 21)	15.93	24.58
	(16, 21)	13.68	27.34
	(17, 22)	11.55	18.73

$\lvert\mathcal{A}^{\text{toll}}\rvert$	OD pair	nominal	robust
4	(8, 20)	10.02	15.90
	(9, 21)	16.44	24.93
	(21, 18)	10.20	20.23
	(16, 21)	14.59	21.47
	(17, 22)	11.55	18.95
6	(8, 20)	10.02	15.90
	(9, 21)	15.93	24.72
	(21, 18)	10.03	21.03
	(16, 21)	13.68	22.04
	(17, 22)	11.55	18.88
8	(8, 20)	10.02	15.89
	(9, 21)	15.93	24.71
	(21, 18)	10.03	21.03
	(16, 21)	13.68	22.40
	(17, 22)	11.55	18.99

larger models is thus even more pronounced in the robust setting. In Table 2, it can be seen that only the smallest of our considered instances, i.e., the one with 4 OD pairs and 4 toll arcs, can be solved within the time limit of 1 h in the robust setting. Given this limitation, we thus refrain from presenting results for larger instances with 6 or more OD pairs.

While only one of the instances considered in the robust setting can be solved, the results presented in Table 2 still provide valuable insights into the impact of robustified travel decisions on the revenues realized by the toll-setting authority. It can be seen that, for the "Sioux Falls East" network, the revenues that are realized by imposing tolls are significantly higher in the robust setting compared to the nominal one. In this context, we emphasize that it is not the toll-setting authority that hedges against uncertain travel costs in a robust way, but the users of the traffic network. In particular, the users of the traffic network decide on their route choices in a "here-and-now" fashion, i.e., before the uncertainty realizes. Viewing the overall toll-setting problem as a single-leader multi-follower game, this means that we consider multiple "here-and-now" followers. Since this problem is considered from the leader's perspective, having higher revenues in the robust setting is thus not in contrast to classic robust optimization theory. To further illustrate this, we show the nominal and the robustified travel costs faced by each commodity in the "Sioux Falls East" network in Table 3. It can be seen that, when hedging against uncertain travel costs in a robust way, users of the traffic network always face increased travel costs to reach their destination. More formally, the previous observations indicate that, while the set of feasible flows do not change in the robust compared to the nominal setting, the set of Wardrop equilibria may change. Hence, for given tolls $\tau \in \mathcal{T}$, neither $S_{\text{rob}}(\tau) \subseteq S(\tau)$ nor $S_{\text{rob}}(\tau) \supseteq S(\tau)$ holds in general.

To further illustrate the impact of robustified travel decisions on the actual route choices and the imposed tolls, we now focus on the "Sioux Falls East" instance with 4 OD pairs and 4 toll arcs, which can be solved in both the nominal and the robust setting. In Figure 2, we show the flows in a nominal Wardrop equilibrium and the tolls imposed by the toll-setting authority. It can be seen that revenues

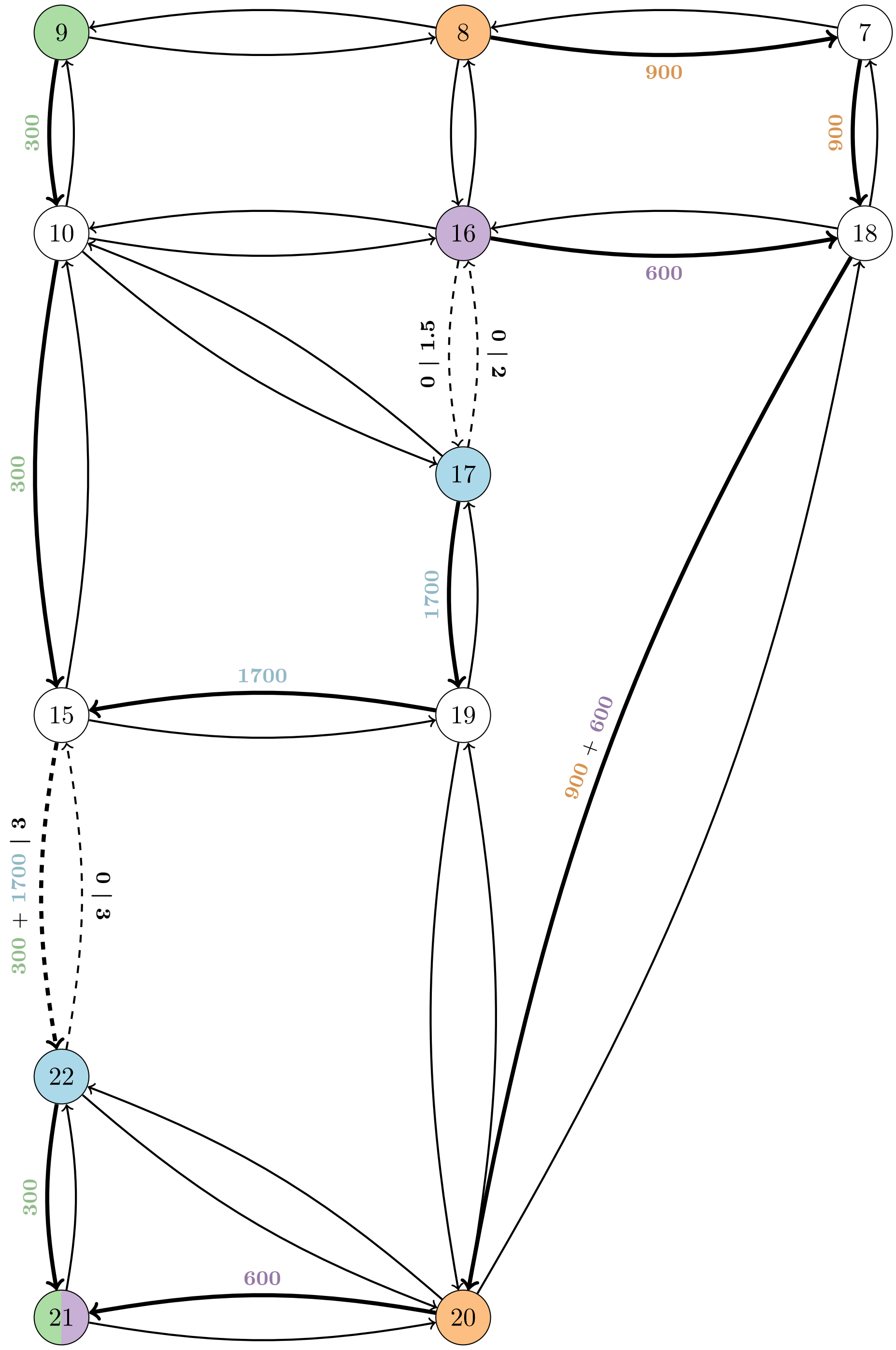

FIGURE 2. The “Sioux Falls East” network with 4 OD pairs and 4 toll arcs. Each OD pair is color-coded (orange, green, blue, purple). Dashed arcs represent toll arcs and solid arcs represent toll-free arcs. Edge labels correspond to commodity flows. For toll arcs, edge labels are given in the format “flow | toll”. If no label is shown, there is no flow on that edge.

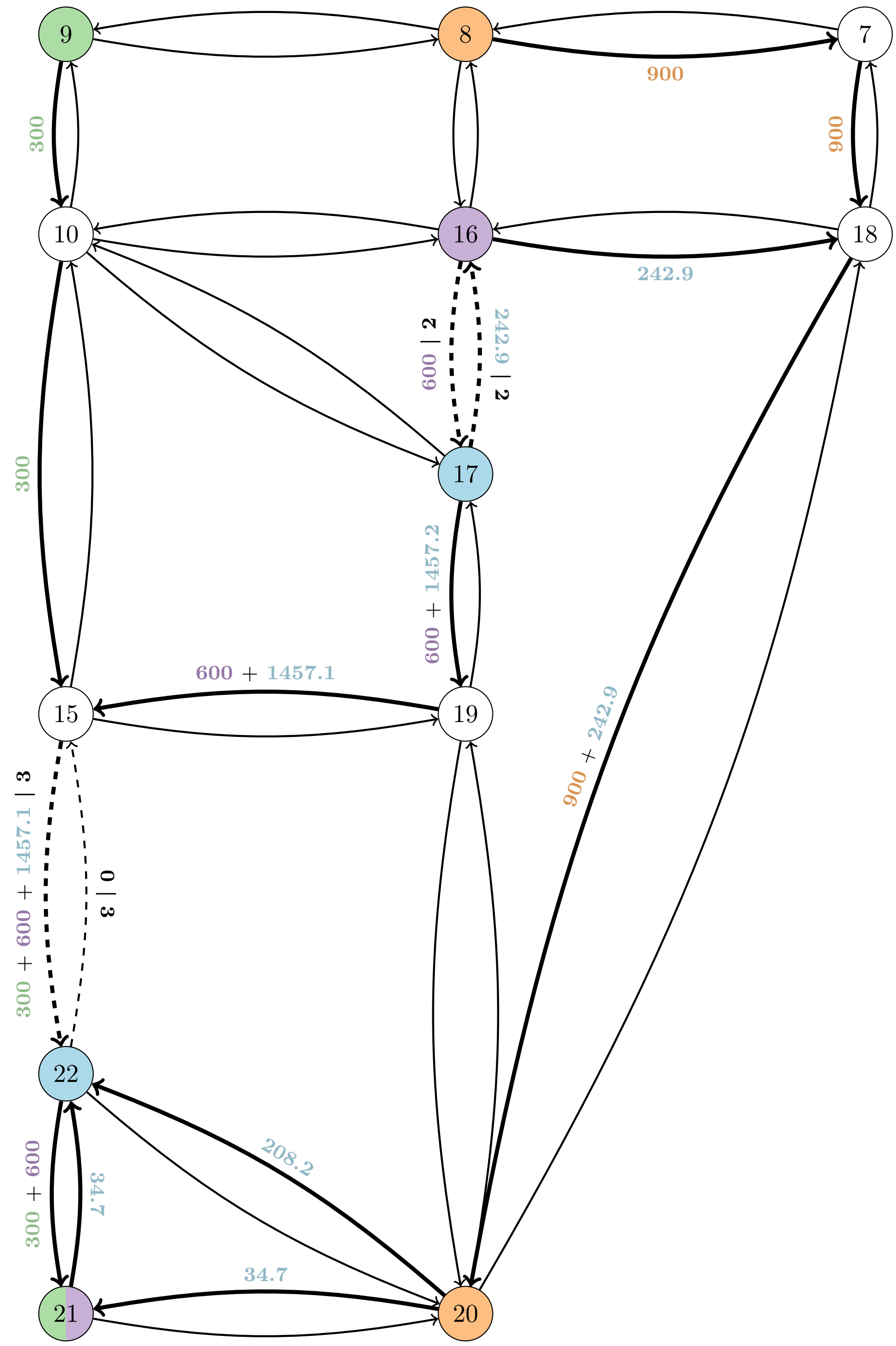

Figure 3. The robust "Sioux Falls East" network with 4 OD pairs and 4 toll arcs. Each OD pair is color-coded (orange, green, blue, purple). Dashed arcs represent toll arcs and solid arcs represent toll-free arcs. Edge labels correspond to commodity flows. For toll arcs, edge labels are given in the format "flow | toll". If no label is shown, there is no flow on that edge.

are only generated by imposing tolls on arc $(15, 22)$. The remaining toll arcs are not used by any commodity. Nevertheless, we emphasize that imposing tolls on arcs with zero flow may still be beneficial for the toll-setting authority, even if no revenues are generated. In this way, the toll-setting authority can influence the travel decisions of the users of the traffic network such as to encourage or discourage the use of specific arcs. This may lead to overall higher revenues. In particular, decreasing the imposed tolls on arcs with zero flow may affect the flows in a Wardrop equilibrium. Let us now consider the specific routes taken by each commodity. In Figure 2, we observe that the green and the blue commodities, i.e., OD pairs $(9, 21)$ and $(17, 22)$, take the most direct route to reach their destination. In doing so, they accept to pay tolls along the way. The orange and the purple commodities, i.e., OD pairs $(8, 20)$ and $(16, 21)$, do not take the most direct route and prefer to take a detour to avoid being charged tolls. However, the situation may change significantly if users of the traffic network hedge against uncertain travel costs in a robust way. In Figure 3, we show the imposed tolls and the flows in a robust Wardrop equilibrium. There are five aspects that we find particularly remarkable. First, revenues are now additionally generated by imposing tolls on the arcs that connect nodes 16 and 17. This is in contrast to the nominal setting. Second, we note that robust travel decisions do not affect the actual tolls charged on the arcs of the network for this instance. Despite the fact that the toll-setting authority imposes higher tolls on arc $(16, 17)$ in the robust setting (2 vs. 1.5), we emphasize that imposing tolls of 2 would be optimal in the nominal setting as well; cf. Proposition 5. Third, the green and the orange commodities do not change their travel decisions when hedging against travel cost uncertainties in a robust way. Also in the robust setting, the green commodity takes the most direct route, accepting the toll charges, while the orange commodity takes a detour to avoid toll arcs. Fourth, the travel decision of the purple commodity changes completely in the robust setting. Instead of taking a toll-free detour, it now takes the most direct route, which includes a toll arc. Finally, we point out that the flow of the blue commodity is split between the most direct tolled route and the toll-free detour. Moreover, the flows of the blue commodity are split between $(20, 21, 22)$ and $(20, 22)$ on the toll-free path.

To sum up, our case study illustrates that making robust travel decisions due to travel cost uncertainties may significantly impact the travel behavior and, thus, the revenues realized by imposing tolls. We have seen that users of the traffic network, who hedge against travel cost uncertainties within their user-specific uncertainty set, may be indifferent to uncertainties, change their travel decisions completely, or decide on something in between. While we have further observed that the actual toll-setting policies do not change in the robust setting for the specific "Sioux Falls East" instance considered in our case study, we note that this may not be the case in general. Nevertheless, given the significant increase in size and computational difficulty of the robustified toll-setting problem compared to the nominal one, an interesting future research question may be to identify situations in which nominal toll-setting policies also provide favorable results in the robust setting.

6. Conclusion

In this paper, we consider a multi-commodity traffic network in which a toll-setting authority aims to maximize revenues by imposing tolls on certain arcs of the network. Users of the traffic network act in the sense of Wardrop's user equilibrium so that their individual travel costs are minimized. We model this setting as a mathematical problem with equilibrium constraints, for which we present a mixed-integer, nonlinear, and nonconvex reformulation that exploits binary variables and big-M constants. We prove existence of solutions to this problem, derive correct big-Ms, and provide valid inequalities. Moreover, we consider the setting in which the network users pursue a robust approach to hedge against uncertainties regarding their travel costs. In this paper, the uncertainties are assumed to vary within a predefined and user-specific uncertainty set, which relates to the notion of Γ-robustness. We reformulate the robustified problem as a mixed-integer, nonlinear, and nonconvex problem and prove the existence of robust solutions. To illustrate the impact of considering robust travel decisions on the revenues realized by the toll-setting authority, we further conduct a case study using a subnetwork of the well-known Sioux Falls network. We observe that addressing uncertainties in the travel costs may significantly impact the travel behavior and, in particular, may lead to increased revenues realized by imposing tolls. However, for the specific instance considered in our case study, we observe that the actual toll-setting policies do not change in the robust compared to the nominal setting. Given that solving the robustified toll-setting problem is significantly more challenging than the nominal one, a potential future research question could thus be to identify situations in which quality guarantees for nominal toll-setting policies in the robust setting are available. Another interesting research question could be to identify properties that ensure the existence of Γ-robust Wardrop equilibria in the classic sense of Bertsimas and Sim (2003) and Sim (2004). Finally, a possible direction of future research may be the development of tailored solution approaches that exploit, e.g., piecewise-linear approximations of the bilinearities in the objective function of the toll-setting problem.

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