PROPERTIES OF TWO-STAGE STOCHASTIC MULTI-OBJECTIVE LINEAR PROGRAMS

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Abstract

We consider a two-stage stochastic multi-objective linear program (TSSMOLP) which is a natural multi-objective generalization of the well-studied two-stage stochastic linear program. The second-stage recourse decision is governed by an uncertain multi-objective linear program whose solution maps to an uncertain second-stage nondominated set. The TSSMOLP then comprises the objective function, which is the Minkowsi sum of a linear term plus the expected value of the second-stage nondominated set, and the constraints, which are linear. Since the second-stage nondominated set is a random set, its expected value is defined through the selection expectation. We prove properties of TSSMOLPs and the multifunctions that arise therein, including that the global Pareto set of a TSSMOLP with two or more objectives is cone-convex on a general probability space. We also prove that two reformulations of the TSSMOLP are nondominance-equivalent to the original; these reformulations facilitate mathematical analysis and the future development of TSSMOLP solution methods.

Key words: stochastic programming; two-stage stochastic linear programming; multi-objective stochastic optimization; multi-objective optimization under uncertainty

1 Introduction

Two-stage stochastic programs refer to optimization problems in which strategic decisions must be made first under uncertainty, followed by recourse decisions made after the uncertainty is revealed. Two-stage stochastic programs arise in a wide variety of application areas, including agriculture, airline management, production planning, water resource modeling, energy planning, supply chain management, and transportation and logistics [Birge and Louveaux, 2011, Infanger, 2011, Shapiro et al., 2009]. In such problems, key strategic decisions, such as the allocation of crops in a field, the allocation of airplanes to certain routes, or the amount of a new product to stock, must be made in the *first stage* before uncertain parameters, such as crop yield, the weather, or product demand, are known. Once the uncertainty is observed, recourse decisions made in the *second stage* enable the system's decision-makers to respond to the observed values of the previously-uncertain parameters. For example, a farmer can adjust purchases and sales of products in response to crop yield, airlines can re-route airplanes in response to weather events, and stores can adjust subsequent product orders in response to observed demand.

Historically, two-stage stochastic programs have been formulated with the goal of optimizing one objective. In the seminal work by Dantzig [1955], the goal is to minimize the total expected *cost*. In practical applications, a single "cost" objective may be formulated in

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a variety of ways: by considering the monetary cost of each decision (e.g., maximize expected profit), or by assigning monetary value to all system design criteria and aggregating the criteria into a single "cost" value, or by deeming one system design criterion to be the "cost" and posing other "non-cost" system criteria as constraints. Either way, classic two-stage stochastic programs usually have a single objective, which results in a single optimal value, and there exists at least one feasible decision that can produce the optimal value.

In this article, we consider two-stage stochastic programs which enable decision-makers to consider multiple simultaneous and conflicting linear objectives across both decision stages. Specifically, the second-stage recourse decision is governed by an uncertain multi-objective linear program (MOLP) whose solution maps to an uncertain second-stage nondominated set. The two-stage stochastic multi-objective linear program (TSSMOLP) then comprises the objective function, which is the Minkowsi sum of a linear term plus the expected value of the second-stage nondominated set, and the constraints, which are linear. Since the second-stage nondominated set is a random set, its expected value is defined through the selection expectation, which employs the Aumann integral [Aumann, 1965]. We view this formulation as the natural extension from a recourse decision which is posed as an *uncertain linear program* to a recourse decision which is posed as an *uncertain multi-objective linear program*. Thus, our interest lies in the multi-objective generalization of the well-studied two-stage stochastic linear program [Dantzig, 1955, Sen and Higle, 1999, Shapiro et al., 2009, Birge and Louveaux, 2011, Infanger, 2011].

1.1 Motivating example

To provide a concrete example of a TSSMOLP, we consider the context of disaster management or humanitarian logistics, in which there are usually multiple stages and conflicting objectives [Celik et al., 2012, Huang et al., 2012, Gutjahr and Nolz, 2016, Kress, 2016]. Inspired by Mete and Zabinsky [2010], suppose there exists an impending disaster, such as a hurricane or earthquake, for which leaders must make logistic deployment decisions. Supplies can be pre-staged by moving them from a centralized warehouse to several localized depots in the first stage. After the disaster occurs and the uncertainty is realized in the form of its exact location and severity, the supplies are then moved from localized depots to community distribution centers to meet demand in the second stage. Throughout the disaster planning and response process, decision-makers want to simultaneously minimize the expected total transport cost and maximize the expected demand coverage.

Given a feasible first-stage decision $x \in \mathbb{R}^{q_1}$, which allocates the supplies to the q_1 localized depots, and after the disaster occurs such that the uncertainty u is known, moving the supplies from q_1 depots to r_2 community distribution centers to minimize the transportation cost and maximize the demand coverage is an MOLP with $p = 2$ objectives. For a second-stage decision $y \in \mathbb{R}^{q_2}$, where $q_2 = q_1 \times r_2$ and y_{ij} represents the amount of supplies to transport from localized depot i to community distribution center j, the second-stage MOLP is

$$\begin{aligned} \text{minimize} \quad & \begin{bmatrix} d_1(u)^\intercal y \\ d_2(u)^\intercal y \end{bmatrix} \\ \text{s.t.} \quad & \textstyle\sum_{j=1}^{m} y_{ij} \leq x_i \text{ for all depots } i = 1, \ldots, q_1, \\ & \textstyle\sum_{i=1}^{q_1} y_{ij} \leq s_j(u) \text{ for all distribution centers } j = 1, \ldots, r_2, \\ & y \in \mathbb{R}^{q_2}, y \geqq 0. \end{aligned} \tag{1}$$

Here, the coefficient on the first objective $d_1(u) \in \mathbb{R}^{q_2}$ is a vector representing the observed transportation costs for moving supplies from depot i to distribution center j, which are subject to uncertainty due to the possibility that certain routes may become impassable or certain modes of transport may become infeasible. The coefficient on the second objective $d_2(u) = (-d_{21}(u), \ldots, -d_{2r_2}(u), \ldots, -d_{21}(u), \ldots, -d_{2r_2}(u))^\intercal \in \mathbb{R}^{q_2}$ is a vector representing the negative observed proportional demand at the community distribution centers, where $\sum_{j=1}^{r_2} d_{2j} = 1$. Notice that the demand coverage objective is equal to $-\sum_{j=1}^{r_2}(d_{2j}(u)\sum_{i=1}^{q_1} y_{ij})$ where $\sum_{i=1}^{q_1} y_{ij}$ is the amount of supply received by distribution center j across all depots $i = 1, \ldots, q_1$ [Rath et al., 2016]. Finally, the constraints ensure that no depot ships more than it has, and no distribution center j accepts more than its capacity $s_j(u)$, where the distribution center's capacity is subject to uncertainty u; e.g., a storage area may be destroyed or unusable.

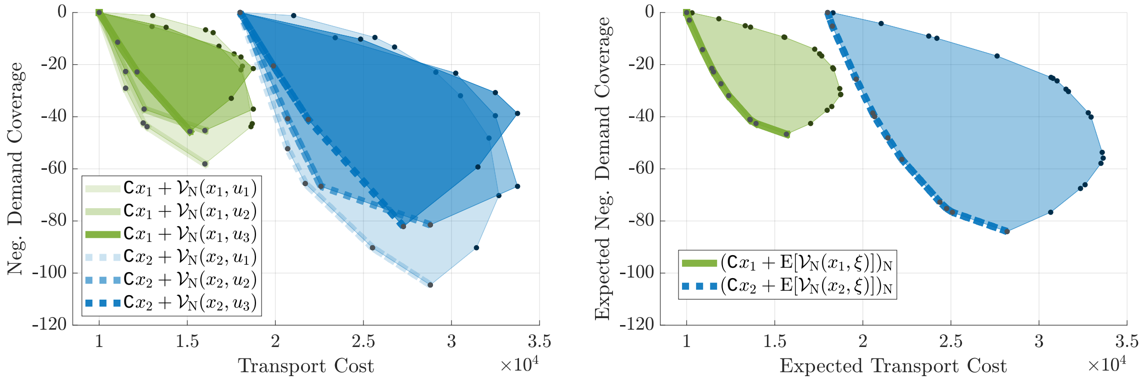
For each feasible first-stage decision x, a solution to the second-stage MOLP maps to the second-stage nondominated set in the objective space, which we define as follows. First, the feasible set of (1) is a polyhedron, which we denote as the set $\mathcal{Y}(x, u) \subset \mathbb{R}^{q_2}$. Likewise, its image set is a polyhedron, $\mathcal{V}(x, u) = \mathsf{D}(u)\mathcal{Y}(x, u)$, where $\mathsf{D}(u) \in \mathbb{R}^{p \times q_2}$ is a matrix representing the second-stage coefficients across p conflicting objectives, $\mathsf{D}(u) = (d_1(u)^\intercal, d_2(u)^\intercal)^\intercal$, and the multiplication is elementwise, $\mathsf{D}(u)\mathcal{Y}(x, u) = \{\mathsf{D}(u)y \colon y \in \mathcal{Y}(x, u)\}$. Then the second-stage nondominated set is

$$\mathcal{V}_{\mathrm{N}}(x, u) = \{z \in \mathcal{V}(x, u) \colon \nexists \tilde{z} \in \mathcal{V}(x, u) \text{ such that } \tilde{z} \leq z\} \subset \mathbb{R}^p,$$

where, consistent with the multi-objective optimization literature, we use $\tilde{z} \leq z$ to denote that $\tilde{z}_k \leq z_k$ for all $k = 1, \ldots, p$ and $\tilde{z} \neq z$; we use $\tilde{z} \leqq z$ when equality is allowed.

While only one second-stage decision can be implemented in practice, the goal of solving the second-stage MOLP in (1) is to present the decision-maker with a characterization of all nondominated outcomes with respect to the simultaneous objectives. Given two feasible first-stage decisions $x_1, x_2 \in \mathcal{X}$ and three possible scenarios u_1, u_2, and u_3, Figure 1a shows the resulting six polyhedral image sets and their corresponding nondominated sets for the second-stage MOLP in (1), which are translated by the first stage costs $\mathsf{C}x_1$ and $\mathsf{C}x_2$ described in problem (2) below. For any given combination of first-stage decision and uncertainty in Figure 1a, the resulting nondominated set provides the decision-maker with perspective on the tradeoffs between the objectives when choosing a final decision for implementation. For example, the decision maker can readily assess how much more it will cost to achieve an incremental improvement in demand coverage under each first-stage decision and scenario. MOLP solution methods are well-studied in the literature, and open-source software exists to solve MOLPs. We refer the reader to Benson [1998], Ehrgott [2005], Löhne [2011], Luc [2016] for MOLPs and to Adeyefa and Luhandjula [2011] for stochastic MOLPs. Figure 1 was created using Bensolve, which is an open-source software for solving vector optimization problems [Löhne and Weißing, 2017].

Having posed the second-stage problem as an uncertain MOLP, we now pose the overall TSSMOLP to determine the first-stage decisions which lead to globally Pareto optimal outcomes with respect to both objectives. Let $\xi \colon \Omega \to \mathbb{R}^m$ denote a random vector in a to-be-discussed probability space that models the uncertainty. The TSSMOLP to minimize



(a) For a given (x, u), the plot shows the translated second-stage image set $\mathsf{C}x + \mathcal{V}(x, u)$ (polyhedron) and translated second-stage nondominated set and $\mathsf{C}x + \mathcal{V}_\mathrm{N}(x, u)$ (line).

(b) For a given x, the plot shows the translated expected second-stage image set $\mathsf{C}x + \mathrm{E}[\mathcal{V}(x, \xi)]$ (polyhedron) and the nondominated translated expected second-stage nondominated set $(\mathsf{C}x + \mathrm{E}[\mathcal{V}_\mathrm{N}(x, \xi)])_\mathrm{N}$ (line).

Figure 1: Example instances of the relevant sets for the TSSMOLP in (2), given two first-stage decisions $x_1, x_2 \in \mathcal{X}$ and scenarios $u_1, u_2, u_3 \in \Xi$ occurring with respective probabilities $\alpha_1, \alpha_2, \alpha_3 > 0$, $\alpha_1+\alpha_2+\alpha_3 = 1$.

the expected total transport cost and maximize the expected demand coverage is

$$
\begin{aligned}
\text{minimize} \quad & \begin{bmatrix} c_1^\intercal x \\ c_2^\intercal x \end{bmatrix} + \mathrm{E}[\mathcal{V}_\mathrm{N}(x, \xi)] \\
\text{s.t.} \quad & \textstyle\sum_{i=1}^{q_1} x_i \leq b_\mathrm{w} \\
& x_i \leq b_i \text{ for all depots } i = 1, \dots, q_1 \\
& x \in \mathbb{R}^{q_1}, x \geqq 0,
\end{aligned}
\tag{2}
$$

where $c_1 \in \mathbb{R}^{q_1}$ is a vector representing the first-stage transportation costs, $c_2 = (0, \dots, 0)$ implies that demand coverage is not a consideration in the first stage (although we could formulate a more complex example with some form of first-stage demand coverage), and the constraints defining the feasible set $\mathcal{X} \subset \mathbb{R}^{q_1}$ imply we do not ship more than the warehouse has or more than the depots can accept. The symbol $+$ in the objective function of (2) denotes a Minkowski sum between a vector and a set; thus, the objective in (2) is a set [Hamel et al., 2015]. The expected value of the random nondominated set $\mathrm{E}[\mathcal{V}_\mathrm{N}(x, \xi)]$ is defined using the selection expectation, which is discussed on a general probability space in Appendix A. To fix ideas, for now, it is sufficient to consider only the case of an atomic probability measure (see Subsection 1.4) in which the random variable ξ takes on a finite set of values called scenarios, $u_1 = \xi(\omega_1), \dots, u_n = \xi(\omega_n)$, $\omega_i \in \Omega$ for all $i = 1, \dots, n$, with respective probabilities $\alpha_1, \dots, \alpha_n \in (0, 1), \sum_i \alpha_i = 1$. In this special case, the expected value under the selection expectation equals

$$\mathrm{E}[\mathcal{V}_\mathrm{N}(x, \xi)] = \sum_{i=1}^{n} \alpha_i \mathcal{V}_\mathrm{N}(x, u_i)$$

where the sum of the nondominated sets employs the Minkowski sum.

Given two feasible first-stage decisions $x_1, x_2 \in \mathcal{X}$ and three possible scenarios u_i occurring with respective probabilities α_i for $i \in \{1, 2, 3\}$, Figure 1b shows the corresponding expected image and nondominated sets from (2). Each first-stage decision x produces an entire set of expected nondominated outcomes in the image space and, together with the uncertainty, governs the nature of the nondominated set available to the decision-maker when solving the second-stage MOLP. We remark here that $\mathrm{E}[\mathcal{V}_{\mathrm{N}}(x,\xi)]$ may contain points dominated by other points in the same set; in particular, if the underlying probability measure is nonatomic, $\mathrm{E}[\mathcal{V}_{\mathrm{N}}(x,\xi)]$ is convex under the selection expectation. Due to the outer minimization in (2), we plot $(\mathsf{C}x+\mathrm{E}[\mathcal{V}_{\mathrm{N}}(x,\xi)])_{\mathrm{N}}$ instead of $\mathsf{C}x+\mathrm{E}[\mathcal{V}_{\mathrm{N}}(x,\xi)]$ in Figure 1b.

Ultimately, our goal in solving (2) is to present the decision-maker with only feasible first-stage decisions that produce expected second-stage nondominated sets containing at least one globally Pareto optimal point. Under the definition of global Pareto optimality which we adopt in Definition 2.1, such feasible decisions are members of the *global efficient set*, $\mathcal{X}_{\mathrm{E}}$. Temporarily supposing that we consider a reduced feasible set $\tilde{\mathcal{X}}=\{x_1,x_2\}$ in Figure 1b, both x_1 and x_2 are globally efficient on $\tilde{\mathcal{X}}$: first, all points in $(\mathsf{C}x_1+\mathrm{E}[\mathcal{V}_{\mathrm{N}}(x_1,\xi)])_{\mathrm{N}}$ are globally Pareto optimal on $\tilde{\mathcal{X}}$, which implies x_1 is efficient. Second, while some points in $(\mathsf{C}x_2+\mathrm{E}[\mathcal{V}_{\mathrm{N}}(x_2,\xi)])_{\mathrm{N}}$ are dominated by points in $(\mathsf{C}x_1+\mathrm{E}[\mathcal{V}_{\mathrm{N}}(x_1,\xi)])_{\mathrm{N}}$, others are not, implying that a subset of $(\mathsf{C}x_2+\mathrm{E}[\mathcal{V}_{\mathrm{N}}(x_2,\xi)])_{\mathrm{N}}$ is globally Pareto optimal on $\tilde{\mathcal{X}}$. Thus, x_2 is also efficient. The first-stage decisions we wish to exclude from consideration are those that produce expected second-stage nondominated sets containing only dominated points; the image sets corresponding to these first-stage decisions are not shown in Figure 1.

Since every feasible decision in the global efficient set produces an entire set of expected second-stage nondominated outcomes, the multi-objective setting may seem prohibitively complex or impractical in a real-world setting. However, the multi-objective optimization literature contains methods which make the solutions to complex or high-dimensional multi-objective optimization problems usable for decision-makers. For example, Sayin [2000] discusses methods for creating discrete representations of Pareto sets which can be easier for the decision-maker to evaluate. In the context of TSSMOLPs, such methods can be adapted to provide the decision-maker with valuable and comprehensible information regarding the expected second-stage nondominated sets across different efficient first-stage decisions. For example, once again comparing the two expected nondominated sets in Figure 1b on the reduced feasible set $\tilde{\mathcal{X}}=\{x_1,x_2\}$, one can see that if an expected demand coverage of 30 is acceptable, decision x_1 is much cheaper in expectation. However, if expected demand coverage must be over 60, only x_2 can achieve it, and the expected nondominated set for x_2 quantifies the increase in expected transport cost as the expected demand coverage increases.

Finally, we remark that reformulating the TSSMOLP in (2) into a single-objective optimization problem and solving it provides only a very small part of the information supplied by the original TSSMOLP: The single-objective problem's optimal solution is only one of the efficient points of the TSSMOLP. The multi-objective setting facilitates providing decision-makers with the broader perspective demonstrated in Figure 1, which enables them to incorporate their judgment and factors external to the mathematical model when choosing among the Pareto alternatives.

1.2 Related literature

While the problem context for TSSMOLPs arises in the literature, general formulations and unified mathematical solution approaches are only beginning to appear. In this regard, Dowson et al. [2022] and Hamel and Löhne [2024] consider TSSMOLPs where the uncertainty is governed by atomic distributions with finite support. First, Dowson et al. [2022] provide an algorithm to solve a bi-objective multistage stochastic program assuming the probability distribution governing the uncertainty is known. The authors employ scalarization in both stages, so that the expected second stage objective value at optimality is a vector rather than a set. Second, the formulation in Hamel and Löhne [2024] employs the selection expectation to handle the second-stage random sets, however, they minimize over the second-stage decision variables outside the Minkowski sum rather than inside. Their formulation would be equivalent to ours in one objective [Shapiro et al., 2009]; however, it is not immediately clear to us that equivalence holds in two or more objectives.

Further related literature appears in the survey by Gutjahr and Pichler [2016], which includes a section on two-stage stochastic multi-objective optimization. They remark that the majority of problems appearing in the literature assume a bi-objective formulation with special dependence structure between the objectives that facilitates analysis through the epsilon-constraint method. Many such papers in disaster relief planning reformulate multiple objectives into a single-objective problem using, e.g., goal programming, compromise programming, or the epsilon-constraint method [Grass and Fisher, 2016, p. 94]; also see the discussion in Rath et al. [2016]. Further, we remark that papers in both areas often employ integer or binary variables which are both useful in practice and beyond the scope of the present work, e.g. Huang et al. [2012], Yang and Bayraksan [2023]. Finally, while we consider the risk-neutral formulation, we remark that risk-averse formulations exist in the literature; see Ç. Ararat et al. [2017], Dentcheva and Wolfhagen [2016], Noyan et al. [2022].

1.3 Contribution and overview of main results

To the best of our knowledge, we are the first to define and study properties of (risk-neutral) TSSMOLPs for two or more objectives on a *general probability space*. By allowing probability measures that are nonatomic or atomic with infinite support, our formulation facilitates modeling a wider variety of real-world settings than existing formulations in Dowson et al. [2022] and Hamel and Löhne [2024]. In the process, we avoid the need to discretize or truncate the support of an otherwise nonatomic probability measure, which introduces error that cannot be overcome by sampling when the probability measure is unknown. However, analyzing and solving such general TSSMOLPs poses new challenges because fundamental results are not yet available. We fill this gap in the literature by proving foundational properties of TSSMOLPs and the set-valued maps, or multifunctions, that arise therein.

We demonstrate the properties of the TSSMOLPs formulated in Section 2 as follows. First, in Section 3, we prove properties of the second-stage image and nondominated random multifunctions, and in Section 4, we prove properties of their corresponding expectations. Importantly, under appropriate regularity conditions, we show that the second-stage image multifunction $\mathcal{V}\colon \mathcal{X} \times \Xi \rightrightarrows \mathbb{R}^p$ is a *set-valued convex normal integrand* (Proposition 3.4), and we show that the expected second-stage image multifunction $\mathrm{E}[\mathcal{V}(\cdot,\xi)]\colon \mathcal{X} \rightrightarrows \mathbb{R}^p$ is outer semicontinuous, graph-convex, and bounded on the feasible set $\mathcal{X}$ (Proposition 4.3). Then, in Section 5, we present two reformulations of the TSSMOLP and prove that they are nondominance-equivalent to the original. First, under the *polyhedral reformulation*, we show that we can replace the expected second-stage nondominated set $\mathrm{E}[\mathcal{V}_{\mathrm{N}}(x,\xi)]$ in the objective function of the original TSSMOLP in Section 2 with the expected second-stage image set $\mathrm{E}[\mathcal{V}(x,\xi)]$ (Theorem 5.2). Thus, we can work only with the second-stage image random multifunction and its corresponding expected value whenever doing so is convenient. Second, under the *full-dimensional reformulation*, we show that we can replace $\mathrm{E}[\mathcal{V}_{\mathrm{N}}(x,\xi)]$ with the expected value of a full-dimensional polyhedron constructed from $\mathcal{V}(x,\xi)$ (Theorem 5.6).

This reformulation stands in analogy to the reformulation facilitating outer approximation methods for MOLPs by Benson [1998]. The polyhedral and full-dimensional reformulations facilitate future mathematical analysis, algorithm development, and computation. In evidence of their usefulness, we employ the polyhedral reformulation and the properties of the second-stage image multifunction to provide a concise proof that the global Pareto set of the TSSMOLP is cone-convex (Theorem 6.1) in Section 6. This result implies that every point in the global Pareto set has a supporting hyperplane, such that outer approximation methods might be employed to solve TSSMOLPs. Section 7 contains concluding remarks.

Given the foundational nature of this work and the breadth of our intended audience spanning multi-objective optimization and stochastic programming, we provide detailed references to the results in four books which we recommend as reading companions: Shapiro et al. [2009] for stochastic programming, Ehrgott [2005] for multi-objective optimization, Rockafellar and Wets [1998] for set-valued mappings (or multifunctions), and Molchanov [2017] for the theory of random sets (also called measurable multifunctions). Both Shapiro et al. [2009] and Molchanov [2017] contain useful appendices detailing key concepts of probability theory on which we rely. For convenience and completeness, preliminary concepts and definitions for working with random closed sets, multifunctions, random multifunctions, and the selection expectation are included in Appendix A.

1.4 Notation and terminology

When comparing two vectors $z, z' \in \mathbb{R}^p$, we write $z' \leqq z$ if $z'_k \leq z_k$ for all $k \in \{1, \ldots, p\}$ and write $z' \leq z$ if $z' \leqq z$ and $z' \neq z$; we use analogous definitions for $\geqq, \geq$. Thus, the set

$$\mathbb{R}^p_{\geqq} := \{z \in \mathbb{R}^p \colon z \geqq 0\}$$

is the non-negative orthant in p dimensions; likewise, $\mathbb{R}^p_{>} := \{z \in \mathbb{R}^p \colon z > 0\}$. The set $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ denotes the extended real numbers where $-\infty \leq z_1 \leq +\infty$ for all $z_1 \in \overline{\mathbb{R}}$. Then $\overline{\mathbb{R}}^p = (z_1, \ldots, z_p)$ where $z_k \in \overline{\mathbb{R}}$ for all $k = 1, \ldots, p$. Unless otherwise indicated, $\|z\|$ is the Euclidean norm of the vector $z \in \mathbb{R}^p$. $\mathcal{B}_1(0_p)$ denotes the p-dimensional unit ball.

For a set $\mathcal{S} \subset \mathbb{R}^p$, $\operatorname{cl}\mathcal{S}$ denotes its closure, $\operatorname{bd}\mathcal{S}$ its boundary, $\operatorname{int}\mathcal{S}$ its interior, and $\operatorname{diam}\mathcal{S} := \sup_{s_1,s_2\in\mathcal{S}}\|s_1 - s_2\|$ its diameter. For sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{R}^p$, the Minkowski sum is $\mathcal{S}_1 + \mathcal{S}_2 = \{s_1 + s_2 \colon s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2\}$, scalar multiplication of $\alpha \in \mathbb{R}$ with $\mathcal{S}$ is $\alpha\mathcal{S} = \{\alpha s \colon s \in \mathcal{S}\}$, and multiplication of a matrix $\mathsf{M} \in \mathbb{R}^{n\times p}$ times a set $\mathcal{S} \subset \mathbb{R}^{p\times 1}$ is $\mathsf{M}\mathcal{S} = \{\mathsf{M}s \colon s \in \mathcal{S}\} \subset \mathbb{R}^{n\times 1}$. For any set $\mathcal{S} \subset \mathbb{R}^p$, let $\mathcal{S}_{\mathrm{N}}$ be the set of all nondominated points in $\mathcal{S}$,

$$\mathcal{S}_{\mathrm{N}} := \{s \in \mathcal{S} \colon \nexists s' \in \mathcal{S} \text{ such that } s' \leq s\}.$$

In general, $\mathcal{S}_{\mathrm{N}}$ may be empty (e.g. if $\mathcal{S}$ is open) [Ehrgott, 2005]. The norm of a set $\mathcal{S}$ is

$$\|\mathcal{S}\| := \sup\{\|s\| \colon s \in \mathcal{S}\}. \tag{3}$$

A set that can be written as the intersection of finitely many half-spaces is a *polyhedral* convex set [Rockafellar, 1970, p. 11] or a *polyhedron* [Ehrgott, 2005, p. 163]. We sometimes refer to a bounded polyhedron as a *polytope*. The *dimension* of a polyhedron $\mathcal{X}$ is the maximal number of affinely independent points of $\mathcal{X}$, minus one [Ehrgott, 2005, p. 164]. A set $\mathcal{S}$ is *not connected* if there exist open sets $\mathcal{O}_1, \mathcal{O}_2$ such that $\mathcal{S} \subset \mathcal{O}_1 \cup \mathcal{O}_2$, $\mathcal{S} \cap \mathcal{O}_1 \neq \emptyset$, $\mathcal{S} \cap \mathcal{O}_2 \neq \emptyset$, and $\mathcal{S} \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$; otherwise, $\mathcal{S}$ is *connected* [Ehrgott, 2005, p. 86].

Given a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$, the acronym *w.p.1* stands for *with probability one*. For any event $\mathcal{A} \in \mathfrak{A}$, stating that $\mathcal{A}$ occurs w.p.1 means $\mathbb{P}(\mathcal{A}) = \mathbb{P}(\{\omega \in \Omega \colon \omega \in \mathcal{A}\}) = 1$. The abbreviations *a.e.* and *a.s.* stand for *almost every* and *almost surely*, respectively. The σ-algebra $\mathfrak{A}$ and the probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ are called *complete* if, for every set $\mathcal{A} \in \mathfrak{A}$ with $\mathbb{P}(\mathcal{A}) = 0$, all subsets of $\mathcal{A}$ are contained in $\mathfrak{A}$ [Molchanov, 2017, p. 587]. For any set $\mathcal{S}$, $\mathfrak{B}(\mathcal{S})$ denotes the σ-algebra of all Borel subsets of $\mathcal{S}$. A measure $\mathbb{P}$, and the probability space $(\Omega, \mathfrak{A}, \mathbb{P})$, is *nonatomic* if any set $\mathcal{A} \in \mathfrak{A}$ such that $\mathbb{P}(\mathcal{A}) > 0$ contains a subset $\mathcal{B} \in \mathfrak{A}$ such that $\mathbb{P}(\mathcal{A}) > \mathbb{P}(\mathcal{B}) > 0$ [Shapiro et al., 2009, p. 367]. If a measure is not nonatomic, then it is atomic; that is, it contains *atoms*: events $\mathcal{A}$ for which $\mathbb{P}(\mathcal{A}) > 0$ but for all $\mathcal{B} \subset \mathcal{A}$, $\mathbb{P}(\mathcal{B}) = \mathbb{P}(\mathcal{A})$ or $\mathbb{P}(\mathcal{B}) = 0$.

2 TSSMOLP problem formulation

Let ξ denote a real-valued random vector $\xi \colon \Omega \to \mathbb{R}^m$, defined with respect to the complete probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. We formulate the TSSMOLP for $p \geq 2$ objectives as

$$\begin{aligned} \text{minimize} \quad & \{\mathsf{C}x + \mathrm{E}[\mathcal{V}_{\mathrm{N}}(x, \xi)]\} \\ \text{s.t.} \quad & x \in \mathcal{X} \coloneqq \big\{x \in \mathbb{R}^{q_1} \colon \mathsf{A}x = b,\, x \geqq 0\big\}, \end{aligned} \tag{M}$$

where the second-stage decision is modeled as a stochastic MOLP with nondominated set

$$\begin{aligned} \mathcal{V}_{\mathrm{N}}(x, \xi) = \min \quad & \mathsf{D}(\xi) y \\ \text{s.t.} \quad & y \in \mathcal{Y}(x, \xi) \coloneqq \big\{y \in \mathbb{R}^{q_2} \colon \mathsf{W}(\xi) y = h(\xi) - \mathsf{T}(\xi) x,\, y \geqq 0\big\}. \end{aligned} \tag{L}$$

The expected value of the second-stage nondominated set in (M) is the selection expectation with respect to the random variable ξ, where the selection expectation is defined using the Aumann integral (Definitions A.9 and A.10). The first-stage feasible set $\mathcal{X}$ is a nonempty, polyhedral, convex subset of $\mathbb{R}^{q_1}$. The matrix $\mathsf{C} \in \mathbb{R}^{p \times q_1}$ is the first-stage cost matrix, $\mathsf{D}(\xi) \in \mathbb{R}^{p \times q_2}$ is the second-stage cost matrix, and there are p linear objectives in each stage,

$$\mathsf{C}x = \begin{bmatrix} c_1^\intercal x \\ \vdots \\ c_p^\intercal x \end{bmatrix}, \quad \mathsf{D}(\xi) y = \begin{bmatrix} d_1(\xi)^\intercal y \\ \vdots \\ d_p(\xi)^\intercal y \end{bmatrix}.$$

The constraints are specified by the matrix $\mathsf{A} \in \mathbb{R}^{r_1 \times q_1}$ and vector $b \in \mathbb{R}^{r_1}$ in the first stage, and by (possibly random) matrices $\mathsf{W}(\xi) \in \mathbb{R}^{r_2 \times q_2}$, $\mathsf{T}(\xi) \in \mathbb{R}^{r_2 \times q_1}$, and vector $h(\xi) \in \mathbb{R}^{r_2}$ in the second stage. We refer to the image set for the second-stage MOLP in (L) as

$$\mathcal{V}(x, \xi) = \mathsf{D}(\xi) \mathcal{Y}(x, \xi) = \{\mathsf{D}(\xi) y \colon y \in \mathcal{Y}(x, \xi)\} \subset \mathbb{R}^p, \tag{4}$$

which is a random polyhedral convex set. Random elements corresponding to uncertainty in the second-stage problem, $(\mathsf{D}(\xi), h(\xi), \mathsf{T}(\xi), \mathsf{W}(\xi))$, are understood to be $\mathfrak{A}$-measurable.

If the second-stage MOLP is infeasible for some $x \in \mathcal{X}$ and $\xi(\omega)$, $\omega \in \Omega$, then the second-stage image set is empty, $\mathcal{V}(x, \xi(\omega)) = \emptyset$, which implies $\mathcal{V}_{\mathrm{N}}(x, \xi(\omega)) = \emptyset$. Further, if there does not exist a point $z \in \mathbb{R}^p$ such that $\mathcal{V}(x, \xi(\omega)) \subset z + \mathbb{R}^p_{\geqq}$, then we say that the second-stage MOLP is unbounded from below. In this case, following the conventions in Ehrgott [2005], the second-stage nondominated set is empty, $\mathcal{V}_{\mathrm{N}}(x, \xi(\omega)) = \emptyset$. For example, if $\mathcal{V}(x, \xi(\omega))$ is the half-space $\{(z_1, z_2) \in \mathbb{R}^2 \colon z_1 \geq 0\}$, then $\mathcal{V}(x, \xi(\omega))$ is unbounded from below and $\mathcal{V}_{\mathrm{N}}(x, \xi(\omega)) = \emptyset$. These events shall occur on a set of measure zero by assumption; we formalize this assumption later.

To define the solution to (M), first, notice that each feasible decision value $x \in \mathcal{X}$ maps to a (possibly empty) subset of $\mathbb{R}^p$ denoted by $\{\mathsf{C}x + \mathrm{E}[\mathcal{V}_{\mathrm{N}}(x, \xi)]\}$. Such a mapping from $\mathcal{X}$ into the set of subsets of $\mathbb{R}^p$ is called a *set-valued mapping* or *multifunction* (Definition A.1). Throughout, we adopt the notation and conventions as discussed in Rockafellar and Wets [1998, p. 149] and let the mapping $\mathcal{Z} \colon \mathcal{X} \rightrightarrows \mathbb{R}^p$ be the multifunction

$$\mathcal{Z}(x) \coloneqq \mathsf{C}x + \mathrm{E}[\mathcal{V}_{\mathrm{N}}(x, \xi)] \text{ for all } x \in \mathcal{X}.$$

Given any set of feasible first-stage decisions $\mathcal{S} \subseteq \mathcal{X}$, the image of $\mathcal{S}$ under $\mathcal{Z}$ is the union of the translated expected second-stage nondominated sets,

$$\mathcal{Z}(\mathcal{S}) \coloneqq \cup_{x \in \mathcal{S}} \mathcal{Z}(x).$$

Thus, $\mathcal{Z}(\mathcal{X})$ denotes the image set corresponding to (M). This notation enables us to define the global Pareto and efficient sets as follows.

Definition 2.1. The *global Pareto set* for the TSSMOLP in (M) is

$$\mathcal{Z}_{\mathrm{P}} \coloneqq \mathcal{Z}_{\mathrm{N}}(\mathcal{X}) = \{z^* \in \mathcal{Z}(\mathcal{X}) \colon \nexists z \in \mathcal{Z}(\mathcal{X}) \text{ such that } z \leq z^*\}.$$

The solution to (M) in the decision space is the *global efficient set*

$$\mathcal{X}_{\mathrm{E}} \coloneqq \{x^* \in \mathcal{X} \colon \exists z^* \in \mathcal{Z}(x^*) \text{ such that } z^* \in \mathcal{Z}_{\mathrm{P}}\}.$$

Hamel et al. [2015, p. 82] refer to this approach for defining the global Pareto set as the "vector approach to set optimization." Under Definition 2.1, we say that a feasible point $x^* \in \mathcal{X}$ is efficient for (M) if at least one of the points in its corresponding translated expected second-stage nondominated set $\mathcal{Z}(x^*)$ belongs to the global Pareto set, $\mathcal{Z}_{\mathrm{P}}$. Notably, under Definition 2.1, the image of the efficient set,

$$\mathcal{Z}(\mathcal{X}_{\mathrm{E}}) = \cup_{x^* \in \mathcal{X}_{\mathrm{E}}} \mathcal{Z}(x^*) = \cup_{x^* \in \mathcal{X}_{\mathrm{E}}} \mathsf{C}x^* + \mathrm{E}[\mathcal{V}_{\mathrm{N}}(x^*, \xi)],$$

is a superset of the global Pareto set $\mathcal{Z}_{\mathrm{P}}$; hence,

$$\mathcal{Z}_{\mathrm{P}} \equiv \mathcal{Z}_{\mathrm{N}}(\mathcal{X}) \subseteq \mathcal{Z}(\mathcal{X}_{\mathrm{E}}) \subseteq \mathcal{Z}(\mathcal{X}).$$

Remark 2.2. In addition to $\mathcal{Z}_{\mathrm{P}}$, a decision-maker may be interested in observing $\mathcal{Z}_{\mathrm{N}}(x^*) = (\mathsf{C}x^* + \mathrm{E}[\mathcal{V}_{\mathrm{N}}(x^*, \xi)])_{\mathrm{N}} \subseteq \mathcal{Z}(\mathcal{X}_{\mathrm{E}})$ for one or more identified values $x^* \in \mathcal{X}_{\mathrm{E}}$, since trade-off rates in the relevant second-stage expected nondominated sets that make up $\mathcal{Z}(\mathcal{X}_{\mathrm{E}})$ may be informative for decision-making; see, for example, Figure 1b.

Remark 2.3. The TSSMOLP in (M) is fundamentally different from the traditional multi-objective simulation optimization (MOSO) formulation discussed in Hunter et al. [2019], in which the minimization acts on a collection of vectors. We remark that in general, (M) is not equivalent to the problem

$$\text{minimize} \quad \left[\mathrm{E}[c_1^\intercal x + V_1(x, \xi)], \ldots, \mathrm{E}[c_p^\intercal x + V_p(x, \xi)]\right]^\intercal \quad \text{s.t.} \quad x \in \mathcal{X},$$

where $V_k(x,\xi)$ is the optimal value of the kth stochastic linear program identical to (L) except having only one objective, $d_k(\xi)^\intercal y$ for each $k = 1,\dots,p$. Modeling each objective with a separate second-stage linear program would decouple the interactions between the second-stage decision variables y.

Remark 2.4. Rather than working exclusively in the original probability space $(\Omega, \mathfrak{A}, \mathbb{P})$, often, it is convenient to work in the probability space that results from completing the probability space induced by ξ. Let the probability space induced by ξ be $(\Xi, \mathfrak{B}(\Xi), P_\xi)$, where Ξ is a Borel subset of $\mathbb{R}^m$ denoting the support of ξ, and $\mathfrak{B}(\Xi)$ denotes the σ-algebra of all Borel subsets of Ξ. The probability space $(\Xi, \mathfrak{B}(\Xi), P_\xi)$ can be completed, and we denote the resulting completed induced probability space as $(\Xi, \mathfrak{M}, \mathbb{P}_\xi)$ [Billingsley, 1995, p. 45]. For clarity here and in what follows, $\xi\colon \Omega \to \mathbb{R}^m$ is always a random vector with support Ξ, $\xi(\omega)$ is its value at $\omega \in \Omega$, and u represents a generic vector value in $\Xi \subseteq \mathbb{R}^m$.

3 Properties of the random closed-valued multifunctions

In addition to the image set mapping $\mathcal{Z}$ defined in Section 2, the TSSMOLP formulation in (M) contains several other relevant multifunctions whose properties we study. In particular, for each $x \in \mathcal{X}$, the mappings $\mathcal{V}(x,\xi)$ and $\mathcal{V}_{\mathrm{N}}(x,\xi)$ represent *random closed sets*, which are also called *measurable closed-valued multifunctions* (Definition A.2). The mappings $\mathcal{V}(\cdot,\xi)$ and $\mathcal{V}_{\mathrm{N}}(\cdot,\xi)$ appearing in (4) and (L), respectively, represent *random closed-valued multifunctions*, which are also called *set-valued integrands* (Definition A.3).

In what follows, we demonstrate the properties of the random multifunctions relevant to TSSMOLPs. Specifically, in Proposition 3.4, we demonstrate that $\mathcal{V}(\cdot,\cdot)$ is a *set-valued convex normal integrand*, and in Proposition 3.6, we demonstrate that $\mathcal{V}_{\mathrm{N}}(\cdot,\cdot)$ is a *set-valued normal integrand* (Definition A.4). Since both random multifunctions arise in the context of an MOLP, we begin by presenting basic properties of MOLPs in the following Lemma 3.1. Then, we present the main results in Subsections 3.1 and 3.2 which follow. We continue to maintain the notation and conventions of Rockafellar and Wets [1998], even though we invoke results from Molchanov [2017] which employs an alternate, but equivalent, representation for multifunctions (see Rockafellar and Wets [1998, Theorem 14.4, p. 645], Molchanov [2017, Theorem 1.3.9, p. 62]). For convenience, in the remainder of Section 3, we work in the completed probability space induced by ξ, $(\Xi, \mathfrak{M}, \mathbb{P}_\xi)$; see Remark 2.4.

Lemma 3.1 (properties of MOLPs)**.** *Consider an MOLP,*

$$\text{minimize} \quad \mathsf{C}x \quad \text{s.t.} \quad x \in \mathcal{X} = \{\mathsf{A}x = b,\, x \geqq 0\},$$

where the feasible set $\mathcal{X} \subset \mathbb{R}^q$ is nonempty, there are p objectives such that $\mathsf{C} \in \mathbb{R}^{p\times q}$, and the image set is $\mathcal{S} = \{\mathsf{C}x\colon x \in \mathcal{X}\} \neq \emptyset$. Then the following statements hold:

1. *The image set $\mathcal{S}$ is a closed polyhedral convex set (see Subsection 1.4).*
2. *If $\mathcal{S}$ is bounded from below, that is, there exists $z \in \mathbb{R}^p$ such that $\mathcal{S} \subset z + \mathbb{R}^p_{\geqq}$, then the nondominated set $\mathcal{S}_{\mathrm{N}}$ is nonempty, connected, and closed.*
3. *If $\mathcal{X}$ is compact, then (a) $\mathcal{S}$ is a nonempty, compact polyhedron, and (b) $\mathcal{S}_{\mathrm{N}}$ is nonempty, connected, and compact.*

Proof sketch. All of Parts 1 and 2 except that $\mathcal{S}_{\mathrm{N}}$ is closed follow from Ehrgott [2005], where connected sets are defined in Subsection 1.4. That $\mathcal{S}_{\mathrm{N}}$ is closed follows because the efficient set for the MOLP is a union of a finite number of closed faces of $\mathcal{X}$ [Benson and Sun, 1999, Yu and Zeleny, 1975], and $\mathcal{S}_{\text{N}}$ is its image under a linear map; see Rockafellar [1970]. Part 3a follows because the image set is the image of the feasible set under a linear map, and Part 3b follows from Parts 2 and 3a. □

3.1 Properties of the second-stage image random multifunction

We begin by considering the properties of the second-stage image mapping $\mathcal{V}\colon \mathcal{X} \times \Xi \rightrightarrows \mathbb{R}^p$ defined in (4). First, under Definition A.3, the map $\mathcal{V}(\cdot,\cdot)$ is a random closed-valued multifunction because for each $x \in \mathcal{X}$, $\mathcal{V}(x,\cdot)$ is a random closed set taking values in $\mathbb{R}^p$. That $\mathcal{V}(x,\cdot)$ is a random closed set for each $x \in \mathcal{X}$ holds by the properties of image sets for MOLPs enumerated in Lemma 3.1, and because for each $x \in \mathcal{X}$, all of the second-stage objective functions are linear (and, thus, continuous) in y, and each $d_k(\cdot)^\intercal y$ is measurable with respect to its argument for $k = 1, \ldots, p$; see Vogel [1992, p. 115], Rockafellar [1976].

Before establishing that $\mathcal{V}(\cdot,\cdot)$ is a set-valued convex normal integrand in Proposition 3.4, first, we detail several properties of $\mathcal{V}(\cdot, u)$ for given $u \in \Xi$ in Lemma 3.2. Then, in Lemma 3.3, we establish joint measurability under Assumption 1, which ensures the second-stage problem has relatively complete recourse [Shapiro et al., 2009, p. 33].

Lemma 3.2. *Consider the second-stage feasible set multifunction $\mathcal{Y}(\cdot,u)\colon \mathcal{X} \rightrightarrows \mathbb{R}^{q_2}$ for given $u \in \Xi$ such that its domain $\operatorname{dom}\mathcal{Y}(\cdot,u) = \{x \in \mathcal{X}\colon \mathcal{Y}(x,u) \neq \emptyset\}$ is nonempty. Then*

1. *$\mathcal{Y}(\cdot,u)$ is polyhedral, graph-convex, and Lipschitz continuous on its domain (Definition A.5), and*
2. *the second-stage image set multifunction $\mathcal{V}(\cdot,u)\colon \mathcal{X} \rightrightarrows \mathbb{R}^p$ is a polyhedral, graph-convex map and, thus, Lipschitz continuous on $\operatorname{dom}\mathcal{Y}(\cdot,u)$; that is, there exists $\kappa_0 \in \mathbb{R}_>$ such that $\mathcal{V}(x',u) \subseteq \mathcal{V}(x,u) + \kappa_0\|x'-x\|\mathcal{B}_1(0_p)$ for all $x, x' \in \operatorname{dom}\mathcal{Y}(\cdot,u)$; recall that $\mathcal{B}_1(0_p)$ denotes the p-dimensional unit ball.*

Proof. For Part 1, $\operatorname{gph}\mathcal{Y}(\cdot,u) := \{(x,y)\colon y \in \mathcal{Y}(x,u)\} \subset \mathbb{R}^{q_1} \times \mathbb{R}^{q_2}$, and notice that it is the intersection of $\mathcal{X} \times \mathbb{R}^{q_2}$ with $\{(x,y)\colon \mathsf{W}(u)y = h(u) - \mathsf{T}(u)x\}$ and $\{(x,y)\colon y \geqq 0\}$; see Rockafellar and Wets [1998, Example 5.8 Detail, p. 154]. Therefore, $\operatorname{gph}\mathcal{Y}(\cdot,u)$ is an intersection of a finite number of convex polyhedral sets, implying it is both convex and polyhedral. Thus, $\mathcal{Y}(\cdot,u)$ is Lipschitz continuous on its domain [Rockafellar and Wets, 1998, Example 9.35, p. 376]. Part 2 follows from Part 1 because the graph of $\mathcal{V}(\cdot,u)$,

$$\operatorname{gph}\mathcal{V}(\cdot,u) := \{(x,z)\colon z \in \mathcal{V}(x,u)\} = \{(x,z)\colon z \in \mathsf{D}(u)\mathcal{Y}(x,u)\} \subset \mathbb{R}^{q_1} \times \mathbb{R}^p,$$

must also be a convex polyhedron (see, e.g., Rockafellar [1970, p. 174]), which implies $\mathcal{V}(\cdot,u)$ is Lipschitz continuous on its domain. □

Assumption 1 (relatively complete recourse)**.** *For every $x \in \mathcal{X}$, the second-stage feasible set $\mathcal{Y}(x,\xi)$ is nonempty w.p.1; that is, $\mathbb{P}\big(\mathcal{Y}(x,\xi) \neq \emptyset\big) = 1$ for all $x \in \mathcal{X}$. Equivalently, in the probability space $(\Xi, \mathfrak{M}, \mathbb{P}_\xi)$,*

$$\mathbb{P}_\xi\left(\cap_{x\in\mathcal{X}}\{u \in \Xi\colon \mathcal{Y}(x,u) \neq \emptyset\}\right) = \mathbb{P}_\xi\big(\{u \in \Xi\colon \operatorname{dom}\mathcal{Y}(\cdot,u) = \mathcal{X}\}\big) = 1.$$

Lemma 3.3. *Under Assumption 1, the random closed-valued multifunction $\mathcal{V}\colon \mathcal{X} \times \Xi \rightrightarrows \mathbb{R}^p$ is jointly measurable with respect to the σ-algebra of $\mathcal{X} \times \Xi$, which is given by the product σ-algebra of $\mathfrak{B}(\mathcal{X})$ and $\mathfrak{M}$.*

Proof. Assumption 1 and Lemma 3.2 imply that $\mathcal{V}(x,\cdot)$ is Lipschitz continuous in x a.s. (Equivalently, in the original probability space, we have $\mathcal{V}(x,\xi)$ is Lipschitz continuous in x a.s.) Therefore, by Molchanov [2017, Proposition 5.1.20, p. 463], the set-valued process is separable. Then, joint measurability holds by Molchanov [2017, Theorem 5.1.21, p. 463]. ☐

Having established Lemmas 3.2 and 3.3, we are now ready to present Proposition 3.4, in which we demonstrate that $\mathcal{V}$ is a set-valued convex normal integrand. This proposition holds under Assumption 1 and the following Assumption 2, which ensures that we can equip the feasible set $\mathcal{X}$ with a uniform distribution.

Assumption 2. *The feasible set $\mathcal{X} \subset \mathbb{R}^{q_1}$ is a nonempty compact polyhedral convex set.*

Proposition 3.4. *Under Assumptions 1 and 2, the random closed-valued multifunction $\mathcal{V}\colon \mathcal{X} \times \Xi \rightrightarrows \mathbb{R}^p$ is a set-valued convex normal integrand.*

Proof. First, $\operatorname{gph}\mathcal{V}(\cdot,u)$ must be closed for every $u \in \Xi$ (Definition A.4 Part (a)). If $u \in \Xi$ is such that $\operatorname{dom}\mathcal{Y}(\cdot,u) \neq \emptyset$, then $\operatorname{gph}\mathcal{V}(\cdot,u)$ is closed because it is polyhedral by Lemma 3.2. If $u \in \Xi$ is such that $\operatorname{dom}\mathcal{Y}(\cdot,u) = \emptyset$, then $\mathcal{V}(x,u) = \emptyset$ for all $x \in \mathcal{X}$. Therefore, $\operatorname{gph}\mathcal{V}(\cdot,u) = \{(x,z)\colon z \in \mathcal{V}(x,u)\} = \emptyset$, which is closed. Thus, $\operatorname{gph}\mathcal{V}(\cdot,u)$ is closed for all $u \in \Xi$. Second, $\operatorname{gph}\mathcal{V} = \{(x,u,z)\colon z \in \mathcal{V}(x,u)\}$ must belong to $\mathfrak{B}(\mathcal{X}) \otimes \mathfrak{M} \otimes \mathfrak{B}(\mathbb{R}^p)$, where $\mathcal{X}$ is a convex, Borel subset of $\mathbb{R}^{q_1}$ (Definition A.4 Part (b)). Given the joint measurability of $\mathcal{V}(\cdot,\cdot)$ in Lemma 3.3 and the fact that $\mathcal{X}$ is a nonempty compact polyhedral convex set under Assumption 2, we can equip $\mathcal{X}$ with a uniform probability measure P_X having full support, i.e. $P_X(\mathcal{X}) = 1$, and specify a joint distribution $P_{(X,\xi)}$ through independence. Then, we can view $\mathcal{V}$ as a random closed set on the joint probability space $(\mathcal{X} \times \Xi, \mathfrak{B}(\mathcal{X}) \otimes \mathfrak{M}, P_{(X,\xi)})$; also see the discussion in Molchanov [2017, p. 464]. This view allows us to invoke Molchanov [2017, Theorem 1.3.3, p. 59] or Rockafellar and Wets [1998, Theorem 14.8, p. 648] so that the joint measurability of the multifunction $\mathcal{V}$ implies the $\mathfrak{B}(\mathcal{X}) \otimes \mathfrak{M} \otimes \mathfrak{B}(\mathbb{R}^p)$-measurability of $\operatorname{gph}\mathcal{V}$. (Note that here, we do not require the probability space to be complete.) Thus, $\mathcal{V}$ is a set-valued normal integrand. Finally, that $\operatorname{gph}\mathcal{V}(\cdot,u)$ is convex for a.e. $u \in \Xi$ follows directly from Lemma 3.2 Part 2 together with Assumption 1. ☐

3.2 Properties of the second-stage nondominated random multifunction

Next, we consider the second-stage nondominated map $\mathcal{V}_{\mathrm{N}}(\cdot,\cdot)$ defined in (L) and demonstrate that it is a set-valued normal integrand under Definition A.4. To begin, under Definition A.3, $\mathcal{V}_{\mathrm{N}}\colon \mathcal{X} \times \Xi \rightrightarrows \mathbb{R}^p$ is a random closed-valued multifunction. To see this, fix an $x \in \mathcal{X}$ and notice that $\mathcal{V}(x,\cdot) + \mathbb{R}^p_{\geqq}$ is a random upper convex set; we say a set $\mathcal{S} \subseteq \mathbb{R}^p$ is *upper* if for all $z_1 \in \mathcal{S}$ and all $z_2 \in \mathbb{R}^p$, $z_1 \leq z_2$ implies $z_2 \in \mathcal{S}$. For all $u \in \Xi$,

$$\mathcal{V}_{\mathrm{N}}(x,u) = \big(\mathcal{V}(x,u) + \mathbb{R}^p_{\geqq}\big)_{\mathrm{N}}$$

by Ehrgott [2005, Proposition 2.3, p. 27]. Thus, $\mathcal{V}_{\mathrm{N}}(x,\cdot)$ is the nondominated set of the random upper convex set $\mathcal{V}(x,\cdot) + \mathbb{R}^p_{\geqq}$ (see Molchanov [2017, Theorem 1.3.25, p. 69]), which implies $\mathcal{V}_{\mathrm{N}}(x,\cdot)$ is a random closed set by Molchanov [2017, Proposition 1.8.27, p. 159].

We now show that $\mathcal{V}_{\mathrm{N}}(\cdot,\cdot)$ is a set-valued normal integrand. First, in Lemma 3.5, we show that $\mathcal{V}_{\mathrm{N}}(\cdot,u)$ is piecewise polyhedral for each $u \in \Xi$; see Rockafellar and Wets [1998, Example 9.57, p. 399] for a discussion of the implications of this fact. Then, Proposition 3.6 establishes the result.

Lemma 3.5. *For given $u \in \Xi$, the multifunction $\mathcal{V}_{\mathrm{N}}(\cdot, u)\colon \mathcal{X} \rightrightarrows \mathbb{R}^p$ is piecewise polyhedral; that is,* $\operatorname{gph} \mathcal{V}_{\mathrm{N}}(\cdot, u) = \{(x, z^*)\colon z^* \in \mathcal{V}_{\mathrm{N}}(x, u)\}$ *can be expressed as the union of finitely many polyhedral sets which are faces of* $\operatorname{gph} \mathcal{V}(\cdot, u)$.

Proof. From Lemma 3.2, $\operatorname{gph} \mathcal{V}(\cdot, u) = \{(x, z)\colon z \in \mathcal{V}(x, u)\}$ is a convex polyhedron, which implies it is closed and has finitely many faces. Each face is itself a convex polyhedron [Rockafellar, 1970, Theorem 19.1, p. 171], and the empty set is polyhedral [Rockafellar, 1970, p. 170]. If $\operatorname{gph} \mathcal{V}(\cdot, u) = \emptyset$, then $\operatorname{gph} \mathcal{V}_{\mathrm{N}}(\cdot, u) = \emptyset$ is a union of finitely many polyhedral sets. Likewise, if $\operatorname{gph} \mathcal{V}(\cdot, u) \neq \emptyset$ and $\operatorname{gph} \mathcal{V}_{\mathrm{N}}(\cdot, u) = \emptyset$, e.g. due to unboundedness, then the result also holds. Thus, henceforth, suppose $\operatorname{gph} \mathcal{V}_{\mathrm{N}}(\cdot, u) \neq \emptyset$, $\operatorname{gph} \mathcal{V}(\cdot, u) \neq \emptyset$, and let $\cup_{j=1}^{\ell} \mathcal{F}_j(u) = \operatorname{bd}(\operatorname{gph} \mathcal{V}(\cdot, u))$ denote the union of the ℓ faces of $\operatorname{gph} \mathcal{V}(\cdot, u)$.

First, suppose $(x, z^*) \in \operatorname{gph} \mathcal{V}_{\mathrm{N}}(\cdot, u)$. Then since $z^* \in \mathcal{V}_{\mathrm{N}}(x, u) \subseteq \operatorname{bd} \mathcal{V}(x, u)$ by Ehrgott [2005, Proposition 2.4, p. 28], we have $z^* \in \operatorname{bd} \mathcal{V}(x, u) \subseteq \operatorname{bd}(\operatorname{gph} \mathcal{V}(\cdot, u))$. Then there exists a face $\mathcal{F}_{j^*}(u)$, $j^* \in \{1, \dots, \ell\}$ such that $(x, z^*) \in \mathcal{F}_{j^*}(u)$. Therefore, $\operatorname{gph} \mathcal{V}_{\mathrm{N}}(\cdot, u) \subseteq \cup_{j=1}^{\ell} \mathcal{F}_j(u)$.

Let $\mathcal{F}_{j^*}(u)$ denote any face of $\operatorname{gph} \mathcal{V}(\cdot, u)$ such that there exists $(x, z^*) \in \operatorname{gph} \mathcal{V}_{\mathrm{N}}(\cdot, u)$ with $(x, z^*) \in \mathcal{F}_{j^*}(u)$. We now show that $\mathcal{F}_{j^*}(u) \subseteq \operatorname{gph} \mathcal{V}_{\mathrm{N}}(\cdot, u)$. If $\mathcal{F}_{j^*}(u) = \{(x, z^*)\}$, the result is trivially true. Thus, henceforth, suppose there exists another point in the same face, $(\tilde{x}, \tilde{z}) \in \mathcal{F}_{j^*}(u)$, where we shall demonstrate $(\tilde{x}, \tilde{z}) \in \operatorname{gph} \mathcal{V}_{\mathrm{N}}(\cdot, u)$. Since $\operatorname{gph} \mathcal{V}(\cdot, u)$ is a polyhedral convex set, Lemma 3.2 implies that there exists a normal vector $\lambda \in \mathbb{R}^{q_1+p}$ which defines a supporting hyperplane of face $\mathcal{F}_{j^*}(u)$ at (x, z^*) [Boyd and Vandenberghe, 2004, p. 51]. By the definition of supporting hyperplane, for any point $(\hat{x}, \hat{z}) \in \operatorname{gph} \mathcal{V}(\cdot, u)$,

$$\lambda_x^{\intercal} x + \lambda_z^{\intercal} z^* \leq \lambda_x^{\intercal} \hat{x} + \lambda_z^{\intercal} \hat{z} \tag{5}$$

where $\lambda = (\lambda_x, \lambda_z)^{\intercal}$. Since $\operatorname{gph} \mathcal{V}(\cdot, u)$ is a polyhedron, the hyperplane defined by λ is the supporting hyperplane of the entire face $\mathcal{F}_{j^*}(u)$. Therefore, (5) also holds by replacing (x, z^*) on the left-hand side with $(\tilde{x}, \tilde{z}) \in \mathcal{F}_{j^*}(u)$. Further, since $(\hat{x}, \hat{z})$ is an arbitrary point in $\operatorname{gph} \mathcal{V}(\cdot, u)$ on the right-hand side, let $\hat{x} = \tilde{x}$, so that

$$\lambda_x^{\intercal} \tilde{x} + \lambda_z^{\intercal} \tilde{z} \leq \lambda_x^{\intercal} \tilde{x} + \lambda_z^{\intercal} \hat{z},$$

which implies $\lambda_z^{\intercal} \tilde{z} \leq \lambda_z^{\intercal} \hat{z}$. To conclude that $\tilde{z} \in \mathcal{V}_{\mathrm{N}}(\tilde{x}, u)$, it only remains to show that $\lambda_z \in \mathbb{R}^p_{>}$; this fact follows because $z^* \in \mathcal{V}_{\mathrm{N}}(x, u)$ (see Isermann [1974, Theorem 1] or Ehrgott [2005, Theorem 6.11, p. 159]). Since $\tilde{z} \in \mathcal{V}_{\mathrm{N}}(\tilde{x}, u)$, then $(\tilde{x}, \tilde{z}) \in \operatorname{gph} \mathcal{V}_{\mathrm{N}}(\cdot, u)$. Thus, $\operatorname{gph} \mathcal{V}_{\mathrm{N}}(\cdot, u)$ can be written as the finite union $\cup_{j^*} \mathcal{F}_{j^*}(u)$. □

Proposition 3.6. *Under Assumptions 1 and 2, the random closed-valued multifunction $\mathcal{V}_{\mathrm{N}}\colon \mathcal{X} \times \Xi \rightrightarrows \mathbb{R}^p$ is a set-valued normal integrand.*

Proof. By Lemma 3.5, the graph of $\mathcal{V}_{\mathrm{N}}(\cdot, u)$ is the union of finitely many (closed) polyhedral sets for each $u \in \Xi$. Therefore, it is closed. For the measurability of $\operatorname{gph} \mathcal{V}_{\mathrm{N}}$, first, recall from the proof of Proposition 3.4 that $\mathcal{V}\colon (\mathcal{X} \times \Xi) \rightrightarrows \mathbb{R}^p$ is a random convex closed set in the joint probability space $(\mathcal{X} \times \Xi, \mathfrak{B}(\mathcal{X}) \otimes \mathfrak{M}, P_{(X,\xi)})$. Molchanov [2017, Theorem 1.3.25, p. 69] implies $\mathcal{V} + \mathbb{R}^p_{\geqq}$ is a random closed set in the same probability space, which is also convex and upper. Then since $\mathcal{V}_{\mathrm{N}}(x, u) = (\mathcal{V}(x, u) + \mathbb{R}^p_{\geqq})_{\mathrm{N}}$ for all $(x, u) \in \mathcal{X} \times \Xi$ [Ehrgott, 2005, Proposition 2.3, p. 27], the proof of Proposition 1.8.27 in Molchanov [2017, p. 159] implies that $\operatorname{gph} \mathcal{V}_{\mathrm{N}}$ is $\mathfrak{B}(\mathcal{X}) \otimes \mathfrak{M} \otimes \mathfrak{B}(\mathbb{R}^p)$-measurable. □

4 Expected value multifunctions

Having established the properties of the random multifunctions in Section 3, we now consider the corresponding expected value multifunctions for the TSSMOLP formulation in (M). In the following Subsections 4.1 and 4.2, we provide properties of the expected value multifunctions from the second-stage MOLP, $\mathrm{E}[\mathcal{V}(\cdot,\xi)]\colon \mathcal{X} \rightrightarrows \mathbb{R}^p$ and $\mathrm{E}[\mathcal{V}_\mathrm{N}(\cdot,\xi)]\colon \mathcal{X} \rightrightarrows \mathbb{R}^p$, respectively. Specifically, we demonstrate that under mild regularity conditions, $\mathrm{E}[\mathcal{V}(\cdot,\xi)]$ is outer semicontinuous, graph-convex, and bounded on $\mathcal{X}$ (Proposition 4.3), and that $\mathrm{E}[\mathcal{V}_\mathrm{N}(\cdot,\xi)]$ is outer semicontinuous and bounded on $\mathcal{X}$ (Proposition 4.4).

While details of the selection expectation appear in Appendix A.2, we include here a useful and intuitive lemma regarding the selection expectations for two random sets defined in the same probability space where one random set is almost surely a subset of the other.

Lemma 4.1 (see Molchanov [2017, Theorem 2.1.31, p. 244])**.** *Let $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathbb{R}^p$ be two integrable random closed sets (Definition A.8) defined on the complete probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ such that $\mathcal{G}_1(\omega) \subseteq \mathcal{G}_2(\omega)$ for all $\omega \in \Omega_1$, where $\Omega_1 \subseteq \Omega$ and $\mathbb{P}(\Omega_1) = 1$. Then $\mathrm{E}[\mathcal{G}_1] \subseteq \mathrm{E}[\mathcal{G}_2]$.*

Proof sketch. Let G_1, G_2 denote integrable selections of $\mathcal{G}_1, \mathcal{G}_2$, respectively (Definition A.7), where $G_1(\omega) \in \mathcal{G}_1(\omega) \subseteq \mathcal{G}_2(\omega)$ for all $\omega \in \Omega_1$ implies that G_1 is an integrable selection of $\mathcal{G}_2$ for all $\omega \in \Omega_1$. Letting $\mathcal{L}^1(\mathcal{G}_1), \mathcal{L}^1(\mathcal{G}_2)$ denote the families of all integrable selections of $\mathcal{G}_1, \mathcal{G}_2$, respectively, and employing Definitions A.9 and A.10 and the fact that $\mathbb{P}(\Omega_1) = 1$,

$$\begin{aligned}\mathrm{E}[\mathcal{G}_1] = \mathrm{cl}\left\{\textstyle\int_{\Omega_1} G_1(\omega) d\,\mathbb{P}(\omega)\colon G_1 \in \mathcal{L}^1(\mathcal{G}_1)\right\} &\subseteq \mathrm{cl}\left\{\textstyle\int_{\Omega_1} G_2(\omega) d\,\mathbb{P}(\omega)\colon G_2 \in \mathcal{L}^1(\mathcal{G}_2)\right\} \\ &= \mathrm{E}[\mathcal{G}_2]. \qquad \square\end{aligned}$$

4.1 Properties of the expected second-stage image multifunction

First, we consider the expected second-stage image multifunction $\mathrm{E}[\mathcal{V}(\cdot,\xi)]\colon \mathcal{X} \rightrightarrows \mathbb{R}^p$. Since we shall invoke Fatou's Lemma for random compact sets in $\mathbb{R}^p$ [Molchanov, 2017, Theorem 2.1.60, p. 263], in Assumption 3, we restrict ourselves to the case in which the closed second-stage feasible sets $\mathcal{Y}(x,\xi)$ are also bounded w.p.1.

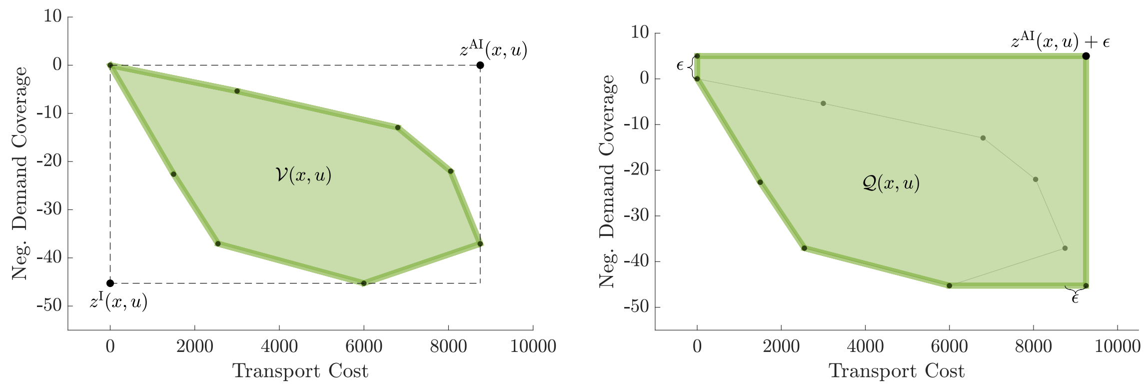
Assumption 3 (compact second-stage feasible set)**.** *For every $x \in \mathcal{X}$, the random closed set $\mathcal{Y}(x,\xi)$ is bounded w.p.1; that is, $\mathbb{P}\left(\mathrm{diam}\,\mathcal{Y}(x,\xi) < \infty\right) = 1$ for all $x \in \mathcal{X}$. Equivalently, in the probability space $(\Xi, \mathfrak{M}, \mathbb{P}_\xi)$, $\mathbb{P}_\xi\left(\cap_{x\in\mathcal{X}} \{u \in \Xi\colon \mathrm{diam}\,\mathcal{Y}(x,u) < \infty\}\right) = 1$.*

Assumptions 1 and 3, together with Lemma 3.2, imply that the second-stage image set $\mathcal{V}(x,\xi)$ is a nonempty random compact polyhedral convex set w.p.1 for all $x \in \mathcal{X}$. To ensure $\mathcal{V}(x,\xi)$ is also integrably bounded uniformly in $x \in \mathcal{X}$, in the following Assumption 4, we make assumptions on the integrability of the ideal and anti-ideal points, defined next.

Definition 4.2 (see Benson [1998, p. 6])**.** Let $(x,u) \in \mathcal{X} \times \Xi$ where $\mathcal{V}(x,u)$ is the second-stage image set. For each objective $k = 1, \ldots, p$, define the minimum and maximum as

$$z_k^\mathrm{I}(x,u) := \min\{z_k(u)\colon z(u) \in \mathcal{V}(x,u)\}, \quad z_k^\mathrm{AI}(x,u) := \max\{z_k(u)\colon z(u) \in \mathcal{V}(x,u)\},$$

where by convention, if $\mathcal{V}(x,u) = \emptyset$, then $z_k^\mathrm{I}(x,u) = +\infty$ and $z_k^\mathrm{AI}(x,u) = -\infty$. If $\mathcal{V}(x,u)$ is unbounded in the relevant direction, then $z_k^\mathrm{I}(x,u) = -\infty$ or $z_k^\mathrm{AI}(x,u) = +\infty$. The points $z^\mathrm{I}(x,u) \in \overline{\mathbb{R}}^p$ and $z^\mathrm{AI}(x,u) \in \overline{\mathbb{R}}^p$ are the *ideal* and *anti-ideal* points for $\mathcal{V}(x,u)$, respectively.



(a) Ideal and anti-ideal points.

(b) $\mathcal{Q}(x,u)$ is constructed from $\mathcal{V}(x,u)$.

Figure 2: For the motivating example in Subsection 1.1, the images show the ideal and anti-ideal points for a second-stage image set $\mathcal{V}(x,u)$, as defined in Definition 4.2, and the construction of the p-dimensional image set $\mathcal{Q}(x,u)$, as discussed in Subsection 5.2.

For the motivating example discussed in Subsection 1.1, Figure 2a depicts the construction of the ideal and anti-ideal points from the second-stage image set $\mathcal{V}(x_1,u_2)$; henceforth, we suppress the subscripts and refer to this set as $\mathcal{V}(x,u)$. As shown in Figure 2a, the ideal and anti-ideal points usually lie outside the image set. Thus, in general, they are not measurable selections of the image set. Instead, they define a bounding hyperrectangle which contains the image set. In Assumption 4, we assume the supremum norm of all such points is integrable.

Assumption 4 (integrability)**.** *The maximum norm of all second-stage ideal and anti-ideal points is integrable; i.e.,* $\mathrm{E}[Z^*]<\infty$ *where* $Z^*\colon\Omega\to\mathbb{R}$ *is the non-negative random variable*

$$Z^* := \sup_{x\in\mathcal{X}}\max\{\|z^{\mathrm{I}}(x,\xi)\|, \|z^{\mathrm{AI}}(x,\xi)\|\}.$$

We are now ready to demonstrate properties of the multifunction $\mathrm{E}[\mathcal{V}(\cdot,\xi)]$ in the following Proposition 4.3; specifically, that it is outer semicontinuous and graph-convex.

Proposition 4.3. *Under Assumptions 1, 3, and 4, the multifunction* $\mathrm{E}[\mathcal{V}(\cdot,\xi)]\colon\mathcal{X}\rightrightarrows\mathbb{R}^p$ *is outer semicontinuous, graph-convex, and bounded on* $\mathcal{X}$.

Proof. First, we demonstrate the outer semicontinuity of $\mathrm{E}[\mathcal{V}(\cdot,\xi)]$ using Fatou's Lemma for random compact sets in $\mathbb{R}^p$, which appears in Molchanov [2017, Theorem 2.1.60, p. 263]. By Lemma 3.2, for any given $u\in\Xi$ such that $\operatorname{dom}\mathcal{Y}(\cdot,u)=\{x\in\mathcal{X}\colon\mathcal{Y}(x,u)\neq\emptyset\}$ is nonempty, $\mathcal{V}(\cdot,u)$ is a Lipschitz continuous multifunction on $\operatorname{dom}\mathcal{Y}(\cdot,u)$. Therefore, it is also outer semicontinuous on $\operatorname{dom}\mathcal{Y}(\cdot,u)$, which implies that its graph is closed and for any $x_0\in\operatorname{dom}\mathcal{Y}(\cdot,u)$,

$$\begin{aligned}\limsup_{x\to x_0}\mathcal{V}(x,u) &:= \bigcup_{x_\nu\to x_0}\limsup_{\nu\to\infty}\mathcal{V}(x_\nu,u)\\ &= \{z\colon \exists x_\nu\to x_0, \exists z_\nu\to z \text{ with } z_\nu\in\mathcal{V}(x_\nu,u)\}=\mathcal{V}(x_0,u),\end{aligned}\tag{6}$$

where the sequence $\{x_\nu\}$ lies in $\operatorname{dom}\mathcal{Y}(\cdot,u)$ [Rockafellar and Wets, 1998, p. 152]. For the random vector $\xi\colon\Omega\to\Xi$, consider the sequence of random sets specified by $\{\mathcal{V}(x_\nu,\xi),\nu=1,2,\ldots\}$ where $\{x_\nu\}$ lies in $\mathcal{X}$ and each set $\mathcal{V}(x_\nu,\xi)$ is nonempty and compact w.p.1 following

Lemma 3.2 with Assumptions 1 and 3. The sequence $\{\|\mathcal{V}(x_\nu,\xi)\|, \nu = 1,2,\ldots\}$ is uniformly integrable because in the original probability space $(\Omega, \mathfrak{A}, \mathbb{P})$, Assumption 4 implies

$$\begin{aligned}\|\mathcal{V}(x_\nu,\xi(\omega))\| &= \sup\{\|z(x_\nu,\xi(\omega))\| \colon z(x_\nu,\xi(\omega)) \in \mathcal{V}(x_\nu,\xi(\omega))\}\\ &\le \sup\nolimits_{x\in\mathcal{X}} \max\{\|z^{\mathrm{I}}(x,\xi(\omega))\|, \|z^{\mathrm{AI}}(x,\xi(\omega))\|\} = Z^*(\omega) \end{aligned} \tag{7}$$

for all $\omega \in \Omega, \nu = 1,2,\ldots$ where $\mathrm{E}[Z^*] < \infty$. Further, Assumption 4 implies $\mathcal{V}(x,\xi)$ is integrably bounded for all $x \in \mathcal{X}$. Since the selection expectation equals the Aumann integral for any integrably bounded random compact set in $\mathbb{R}^p$ [Molchanov, 2017, p. 264], by Fatou's Lemma, we have

$$\begin{aligned}\limsup_{x\to x_0} \mathrm{E}[\mathcal{V}(x,\xi)] &= \limsup_{x\to x_0} \int_\Xi \mathcal{V}(x,u)d\,\mathbb{P}_\xi(u) \subseteq \int_\Xi \limsup_{x\to x_0} \mathcal{V}(x,u)d\,\mathbb{P}_\xi(u)\\ &= \int_\Xi \mathcal{V}(x_0,u)d\,\mathbb{P}_\xi(u) = \mathrm{E}[\mathcal{V}(x_0,\xi)] \end{aligned} \tag{8}$$

where (8) follows from (6) and the implicit sequence $\{x_\nu\}$ and its limit x_0 lie in $\mathcal{X}$. Therefore, $\mathrm{E}[\mathcal{V}(\cdot,\xi)]$ is outer semicontinuous relative to $\mathcal{X}$.

Now, we show that $\operatorname{gph}\mathrm{E}[\mathcal{V}(\cdot,\xi)] = \{(x,z)\colon z \in \mathrm{E}[\mathcal{V}(x,\xi)]\}$ is a convex set. From Rockafellar and Wets [1998, p. 155], this result holds if and only if for all $x_1, x_2 \in \mathcal{X}$,

$$(1-\beta)\,\mathrm{E}[\mathcal{V}(x_1,\xi)] + \beta\,\mathrm{E}[\mathcal{V}(x_2,\xi)] \subseteq \mathrm{E}[\mathcal{V}((1-\beta)x_1 + \beta x_2,\xi)] \;\text{ for }\; \beta \in (0,1). \tag{9}$$

Given $u \in \Xi$, the graph-convexity of $\mathcal{V}(\cdot,u)$ in Lemma 3.2 implies that for $x_1, x_2 \in \operatorname{dom}\mathcal{Y}(\cdot,u)$,

$$(1-\beta)\mathcal{V}(x_1,u) + \beta\mathcal{V}(x_2,u) \subseteq \mathcal{V}((1-\beta)x_1 + \beta x_2, u) \;\text{ for }\; \beta \in (0,1). \tag{10}$$

Under Assumption 1, for random vector $\xi \in \Xi$ and $x_1, x_2 \in \mathcal{X}$, (10) implies

$$(1-\beta)\mathcal{V}(x_1,\xi) + \beta\mathcal{V}(x_2,\xi) \subseteq \mathcal{V}((1-\beta)x_1 + \beta x_2, \xi) \;\text{ for }\; \beta \in (0,1) \;\text{ a.s.} \tag{11}$$

First, consider the expected value of the left side of (11), where henceforth, $\beta \in (0,1)$ and $x_1, x_2 \in \mathcal{X}$. By Molchanov [2017, Proposition 2.1.32, p. 244],

$$\mathrm{E}[(1-\beta)\mathcal{V}(x_1,\xi) + \beta\mathcal{V}(x_2,\xi)] = \operatorname{cl}\big((1-\beta)\,\mathrm{E}[\mathcal{V}(x_1,\xi)] + \beta\,\mathrm{E}[\mathcal{V}(x_2,\xi)]\big).$$

Under the selection expectation and since $\mathcal{V}(x,\xi)$ is almost surely convex for each $x \in \mathcal{X}$, $(1-\beta)\,\mathrm{E}[\mathcal{V}(x_1,\xi)]$ and $\beta\,\mathrm{E}[\mathcal{V}(x_2,\xi)]$ are closed and convex [Molchanov, 2017, p. 238f.]. Further, under Assumption 4, $(1-\beta)\,\mathrm{E}[\mathcal{V}(x_1,\xi)]$ and $\beta\,\mathrm{E}[\mathcal{V}(x_2,\xi)]$ are bounded [Molchanov, 2017, Proposition 2.1.39, p. 250]; therefore, both sets are compact convex sets. Since the Minkowski sum of compact convex sets is compact,

$$\mathrm{E}[(1-\beta)\mathcal{V}(x_1,\xi) + \beta\mathcal{V}(x_2,\xi)] = (1-\beta)\,\mathrm{E}[\mathcal{V}(x_1,\xi)] + \beta\,\mathrm{E}[\mathcal{V}(x_2,\xi)]. \tag{12}$$

Then, the desired result in (9) holds by applying Lemma 4.1 to (10) and (11) and combining this result with (12).

Finally, by definition, $\mathrm{E}[\mathcal{V}(\cdot,\xi)]$ is bounded if its range $\mathrm{E}[\mathcal{V}(\mathcal{X},\xi)] \coloneqq \cup_{x\in\mathcal{X}}\,\mathrm{E}[\mathcal{V}(x,\xi)]$ is a bounded subset of $\mathbb{R}^p$ [Rockafellar and Wets, 1998, Definition 5.14, p. 157f.]. To show

$\mathrm{E}[\mathcal{V}(\mathcal{X},\xi)]$ is bounded, we demonstrate that $\|\mathrm{E}[\mathcal{V}(\mathcal{X},\xi)]\| < \infty$. Under Assumption 4, using the set norm in (3) and applying inequalities analogous to (7), we have

$$\begin{aligned}\|\mathrm{E}[\mathcal{V}(\mathcal{X},\xi)]\| &= \sup\{\|z\|\colon z \in \cup_{x\in\mathcal{X}} \mathrm{E}[\mathcal{V}(x,\xi)]\}\\ &= \sup_{x\in\mathcal{X}} \big\{ \sup\{\|z(x)\|\colon z(x) \in \mathrm{E}[\mathcal{V}(x,\xi)]\}\big\} \leq \mathrm{E}[Z^*] < \infty.\end{aligned}$$ □

4.2 Properties of the expected second-stage nondominated multifunction

We conclude our study of the expected value of second-stage multifunctions relevant to (M) with the following Proposition 4.4 regarding the expected second-stage nondominated multifunction.

Proposition 4.4. *Under Assumptions 1–4, the multifunction* $\mathrm{E}[\mathcal{V}_{\mathrm{N}}(\cdot,\xi)]\colon \mathcal{X} \rightrightarrows \mathbb{R}^p$ *is outer semicontinuous and bounded on* $\mathcal{X}$.

Proof sketch. By Lemma 3.5, the graph of $\mathcal{V}_{\mathrm{N}}(\cdot,u)$ is the union of finitely many (closed) polyhedral sets for each $u \in \Xi$. Therefore, it is closed, which implies $\mathcal{V}_{\mathrm{N}}(\cdot,u)$ is outer semicontinuous on $\operatorname{dom}\mathcal{Y}(\cdot,u)$ [Rockafellar and Wets, 1998, Theorem 5.7, p. 154]. Now, consider the sequence of random sets specified by $\{\mathcal{V}_{\mathrm{N}}(x_\nu,\xi), \nu = 1,2,\ldots\}$ where $\{x_\nu\}$ lies in $\mathcal{X}$ and each $\mathcal{V}_{\mathrm{N}}(x_\nu,\xi)$ is nonempty and compact w.p.1 following Lemma 3.1 under Assumptions 1 and 3. By the same arguments as in the proof of Proposition 4.3, $\{\|\mathcal{V}_{\mathrm{N}}(x_\nu,\xi)\|, \nu = 1,2,\ldots\}$ is uniformly integrable under Assumption 4, outer semicontinuity holds by applying Fatou's Lemma, and boundedness follows under Assumption 4. □

5 Nondominance-equivalent reformulations

For a given multi-objective optimization problem, a nondominance-equivalent reformulation is a new optimization problem which has the same global Pareto set as the original. The TSSMOLP in (M) can be reformulated into two nondominance-equivalent optimization problems which do not rely on the second-stage random multifunction $\mathcal{V}_{\mathrm{N}}(\cdot,\cdot)$. The first nondominance-equivalent reformulation we present, called the *polyhedral reformulation*, replaces the random nondominated set of the second-stage MOLP, $\mathcal{V}_{\mathrm{N}}(x,\xi)$, with the random image set $\mathcal{V}(x,\xi)$ for all $x \in \mathcal{X}$. This reformulation facilitates some aspects of mathematical analysis, including the main result that the global Pareto set $\mathcal{Z}_{\mathrm{P}}$ is cone-convex in Section 6. The second nondominance-equivalent reformulation, called the *full-dimensional reformulation*, goes a step further by ensuring that the random image set of the second-stage MOLP is p-dimensional w.p.1. This reformulation enables complexity analysis for computing the expected value of the second-stage image set when the distribution is discrete and may facilitate the future development and analysis of outer approximation methods for TSSMOLPs (see, e.g., Benson [1998]).

5.1 Polyhedral reformulation

For the first reformulation, we demonstrate that the optimization problem

$$\begin{aligned}\text{minimize}\quad & \{\mathsf{C}x + \mathrm{E}[\mathcal{V}(x,\xi)]\}\\ \text{s.t.}\quad & x \in \mathcal{X} = \big\{x \in \mathbb{R}^{q_1}\colon \mathsf{A}x = b,\ x \geqq 0\big\}\end{aligned} \tag{P}$$

is a nondominance-equivalent problem for the original TSSMOLP in (M). Let the expected value multifunction corresponding to (P) be $\phi\colon \mathcal{X} \rightrightarrows \mathbb{R}^p$ where

$$\phi(x) \coloneqq \mathsf{C}x + \mathrm{E}[\mathcal{V}(x,\xi)] \text{ for all } x \in \mathcal{X}.$$

Before we prove Theorem 5.2, which states the nondominance-equivalence of (P) and (M), we require the following lemma.

Lemma 5.1. *Under Assumptions 1, 3, and 4,* $\mathrm{E}[\mathcal{V}_{\mathrm{N}}(x,\xi)] + \mathbb{R}^p_{\geqq} = \mathrm{E}[\mathcal{V}(x,\xi)] + \mathbb{R}^p_{\geqq}$ *for each* $x \in \mathcal{X}$.

Proof. Let $x \in \mathcal{X}$. By Lemma 3.2 and under Assumption 1, $\mathcal{V}(x,\xi)$ is a nonempty polyhedral convex set w.p.1. Therefore, using Ehrgott [2005, Proposition 2.4, p. 28], we have $\mathcal{V}_{\mathrm{N}}(x,\xi) \subseteq \mathrm{bd}\,\mathcal{V}(x,\xi) \subseteq \mathcal{V}(x,\xi)$, and thus $\mathcal{V}_{\mathrm{N}}(x,\xi) + \mathbb{R}^p_{\geqq} \subseteq \mathcal{V}(x,\xi) + \mathbb{R}^p_{\geqq}$, w.p.1. For the other direction, under Assumptions 1 and 3, $\mathcal{V}(x,\xi)$ is nonempty and compact w.p.1. Then $\mathcal{V}_{\mathrm{N}}(x,\xi)$ is externally stable and $\mathcal{V}(x,\xi) \subset \mathcal{V}_{\mathrm{N}}(x,\xi) + \mathbb{R}^p_{\geqq}$ w.p.1; see Ehrgott [2005, Theorem 2.21, p. 33]. Adding the positive orthant to both sides yields $\mathcal{V}(x,\xi) + \mathbb{R}^p_{\geqq} \subseteq \mathcal{V}_{\mathrm{N}}(x,\xi) + \mathbb{R}^p_{\geqq} + \mathbb{R}^p_{\geqq} = \mathcal{V}_{\mathrm{N}}(x,\xi) + \mathbb{R}^p_{\geqq}$. Then, under Assumption 4, apply Lemma 4.1 in both directions. ☐

Theorem 5.2. *Under Assumptions 1, 3, and 4,* $\phi(\mathcal{X})$ *is a nondominance-equivalent set for* $\mathcal{Z}(\mathcal{X})$*; that is,* $\phi_{\mathrm{N}}(\mathcal{X}) = \mathcal{Z}_{\mathrm{N}}(\mathcal{X}) \equiv \mathcal{Z}_{\mathrm{P}}$.

Proof. Using Lemma 5.1 and Ehrgott [2005, Proposition 2.3, p. 27], we have

$$\begin{aligned}\mathcal{Z}_{\mathrm{N}}(\mathcal{X}) &= \Big(\mathcal{Z}(\mathcal{X}) + \mathbb{R}^p_{\geqq}\Big)_{\mathrm{N}} = \Big(\cup_{x\in\mathcal{X}} \left(\mathsf{C}x + \mathrm{E}[\mathcal{V}_{\mathrm{N}}(x,\xi)] + \mathbb{R}^p_{\geqq}\right)\Big)_{\mathrm{N}} \\ &= \Big(\cup_{x\in\mathcal{X}} \left(\mathsf{C}x + \mathrm{E}[\mathcal{V}(x,\xi)] + \mathbb{R}^p_{\geqq}\right)\Big)_{\mathrm{N}} = \Big(\phi(\mathcal{X}) + \mathbb{R}^p_{\geqq}\Big)_{\mathrm{N}} = \phi_{\mathrm{N}}(\mathcal{X}).\end{aligned}$$ ☐

Now, using the properties of the expected second-stage image multifunction from Subsection 4.1, along with the assumption that the feasible set $\mathcal{X}$ is compact, the following Proposition 5.3 holds for the expected value multifunction ϕ. Working with ϕ may be preferable to working with $\mathcal{Z}$ because ϕ is graph-convex, regardless of whether the probability space is nonatomic.

Proposition 5.3. *Under Assumptions 1–4, the multifunction* $\phi\colon \mathcal{X} \rightrightarrows \mathbb{R}^p$ *is outer semicontinuous, graph-convex, and bounded on* $\mathcal{X}$.

Proof. To see that ϕ is outer semicontinuous, first, define the mapping $T\colon \mathcal{X} \rightrightarrows \mathbb{R}^p$ where $T(x) = \{\mathsf{C}x\}$ for all $x \in \mathcal{X}$. Then T is bounded on $\mathcal{X}$ because $\|T(\mathcal{X})\| = \|\mathsf{C}\mathcal{X}\| < \infty$ since $\mathcal{X}$ is compact under Assumption 2. Further, T is continuous on $\mathcal{X}$. Then since T is continuous and bounded and $\mathrm{E}[\mathcal{V}(\cdot,\xi)]$ is outer semicontinuous on $\mathcal{X}$ by Proposition 4.3, it follows that $T + \mathrm{E}[\mathcal{V}(\cdot,\xi)]$ is also outer semicontinuous on $\mathcal{X}$ [Rockafellar and Wets, 1998, Exercise 5.24, p. 162]. The proof for the graph convexity of ϕ follows along the lines of the proof in Proposition 4.3; specifically, for $x_1, x_2 \in \mathcal{X}$ and $\beta \in (0,1)$, we have

$$\begin{aligned}&(1-\beta)\left(\mathsf{C}x_1 + \mathrm{E}[\mathcal{V}(x_1,\xi)]\right) + \beta\left(\mathsf{C}x_2 + \mathrm{E}[\mathcal{V}(x_2,\xi)]\right) \\ &\quad = \mathsf{C}\left((1-\beta)x_1 + \beta x_2\right) + (1-\beta)\,\mathrm{E}[\mathcal{V}(x_1,\xi)] + \beta\,\mathrm{E}[\mathcal{V}(x_2,\xi)] \\ &\quad \subseteq \mathsf{C}\left((1-\beta)x_1 + \beta x_2\right) + \mathrm{E}[\mathcal{V}((1-\beta)x_1 + \beta x_2,\xi)] \qquad (13)\end{aligned}$$

where (13) follows from the graph convexity of $\mathrm{E}[\mathcal{V}(\cdot,\xi)]$ which was shown in Proposition 4.3 via (9). Finally, that ϕ is bounded holds because T and $\mathrm{E}[\mathcal{V}(\cdot,\xi)]$ are bounded on $\mathcal{X}$. □

5.2 Full-dimensional reformulation

Even though the TSSMOLP in (M) has p objectives in each stage, the dimension of the image set is such that $\dim(\mathcal{V}(x,\xi)) \leq \operatorname{rank}(\mathsf{D}(\xi)) \leq p$ w.p.1, which implies $\mathcal{V}(x,\xi)$ may not be full-dimension [Benson, 1998, Proposition 2.1, p. 5]. Therefore, to facilitate analysis and the future development of outer approximation methods, we construct a random set $\mathcal{Q}(x,\xi)$ which is a full-dimension polyhedron and a nondominance-equivalent set for $\mathcal{V}(x,\xi)$ w.p.1 for every $x \in \mathcal{X}$; our construction closely follows that of Benson [1998].

To begin, given $\epsilon \in \mathbb{R}^p_>$, for every $(x,u) \in \mathcal{X} \times \Xi$, use the anti-ideal point in Definition 4.2 to define the polyhedron

$$\begin{aligned}\mathcal{Q}(x,u) &:= \big(\mathcal{V}(x,u) + \mathbb{R}^p_{\geqq}\big) \cap \big(z^{\mathrm{AI}}(x,u) + \epsilon - \mathbb{R}^p_{\geqq}\big) \\ &= \{z \in \mathbb{R}^p \colon \mathsf{D}(u)y \leqq z \leqq z^{\mathrm{AI}}(x,u) + \epsilon \text{ for some } y \in \mathcal{Y}(x,u)\},\end{aligned} \tag{14}$$

whose construction is shown in Figure 2b for a second-stage image set from the motivating example in Subsection 1.1. Under the definition of $\mathcal{Q}(x,u)$ in (14), Lemma 5.4 states the almost sure nondominance-equivalence of the random sets $\mathcal{Q}(x,\xi)$ and $\mathcal{V}(x,\xi)$.

Lemma 5.4. *Under Assumptions 1 and 3, the set $\mathcal{Q}(x,\xi)$ is almost surely a nondominance-equivalent polyhedron for $\mathcal{V}(x,\xi)$; that is, $\mathcal{Q}_{\mathrm{N}}(x,\xi) = \mathcal{V}_{\mathrm{N}}(x,\xi)$ a.s. for all $x \in \mathcal{X}$.*

Proof. Let $(x,u) \in \mathcal{X} \times \Xi$ be such that $\mathcal{Y}(x,u) \neq \emptyset$ and $\operatorname{diam}\mathcal{Y}(x,u) < \infty$, which implies $\mathcal{Y}(x,u)$ is compact, $\mathcal{Q}(x,u)$ is a nonempty, compact polyhedron, and $\mathcal{Q}_{\mathrm{N}}(x,u)$ is nonempty, connected, and compact by Lemma 3.1. That $\mathcal{Q}_{\mathrm{N}}(x,u) = \mathcal{V}_{\mathrm{N}}(x,u)$ follows directly from Benson [1998, Theorem 2.1, p. 6]. Then, the set of outcomes for which $\mathcal{Q}_{\mathrm{N}}(x,u) \neq \mathcal{V}_{\mathrm{N}}(x,u)$ occurs on a set of measure zero for all $x \in \mathcal{X}$ under Assumptions 1 and 3. □

The random set $\mathcal{Q}(x,\xi)$ inherits several properties of $\mathcal{V}(x,\xi)$; specifically, that it is nonempty and compact w.p.1 under Assumptions 1 and 3 and integrably bounded uniformly in $x \in \mathcal{X}$ under Assumption 4, which is formalized in the following Lemma 5.5.

Lemma 5.5. *Under Assumptions 1, 3, and 4, the following hold:*

1. *For each $x \in \mathcal{X}$, $\mathcal{Q}(x,\xi)$ is a nonempty random compact polyhedral convex set in $\mathbb{R}^p$ of dimension p w.p.1.*
2. *$\mathcal{Q}(x,\xi)$ is integrably bounded uniformly in $x \in \mathcal{X}$.*
3. *For each $x \in \mathcal{X}$, $\mathrm{E}[\mathcal{Q}_{\mathrm{N}}(x,\xi)] + \mathbb{R}^p_{\geqq} = \mathrm{E}[\mathcal{Q}(x,\xi)] + \mathbb{R}^p_{\geqq}$.*

Proof sketch. Let $x \in \mathcal{X}$ and fix a value of $\epsilon \in \mathbb{R}^p_>$ in (14). For Part 1, Lemma 3.2 and Assumptions 1 and 3 imply $\mathcal{V}(x,\xi(\omega))$ is a nonempty, compact polyhedron and all components of $z^{\mathrm{AI}}_k(x,\xi(\omega))$ are finite for a.e. $\omega \in \Omega$. Therefore, that $\mathcal{Q}(x,\xi(\omega))$ is a nonempty, compact polyhedral convex set in $\mathbb{R}^p$ of dimension p for a.e. $\omega \in \Omega$ holds by applying Benson [1998, Proposition 2.2, p. 6]; measurability of the corresponding multifunction $\mathcal{Q}\colon \mathcal{X} \times \Xi \rightrightarrows \mathbb{R}^p$ follows by Molchanov [2017, p. 69]. Since ϵ is a finite constant, Part 2 follows from Assumption 4. Finally, Part 3 follows by applying the same logic as in the proof of Lemma 5.1. □

Under Lemma 5.5, $\mathcal{Q}(x,\xi)$ is an integrable random compact set for each $x \in \mathcal{X}$. Thus, we can pose the optimization problem

$$\begin{aligned} \text{minimize} \quad & \{\mathsf{C}x + \mathrm{E}[\mathcal{Q}(x,\xi)]\} \\ \text{s.t.} \quad & x \in \mathcal{X} = \big\{x \in \mathbb{R}^{q_1} \colon \mathsf{A}x = b,\ x \geqq 0\big\}. \end{aligned} \tag{F}$$

In the following Theorem 5.6, we show that the optimization problem in (F) is a nondominance-equivalent reformulation for the original TSSMOLP in (M). Let the expected value multifunction corresponding to (F) be $\psi\colon \mathcal{X} \rightrightarrows \mathbb{R}^p$ where

$$\psi(x) \coloneqq \mathsf{C}x + \mathrm{E}[\mathcal{Q}(x,\xi)] \text{ for each } x \in \mathcal{X}.$$

Theorem 5.6. *Under Assumptions 1, 3, and 4, the following hold:*
1. *$\psi(\mathcal{X})$ is a nondominance-equivalent set for $\mathcal{Z}(\mathcal{X})$; that is, $\psi_{\mathrm{N}}(\mathcal{X}) = \mathcal{Z}_{\mathrm{N}}(\mathcal{X}) \equiv \mathcal{Z}_{\mathrm{P}}$.*
2. *The set $\psi(\mathcal{X}) \subset \mathbb{R}^p$ is full-dimensional; that is, it contains a p-dimensional ball.*

Proof. First, apply Lemma 4.1 to Lemma 5.4 and use the result with Lemmas 5.1 and 5.5 to yield that for each $x \in \mathcal{X}$,

$$\mathrm{E}[\mathcal{Q}(x,\xi)] + \mathbb{R}^p_{\geqq} = \mathrm{E}[\mathcal{Q}_{\mathrm{N}}(x,\xi)] + \mathbb{R}^p_{\geqq} = \mathrm{E}[\mathcal{V}_{\mathrm{N}}(x,\xi)] + \mathbb{R}^p_{\geqq} = \mathrm{E}[\mathcal{V}(x,\xi)] + \mathbb{R}^p_{\geqq}. \tag{15}$$

Then, Part 1 of the theorem follows by the same logic as the proof of Theorem 5.2.

For Part 2, since $\psi(\mathcal{X}) = \cup_{x\in\mathcal{X}} \mathsf{C}x + \mathrm{E}[\mathcal{Q}(x,\xi)]$, it is sufficient to show for a given $x \in \mathcal{X}$, $\mathrm{E}[\mathcal{Q}(x,\xi)]$ is full-dimensional. Let $x \in \mathcal{X}$, and let $\mathcal{B}_R(Y)$ denote a p-dimensional random ball of nonnegative random radius $R \in \mathbb{R}_{\geqq}$ centered at a random point $Y \in \mathbb{R}^p$. By Lemma 5.5 and recalling that $\epsilon > 0$ in (14), $\mathcal{Q}(x,\xi)$ is a nonempty integrable random compact polyhedral convex set of dimension p w.p.1. Further, $\operatorname{bd}\mathcal{Q}(x,\xi)$ is an integrable random closed set [Molchanov, 2017, Theorem 1.3.25, p. 69], $\operatorname{int}\mathcal{Q}(x,\xi)$ is a random open set [Molchanov, 2017, Proposition 1.3.36, p. 76], and $\operatorname{int}\mathcal{Q}(x,\xi) \neq \emptyset$ w.p.1. Since $\operatorname{bd}\mathcal{Q}(x,\xi(\omega)) \subset \mathcal{Q}(x,\xi(\omega))$ and $\operatorname{bd}\mathcal{Q}(x,\xi(\omega)) \neq \mathcal{Q}(x,\xi(\omega))$ for all $\omega \in \operatorname{dom}\mathcal{Q}$, we have that the collection of all integrable selections $\mathcal{L}^1\big(\operatorname{bd}\mathcal{Q}(x,\xi)\big) \subset \mathcal{L}^1\big(\mathcal{Q}(x,\xi)\big)$. Therefore, there exists an integrable selection of $\mathcal{Q}(x,\xi(\omega))$ which is not an integrable selection of $\operatorname{bd}\mathcal{Q}(x,\xi(\omega))$ for all $\omega \in \operatorname{dom}\mathcal{Q}$. Let this selection be $Y(\omega) \in \operatorname{int}\mathcal{Q}(x,\xi(\omega))$ for all $\omega \in \operatorname{dom}\mathcal{Q}$. Then there exists an integrable random ball centered at Y having random radius R such that $\mathcal{B}_{R(\omega)}(Y(\omega)) \subseteq \mathcal{Q}(x,\xi(\omega))$ with $R(\omega) > 0$ for every $\omega \in \operatorname{dom}\mathcal{Q}$, whence $\mathrm{E}[\mathcal{B}_R(Y)] = \mathcal{B}_{\mathrm{E}[R]}(\mathrm{E}[Y]) \subseteq \mathrm{E}[\mathcal{Q}(x,\xi)]$ under Lemma 4.1; also see Molchanov [2017, Example 2.1.41, p. 251]. Since $R > 0$ w.p.1, $\mathrm{E}[R] > 0$, and the result follows. □

5.2.1 Properties of the multifunctions arising from the full-dimensional reformulation

The relevant multifunctions in the reformulation (F) and the expected value multifunction ψ inherit nice properties from their constructions. We enumerate these properties below, which follow from results already proven.

To begin, given a scenario $u \in \Xi$, the analogue of Lemma 3.2 Part 2 holds for the multifunction $\mathcal{Q}(\cdot,u)\colon \mathcal{X} \rightrightarrows \mathbb{R}^p$.

Lemma 5.7. *Let $u \in \Xi$ be such that $\operatorname{dom}\mathcal{Y}(\cdot,u) \neq \emptyset$. Then the multifunction $Q(\cdot,u)\colon \mathcal{X} \rightrightarrows \mathbb{R}^p$ is a polyhedral, graph-convex map and thus, Lipschitz continuous on $\operatorname{dom}\mathcal{Y}(\cdot,u)$.*

Proof. This result follows because gph $\mathcal{Q}(\cdot, u) := \{(x, z) \colon z \in \mathcal{Q}(x, u)\} \subset \mathbb{R}^{q_1} \times \mathbb{R}^p$ is the intersection of $\mathcal{X} \times \mathbb{R}^p$ with the polyhedral sets $\left(\mathcal{V}(x, u) + \mathbb{R}^p_{\geqq}\right)$ and $\left(z^{\mathrm{AI}}(x, u) + \epsilon - \mathbb{R}^p_{\geqq}\right)$. Therefore, gph $\mathcal{Q}(\cdot, u)$ is an intersection of a finite number of convex polyhedral sets, implying it is both convex and polyhedral. □

Employing Lemma 5.7, the analyses for the multifunctions $\mathcal{Q}(\cdot,\cdot)$, $\mathrm{E}[\mathcal{Q}(\cdot, \xi)]$, and $\psi(\cdot)$ follow along identical lines to the previous analyses for $\mathcal{V}(\cdot,\cdot)$, $\mathrm{E}[\mathcal{V}(\cdot, \xi)]$, and $\phi(\cdot)$, in Proposition 3.4, Proposition 4.3, and Proposition 5.3, respectively. Therefore, we state the results in the following corollary.

Corollary 5.8. *The following hold:*

1. *Under Assumptions 1 and 2, the random closed-valued multifunction* $Q \colon \mathcal{X} \times \Xi \rightrightarrows \mathbb{R}^p$ *is a set-valued convex normal integrand.*
2. *Under Assumptions 1, 3, and 4, the multifunction* $\mathrm{E}[\mathcal{Q}(\cdot, \xi)] \colon \mathcal{X} \rightrightarrows \mathbb{R}^p$ *is outer semicontinuous, graph-convex, and bounded on* $\mathcal{X}$*.*
3. *Under Assumptions 1–4, the multifunction* $\psi \colon \mathcal{X} \rightrightarrows \mathbb{R}^p$ *is outer semicontinuous, graph-convex, and bounded on* $\mathcal{X}$*.*

5.2.2 Complexity under atomic probability measures with finite support

As discussed in Remark 2.2, a decision-maker may be interested in

$$\mathcal{Z}_{\mathrm{N}}(x^*) = (\mathsf{C}x^* + \mathrm{E}[\mathcal{V}_{\mathrm{N}}(x^*, \xi)])_{\mathrm{N}} \subseteq \mathcal{Z}(\mathcal{X}_{\mathrm{E}})$$

for one or more identified values $x^* \in \mathcal{X}_{\mathrm{E}}$ to assess the trade-off rates in the second-stage expected nondominated set. Equation (15) and Ehrgott [2005, Proposition 2.3, p. 27] imply that given $x^* \in \mathcal{X}$,

$$\begin{aligned}\mathcal{Z}_{\mathrm{N}}(x^*) = (\mathsf{C}x^* + \mathrm{E}[\mathcal{V}_{\mathrm{N}}(x^*, \xi)])_{\mathrm{N}} &= (\mathsf{C}x^* + \mathrm{E}[\mathcal{V}(x^*, \xi)])_{\mathrm{N}} = \phi_{\mathrm{N}}(x^*) \\ &= (\mathsf{C}x^* + \mathrm{E}[\mathcal{Q}(x^*, \xi)])_{\mathrm{N}} = \psi_{\mathrm{N}}(x^*).\end{aligned}$$

Given this relationship and assuming we are able to calculate $\mathcal{Z}_{\mathrm{N}}(x^*)$, we can choose the least computationally complex formulation.

Under the full-dimensional reformulation, and assuming the random variable ξ takes only a finite set of values described by the scenarios $\{u_i, i = 1, \ldots, n\}$ occurring with respective known probabilities $\alpha_i > 0, i = 1, \ldots, n$, $\sum_{i=1}^n \alpha_i = 1$, then computing $\mathrm{E}[Q(x, \xi)]$ for given $x \in \mathcal{X}$ corresponds to calculating the weighted Minkowski sum of $n < \infty$ bounded p-dimensional polyhedral convex sets according to (16) as $\mathrm{E}[\mathcal{Q}(x, \xi)] = \sum_{i=1}^n \alpha_i \mathcal{Q}(x, u_i)$; consistent with the relevant literature, in this section, we refer to bounded p-dimensional polyhedral convex sets as convex polytopes. The finite Minkowski sum of convex polytopes is itself a convex polytope [Sanyal, 2009]. Therefore, we obtain an upper bound on the complexity of storing the expected value polytope $\mathrm{E}[Q(x, \xi)]$ as a list of its vertices as follows. Let $\{\mathcal{Q}_i, i = 1, \ldots, n\}$ be a set of p-dimensional convex polytopes, $p \geq 2$, each containing at least $v_i \geq p + 1$ vertices for $i \in \{1, \ldots, n\}$. If $n \leq p - 1$, Fukuda and Weibel [2007] show that the number of vertices of $\sum_{i=1}^n \mathcal{Q}_i$ is bounded above by the product of the number of vertices of the individual polytopes, $\prod_{i=1}^n v_i$. If $n \geq p$, Sanyal [2009] shows that a smaller upper bound holds, specifically, $(1 - (p+1)^{-p}) \prod_{i=1}^n v_i$. The complexity is scale independent, so that computing $\sum_{i=1}^n \mathcal{Q}_i$ has the same complexity as computing $\sum_{i=1}^n \alpha_i \mathcal{Q}_i$ for weights

$\alpha_i > 0, \sum_{i=1}^{n} \alpha_i = 1$ [Karavelas and Tzanaki, 2016]. Therefore, given $x \in \mathcal{X}$, these upper bounds apply to the storage complexity for computing $\mathrm{E}[\mathcal{Q}(x, \xi)]$.

Controlling the computational complexity is likely to be crucial for future TSSMOLP applications when the number of scenarios is large. We suspect that in some cases, computing $\mathrm{E}[\mathcal{Q}(x, \xi)]$ results in fewer vertices than computing $\mathrm{E}[\mathcal{V}(x, \xi)]$ because the number of dominated vertices may be reduced. This concept is illustrated in Figure 2, where $\mathcal{V}(x, u)$ in Figure 2a has 8 vertices, while $\mathcal{Q}(x, u)$ in Figure 2b has 7 vertices. Other approaches, such as constructing representations of the second-stage image or nondominated sets, may also provide further reduction of the computational complexity. See, e.g., Sayin [2000] for further reading on representations of nondominated sets.

6 Cone-convexity of the global Pareto set

Given the properties of TSSMOLPs in Sections 3–5, we are now ready to present a main result of the paper: that the global Pareto set for the TSSMOLP in (M) is cone-convex. More specifically, we show that the upper image of the global Pareto set, $\mathcal{Z}_{\mathrm{P}} + \mathbb{R}^p_{\geqq}$, is nonempty, closed, and convex. This structural property is especially useful for the future development of solution methods for TSSMOLPs.

Theorem 6.1. *Under Assumptions 1–4, the global Pareto set $\mathcal{Z}_{\mathrm{P}}$ is nonempty and bounded, and $\mathcal{Z}_{\mathrm{P}} + \mathbb{R}^p_{\geqq}$ is a closed, convex set.*

Proof. First, we show that $\mathcal{Z}_{\mathrm{P}}$ is nonempty and bounded. By Theorem 5.2, $\mathcal{Z}_{\mathrm{P}} = \phi_{\mathrm{N}}(\mathcal{X})$, where ϕ is outer semicontinuous, graph-convex, and bounded on $\mathcal{X}$ by Proposition 5.3. The outer semicontinuity of ϕ and the fact that $\mathcal{X}$ is compact under Assumption 2 imply that $\phi(\mathcal{X})$ closed [Molchanov, 2017, Lemma E.3, p. 580]. Since $\phi(\mathcal{X})$ is closed and bounded, $\phi(\mathcal{X})$ is compact; under our assumptions, $\phi(\mathcal{X})$ is also nonempty. Therefore, $\mathcal{Z}_{\mathrm{P}} = \phi_{\mathrm{N}}(\mathcal{X})$ is externally stable [Ehrgott, 2005, Theorem 2.21, p. 33], which implies $\phi(\mathcal{X}) \subset \mathcal{Z}_{\mathrm{P}} + \mathbb{R}^p_{\geqq}$ and, thus, $\mathcal{Z}_{\mathrm{P}} \neq \emptyset$. Since $\phi(\mathcal{X})$ is closed, $\mathcal{Z}_{\mathrm{P}} \subseteq \mathrm{bd}\, \phi(\mathcal{X}) \subseteq \phi(\mathcal{X})$ [Ehrgott, 2005, Proposition 2.4, p. 28], which implies $\mathcal{Z}_{\mathrm{P}}$ is bounded.

To see that $\mathcal{Z}_{\mathrm{P}} + \mathbb{R}^p_{\geqq}$ is a closed, convex set, first, note that $\mathcal{Z}_{\mathrm{P}} + \mathbb{R}^p_{\geqq} = \phi(\mathcal{X}) + \mathbb{R}^p_{\geqq}$. The graph convexity of ϕ implies $\phi(\mathcal{X})$ is convex [Rockafellar and Wets, 1998, p. 155]. Therefore, $\phi(\mathcal{X}) + \mathbb{R}^p_{\geqq}$ is convex. Since $\phi(\mathcal{X})$ is compact and $\mathbb{R}^p_{\geqq}$ is closed, $\phi(\mathcal{X}) + \mathbb{R}^p_{\geqq}$ is closed. □

7 Concluding remarks

We consider a formulation for two-stage stochastic programs which enables the modeler to pose multiple simultaneous linear objectives in each stage with uncertainty defined on a general probability space. We study the properties of the resulting TSSMOLP and the multifunctions that arise therein, provide two nondominance-equivalent reformulations, and use one of the reformulations to show that the TSSMOLP has a cone-convex global Pareto set. Directly solving a TSSMOLP, rather than a single-objective reformulation, has the potential to provide valuable new insights and perspective to decision-makers facing complex real-world problems which are subject to conflict and uncertainty.

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A Preliminary concepts and definitions

In Appendix A.1 we include concepts and definitions relevant to multifunctions, and in Appendix A.2 we include concepts and definitions relevant to the selection expectation. Throughout, let $(\Omega, \mathfrak{A}, \mathbb{P})$ denote a complete probability space (see Subsection 1.4).

A.1 Multifunctions

To begin, we define multifunctions, which are set-valued maps.

Definition A.1 (Shapiro et al. [2009, p. 365])**.** A *multifunction* $\mathcal{G}$ is a mapping from a set Ω into the set of subsets of $\mathbb{R}^p$, written $\mathcal{G}\colon \Omega \rightrightarrows \mathbb{R}^p$, such that $\mathcal{G}$ assigns to each $\omega \in \Omega$ a (possibly empty) subset of $\mathbb{R}^p$. The multifunction $\mathcal{G}$ is *closed-valued* if $\mathcal{G}(\omega)$ is a closed subset of $\mathbb{R}^p$ for every $\omega \in \Omega$.

Importantly, the *graph* of the multifunction $\mathcal{G}$ is defined as

$$\operatorname{gph} \mathcal{G} \coloneqq \{(\omega, y)\colon y \in \mathcal{G}(\omega)\} \subseteq \Omega \times \mathbb{R}^p;$$

see Figure 5-1 in Rockafellar and Wets [1998, p. 149].

Next, in Definition A.2, we define a random closed set which takes on values in $\mathbb{R}^p$. Given a complete probability space, several measurability concepts are equivalent (see, e.g., Molchanov [2017, Theorem 1.3.3, p. 59], Rockafellar and Wets [1998, Theorem 14.3, p. 644]); therefore, we adopt a definition which is convenient for our purposes.

Definition A.2 (Shapiro et al. [2009, p. 365])**.** A closed-valued multifunction $\mathcal{G}\colon \Omega \rightrightarrows \mathbb{R}^p$ is a *measurable (closed-valued) multifunction* or *random closed set* if for every closed set $\mathcal{A} \subset \mathbb{R}^p$, the set $\mathcal{G}^{-1}(\mathcal{A}) \coloneqq \{\omega \in \Omega\colon \mathcal{G}(\omega) \cap \mathcal{A} \neq \emptyset\}$ is a measurable set in $\mathfrak{A}$.

Because the probability space is complete and the sets we consider take values in $\mathbb{R}^p$, the measurability of a multifunction can be deduced from its graph. Specifically, in a complete probability space, a multifunction is measurable if and only if its graph is measurable; i.e., if $\operatorname{gph} \mathcal{G} \in \mathfrak{A} \otimes \mathfrak{B}(\mathbb{R}^p)$ where $\mathfrak{A} \otimes \mathfrak{B}(\mathbb{R}^p)$ denotes the product σ-algebra of $\mathfrak{A}$ and $\mathfrak{B}(\mathbb{R}^p)$; see Molchanov [2017, Theorem 1.3.3, p. 59], Rockafellar and Wets [1998, Theorem 14.8, p. 648].

Next, we define a random (closed-valued) multifunction, which may also be considered a *set-valued stochastic process* indexed by the continuous parameter $x \in \mathbb{R}^q$ (Molchanov [2017, p. 462]; see also Kisielewicz [2013, 2020]).

Definition A.3 (see Molchanov [2017, p. 462])**.** A mapping $\mathcal{H}\colon \mathbb{R}^q \times \Omega \rightrightarrows \mathbb{R}^p$ is a *random (closed-valued) multifunction* or *set-valued integrand* if for every fixed $x \in \mathbb{R}^q$, the multifunction $\mathcal{H}(x, \cdot)$ is a random closed set; that is, it is $\mathfrak{A}$-measurable according to Definition A.2.

In analogy to random lower semicontinous functions, also called normal integrands [Shapiro et al., 2009, p. 366], we consider random outer semicontinuous multifunctions, also called *set-valued normal integrands*.

Definition A.4 (see Mordukhovich and Pérez-Aros [2021, p. 3217])**.** A mapping $\mathcal{H}\colon \mathbb{R}^q \times \Omega \rightrightarrows \mathbb{R}^p$ is a *set-valued normal integrand* on a complete probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ if (a) for all $\omega \in \Omega$ the multifunction $\mathcal{H}(\cdot, \omega)$ has a closed graph, and (b) the graph of $\mathcal{H}$ belongs to $\mathfrak{B}(\mathbb{R}^q) \otimes \mathfrak{A} \otimes \mathfrak{B}(\mathbb{R}^p)$. If, in addition, the set $\operatorname{gph} \mathcal{H}(\cdot, \omega)$ is convex for a.e. $\omega \in \Omega$, then $\mathcal{H}$ is a *set-valued convex normal integrand*.

We remark that Mordukhovich and Pérez-Aros [2021, p. 3217] define set-valued normal integrands in the context of a mapping $\Phi\colon \Omega \times \mathbb{R}^q \rightrightarrows \mathbb{R}^m$ and require that $\operatorname{gph} \Phi$ belongs to $\mathfrak{A} \otimes \mathfrak{B}(\mathbb{R}^q \times \mathbb{R}^p)$. Since we consider a mapping with arguments in the opposite order, $\mathcal{H}\colon \mathbb{R}^q \times \Omega \rightrightarrows \mathbb{R}^m$, and since $\mathfrak{B}(\mathbb{R}^q) \otimes \mathfrak{B}(\mathbb{R}^p) = \mathfrak{B}(\mathbb{R}^q \times \mathbb{R}^p)$ [Jones, 2001, p. 256], the condition of graph measurability in Definition A.4 is equivalent to that in Mordukhovich and Pérez-Aros [2021]. Notice that condition (a) in Definition A.4, that the graph of $\mathcal{H}(\cdot, \omega)$ is closed, holds if and only if the multifunction $\mathcal{H}(\cdot, \omega)\colon \mathbb{R}^q \rightrightarrows \mathbb{R}^p$ is outer semicontinuous [Rockafellar and Wets, 1998, Theorem 5.7, p. 154], where we refer the reader to Rockafellar and Wets [1998, p. 152ff.] for a definition and discussion of outer semicontinuous, inner semicontinuous, and continuous multifunctions. We conclude with a definition of Lipschitz continuity in Definition A.5.

Definition A.5 (see Rockafellar and Wets [1998, p. 368f.])**.** A mapping $\mathcal{H}(\cdot, \omega)\colon \mathbb{R}^q \rightrightarrows \mathbb{R}^p$ is Lipschitz continuous on $\mathcal{X} \subset \mathbb{R}^q$ if it is nonempty, closed-valued on $\mathcal{X}$ and there exists a constant $\kappa \in \mathbb{R}_>$ such that $\mathcal{H}(x', \omega) \subseteq \mathcal{H}(x, \omega) + \kappa \|x' - x\| \mathcal{B}_1(0_p)$ for all $x, x' \in \mathcal{X}$.

A.2 The selection expectation

Loosely speaking, the selection expectation of a random closed set is the closure of its Aumann integral, and the Aumann integral is the set of expected values of all the measurable selections which are also integrable. Thus, in what follows, we discuss measurable selections, integrable selections, the Aumann integral, and finally, the selection expectation.

First, in Definition A.6, we define the concept of *measurable selections* of random closed sets; that is, real-valued random vectors that "fit inside" the random closed set; see Figure 1.3.1 in Molchanov [2017, p. 58].

Definition A.6 (Shapiro et al. [2009, p. 365])**.** Let $\mathcal{G}\colon \Omega \rightrightarrows \mathbb{R}^p$ be a random closed set (Definition A.2), and let its domain be $\operatorname{dom} \mathcal{G} := \{\omega \in \Omega\colon \mathcal{G}(\omega) \neq \emptyset\}$. A mapping $G\colon \operatorname{dom} \mathcal{G} \to \mathbb{R}^p$ is called a *measurable selection* of $\mathcal{G}$ if $G(\omega) \in \mathcal{G}(\omega)$ for all $\omega \in \operatorname{dom} \mathcal{G}$ and G is measurable. Let the family of all measurable selections of $\mathcal{G}$ be $\mathcal{L}^0(\mathcal{G})$.

As a result of Definition A.6, if the random closed set $\mathcal{G}$ is nonempty w.p.1 and G is a measurable selection of $\mathcal{G}$, then $G \in \mathcal{G}$ w.p.1. Since our interest is in the expected value of a random closed set, we are also concerned with whether the measurable selections are integrable, and whether the random closed set is integrably bounded.

Definition A.7 (Shapiro et al. [2009, p. 367], also Molchanov [2017, p. 226])**.** Let $G \in \mathcal{L}^0(\mathcal{G})$ be a measurable selection of the random closed set $\mathcal{G}$. If $\int_\Omega \|G(\omega)\| d\,\mathbb{P}(\omega) < \infty$, then G is an *integrable selection* of $\mathcal{G}$. Let the family of all integrable selections of $\mathcal{G}$ be $\mathcal{L}^1(\mathcal{G})$.

Definition A.8 (Molchanov [2017, p. 227])**.** A random closed set $\mathcal{G}$ is *integrable* if the family of all integrable selections is nonempty, $\mathcal{L}^1(\mathcal{G}) \neq \emptyset$. It is *integrably bounded* if the expected value of the random variable $\|\mathcal{G}\|$ is finite, where $\|\mathcal{G}\| := \sup\{\|z\|\colon z \in \mathcal{G}\}$.

Note that if $\mathcal{G}$ is an integrably bounded random closed set, then $\mathcal{L}^0(\mathcal{G}) = \mathcal{L}^1(\mathcal{G})$ [Molchanov, 2017, p. 227]. With integrable random closed sets defined, we are now ready to define the Aumann integral, originally due to Aumann [1965].

Definition A.9 (Shapiro et al. [2009, p. 367], Molchanov [2017, p. 238])**.** Let $\mathcal{G}$ be a random closed set. If $\Omega = \{\omega_1, \dots, \omega_n\}$ is finite with respective probabilities $\alpha_1, \dots, \alpha_n$, then

$$\int_\Omega \mathcal{G}(\omega) d\,\mathbb{P}(\omega) \coloneqq \sum_{i=1}^{n} \alpha_i \mathcal{G}(\omega_i). \tag{16}$$

For a general measure $\mathbb{P}$ on $(\Omega, \mathfrak{A})$, the integral of $\mathcal{G}$ is the *Aumann integral*,

$$\int_\Omega \mathcal{G}(\omega) d\,\mathbb{P}(\omega) \coloneqq \left\{ \int_\Omega G(\omega) d\,\mathbb{P}(\omega) \colon G \in \mathcal{L}^1(\mathcal{G}) \right\} = \{\mathrm{E}[G] \colon G \in \mathcal{L}^1(\mathcal{G})\}.$$

The selection expectation, which is the closure of the Aumann integral, provides an intuitive way to think about expectations of random sets in terms of the vector valued random variables that form the set of all integrable selections.

Definition A.10 (Molchanov [2017, p. 238, p. 250], Shapiro et al. [2009, p. 367])**.** The *selection expectation* of an integrable random closed set $\mathcal{G}$ is the closure of the set of expectations of all integrable selections of $\mathcal{G}$, i.e., $\mathrm{E}[\mathcal{G}] = \mathrm{cl}\{\mathrm{E}[G] \colon G \in \mathcal{L}^1(\mathcal{G})\}$.

Key to intuition surrounding the selection expectation is that an integrable random closed set $\mathcal{G}$ has a Castaing representation composed of integrable selections [Molchanov, 2017, p. 228]. A *Castaing representation* is a countable family $\{G_i, i \in \mathbb{N}\}$ of measurable selections of $\mathcal{G}$ such that $\mathcal{G}(\omega) = \mathrm{cl}(\{G_i(\omega), i \in \mathbb{N}\})$ for every $\omega \in \Omega$, that is, the set $\{G_i(\omega), i \in \mathbb{N}\}$ is dense in $\mathcal{G}(\omega)$; see also Rockafellar [1976], Shapiro et al. [2009, p. 365], Molchanov [2017, p. 60], Rockafellar and Wets [1998, p. 646]. However, the selection expectation of a random set depends on the underlying probability space in a way that the expectation of a random vector does not. See Molchanov [2017, Example 2.1.23, p. 238] for an example of a deterministic set whose selection expectation differs when calculated with respect to two different probability spaces. In addition, if $\mathcal{G}$ is an integrable random closed set in $\mathbb{R}^p$ defined with respect to a nonatomic probability space (see Subsection 1.4), then $\mathrm{E}[\mathcal{G}]$ is convex [Molchanov, 2017, Theorem 2.1.26, p. 239].

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