Cover-based inequalities for the single-source capacitated facility location problem with customer preferences

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Abstract

The *single-source capacitated facility location problem with customer preferences (SSCFLPCP)* is known to be strongly NP-hard and computational tests imply that state-of-the-art solvers struggle with computing exact solutions. In this paper, we contribute two novel preprocessing methods, which reduce the size of the considered integer programming formulation, and introduce sets of valid inequalities, which decrease the integrality gap. Each of the introduced results utilises structural synergies between capacity constraints and customer preferences in the SSCFLPCP. First, we derive two preprocessing methods, where the first method fixes location variables and the second method fixes allocation variables. Afterwards, we study cover-based inequalities. Here, we first strengthen the well-known cover inequalities: when determining covers, we also consider demands of customers not in the cover that must be assigned to the covered facility if a customer in the cover is assigned to it. We further strengthen these inequalities by including information on the assignments of customers in a cover if they are not assigned to the covered facility. Afterwards, we derive a new family of valid inequalities which expresses the relation of open facilities based on sets of customers covering a facility. We then discuss solution methods for the corresponding separation problems and, finally, test our results for two preference types in a computational study. Our results show a clear positive impact of the new preprocessing methods and inequalities, in particular when preferences coincide with closest assignments.

Keywords: facility location, customer preferences, valid inequalities, preprocessing

1 Introduction

Facility location problems (FLPs) play an essential role in operations research literature (Laporte et al., 2019, Celik Turkoglu and Erol Genevois, 2020). In their basic version, facilities providing some sort of service for customers need to be opened at potential sites and customers are assigned to these facilities. The aim is to minimise the total cost consisting of opening costs for the facilities and costs for assigning the customers. We refer to this problem as the *uncapacitated facility location problem* (*UFLP*). Obviously, each customer is assigned to the open facility with smallest assignment cost. The class of FLPs is widely applicable to many real-world problems such as locating health care institutions (Ahmadi-Javid et al., 2017), charging stations for electrical cars (Ahmad et al., 2022) or the location of bike rental stations (Kang et al., 2023) and many more (Celik Turkoglu and Erol Genevois, 2020).

There are no limitations on the service a facility can provide in UFLPs. However, such limitations have to be taken into account for many real-world problems, for example, when locating hospitals (Mestre et al.,

2015) or bike rental stations (Kang et al., 2023). Often, customers incur a certain demand and facilities have a certain capacity for serving customers' demands. Such problems belong to the class of *capacitated facility location problems* (*CFLP*s). It is no longer guaranteed that each customer will be assigned to the open facility with lowest assignment cost when considering capacities. The problem is called *single-source* CFLP if each customer has to be assigned to exactly one open facility.

If customers have an individual agenda regarding the facilities they want to be served at, then their assignment in the solution to the UFLP or CFLP might be in conflict with their preferences. In reality, however, customers then deviate to one of their most preferred open facilities. While this yields a feasible, although possibly worse, solution in the UFLP, it might induce an infeasible solution in the capacitated version. A real-world application for this class of location problems lies in the location of health care emergency centers. Here, facilities offer medical services, which are limited due to time and space. Customers' demands have to be served at these facilities without violating the facilities' capacities. In an emergency, it can be assumed that customers will seek the service of their closest open facility.

We refer to the problem in which each customer has to be assigned to one of their most preferred open facilities as the *(single-source capacitated) facility location problem with customer preferences* ((SSC)FLPCP). In this work, we study the SSCFLPCP, for which determining a feasible solution is already strongly NP-complete (e.g., Büsing et al. (2022)). Computational tests show that state-of-the-art solvers struggle with computing exact solutions. We contribute two novel preprocessing methods, which reduce the size of the considered integer programming formulation, and introduce sets of valid inequalities, which decrease the integrality gaps. More specifically:

- We propose two preprocessing methods, which fix location and allocation decisions, respectively.
- We propose the concept of implied demand covers, which generalise normal covers by simultaneously considering preferences of customers and capacities of facilities.
- We study the complexity of finding implied-demand covers.
- We propose and analyse inequalities for the SSCFLPCP based on implied-demand covers, which incorporate information on assignments of customers in the cover that are not assigned to the covered facility.
- We perform a computational study in which our results are tested for the case that preferences coincide with closest assignments as well as perturbed closest assignments. Our results indicate that our approaches work well for the former case and show potential for the latter one.

With these contributions, we aim to offer results for improving the performance of exact solvers. The remainder of this article is organized as follows. Section 2 provides an overview over related literature and adresses the research gap we aim to close with this paper. A formal problem definition is given in Section 3, in which we also provide an integer linear programming formulation and summarize used notation. Section 4 introduces the novel preprocessing routines, while Section 5 introduces several sets of new valid inequalities that utilise information from sets of customers covering a facility. In Section 6, we briefly discuss algorithms for solving the separation problems of our newly developed valid inequalities. Finally, we discuss the results of our computational study in Section 7.

2 Related work

Facility location problems are thoroughly studied in the literature, see, e.g., Laporte et al. (2019) or Celik Turkoglu and Erol Genevois (2020) for recent surveys. From the complexity-theoretical point of view, feasible solutions for UFLP instances can be computed in polynomial time while the computation of a cost-minimising solution is strongly NP-hard (Mirchandani and Francis, 1990). Conversely to the UFLP, finding a feasible solution for the single-source capacitated facility location problem is already strongly NP-hard.

Even if the set of open facilities is already known, it might be hard to find a feasible assignment that meets the capacity constraints. This can be seen via a reduction from 3-*partition* (Garey and Johnson, 1979). The relevance and complexity of FLPs has triggered a large number of scientific articles related to different aspects such as, exact solution methods (see, e.g., Avella and Boccia (2009), Görtz and Klose (2012) and Fischetti et al. (2016)), the investigation of their polyhedral structure (see, e.g., Leung and Magnanti (1989), Aardal et al. (1995), Avella and Boccia (2009) and Avella et al. (2021)), and heuristic solution methods (see, e.g., Mirchandani and Francis (1990) and Korte and Vygen (2018)).

Most articles considering facility location problems with customer preferences focus on the uncapacitated case. The work of Hanjoul and Peeters (1987) is considered to be the first occurrence of preference constraints in the context of facility location problems. The authors propose an exact algorithm, which utilises a branch-and-bound procedure, as well as two heuristics for solving the UFLP with customer preferences. Several articles focus on preprocessing strategies and valid inequalities that are used within exact methods, see, e.g., Cánovas et al. (2007), Espejo et al. (2012) and Vasilyev et al. (2013). A semi-Lagrangian relaxation heuristic approach is proposed by Cabezas and García (2022).

Several articles focus on the special case of *closest assignment constraints*, see, e.g., Rojeski and ReVelle (1970), Wagner and Falkson (1975) and Gerrard and Church (1996). Here, customer preferences are defined by assignment costs, i.e., for any two facilities a customer always prefers the one with less assignment costs. Since closest assignment constraints are automatically satisfied in cost-minimising solutions to the UFLP, these articles incorporate additional aspects. These aspects include the consideration of capacities where splitting demands of customers is allowed and capacities may be expanded (Rojeski and ReVelle (1970)) and the consideration of price-elastic demand functions, which affect the objective function, and the objective to maximise the joint surplus of consumers' producers in order to construct a public facility location model (Wagner and Falkson, 1975). Gerrard and Church (1996) review constraints for integer linear programming formulations for enforcing closest assignments and identify applications for facility location problems with closest assignment constraints.

Adding closest assignment constraints to the classical single-source capacitated facility location problem yields surprising complexity results. If the assignment costs correspond to distances in an underlying graph, then an optimal solution can be computed in polynomial time if the graph is a path or cycle (Büsing et al., 2022). In contrast, the single-source CFLP is strongly NP-hard independent of the structure of assignment costs. However, less research has been conducted on the capacitated problem compared to its uncapacitated counterpart.

Most research on capacitated FLPs with customer preferences revolves around a relaxed version of the problem in which customers are allowed to be assigned to less-preferred open facilities (Casas-Ramírez et al., 2018, Calvete et al., 2020, Polino et al., 2023). Here, the authors consider variations of a bilevel setting where the leader opens facilities with the aim to minimise the total sum of opening and assignment costs and a guarantee that each customer will be served; the follower assigns each customer, for which a ranking of all potential facilities is known, to the open facilities with the objective to optimise the sum of achieved preference rankings of the customers. Calvete et al. (2020) compare computational results of this relaxed problem with results of the SSCFLPCP. Caramia and Mari (2016) and Kang et al. (2023) study a variation of the capacitated facility location problem with customer preferences in which the capacities installed at facilities are part of the decision process. Conversely to the previous works, these two articles allow for customers to split their demands among multiple facilities.

The consideration of capacities in facility location problems induces a knapsack-like structure in the CFLP. For knapsack problems, cover inequalities are well studied (Crowder et al., 1983, Gu et al., 1999) and they can be strengthened to so-called extended cover inequalities and lifted cover inequalities (Balas, 1975). Klabjan et al. (1998) show that the cover inequality separation problem is weakly NP-hard even if the binary integer program is a knapsack problem. Kaparis and Letchford (2010) present exact and heuristic separation algorithms. Due to the knapsack-like structure occurring in capacitated facility location problems, cover inequalities may serve as useful valid inequalities in these problems as well (Aardal et al., 1995).

To the best of our knowledge, there has almost no research been done on the SSCFLPCP; polynomial time algorithms for special cases suggest further advantages arising from the combination of capacities and closest assignments. With this work, we aim to shed light on cover-based valid inequalities in the context of single-source capacitated facility location problems with customer preferences.

3 Problem definition, notation, and an integer linear programming formulation

In the single-source capacitated facility location problem with customer preferences (SSCFLPCP), we are given a set of customers I with demands $d_i \in \mathbb{N}_0$ for each $i \in I$ and a set of potential facilities J. A capacity $Q_j \in \mathbb{N}_0$ and opening costs $f_j \geq 0$ are associated with each potential facility $j \in J$. Assigning a customer $i \in I$ to a facility $j \in J$ yields assignment costs $c_{ij} \geq 0$. Furthermore, each customer has ranked all potential facilities inducing a complete, weak ordering of all facilities for every customer. To that end, $j <_i k$ indicates that customer $i \in I$ strictly prefers potential facility $j \in J$ over potential facility $k \in J$, while $j =_i k$ indicates that customer $i \in I$ is indifferent about facilities $j, k \in J$, i.e., if neither $j <_i k$ nor $k <_i j$ holds. Similarly, $j \leq_i k$ indicates that customer $i \in I$ either strictly prefers facility $j \in J$ over facility $k \in J$ or is indifferent about them.

A solution to the SSCFLPCP consists of a subset $F \subseteq J$ of facilities that are opened and an assignment $\Lambda : I \to F$ of customers to these facilities, which respects the capacity limits of all open facilities. That is, $\sum_{i \in \Lambda^{-1}(j)} d_i \leq Q_j$ must hold for each $j \in F$ with $\Lambda^{-1}(j)$ the set of customers assigned to facility j. Furthermore, each customer must be assigned to a facility they prefer most among all open facilities, i.e., $\Lambda(i) \leq_i j$ holds for all $i \in I$ and $j \in F$. The cost of a solution (F, Λ) equals the total opening and assignment costs, i.e., $\sum_{j \in F} f_j + \sum_{i \in I} c_{i\Lambda(i)}$. The objective of the SSCFLPCP is to find a solution with minimum cost.

The SSCFLPCP is strongly NP-hard as it contains the single-source capacitated facility location problem as a special case when each customer is indifferent about all facilities. For ease of readability, we introduce further notation for preference sets.

Notation In the remainder of this article, we use notations $J_{ij}^{<} = \{k \in J : k <_i j\}$, $J_{ij}^{=} = \{k \in J : k =_i j\}$, $J_{ij}^{\leq} = J_{ij}^{<} \cup J_{ij}^{=}$, $J_{ij}^{\geq} = J \setminus J_{ij}^{<}$, and $J_{ij}^{>} = J \setminus J_{ij}^{\leq}$ for each customer $i \in I$ and facility $j \in J$ to indicate the sets of facilities a customers strictly prefers over j, is indifferent compared to j, prefers at least as much as j, does not prefer more than j, and prefers less than j, respectively.

We will use formulation (1) as a basis for our study of preprocessing rules and valid inequalities. The formulation uses decision variables $y_j \in \{0, 1\}$, which indicate whether facility $j \in J$ is opened ($y_j = 1$), and variables $x_{ij} \in \{0, 1\}$, which indicate whether customer $i \in I$ is assigned to facility $j \in J$ ($x_{ij} = 1$).

$$
\begin{aligned}
\min \quad & \sum_{j \in J} f_j y_j + \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} && && \text{(1a)} \\
\text{s.t.} \quad & \sum_{j \in J} x_{ij} = 1 && \forall i \in I && \text{(1b)} \\
& \sum_{i \in I} d_i x_{ij} \leq Q_j y_j && \forall j \in J && \text{(1c)} \\
& \sum_{k \in J_{ij}^{>}} x_{ik} + y_j \leq 1 && \forall i \in I,\ j \in J && \text{(1d)} \\
& x_{ij} \leq y_j && \forall i \in I,\ j \in J && \text{(1e)} \\
& x_{ij} \in \{0, 1\} && \forall i \in I,\ j \in J && \text{(1f)} \\
& y_j \in \{0, 1\} && \forall j \in J && \text{(1g)}
\end{aligned}
$$

Objective function (1a) minimises the sum of total opening costs and the cost of assigning each customer. Constraints (1b) guarantee that each customer is assigned to a facility. Constraints (1c) ensure that the total demand assigned to an open facility respects its capacity and that customers can only be assigned to open facilities. Constraints (1d), which have first been introduced by Wagner and Falkson (1975), guarantee that customer preferences are respected. More specifically, these constraints ensure that, if facility j is open, then customer $i \in I$ cannot be assigned to a facility $k \in J$, which they prefer less than j. Linking constraints (1e) are redundant, but are well known to significantly strengthen the linear programming relaxation of capacitated facility location problems.

4 Preprocessing

In this section, we introduce preprocessing methods that allow to fix or eliminate variables from formulation (1). Combining arguments related to maximum capacities of facilities and customer preferences, they are specific for the SSCFLPCP. Similar ideas could, however, be used for related problems with capacity constraints in which assignment decisions cannot be (fully) controlled by a central decision maker. The first procedure was observed by Cánovas et al. (2007) for the uncapacitated version with strict preferences and is based on the fact that any open facility $j \in J$ must always serve all customer who prefer j over all other facilities.

Proposition 1 (Cánovas et al. (2007))**.** *Consider an arbitrary facility $j \in J$ and a customer $i \in I$. If there is no facility customer i prefers over j or is indifferent to, customer i must be assigned to facility j if the facility is open, i.e., inequality $y_j \leq x_{ij}$ holds.*

This preprocessing is also valid for the uncapacitated problem. When considering capacities, we can further strengthen this result. As Proposition 2 states, a facility that cannot accommodate the demand of all such customers can, therefore, not be opened in any feasible solution.

Proposition 2 (*CloseViolatedFacs*)**.** *Consider an arbitrary facility $j \in J$ and the set of customers $I(j) = \{i \in I : J_{ij}^{\leq} = \{j\}\}$ that prefer j over all other facilities. If the total demand of these customers exceeds the capacity of facility j, i.e., if $\sum_{i \in I(j)} d_i > Q_j$, then facility j cannot be opened. Consequently, variables y_j and x_{ij} are equal to zero for all $i \in I$ in any solution.*

Notice that closing one or more facilities according to this preprocessing, i.e., forcing the associated variables to zero, may increase the demand induced by customers preferring a facility most at other facilities - since the most preferred facilities of some customers are no longer available. Consequently, preprocessing based on Proposition 2 is applied iteratively, until no new facilities are eliminated.

Cánovas et al. (2007) also observed a property for the uncapacitated problem that allows to fix some allocation variables between customers and facilities. Their result works for the case that customers are not indifferent between multiple facilities. In the following, we strengthen their result so that it also allows indifferent customers.

Proposition 3 (Cánovas et al. (2007))**.** *Consider an arbitrary facility $j \in J$ and two customers $i, l \in I$ such that the set of facilities customer l prefers over j or is indifferent to, i.e., $J_{lj}^{\leq}$, is subset of the union of facility j with the set of facilities customer i prefers over j, i.e., $J_{lj}^{\leq} \subseteq J_{ij}^{<} \cup \{j\}$. Then customer l must be assigned to facility j if customer i is assigned to it, i.e., the inequality $x_{ij} \leq x_{lj}$ holds.*

The observations stated in the latter two propositions reveal that assigning one or more customers to a specific facility may imply the assignment of other customers to the same facility. For the capacitated problem variant, such a set of implied customers also implies a certain demand at this facility. These concepts of *implied customers* and *implied demand* are crucial for the preprocessing rule stated in Proposition 4 and several of the valid inequalities proposed in Section 5. We formally introduce them in Definitions 1 and 2.

Definition 1 (Implied Customers). *The set of customers $\mathcal{I}(i,j) \subseteq I$ implied from customer $i \in I$ at facility $j \in J$ is the set of customers that must be assigned to facility $j \in J$ if customer $i \in I$ is assigned to it, i.e., $\mathcal{I}(i,j) = \{i\} \cup \{l \in I : J^{\leq}_{lj} \subseteq (J^{<}_{ij} \cup \{j\})\}$. Similarly, the set of customers $\mathcal{I}(I',j) \subseteq I$ implied from customer set $I' \subseteq I$ at facility $j \in J$ is the set of customers that must be assigned to facility $j \in J$ if all customer $i \in I'$ are assigned to it, i.e., $\mathcal{I}(I',j) = \cup_{i \in I'} \mathcal{I}(i,j)$.*

Definition 2 (Implied Demand). *The demand implied from customer $i \in I$ at facility $j \in J$ is the total demand of all customers implied by customer $i \in I$ at facility j, i.e., $\mathcal{D}(i,j) = \sum_{l \in \mathcal{I}(i,j)} d_l$. Similarly, the demand implied from customer set $I' \subseteq I$ at facility $j \in J$ is the total demand of all customers implied from customer set I' at facility j, i.e., $\mathcal{D}(I',j) = \sum_{l \in \mathcal{I}(I',j)} d_l$.*

Proposition 4, which can be seen as an extension of Proposition 3, uses the concept of implied demand to derive conditions under which a customer can never be assigned to a facility.

Proposition 4 (*ImplDem*). *Consider an arbitrary facility $j \in J$ and a customer $i \in I$. If the total demand implied from customer i at facility j exceeds the capacity of j, i.e., if $\mathcal{D}(i,j) > Q_j$, then customer i cannot be assigned to facility j. Consequently, variable x_{ij} is equal to zero in any solution and can be set to zero in formulation* (1).

The preprocessing rule following from Proposition 5 has been proposed by Cánovas et al. (2007). It considers the case when the sets of facilities two customers i and l prefer over a given facility j are identical. If there are no facilities to which i and l are indifferent to compared to j, then either both of these customers are assigned to j or none of them.

Proposition 5 (Cánovas et al. (2007)). *Consider an arbitrary facility $j \in J$ and two customers $i, l \in I$. If both customers prefer the same facilities over j, i.e., if $J^{<}_{ij} = J^{<}_{lj}$, and there does not exist a facility $k \in J$, $k \neq j$, that either i or l is indifferent to with respect to j, i.e., if $J^{=}_{ij} = J^{=}_{lj} = \{j\}$, then either both customers i and l are assigned to j or none of them. Consequently, the equation $x_{ij} = x_{lj}$ holds and can be added to formulation* (1).

Note that this result is a direct implication of Proposition 3. We study the performance of the various preprocessing methods in Section 7. We will see that method *ImplDem* (Proposition 4) performs best if preferences coincide with closest assignments. If preferences coincide with perturbed closest assignments, all introduced methods fix variables or add constraints - yet they do not yield an improvement of the optimality gap.

5 Cover-based inequalities

In this section, we propose several sets of valid inequalities for the SSCFLPCP that are based on utilising the combination of customer preferences and capacity constraints of facilities. One basic concept used in these inequalities are customer sets whose (implied) demand cannot be served by a particular facility, i.e., (implied-demand) covers that are formally introduced in Definition 3.

Definition 3 ((Implied-Demand) Cover). *A* cover *for a facility $j \in J$ is a set of customers $I' \subseteq I$ whose demand exceeds the capacity of j, i.e., for which $\sum_{i \in I'} d_i > Q_j$ holds. An* implied-demand cover *for a facility $j \in J$ is a set of customers $\tilde{I} \subseteq I$ whose implied demand exceeds the capacity of j, i.e., for which $\mathcal{D}(\tilde{I}, j) > Q_j$ holds.*

From this definition, we can immediately observe the following relation between covers and implied-demand covers which will be helpful to derive relations between different inequalities introduced below.

Corollary 1. *Consider an arbitrary facility $j \in J$. Then, for every cover I' for j there exists an implied-demand cover $\tilde{I}$ for j such that $\tilde{I} \subseteq I'$.*

An (implied-demand) cover $I' \subseteq I$ is called *minimal* if there does not exist a customer $i \in I'$ such that $I' \setminus \{i\}$ is still an (implied-demand) cover. It is well known that a classical minimal cover can be found in polynomial time. The special case of a *minimum cover*, i.e., a minimal cover with a minimum number of elements, can be identified by solving a $0-1-$ knapsack problem. As a special case with cost of one for each item, minimum covers can be computed in polynomial time by ordering the items by their weight and adding them starting from the heaviest to the lightest until there is no capacity left. As Theorem 1 shows, finding a minimum implied-demand cover is, on the contrary, NP-complete in the strong sense.

Theorem 1. *Finding a minimum implied-demand cover $\tilde{I} \subseteq I$ for a facility $j \in J$ is strongly NP-complete. A minimal implied-demand cover can be computed in polynomial time.*

Proof. In order to prove the first statement of the theorem, we show that the decision version of the minimum implied-demand cover problem is strongly NP-complete via a reduction from minimum set cover. Given a minimum set cover instance $\mathcal{I}$ consisting of a set of elements $U = \{1, 2, \ldots, n\}$ to which we refer to as the universe and a collection S of m subsets of universe U, the goal is to decide whether there is a sub-collection of S of size $K \in \mathbb{Z}_{\geq 0}$ whose union equals the universe. W.l.o.g. we assume that collection S does not contain subsets s_1, s_2 with $s_1 \subseteq s_2$ and that each element of the collection S contains at least two elements.

We transform $\mathcal{I}$ into an instance $\mathcal{I}'$ of the minimum implied-demand cover problem as follows. Introduce one customer i for each element $u \in U$ in the universe and one customer k for each element $s \in S$. Denote the set of customers with I and set the cost and weight of a customer $i \in I$ to one, i.e., it is $c_i = d_i = 1$. Define the set of facilities as $J = \cup_{i \in I}\{j_i\} \cup \{j\}$. Note that in this setting, we are interested in a minimum implied demand cover for facility j. Hence, it suffices to define $J_{ij}^{<}$ and $J_{ij}^{=}$ for all customers $i \in I$. The set of facilities a customer i corresponding to an element $u \in U$ prefers over j is defined as $J_{ij}^{<} = \{j_i\}$ and $J_{ij}^{=} = \{j\}$. The set of facilities a customer k corresponding to a subset $s \in S$ prefers over j is defined as $J_{kj}^{<} = \bigcup_{e \in s} J_{ej}^{<} \cup \{j_k\}$ and $J_{kj}^{=} = \{j\}$. Note that all customers corresponding to elements $s \in S$ imply the assignment of all customers corresponding to items occurring in considered set s to j. We set the lower bound for the demands, that is, the smallest number of customers violating the capacity of facility j, to $Q_j + 1 = K + |U|$. The goal is to decide whether there is a set $\bar{I} \subseteq I$ containing K customers whose assignment is not implied by other customers in set $\bar{I}$; these K customers correspond to a minimum implied-demand cover $\tilde{I}$. Instance $\mathcal{I}'$ can clearly be constructed in polynomial time.

It remains to prove that there is a minimum cover for instance $\mathcal{I}$ of size K if and only if there is a solution $\bar{I} \subseteq I$ for instance $\mathcal{I}'$ containing K customers whose assignment is not implied by other customers in set $\bar{I}$.

Suppose there is a minimum set cover for instance $\mathcal{I}$ of size K and denote this sub-collection with S'. From this, we construct our solution for instance $\mathcal{I}'$ of the minimum implied-demand cover problem as follows. Assign all customers i corresponding to elements in set $S' \cup U$ to facility j and denote this set with $\bar{I}$. Then, facility j, with a capacity of $Q_j = K + |U| - 1$, serves a demand of $K + |U|$; thus, the set of customers corresponding to S' defines an implied-demand cover. Due to the construction of instance $\mathcal{I}'$, set $\bar{I}$ consists of non-implied customers as well as all customers implied by them. In particular, all customers corresponding to an element in universe U are implied by one of the K customers corresponding to elements in collection S'. In total, this yields an implied-demand cover of size $(K + |U|) - |U| = K$.

Conversely, suppose $\bar{I}$ corresponds to a solution for instance $\mathcal{I}'$ consisting of K non-implied customers. First, we observe that all customers corresponding to an element in universe U are added to the cover. Suppose there is at least one such customer not added to the cover. In order to still satisfy the constraint regarding the lower bound on the demand, we have to assign at least $K + 1$ customers who correspond to items in collection S. Then, we add at least $K + |U|$ units to the objective function but can only deduct less or equal to $|U| - 1$ units. This yields an objective value of $z(I') \geq K + |U| - (|U| - 1) = K + 1 > K$, a contradiction to our assumption. Second, we observe that all customers corresponding to elements in U are implied by at least one customer in set $\bar{I}$ corresponding to an element in set S each. Suppose at least one customer corresponding to an element in universe U is added to cover $\bar{I}$ and is not implied by another customer in $\bar{I}$. In this case, the solution has an objective value of $z(I') \geq (K + |U|) - (|U| - 1) = K + 1 > K$ - a contradiction to the solution having objective value K. Given these results, we construct a solution for instance $\mathcal{I}$ as follows. Add all customers in set $\bar{I}$ that correspond to sets in collection S to the minimum cover. Thus, the constructed minimum cover has size K and it follows immediately that all elements in U are covered by at least one set in the minimum cover. Hence, the constructed solution is a minimum cover for instance $\mathcal{I}$ and our claim follows.

We can compute a minimal implied demand cover $\tilde{I}$ for a facility j by first adding all customers to cover $\bar{I}$ and then removing customers that are not implied by any other customer one by one from $\bar{I}$ until the budget constraint would be violated. Define the minimal implied-demand cover as the set of customers not implied by another customer in set $\bar{I}$. □

Theorem 1 shows that the occurrence of customers in several sets of implied customers adds a new complexity to the problem of finding implied-demand covers. Note that the problem of finding a minimum implied-demand cover is closely related to *knapsack problems with partial order* (KPO) (Johnson and Niemi, 1983). However, in minimum implied-demand covers, we are interested in minimising the number of non-implied customers; in KPOs, this property is not measured.

Next, we analyse classical cover inequalities and introduce extensions which incorporate implied-demands.

5.1 Cover inequalities

Cover inequalities are known to be valid for the capacitated facility location problem due to the knapsack structure of the capacity constraints (1c). As the considered user preferences do not impact their validity, classic cover inequalities (2) are also valid for the SSCFLPCP.

$$\sum_{i \in I'} x_{ij} \leq |I'| - 1 \qquad \forall j \in J, I' \subseteq I : \sum_{i \in I'} d_i > Q_j \tag{2}$$

We denote this set of inequalities with $\boldsymbol{C}$. It is not difficult to see that the same arguments as for inequalities (2) can be used to to show validity of inequalities (3) that instead consider an implied-demand cover $I' \subseteq I$ for facility $j \in J$:

$$\sum_{i \in I'} x_{ij} \leq |I'| - 1 \qquad \forall j \in J, I' \subseteq I : \mathcal{D}(I', j) > Q_j \tag{3}$$

We denote this set of inequalities with $\boldsymbol{IC}$. Corollary 1 implies that for each inequality (2) there exists at least one inequality (3) whose set of left-hand side variables is a (potentially identical) subset of the one of the former. Thus, Corollary 2 holds.

Corollary 2. *Inequalities* (3) *imply inequalities* (2).

It is well known, that classical cover inequalities can be strengthened to so-called extended or lifted cover inequalities. Next, we use analogous ideas to lift implied-demand cover inequalities (3). To ease notation and arguments in the proofs of the following two theorems, we first define the concepts of a customer's individual and overall largest contribution to the total demand implied by a set of customers.

Definition 4 (Individual and Maximum Contribution)**.** *For each set of customers $I' \subseteq I$, every customer $i \in I'$ and facility $j \in J$, we denote by $r_i(I', j) = \mathcal{D}(I', j) - \mathcal{D}(I' \setminus \{i\}, j)$ the individual contribution of customer $i \in I'$ to the demand implied by I' for facility $j \in J$. Furthermore, $r_{\max}(I', j) = \max_{i \in I'} r_i(I', j)$ is used to denote the largest individual contribution of any customer from I'.*

With this concept at hand, we define extended implied-demand cover inequalities as follows.

Theorem 2 (**EIC**)**.** *Let $I' \subseteq I$ be an implied-demand cover for facility $j \in J$ and $E \neq \emptyset$ be a set of customers not implied by I', i.e., $E \subseteq I \setminus \mathcal{I}(I', j)$. Then, the* extended implied-demand cover inequality (**EIC**)

$$\sum_{i \in I'} x_{ij} + \sum_{i \in E} x_{ij} \leq |I'| - 1 \tag{4}$$

is valid and dominates implied-demand cover inequality (3) *defined on* I' *and* j *if there exists a sequence* $(s_1, \ldots, s_{|E|})$ *of* E *such that* $r_{s_\ell}(I' \cup \{s_1, \ldots s_\ell\}, j) \geq r_{\max}(I', j)$ *holds for all* $\ell \in \{1, \ldots, |E|\}$.

Proof. We show the theorem by contradiction. Assume, that the theorem holds for the first $\ell - 1$ customers from the considered sequence of E, i.e., inequality

$$\sum_{i \in I'} x_{ij} + \sum_{i \in \{s_1, \ldots, s_{\ell-1}\}} x_{ij} \leq |I'| - 1 \tag{5}$$

is valid but

$$\sum_{i \in I'} x_{ij} + \sum_{i \in \{s_1, \ldots, s_{\ell-1}\}} x_{ij} + x_{s_\ell, j} \leq |I'| - 1$$

does not hold. Thus, there exists a set $I'' \subset I' \cup \{s_1, s_2, \ldots, s_\ell\}$ of cardinality at least $|I'|$ whose implied demand does not exceed the facility's capacity, i.e., $\mathcal{D}(I'', j) \leq Q_j$. Clearly, $s_\ell \in I''$ as inequality (5) is assumed to be valid. Thus, the implied demand of subset $I'' \setminus \{s_\ell\}$, with cardinality of at most $|I'| - 1$, can be at most $Q_j - r_{s_\ell}(I'', j)$. Consider an arbitrary customer $a \in I' \cup \{s_1, \ldots, s_{\ell-1}\}$ such that $r_a(I' \cup \{s_1, \ldots, s_{\ell-1}\}, j) = r_{\max}(I' \cup \{s_1, \ldots, s_{\ell-1}\}, j)$. If $a \notin I''$, then there exists at least one customer $b \in I' \setminus I''$ (otherwise I' cannot be a cover) with $r_b(I', j) \leq r_{\max}(I', j)$. Per assumption it is $r_{\max}(I', j) \leq r_{s_\ell}(I' \cup \{s_1, s_2, \ldots, s_\ell\}, j)$ and it follows that $\mathcal{D}((I'' \cup \{b\}) \setminus \{s_\ell\}, j) \leq \mathcal{D}(I'', j) \leq Q_j$. Since $|(I'' \cup \{b\}) \setminus \{s_\ell\}| \geq |I'|$ this contradicts the original assumption that (5) holds. If, on the other hand, $a \in I''$, then there exists at least one customer $b \in (I' \cup \{s_1, \ldots, s_{\ell-1}\}) \setminus I''$ (otherwise I' cannot be a cover) whose individual contribution to set $I' \cup \{s_1, \ldots, s_{\ell-1}\}$ is at most $r_{\max}(I', j)$. As before, this implies that the implied demand of set $(I'' \cup \{b\}) \setminus \{s_\ell\}$ (of cardinality at least $|I'|$) does not exceed Q_j contradicting validity of inequality (5). $\square$

Note that there may be two customers in set E where the assignment of one of the customers to j implies the assignment of the other customer to j. This may be counter-intuitive as we have to avoid this property in minimal implied-demand covers. For extended implied-demand covers, it is an advantage to have two such customers in set E.

Next, we introduce a set of lifted implied-demand cover inequalities that are conceptually similar to those known for standard covers (Conforti et al., 2014) and dominate extended implied-demand cover inequalities (4).

Theorem 3 (**LIC**)**.** *Consider a facility* $j \in J$*, an implied-demand cover* $I' \subseteq I$ *for this facility, and an ordering* $(s_1, \ldots, s_{|I \setminus \mathcal{I}(I', j)|})$ *of all customers not implied by* I'*. Then, the* lifted implied-demand cover inequality (**LIC**)

$$\sum_{i \in I'} x_{ij} + \sum_{i=1}^{|I \setminus \mathcal{I}(I', j)|} \alpha_{s_i j} x_{s_i j} \leq |I'| - 1 \tag{6}$$

is valid and dominates inequality (4) *obtained for the same sequence of customers not in* I' *if* $\alpha_{s_i j} = |I'| - 1 - \max_{C \subseteq I' \cup \{s_1, \ldots, s_{i-1}\}} \{\sum_{k \in C} \alpha_{kj} : \mathcal{D}(C, j) \leq Q_j - r_{s_i}(C \cup \{s_i\}, j),\ (I' \cup \{s_1, \ldots, s_{i-1}\}) \cap \mathcal{I}(s_i, j) \subseteq C\}$ *holds for all* $i \in \{1, \ldots, |I \setminus \mathcal{I}(I', j)|\}$*. Value* $\alpha_{s_i j}$ *corresponds to the number of elements in set* $I' \cup \{s_1, \ldots, s_{i-1}\}$ *that can be replaced in the cover by adding element* s_i *with* $i \in \{1, \ldots, |I \setminus \mathcal{I}(I', j)|\}$ *due to* s_i*'s (implied) demand.*

Proof. Following the proof for extended cover inequalities given in Wolsey (2020) we consider $\ell \in \{1, \ldots, |I \setminus \mathcal{I}(I', j)|\}$ and let

$$\sum_{i \in I'} x_{ij} + \sum_{i=1}^{\ell-1} \alpha_{s_i j} x_{s_i j} \leq |I'| - 1$$

be the inequality obtained so far. We observe that the value of $\alpha_{s_\ell j}$ is irrelevant if $x_{s_\ell j} = 0$. If, on the other hand, customer s_ℓ is assigned to j it consumes at least $r_{s_\ell}(I' \cup \{s_1, \dots, s_\ell\}, j)$ units of demand and implies the assignment of all customers in $(I' \cup \{s_1, \dots, s_{\ell-1}\}) \cap \mathcal{I}(s_\ell, j)$ to j; thus, the maximum number of customers from $I' \cup \{s_1, \dots, s_{\ell-1}\}$ that can still be assigned to j is therefore equal to $\max_{C \subseteq I' \cup \{s_1, \dots, s_{\ell-1}\}} \{\sum_{k \in C} \alpha_{kj} : \mathcal{D}(C, j) \leq Q_j - r_{s_\ell}(I' \cup \{s_1, \dots, s_\ell\}, j),\ (I' \cup \{s_1, \dots, s_{\ell-1}\}) \cap \mathcal{I}(s_\ell, j) \subseteq C\}$, and $\alpha_{s_\ell j}$ can therefore be set to $|I'| - 1$ minus this value. We also observe, that $\alpha_{s_\ell j} \geq 1$ holds if $r_{s_\ell}(I' \cup \{s_1, \dots, s_\ell\}, j) \geq r_{\max}(I', j)$ and the resulting lifted implied-demand cover inequality therefore dominates the extended one. ☐

We discuss potential lifting-sequences in Section 6. Next, we study further inequalities that can be derived due to the presence of both capacity and preference constraints.

5.2 Strengthened implied-demand cover inequalities

We next introduce two additional sets of inequalities (7) and (8) that are based on further observations concerning customer preferences. Notice that we introduce and discuss these inequalities based on implied-demand covers. Similar to above, weaker counterparts based on traditional covers are valid too. The first set of inequalities is defined in Theorem 4.

Theorem 4 (**RemElem**$_{IC}$)**.** *Consider a facility $j \in J$, an implied-demand cover $I' \subseteq I$ for this facility consisting of at least two customers, i.e., $|I'| \geq 2$, and let $l \in I'$. Then, implied-demand cover inequality (3) can be strengthened to valid inequality*

$$\sum_{i \in I' \setminus \{l\}} x_{ij} \leq \sum_{k \in J_{lj}^{\leq} \setminus (\cup_{i \in I' \setminus \{l\}} J_{ij}^{<} \cup \{j\})} x_{lk} + (|I'| - 2) \tag{7}$$

and we denote this set of inequalities with **RemElem**$_{IC}$.

Proof. Note that any considered implied-demand cover I' must consist of at least two customers. Suppose cover $I' = \{k\}$ consists of one customer. If facility j is closed and customer k does not prefer any facility over j, inequality (7) turns into $0 \leq 0 + 1 - 2 = -1$, which is infeasible; if facility j is open and i assigned to j, we have $1 \leq 0 + 1 - 2 = -1$, which is infeasible.

In the following, we assume that each implied-demand cover consists of at least two customers. Consider an implied-demand cover $I' \subseteq I$ for facility $j \in J$ and the associated implied-demand cover inequality $\sum_{i \in I'} x_{ij} \leq |I'| - 1$. Let $l \in I'$ be an arbitrary customer from I' and recall that every customer must be assigned to some facility, i.e., $\sum_{j \in J} x_{lj} = 1$. Using the equation $J = (J_{lj}^{\leq} \setminus \{j\}) \cup \{j\} \cup J_{lj}^{>}$, we can rewrite the left-hand side of the initial cover inequality as follows:

$$\sum_{i \in I'} x_{ij} = x_{lj} + \sum_{i \in I' \setminus \{l\}} x_{ij} = 1 - (\sum_{k \in J_{lk}^{\leq} \setminus \{j\}} x_{lk} + \sum_{k \in J_{lk}^{>}} x_{lk}) + \sum_{i \in I' \setminus \{l\}} x_{ij}.$$

Thus, the initial cover inequality is equivalent to

$$\sum_{i \in I' \setminus \{l\}} x_{ij} \leq |I'| - 2 + \sum_{k \in J_{lk}^{\leq} \setminus \{j\}} x_{lk} + \sum_{k \in J_{lk}^{>}} x_{lk}$$

which states that all customer in $I' \setminus \{l\}$ can be assigned to facility j only if customer l is assigned to another facility.

Observe that customer l cannot be assigned to a facility $k \in J_{lj}^{>}$ if j is open and that the cover inequality is redundant if j is not open - in which case the left-hand side is equal to zero. Thus, all assignment variables considering facilities in $J_{lj}^{>}$ can be removed from the right-hand side leading to the stronger inequality

$$\sum_{i \in I' \setminus \{l\}} x_{ij} \leq |I'| - 2 + \sum_{k \in J_{lk}^{\leq} \setminus \{j\}} x_{lk}.$$

Further notice that the variable term on the right-hand side is only relevant for the validity of the inequality if all customers in $I' \setminus \{l\}$ are assigned to facility j. Otherwise the left-hand side value is at most $|I'| - 2$. In this case, no facility that is preferred over j by any of these customers may be open and the previous inequality can therefore be further strengthened to

$$\sum_{i \in I' \setminus \{l\}} x_{ij} \leq |I'| - 2 + \sum_{k \in J^{\leq}_{lj} \setminus (\cup_{i \in I' \setminus \{l\}} J^{<}_{ij} \cup \{j\})} x_{lk}.$$

□

The following corollary can be shown by repeating the steps in the proof of Theorem 4 using a standard cover instead of an implied-demand cover as starting point.

Corollary 3. *A set of valid inequalities is obtained from inequalities* (7) *by considering covers instead of implied-demand covers. The resulting set of inequalities dominates cover inequalities* (2)*.*

Further utilising the idea of removing customers included in an (implied-demand) cover from the left-hand side of a strengthened cover inequality (7) we can also derive the next result.

Theorem 5 (**RemAll**$_{IC}$)**.** *Consider a facility $j \in J$, an implied-demand cover $I' \subseteq I$ for this facility and let $i \in I'$. Then, implied-demand cover inequality* (3) *can be strengthened to valid inequality*

$$x_{ij} \leq \sum_{l \in I' \setminus \{i\}} \sum_{a \in J^{\leq}_{lj} \setminus (J^{<}_{ij} \cup \{j\})} x_{la} \tag{8}$$

and we denote this set of inequalities with **RemAll**$_{IC}$*.*

Proof. Note that inequality (8) is valid for implied-demand covers of size one. In this case, the inequality reduces to $x_{ij} \leq 0$, which is valid since customer i cannot be assigned to facility j as its implied demand is too high.

Let $\sum_{i \in I'} x_{ij} \leq |I'| - 1$ be the implied-demand cover inequality associated to implied-demand cover I'. We substitute all left-hand side variables x_{lj} corresponding to $l \in I' \setminus \{i\}$ for some $i \in I'$ by $1 - (\sum_{k \in J^{\leq}_{lj} \setminus \{j\}} x_{lk} + \sum_{k \in J^{>}_{lj}} x_{lk})$ and obtain

$$x_{ij} \leq \sum_{l \in I' \setminus \{i\}} \sum_{a \in J^{\leq}_{lj} \setminus (\{j\})} x_{la} + \sum_{l \in I' \setminus \{i\}} \sum_{a \in J^{>}_{lj}} x_{la}$$

Like in the previous proof, no customer $l \in I' \setminus \{i\}$ can be assigned to a facility $k \in J^{>}_{lj}$ if j is open and the cover inequality is redundant if j is closed. Thus, we can, again, remove the facilities in $J^{>}_{lj}$, for $l \in I' \setminus \{i\}$, from the right hand side. Furthermore, the term on the right-hand side is only relevant for the validity of the inequality if customer i is assigned to facility j. In this case, no facility that i strictly prefers over j may be open. The combination of these two arguments leads to the stronger inequality

$$x_{ij} \leq \sum_{l \in I' \setminus \{i\}} \sum_{a \in J^{\leq}_{lj} \setminus (J^{<}_{ij} \cup \{j\})} x_{la}$$

□

Theorem 6 analyses the impact of removing customers included in an (implied-demand) cover one by one from the left-hand side of a cover-like inequality.

Theorem 6 (**RemOBO**$_{IC}$)**.** *Consider a facility $j \in J$, an implied-demand cover $I' \subseteq I$ and a customer $i \in I'$. Let $\nu = (\pi(I' \backslash \{i\}), i) \in I^{|I|}$ be an ordering of customers in set I' and $\pi : (i)_{i \in I} \mapsto \pi((i)_{i \in I})$ a permutation. Then, inequality*

$$x_{ij} \leq \sum_{m=2}^{|I'|} \sum_{\substack{a \in J^{\leq}_{\nu(m-1)j} \backslash \\ (\cup_{i'=\nu(m)}^{\nu(|I'|)} J^{<}_{i'j} \cup \{j\})}} x_{\nu(m-1)a} \tag{9}$$

is valid and dominates inequalities (7) and (8). We denote this set of inequalities with ***RemOBO***$_{IC}$.

Proof. We first observe, that inequalities (9) are valid for implied-demand covers of size one in which case they reduce to $x_{ij} \leq 0$, which is clearly valid since customer i cannot be assigned to this facility as its implied demand is too high.

Thus, we restrict our attention to implied-demand covers $I' \subseteq I$ at facilities $j \in J$ such that $|I'| \geq 2$. In this case inequality (7)

$$\sum_{i \in I' \backslash \{l\}} x_{ij} \leq \sum_{k \in J^{\leq}_{lj} \backslash (\cup_{i \in I' \backslash \{l\}} J^{<}_{ij} \cup \{j\})} x_{lk} + (|I'| - 2)$$

is valid for any $l \in I'$, cf. Theorem 4. Next, we use use relation $x_{kj} = 1 - \sum_{a \in J^{\leq}_{kj} \backslash \{j\}} x_{ka} - \sum_{a \in J^{>}_{kj}} x_{ka}$ to substitute x_{kj} for a $k \in I' \setminus \{l\}$. Rearranging the terms, we obtain

$$\begin{aligned}
\sum_{i \in I' \backslash \{k,l\}} x_{ij} &\leq \sum_{a \in J^{\leq}_{lj} \backslash (\cup_{i' \in I' \backslash \{l\}} J^{<}_{i'j} \cup \{j\})} x_{lk} + (|I'| - 2) - (1 - \sum_{a \in J^{\leq}_{kj} \backslash \{j\}} x_{ka} - \sum_{a \in J^{>}_{kj}} x_{ka}) \\
&= \sum_{a \in J^{\leq}_{lj} \backslash (\cup_{i' \in I' \backslash \{l\}} J^{<}_{i'j} \cup \{j\})} x_{la} + (|I'| - 3) + \sum_{a \in J^{\leq}_{kj} \backslash \{j\}} x_{ka} + \sum_{a \in J^{>}_{kj}} x_{ka}
\end{aligned}$$

We observe that the rightmost term $\sum_{a \in J^{>}_{kj}} x_{ka}$ can only be non-zero if facility j is closed as these assignment variables all relate to assigning some customer to a facility customer k prefers less than j. Thus, we can drop this term from the inequality as the left-hand side $\sum_{i \in I' \backslash \{k,l\}} x_{ij}$ must be equal to zero if this rightmost expression is greater than zero. The inequality above is only of interest if all customers $i \in I' \backslash \{k, l\}$ are assigned to j. Thus, all facilities such customers strictly prefer over j have to be closed and we may exclude all assignments of k to such facilities in the term $\sum_{a \in J^{\leq}_{kj} \backslash \{j\}} x_{ka}$. Then, we can strengthen this term to $\sum_{a \in J^{\leq}_{kj} \backslash (\cup_{i' \in I' \backslash \{k,l\}} J^{<}_{i'j} \cup \{j\})} x_{ka}$, leading to

$$\begin{aligned}
\sum_{i \in I' \backslash \{k,l\}} x_{ij} &\leq (|I'| - 3) + \sum_{a \in J^{\leq}_{lj} \backslash (\cup_{i' \in I' \backslash \{l\}} J^{<}_{i'j} \cup \{j\})} x_{la} + \sum_{a \in J^{\leq}_{kj} \backslash (\cup_{i' \in I' \backslash \{k,l\}} J^{<}_{i'j} \cup \{j\})} x_{ka} \\
&= (|I'| - 3) + \sum_{m=2}^{3} \sum_{a \in J^{\leq}_{\nu(m-1)j} \backslash (\cup_{i'=\nu(m)}^{\nu(|I'|)} J^{<}_{i'j} \cup \{j\})} x_{\nu(m-1)a}
\end{aligned}$$

for $\nu = (l, k, \pi(I' \setminus \{k, l\}))$ an ordering of the customers with customers l, k at the first and second position. Repeating this procedure for $|I'| - 3$ further elements and defining ordering ν according to the order the respective elements are moved to the right-hand side yields the claim.

It follows immediately that inequality (9) dominates inequalities (7) and (8). □

We can also apply this procedure to extended implied-demand cover inequalities. Consider an extended implied-demand cover I', which is an extension of an implied-demand cover $\tilde{I}$ with $\tilde{I} \subsetneq I'$. Then, we can move at most $|\tilde{I}| - 1$ elements from the left-hand side to the right-hand side in order to strengthen implied-demand cover inequalities (3). Otherwise, assigning no customer in extended implied-demand cover I' to facility j would result in inequality $0 \leq -1$.

Finally, we consider another approach to utilise information provided by covers.

5.3 Location-centered inequalities

In the following, we study a different approach to derive valid inequalities from (implied-demand) covers. First, consider a cover $I' \subseteq I$ for some facility $j \in J$. From the definition of a cover it is immediate that facility j can only be opened if at least one customer from I' is assigned to a different facility. The latter is, however, only possible if at least one facility that is preferred by some customer from I' over j or to which such a customer is indifferent is opened too. Thus, the set of inequalities

$$y_j \leq \sum_{k \in (\cup_{i \in I'} J_{ij}^{\leq}) \setminus \{j\}} y_k \qquad \forall j \in J,\ I' \subseteq I : \sum_{i \in I'} d_i > Q_j \tag{10}$$

is valid. We denote this set of inequalities with ***ImplFac***. It is not difficult to see, that the same arguments as for inequalities (10) can be used to to show validity of inequalities (11) that instead consider an implied-demand cover $I' \subseteq I$ for facility $j \in J$:

$$y_j \leq \sum_{k \in (\cup_{i \in I'} J_{ij}^{\leq}) \setminus \{j\}} y_k \qquad \forall j \in J,\ I' \subseteq I : \mathcal{D}(I', j) > Q_j \tag{11}$$

For this family of valid inequalities, however, the consideration of implied-demands does not provide an advantage.

Corollary 4. *Inequalities* (10) *and inequalities* (11) *are equivalent.*

Proof. Consider a valid inequality (10) based on cover C. Define a set C' which consists of all non-implied customers in C. Then, it is $C \subseteq \mathcal{I}(C', j)$ and set C' is an implied-demand cover. The right-hand side of inequalities (10) and (11) is defined by the preference sets of non-implied customers. Hence, the right-hand sides of both inequalities coincides.

Conversely, consider an inequality (11) based on implied-demand cover C'. Build a solution C for inequalities (10) as follows: Set $C = \mathcal{I}(C', j)$. Obviously, set C is still a cover. Due to the definition of implied customers, it is $\cup_{i \in C} J_{ij}^{\leq} = \cup_{k \in C'} J_{kj}^{\leq}$ and the value on the right-hand side coincides for implied-demand cover C' and cover C. $\square$

We only consider inequalities (10) instead of inequalities (11) since the former inequalities don't demand the computation of implied-demand covers - which is likely to be more complex than the computation of traditional covers. We study the performance of all valid inequalities derived above in Section 7. The exact and heuristic separation methods needed for the computational study are discussed in the subsequent section.

6 Separation

In order to evaluate the impact of the valid inequalities above, we have to solve the corresponding separation problems. The considered valid inequalities are summarised in Table 1. Since inequalities **C**, **EC** and **LC** are already well studied in the literature, we mainly focus on the remaining inequalities. Throughout this section, we assume that $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in [0,1]^{|I||J|+|J|}$ is the current solution to the LP relaxation of (1).

6.1 Implied-demand cover inequalities

In the following, we first provide an integer programming formulation that solves the separation problem for implied-demand cover inequalities. Then, we introduce heuristics for extending an implied demand-cover

Name	Number	Inequality	$I' \subseteq I$ s.t.
C, **IC**	(2), (3)	$\sum_{i \in I'} x_{ij} \leq \lvert I' \rvert - 1$	$\sum_{i \in I'} d_i > Q_j,\ \mathcal{D}(I', j) > Q_j$
EIC	(4)	$\sum_{i \in I' \cup E} x_{ij} \leq \lvert I' \rvert - 1$	$\mathcal{D}(I', j) > Q_j$, $E \subseteq I$ as described above
LIC	(6)	$\sum_{i \in I'} x_{ij} + \sum_{i=1}^{\lvert I \setminus \mathcal{I}(I', j) \rvert} \alpha_{ij} x_{ij} \leq \lvert I' \rvert - 1$	$\mathcal{D}(I', j) > Q_j$, α_{ij} as described above for $i \in I$
RemElem$_{IC}$	(7)	$\sum_{i \in I' \setminus \{l\}} x_{ij} \leq \sum_{k \in J_{lj}^{\leq} \setminus (\cup_{i \in I' \setminus \{l\}} J_{ij}^{<} \cup \{j\})} x_{lk} + (\lvert I' \rvert - 2)$	$\mathcal{D}(I', j) > Q_j, l \in I', \lvert I' \rvert \geq 2$
RemAll$_{IC}$	(8)	$x_{ij} \leq \sum_{l \in I' \setminus \{i\}} \sum_{a \in J_{lj}^{\leq} \setminus (J_{ij}^{<} \cup \{j\})} x_{la}$	$\mathcal{D}(I', j) > Q_j, i \in I'$
RemOBO$_{IC}$	(9)	$x_{ij} \leq \sum_{m=2}^{\lvert I' \rvert} \sum_{a \in J_{\nu(m-1)j}^{\leq} \setminus (\cup_{i'=\nu(m)}^{\nu(\lvert I' \rvert)} J_{i'j}^{<} \cup \{j\})} x_{\nu(m-1)a}$	$\mathcal{D}(I', j) > Q_j, i \in I', \nu \in I^{\lvert I' \rvert}$
ImplFac	(10)	$y_j \leq \sum_{k \in (\cup_{i \in I'} J_{ij}^{\leq}) \setminus \{j\}} y_k$	$\sum_{i \in I'} d_i > Q_j$

Table 1: Valid inequalities for facilities $j \in J$. Classical extended and lifted covers are omitted in this table; in the following, we refer to them as **EC** and **LC** respectively. If inequalities (7), (8) or (9) are constructed from classical covers, we drop index IC in the name.

inequality to an extended and a lifted one, respectively.

Formulation (12) solves the separation problem for implied-demand cover inequalities for each facility $j \in J$. Here, we consider binary decision variables $z_i, u_i \in \{0, 1\}$ such that $z_i = 1$ if customer i is either in the implied-demand cover or implied by an element of it ($z_i = 0$ otherwise) and $u_i = 1$ if customer i is implied by another customer in the implied-demand cover ($u_i = 0$ otherwise). Then, a customer $i \in I$ is in the implied-demand cover if $z_i = 1$ and $u_i = 0$.

$$
\begin{aligned}
\min \quad & \sum_{i \in I} (1 - \bar{x}_{ij}) \cdot (z_i - u_i) & & & & \text{(12a)} \\
\text{s.t.} \quad & \sum_{i \in I} d_i z_i & \geq \quad & Q_j + 1 & & \text{(12b)} \\
& z_i & \leq \quad & z_k & \forall i, k \in I : k \in \mathcal{I}(i, j) & \quad \text{(12c)} \\
& u_k & \leq \quad & \sum_{i \in I : k \in \mathcal{I}(i, j) \setminus \{i\}} (z_i - u_i) & \forall k \in I & \quad \text{(12d)} \\
& z_i, u_i & \in \quad & \{0, 1\} & \forall i \in I & \quad \text{(12e)}
\end{aligned}
$$

Here, inequality (12b) guarantees that the found solution is a cover. Inequalities (12c) ensure that implied demands are considered, i.e., if a customer is considered in the cover, we also consider the demands of all customers implied by said customer. Constraints (12d) ensure that we only consider values of non-implied customers in the objective function. In order to better understand this inequality, consider two cases. First, suppose that a customer $k \in I$ is implied by another customer in the implied-demand cover. Let $i \in I \setminus \{k\}$ be that customer. Then, it is $k \in \mathcal{I}(i, j)$ and $z_i = 1$ and $u_i = 0$. Hence, the right-hand side of inequality (12d) is greater or equal to one. Assigning variable u_k the value of one decreases the objective value and is therefore advantageous. Second, suppose that customer $k \in I$ is not implied by another customer in the implied-demand cover. In this case, there is no customer $i \in I \setminus \{k\}$ with $k \in \mathcal{I}(i, j)$ and $z_i = 1 - u_i = 1$. Therefore, the right-hand side of inequality (12d) is equal to zero for customer k and it immediately follows that variable u_k is also equal to zero. In conclusion, inequalities (12d) ensure that variable u_k is equal to one if customer k is not an element of the implied-demand cover but implied by at least one customer of it; it is equal to zero either if its demand is not considered in the implied-demand cover or if it is considered in the implied-demand cover. Objective function (12a) ensures that the computed cover minimises the right-hand side of reformulated inequalities **IC**.

If the objective value is lower than one, there is a violated implied-demand cover inequality. Otherwise, current LP solution $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ does not violate these inequalities. In order to compute a classical cover inequality, remove inequalities (12d) and set $u_i = 0$ for all customers $i \in I$.

Extended implied-demand cover inequalities Extended implied-demand cover inequalities are separated heuristically using a two-phase approach. For each facility j, we first use formulation (12) to identify a maximally violated implied-demand cover I' and compute the individual contributions of each customer in I' to facility j. This can be done in $\mathcal{O}(|I|^2)$ steps by computing for each customer in I' the set of implied customers, then determining the set of customers only implied by the currently considered customer and finally computing their individual contribution.

In the second phase, we choose the set of customers in the extended cover in a greedy manner. Here, we only consider *customers of interests*, i.e., customers with an individual contribution greater than or equal to the maximum individual contribution in set I'. We consider two rules for deciding the next customer to be added.

1. The customer with smallest individual contribution among all remaining customers of interest is chosen as the next customer to be added in each iteration.

2. The customer with greatest LP-value among all remaining customers of interest is chosen as the next customer to be added in each iteration.

The first approach is similar to the standard procedure used to determine traditional extended covers and coincides with the latter if there are no implications between the customers. This approach has, however, disadvantages if the implied demand and the value $\bar{x}_{ij}$ of the allocation variable of some customer $i \in I$ is very small. Due to the former property, customer i is added to the extended implied-demand cover. Now, customer i might block another customer $k \in I \setminus I'$ with a higher assigned allocation-value from being added. This wrong choice potentially weakens inequalities **EIC**.

The second approach aims to prevent this behaviour. Now, customers with higher assigned allocation-values are preferably added to the extended implied-demand cover. The hope is to increase the left-hand side in inequalities **EIC** by focusing on customers with higher allocation-values. Due to Proposition 3, the assigned allocation value of a customers $l \in I$ who prefers less facilities over the considered facility $j \in J$ is greater than of a customer $i \in I$ who prefers more facilities over the considered facility if $J_{lj}^{\leq} \subseteq J_{ij}^{<} \cup \{j\}$. Hence, the implied demand of customers with greater assigned allocation values is likely to be smaller than the implied demand of customers with lower assigned allocation value and we partially incorporate the first approach.

Once the next customer to be added to the extended implied-demand cover is chosen, we update the set of customers of interest by excluding implied demands of implied customers by customers in the extended set and repeat this procedure until no new customers are found.

Lifted implied-demand cover inequalities We proceed with a heuristic for solving the separation problem of inequalities **LIC** in a similar manner to before. That is, given a LP-solution $(\bar{x}, \bar{y})$ for the relaxation of integer linear program (1), we start by computing a maximum violated implied-demand cover I' at a facility $j \in J$ by solving separation problem (12) for inequalities **IC**. For the remaining elements not in the implied-demand cover - or implied by an element in said cover, i.e., elements in set $I \setminus \mathcal{I}(I', j)$, we compute lifting coefficients. For that, we define a lifting-sequence, denoted with H. Before we define said sequence, we briefly recall the process of computing the corresponding lifting coefficients.

As pointed out in Theorem 3, each element $p \in I \setminus \mathcal{I}(I', j)$ in lifting-sequence H is multiplied by a non-negative factor α_{pj}, which expresses how many elements in set $I' \cup \{1, \ldots, p-1\}$ element p can replace. This value for customer p and facility j can be computed by solving $|I'| - 1 - \max_{C \subseteq I' \cup \{1,\ldots,p-1\}} \{\sum_{k \in C} \alpha_{kj} : \mathcal{D}(C, j) \leq Q_j - r_p(C \cup \{p\}, j),\ (I' \cup \{1, \ldots, p-1\}) \cap \mathcal{I}(p, j) \subseteq C\}$.

Note that the maximisation problem includes the constraint that all customers implied by p have to be considered in set C. Hence, if an element i is in front of an element k with $i \in \mathcal{I}(k,j)$ in sequence H, we have to consider element i in set C when computing the lifting coefficient for k. This forced consideration of customer i might block better solutions to the maximisation problem and there is a chance that this yields a higher lifting coefficient. In conclusion, we derive the relevant lifting sequence based on the size of the set of implied customers at the considered facility for each customer in set $I \setminus \mathcal{I}(I',j)$. Here, the customer who implies most customers is considered as the last element in the lifting sequence; the customer implying the least number of customers is considered as the first element in the sequence.

In order to compute the desired coefficients α_{pj} for a customer $p \in I \setminus \mathcal{I}(I',j)$ in sequence H and a facility $j \in J$, we compute $\max_{C \subseteq I' \cup \{1,\dots,p-1\}} \{\sum_{k \in C} \alpha_{kj} : \mathcal{D}(C,j) \leq Q_j - r_p(C \cup \{p\}, j), (I' \cup \{1,\dots,p-1\}) \cap \mathcal{I}(p,j) \subseteq C\}$ via an integer linear program. For that, introduce binary decision variables $z_i \in \{0,1\}$ indicating whether customer $i \in C \cup \{1,2,\dots,p-1\}$ can be assigned to facility j ($z_i = 1$) or not ($z_i = 0$). Binary decision variables $u_k \in \{0,1\}$ indicate whether customer $k \in \mathcal{I}(C \cup \{1,2,\dots,p-1\}, j)$ is implied by a customer $i \in C \cup \{1,2,\dots,p-1\}$ ($u_k = 1$) or not ($u_k = 0$). Based on lifting-sequence H, we solve the following integer linear program for each element $p \in H$.

$$\max \sum_{i \in I' \cup \{1,2,\dots,p-1\}} \alpha_{ij} z_i \tag{13a}$$

$$\text{s.t.} \sum_{\substack{i \in \mathcal{I}(I' \cup \{1,2,\dots,p-1\},j) \setminus \\ \mathcal{I}(p,j)}} d_i u_i \leq Q_j - \mathcal{D}(p,j) \tag{13b}$$

$$z_i \leq u_k \qquad \forall i \in I' \cup \{1,2,\dots,p-1\}, k \in \mathcal{I}(i,j) \tag{13c}$$

$$z_i = 1 \qquad \forall i \in (I' \cup \{1,2,\dots,p-1\}) \cap \mathcal{I}(p,j) \tag{13d}$$

$$z_i, u_k \in \{0,1\} \qquad \forall i \in I' \cup \{1,2,\dots,p-1\}, k \in \mathcal{I}(I' \cup \{1,2,\dots,p-1\}, j) \tag{13e}$$

Objective function (13a) sums up the weighted value of all decision variables in sequence H that are considered before element p and the goal is to find a solution that maximises this sum. Constraint (13b) combined with constraints (13c) limit the number of variables z_i that can be set to one. That is, they ensure that the joint implied demand of customers from set $(C \cup \{1,2,\dots,p-1\}) \setminus \mathcal{I}(p,j)$ that are considered in the lifted cover does not exceed the remaining capacity of facility j after assigning customer p to j. Constraints (13d) ensure that each customer in set $C \cup \{1,2,\dots,p-1\}$ that is implied by customer p is considered in the lifted cover as well. Given a solution (z^*, u^*) with objective value $c(z^*, u^*)$ to this integer linear program, set coefficient $\alpha_{pj} = |I'| - 1 - c(z^*, u^*)$.

Next, we introduce solution approaches for the separation problems of the strengthened implied-demand cover inequalities.

6.2 Strengthened implied-demand cover inequalities

In the following, we consider methods to solve the separation problems for inequalities $\textbf{RemElem}_{IC}$, $\textbf{RemAll}_{IC}$ as well as $\textbf{RemOBO}_{IC}$ and their versions without implied demands. We start with inequalities $\textbf{RemElem}_{IC}$.

Instead of inequalities (7) for $\textbf{RemElem}_{IC}$, we consider reformulation $1 \leq \sum_{i \in I' \setminus \{l\}} (1 - x_{ij}) + \sum_{k \in J^{\leq}_{lj} \setminus (\cup_{i \in I' \setminus \{l\}} J^{\leq}_{ij} \cup \{j\}} x_{lk}$. The separation problem for finding a (maximum) violated reformulated inequality $\textbf{RemElem}_{IC}$ for a facility $j \in J$ is stated in (14). Here, binary decision variables $z_i \in \{0,1\}$ indicate whether the demand of customer $i \in I$ is considered in the implied demand cover ($z_i = 1$) or not ($z_i = 0$). Binary decision variables $u_i \in \{0,1\}$ state whether customer i is implied by a customer in the cover ($u_i = 1$) or not ($u_i = 0$). We model the decision regarding which customer is to be considered on the right-hand side of inequalities (7) through binary decision variables $\hat{w}_i, w_{ik}, \tilde{w}_k \in \{0,1\}$ for $i \in I$ and $k \in J$. Variable $\hat{w}_i$ indicates whether customer $i \in I$ is considered on the right-hand side ($\hat{w}_i = 1$) or not ($\hat{w}_i = 0$). Variable w_{ik}

indicates whether customer i is considered on the right-hand side and prefers a facility k, that is not strictly preferred by a customer remaining on the left-hand side, over j ($w_{ik} = 1$) or not ($w_{ik} = 0$). Variable $\tilde{w}_k$ indicates whether there is another customer in the cover besides the customer on the right-hand side who strictly prefers facility k over j ($\tilde{w}_k = 1$) or not ($\tilde{w}_k = 0$).

$$
\begin{aligned}
\min \quad & \sum_{i \in I} (1 - \bar{x}_{ij}) \cdot (z_i - u_i - \hat{w}_i) \;+\; \sum_{i \in I} \sum_{k \in J} \bar{x}_{ik} \cdot w_{ik} && && \text{(14a)} \\
\text{s.t.} \quad & \sum_{i \in I} d_i \cdot z_i \geq Q_j + 1 && && \text{(14b)} \\
& z_i \leq z_k && \forall i \in I, k \in \mathcal{I}(i,j) && \text{(14c)} \\
& u_k \leq \sum_{i \in I : k \in \mathcal{I}(i,j) \setminus \{i\}} (z_i - u_i) && \forall k \in I && \text{(14d)} \\
& \sum_{i \in I} \hat{w}_i = 1 && && \text{(14e)} \\
& \hat{w}_i + u_i \leq z_i && \forall i \in I && \text{(14f)} \\
& \hat{w}_i \leq w_{ik} + \tilde{w}_k && \forall i \in I, k \in J^{\leq}_{ij} \setminus \{j\} && \text{(14g)} \\
& \tilde{w}_k \leq \sum_{i \in I : k \in J^{<}_{ij}} (z_i - u_i - \hat{w}_i) && \forall k \in J && \text{(14h)} \\
& 2 \leq \sum_{i \in I} (z_i - u_i) && && \text{(14i)} \\
& z_i, u_i, \hat{w}_i, w_{ik}, \tilde{w}_k \in \{0,1\} && \forall i \in I, k \in J. && \text{(14j)}
\end{aligned}
$$

In the objective function (14a), we add $1 - \bar{x}_{ij}$ to the objective value for each non-implied customer $i \in I$ in computed cover that is on the left-hand side of the inequality. Furthermore, we add value $\bar{x}_{lk}$ for customer $l \in I$ that is considered on the right-hand side of the inequality and all facilities only customer l prefers over j, i.e., facilities $k \in J^{\leq}_{lj} \setminus (\cup_{i \in I' \setminus \{l\}} J^{<}_{ij} \cup \{j\})$. Constraint (14b) combined with constraints (14c) ensure that the implied-demand of the computed cover violates the capacity of facility j. Constraints (14d) indicate whether a customer is a non-implied customer in the cover or an implied customer. Constraint (14e) ensures that exactly one customer is considered on the right-hand side. Constraints (14f) guarantee that a customer can either be implied or considered on the right-hand side or none of both if the corresponding customer is considered in the cover. Inequalities (14g) and (14h) enforce that only assignment values $\bar{x}_{lk}$ of the customer l on the right-hand side to facilities k this customer prefers over j are considered in the sum on the right-hand side of inequality (7) - without those facilities at least one of the remaining customers on the left-hand side strictly prefers over j. Last but not least, constraint (14i) ensure that the implied-demand cover consists of at least two customers.

There is a violated inequality if the objective value is strictly lower than one. When solving the separation problem of inequalities **RemElem** without implied-demands, remove constraints (14c) and (14d) and set $u_i = 0$ for all $i \in I$.

We can formulate the separation problem for finding a maximum violated inequality $\textbf{RemAll}_{IC}$ by reformulating inequalities (8) in a similar manner to inequalities (7). Then, we aim to solve integer linear program (15) for each facility $j \in J$. Here, we consider decision variables $z_i, u_i, \hat{w}_i, w_{ik}, \tilde{w}_k \in \{0,1\}$ for $i \in I$ and $k \in J$. Just like in formulation (14), decision variable $z_i \in \{0,1\}$ indicates whether the demand of customer $i \in I$ is considered in the implied demand cover ($z_i = 1$) or not ($z_i = 0$) and decision variable $u_i \in \{0,1\}$ indicates whether customer $i \in I$ is implied by a customer in the cover ($u_i = 1$) or not ($u_i = 0$). Similar to above, binary decision variable $\hat{w}_i \in \{0,1\}$ indicates whether customer $i \in I$ is considered on the left-hand side ($\hat{w}_i = 1$) or not ($w_i = 0$). Decision variable w_{ik} indicates whether customer $i \in I$ is considered on the right-hand side in inequality $\textbf{RemAll}_{IC}$ and prefers facility $k \in J$ over j while the customer remaining on the left-hand side does not strictly prefer k over j ($w_{ik} = 1$) or not ($w_{ik} = 0$). Decision variable $\tilde{w}_k \in \{0,1\}$

indicates whether facility $k \in J$ is strictly preferred by the customer remaining on the left-hand side of inequality $\mathbf{RemAll}_{IC}$ ($\tilde{w}_k = 1$) or not ($\tilde{w}_k = 0$).

$$
\begin{aligned}
\min \quad & \sum_{i \in I} (1 - \bar{x}_{ij}) \cdot \hat{w}_i && + \sum_{i \in I} \sum_{k \in J^{\leq}_{ij} \setminus \{j\}} \bar{x}_{ik} \cdot w_{ik} && && \text{(15a)} \\
\text{s.t.} \quad & \sum_{i \in I} d_i \cdot z_i && \geq Q_j + 1 && && \text{(15b)} \\
& z_i && \leq z_k && \forall i \in I, k \in \mathcal{I}(i,j) && \text{(15c)} \\
& u_k && \leq \sum_{i \in I : k \in \mathcal{I}(i,j) \setminus \{i\}} (z_i - u_i) && \forall k \in I && \text{(15d)} \\
& \sum_{i \in I} \hat{w}_i && = 1 && && \text{(15e)} \\
& \hat{w}_i + u_i + w_{ik} && \leq z_i && \forall i \in I, k \in J^{\leq}_{ij} \setminus \{j\} && \text{(15f)} \\
& z_i && \leq \hat{w}_i + u_i + w_{ik} + \tilde{w}_k && \forall i \in I, k \in J^{\leq}_{ij} \setminus \{j\} && \text{(15g)} \\
& \tilde{w}_k && \leq \sum_{i \in I : k \in J^{<}_{ij}} \hat{w}_i && \forall k \in J && \text{(15h)} \\
& z_i, u_i, \hat{w}_i, w_{ik}, \tilde{w}_k && \in \{0,1\} && \forall i \in I, k \in J. && \text{(15i)}
\end{aligned}
$$

Constraints (15b) – (15e) coincide with constraints (14b) – (14e) in formulation (14). However, constraint (15e) indicates that a customer stays on the left-hand side in inequality (15) instead of being the only customer pulled to the right-hand side.

In the objective function (15a), we consider value $1 - \bar{x}_{ij}$ of the customer $i \in I$ on the left-hand side of the corresponding valid inequality and add assignment values $\bar{x}_{ik}$ for all customers $i \in I$ considered on the right-hand side and facilities $k \in J^{\leq}_{ij} \setminus \{j\}$, which the customer on the left-hand side not strictly prefers over j. Constraints (15f) ensure that a customer $i \in I$ can be either on the left-hand side of considered valid inequality or can be an implied customer or their assignment to facility $k \in J$ is of interest if their demand is considered in the implied-demand cover or they are not considered in the cover at all. Constraints (15g) enforce that a customer $i \in I$ whose demand is considered in the implied-demand cover is either considered on the left-hand side of the valid inequality or is an implied customer or their assignment to facility $k \in J^{\leq}_{ij}$ is considered or facility k is strictly preferred by the customer on the left-hand side. Constraints (15h) allow the assignment of value one to variable $\tilde{w}_k$ for $k \in J$ if facility k is strictly preferred by the customer considered on the left-hand side of valid inequality $\mathbf{RemAll}_{IC}$.

There is a violated inequality if the objective value is strictly lower than one. When considering the separation problem of inequalities **RemAll** without implied demands, remove constraints (15c) and (15d) and set $u_i = 0$ for all $i \in I$.

Due to the problem's increased complexity, we solve the separation problem for inequalities $\mathbf{RemOBO}_{IC}$ heuristically through a modification of the solution for inequalities $\mathbf{RemAll}_{IC}$. Given a feasible solution $(\bar{x}, \bar{y})$ for the relaxation of formulation (1), the procedure is as follows for each facility $j \in J$.

1. Compute a maximum violated inequality $\mathbf{RemAll}_{IC}$ for a given LP-solution $(\bar{x}, \bar{y})$. Denote the found cover with $I' \subseteq I$. Let $l \in I'$ be the customer on the left-hand side of the inequality.

2. Sort the items on the right-hand side in inequality $\mathbf{RemAll}_{IC}$, i.e., the customers in set $I' \setminus \{l\}$, according to the sum of the assignments to facilities each customer likes less than j in LP-solution $(\bar{x}, \bar{y})$. Sort them in increasing order.

3. In order to construct a valid inequality $\mathbf{RemOBO}_{IC}$, consider the customer with smallest assignment of allocation values in that ordering and denote them with i, that is, the customer where most of their

demand is assigned to facilities they prefer over j. This will be the customer where we only consider the assignment to preferred facilities none of the remaining customers in found cover prefer over j. For the second customer in considered ordering, who we denote with i', we only consider the assignment to preferred facilities none of the customers in $I' \setminus \{i, i'\}$ prefer over j. This continues until all assignments of customers in the order are considered in the objective function.

4. Check whether this newly constructed valid inequality is violated.

If we don't consider implied demands, we solve the separation problem of inequalities **RemAll** in step 1. The remaining steps coincide with the separation for inequalities $\mathbf{RemOBO}_{IC}$.

6.3 Location-centered inequalities

Finally, we study the separation problem of inequalities **ImplFac** based on classical covers. To identify a violated inequality of type (10), we are interested in finding the most violated one, i.e., the inequality that minimises $\sum_{k \in (\cup_{i \in I'} J_{ij}^{\leq}) \setminus \{j\}} \bar{y}_k - \bar{y}_j$ with I' a cover. If this minimum is non-negative, we know that there are no violated inequalities.

In order to compute such a most violated inequality, we solve integer linear program (16) for each facility $j \in J$. For that, introduce variables z_i for each customer $i \in I$, which indicate whether customer i is considered in the cover or the assignment of i is implied by a customer in the cover ($z_i = 1$) or not ($z_i = 0$). Furthermore, introduce variables v_k for each $k \in J$, which indicate if at least customer $i \in I$ with $k \in J_{ij}^{\leq} \setminus \{j\}$ is added to the cover ($v_k = 1$) or not ($v_k = 0$).

$$
\begin{aligned}
\min \quad & \sum_{k \in J \setminus \{j\}} \bar{y}_k v_k & & & & \text{(16a)} \\
\text{s.t.} \quad & \sum_{i \in I} d_i z_i & \geq \quad & Q_j + 1 & & \text{(16b)} \\
& z_i & \leq \quad & v_k & \forall i \in I, k \in J_{ij}^{\leq} \setminus \{j\} \quad & \text{(16c)} \\
& v_k, z_i & \in \quad & \{0, 1\} & \forall i \in I, k \in J \quad & \text{(16d)}
\end{aligned}
$$

Inequality (16b) enforces that the computed set is indeed a cover. Inequalities (16c) ensure that the allocation value of a facility is counted in the objective function if the facility is preferred by at least one customer in the computed cover. Objective function (16a) minimises the total sum of all values assigned to allocation variables in the relaxation of integer linear program (1) which are preferred by customers in the computed cover.

6.4 Complexity of separating cover-based inequalities

In the following, we discuss the complexity of the separation problems for the inequalities introduced in this article. Note that all results proving strong NP-hardness depend on solutions for the relaxation of (1) which are not extreme points. In this work, we do not further explore the complexity for extreme points and leave these problems for future work. We start with the analysis of inequalities **IC**.

Implied-demand cover inequalities Observe first that the CFLP is a special case of the SSCFLPCP in which each customer is indifferent between all available facilities; here, the implied demand of each customer coincides with their own demand. In this case, the separation problem corresponds to the problem of separating classical cover inequalities and is known to be weakly NP-hard (Klabjan et al., 1998). This relation provides a lower bound on the complexity of separating inequalities **IC**. Theorem 1, however, suggests a higher complexity for computing minimum implied-demand covers - note, though, that the separation problem is a special case of computing a minimum implied-demand cover and is not necessarily strongly NP-hard itself. In Proposition 6, we show that there are optimal LP solutions for which the separation problem is strongly NP-hard.

Proposition 6. *Let $j \in J$ be a facility. There exists an optimal solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in [0,1]^{|I||J|+|J|}$ for the linear relaxation of* (1) *for which finding an implied-demand cover $I' \subseteq I$ such that inequality $\sum_{i \in I'} x_{ij} \leq |I'| - 1$ is maximum violated is strongly NP-hard.*

We prove this result via a reduction from *minimum set cover*. For the sake of readability, we refer to the Appendix for the proof of this proposition. Next, we study the computational complexity of deriving an extended implied-demand cover from a given implied-demand cover.

Extended implied-demand cover inequalities When considering traditional covers, it is straightforward to compute the corresponding extended cover, for which the corresponding extended cover inequality is maximally violated: add all items with greater weight than the maximum weight in the cover. This extension appears to be more difficult when considering implied-demand covers. Here, the choice of a customer to be added to the extended cover as well as their position in the sequence, cf. Theorem 2, defines the set of customers considered in an extended cover. Proposition 7 shows that there are solutions for the relaxed formulation of (1) for which the extension of a given implied-demand cover to an extended implied-demand cover, with maximally violated corresponding extended implied-demand cover inequality, is indeed more difficult.

Proposition 7. *Let $I' \subseteq I$ be an implied-demand cover for a facility $j \in J$ and suppose the goal is to find a (maximally) violated extended implied-demand cover inequality based on I'. There exists an optimal solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in [0,1]^{|I||J|+|J|}$ for the linear relaxation of* (1) *for which finding an extended implied-demand cover maximising the total violation is strongly NP-hard.*

We prove this result via a reduction from *exact cover by 3-sets*. The proof is given in the Appendix. Next, we analyse the strengthened implied-demand cover inequalities.

Strengthened implied-demand cover inequalities In the following, we study the complexity of strengthening a given maximally violated (implied-demand) cover to maximally violated inequalities $\mathbf{RemElem}_{IC}$, $\mathbf{RemAll}_{IC}$ and $\mathbf{RemOBO}_{IC}$.

First, consider inequalities $\mathbf{RemElem}_{IC}$ or $\mathbf{RemAll}_{IC}$ and let I' be an implied-demand cover which maximally violates implied-demand cover inequality **IC**. We can easily derive maximally violated inequalities $\mathbf{RemElem}_{IC}$ and $\mathbf{RemAll}_{IC}$ from I'. For the former inequalities, determine the customer $i \in I'$ for which the difference between one minus the assignment value to facility j minus the sum of assignments to facilities they either prefer or are indifferent to regarding j, without facility j or those facilities strictly preferred by at least one customer in $I' \setminus \{i\}$ over j, i.e., the sum of assignments to facilities in set $J_{ij}^{\leq} \setminus (\cup_{l \in I' \setminus \{i\}} J_{lj}^{<} \cup \{j\})$, is greatest. Let this customer be the one considered on the right-hand side of inequality (7). This yields a maximally violated inequality of type $\mathbf{RemElem}_{IC}$ based on an implied-demand cover I' and also holds for inequalities **RemElem**. When computing a maximally violated inequality of type $\mathbf{RemAll}_{IC}$ based on a given implied-demand cover I', determine the customer $i \in I$ for which the difference between the right-hand side according to inequality (8), ignoring facilities strictly preferred by i, minus the assignment value of i to j is smallest. This will be the customer staying on the left-hand side of inequality (8) based on implied-demand cover I'. These considerations also hold for inequalities **RemAll**.

Second, consider inequalities $\mathbf{RemOBO}_{IC}$. If inequalities $\mathbf{RemOBO}_{IC}$ are to be constructed from an implied-demand cover I', the computation of a maximally violated inequality is no longer easy. There are solutions for the linear relaxation of integer linear program (1), which are not extreme points, for which it is strongly NP-hard to determine a maximally violated inequality from a given (implied-demand) cover I'.

Proposition 8. *Let $I' \subseteq I$ be an (implied-demand) cover for a facility $j \in J$. There exists an optimal solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in [0,1]^{|I||J|+|J|}$ for the linear relaxation of* (1) *for which determining the order in which customers in I' are to be considered on the right-hand side of inequality $\boldsymbol{RemOBO_{IC}}$ in order to find a (maximum) violated inequality is strongly NP-hard.*

This result can be shown via a reduction from *minimum set cover*. The proof is given in the Appendix.

Location-centered inequalities Last but not least, we analyse the complexity of solving the separation problem for inequalities **ImplFac**. There are solutions for formulation (16) for which it is strongly NP-hard to determine whether there is an (implied-demand) cover which violates inequality (11).

Proposition 9. *Let $j \in J$ be a facility. There exists an optimal solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in [0,1]^{|I||J|+|J|}$ for the linear relaxation of* (1) *for which finding a maximum violated inequality $y_j \leq \sum_{k \in (\cup_{i \in I'} J_{ij}^{\lesssim}) \setminus \{j\}} y_k$ with $I' \subseteq I$ an (implied-demand) cover is strongly NP-hard.*

We prove this result via a reduction from *3-Sat*. For the sake of readability, we refer to the Appendix for the proof of this proposition.

Given this analysis of the computational complexity for solving the separation problems of the inequalities discussed in Section 5, we proceed with the computational study. Here, we test the computational performance of all approaches derived above.

7 Computational study

In this section, we report on the results of our computational study. Our goal is to analyse the impact of (a) the preprocessing introduced in Section 4 and (b) the valid inequalities introduced in Section 5. This study focuses on the potential of the introduced preprocessing and valid inequalities to reduce the size of the formulation and decrease the integrality gaps. In order to study the impact on the computational performance, further research on heuristics for the occurring separation problems is needed.

7.1 Set-up

In our tests, we consider 20 instances developed for the classical capacitated facility location problem with metric assignment costs (Avella and Boccia, 2009), which can be currently accessed at Università degli Studi di Brescia (2015). Each of the considered instances consists of 300 customers and 300 facilities.

We consider two preference types. In the first type, preferences correspond to the assignment costs and a customer $i \in I$ prefers a facility $j \in J$ over a facility $k \in J$ if the assignment costs of i to j are lower than the costs of assigning i to k. In this study, preferences are determined by underlying metric costs. In the second preference type, we slightly perturb the first preference type. Like before, facilities with small assignment costs are generally preferred over facilities with great assignment costs. Yet, a customer's preference ordering regarding facilities with similar assignment costs might deviate from the preference ordering according to lowest assignment costs. In order to determine the second preference ordering, we follow the procedure introduced by Cánovas et al. (2007). In the former preference type, instances occur in which customers are indifferent between several potential facilities; in the latter preference type, each customer has a strict preference ordering of the potential facilities. Note that feasible instances for the capacitated facility location problem might turn infeasible when considering customer preferences due to more restrictive assignment rules. Among the instances studied here, only instance $i300 - 5$ is infeasible if preferences correspond to closest assignments. We exclude this case from our study. In conclusion, we study 19 instances for the case that preferences correspond to closest assignments and 20 instances when considering perturbed closest assignments as preferences.

All algorithms have been implemented in python 3.10.4 and all experiments have been performed on a Linux machine (Rocky Linux 8.9 Green Obsidian) with CPU clock 2.1 GHz and 3 GB RAM. We use Gurobi version 10.0.0 with default settings to solve integer linear programs if not stated otherwise.

7.2 Preprocessing

We first analyse the impact of the preprocessing methods introduced in Section 4 on the computational performance.

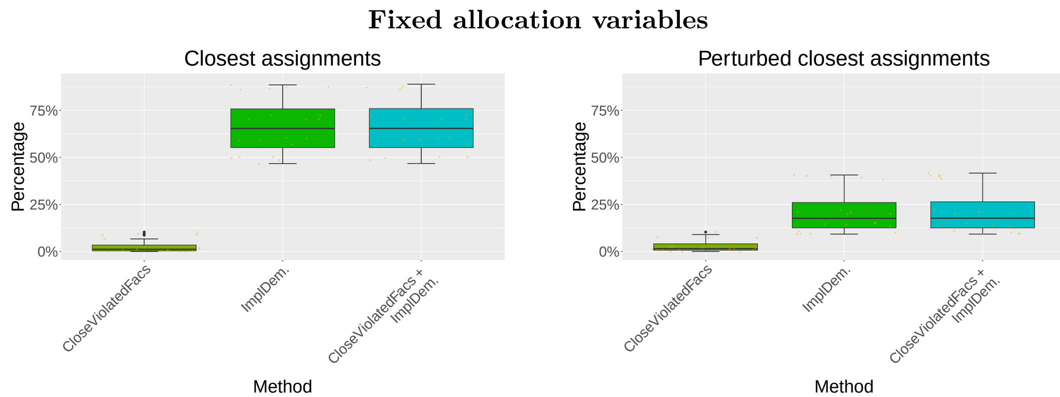

Figure 1: Percentages of allocation variables fixed to zero by methods *CloseViolatedFacs*, *ImplDem* and their combination.

7.2.1 Decision variables

Figure 1 shows the percentage of allocation variables fixed to zero by methods *CloseViolatedFacs* and *ImplDem* as well as their combination. The percentage of fixed location variables by *CloseViolatedFacs* corresponds to the percentage of fixed assignment variables; thus, we only focus on assignment variables.

Method *CloseViolatedFacs* performs weakest for both preference types and the percentage of fixed variables is not visibly affected by the preference type. Method *ImplDem* on its own as well as its combination with method *CloseViolatedFacs* perform similar within the same preference type. If preferences correspond to closest assignments, more than 50% of all allocation variables can be fixed for nearly all instances. If preferences correspond to perturbed closest assignments, the performance decreases visibly and is only slightly better than method *CloseViolatedFacs*.

The similarity between the computational results of method *ImplDem* as well as its combination with method *CloseViolatedFacs* is due to the following. Suppose a facility j has to be closed since it can not serve the demands of all customers that prefer it most. Then, the implied demand of any customer at facility j includes the demand of customers that prefer j most. Hence, no customer can be assigned to j even if j's location variable is not specifically set to zero. This does not make Proposition 2 useless, though: this similarity only holds for decisions made in the first iteration of applying preprocessing *CloseViolatedFacs*. Method *CloseViolatedFacs* is, however, reiterated multiple times since closing one facility changes the set of customers who prefer a facility most for the remaining facilities. This potentially leads to facilities closed in *CloseViolatedFacs* which can not be found by preprocessing *ImplDem*. This impact, however, is not strong in the considered instances. Note that we first apply *CloseViolatedFacs* and afterwards *ImplDem*. By first closing violated facilities, we decrease the number of operations in *ImplDem*.

The drop in the performance of method *ImplDem* and its combination with *CloseViolatedFacs* between the two preference types is likely due to a decrease in the size of implied customer sets when considering perturbed closest assignments. Method *ImplDem* depends highly on these sets, which occur naturally when preferences correspond to metric assignment costs.

In conclusion, method *CloseViolatedFacs* is not affected by the choice of preferences but does not perform strongly in the first place; method *ImplDem* is affected by the choice of preferences and its impact is much stronger when considering preferences corresponding to assignment costs.

7.2.2 Additional constraints

In the following, we analyse the impact of the combined preprocessing methodes introduced by Cánovas et al. (2007) both on their own and after applying methods *CloseViolatedFacs* or *ImplDem*. According to Cánovas et al. (2007), further constraints are added to the integer programming formulation if (a) a

Added constraints based on maximally contained preference sets

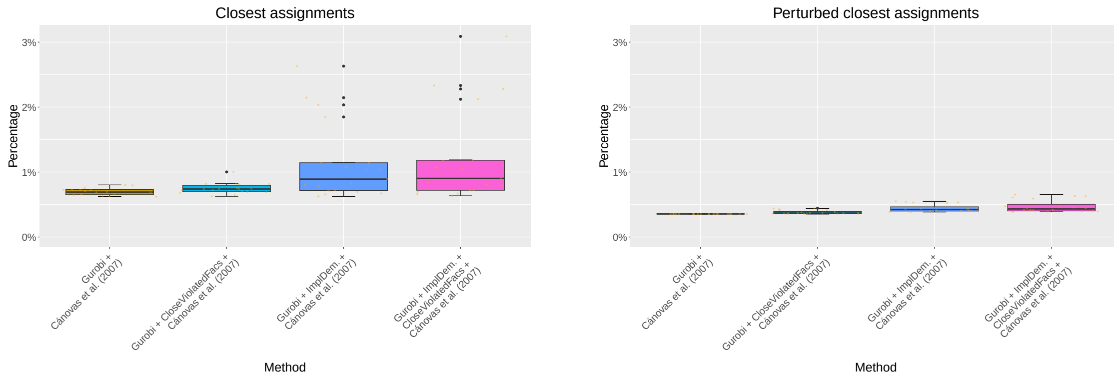

Figure 2: Percentages of added implication constraints derived from maximally contained customers of all potential constraints, cf. Proposition 3.

customer prefers a facility most, cf. Proposition 1; (b) the preference set of a customer $i \in I$ regarding a facility $j \in J$ is *maximally* contained in the preference set of another customer $k \in I \setminus \{i\}$, i.e., there is no third customer $l \in I \setminus \{i, k\}$ with $J_{ij}^{\leq} \setminus \{j\} \subset J_{lj}^{\leq} \setminus \{j\} \subset J_{kj}^{\leq} \setminus \{j\}$, cf. Proposition 3; or (c) if two customers have coinciding preference sets, cf. Proposition 5. Since method (a) does not tell us much about the interaction about the impact, we focus on the analysis of methods (b) and (c).

Figure 2 depicts the percentage of additional constraints due to (b). More specifically, the graphic shows the values resulting from dividing the total number of added constraints by the number of potential constraints in the model before applying (b). We observe a slight increase in the percentage of added constraints when applying Cánovas et al. (2007) after applying *CloseViolatedFacs* or *ImplDem*. This observation is independent of the preference type although it's more visible if preferences correspond to closest assignments.

This indicates that we add more constraints of type (b) to formulation (1) when combining the methods by Cánovas et al. (2007) with the newly derived methods. This behaviour is most likely due to the percentage of location or allocation variables fixed to zero by methods *CloseViolatedFacs* and *ImplDem*: it changes the preference sets of customers due to closed facilities and decreases the number of potentially available relations regarding maximally contained preference sets. Closing some facilities implies that the number of maximally contained preferences might increase; neglecting customers that are never to be assigned to a facility implies a decrease of the denominator for determining the percentages of added constraints.

Figure 3 shows the percentage of additionally considered constraints due to (c). More specifically, the graphic shows the values resulting from dividing the number of added inequalities by the number of potential inequalities in the model before applying (c). This method has no visible advantage when considering perturbed closest assignments. If preferences correspond to closest assignments, applying *CloseViolatedFacs* before applying Cánovas et al. (2007) yields a slight raise in the relative number of added constraints. Applying *ImplDem* first, however, yields to a drop in the relative number of added constraints.

An explanation for the raise in the percentage of considered constraints after applying *CloseViolatedFacs* is that keeping certain facilities closed can turn more preference sets equal; furthermore, closing facilities decreases the denominator when computing the values depicted in the graphic. The decrease of the percentages after applying method *ImplDem* is likely due to neglecting customers whose implied demand can not be served at a considered facility. Especially in the case of preferences corresponding to closest assignments, there seems to be a correlation between the number of implied customers and the ranking of a facility: if a customer prefers a facility less, it is likely that their implied demand at that facility is violating the facility's capacity and the respective assignment variable is fixed to zero. Moreover, if a customer prefers a certain facility less it is likely that their preference set coincides with the preference set of a customer who

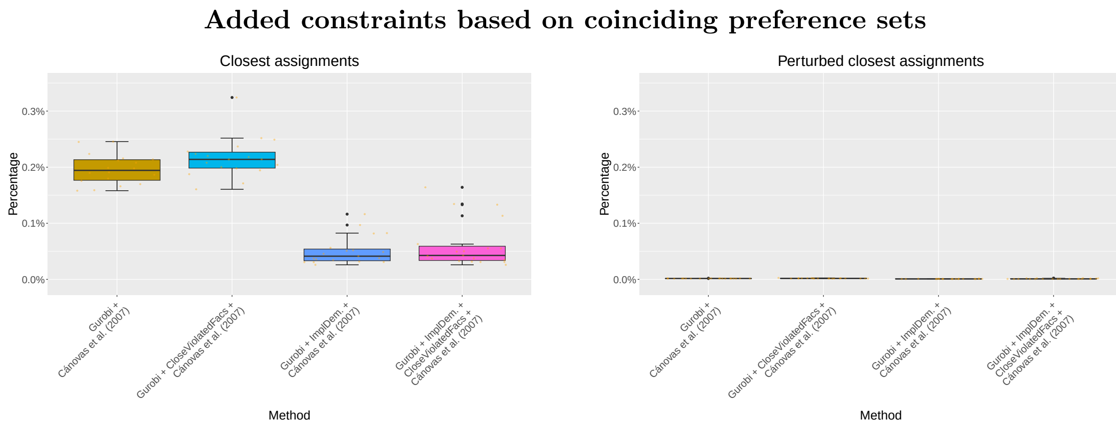

Figure 3: Percentages of added constraints derived from customers sharing the same preference sets regarding a facility $j \in J$ of all potential constraints, cf. Proposition 5.

likes the respective facility equally less.

Lastly, note that Cánovas et al. (2007) also introduce a fourth preprocessing method in which they modify preference constraint (1d) in the integer linear program so that all customers with disjunct sets of facilities they like less than the considered facility are summed up on the left-hand side of the inequality. We restrain from considering this approach for two reasons. First, we don't propose an extension of this method in this work; we only propose extensions for preprocessing methods (a), (b), (c). Hence, the incorporation of this method will not give us any information on the impact of our extensions. Second, it is computationally expensive to compute sets of customers with disjunct sets of facilities they like less than the considered facility. Cánovas et al. (2007) propose a transformation of the problem into an undirected graph with the aim to compute cliques. However, the inequalities introduced in this paper are already difficult to solve and we prioritise them in the available computational time.

We conclude that combining Cánovas et al. (2007) with our newly developed preprocessing methods does not yield a clear advantage if preferences coincide with perturbed closest assignments. If preferences coincide with closest assignments, only *CloseViolatedFacs* yields a relative increase of additionally considered constraints from Cánovas et al. (2007); for the combination with method *ImplDem*, no clear statement is possible. In the following, we discuss the solution process in Gurobi depending on the various preprocessing combinations.

7.2.3 Impact of the preprocessing methods on Gurobi's performance

In the following, we discuss whether the preprocessing methods from Section 4 have a positive impact on the performance when solving the linear programming formulations of the considered instances with Gurobi. We first analyse the time needed to build the model in Gurobi and to perform the preprocessing methods from Section 4 up to the point that Gurobi starts its optimisation, including its own preprocessing; we refer to this whole process as *build-up time*. Afterwards, we compare the optimality gaps of the various approaches achieved by Gurobi within one hour minus the build-up time and determine the best-performing preprocessing methods depending on the considered preference type.

Figure 4 shows the build-up times for the various preprocessing combinations and both preference types. First, note that the build-up times for both preference types are very similar and they take at most 13 minutes for all instances and all methods. The build-up times for method *CloseViolatedFacs* are the fastest; method Cánovas et al. (2007) as well as method *CloseViolatedFacs* combined with Cánovas et al. (2007) need

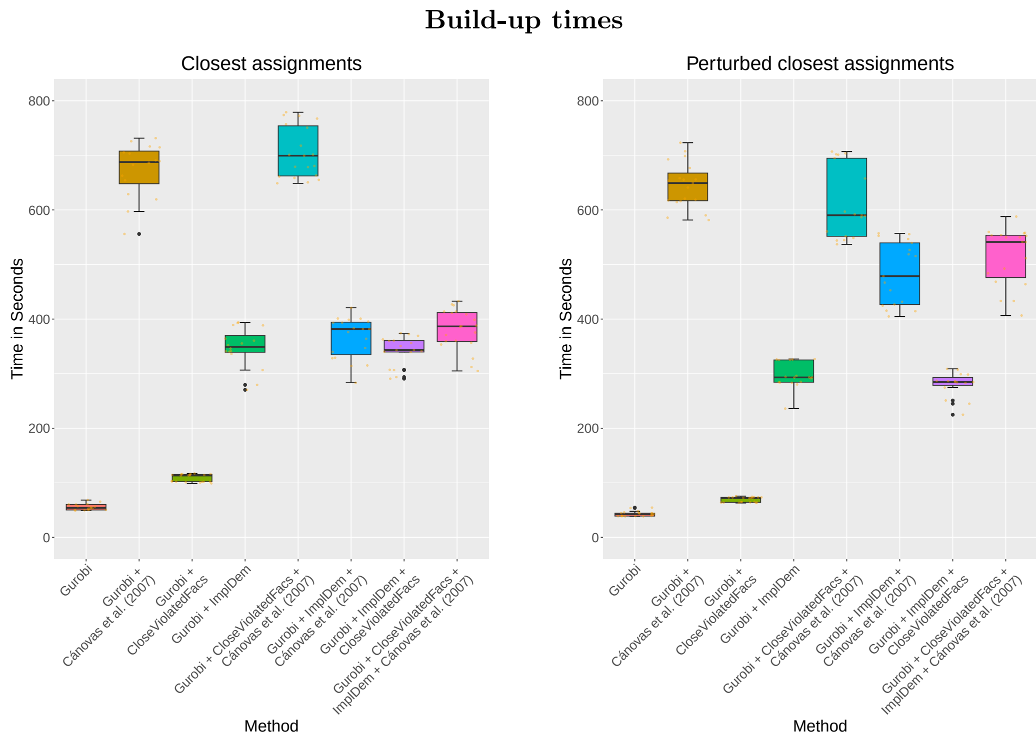

Figure 4: Build-up times for different preprocessing methods and preference types

the largest amount of time. For the case that preferences correspond to closest assignments the remaining preprocessing approaches yield similar build-up times of at most seven minutes; if preferences correspond to perturbed closest assignments, methods *ImplDem* as well as *CloseViolatedFacs* combined with *ImplDem* yield second-best results of at most six minutes. Interestingly, the build-up time for applying methods *CloseViolatedFacs* and *ImplDem* does not increase compared to just applying *ImplDem*.

The short build-up times for methods *CloseViolatedFacs* and *ImplDem* are most likely do to the few operations that have to be performed. The lack in the increase of the build-up times when combining methods *CloseViolatedFacs* and *ImplDem* is likely due to the operations we don't have to perform in the latter method after fixing the location values to zero after performing the former method. The long build-up times for methods Cánovas et al. (2007) and *CloseViolatedFacs* combined with Cánovas et al. (2007) is likely due to the needed comparisons of all customers and all facilities for Cánovas et al. (2007).

Figure 5 depicts the remaining optimality gap in percent achieved by Gurobi after one hour and after performing the various preprocessing combinations. This time includes the corresponding build-up times. Within Gurobi, we set the number of threads to one; besides this, we do not modify any other default parameters.

First, note that all instances are hard to solve by applying plain Gurobi. Here, Gurobi manages preferences corresponding to perturbed closest assignments better than preferences corresponding to closest assignments. Second, note that there is a difference in the performance of the various preprocessing methods depending on the preference type. If preferences correspond to closest assignments, approaches involving method *ImplDem* perform best. The approach combining *ImplDem* with Cánovas et al. (2007), in particular, nearly halves the median achieved by applying plain Gurobi. Plain Gurobi and method *CloseViolatedFacs* perform worst. Approaches involving method Cánovas et al. (2007) yield comparable results to plain Gurobi even though their build-up times are greater than the build-up times of the remaining approaches. If preferences correspond to perturbed closest assignments, all approaches yield similar results and the impact of the

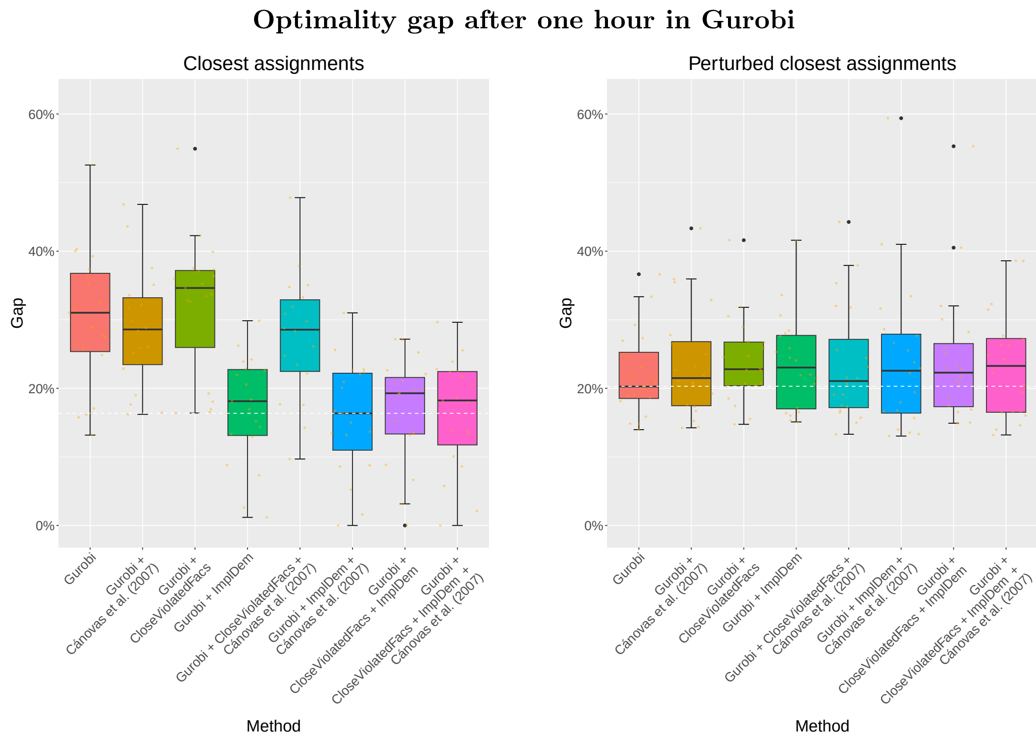

Figure 5: Remaining optimality gaps in percent for different preprocessing methods and preference types. The white dotted line depicts the level of the lowest median amongst all approaches.

methods introduced in Section 4 does not seem to be relevant. Yet, plain Gurobi works better with this preference type compared to the former type. Furthermore, under consideration of the interquartile range and the lowest and highest values, plain Gurobi performs best.

The poor performance of the introduced preprocessing methods and their combinations if preferences correspond to perturbed closest assignments is likely due to the small impact of the preprocessing, which is outweighted by the built-up time needed. There might be two reasons for the smaller impact. First, as mentioned before, a lot of structure disappears when considering perturbed preferences, as seen for method *ImplDem* in Figure 1.

In conclusion, applying *ImplDem* combined with Cánovas et al. (2007) as preprocessing yields the best performance if preferences correspond to closest assignments. Otherwise, none of our approaches yields clear improvement. Hence, for the case of perturbed closest assignments, future work might revolve around new preprocessing ideas. Next, we consider the valid inequalities from Section 5.

7.3 Valid inequalities

In the following, we evaluate the relevance of valid inequalities **(I)C**, **E(I)C**, **L(I)C** as well as inequalities **RemElem**, **RemAll**, **RemOBO** with and without implied-demands and **ImplFac**, cf. Table 1. We consider the performance of each of these inequalities on their own and, for selected inequalities, their combinations. Here, we utilise the findings from the analysis of the preprocessing approaches from before and consider the best-performing preprocessing approaches for each preference type. Hence, if preferences correpond to closest assignments, we consider preprocessing *ImplDem* combined with Cánovas et al. (2007). Otherwise, we do not consider any preprocessing from Section 4.

Closest assignments

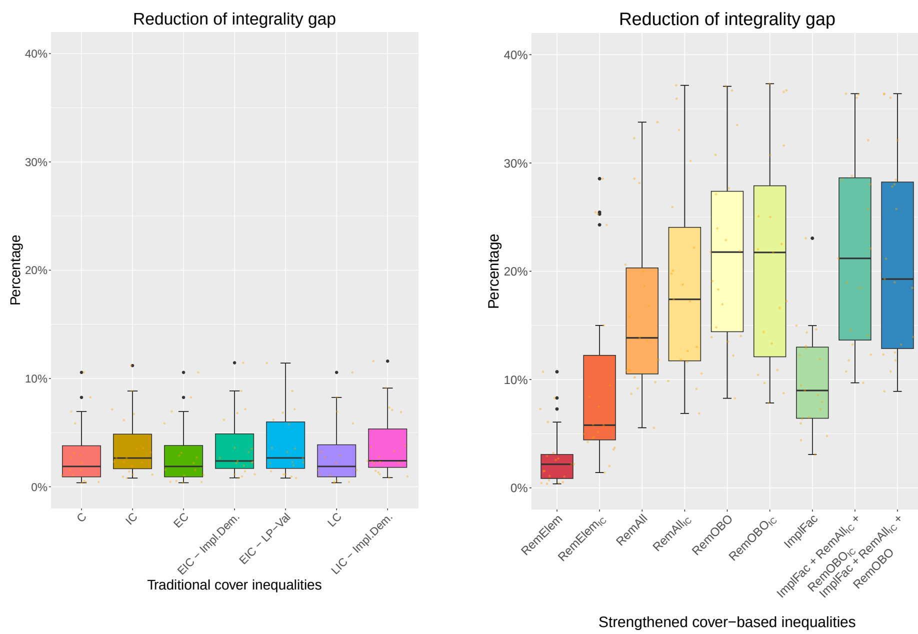

Figure 6: Reduction of the integrality gaps for the valid inequalities discussed in Section 5. Due to lack of memory, no bounds could be computed for instances $i300 - \{10, 14, 18\}$ with methods **C**, **EC** and **LC**.

Before we discuss the results of our computational tests, we briefly describe the setting. We measure the performance of the considered inequalities through the *reduction of the integrality gap*. In order to compute this value, we start by considering a solution for the relaxation of integer linear program (1), which provides a lower bound on the optimal integer solution. We gradually add violated inequalities to integer program (1); these inequalities are found by solving the separation problem for the considered inequality-family. If we don't find further violated inequalities, we compute new lower bounds. This iteration ends when there is no improvement in the computed lower bound anymore. We denote this final lower bound with $x^*_{LP'}$. Denote the best known IP-solution computed in the previous subsection with x^*_{IP} and the lower bound corresponding to the solution of the relaxation of formulation (1) with x^*_{LP}. Then, we measure the reduction of the integrality gap through $r = (1 - (x^*_{IP} - x^*_{LP'})/(x^*_{IP} - x^*_{LP})) \cdot 100$.

To assess the impact of our inequalities without side effects, we deactivate Gurobi's presolve and heuristics. Furthermore, we provide one hour for the process of computing the reduction of the integrality gap and refer to Section 6 for further information on solution approaches for the separation problems to be solved in this section. If we combine inequalities, the ordering of the combination is reflected in the name. Furthermore, we distribute the given time equally among all considered inequalities in the combination. If no more violated inequalities are detected within the given time interval, we distribute the remaining time equally amongst the separation of the remaining inequalities. As an example, suppose four inequalities are combined. Then, we determine and add violated inequalities for the first inequality for 15 minutes. Based on the formulation after 15 minutes, we detect and add violated inequalities for the second inequality for 15 minutes. If no further violated inequalities are found after 10 minutes, each of the remaining two inequalities will be considered for 17.5 minutes.

Perturbed closest assignments

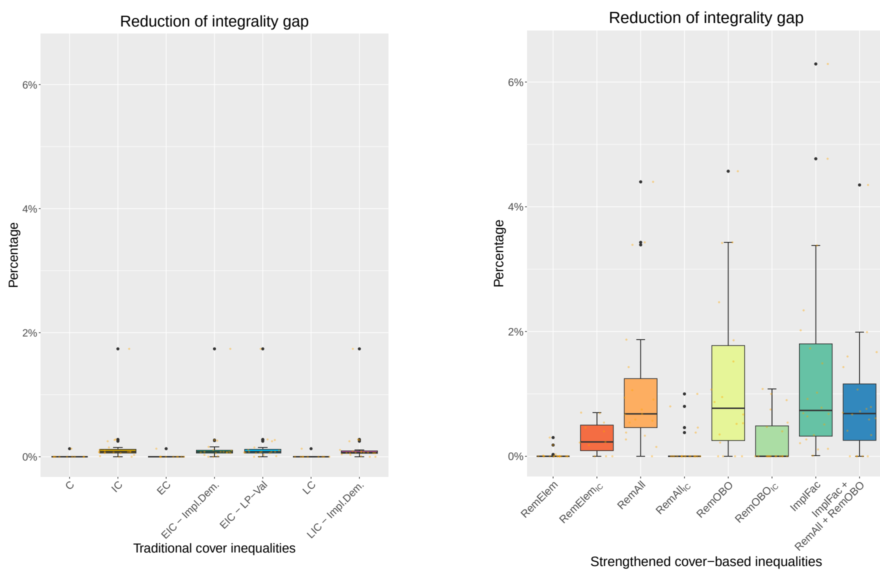

Figure 7: Reduction of the integrality gaps for the valid inequalities discussed in Section 5. Due to timeouts or lack of memory, some bounds could not be computed for some instances for methods **C**, **EC**, **LC** and **EIC - Impl.Dem.**; furthermore, bounds for **RemElem**$_{IC}$ could only be computed for 50% of the considered instances, in the remaining instances, no improvement was done - this is not depicted in the graphic.

7.3.1 Reduction of the integrality gap

Figure 6 depicts the reduction of the integrality gap for the inequalities discussed in Section 5 in case that preferences correspond to closest assignments. The graphic on the left-hand-side shows the results of cover inequalities and their extensions; the graphic on the right-hand-side shows the computational results for the remaining inequalities.

First, note that the traditional cover inequalities do not have a high impact. Moreover, for inequalities **C**, **EC** and **LC** no bounds could be computed for instances $i300-\{10,14,18\}$. On the contrary, strengthened cover-based inequalities **RemOBO** and **RemOBO**$_{IC}$ as well as the combination of **RemOBO**$_{IC}$ with **ImplFac** and **RemAll**$_{IC}$ perform best. Their median is at least seven times higher than the median received from applying traditional cover inequalities. Second, the idea of implied-demand covers yields a visible improvement for nearly all inequalities and none of the inequalities depending on implied-demand covers run into memory problems. Third, note that inequalities **ImplFac**, which do not rely on customer assignments, perform better than cover inequalities and their extensions but worse than nearly all of the strengthened cover inequalities. Lastly, combining the best-performing valid inequalities does not yield clear improvements.

While the classical cover inequalities and their extensions do not seem to perform well, strengthening them yields strong results and their effectiveness in a branch-and-cut approach should be tested in future work. Furthermore, considering implied-demand covers eases the computations and allows for more instances to be solved.

Figure 7 shows the reduction of the integrality gap for the inequalities discussed in Section 5 if preferences correspond to perturbed closest assignments. Like before, the graphic on the left-hand-side corresponds to the cover inequalities and their extensions, the graphic on the right-hand-side corresponds to the strengthened cover-based inequalities.

Closest assignments

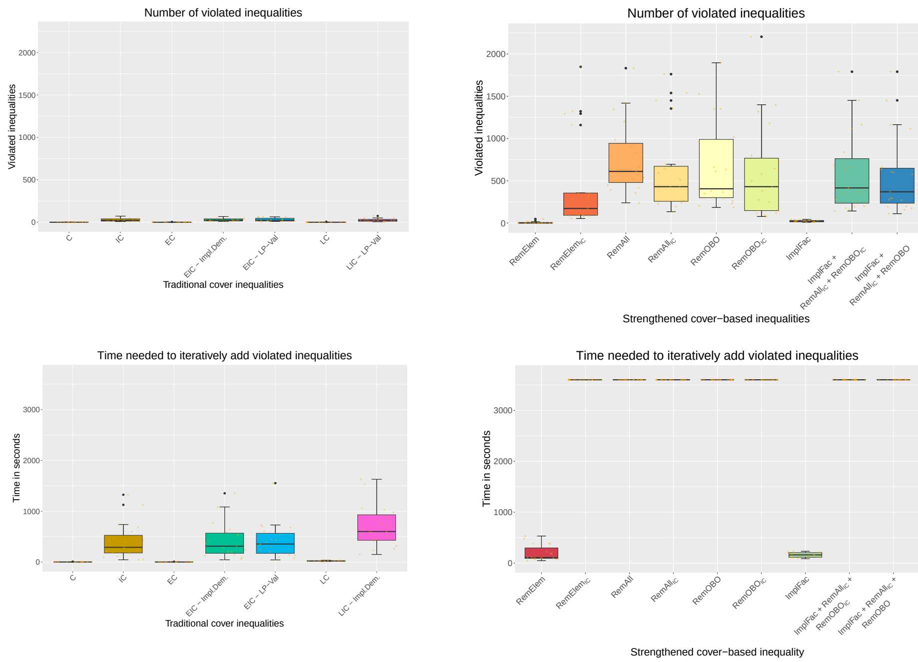

Figure 8: Found violated inequalities and time needed for computing the integrality gap.

First, note that traditional cover inequalities have close to no impact. Furthermore, none of the strengthened cover-based inequalities has a strong impact compared to their performance in the former preference type. Here, inequalities **RemOBO** and **ImplFac** perform best and combining the inequalities does not yield improvements. Second, note that the consideration of implied-demand covers does not offer much improvement. In some cases, considering implied-demand covers worsens the performance. Third, note that not all approaches managed to yield results for all inequalities: due to timeouts or lack of memory, methods **C**, **EC** and **LC** did not yield bounds for instances $i300 - \{1, 2, 4, 7, 8, 9, 10, 15, 18, 19\}$; approach **EIC - Impl.Dem.** was not able to return bounds for instance $i300 - 5$, approach **RemElem**$_{IC}$ did not succeed for instances $i300 - \{1, 2, 3, 4, 5, 7, 8, 9, 10\}$.

These computations underline that the choice of preference constraints has a strong impact on the performance of the inequalities discussed in Section 5. Furthermore, they reveal that the structures utilised by inequalities **ImplFac** are less affected by changes in the preference constraints. Yet, they lead to a higher complexity when considering inequalities **RemElem**$_{IC}$. Lastly, note that while the computational results look dire in case that preferences correspond to perturbed closest assignments, the strengthened cover-inequalities have a positive potential. If we compare their performance to the performance of traditional cover inequalities, we manage to close the integrality gap by a lot. However, as mentioned before, these results emphasise the need for new approaches to deal with general preference constraints.

7.3.2 Found violated inequalities and time needed

Lastly, we take a closer look at the number of detected violated inequalities during the process of computing the reduction of the integrality gap as well as the time needed to detect all violated inequalities. As mentioned before, we let Gurobi compute the reduction of the integrality gap for one hour.

Figure 8 shows the number of violated inequalities found during the computation of the reduction of the

Perturbed closest assignments

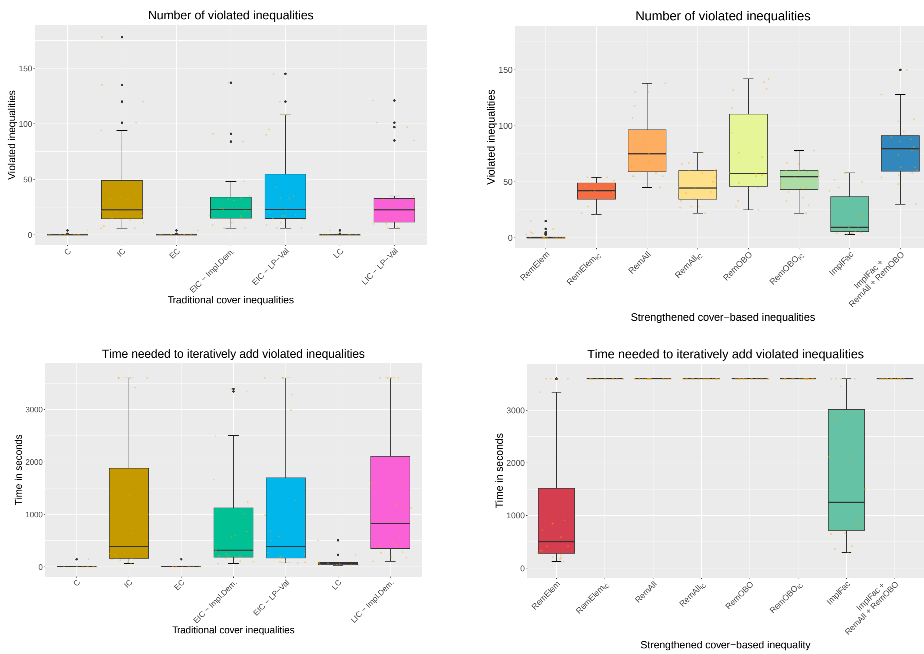

Figure 9: Found violated inequalities and time needed for computing the integrality gap.

integrality gaps as well as the time needed to find these inequalities for the case that preferences correspond to closest assignments. First, note that a lot of violated inequalities are detected for all strengthened cover-based approaches except for **RemElem** and **ImplFac**. In comparison, only a few violated cover inequalities and their extensions are not found. Second, note that the number of detected violated inequalities decreases for most of the strengthened cover-based inequalities when considering implied-demand covers. Third, all strengthened cover-based inequalities except for **RemElem** and **ImplFac** don't finish within the given time while the computations for the traditional cover inequalities and their extensions finish after 2 000 seconds at the latest. Yet, there is a slight raise in the number of violated inequalities detected for strengthened cover-based inequalities. Here, considering implied-demand covers yields a raise in the running time.

The raise in the number of detected strengthened cover-based inequalities, except for **ImplFac**, is responsible for the improved value of the reduction of the integrality gap. Since not all violated inequalities are found within the time limit, these inequalities have further potential. All violated inequalities are found for the remaining inequalities. Here, it is worth mentioning the performance of inequalitis **ImplFac**. While not a lot of violated inequalities **ImplFac** are detected, these inequalities yield a noteworthy increase of the reduction of the integrality gap compared to traditional cover inequalities - for which a similar number of violated inequalities is found. The poor performance of the traditional cover inequalities and their extensions might be due to the presence of the preference constraints. These new constraints do not allow to add just any customer to the cover - conversely to cover inequalities occurring in traditional capacitated facility location problems. Hence, considering preference constraints seem to weaken the impact of cover inequalities.

Figure 9 depicts the number of violated inequalities found during the computation of the reduction of the integrality gaps as well as the time needed to find these inequalities for the case that preferences correspond to perturbed closest assignments. Here, the number of found violated inequalities is more homogeneous across the considered inequalities compared to the previous preference type. The running times for computing the reduction of the integrality gaps behaves similar to the former preference type. Yet, there is a raise in the time needed if preferences correspond to perturbed closest assignments.

The computational results suggest that strengthened cover-based inequalities are stronger than traditional cover inequalities. Since none of the strengthened cover-based inequalities, besides **RemElem** and **ImplFac** detect all violated inequalities within one hour, their potential might be even higher.

In general, future work might work on fast solution methods for solving strengthened cover-based inequalities that did not terminate within one hour. Furthermore, valid inequalities utilising different underlying structures should be explored.

8 Conclusion

In this paper, we study preprocessig methods and valid inequalities for the single-source capacitated facility location problem with customer preferences (SSCFLPCP). While (capacitated) facility location problems are already well studied, less research has been conducted on the SSCFLPCP. Considering customer preferences, however, is important in settings, where customers can choose a facility which serves their demand.

The main contributions in this work are two novel preprocessing methods, which reduce the size of the considered integer programming formulation, the introduction of the concept of implied-demand covers, which generalise normal covers by simultaneously considering preferences of customers and capacities of facilities, a study of the complexity of finding implied-demand covers as well as new, cover-based valid inequalities, which decrease the integrality gaps. More specifically, the first preprocessing method fixes location variables if the corresponding facility is not capable of serving the customers who prefer it most. The second preprocessing method fixes allocation variables if a facility is not able to serve the demand of a customer assigned to it as well as the demands of customers who strictly prefer the same facilities over the considered facility as the assigned customer. We propose and analyse inequalities for the SSCFLPCP based on implied-demand covers, which incorporate information on assignments of customers in the cover that are not assigned to the covered facility. We discuss solution approaches for the separation problems belonging to our new inequalities. All of our new inequalities are also valid, yet weaker, for traditional covers. Finally, we test the computational potential of our findings in a computational study. Here, we discover that the choice of preferences has a great impact on the performance of our preprocessing and valid inequalities. While our approaches work well if preferences coincide with closest assignments, their impact is much smaller if preferences coincide with perturbed closest assignments. Yet, within computations for the latter preference type, our approaches perform better than traditional cover inequalities.

This work reveals that there are a lot of open research questions waiting. First, the complexity study of the separation problems shows that there are optimal solutions to the studied linear relaxation of formulation (1) for which the separation problems are strongly NP-hard. These solutions are no extreme points, though, and state-of-the-art solvers, who compute optimal solutions for the relaxation of (1) via the simplex algorithm, will not find them. Hence, it is yet to be determined to which complexity class the separation problems for extreme points belong. Second, the computational study shows that there is potential in the strengthened cover inequalities. This potential is held back by slow methods for solving the separation problems. Here, it is important to develop faster, well-performing heuristics for solving these problems. This is especially important if the inequalities are to be used in competitive branch-and-cut approaches. Third, we observe that our preprocessing methods and inequalities perform worse when preferences coincide with perturbed closest assignments. Future work might also include the study of preprocessing methods and inequalities which perform better for this preference type.

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Appendix

In this appendix, we give omitted proofs for the complexity results in Section 6. For implied-demand covers, the result is as follows.

Proposition 6. *Let $j \in J$ be a facility. There exists an optimal solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in [0,1]^{|I||J|+|J|}$ for the linear relaxation of* (1) *for which finding an implied-demand cover $I' \subseteq I$ such that inequality $\sum_{i \in I'} x_{ij} \leq |I'| - 1$ is maximum violated is strongly NP-hard.*

Proof. We reduce from the strongly NP-hard *minimum set cover* problem. We consider a minimum set cover instance $\mathcal{I}$ consisting of an universe $U = \{1, 2, \ldots, n\}$ and a collection S of m subsets of universe U and the goal is to decide whether there is a sub-collection of S of size $K \in \mathbb{Z}_{\geq 0}$ whose union equals the universe. W.l.o.g., we assume that collection S does not contain subsets s_1, s_2 with $s_1 \subseteq s_2$ and that each element of the collection S contains at least two elements.

For the reduction, we can transform instance $\mathcal{I}$ into an SSCFLPCP-instance $\mathcal{I}'$ in a similar manner to the proof of Theorem 1. That is, we introduce one customer i for each element $u \in U$ in the universe, one customer k for each element $s \in S$ and denote the set of customers with I. We set the demand of each customer $i \in I$ to one, i.e., it is $d_i = 1$. Define the set of facilities as $J = \cup_{i \in I}\{j_i\} \cup \{j\}$. The set of facilities customer $i \in I$ corresponding to an element $u \in U$ prefers over j is defined as $J_{ij}^{<} = \{j_i\}$ and it is $J_{ij}^{=} = \{j\}$. The set of facilities customer $k \in I$ corresponding to an element $s \in S$ prefers over j is defined as $J_{kj}^{<} = \cup_{e \in s} J_e j^{<} \cup \{j_k\}$ and it is $J_{kj}^{=} = \{j\}$. Set the opening cost of facility j to zero and its capacity to $Q_j = K + |U|$. Furthermore, set the capacities of facilities $a \in J \setminus \{j\}$ to one and their opening costs to $(|U| + |S|)^2$, i.e., it is $Q_a = 1$ and $f_a = (|U| + |S|)^2$. We set the assignment costs of a customer $i \in I$ to all facilities $a \in J_{ij}^{\leq}$ to one and the remaining assignment costs to two. Then, we extended instance $\mathcal{I}'$ to a complete instance of SSCFLPCP. An LP-solution is defined as

$$\bar{y}_k = \begin{cases} 1 & \text{if } k = j \\ (|S| - K)/(|S| + |U|) & \text{otherwise} \end{cases}$$

as well as

$$\bar{x}_{ik} = \begin{cases} (K + |U|)/(|S| + |U|) & \text{if } k = j \\ (|S| - K)/(|S| + |U|) & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}$$

for each customer $i \in I$ and facility $k \in J$ corresponds to a feasible solution for considered SSCFLPCP-instance as we see next.

- The sum over assignment-variables to facilities for each customer is equal to one, i.e., $\sum_{k \in J} \bar{x}_{ik} = (K + |U|)/(|S| + |U|) + (|S| - K)/(|S| + |U|) = 1$ for all customers $i \in I$.
- The capacity of each facility is not violated, i.e., $\sum_{i \in I} \bar{x}_{ij} = \sum_{i=1}^{|S|+|U|} (K + |U|)/(|S| + |U|) = K + |U|$ for facility j and $\sum_{i \in I} \bar{x}_{ia} = (|S| - K)/(|S| + |U|) + \sum_{i \in I \setminus \{a\}} 0 = (|S| - K)/(|S| + |U|) \leq 1 = Q_j$ for facility $a \in J \setminus \{j\}$.
- The preference constraint is met for each customer $i \in I$ and facility $a \in J$. Since facility j is fully open, no customer may be assigned to a facility they like less than j, which is true in the considered solution. Consider a facility $a \in J \setminus \{j\}$ which has LP-value $\bar{y}_a = (|S| - K)/(|S| + |U|)$. Suppose i corresponds to a customer who prefers one facility over j; then, $1 - (|S| - K)/(|S| + |U|) = (K + |U|)/(|S| + |U|)$ of their demand may be assigned to a facility they like less. We can argue in a similar manner that the preference constraint is not violated for customers who prefer more than one facility over j.
- The constraint connecting the value of the assignment variable and the location variable is also met.

Hence, this constructed solution is feasible. The solution is also of minimum cost. Since opening facility j is cheap, it will be open in any feasible solution. However, facility j can only serve a demand of $K + |U|$, thus leaving the demand to be served at different facilities by $|S| - K$. Since all remaining facilities have the same opening costs, opening a few completely or all facilities partially does not have an impact on the objective value. Since all customers have the same travel-distance to facility j or the facilities they prefer over j, their assignment does not have an impact on the assignment costs. In conclusion, this solution is of minimum cost.

Multiplying the cost-parameters of each customer $i \in I$ with factor $1 - (K + |U|)/(|S| + |U|)$ in the proof of Theorem 1 and looking for a solution with cost of at most $K \cdot (1 - (K + |U|)/(|S| + |U|))$ turns instance $\mathcal{I}'$ into an instance of the separation problem for inequalities (3).

Note, however, that this result only implies the existence of a LP-solution for which the separation problem is NP-hard. It is still unknown whether the separation problem is strongly NP-hard if the considered LP solution is an extreme point. $\square$

The complexity of determining an extended implied-demand cover, given a cover, is studied in the following.

Proposition 7. *Let $I' \subseteq I$ be an implied-demand cover for a facility $j \in J$ and suppose the goal is to find a (maximally) violated extended implied-demand cover inequality based on I'. There exists an optimal solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in [0,1]^{|I||J|+|J|}$ for the linear relaxation of* (1) *for which finding an extended implied-demand cover maximising the total violation is strongly NP-hard.*

Proof. Let $(\bar{x}, \bar{y}) \in [0,1]^{|J|+|I|\cdot|J|}$ a solution to the LP relaxation of (1). Suppose we are given a implied-demand cover. In order to find a maximum violated extended implied-demand cover inequality, our goal is to determine a set $E \subseteq I \backslash \mathcal{I}(I', j)$ of customers with maximum sum of relaxed assignment values for which there exists a sequence $(1, \dots |E|)$ of E such that $r_\ell(I' \cup \{1, \dots \ell\}, j) \geq r_{\max}(I', j)$ holds for all $\ell \in \{1, \dots, |E|\}$. For the sake of simplicity, we consider the corresponding decision problem. That is, we want to decide whether there is a set E as described above with maximum sum of relaxed assignment values of at least a certain value.

We prove our claim via a reduction from the NP-complete problem *exact cover by 3-sets* (Garey and Johnson, 1979). Consider an instance $\mathcal{I}$ of exact cover by 3-sets consisting of a universe U with $|U| = 3 \cdot q$, for a $q \in \mathbb{Z}_{>0}$, and a collection S of subsets of universe U with $|s| = 3$ for all $s \in S$. The goal is to decide whether there is a subcollection $S' \subseteq S$ that satisfies the following two conditions. First, the intersection of any two distinct subsets in subcollection S' is empty; second, the union of the elements in S' is equal to set U. W.l.o.g. we assume that collection S does not contain subsets s_1, s_2 with $s_1 \subseteq s_2$.

We transform instance $\mathcal{I}$ into an instance $\mathcal{I}'$ of the separation problem as follows. Introduce one customer for each element in set U as well as one customer for each element in collection S. We denote this set of customers with I. Furthermore, we introduce two extra customers ι, ι', which correspond to implied-demand cover I'. We set the demand of each customer $i \in I$ to $d_i = 1$ and the demand of customers ι, ι' to $d_\iota = d_{\iota'} = 4$. Define the set of facilities as $J = \cup_{i \in I}\{j_i\} \cup \{j_\iota, j_{\iota'}\} \cup \{j\}$. Note that in this setting we are interested in a cover for facility j. Hence, it suffices to define sets $J^{<}_{ij}$ and $J^{=}_{ij}$ for all customers $i \in I$ and customers ι, ι'. We set the set of facilities a customer $i \in I$ is indifferent to $J^{=}_{ij} = \{j\}$ for all customers in set I and set the set of facilities a customer $i \in I$ strictly prefers over j to $J^{<}_{ij} = \{j_i\}$ if customer i corresponds to an element in universe U and set $J^{<}_{ij} = \cup_{e \in s} J^{<}_{ej} \cup \{j_i\}$ if customer i corresponds to set $s \in S$. Note that a customer corresponding to elements $s \in S$ implies the assignment to j of all other customers corresponding to items in set s. For customers ι, ι', define the preference sets as $J^{=}_{\iota j} = J^{=}_{\iota' j} = \{j\}$ and $J^{<}_{\iota j} = \{j_\iota\}$, $J^{<}_{\iota' j} = \{j_{\iota'}\}$. We set the capacity of facility j to $Q_j = 7$ with opening costs of $f_j = 7$ and the capacity of the remaining facilities to $Q_a = 1$ if $a \in J \setminus \{j, j_\iota, j_{\iota'}\}$ with opening costs of $f_a = 1$ and $Q_a = 4$ with opening costs of $f_a = 4$ otherwise.

We set the LP-value regarding the assignment to facility j of each customer $i \in I$ to $x_{ij} = 1/(|U| + |S|)$ and the LP-values of customers ι, ι' to $x_{\iota j} = x_{\iota' j} = 3/4$. It follows immediately facility j serves a total demand of 7. We assign the remaining demand of each customer to facilities they strictly prefer over j. This is indeed possible per construction of the instance. We can construct a full SSCFLPCP-instance in which this solution corresponds to an actual LP solution in a similar manner to the proof of Theorem 9. We omit this step for the sake of simplicity.

Per construction, the maximum contribution of a customer in given implied-demand cover I' is equal to 4. Our goal is to find a subset E of customers in I such that (a) there exists a sequence $(1, \dots, |E|)$ of E such that $r_\ell(I' \cup \{1, \dots, l\}, j) \geq r_{\max}(I', j) = 4$ holds for all $\ell \in \{1, \dots, |E|\}$ and (b) the sum of assignment values of elements in E is above a certain value.

In conclusion, our goal is to prove that there is an exact cover by 3-sets for instance $\mathcal{I}$ of at least q if and only if there is an extended implied-demand cover with set E having a sum of LP-values regarding the assignment to facility j of at least $q/(|U|+|S|)$.

Let S' be a solution for instance $\mathcal{I}$ consisting of at least q sets. We can transform this solution into a feasible solution for instance $\mathcal{I}'$ as follows. Add all customers in set I corresponding to an item in subcollection S' to set E. Then, the sum of the LP-values regarding the assignment to facility j of customers in set E is at least $q/(|U|+|S|)$. Moreover, the individual contribution of each customer in set E is at least four: each of the customers has a demand of one, each of them implies exactly three customers corresponding to elements in universe U and no two customers in set E imply the same customers per construction. Since the individual contribution of customers in set E is independent of one another, we can add them in an arbitrary sequence.

Conversely, let E be a solution for instance $\mathcal{I}'$ with sum of LP-values of at least $q/(|U|+|S|)$. Note that set E contains only customers corresponding to sets in collection S. Otherwise, the individual contribution of the customers would be too small. Hence, add the sets corresponding to the customers in set E to the solution of instance $\mathcal{I}$. It remains to show that the constructed solution covers all customers in universe U and no two sets in the solution share the same item. The latter claim holds true - otherwise, at least one of the corresponding customers in set E would have an individual contribution strictly lower than four; in this case, the customer would not have been considered in set E in the first place. Suppose the solution for instance $\mathcal{I}$ does not cover all elements in universe U. Then, due to the previous argument, at least three elements in universe U are uncovered. Since at least three customers are uncovered and no two sets in the solution share the same customer, we cover at most $3 \cdot (q-1)$ customers with at most $q-1$ sets - since universe U consists of $3 \cdot q$ elements. In this case, set E consists of at most $q-1$ customers and the sum of assignment values adds up to at most $(q-1)/(|U|+|S|)$. This value is strictly lower than the demanded value of $q/(|U|+|S|)$ - a contradiction. Thus, the constructed solution is indeed a feasible solution for instance $\mathcal{I}$ and our claim follows. $\square$

For finding an optimal sequence for pulling elements to the right-hand side in inequalities **RemOBO**, the following result holds.

Proposition 8. *Let $I' \subseteq I$ be an (implied-demand) cover for a facility $j \in J$. There exists an optimal solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in [0,1]^{|I||J|+|J|}$ for the linear relaxation of* (1) *for which determining the order in which customers in I' are to be considered on the right-hand side of inequality **RemOBO$_{IC}$** in order to find a (maximum) violated inequality is strongly NP-hard.*

Proof. We prove our claim via a reduction from *minimum set cover* and consider the corresponding decision problems. Given a minimum set cover instance $\mathcal{I}$ consisting of an universe $U = \{1, 2, \ldots, n\}$ and a collection S of m subsets of universe U, the goal is to decide whether there is a sub-collection of S of size $K \in \mathbb{Z}_{\geq 0}$ whose union equals the universe. W.l.o.g. we assume that collection S does not contain subsets s_1, s_2 with $s_1 \subseteq s_2$ and that each element of collection S contains at least two elements.

We transform $\mathcal{I}$ into an instance $\mathcal{I}'$ of the decision version of the separation problem for inequalities (9). Introduce one customer i for each element $u \in U$ in the universe and one customer k for each element $s \in S$. Denote the set of customers with I and set the demand of each customer $i \in I$ to one, i.e., it is $d_i = 1$. Define the set of facilities as $J = \cup_{i \in I}\{j_i\} \cup \{j\}$. The set of facilities a customer $i \in I$ corresponding to an element $u \in U$ prefers over j is defined as $J_{ij}^{<} = \{j_i\}$ and $J_{ij}^{=} = \{j\}$. The set of facilities a customer $k \in I$ corresponding to a subset $s \in S$ prefers over j is defined as $J_{kj}^{<} = \cup_{e \in s} J_{ej}^{<} \cup \{j_k\}$ and $J_{kj}^{=} = \{j\}$. All customers corresponding to elements in collection S imply the assignment of customers corresponding to universe-items occurring in the considered subset to j and we don't have any implication cycles. Like in the proof of Proposition 6, we set the opening cost of facility j to zero, i.e., $f_j = 0$, and its capacity to $Q_j = |U| + K$. Set the opening costs of facilities $a \in J \setminus \{j\}$ to $f_a = (|U|^2 + |S|^2)$ and the capacities to one, i.e., $C_a = 1$. We set the assignment costs of each customer to facility j as well as to each facility they prefer over j to one. We set the assignment costs to all remaining facilities to two. Then, instance $\mathcal{I}'$ corresponds to a full instance of the SSCFLPCP.

We construct a fractional solution for the LP-relaxation of (1) as follows. We set the values of the assignment variables of customer $i \in I$ to facility j to $\bar{x}_{ij} = (K+|U|)/(|S|+|U|)$ and the sum of values of the assignment variables to a facility in $J_{ij}^{<}$ to $\sum_{a \in J_{ij}^{<}} \bar{x}_{ia} = (|S|-K)/(|S|+|U|)$, i.e., it is $c_i = (|S|-K)/(|S|+|U|)$, and assume that the value of the assignment variable of a customer $k \in I$ corresponding to a subset $s \in S$ to a facility in $J_{kj}^{<} \setminus \{j_k\}$ is equal to zero. Hence, we have

$$\bar{x}_{ia} = \begin{cases} (K+|U|)/(|S|+|U|) & \text{if } a = j \\ (|S|-K)/(|S|+|U|) & \text{if } a = i \\ 0 & \text{otherwise} \end{cases}$$

for each $i \in I, a \in J$. We set the lower bound for the demands to $Q_j + 1 = K + |U|$. In order to extend the solution to a complete LP solution, set the decision variable regarding whether facility j is open to one, i.e., $\bar{y}_j = 1$, and the decision variables regarding whether each facility at least one customer in I strictly prefers over j is open to $(|S|-K)/(|S|+|U|)$. It is not hard to see that the choice of assignment variables satisfies all constraints of IP (1) and minimises the objective function.

The goal is to decide whether there is a set of customers $\bar{I} \subseteq I$ with implied demand of at least $Q_j + 1$ so that the sum of assignments of non-implied customers in $\bar{I}$ to facilities they strictly prefer over j is equal to $K \cdot (|S|-K)/(|S|+|U|)$. First, note that through our choice of values to the assignment-variables, we don't have to take overlapping preference-sets into account when considering customers corresponding to elements in collection S. Second, note that we don't subtract anything in the objective function of instance $\mathcal{I}'$ even though this is part of the separation problem. This, however, is not problematic. We will show that any cover of interest must consist of customers corresponding to elements in collection S. Per construction, each customer corresponding to an element in collection S has the same pattern behind their assignment value to facility j and facilities they strictly prefer over j. Then, it suffices to compute a solution with minimum cost. Afterwards, we pick a random, non-implied customer and subtract their assignment-value to a facility they strictly prefer over j as well as their assignment-value to facility j from the objective value. This yields a value of $(K \cdot |S| - K^2 - |S| - |U|)/(|S|+|U|) \in \mathbb{R}$. If this value is smaller than zero, the implied-cover we found implies a maximum violated inequality (9). Instance $\mathcal{I}'$ can clearly be constructed in polynomial time.

In order to prove our claim, we show that there is indeed a cover of size K for instance $\mathcal{I}$ if and only if there is a solution $\bar{I} \subseteq I$ for instance $\mathcal{I}'$ where the sum of assignment-values of customers in $\bar{I}$ that are not implied by other customers in $\bar{I}$ to facilities they strictly prefer over j corresponds to $K \cdot (|S|-K)/(|S|+|U|)$.

Suppose there is a minimum cover for instance $\mathcal{I}$ consisting of K elements and denote this sub-collection with S'. In order to construct a solution for instance $\mathcal{I}'$, assign all customers corresponding to elements in set $S' \cup U$ to the cover $\bar{I}$. Then, the demand of the customers corresponds to $K + |U|$ on the left-hand side of the capacity constraint, which satisfies the constraint. Due to the construction of instance $\mathcal{I}'$, the customers in set $\bar{I}$ are not implied by other customers correspond to items $s \in S'$. Hence, we have a solution where the sum of assignment-costs of non-implied customers in $\bar{I}$ to a facility they prefer over j corresponds to $K \cdot (|S|-K)/(|S|+|U|)$.

Conversely, suppose set $\bar{I}$ corresponds to a solution for instance $\mathcal{I}'$ with the sum of assignment-costs of non-implied customers in $\bar{I}$ to a facility they prefer over j having value $K \cdot (|S|-K)/(|S|+|U|)$. Observe first that all customers corresponding to an element in universe U are added to the cover. Suppose there is at least one such customer not added to the cover. In order to still satisfy the budget constraint, we have to assign at least $K+1$ customers corresponding to items in collection S. However, this yields an objective value of $(K+1) \cdot (|S|-K)/(|S|+|U|) > K \cdot (|S|-K)/(|S|+|U|)$, a contradiction to our underlying assumption. Second, we observe that all customers corresponding to elements in U are implied by at least one customer in set $\bar{I}$ corresponding to an element in set S each. Suppose at least one customer corresponding to an element in universe U is added to cover $\bar{I}$ and is not implied by another customer in $\bar{I}$. In this case, $K+1$ customers are not implied by another customer and the solution has an objective value of $(K+1) \cdot (|S|-K)/(|S|+|U|) > K \cdot (|S|-K)/(|S|+|U|)$, which is again a contradiction to the solution having objective value $K \cdot (|S|-K)/(|S|+|U|)$. Given these results, we construct a solution for instance $\mathcal{I}$

as follows. Add all sets in collection S corresponding to customers in set $\bar{I}$ to the minimum cover. Thus, the constructed minimum cover has size K and it follows immediately that all elements in U are covered by at least one set in the minimum cover. Hence, the constructed solution is a minimum cover for instance $\mathcal{I}$ and our claim follows. $\square$

For the cover-based inequalities that are not based on cover inequalities, the following result holts.

Proposition 9. *Let $j \in J$ be a facility. There exists an optimal solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in [0,1]^{|I||J|+|J|}$ for the linear relaxation of* (1) *for which finding a maximum violated inequality $y_j \leq \sum_{k \in (\cup_{i \in I'} J^{\leq}_{ij}) \setminus \{j\}} y_k$ with $I' \subseteq I$ an (implied-demand) cover is strongly NP-hard.*

Proof. We prove that the decision version of the separation problem is strongly NP-complete via a reduction from 3-Sat (Garey and Johnson, 1979). The proof is similar to the reduction from 3-Sat to Clique in Sipser (1996). Consider an instance $\mathcal{I}$ of 3-Sat consisting of a set U of variables and a collection C of clauses over U such that clause $c \in C$ consists of three literals. Our goal is to decide whether there is a satisfying truth assignment for C. In this proof, we assume that set C consists of $K > 2$ clauses.

We transform instance $\mathcal{I}$ into an instance of the separation problem for inequalities (11) as follows. Introduce one customer i for any two clause-literal-pairs $\{(c,u),(d,v)\}$ with clauses $c, d \in C$ with $c \neq d$ and literals $u \in c, v \in d$ with u, v are not each other's negations. Denote the set of customers with I. The number of customers lies in $\mathcal{O}(9 \cdot |C|^2)$. We set the demand of each customer to one, i.e., it is $d_i = 1$ for all $i \in I$. We define the set of facilities J as follows. Introduce one facility j. Furthermore, introduce one facility j_{cu} for each clause-literal-pair (c,u) with $c \in C, u \in c$ and one customer-specific facility j_i for each $i \in I$; this yields $\mathcal{O}(|C|^2)$ facilities. We define the preference set of a customer $i \in I$ corresponding to clause-literal pair $\{(c,u),(d,v)\}$ as $J^{=}_{ij} = \{j\}$ and $J^{<}_{ij} = \{j_i, j_{cu}, j_{dv}\}$. Note that no customer is implied by another customer regarding facility j in this instance; hence, this proof also holds for the case where classical covers are considered. Set the capacity of facility j to $K \cdot (K-1)/2$, i.e., $Q_j = K \cdot (K-1)/2$, and the capacity of the remaining facilities $a \in J \setminus \{j\}$ to $Q_a = |\{i \in I : \ a \in J^{<}_{ij}\}|$.

Last but not least we construct a fractional solution for the LP-relaxation of (1). Set the values of the location variables to

$$\bar{y}_a = \begin{cases} 1 & \text{if } a = j, \\ (|I| - Q_j)/(2 \cdot |I|) & \text{if } a \in J \setminus (\cup_{i \in I}\{j_i\} \cup \{j\}), \\ 0 & \text{if } a \in I \end{cases}$$

and the values of the allocation variables to

$$\bar{x}_{ia} = \begin{cases} Q_j/|I| & \text{if } a = j, \\ (|I| - Q_j)/(2 \cdot |I|) & \text{if } a \in J^{<}_{ij} \setminus \{j_i\}, \\ 0 & \text{otherwise.} \end{cases}$$

for each customer $i \in I$. This solution corresponds to a feasible solution with optimum value.

First, observe that the values of all decision variables lie in interval $[0,1]$. This claim is obviously true for the location variables. It also holds for the allocation variables. In order to prove the claim we show that inequality $Q_j \leq |I|$ holds. Given K clauses, there needs to be at least one pair $\{(c,u),(d,v)\}$ for any two clauses $c, d \in C$ with $c \neq d$ with at least one literal $u \in c, v \in d$ in each clause such that u, v are not each other's negations. Otherwise, we can determine in polynomial time that the instance is infeasible. This yields at least $K \cdot (K-1)/2$ customers in set I and it is $\bar{x}_{ik} \in [0,1]$ for any $i \in I, k \in J$.

Second, we show that the constructed LP solution meets all constraints. Each customer is assigned completely to open facilities since $\sum_{a \in J} \bar{x}_{ia} = Q_j/|I| + 2 \cdot (|I| - Q_j)/(2 \cdot |I|) = 1$ holds. Per construction, the capacity of each facility is met. Furthermore, the preference constraints hold. Since facilities each customer $i \in I$ strictly prefers over j is open with $(|I| - Q_j)/(2 \cdot |I|)$, the demand of customer i assigned to j may be at most $1 - (|I| - Q_j)/(2 \cdot |I|) = (|I| + Q_j)/(2 \cdot |I|)$, which is greater than $Q_j/|I|$ due to valid inequality $Q_j \leq |I|$. Hence, this solution meets the preference constraints. Finally, the linking constraints, i.e., $\bar{x}_{ij} \leq \bar{y}_j$ for $i \in I, j \in J$, obviously hold true.

Last but not least, observe that fractional solution $(\bar{x}, \bar{y}) \in [0,1]^{|I|\cdot|J|+|J|}$ is indeed a feasible, cost-minimising solution for instance $\mathcal{I}$. If we set the opening costs so that opening facility j induces a cost of zero and opening one of the facilities $a \in J \setminus \{j\}$ induces a cost of $|I|$ each and define that the distance between any customer $i \in I$ and any facility $k \in J$ is equal to one, solution $(\bar{x}, \bar{y})$ corresponds to a cost-minimising LP solution. Note, though, that this solution is not an extreme point.

It remains to prove that there is a satisfying truth assignment for C in instance $\mathcal{I}$ if and only if there is a subset $I' \subseteq I$ of customers with weight of at least $Q_j + 1$ and value $\sum_{a \in \cup_{i \in I'}} y_a = K \cdot (|I| - Q_j)/(2 \cdot |I|)$ for instance $\mathcal{I}'$.

Suppose C has a satisfying assignment in instance $\mathcal{I}$, i.e., at least one literal is true in each clause. For each clause $c \in C$ consider exactly one arbitrary literal $u \in c$ with value TRUE and add the facility corresponding to this clause-literal-pair (c, u) to the set of facilities strictly preferred by at least one customer in set I' over j. We select one literal with value TRUE per clause; this yields K distinct facilities, which are supposed to be strictly preferred by customers in set I' over j, and the sum of the fractional LP-values for those facilities adds up to $K \cdot (|I| - Q_j)/(2 \cdot |I|)$. Denote this set of facilities with J'. It remains to show that there is a subset $I' \subseteq I$ of customers in instance $\mathcal{I}'$ with $|I'| \geq Q_j + 1$ and $\cup_{a \in \cup_{i \in I'}} J^<_{ij} = J'$. We add a customer for each pair of clause-literal-pairs corresponding to selected facilities in J' to set I'. Such a customer exists because the corresponding clause-literal-pairs do not satisfy one of the exceptions described before: we only add one facility per clause to set J' and the literals in the corresponding clause-literal-pairs don't have contradictory truth-values since we only select among facilities corresponding to a clause-literal-pair where the literal has value TRUE. This yields a total of $K \cdot (K-1)/2$ customers in set I' and the constructed solution satisfies the budget constraint. Furthermore, per construction, we count the LP-value regarding opening those facilities at least one customer in set I' strictly prefers over j. Hence, set I' corresponds to a solution of instance $\mathcal{I}'$.

Conversely, consider a feasible solution for instance $\mathcal{I}'$ consisting of a subset of customers I' with demand of at least $K \cdot (K-1)/2$ and the set of facilities at least one customer in I' strictly prefers over j, denoted with J', consisting of K facilities with objective value $K \cdot (|I| - Q_j)/(2 \cdot |I|)$. Assign TRUE to literal u in clause c if there is a customer in set I' who strictly prefers the facility corresponding to clause-literal-pair (c, u) over facility j. Observe that there is one literal with value TRUE in each clause. Suppose there is a clause containing two literals with value TRUE. Then, there is no customer corresponding to a pair of clause-literal-pairs where the clause is the same in both pairs and we only consider a set I' consisting of at most $K \cdot (K-1)/2 - 1$ customers. This would yield a violation of the budget constraint in instance $\mathcal{I}'$. Thus, each of the K clauses has exactly one literal set to TRUE. We assign the truth-values to the variables in U so that each clause-literal-pair (c, u) corresponding to a facility considered in set J' is made true. Doing so is always possible because there is no customer in set I who strictly prefers two facilities with contradictory literals simultaneously over facility j; hence, such customers do not exist in set I'. This assignment to the variables satisfies a truth-assignment because for each clause there is one clause-literal-pair corresponding to a facility in J' and hence each clause contains a literal that is assigned a TRUE-value. Thus, our claim follows. $\square$