Exact SDP relaxations for a class of quadratic programs with finite and infinite quadratic constraints

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Abstract

We investigate exact semidefinite programming (SDP) relaxations for the problem of minimizing a nonconvex quadratic objective function over a feasible region defined by both finitely and infinitely many nonconvex quadratic inequality constraints (semi-infinite QCQPs). Specifically, we present two sufficient conditions on the feasible region under which the QCQP, with any quadratic objective function over the feasible region, is equivalent to its SDP relaxation. The first condition is an extension of a result recently proposed by the authors (arXiv:2308.05922, to appear in *SIAM J. Optim.*) from finitely constrained quadratic programs to semi-infinite QCQPs. The newly introduced second condition offers a clear geometric characterization of the feasible region for a broad class of QCQPs that are equivalent to their SDP relaxations. Several illustrative examples, including quadratic programs with ball-, parabola-, and hyperbola-based constraints, are also provided.

Key words. Geometric conic optimization problem, finite and semi-infinite quadratically constrained quadratic program, exact semidefinite programming relaxations, ball-, parabola- and hyperbola-based constraints.

AMS Classification. 90C20, 90C22, 90C25, 90C26,

1 Introduction

We introduce a conic optimization problem (COP) for quadratic programs with both finitely and infinitely many nonconvex quadratic inequality constraints. Let $\mathbb{S}^n$ be the linear space of $n \times n$ symmetric matrices with an inner product $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \text{trace}\boldsymbol{A}\boldsymbol{B}$ for every $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{S}^n$, and $\mathbb{S}^n_+$ the cone of $n \times n$ positive semidefinite matrices. For every cone $\mathbb{K} \subset \mathbb{S}^n_+$, $\boldsymbol{Q} \in \mathbb{S}^n$ and $\boldsymbol{H} \in \mathbb{S}^n$, COP$(\mathbb{K}, \boldsymbol{Q}, \boldsymbol{H})$ denotes the problem of minimizing $\langle \boldsymbol{Q}, \boldsymbol{X} \rangle$ subject to $\boldsymbol{X} \in \mathbb{K}$ and $\langle \boldsymbol{H}, \boldsymbol{X} \rangle = 1$, *i.e.*,

$$\eta(\mathbb{K}, \boldsymbol{Q}, \boldsymbol{H}) \;=\; \inf \left\{ \langle \boldsymbol{Q}, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{K},\ \langle \boldsymbol{H}, \boldsymbol{X} \rangle = 1 \right\},$$

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where $\mathbb{K} \subset \mathbb{S}^n_+$ is a cone if $\lambda \boldsymbol{X} \in \mathbb{K}$ holds for every $\boldsymbol{X} \in \mathbb{K}$ and $\lambda \geq 0$. We note that a cone $\mathbb{K}$ is not necessarily convex. When $\boldsymbol{Q}, \boldsymbol{H} \in \mathbb{S}^n$ are unspecified and arbitrary, we denote as COP($\mathbb{K}$) and $\eta(\mathbb{K})$. If COP($\mathbb{K}$) is infeasible, we assume that $\eta(\mathbb{K}) = +\infty$.

Let

$$\begin{aligned} \mathbb{R}^n &= \text{the } n\text{-dimensional Euclidean space of column vectors } \boldsymbol{x} = (x_1, \ldots, x_n), \\ \boldsymbol{\Gamma}^n &= \{\boldsymbol{x}\boldsymbol{x}^T \in \mathbb{S}^n : \boldsymbol{x} \in \mathbb{R}^n\} = \{\boldsymbol{X} \in \mathbb{S}^n_+ : \text{rank}\boldsymbol{X} = 1\}, \end{aligned}$$

where $\boldsymbol{x}^T$ denotes the row vector obtained by transposing $\boldsymbol{x} \in \mathbb{R}^n$. We mention that $\mathbb{S}^n_+$ is described as co$\boldsymbol{\Gamma}^n$ (the convex hull of $\boldsymbol{\Gamma}^n$). For every closed convex cone $\mathbb{J} \subseteq \mathbb{S}^n_+$, $\boldsymbol{Q} \in \mathbb{S}^n$ and $\boldsymbol{H} \in \mathbb{S}^n$, we consider the problems COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}$) and COP($\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}$), which corresponds to a geometric form of quadratically constrained quadratic program (QCQP) of the form:

$$\begin{aligned} \eta(\mathbb{J} \cap \boldsymbol{\Gamma}^n, \boldsymbol{Q}, \boldsymbol{H}) &= \{\langle \boldsymbol{Q}, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{J} \cap \boldsymbol{\Gamma}^n, \ \langle \boldsymbol{H}, \boldsymbol{X} \rangle = 1\} \\ &= \left\{\langle \boldsymbol{Q}, \boldsymbol{x}\boldsymbol{x}^T \rangle : \boldsymbol{x} \in \mathbb{R}^n, \ \boldsymbol{x}\boldsymbol{x}^T \in \mathbb{J}, \ \langle \boldsymbol{H}, \boldsymbol{x}\boldsymbol{x}^T \rangle = 1\right\}, \end{aligned}$$

and to its SDP relaxation, respectively:

$$\eta(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}) = \{\langle \boldsymbol{Q}, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{J}, \ \langle \boldsymbol{H}, \boldsymbol{X} \rangle = 1\}.$$

A standard inequality-form QCQP (2) (see also (3)) will be presented below as a special case of the geometric form-QCQP introduced above. If $\eta(\mathbb{J} \cap \boldsymbol{\Gamma}^n, \boldsymbol{Q}, \boldsymbol{H}) = \eta(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H})$, we say that QCQP COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}$) and its SDP relaxation COP($\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}$) are *equivalent*, or that COP($\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}$) is *an exact SDP relaxation* of QCQP COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}$).

The equivalence of QCQPs and their SDP relaxation has been extensively studied in the literature [1, 2, 5, 10, 11, 12, 20, 21]. The QCQPs studied in the literature can be classified into two categories. The first category involves QCQPs where specific sign patterns of the data matrices are required [3, 12, 18, 22]. The second category focuses on characterizing the equality and inequality constraints of QCQPs that are equivalent to its SDP relaxation for any objective function [1, 2, 10, 17, 19, 20, 21]. This paper falls into the second category, with the main focus on the theoretical characterization of $\mathbb{J}$ such that COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}$) and COP($\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}$) are equivalent for every $\boldsymbol{Q}$ and $\boldsymbol{H}$. We also mention that the recent paper [13] summarizes various sufficient conditions for global optimality in general QCQPs, which include a characterization through their equivalent SDP relaxation.

Regarding the equivalence of QCQP and SDP mentioned above, the following result is known. Let

$$\widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n) = \text{the family of closed convex cones } \mathbb{J} \subset \mathbb{S}^n_+ \text{ such that co}(\mathbb{J} \cap \boldsymbol{\Gamma}^n) = \mathbb{J}.$$

Theorem 1.1. *Let $\boldsymbol{Q}$, $\boldsymbol{H} \in \mathbb{S}^n$ and $\mathbb{J} \subseteq \mathbb{S}^n_+$ is a closed convex cone.*

(i) Assume that $\mathbb{J} \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$. Then

$$-\infty < \eta(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}) \textit{ if and only if } -\infty < \eta(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}) = \eta(\mathbb{J} \cap \boldsymbol{\Gamma}^n, \boldsymbol{Q}, \boldsymbol{H}). \quad (1)$$

(ii) If $\mathbb{J}$ is a face of $\mathbb{S}^n_+$, then $\mathbb{J} \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$.

(iii) $\mathbb{J}' \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$ for every face $\mathbb{J}'$ of $\mathbb{J} \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$.

(iv) Assume that $\boldsymbol{H} \in \mathbb{S}^n$ is positive definite. Then $\mathbb{J} \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$ if and only if $\eta(\mathbb{J} \cap \boldsymbol{\Gamma}^n, \boldsymbol{Q}, \boldsymbol{H}) = \eta(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H})$ for every $\boldsymbol{Q} \in \mathbb{S}^n$.

All assertions in Theorem 1.1 are special cases of more general results presented in [2, 14]. For assertion (i), we refer to [14, Theorem 3.1] and [2, Corollary 2.2]; for (ii) and (iii), to [14, Lemma 2.1]; and for (iv), to [2, Theorem 1.2]. Throughout the paper, we will focus on $\mathbb{J} \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$ as the key property to ensure the equivalence relation (1) between COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}$) and COP($\mathbb{J}$).

Among the assertions of Theorem 1.1, (ii) and (iii) are useful for incorporating equality constraints into COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})$) and its exact SDP relaxation. For linear equality constraints, see Example 4.9 of [2], and for general quadratic equality constraints, refer to Section 4.4 of [2]. Assertion (iv) played an essential role in proving a special case of Theorem 1.2 below in [2].

1.1 Main results

We commonly represent $\mathbb{J}$ using inequalities. For every $\boldsymbol{B} \in \mathbb{S}^n$, let

$$\mathbb{J}_+(\boldsymbol{B}),\ \mathbb{J}_0(\boldsymbol{B}) \text{ or } \mathbb{J}_-(\boldsymbol{B}) = \left\{\boldsymbol{X} \in \mathbb{S}^n_+ : \langle \boldsymbol{B},\, \boldsymbol{X}\rangle \geq,\ = \text{or}\ \leq 0, \quad \text{respectively}\right\}$$

and $\mathbb{J}_+(\mathcal{B}) = \left\{\boldsymbol{X} \in \mathbb{S}^n_+ : \langle \boldsymbol{B},\, \boldsymbol{X}\rangle \geq 0\ (\boldsymbol{B} \in \mathcal{B})\right\}$ for every $\mathcal{B} \subseteq \mathbb{S}^n$. Since $\mathbb{J} \subseteq \mathbb{S}^n_+$ is a closed convex cone, $\mathbb{J}$ can be represented as the intersection of (possibly infinitely many) half spaces and $\mathbb{S}^n_+$ such that $\mathbb{J} = \mathbb{J}_+(\mathcal{B})$ for some $\mathcal{B} \subseteq \mathbb{S}^n$. Then QCQP COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}$) and its SDP relaxation COP($\mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}$) are written as

$$\begin{aligned}
\eta(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}) &= \inf\left\{\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} : \begin{array}{l} \boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x} \geq 0\ (\boldsymbol{B} \in \mathcal{B}), \\ \boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} = 1 \end{array}\right\}, \qquad (2) \\
\eta(\mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}) &= \inf\left\{\langle \boldsymbol{Q},\, \boldsymbol{X}\rangle : \begin{array}{l} \boldsymbol{X} \in \mathbb{S}^n_+,\ \langle \boldsymbol{B},\, \boldsymbol{X}\rangle \geq 0\ (\boldsymbol{B} \in \mathcal{B}), \\ \langle \boldsymbol{H},\, \boldsymbol{X}\rangle = 1 \end{array}\right\}.
\end{aligned}$$

We should mention that there are many choices for such a $\mathcal{B}$. For example, we can take $\mathcal{B} = \mathbb{J}^*$, where $\mathbb{J}^* = \{\boldsymbol{Y} \in \mathbb{S}^n : \langle \boldsymbol{X},\, \boldsymbol{Y}\rangle \geq 0 \text{ for every } \boldsymbol{X} \in \mathbb{J}\}$ (the dual of $\mathbb{J}$). This trivial choice of $\mathcal{B}$, however, involves many redundant matrices to represent $\mathbb{J}$. Without loss of generality, we may assume that

- (A-1) $\mathcal{B}$ is bounded.
- (A-2) $\mathcal{B}$ is closed.

In fact, given $\mathcal{B}' \subseteq \mathbb{S}^n_+$ such that $\mathbb{J} = \mathbb{J}_+(\mathcal{B}')$, we can always replace $\mathcal{B}'$ with $\mathcal{B} = \text{cl}\{\boldsymbol{B}'/ \parallel \boldsymbol{B}' \parallel : \boldsymbol{B}' \in \mathcal{B}, \boldsymbol{B}' \neq \boldsymbol{O}\}$, where $\parallel \boldsymbol{B}' \parallel$ denotes the Frobenius norm of $\boldsymbol{B}' \in \mathbb{S}^n$. Throughout the paper, condition (A-1) is assumed, while condition (A-2) is not, as it contradicts another crucial condition, condition (A-5) described below, which strengthens our main theorem, Theorem 1.2, as shown in Example 6.3.

As a sufficient condition for the equivalence of COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}$) and its SDP relaxation COP($\mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}$), Arima et al. [2] introduced the following condition:

- (B) For every distinct pair $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}$, $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$ holds.

The following theorem extends their result to cases involving infinite $\mathcal{B}$.

Theorem 1.2. *Let $\mathcal{B} \subseteq \mathbb{S}^n$. Assume that condition (B) holds. Then $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$.*

A proof of Theorem 1.2 is given in Section 2. Obviously, whether $\mathbb{J} \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$ or $\mathbb{J} \notin \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$ is independent of how $\mathbb{J}$ is represented by $\mathcal{B} \subseteq \mathbb{S}^n$. Whether condition (B) is satisfied or not, however, depends on $\mathcal{B}$. Thus, when applying Theorem 1.2 to determine the equivalence between COP($\mathbf{\Gamma}^n \cap \mathbb{J}$) and its SDP relaxation COP($\mathbb{J}$) for a given closed convex cone $\mathbb{J} \subseteq \mathbb{S}^n_+$, the choice of representation of $\mathbb{J}$ by $\mathcal{B}$ is critically important.

Given COP($\mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}'$) with a closed convex cone $\mathbb{J}' \subseteq \mathbb{S}^{n'}_+$, $\boldsymbol{Q}' \in \mathbb{S}^{n'}$ and $\boldsymbol{H}' \in \mathbb{S}^{n'}$, we propose a two-step approach:

- first, reduce COP($\mathbf{\Gamma}^{n'} \cap \mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}'$) to an equivalent COP($\mathbf{\Gamma}^n \cap \mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}$) that satisfies Slater's constraint qualification for some $n \leq n'$, closed convex cone $\mathbb{J} \subseteq \mathbb{S}^n_+$, $\boldsymbol{Q} \in \mathbb{S}^n$ and $\boldsymbol{H} \in \mathbb{S}^n$ using the facial reduction technique [4],
- second, represent $\mathbb{J}$ by a less redundant $\mathcal{B} \subseteq \mathbb{S}^n$.

As a result, we obtain COP($\mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}$) (equivalent to COP($\mathbf{\Gamma}^{n'} \cap \mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}'$)) to which Theorem 1.2 can be applied. The resulting $\mathcal{B}$ satisfies

(A-3) $\mathbb{J}_+(\mathcal{B}) \cap \mathbb{S}^n_{++} \neq \emptyset$, where $\mathbb{S}^n_{++}$ denotes the set of $n \times n$ positive definite matrices. (Slater's constraint qualification).

(A-4) Either $\mathcal{B} \cap \mathbb{S}^n_+ = \emptyset$ or $\mathcal{B} = \{\boldsymbol{O}\}$; the latter is a trivial case where $\mathbb{J}_+(\{\boldsymbol{O}\}) = \mathbb{S}^n_+$.

(A-5) $\mathbb{J}_+(\boldsymbol{B}) \not\subseteq \mathbb{J}_+(\boldsymbol{A})$ for every distinct $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}$.

(as shown in Theorem 5.2).

It should be noted that $\mathcal{B}$ may not satisfy condition (A-2) in general as illustrated in Example 6.3. More details will be discussed in Section 5. In the subsequent discussion, given a $\mathcal{B}' \subseteq \mathbb{S}^{n'}$ for some n', we simply say that $\mathcal{B}'$ satisfies conditions (A-1), (A-2), (A-3), (A-4), (A-5) and/or (B) if $\mathcal{B} = \mathcal{B}'$ satisfies these conditions. This convention also applies to conditions (B)' and (C)' introduced below.

While conditions (A-3), (A-4) and (A-5) are not necessary for applying Theorem 1.2, the above reduction process not only improves the effectiveness of condition (B) but also simplifies and enriches its application. By definition, $\mathcal{B}'$ satisfies conditions (A-3), (A-4), (A-5) and/or (B) if and only if every nonempty subset $\mathcal{B}$ of $\mathcal{B}'$ does. However, it is possible for $\mathbb{J}_+(\mathcal{B}') \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$ to hold even if $\mathbb{J}_+(\mathcal{B}' \backslash \{\boldsymbol{A}\}) \notin \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$ for some $\boldsymbol{A} \in \mathcal{B}'$. Specifically, this often occurs when $\mathbb{J}_+(\mathcal{B}')$ is contained in a proper face of $\mathbb{S}^n_+$. To see this, suppose that $\mathcal{B}$ is an arbitrary nonempty subset of $\mathbb{S}^n$ such that $\mathbb{J}_+(\mathcal{B}) \notin \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$ and $\mathbb{F}$ is a nonempty face of $\mathbb{S}^n_+$ contained in $\mathbb{J}_+(\mathcal{B})$; at least one such face always exists since $\{\boldsymbol{O}\}$ is a face of $\mathbb{J}_+(\mathcal{B})$. Represent the face $\mathbb{F}$ such that $\mathbb{F} = \{\boldsymbol{X} \in \mathbb{S}^n_+ : \langle \boldsymbol{A}, \boldsymbol{X} \rangle \geq 0\}$ for some negative semidefinite matrix $\boldsymbol{A} \in \mathbb{S}^n$. Let $\mathcal{B}' = \mathcal{B} \cup \{\boldsymbol{A}\}$. Then $\mathbb{J}_+(\mathcal{B}') = \mathbb{J}_+(\mathcal{B}) \cap \mathbb{F} = \mathbb{F}$. By Theorem 1.1 (ii), we know that $\mathbb{J}_+(\mathcal{B}') \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$. In this case, however, the application of Theorem 1.2 to $\mathcal{B}'$ would fail to determine whether $\mathbb{J}_+(\mathcal{B}') \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$ because $\mathcal{B} = \mathcal{B}' \backslash \{\boldsymbol{F}\}$ does not satisfy condition (B). A more non-trivial case justifying the reduction mentioned above from COP($\mathbf{\Gamma}^{n'} \cap \mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}'$)) to COP($\mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}$) will be presented in Example 6.1. Condition (A-4) is natural because if $\boldsymbol{A} \in \mathcal{B} \cap \mathbb{S}^n_+$ existed, then $\boldsymbol{A}$ would be redundant as $\mathbb{J}_+(\mathcal{B}) = \mathbb{J}_+(\mathcal{B} \backslash \{\boldsymbol{A}\})$. Also if $\mathbb{J}_+(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$ for some distinct $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}$, then $\boldsymbol{A}$ would be redundant. Hence condition (A-5) is also reasonable. Some fundamental properties on $\mathcal{B}$ satisfying conditions (A-3), (A-4), (A-5) and/or (B) will be discussed in Section 3. In particular, Theorem 3.4 provides algebraic criteria equivalent to $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$ in condition (B) under conditions (A-3), (A-4) and (A-5).

Condition (B) is defined in $\mathbb{S}^n$, the space of the variable matrix $\boldsymbol{X}$ of $\text{COP}(\mathbb{J}_+(\mathcal{B}))$. For practical applications, It is more convenient to provide a direct characterization of $\text{COP}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$ which involves $\mathcal{B}$ satisfying condition (B). We examine the special case where $\boldsymbol{H} \in \mathbb{S}^n$ is an $n \times n$ matrix with 1 in the (n,n)th element and 0 elsewhere. To adapt condition (B) to this special case, we introduce some notation and additional conditions. Define

$$q(\boldsymbol{u}, z, \boldsymbol{Q}) \;=\; \begin{pmatrix}\boldsymbol{u}\\ z\end{pmatrix}^T \boldsymbol{Q} \begin{pmatrix}\boldsymbol{u}\\ z\end{pmatrix} \text{ for every } (\boldsymbol{u}, z, \boldsymbol{Q}) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{S}^n.$$

If $\boldsymbol{Q} = \begin{pmatrix}\boldsymbol{C} & \boldsymbol{c}^T\\ \boldsymbol{c} & \gamma\end{pmatrix}$ is denoted with $\boldsymbol{C} \in \mathbb{S}^{n-1}$, $\boldsymbol{c} \in \mathbb{R}^{n-1}$ and $\gamma \in \mathbb{R}$, then $q(\boldsymbol{u}, 1, \boldsymbol{Q})$ is a quadratic function of the form $\boldsymbol{u}^T\boldsymbol{C}\boldsymbol{u} + 2\boldsymbol{c}^T\boldsymbol{u} + \gamma$ in $\boldsymbol{u} \in \mathbb{R}^{n-1}$. Thus, $\text{COP}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$ corresponds to the standard inequality form of (semi-infinite) QCQP:

$$\eta(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}) \;=\; \inf\left\{q(\boldsymbol{u}, 1, \boldsymbol{Q}) : \boldsymbol{u} \in \mathbb{R}^{n-1},\ q(\boldsymbol{u}, 1, \boldsymbol{B}) \geq 0 \ (\boldsymbol{B} \in \mathcal{B})\right\}. \quad (3)$$

For every $z \in [0, 1]$, $\boldsymbol{B} \in \mathbb{S}^n$ and $\mathcal{B} \subseteq \mathbb{S}^n$, define

$$\begin{aligned}
&\mathbb{F}_+(z, \boldsymbol{B}),\ \mathbb{F}_0(z, \boldsymbol{B}),\ \mathbb{F}_-(z, \boldsymbol{B}) \text{ or } \mathbb{F}_{--}(z, \boldsymbol{B}) \\
&\qquad = \left\{\boldsymbol{u} \in \mathbb{R}^{n-1} : q(\boldsymbol{u}, z, \boldsymbol{B}) \geq 0,\ = 0,\ \leq 0 \text{ or } < 0, \text{ respectively}\right\}, \\
&\mathbb{F}_+(z, \mathcal{B}) = \bigcap_{\boldsymbol{B} \in \mathcal{B}} \mathbb{F}_+(z, \boldsymbol{B}) = \left\{\boldsymbol{u} \in \mathbb{R}^{n-1} : q(\boldsymbol{u}, z, \boldsymbol{B}) \geq 0 \ (\boldsymbol{B} \in \mathcal{B})\right\}.
\end{aligned}$$

We consider the following conditions on the feasible region of QCQP (3), which can be written as $\mathbb{F}_+(1, \mathcal{B})$.

(B)’ $\mathbb{F}_-(1, \boldsymbol{B}) \subseteq \mathbb{F}_+(1, \boldsymbol{A})$ (or equivalently $\mathbb{F}_-(1, \boldsymbol{B}) \cap \mathbb{F}_{--}(1, \boldsymbol{A}) = \emptyset$) for every distinct $\boldsymbol{A}$, $\boldsymbol{B} \in \mathcal{B}$.

(C)’ For every $\boldsymbol{B} \in \mathcal{B}$, $\mathbb{F}_{--}(1, \boldsymbol{B}) \neq \emptyset$.

Theorem 1.3. *Assume conditions (B)’ and (C)’. Then* $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$.

A proof of Theorem 1.3 is given in Section 4. We note that condition (C)’ is fairly reasonable because if it is not satisfied for some $\boldsymbol{B} \in \mathcal{B}$, then $\mathbb{F}_+(1, \boldsymbol{B}) = \{\boldsymbol{u} \in \mathbb{R}^{n-1} : q(\boldsymbol{u}, 1, \boldsymbol{B}) \geq 0\} = \mathbb{R}^{n-1}$; hence $q(\boldsymbol{u}, 1, \boldsymbol{B}) \geq 0$ is a redundant constraint. The geometry of conditions (B)’ and (C)’ is particularly clear when $n - 1 = 2$. Figure 1 illustrates 6 cases where $\boldsymbol{B} \in \mathbb{S}^3$ satisfies $\mathbb{F}_{--}(1, \boldsymbol{B}) \neq \emptyset$. We can combine some of them for $\mathcal{B}$ satisfying conditions (B)’ and (C)’. In particular, we see that $\{\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3\}$, $\{\boldsymbol{B}_1, \boldsymbol{B}_6\}$, $\{\boldsymbol{B}_1, \boldsymbol{B}_3, \boldsymbol{B}_5\}$ and $\{\boldsymbol{B}_2, \boldsymbol{B}_4\}$ satisfy both conditions. We also note that $\mathbb{F}_+(1, \{\boldsymbol{B}_1, \boldsymbol{B}_6\})$ forms a unit disk with a hole if $0 < r < 1$ and a unit circle if $r = 1$. Figure 2 illustrates another example of $\mathcal{B}$ that satisfies conditions (B)’ and (C)’.

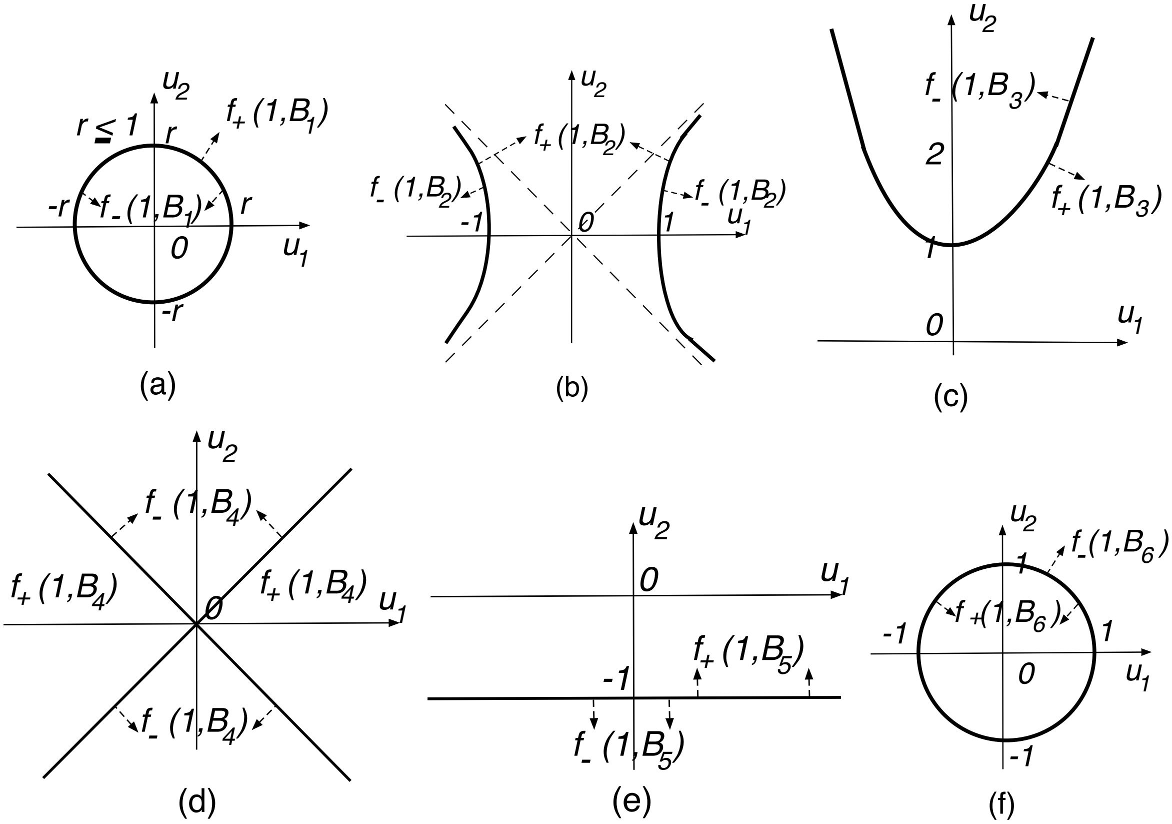

Figure 1: Illustration of $\mathbb{F}_+(1, \boldsymbol{B})$ and $\mathbb{F}_-(1, \boldsymbol{B})$. (a) $q(\boldsymbol{u}, z, \boldsymbol{B}_1) = u_1^2 + u_2^2 - rz^2$, where $0 < r \leq 1$. (b) $q(\boldsymbol{u}, z, \boldsymbol{B}_2) = -u_1^2 + u_2^2 + z^2$. (c) $q(\boldsymbol{u}, z, \boldsymbol{B}_3) = u_1^2 - u_2 z + z^2$. (d) $q(\boldsymbol{u}, z, \boldsymbol{B}_4) = u_1^2 - u_2^2$. (e) $q(\boldsymbol{u}, z, \boldsymbol{B}_5) = u_1 z + z^2$. (f) $q(\boldsymbol{u}, z, \boldsymbol{B}_6) = -u_1^2 - u_2^2 + z^2$. $\mathbb{F}_+(1, \boldsymbol{B}_k) = \{\boldsymbol{u} \in \mathbb{R}^2 : q(\boldsymbol{u}, 1, \boldsymbol{B}_k) \geq 0\}$ $(k = 1, 2, 3, 4, 5, 6)$.

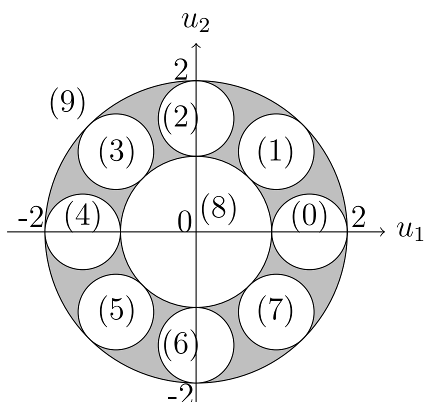

Figure 2: An example $\mathcal{B} = \{\boldsymbol{B}_k : k = 0, 1, \ldots, 9\}$ that satisfies conditions (B)' and (C)'. $\mathbb{F}_-(1, \boldsymbol{B}_k)$ represents the disk (k) with center $1.5(\cos(k\pi/4), \sin(k\pi/4))$ and radius 0.5 $(k = 0, 1, \ldots, 7)$. $\mathbb{F}_-(1, \boldsymbol{B}_8)$ is depicted as the disk (8) with center $(0, 0)$ and radius 1. $\mathbb{F}_-(1, \boldsymbol{B}_9)$ represents the exterior (9) (including the boundary) of the disk with center $(0, 0)$ and radius 2. The gray region denotes the feasible region $\mathbb{F}_+(1, \mathcal{B})$. We see that conditions (B)' and (C)' are satisfied.

1.2 Main contribution and comparison to some existing results

By presenting Theorem 1.2, which extends the results from the finite $\mathcal{B}$ case to the infinite $\mathcal{B}$ case, we demonstrate for the first time that a broad class of semi-infinite QCQPs is equivalent to their SDP relaxation. This result is achieved by extending the authors' previous work [14, 2]. Specifically, we show that quadratic programs with ball-, parabola-, and hyperbola-based constraints can be solved exactly using their SDP relaxation, with illustrative examples.

Each $\mathbb{J} \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$ is called *rank-one generated (ROG)* in the literature [1, 6]. Specifically, Proposition 1 in [1] is related to Theorem 1.2. The detailed comparison between these two results are provided in Section 3, where we show that Theorem 1.2 together with the proposed reduction is more general than [1, Proposition 1]. Example 6.1 illustrates a case addressed by Theorem 1.2 with the reduction but not covered by [1, Proposition 1].

Another key contribution is Theorem 1.3, which provides a direct and geometric characterization of the feasible region for a wide class of semi-infinite QCQPs that can be reformulated as their SDP relaxations. This characterization is straightforward and clear, and enhances the reader's understanding of this class of QCQPs.

1.3 Outline of the paper

In Section 2, we present a proof of Theorem 1.2. Section 3 is devoted to fundamental properties and implications of conditions (A-3), (A-4), (A-5) and (B). Proposition 1 in [1] referred above is restated as Theorem 3.5, and then its detailed comparison with Theorem 1.2 is described. Section 4 presents the proof of Theorem 1.3. The reduction of a general COP($\mathbf{\Gamma}^{n'} \cap \mathbb{J}'$) to a COP($\mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})$) with $\mathcal{B}$ satisfying conditions (A-3), (A-4) and (A-5) is described in Section 5. In Section 6, we present five illustrative examples. Finally, Section 7 contains our concluding remarks.

2 Proof of Theorem 1.2

We present three lemmas for the proof of Theorem 1.2. The first lemma is also used in Section 4 to prove Theorem 1.3.

Lemma 2.1. *Let* $\{\mathcal{B}_k \subseteq \mathbb{S}^n : k = 1, 2, \ldots\}$ *be a sequence such that* $\bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}_k) = \mathbb{J}$ *for some closed convex cone* $\mathbb{J} \subseteq \mathbb{S}^n_+$. *Then* $\bigcap_{m=1}^{\infty} co\big(\mathbf{\Gamma}^n \cap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k)\big)\big) = co\big(\mathbf{\Gamma}^n \cap \mathbb{J}\big)$.

Proof. Since $\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k) \supseteq \mathbb{J}$ $(m = 1, 2, \ldots)$, $\bigcap_{m=1}^{\infty} \mathrm{co}\big(\mathbf{\Gamma}^n \cap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k)\big)\big) \supseteq \mathrm{co}\big(\mathbf{\Gamma}^n \cap \mathbb{J}\big)$ follows. To prove the converse inclusion, let $\overline{\boldsymbol{X}} \in \mathrm{co}\big(\mathbf{\Gamma}^n \bigcap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k)\big)\big)$ $(m = 1, 2, \ldots)$. Then, for each $m = 1, 2, \ldots$, there exist $\boldsymbol{X}^p_m \in \mathbf{\Gamma}^n \bigcap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k)\big)$ $(p = 1, 2, \ldots, \ell)$ for some $\ell \leq \dim \mathbb{S}^n = n(n-1)/2$ such that $\overline{\boldsymbol{X}} = \sum_{p=1}^{\ell} \boldsymbol{X}^p_m$. Let $q \in \{1, 2, \ldots, \ell\}$ be fixed arbitrarily. Consider the sequence $\{\boldsymbol{X}^q_m : m = 1, 2, \ldots, \}$. The sequence is bounded since $\overline{\boldsymbol{X}} \in \mathbb{S}^n_+$, $\boldsymbol{X}^p_m \in \mathbb{S}^n_+$ $(p = 1, \ldots, \ell)$ and $\langle \boldsymbol{I}, \overline{\boldsymbol{X}} \rangle = \langle \boldsymbol{I}, \sum_{p=1}^{\ell} \boldsymbol{X}^p_m \rangle \geq \langle \boldsymbol{I}, \boldsymbol{X}^q_m \rangle$. Hence we may assume without loss of generality that the sequence converges to some $\overline{\boldsymbol{X}}^q$ as $m \to \infty$. Then $\overline{\boldsymbol{X}}^q \in \mathrm{cl}\big(\bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}_k)\big) = \mathbb{J}$. Since $\mathbf{\Gamma}^n$ is closed, we also see $\overline{\boldsymbol{X}}^q \in \mathbf{\Gamma}^n$. Therefore, taking the limit of the identity $\overline{\boldsymbol{X}} = \sum_{p=1}^{\ell} \boldsymbol{X}^p_m$ as $m \to \infty$, we obtain that $\overline{\boldsymbol{X}} = \sum_{p=1}^{\ell} \overline{\boldsymbol{X}}^p$ and $\overline{\boldsymbol{X}}^p \in \mathbf{\Gamma}^n \cap \mathbb{J}$ $(p = 1, \ldots, \ell)$. Therefore, we have shown that $\overline{\boldsymbol{X}} \in \mathrm{co}\big(\mathbf{\Gamma}^n \cap \mathbb{J}\big)$ and $\bigcap_{m=1}^{\infty} \mathrm{co}\big(\mathbf{\Gamma}^n \cap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k)\big)\big) \subseteq \mathrm{co}\big(\mathbf{\Gamma}^n \cap \mathbb{J}\big)$. □

We may assume without loss of generality that $\mathcal{B}$ is bounded (condition (A-1)). For each $\epsilon > 0$, define an open neighborhood $U(\boldsymbol{B}, \epsilon) = \{\boldsymbol{A} \in \mathbb{S}^n : \| \boldsymbol{A} - \boldsymbol{B} \| < \epsilon\}$ of each $\boldsymbol{B} \in \mathrm{cl}\mathcal{B}$. Let $\{\epsilon_k\}$ be a sequence of positive numbers which converges to 0. Let k be fixed. Since $\mathcal{B}$ satisfies (A-1), cl$\mathcal{B}$ is a compact subset of $\mathbb{S}^n$. Hence we can choose a finite subset $\mathcal{A}_k$ of cl$\mathcal{B}$ such that $\bigcup_{\boldsymbol{A}\in\mathcal{A}_k} U(\boldsymbol{A}, \epsilon_k/2) \supseteq \mathrm{cl}\mathcal{B}$ and $U(\boldsymbol{A}, \epsilon_k/2)$ contains a $\boldsymbol{B} \in \mathcal{B}$ ($\boldsymbol{A} \in \mathcal{A}_k$). Let $\mathcal{B}_k$ be the set of such $\boldsymbol{B}$'s. Then the sequence $\{\mathcal{B}_k\ (k = 1, 2, \ldots)\}$ satisfies that

$$\mathcal{B}_k \subseteq \mathcal{B},\ \mathbb{J}_+(\mathcal{B}_k) \supseteq \mathbb{J}_+(\mathcal{B})\ (k = 1, 2, \ldots),\ \bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}_k) \supseteq \mathbb{J}_+(\mathcal{B}), \tag{4}$$

$$\forall \boldsymbol{B} \in \mathcal{B},\ \exists \boldsymbol{B}' \in \mathcal{B}_k; \| \boldsymbol{B}' - \boldsymbol{B} \| < \epsilon_k\ (k = 1, 2, \ldots),. \tag{5}$$

Lemma 2.2. $\bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}_k) = \mathbb{J}_+(\mathcal{B})$.

Proof. By (4), it suffices to show that $\bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}_k) \subseteq \mathbb{J}_+(\mathcal{B})$. Let $\overline{\boldsymbol{X}} \in \bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}_k)$. To prove $\overline{\boldsymbol{X}} \in \mathbb{J}_+(\mathcal{B})$, we show that $\langle \overline{\boldsymbol{B}}, \overline{\boldsymbol{X}} \rangle \geq 0$ for an arbitrary chosen $\overline{\boldsymbol{B}} \in \mathcal{B}$. By (5), there exists a sequence $\{\boldsymbol{B}_k \in \mathcal{B}_k\}$ which converges $\overline{\boldsymbol{B}} \in \mathcal{B}$ as $k \to \infty$. Since $\overline{\boldsymbol{X}} \in \mathbb{J}_+(\mathcal{B}_k)$, we see that $\langle \boldsymbol{B}_k, \overline{\boldsymbol{X}} \rangle \geq 0$ ($k = 1, 2, \ldots$). Hence we obtain $\langle \overline{\boldsymbol{B}}, \overline{\boldsymbol{X}} \rangle \geq 0$ by taking the limit as $k \to \infty$. □

Lemma 2.3. $co\big(\boldsymbol{\Gamma}^n \cap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k)\big)\big) = \bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k)$ ($m = 1, 2, \ldots$).

Proof. Let $m \in \{1, 2, \ldots\}$ be fixed. We note that $\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k) = \mathbb{J}_+(\bigcup_{k=1}^{m} \mathcal{B}_k)$. Each $\bigcup_{k=1}^{m} \mathcal{B}_k$ satisfies condition (B) since it is a subset of $\mathcal{B}$. Since $\bigcup_{k=1}^{m} \mathcal{B}_k$ is finite, $\mathbb{J}_+(\bigcup_{k=1}^{m} \mathcal{B}_k) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$ by [2, Theorem 4.1]. Therefore $\mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k)\big)\big) = \bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k)$ □

Now we show $\mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})\big) = \mathbb{J}_+(\mathcal{B})$, which is equivalent to $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$ by definition. By Lemmas 2.2 and 2.3, we see that

$$\mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k)\big)\big) = \bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k) \supseteq \bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}_k) = \mathbb{J}_+(\mathcal{B})\ (m = 1, 2, \ldots),$$

which implies that $\bigcap_{m=1}^{\infty} \mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k)\big)\big) \supseteq \mathbb{J}_+(\mathcal{B})$. By Lemma 2.1, $\bigcap_{m=1}^{\infty} \mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k)\big)\big) = \mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})\big)$. Therefore, we have shown that $\mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})\big) \supseteq \mathbb{J}_+(\mathcal{B})$. The converse inclusion $\mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})\big) \subseteq \mathbb{J}_+(\mathcal{B})$ is straightforward since $\mathbb{J}_+(\mathcal{B})$ is convex. □

3 Some characterizations of conditions (A-3), (A-4), (A-5) and (B)

We begin by presenting three lemmas on the fundamental properties on $\mathbb{J}_+(\boldsymbol{B})$, $\mathbb{J}_-(\boldsymbol{B})$ and $\mathbb{J}_0(\boldsymbol{B})$, each interesting in its own right. Some of these lemmas are used in the subsequent discussion. Following the lemmas, we present five theorems, Theorems 3.4, 3.5, 3.6, 3.7, and 3.8. Theorems 3.4 provides algebraic characterizations of condition (B) under conditions (A-3), (A-4) and (A-5). The characterizations there relates condition (B) to [1, Proposition 1] stated as Theorem 3.5. We discuss some details about their relation. Theorem 3.6 provides an interesting property of $\mathcal{B} \subseteq \mathbb{S}^n$ satisfying conditions (A-5) and (B). Theorem 3.7 shows that condition (B) is preserved under a linear transformation of the variable vector $\boldsymbol{x}$ of $\mathrm{COP}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}))$. Theorem 3.8 deals with the case where the linear transformation is one-to-one.

Lemma 3.1. *Let $\boldsymbol{B} \in \mathbb{S}^n$. Then $\mathbb{J}_+(\boldsymbol{B}),\ \mathbb{J}_-(\boldsymbol{B}),\ \mathbb{J}_0(\boldsymbol{B}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$.*

Proof. $\mathbb{J}_+(\{\boldsymbol{B}\})$ and $\mathbb{J}_-(\{\boldsymbol{B}\}) = \mathbb{J}_+(\{-\boldsymbol{B}\})$ satisfy condition (B). Hence $\mathbb{J}_+(\boldsymbol{B}),\ \mathbb{J}_-(\boldsymbol{B}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$ by Theorem 1.2. $\mathbb{J}_0(\boldsymbol{B})$ is a face of $\mathbb{J}_+(\boldsymbol{B}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$. Hence $\mathbb{J}_0(\boldsymbol{B}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$ by Theorem 1.1 (iii). □

Lemma 3.2. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{S}^n$ and $\boldsymbol{A} \neq \boldsymbol{B}$. Then*

(i) $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x} \geq 0\} \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \geq 0\} \Leftrightarrow \boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\boldsymbol{B}) \subseteq \boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\boldsymbol{A}) \Leftrightarrow \mathbb{J}_+(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$.

(ii) $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x} = 0\} \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \geq 0\} \Leftrightarrow \boldsymbol{\Gamma}^n \cap \mathbb{J}_0(\boldsymbol{B}) \subseteq \boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\boldsymbol{A}) \Leftrightarrow \mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$.

(iii) $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x} \leq 0\} \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \geq 0\} \Leftrightarrow \boldsymbol{\Gamma}^n \cap \mathbb{J}_-(\boldsymbol{B}) \subseteq \boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\boldsymbol{A}) \Leftrightarrow \mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$.

Proof. (i) By definition, we know that

$$\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\boldsymbol{B}) = \{\boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x} \geq 0\},\ \boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\boldsymbol{A}) = \{\boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \geq 0\}.$$

Hence the first $\Leftrightarrow$ follows. For the second $\Leftrightarrow$, $\Leftarrow$ is straightforward. By Lemma 3.1, $\mathrm{co}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\boldsymbol{B})) = \mathbb{J}_+(\boldsymbol{B})$ and $\mathrm{co}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\boldsymbol{A})) = \mathbb{J}_+(\boldsymbol{A})$. Hence

$$\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\boldsymbol{B}) \subseteq \boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\boldsymbol{A}) \Rightarrow \mathrm{co}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\boldsymbol{B})) \subseteq \mathrm{co}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\boldsymbol{A})) \Rightarrow \mathbb{J}_+(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}).$$

Assertions (ii) and (iii) can be proved similarly, and their proofs are omitted. □

Lemma 3.3. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}$ and $\boldsymbol{A} \neq \boldsymbol{B}$. Then $\mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}) \Rightarrow \mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$ If, in addition, condition (A-5) is satisfied, then $\mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}) \Leftrightarrow \mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$.*

Proof. The first $\Rightarrow$ follows from $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_-(\boldsymbol{B})$. To prove $\Leftarrow$ in the second assertion, assume on the contrary that $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$ but $\mathbb{J}_-(\boldsymbol{B}) \not\subseteq \mathbb{J}_+(\boldsymbol{A})$ or equivalently that $\langle \boldsymbol{B},\, \overline{\boldsymbol{X}} \rangle < 0$ and $\langle \boldsymbol{A},\, \overline{\boldsymbol{X}} \rangle < 0$ for some $\overline{\boldsymbol{X}} \in \mathbb{S}^n_+$. By condition (A-5), $\mathbb{J}_+(\boldsymbol{B}) \not\subseteq \mathbb{J}_+(\boldsymbol{A})$, which implies that $\langle \boldsymbol{B},\, \widetilde{\boldsymbol{X}} \rangle \geq 0$ and $\langle \boldsymbol{A},\, \widetilde{\boldsymbol{X}} \rangle < 0$ for some $\widetilde{\boldsymbol{X}} \in \mathbb{S}^n_+$. If $\langle \boldsymbol{B},\, \widetilde{\boldsymbol{X}} \rangle = 0$, then $\langle \boldsymbol{A},\, \widetilde{\boldsymbol{X}} \rangle \geq 0$ by $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$. Hence $\langle \boldsymbol{B},\, \widetilde{\boldsymbol{X}} \rangle > 0$, and there exists $\lambda \in (0, 1)$ such that

$$\langle \boldsymbol{B},\, \lambda\overline{\boldsymbol{X}} + (1-\lambda)\widetilde{\boldsymbol{X}} \rangle = 0,\ \langle \boldsymbol{A},\, \lambda\overline{\boldsymbol{X}} + (1-\lambda)\widetilde{\boldsymbol{X}} \rangle < 0,\ \lambda\overline{\boldsymbol{X}} + (1-\lambda)\widetilde{\boldsymbol{X}} \in \mathbb{S}^n_+.$$

This contradicts $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$. □

Theorem 3.4. *Assume that conditions (A-3), (A-4) and (A-5) are satisfied. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}$ and $\boldsymbol{A} \neq \boldsymbol{B}$. Then*

$$\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}) \ \Leftrightarrow\ \mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}) \ \Leftrightarrow\ (7) \ \Leftrightarrow\ (8), \tag{6}$$

where

$$\alpha\boldsymbol{A} + \beta\boldsymbol{B} \in \mathbb{S}^n_+ \text{ for some } \alpha > 0 \text{ and } \beta > 0 \tag{7}$$

$$\alpha\boldsymbol{A} + \beta\boldsymbol{B} \in \mathbb{S}^n_+ \text{ for some } (\alpha, \beta) \neq \boldsymbol{0}. \tag{8}$$

Proof. The first '⇔' in (6) is already shown in Lemma 3.3. To prove the second '⇔', consider the primal-dual pair of SDPs

$$\begin{aligned} \zeta_{\rm p} &= \inf\{\langle \boldsymbol{A},\, \boldsymbol{X}\rangle : \boldsymbol{X} \in \mathbb{S}^n_+,\ \langle \boldsymbol{B},\, \boldsymbol{X}\rangle \le 0\} = \inf\{\langle \boldsymbol{A},\, \boldsymbol{X}\rangle : \boldsymbol{X} \in \mathbb{J}_-(\boldsymbol{B})\}, \\ \zeta_{\rm d} &= \sup\{0 : \boldsymbol{A} + \tau \boldsymbol{B} \in \mathbb{S}^n_+,\ \tau \ge 0\}. \end{aligned} \tag{9}$$

Obviously, $\zeta_{\rm p} = 0$ if and only if $\mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$. On the one hand, by (A-3), there exists a feasible solution $\widetilde{\boldsymbol{X}}$ of the primal SDP such that $\widetilde{\boldsymbol{X}} \in \mathbb{S}^n_{++}$. By the duality theorem, $\zeta_p = \zeta_d = 0$ if and only if the dual SDP is feasible, *i.e.*, $\boldsymbol{A} + \tau\boldsymbol{B} \in \mathbb{S}^n_+$ for some $\tau \ge 0$. If $\tau = 0$, then $\boldsymbol{A} \in \mathbb{S}^n_+$, which contradicts (A-4). Therefore $\mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}) \ \Leftrightarrow\ $ (7) follows. In (6), (7) ⇒ (8) is obvious. To prove the converse '⇐', by assuming (8), we show that $\alpha > 0$ and $\beta > 0$. We may assume without loss of generality that $\alpha \ne 0$. Then, one of the following cases occurs.

(a) $\alpha > 0$, $\beta > 0$. In this case (7) holds.

(b) $\alpha\beta < 0$. Say $\alpha > 0$ and $\beta < 0$. Then $\boldsymbol{A} = (-\beta/\alpha)\boldsymbol{B} + \boldsymbol{Y}$ for some $\boldsymbol{Y} \in \mathbb{S}^n_+$, which contradicts condition (A-5). Hence this case cannot occur.

(c) $\alpha > 0$, $\beta = 0$. In this case, $\boldsymbol{A} \in \mathbb{S}^n_+$, which contradicts condition (A-4). Hence this case cannot occur.

(d) $\alpha < 0$, $\beta < 0$. In this case, we observe that

$$\begin{aligned} \mathbb{J}_+(\mathcal{B}) &\subseteq \left\{\boldsymbol{X} \in \mathbb{S}^n_+ : \langle \boldsymbol{A},\, \boldsymbol{X}\rangle \ge 0,\ \langle \boldsymbol{B},\, \boldsymbol{X}\rangle \ge 0\right\} \\ &= \left\{\boldsymbol{X} \in \mathbb{S}^n_+ : \langle \boldsymbol{A},\, \boldsymbol{X}\rangle \ge 0,\ \langle \boldsymbol{B},\, \boldsymbol{X}\rangle \ge 0,\ -\langle \alpha\boldsymbol{A} + \beta\boldsymbol{B},\, \boldsymbol{X}\rangle \ge 0\right\} \\ &\quad \text{(since } \alpha < 0 \text{ and } \beta < 0\text{)} \\ &\subseteq \left\{\boldsymbol{X} \in \mathbb{S}^n_+ : -\langle \alpha\boldsymbol{A} + \beta\boldsymbol{B},\, \boldsymbol{X}\rangle = 0\right\} \ \text{(since } \alpha\boldsymbol{A} + \beta\boldsymbol{B} \in \mathbb{S}^n_+\text{)}. \end{aligned}$$

Hence $\mathbb{J}_+(\mathcal{B})$ is included in a face $\{\boldsymbol{X} \in \mathbb{S}^n_+ : \langle \alpha\boldsymbol{A} + \beta\boldsymbol{B},\, \boldsymbol{X}\rangle = 0\}$ of $\mathbb{S}^n_+$. If $\alpha\boldsymbol{A} + \beta\boldsymbol{B} \ne \boldsymbol{O}$, then the face including $\mathbb{J}_+(\mathcal{B})$ is a proper face of $\mathbb{S}^n_+$, which contradicts condition (A-3). Therefore $-\alpha\boldsymbol{A} - \beta\boldsymbol{B} = \boldsymbol{O} \in \mathbb{S}^n_+$. Since $-\alpha > 0$ and $-\beta > 0$, (7) holds. □

As a result of Theorem 3.4, $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$ in condition (B) can be replaced with any of $\mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$, (7) and (8) under conditions (A-3), (A-4) and (A-5). We also note that (7) implies $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_-\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$ even when none of the conditions (A-3), (A-4) and (A-5) hold, and that whether (7) holds is checked numerically by solving the simple primal SDP (9) or its dual; if $\zeta_p = 0$ or the dual SDP is feasible, then (7) holds.

The following result due to [1] also provides a sufficient condition for $\mathbb{J}_+(\mathcal{B}') \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^{n'})$.

Theorem 3.5. *([1, Proposition 1]) Let $\mathcal{B}'$ be a nonempty finite subset of $\mathbb{S}^{n'}$. Assume that (8) holds for every distinct $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}'$. Then $\mathbb{J}_+(\mathcal{B}') \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^{n'})$.*

We compare our main theorem, Theorem 1.2 with Theorem 3.5. Let the assumption of Theorem 3.5 is satisfied. As in the proof of Theorem 3.4, we may assume without loss of generality that $\alpha \ne 0$, and that one of cases (a), (b), (c), and (d) occurs with replacing n by n' and $\mathcal{B}$ by $\mathcal{B}'$. If (a) occurs, then $\mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$, which implies $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$, follows from the weak duality of the primal-dual pair of SDPs stated in the proof of Theorem 3.4. If (b) occurs, then either of $\boldsymbol{A}$ or $\boldsymbol{B}$ is redundant; $\mathbb{J}_+(\mathcal{B}') = \mathbb{J}_+(\mathcal{B}' \backslash \boldsymbol{A})$ or $\mathbb{J}_+(\mathcal{B}' \backslash \boldsymbol{B})$ holds. If (c) occurs, then $\boldsymbol{A} \in \mathbb{S}^{n'}_+$; hence $\mathbb{J}_+(\mathcal{B}') = \mathbb{J}_+(\mathcal{B}' \backslash \boldsymbol{A})$ holds. If (d) occurs, we have either

$\boldsymbol{O} = \alpha\boldsymbol{A} + \beta\boldsymbol{B} \in \mathbb{S}^{n'}$ or $\boldsymbol{O} \neq \alpha\boldsymbol{A} + \beta\boldsymbol{B} \in \mathbb{S}^{n'}$. In the former case, $-\alpha\boldsymbol{A} - \beta\boldsymbol{B} = \boldsymbol{O} \in \mathbb{S}^n_+$ with $-\alpha > 0$ and $-\beta > 0$; hence $\mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$ follows as in case (a). In the latter case, $\mathbb{J}_+(\mathcal{B}')$ lies in a proper face of $\mathbb{S}^n_+$ and $\mathcal{B}'$ does not satisfy condition (A-3). Therefore, the essential difference between Theorem 1.2 and Theorem 3.5 lies in the last case where $\mathbb{J}_+(\mathcal{B}')$ is contained in a proper face of $\mathbb{S}^n_+$ or condition (A-3) is violated. Theorem 3.5 deals with such a $\mathcal{B}'$ directly or the equivalence of COP$(\mathbb{J}_+(\boldsymbol{\Gamma}^{n'} \cap \mathcal{B}')$ and its SDP relaxation COP$(\mathcal{B}')$. As stated in Section 1, in our approach, we reduce COP$(\mathbb{J}_+(\boldsymbol{\Gamma}^{n'} \cap \mathcal{B}'))$ to an equivalent COP$(\mathbb{J}_+(\boldsymbol{\Gamma}^n \cap \mathcal{B}))$ with $\mathcal{B} \subseteq \mathbb{S}^{n'}$ satisfying condition (A-3), (A-4) and (A-5), and then apply Theorem 1.2 to the resulting $\mathcal{B}$. More details will be discussed in Section 5. We will see in Example 6.1 that our approach is often more effective than directly applying Theorem 3.5 to the original $\mathcal{B}'$.

An important property of $\mathcal{B}$ satisfying conditions (A-5) and (B)

Theorem 3.6. *Define $\mathcal{B}_0 = \{\boldsymbol{B} \in \mathcal{B} : \langle \boldsymbol{B}, \boldsymbol{X} \rangle = 0$ for every $\boldsymbol{X} \in \mathbb{J}_+(\mathcal{B})\}$. Assume conditions (A-5) and (B). Then, one of the following two cases occurs.*

(a) $\overline{\boldsymbol{B}} \in \mathcal{B}_0 \neq \emptyset$ for some $\overline{\boldsymbol{B}} \in \mathcal{B}$. In this case, $\mathbb{J}_+(\mathcal{B}) = \mathbb{J}_0(\overline{\boldsymbol{B}})$.

(b) $\mathcal{B}_0 = \emptyset$. In this case, $\mathbb{J}_-(\boldsymbol{B})$ is a proper subset of $\mathbb{J}_+(\boldsymbol{A})$ for every distinct $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}$.

Proof. Obviously, case (a) $\overline{\boldsymbol{B}} \in \mathcal{B}_0 \neq \emptyset$ for some $\overline{\boldsymbol{B}} \in \mathcal{B}$ or case (b) $\mathcal{B}_0 = \emptyset$ occurs exclusively. In case (a), we see that $\mathbb{J}_+(\mathcal{B}) \subseteq \mathbb{J}_0(\overline{\boldsymbol{B}}) \subseteq \mathbb{J}_+(\mathcal{B})$, where the latter inclusion is from condition (B). Hence $\mathbb{J}_+(\mathcal{B}) = \mathbb{J}_0(\overline{\boldsymbol{B}})$. In case (b), by Lemma 3.3, $\mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$ holds for every distinct $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}$. Assume on the contrary that $\mathbb{J}_-(\overline{\boldsymbol{B}}) = \mathbb{J}_+(\overline{\boldsymbol{A}})$ for some distinct $\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}} \in \mathcal{B}$. Then

$$\mathbb{J}_+(\mathcal{B}) \;\subseteq\; \mathbb{J}_+(\overline{\boldsymbol{A}}) \cap \mathbb{J}_+(\overline{\boldsymbol{B}}) = \mathbb{J}_-(\overline{\boldsymbol{B}}) \cap \mathbb{J}_+(\overline{\boldsymbol{B}}) = \mathbb{J}_0(\overline{\boldsymbol{B}}).$$

Hence $\overline{\boldsymbol{B}} \in \mathcal{B}_0$, which contradicts $\mathcal{B}_0 = \emptyset$. □

Case (a) is a trivial case where $\mathbb{J}_+(\mathcal{B})$ is described as $\mathbb{J}_+(\mathcal{B}) = \{\boldsymbol{X} \in \mathbb{S}^n_+ : \langle \overline{\boldsymbol{B}}, \boldsymbol{X} \rangle = 0\}$ for some $\overline{\boldsymbol{B}} \in \mathcal{B}$. Except this trivial case, we always have

$$\mathbb{J}_-(\boldsymbol{B}) \subset \mathbb{J}_+(\boldsymbol{A});\ \text{hence}\ \{\boldsymbol{X} \in \mathbb{S}^n_+ : \langle \boldsymbol{B}, \boldsymbol{X} \rangle \leq 0\} \cap \{\boldsymbol{X} \in \mathbb{S}^n_+ : \langle \boldsymbol{A}, \boldsymbol{X} \rangle < 0\} = \emptyset \quad \text{for every distinct } \boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}$$

under conditions (A-5) and (B).

Linear transformation of COP$(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$

Now we consider a linear transformation $\boldsymbol{x} = \boldsymbol{L}\boldsymbol{y}$ of the variable vector $\boldsymbol{x}$ in COP$(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$ (see (2)), where $\boldsymbol{L}$ denotes a nonzero $n \times n'$ matrix with $n' \geq 1$ and $\boldsymbol{y} \in \mathbb{R}^{n'}$ a variable vector. Then, COP$(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$ and its SDP relaxation COP$(\mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$ are transformed into COP$(\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}_+(\mathcal{B}'), \boldsymbol{Q}', \boldsymbol{H}')$ and its SDP relaxation COP$(\mathbb{J}_+(\mathcal{B}'), \boldsymbol{Q}', \boldsymbol{H}')$, respectively, where $\mathcal{B}' = \{\boldsymbol{L}^T\boldsymbol{B}\boldsymbol{L} : \boldsymbol{B} \in \mathcal{B}\}$, $\boldsymbol{Q}' = \boldsymbol{L}^T\boldsymbol{Q}\boldsymbol{L}$ and $\boldsymbol{H}' = \boldsymbol{L}^T\boldsymbol{H}\boldsymbol{L}$.

Theorem 3.7. *$\mathcal{B}'$ satisfies condition (B) if $\mathcal{B}$ satisfies condition (B).*

Proof. Let $\boldsymbol{A},\ \boldsymbol{B} \in \mathcal{B}$ and $\boldsymbol{A} \neq \boldsymbol{B}$. Assuming $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$, we will show $\mathbb{J}_0(\boldsymbol{L}^T\boldsymbol{B}\boldsymbol{L}) \subseteq \mathbb{J}_+(\boldsymbol{L}^T\boldsymbol{A}\boldsymbol{L})$. Let $\boldsymbol{Y} \in \mathbb{J}_0(\boldsymbol{L}^T\boldsymbol{B}\boldsymbol{L})$. Then $\boldsymbol{Y} \in \mathbb{S}^{n'}_+$ and $0 = \langle \boldsymbol{L}^T\boldsymbol{B}\boldsymbol{L},\, \boldsymbol{Y}\rangle$, which imply that $\boldsymbol{L}\boldsymbol{Y}\boldsymbol{L}^T \in \mathbb{S}^n_+$ and $0 = \langle \boldsymbol{B},\, \boldsymbol{L}\boldsymbol{Y}\boldsymbol{L}^T\rangle$. Hence $\boldsymbol{L}\boldsymbol{Y}\boldsymbol{L}^T \in \mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$. It follows that $\boldsymbol{Y} \in \mathbb{S}^{n'}_+$ and $0 \leq \langle \boldsymbol{A},\, \boldsymbol{L}\boldsymbol{Y}\boldsymbol{L}^T\rangle = \langle \boldsymbol{L}^T\boldsymbol{A}\boldsymbol{L},\, \boldsymbol{Y}\rangle$. Therefore $\boldsymbol{Y} \in \mathbb{J}_+(\boldsymbol{L}^T\boldsymbol{A}\boldsymbol{L})$. □

Theorem 3.8. *Assume that $n' = n$ and $\boldsymbol{L}$ is nonsingular.*

- *(i) COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}'), \boldsymbol{Q}', \boldsymbol{H}'$) and its SDP relaxation COP($\mathbb{J}_+(\mathcal{B}'), \boldsymbol{Q}', \boldsymbol{H}'$) are equivalent to COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}$) and its SDP relaxation COP($\mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}$), respectively.*
- *(ii) $\mathcal{B}'$ satisfies condition (B) if and only if $\mathcal{B}$ does.*
- *(iii) $\mathbb{J}_+(\mathcal{B}') \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^{n'})$ if and only if $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$.*

Proof. Assertion (i) is straightforward, and (ii) follows from $\mathcal{B} = \{\boldsymbol{L}^{-T}\boldsymbol{B}'\boldsymbol{L}^{-1} : \boldsymbol{B}' \in \mathcal{B}'\}$ and Theorem 3.7. To prove assertion (iii), we observe that

$$
\begin{aligned}
\boldsymbol{L}^{-1}\mathbb{J}_+(\mathcal{B})\boldsymbol{L}^{-T} &= \{\boldsymbol{Y} \in \mathbb{S}^n : \boldsymbol{Y} = \boldsymbol{L}^{-1}\boldsymbol{X}\boldsymbol{L}^{-T},\ \boldsymbol{X} \in \mathbb{S}^n_+,\ \langle \boldsymbol{B},\, \boldsymbol{X}\rangle \geq 0\ (\boldsymbol{B} \in \mathcal{B})\} \\
&= \{\boldsymbol{Y} \in \mathbb{S}^n : \boldsymbol{X} = \boldsymbol{L}\boldsymbol{Y}\boldsymbol{L}^T,\ \boldsymbol{X} \in \mathbb{S}^n_+,\ \langle \boldsymbol{B},\, \boldsymbol{X}\rangle \geq 0\ (\boldsymbol{B} \in \mathcal{B})\} \\
&= \{\boldsymbol{Y} \in \mathbb{S}^n_+ : \langle \boldsymbol{B},\, \boldsymbol{L}\boldsymbol{Y}\boldsymbol{L}^T\rangle \geq 0\ (\boldsymbol{B} \in \mathcal{B})\} \\
&= \{\boldsymbol{Y} \in \mathbb{S}^n_+ : \langle \boldsymbol{L}^T\boldsymbol{B}\boldsymbol{L},\, \boldsymbol{Y}\rangle \geq 0\ (\boldsymbol{B} \in \mathcal{B})\} \\
&= \{\boldsymbol{Y} \in \mathbb{S}^n_+ : \langle \boldsymbol{B}',\, \boldsymbol{Y}\rangle \geq 0\ (\boldsymbol{B}' \in \mathcal{B}')\} \\
&= \mathbb{J}_+(\mathcal{B}'),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{co}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})) = \mathbb{J}_+(\mathcal{B}) \ &\Leftrightarrow\ \boldsymbol{L}^{-1}\big(\mathrm{co}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})\big)\boldsymbol{L}^{-T} = \boldsymbol{L}^{-1}\mathbb{J}_+(\mathcal{B})\boldsymbol{L}^{-T} \\
&\Leftrightarrow\ \mathrm{co}\big((\boldsymbol{L}^{-1}\boldsymbol{\Gamma}^n\boldsymbol{L}^{-T}) \cap (\boldsymbol{L}^{-1}\mathbb{J}_+(\mathcal{B})^{-T})\big) = \boldsymbol{L}^{-1}\mathbb{J}_+(\mathcal{B})\boldsymbol{L}^{-T} \\
&\Leftrightarrow\ \mathrm{co}\big(\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}_+(\mathcal{B}')\big) = \mathbb{J}_+(\mathcal{B}').
\end{aligned}
$$

Therefore, assertion (iii) follows. □

Let $S = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} = \boldsymbol{L}\boldsymbol{y} \text{ for some } \boldsymbol{y} \in \mathbb{R}^{n'}\}$. Obviously, COP($\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}_+(\mathcal{B}'), \boldsymbol{Q}', \boldsymbol{H}'$) is equivalent to the problem

$$
\zeta \ = \ \inf\left\{\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} : \boldsymbol{x} \in S,\ \boldsymbol{x}\boldsymbol{x}^T \in \mathbb{J}_+(\mathcal{B}),\ \boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} = 1\right\} \tag{10}
$$

in the sense that $\zeta = \eta(\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}_+(\mathcal{B}'), \boldsymbol{Q}', \boldsymbol{H}')$ and that an optimal solution $\boldsymbol{y} \in \mathbb{R}^{n'}$ of COP($\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}_+(\mathcal{B}'), \boldsymbol{Q}', \boldsymbol{H}'$) corresponds to an optimal solution of (10). In particular, if rank$\boldsymbol{L} = n$, then both problems are equivalent to COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}$). But each optimal solution of COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}$) corresponds to multiple optimal solutions when $n' > n$. We also mention that Theorem 3.7 is not relevant to $\boldsymbol{Q}' \in \mathbb{S}^n$ and $\boldsymbol{H}' \in \mathbb{S}^n$. Hence we can take any $\boldsymbol{Q}' \in \mathbb{S}^{n'}$ and $\boldsymbol{H}' \in \mathbb{S}^{n'}$ independently from $\boldsymbol{Q} \in \mathbb{S}^n$ and $\boldsymbol{H} \in \mathbb{S}^n$.

Now assume that $1 \leq \ell = \mathrm{rank}\boldsymbol{L} \leq n-1$. Then, we can rewrite S as

$$
S \ = \ \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{M}\boldsymbol{x} = 0\} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}(-\boldsymbol{M}^T\boldsymbol{M})\boldsymbol{x} \geq 0\}
$$

for some row full rank $(n-\ell) \times n$ matrix $\boldsymbol{M}$. Therefore COP($\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}_+(\mathcal{B}'), \boldsymbol{Q}', \boldsymbol{H}'$) and its SDP relaxation COP($\mathbb{J}_+(\mathcal{B}'), \boldsymbol{Q}', \boldsymbol{H}'$) are equivalent to COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}''), \boldsymbol{Q}, \boldsymbol{H}$) and

$\text{COP}(\mathbb{J}_+(\mathcal{B}''), \boldsymbol{Q}, \boldsymbol{H})$, respectively, where $\mathcal{B}'' = \mathcal{B} \cup \{-\boldsymbol{M}^T\boldsymbol{M}\}$. It follows from $\boldsymbol{M}^T\boldsymbol{M} \in \mathbb{S}^n_+$ that $\mathbb{J}_+(-\boldsymbol{M}^T\boldsymbol{M})$, which contains $\mathbb{J}_+(\mathcal{B}'')$, forms a proper face of $\mathbb{S}^n_+$. We see by Theorem 1.1 that if $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$, then $\mathbb{J}_+(\mathcal{B}'') \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$ as well. But $\mathcal{B}''$ does not satisfy condition (A-3). In Section 5, we will discuss how the size of $\text{COP}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}''), \boldsymbol{Q}, \boldsymbol{H})$ with such a $\mathcal{B}''$ can be reduced to $\text{COP}(\mathbb{J}_+(\mathcal{B}'''), \boldsymbol{Q}''', \boldsymbol{H}''')$ with $\mathcal{B}'''$ satisfying conditions (A-1), (A-3), (A-4), and (A-5).

4 Proof of Theorem 1.3

We present four lemmas which lead to the proof of Theorem 1.3.

Lemma 4.1. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{S}^n$ and $\boldsymbol{A} \neq \boldsymbol{B}$. Assume that $\mathit{f\!f}_-(\cdot, \boldsymbol{B}) : [0,1] \to 2^{\mathbb{R}^{n-1}}$ is lower semi-continuous at $\bar{z} = 0$. Then*

$$\mathit{f\!f}_-(1, \boldsymbol{B}) \subseteq \mathit{f\!f}_+(1, \boldsymbol{A}) \Leftrightarrow \boldsymbol{\Gamma}^n \cap \mathbb{J}_-(\boldsymbol{B}) \subseteq \boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\boldsymbol{A}) \Leftrightarrow \mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}). \tag{11}$$

Here a point-to-set-map $\varphi : [0,1] \to 2^{\mathbb{R}^{n-1}}$ is lower semicontinuous at $\bar{z} = 0$ *if for every open subset W of $\mathbb{R}^{n-1}$ with $\varphi(0) \cap W \neq \emptyset$, there exists an $\epsilon \in (0,1]$ such that $\varphi(z) \cap W \neq \emptyset$ for every $z \in [0, \epsilon)$. See [7, 15] for the semicontinuity of a general point-to-set map.*

Proof. Since the second $\Leftrightarrow$ has been already shown in Lemma 3.2 (iii), we only show the first $\Leftrightarrow$. We can rewrite the left inclusion relation of (11) as

$$\begin{aligned} \mathit{f\!f}_-(1, \boldsymbol{B}) &= \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix}^T \boldsymbol{B} \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix} \leq 0 \right\} \\ &\subseteq \mathit{f\!f}_+(1, \boldsymbol{A}) = \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix}^T \boldsymbol{A} \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix} \geq 0 \right\}. \end{aligned} \tag{12}$$

By Lemma 3.2 (iii), the inclusion relation of (11) in the middle is equivalent to

$$\left\{ \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix} \in \mathbb{R}^n : \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix}^T \boldsymbol{B} \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix} \leq 0 \right\} \subseteq \left\{ \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix} \in \mathbb{R}^n : \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix}^T \boldsymbol{A} \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix} \geq 0 \right\} \tag{13}$$

Hence it suffices to show (12) $\Leftrightarrow$ (13). (12) $\Leftarrow$ (13) is straightfoward. To show (12) $\Rightarrow$ (13), assume (12) and let $\begin{pmatrix} \bar{\boldsymbol{u}} \\ \bar{z} \end{pmatrix} \in \left\{ \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix} \in \mathbb{R}^n : \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix}^T \boldsymbol{B} \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix} \leq 0 \right\}$. If $\bar{z} \neq 0$, then $\bar{\boldsymbol{u}}/\bar{z} \in \mathit{f\!f}_-(1, \boldsymbol{B}) \subseteq \mathit{f\!f}_+(1, \boldsymbol{A})$, which implies $\begin{pmatrix} \bar{\boldsymbol{u}} \\ \bar{z} \end{pmatrix} \in \left\{ \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix} \in \mathbb{R}^n : \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix}^T \boldsymbol{A} \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix} \geq 0 \right\}$. Now assume that $\bar{z} = 0$. By the lower semi-continuity of $\mathit{f\!f}_-(\cdot, \boldsymbol{B})$ at $\bar{z} = 0$, for every open neighborhood W of $\bar{\boldsymbol{u}} \in \mathit{f\!f}_-(0, \boldsymbol{B})$, there is an $\epsilon > 0$ such that $W \cap \mathit{f\!f}_-(z, \boldsymbol{B}) \neq \emptyset$ for every $z \in [0, \epsilon)$. Hence, there exists a sequence $\left\{ \begin{pmatrix} \boldsymbol{u}_k \\ z_k \end{pmatrix} \in (0,1] \times \mathbb{R}^{n-1} \right\}$ such that $0 < z_k \to 0$ and $\mathit{f\!f}_-(\boldsymbol{z}_k, \boldsymbol{B}) \ni \boldsymbol{u}_k \to \bar{\boldsymbol{u}}$ as $k \to \infty$. By the discussion above for the case $\bar{z} \neq 0$, each $\begin{pmatrix} \boldsymbol{u}_k \\ z_k \end{pmatrix}$ lies in the set $\left\{ \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix} \in \mathbb{R}^n : \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix}^T \boldsymbol{A} \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix} \geq 0 \right\}$. Since the set is closed, $\begin{pmatrix} \bar{\boldsymbol{u}} \\ \bar{z} \end{pmatrix}$ lies in the set. Thus we have shown (12) $\Rightarrow$ (13). $\square$

Lemma 4.2. *Let $\emptyset \neq \mathcal{B} \subseteq \mathbb{S}^n$. Assume condition (B)', and that for every $\boldsymbol{B} \in \mathcal{B}$,*

$$\mathbb{f}_-(\cdot, \boldsymbol{B}) : [0,1] \to 2^{\mathbb{R}^{n-1}} \text{ is lower semi-continuous at } z = 0. \tag{14}$$

Then $\mathcal{B}$ satisfies condition (B); hence $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$ by Theorem 1.2.

Proof. Let $\boldsymbol{A},\ \boldsymbol{B} \in \mathcal{B}$ and $\boldsymbol{A} \neq \boldsymbol{B}$. Then

$$\begin{aligned} \mathbb{f}_-(1, \boldsymbol{B}) \subseteq \mathbb{f}_+(1, \boldsymbol{A}) \ &\Leftrightarrow\ \mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}) \text{ (by Lemma 4.1)} \\ &\Rightarrow\ \mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}) \text{ (by Lemma 3.3)}. \end{aligned}$$

Therefore condition (B) is satisfied. □

For every $\boldsymbol{B} \in \mathbb{S}^n$, let $[\boldsymbol{B}]_{\boldsymbol{\nu}}$ denote the $(n-1) \times (n-1)$ submatrix of $\boldsymbol{B}$ obtained by deleting the nth row and nth column of $\boldsymbol{B}$; if $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{c} \\ \boldsymbol{c}^T & \gamma \end{pmatrix} \in \mathbb{S}^n$ then $[\boldsymbol{B}]_{\boldsymbol{\nu}} = \boldsymbol{C}$.

Lemma 4.3. *Let $\emptyset \neq \mathcal{B} \subseteq \mathbb{S}^n$. Assume condition (B)', and that for every $\boldsymbol{B} \in \mathcal{B}$,*

$$\mathbb{f}_-(1, \boldsymbol{B}) \neq \emptyset \text{ and } [\boldsymbol{B}]_{\boldsymbol{\nu}} \text{ is nonsingular.} \tag{15}$$

Then $\mathcal{B}$ satisfies condition (B); hence $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$ by Theoem 1.2.

Proof. By Lemma 4.2, it is sufficient to show (15) $\Rightarrow$ (14) for every $\boldsymbol{B} \in \mathbb{S}^n$. Let $\boldsymbol{B} = \begin{pmatrix} [\boldsymbol{B}]_{\boldsymbol{\nu}} & \boldsymbol{c} \\ \boldsymbol{c}^T & \gamma \end{pmatrix} \in \mathbb{S}^n$, and $\boldsymbol{P}$ an $(n-1) \times (n-1)$ orthogonal matrix which diagonalizes $[\boldsymbol{B}]_{\boldsymbol{\nu}}$ such that $\boldsymbol{P}^T [\boldsymbol{B}]_{\boldsymbol{\nu}} \boldsymbol{P} = \text{diag}(\lambda_1, \ldots, \lambda_{n-1})$, where $\lambda_1, \ldots, \lambda_{n-1}$ denote the eigenvalues of $[\boldsymbol{B}]_{\boldsymbol{\nu}}$. Let

$$\begin{aligned} \boldsymbol{B}' &= \begin{pmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix}^T \boldsymbol{B} \begin{pmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{P}^T [\boldsymbol{B}]_{\boldsymbol{\nu}} \boldsymbol{P} & \boldsymbol{P}^T \boldsymbol{c} \\ \boldsymbol{c}^T \boldsymbol{P} & \gamma \end{pmatrix} = \begin{pmatrix} \text{diag}(\lambda_1, \ldots, \lambda_{n-1}) & \boldsymbol{P}^T \boldsymbol{c} \\ \boldsymbol{c}^T \boldsymbol{P} & \gamma \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \mathbb{f}_-(z, \boldsymbol{B}') &= \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : \sum_{i=1}^{n-1} \lambda_i u_i^2 + 2\boldsymbol{c}^T \boldsymbol{P} \boldsymbol{u} z + \gamma z^2 \leq 0 \right\} \\ &= \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix}^T \boldsymbol{B}' \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix} \leq 0 \right\} \\ &= \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix}^T \begin{pmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix}^T \boldsymbol{B} \begin{pmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix} \leq 0 \right\} \\ &= \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : \begin{pmatrix} \boldsymbol{P}\boldsymbol{u} \\ z \end{pmatrix}^T \boldsymbol{B} \begin{pmatrix} \boldsymbol{P}\boldsymbol{u} \\ z \end{pmatrix} \leq 0 \right\} \\ &= \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : \boldsymbol{u} = \boldsymbol{P}^T \boldsymbol{v},\ \begin{pmatrix} \boldsymbol{v} \\ z \end{pmatrix}^T \boldsymbol{B} \begin{pmatrix} \boldsymbol{v} \\ z \end{pmatrix} \leq 0 \right\} \\ &= \boldsymbol{P}^T \mathbb{f}_-(z, \boldsymbol{B}). \end{aligned}$$

Hence we see from the assumption of Theorem 1.3 that

$$\tilde{\boldsymbol{u}} \in \mathbb{f}_-(1, \boldsymbol{B}') \neq \emptyset \text{ for some } \tilde{\boldsymbol{u}} \text{ and } \lambda_i \neq 0 \ (1 \leq i \leq n-1).$$

Therefore, $\mathbb{f}_-(\cdot, \boldsymbol{B}) : [0,1] \to 2^{\mathbb{R}^{n-1}}$ is lower semi-continuous at $z = 0$ if and only if $\mathbb{f}_-(\cdot, \boldsymbol{B}') : [0,1] \to 2^{\mathbb{R}^{n-1}}$ is. The following two cases (a) and (b) occur.

(a) $\lambda_i > 0$ $(1 \leq i \leq n-1)$. In this case, $\mathbb{f}_-(0, \boldsymbol{B}') = \{\boldsymbol{0}\}$. Let W be an arbitrary open subset of $\mathbb{R}^{n-1}$ that intersects with $\mathbb{f}_-(0, \boldsymbol{B}')$. Then $\boldsymbol{0} \in W$. We see from $\tilde{\boldsymbol{u}} \in \mathbb{f}_-(1, \boldsymbol{B}')$ that $\tilde{\boldsymbol{u}} z \in \mathbb{f}_-(z, \boldsymbol{B}')$ for every $z \in [0,1]$. Hence there exists an $\epsilon > 0$ such that $\mathbb{f}_-(z, \boldsymbol{B}') \cap W \neq \emptyset$ for every $z \in [0, \epsilon)$, and $\mathbb{f}_-(\cdot, \boldsymbol{B}')$ is lower semi-continuous at $z = 0$.

(b) $\lambda_i < 0$ for some i, say $\lambda_1 < 0$. Let W be an arbitrary open subset of $\mathbb{R}^{n-1}$ that intersects with $\mathbb{f}_-(0, \boldsymbol{B}')$. Let $\bar{\boldsymbol{u}} \in W \cap \mathbb{f}_-(0, \boldsymbol{B}')$. Then $\sum_{i=1}^{n-1} \lambda_i \bar{u}_i^2 \leq 0$. Since W is an open subset of $\mathbb{R}^{n-1}$ and $\lambda_1 < 0$, we may perturb $\bar{u}_1$ so that the resulting $\hat{\boldsymbol{u}}$ remains in W and satisfies $\sum_{i=1}^{n-1} \lambda_i \hat{u}_i^2 < 0$. By the continuity, $\sum_{i=1}^{n-1} \lambda_i \hat{u}_i^2 + 2\boldsymbol{c}^T \boldsymbol{P} \hat{\boldsymbol{u}} z + \gamma z^2 \leq 0$ *i.e.*, $\hat{\boldsymbol{u}} \in \mathbb{f}_-(z, \boldsymbol{B}')$ for every sufficiently small $z > 0$, which implies $\mathbb{f}_-(z, \boldsymbol{B}') \cap W \neq \emptyset$ for every $z \in [0, \epsilon)$ and some $\epsilon > 0$, and $\mathbb{f}_-(\cdot, \boldsymbol{B}')$ is lower semi-continuous at $z = 0$. □

Lemma 4.4. *Assume that $\mathcal{B} \subseteq \mathbb{S}^n$ satisfies conditions (B)' and (C)'. Then, for every $\boldsymbol{B} \in \mathcal{B}$ and $\epsilon > 0$, there exists an $\epsilon_B \in (0, \epsilon]$ such that*

$$\mathbb{f}_-(1, \boldsymbol{B} + \epsilon_B \boldsymbol{I}) \neq \emptyset \text{ and } [\boldsymbol{B} + \epsilon_B \boldsymbol{I}]_{\boldsymbol{\nu}} \text{ is nonsingular.} \tag{16}$$

Proof. Let $\epsilon > 0$ and $\boldsymbol{B} \in \mathcal{B}$. By condition (C)', $\mathbb{f}_-(1, \boldsymbol{B} + \epsilon_B \boldsymbol{I}) \neq \emptyset$ for every sufficiently small $\epsilon_B > 0$. Let $\mu_1, \ldots, \mu_{n-1}$ be the eigenvalues of $[\boldsymbol{B}]_{\boldsymbol{\nu}}$. Then $[\boldsymbol{B} + \epsilon_B \boldsymbol{I}]_{\boldsymbol{\nu}}$ becomes singular if and only if $\epsilon_B = -\mu_i$ for some i. Therefore, we can choose $\epsilon_B \in (0, \epsilon]$ satisfying (16). □

Now we are ready to prove Theorem 1.3. Let $\{\epsilon_k : k = 1, 2, \ldots\}$ be a sequence of positive numbers which converges 0 as $k \to \infty$. By Lemma 4.4, we can consistently define

$$\mathcal{B}_k = \left\{ \boldsymbol{B} + \epsilon_{kB} \boldsymbol{I} : \begin{array}{l} \boldsymbol{B} \in \mathcal{B}, \text{ some } \epsilon_{kB} \in (0, \epsilon_k] \text{ such that } [\boldsymbol{B} + \epsilon_{kB} \boldsymbol{I}]_{\boldsymbol{\nu}} \\ \text{is nonsingular and } \mathbb{f}_-(1, \boldsymbol{B} + \epsilon_{kB} \boldsymbol{I}) \neq \emptyset \end{array} \right\}$$

$(k = 1, 2, \ldots)$. In addition, we may impose that $\epsilon_{kB} \geq \epsilon_{(k+1)B}$ $(k = 1, 2, \ldots)$ for each $\boldsymbol{B} \in \mathcal{B}$. Hence

$$\begin{array}{l} \mathbb{J}_+(\boldsymbol{B} + \epsilon_{kB} \boldsymbol{I}) \supseteq \mathbb{J}_+(\boldsymbol{B} + \epsilon_{(k+1)B} \boldsymbol{I}) \supseteq \mathbb{J}_+(\boldsymbol{B}) \text{ for every } \boldsymbol{B} \in \mathcal{B}, \\ \mathbb{J}_-(\boldsymbol{A} + \epsilon_{kA} \boldsymbol{I}) \subseteq \mathbb{J}_-(\boldsymbol{A} + \epsilon_{(k+1)A} \boldsymbol{I}) \subseteq \mathbb{J}_-(\boldsymbol{A}) \text{ for every } \boldsymbol{A} \in \mathcal{B} \end{array}$$

$(k = 1, 2, \ldots)$. It follows that

$$\begin{array}{l} \mathbb{J}_+(\mathcal{B}_k) \supseteq \mathbb{J}_+(\mathcal{B}_{k+1}) \supseteq \mathbb{J}_+(\mathcal{B}) \ (k = 1, 2, \ldots), \\ \mathbb{J}_+(\mathcal{B}_m) = \displaystyle\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k) \supseteq \bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}_k) \supseteq \mathbb{J}_+(\mathcal{B}) \ (m = 1, 2, \ldots)). \end{array}$$

To show $\displaystyle\bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}_k) \subseteq \mathbb{J}_+(\mathcal{B})$, assume on the contrary that $\displaystyle\bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}_k) \not\subseteq \mathbb{J}_+(\mathcal{B})$. Then there exists an $\overline{\boldsymbol{X}} \in \bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}_k)$ such that $\overline{\boldsymbol{X}} \notin \mathbb{J}_+(\mathcal{B})$. It follows from $\overline{\boldsymbol{X}} \notin \mathbb{J}_+(\mathcal{B})$ that

$\langle \boldsymbol{B}, \overline{\boldsymbol{X}} \rangle < 0$ for some $\boldsymbol{B} \in \mathcal{B}$. Since $\epsilon_{mB} \in (0, \epsilon_m]$ converges 0 as $m \to \infty$, if m is sufficiently large, then $\langle \boldsymbol{B} + \epsilon_{mB}\boldsymbol{I}, \overline{\boldsymbol{X}} \rangle < 0$; hence $\overline{\boldsymbol{X}} \notin \mathbb{J}_+(\mathcal{B}_m)$. This contradicts $\overline{\boldsymbol{X}} \in \bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}_k)$. Thus, we have shown that $\bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}_k) = \mathbb{J}_+(\mathcal{B})$. Since each $\mathcal{B}_m$ $(m = 1, 2, \ldots)$ satisfies condition (B) by Lemma 4.3, we obtain $\mathbb{J}_+(\mathcal{B}_m) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$, *i.e.*, $\mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}_m)\big) = \mathbb{J}_+(\mathcal{B}_m)$ by Theorem 1.2. Hence,

$$
\begin{aligned}
\mathrm{co}\Big(\boldsymbol{\Gamma}^n \cap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k)\big)\Big) &= \mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}_m)\big) = \mathbb{J}_+(\mathcal{B}_m) \\
&= \bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k) \supseteq \bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}_k) = \mathbb{J}_+(\mathcal{B}) \ (m = 1, 2, \ldots).
\end{aligned}
$$

Thus, $\bigcap_{m=1}^{\infty} \mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k)\big)\big) \supseteq \mathbb{J}_+(\mathcal{B})$. By Lemma 2.1, $\bigcap_{m=1}^{\infty} \mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}_k)\big)\big)$ $= \mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})\big)$. Consequently, $\mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})\big) \supseteq \mathbb{J}_+(\mathcal{B})$. The converse inclusion relation $\mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})\big) \subseteq \mathbb{J}_+(\mathcal{B})$ follows from the convexity of $\mathbb{J}_+(\mathcal{B})$. Therefore, we have shown that $\mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})\big) = \mathbb{J}_+(\mathcal{B})$. □

5 Reduction of $\mathrm{COP}(\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}')$ to $\mathrm{COP}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}))$ satisfying conditions (A-3), (A-4) and (A-5)

Let $\mathbb{J}' \subseteq \mathbb{S}^{n'}_+$ be a nonempty closed convex cone and $\boldsymbol{Q}', \boldsymbol{H}' \in \mathbb{S}^{n'}$. For a QCQP $\mathrm{COP}(\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}')$ and its SDP relaxation $\mathrm{COP}(\mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}')$, we will show that they are equivalent to $\mathrm{COP}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$ and its SDP relaxation $\mathrm{COP}(\mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$, respectively, for some $n \leq n'$, $\boldsymbol{Q}, \boldsymbol{H} \in \mathbb{S}^n$ and $\mathcal{B} \subset \mathbb{S}^n_+$ satisfying conditions (A-1), (A-3), (A-4) and (A-5).

We first apply the facial reduction [4] to $\mathrm{COP}(\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}')$ and its SDP relaxation $\mathrm{COP}(\mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}')$. Let $\mathbb{F}$ be the minimal face of $\mathbb{S}^{n'}_+$ that contains $\mathbb{J}'$. Then $\mathrm{COP}(\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}')$ and $\mathrm{COP}(\mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}')$ are equivalent to $\mathrm{COP}(\boldsymbol{\Gamma}^{n'} \cap \mathbb{F} \cap \mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}')$ and $\mathrm{COP}(\mathbb{F} \cap \mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}')$, respectively. It is well-known that every face of $\mathbb{S}^{n'}_+$ is isomorphic to $\mathbb{S}^n_+$ for some $n \leq n'$ [16]. Therefore, we can further transform them into $\mathrm{COP}(\boldsymbol{\Gamma}^n \cap \mathbb{J}, \boldsymbol{Q}, \boldsymbol{H})$ and $\mathrm{COP}(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H})$, respectively, for some cone $\mathbb{J} \subset \mathbb{S}^n_+$ such that $\mathbb{S}^n_+$ itself is the minimal face of $\mathbb{S}^n_+$ containing $\mathbb{J}$ (*i.e.*, $\mathbb{J} \cap \mathbb{S}^n_{++} \neq \emptyset$, condition (A-3)), and some $\boldsymbol{Q}, \boldsymbol{H} \in \mathbb{S}^n$.

If $\mathbb{F} = \mathbb{S}^{n'}_+$ then take $n = n'$, $\mathbb{J} = \mathbb{J}'$, $\boldsymbol{Q} = \boldsymbol{Q}'$ and $\boldsymbol{H} = \boldsymbol{H}'$. Assume that $\mathbb{F}$ is a proper face of $\mathbb{S}^{n'}_+$. As every proper face of $\mathbb{S}^n_+$ is exposed, there exists a nonzero $\boldsymbol{F} \in \mathbb{S}^n_+$ such that $\mathbb{F} = \{\boldsymbol{U} \in \mathbb{S}^{n'}_+ : \langle \boldsymbol{F}, \boldsymbol{U} \rangle = 0\}$. Let $n = n' - \mathrm{rank}\boldsymbol{F} \leq n' - 1$, and $\boldsymbol{P} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{n'})$ be an orthogonal matrix which diagonalizes $\boldsymbol{F}$ such that $\boldsymbol{P}^T \boldsymbol{F} \boldsymbol{P} = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n'})$, where each λ_i denotes the eigenvalue of $\boldsymbol{F}$ associated with eigenvector $\boldsymbol{p}_i$ $(i = 1, \ldots, n')$. With $n = n' - \mathrm{rank}\boldsymbol{F} \leq n' - 1$, we may assume that $\lambda_i = 0$ $(i = 1, \ldots, n)$ and $\lambda_i > 0$ $(i = n+1, \ldots, n')$. Then

$$
\begin{aligned}
\boldsymbol{P}^T \mathbb{F} \boldsymbol{P} &= \{\boldsymbol{Y} \in \mathbb{S}^{n'}_+ : \boldsymbol{Y} = \boldsymbol{P}^T \boldsymbol{U} \boldsymbol{P},\ \langle \boldsymbol{F}, \boldsymbol{U} \rangle = 0\} \\
&= \{\boldsymbol{Y} \in \mathbb{S}^{n'}_+ : \langle \boldsymbol{P}^T \boldsymbol{F} \boldsymbol{P}, \boldsymbol{Y} \rangle = 0\} \\
&= \{\boldsymbol{Y} \in \mathbb{S}^{n'}_+ : \langle \mathrm{diag}(\lambda_1, \ldots, \lambda_{n'}), \boldsymbol{Y} \rangle = 0\} \\
&= \left\{ \begin{pmatrix} \boldsymbol{X} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} \in \mathbb{S}^{n'}_+ : \boldsymbol{X} \in \mathbb{S}^n_+ \right\}.
\end{aligned}
$$

Thus $\mathbb{F}$ is transformed onto $\mathbb{S}^n_+$ by the automorphism $\boldsymbol{X} \rightarrow \boldsymbol{Y} = \boldsymbol{P}^T\boldsymbol{X}\boldsymbol{P}$. Since $\mathbb{J}' = \mathbb{J}' \cap \mathbb{F}$ by the definition of $\mathbb{F}$, both COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}'$) and COP($\mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}'$) can be transformed into problems in $\mathbb{S}^n_+$ as we see below.

Let

$$\begin{array}{l} \widetilde{\boldsymbol{P}} = \begin{pmatrix} \boldsymbol{p}_1 & \ldots & \boldsymbol{p}_n \end{pmatrix} \in \mathbb{R}^{n' \times n}, \ \boldsymbol{Q} = \widetilde{\boldsymbol{P}}^T \boldsymbol{Q}' \widetilde{\boldsymbol{P}} \in \mathbb{S}^n, \ \boldsymbol{H} = \widetilde{\boldsymbol{P}}^T \boldsymbol{H}' \widetilde{\boldsymbol{P}} \in \mathbb{S}^n, \\ \mathbb{J} = \widetilde{\boldsymbol{P}}^T \mathbb{J}' \widetilde{\boldsymbol{P}} = \left\{ \widetilde{\boldsymbol{P}}^T \boldsymbol{Y} \widetilde{\boldsymbol{P}} : \boldsymbol{Y} \in \mathbb{J}' \right\}. \end{array}$$

$\boldsymbol{Q}$, $\boldsymbol{H}$ and $\mathbb{J}$ can be regarded as 'projections' of $\boldsymbol{Q}'$, $\boldsymbol{H}'$ and $\mathbb{J}'$ onto the face $\mathbb{F}$ of $\mathbb{S}^{n'}$, respectively. Then

$$\begin{aligned} \boldsymbol{P}^T(\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}')\boldsymbol{P} &= \boldsymbol{P}^T(\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}' \cap \mathbb{F})\boldsymbol{P} \\ &= (\boldsymbol{P}^T\boldsymbol{\Gamma}^{n'}\boldsymbol{P}) \cap (\boldsymbol{P}^T\mathbb{J}'\boldsymbol{P}) \cap (\boldsymbol{P}^T\mathbb{F}\boldsymbol{P}) \\ &= (\boldsymbol{P}^T\boldsymbol{\Gamma}^{n'}\boldsymbol{P}) \cap (\boldsymbol{P}^T\mathbb{J}'\boldsymbol{P}) \cap \left\{ \begin{pmatrix} \boldsymbol{X} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} \in \mathbb{S}^{n'}_+ : \boldsymbol{X} \in \mathbb{S}^n_+ \right\} \\ &= \left\{ \begin{pmatrix} \boldsymbol{X} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} \in \mathbb{S}^{n'}_+ : \boldsymbol{X} \in \boldsymbol{\Gamma}^n \cap \mathbb{J} \right\}. \end{aligned} \tag{17}$$

Hence

$$\begin{aligned} \eta(\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}') &= \inf \left\{ \langle \boldsymbol{Q}', \boldsymbol{U} \rangle : \boldsymbol{Y} = \boldsymbol{P}^T\boldsymbol{U}\boldsymbol{P}, \ \boldsymbol{U} \in \boldsymbol{\Gamma}^{n'} \cap \mathbb{J}', \ \langle \boldsymbol{H}', \boldsymbol{U} \rangle = 1 \right\} \\ &= \inf \left\{ \langle \boldsymbol{P}^T\boldsymbol{Q}'\boldsymbol{P}, \boldsymbol{Y} \rangle : \boldsymbol{Y} \in \boldsymbol{P}^T(\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}')\boldsymbol{P}, \ \langle \boldsymbol{P}^T\boldsymbol{H}'\boldsymbol{P}, \boldsymbol{Y} \rangle = 1 \right\} \\ &= \inf \left\{ \langle \boldsymbol{P}^T\boldsymbol{Q}'\boldsymbol{P}, \left(\begin{smallmatrix} \boldsymbol{X} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{smallmatrix}\right) \rangle : \boldsymbol{X} \in \boldsymbol{\Gamma}^n \cap \mathbb{J}, \ \langle \boldsymbol{P}^T\boldsymbol{H}'\boldsymbol{P}, \left(\begin{smallmatrix} \boldsymbol{X} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{smallmatrix}\right) \rangle = 1 \right\} \\ &= \inf \left\{ \langle \boldsymbol{Q}, \boldsymbol{X} \rangle : \boldsymbol{X} \in \boldsymbol{\Gamma}^n \cap \mathbb{J}, \ \langle \boldsymbol{H}, \boldsymbol{X} \rangle = 1 \right\} = \eta(\boldsymbol{\Gamma}^n \cap \mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}). \end{aligned}$$

Similarly, we see that

$$\begin{aligned} \boldsymbol{P}^T(\mathbb{J}')\boldsymbol{P} &= \left\{ \begin{pmatrix} \boldsymbol{X} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} : \boldsymbol{X} \in \mathbb{J} \right\}, \qquad (18) \\ \eta(\mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}) &= \eta(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}). \end{aligned}$$

Thus we have shown that COP($\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}$) and COP($\mathbb{J}', \boldsymbol{Q}', \boldsymbol{H}$) are equivalent to COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}$) and COP($\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}$), respectively. It follows from the definition of $\mathbb{F}$ that $\mathbb{J} \cap \mathbb{S}^n_{++} \neq \emptyset$; hence condition (A-3) holds.

We can show that $\mathbb{J}' \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^{n'})$ if and only if $\mathbb{J} \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$. In fact, we observe that

$$\begin{array}{l} \mathrm{co}(\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}') = \mathbb{J}' \\ \quad \Leftrightarrow \ \boldsymbol{P}^T\big(\mathrm{co}(\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}')\boldsymbol{P} = \boldsymbol{P}^T\mathbb{J}'\boldsymbol{P} \\ \quad \Leftrightarrow \ \mathrm{co}\big(\boldsymbol{P}^T(\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}')\boldsymbol{P}\big) = \boldsymbol{P}^T\mathbb{J}'\boldsymbol{P} \\ \quad \Leftrightarrow \ \mathrm{co}\left(\left\{ \begin{pmatrix} \boldsymbol{X} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} \in \mathbb{S}^{n'}_+ : \boldsymbol{X} \in \boldsymbol{\Gamma}^n \cap \mathbb{J} \right\} \right) = \left\{ \begin{pmatrix} \boldsymbol{X} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} : \boldsymbol{X} \in \mathbb{J} \right\} \\ \qquad \text{(by (17) and (18))}. \end{array}$$

Therefore, $\mathbb{J}' \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^{n'}) \Leftrightarrow \mathbb{J} \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$ follows.

If $\mathbb{J} = \mathbb{S}^n_+$, then we simply take $\mathcal{B} = \{\boldsymbol{O}\}$ for $\mathbb{J} = \mathbb{J}_+(\mathcal{B})$. Thus, in the following discussion on constructing $\mathcal{B}$ satisfying conditions (A-1), (A-3), (A-4), and (A-5) for $\mathbb{J} = \mathbb{J}_+(\mathcal{B})$, we assume that $\mathbb{J} \neq \mathbb{S}^n_+$. If $\mathbb{J}' \subset \mathbb{S}^{n'}_+$ is described as $\mathbb{J}' = \mathbb{J}_+(\mathcal{B}')$ for some closed bounded set $\mathcal{B}' \subset \mathbb{S}^{n'}$, we let $\mathcal{B}^0 = \{\widetilde{\boldsymbol{P}}^T\boldsymbol{B}'\widetilde{\boldsymbol{P}} : \boldsymbol{B}' \in \mathcal{B}'\}$. Otherwise, let $\mathcal{B}^0 = \{\boldsymbol{Y} : \boldsymbol{Y} \in \mathbb{J}^*, \ \| \boldsymbol{Y} \| = 1\}$. In both cases, $\mathbb{J} = \mathbb{J}_+(\mathcal{B}^0)$ holds and $\mathcal{B}^0$ satisfies conditions (A-1), (A-2) and (A-3) (recall $\mathbb{J}$ has been constructed such that $\mathbb{J} \cap \mathbb{S}^n_{++} \neq \emptyset$). We may assume that $\| \boldsymbol{B} \| = 1$ for every $\boldsymbol{B} \in \mathcal{B}^0$. We will focus on how to remove redundant matrices from $\mathcal{B}^0$ to satisfy conditions (A-4) and (A-5) under conditions (A-1), (A-2) and (A-3). It should be noted that condition (A-2) may be lost. See Example 6.3.

For every $\boldsymbol{A}, \ \boldsymbol{B} \in \mathbb{S}^n$, define

$$\boldsymbol{A} \succeq_* \boldsymbol{B} \text{ if } \mathbb{J}_+(\boldsymbol{A}) \supseteq \mathbb{J}_+(\boldsymbol{B}) \text{ and } \boldsymbol{A} =_* \boldsymbol{B} \text{ if } \mathbb{J}_+(\boldsymbol{A}) = \mathbb{J}_+(\boldsymbol{B}).$$

Then $\boldsymbol{A} \succeq_*$ defines a *partial order* on $\mathbb{S}^n$ and $=_*$ establishes an *equivalence relation* on $\mathbb{S}^n$, satisfying the properties that

$$\begin{aligned}
&\boldsymbol{A} \succeq_* \boldsymbol{A}, \ \boldsymbol{A} =_* \boldsymbol{A} \text{ for every } \boldsymbol{A} \in \mathbb{S}^n \text{ (reflexive)}, \\
&\boldsymbol{A} =_* \boldsymbol{B} \text{ if } \boldsymbol{A} \succeq_* \boldsymbol{B} \text{ and } \boldsymbol{B} \succeq_* \boldsymbol{A} \text{ (antisymmetric)}, \\
&\boldsymbol{B} =_* \boldsymbol{A} \text{ if } \boldsymbol{A} =_* \boldsymbol{B} \text{ (symmetric)}, \\
&\boldsymbol{A} \succeq_* \boldsymbol{C} \text{ if } \ \boldsymbol{A} \succeq_* \boldsymbol{B} \text{ and } \boldsymbol{B} \succeq_* \boldsymbol{C} \text{ (transitive)}, \\
&\boldsymbol{A} =_* \boldsymbol{C} \text{ if } \boldsymbol{A} =_* \boldsymbol{B} \text{ and } \boldsymbol{B} =_* \boldsymbol{C} \text{ (transitive)}.
\end{aligned}$$

It is important to note that $\boldsymbol{A} - \boldsymbol{B} \in \mathbb{S}^n_+$ is only a sufficient condition for $\boldsymbol{A} \succeq_* \boldsymbol{B}$. For example, if $\boldsymbol{B} \in \mathbb{S}^n$ is negative definite, then $\boldsymbol{A} \succeq_* \boldsymbol{B}$ holds for any $\boldsymbol{A} \in \mathbb{S}^n$. $\boldsymbol{A}$ is identified with $\boldsymbol{A}'$ if $\boldsymbol{A} =_* \boldsymbol{A}'$. Let $[\boldsymbol{A}]$ denote a representative matrix of the class of matrices equivalent to $\boldsymbol{A} \in \mathbb{S}^n$; if $\boldsymbol{A} =_* \boldsymbol{B}$ then $[\boldsymbol{A}] = [\boldsymbol{B}]$. Let $\mathcal{C} \subseteq \mathbb{S}^n$. We call $[\boldsymbol{B}] \in \mathcal{C}$ a minimal element of $\mathcal{C}$ if there is no $[\boldsymbol{A}] \in \mathcal{C}$ such that $[\boldsymbol{B}] \succeq_* [\boldsymbol{A}]$ and $[\boldsymbol{B}] \neq [\boldsymbol{A}]$, and $[\boldsymbol{C}]$ a lower bound of $\mathcal{C}$ if $[\boldsymbol{A}] \succeq_* [\boldsymbol{C}]$ holds for every $[\boldsymbol{A}] \in \mathcal{C}$, where $[\boldsymbol{C}] \in \mathcal{C}$ is not required. If $[\boldsymbol{A}] \succeq_* [\boldsymbol{B}]$ or $[\boldsymbol{B}] \succeq_* [\boldsymbol{A}]$ holds for every $[\boldsymbol{A}], \ [\boldsymbol{B}] \in \mathcal{C}$, then we call $\mathcal{C}$ *totally ordered*. Define $\mathcal{B}^0 = \{[\boldsymbol{B}] : \boldsymbol{B} \in \mathcal{B}\}$.

Lemma 5.1. *For each $\boldsymbol{A} \in \mathcal{B}^0$, $\mathcal{B}^0(\boldsymbol{A}) \equiv \{\boldsymbol{B} \in \mathcal{B}^0 : \boldsymbol{A} \succeq_* \boldsymbol{B}\}$ has a minimal element $\boldsymbol{C} \in \mathcal{B}^0(\boldsymbol{A})$.*

Proof. By Zorn's Lemma (see, for example, [8, 9]), it suffices to show that every totally ordered subset $\mathcal{C}$ of the set $\mathcal{B}^0(\boldsymbol{A})$ has a lower bound $\overline{\boldsymbol{B}}$ in the set. If $\mathcal{C}$ itself has a minimal element $\overline{\boldsymbol{B}}$, then it is a lower bound of $\mathcal{C}$. Now we assume that $\mathcal{C}$ has no minimal element. Then we can take an infinite sequence $\{\boldsymbol{B}_k \in \mathcal{C} : k = 1, 2, \ldots\}$ satisfying

$$\boldsymbol{A} \succeq_* \boldsymbol{B}_k \succeq_* \boldsymbol{B}_{k+1}, \ \boldsymbol{B}_k \neq \boldsymbol{B}_{k+1} \ (k = 1, 2, \ldots) \text{ and } \forall \boldsymbol{C} \in \mathcal{C}, \ \exists k; \boldsymbol{C} \succeq_* \boldsymbol{B}_k.$$

Subsequently,

$$\begin{aligned}
&\mathbb{J}_+(\boldsymbol{B}_k) \supseteq \mathbb{J}_+(\boldsymbol{B}_{k+1}) \ (k = 1, 2, \ldots), \\
&\mathbb{J}_+(\boldsymbol{B}_m) = \bigcap_{k=1}^{m} \mathbb{J}_+(\boldsymbol{B}_k) \supset \bigcap_{k=1}^{\infty} \mathbb{J}_+(\boldsymbol{B}_k).
\end{aligned} \quad (19)$$

Since $\mathcal{B}$ satisfies conditions (A-1) and (A-2), we may assume that $\boldsymbol{B}_k$ converges some $\widetilde{\boldsymbol{B}} \in \mathcal{B}$. We will show that $\bigcap_{k=1}^{\infty} \mathbb{J}_+(\boldsymbol{B}_k) \supseteq \mathbb{J}_+(\widetilde{\boldsymbol{B}})$. Assume on the contrary that $\widetilde{\boldsymbol{X}} \notin \bigcap_{k=1}^{\infty} \mathbb{J}_+(\boldsymbol{B}_k)$ for some $\widetilde{\boldsymbol{X}} \in \mathbb{J}_+(\widetilde{\boldsymbol{B}})$. Then

$$\langle \boldsymbol{B}_m, \widetilde{\boldsymbol{X}} \rangle < 0 \text{ for some } m \text{ and } \widetilde{\boldsymbol{X}} \in \mathbb{S}^n_+, \ \langle \widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{X}} \rangle \geq 0.$$

If $\widetilde{\boldsymbol{B}} \in \mathcal{B}$ with $\| \widetilde{\boldsymbol{B}} \| = 1$ was negative semidefinite, then $\mathbb{J}_+(\widetilde{\boldsymbol{B}})$ would form a proper face of $\mathbb{S}^n_+$. Since $\mathbb{J} \subseteq \bigcap_{k=1}^{\infty} \mathbb{J}_+(\boldsymbol{B}_k) \subseteq \mathbb{J}_+(\widetilde{\boldsymbol{B}})$, $\mathbb{J}$ would be contained in the proper face of $\mathbb{S}^n_+$, which would contradict condition (A-3). Hence $\langle \widetilde{\boldsymbol{B}}, \widehat{\boldsymbol{X}} \rangle > 0$ for some $\widehat{\boldsymbol{X}} \in \mathbb{S}^n_{++}$. We can take sufficiently small $\epsilon > 0$ such that

$$\boldsymbol{X}(\epsilon) \in \mathbb{S}_{++}, \ \langle \boldsymbol{B}_m, \boldsymbol{X}(\epsilon) \rangle < 0 \text{ and } \langle \widetilde{\boldsymbol{B}}, \boldsymbol{X}(\epsilon) \rangle > 0, \tag{20}$$

where $\boldsymbol{X}(\epsilon) = \widetilde{\boldsymbol{X}} + \epsilon \widehat{\boldsymbol{X}}$. We then see from (19) that $\langle \boldsymbol{B}_k, \boldsymbol{X}(\epsilon) \rangle < 0$ $(k \geq m)$. Taking the limit as $k \to \infty$, we then obtain $\langle \widetilde{\boldsymbol{B}}, \boldsymbol{X}(\epsilon) \rangle \leq 0$, which contradicts the last inequality of (20). Thus we have shown $\bigcap_{k=1}^{\infty} \mathbb{J}_+(\boldsymbol{B}_k) \supseteq \mathbb{J}_+(\widetilde{\boldsymbol{B}})$. This inclusion relation together with (19) implies $\mathbb{J}_+(\boldsymbol{B}_k) \supseteq \mathbb{J}_+(\widetilde{\boldsymbol{B}})$ or $\boldsymbol{B}_k \succeq_* \widetilde{\boldsymbol{B}}$ $(k = 1, 2, \ldots)$. Therefore $\overline{\boldsymbol{B}} = [\widetilde{\boldsymbol{B}}] \in \mathcal{B}^0(\boldsymbol{A})$ is a lower bound of $\mathcal{C}$. □

By Lemma 5.1, we can consistently define

$$\mathcal{B} \ = \ \begin{cases} \{\boldsymbol{O}\} & \text{if } \mathbb{J} = \mathbb{S}^n_+, \\ \left\{[\boldsymbol{B}] \in \mathcal{B}^0 : [\boldsymbol{B}] \text{ is a minimal element of } \mathcal{B}^0\right\} & \text{otherwise.} \end{cases}$$

Theorem 5.2. $\mathbb{J}_+(\mathcal{B}) = \mathbb{J}$, *and conditions (A-1), (A-3), (A-4), and (A-5) hold.*

Proof. If $\mathbb{J} = \mathbb{S}^n_+$, then $\mathcal{B} = \{\boldsymbol{O}\}$ and all the conditions are obviously satisfied. So we assume that $\mathbb{J} \neq \mathbb{S}^n_+$. Since $\mathcal{B} \subseteq \mathcal{B}^0$, $\mathbb{J}_+(\mathcal{B}) \supseteq \mathbb{J}_+(\mathcal{B}^0) = \mathbb{J}$ follows. To see the converse inclusion $\mathbb{J}_+(\mathcal{B}) \subseteq \mathbb{J}_+(\mathcal{B}^0)$, assume that $\boldsymbol{X} \in \mathbb{J}_+(\mathcal{B})$. By Lemma 5.1, for every $\boldsymbol{A} \in \mathcal{B}^0$, there exists $\boldsymbol{B} \in \mathcal{B}$ such that $\boldsymbol{A} \succeq_* \boldsymbol{B}$. Hence $0 \leq \langle \boldsymbol{B}, \boldsymbol{X} \rangle \leq \langle \boldsymbol{A}, \boldsymbol{X} \rangle$ holds. Therefore, $\boldsymbol{X} \in \mathbb{J}_+(\mathcal{B}^0)$ and $\mathbb{J}_+(\mathcal{B}) \subseteq \mathbb{J}_+(\mathcal{B}^0) = \mathbb{J}$ have been shown. Since $\mathcal{B} \subset \mathcal{B}^0$ and $\mathcal{B}^0$ is bounded, $\mathcal{B}$ is also bounded. Hence (A-1) is satisfied. Since $\mathbb{J} \cap \mathbb{S}^n_{++} \neq \emptyset$ and $\mathbb{J}_+(\mathcal{B}) = \mathbb{J}$, (A-3) is satisfied. By the construction of $\mathcal{B}$, $\mathcal{B}$ satisfies (A-5). It follows from $\mathbb{J} \neq \mathbb{S}^n_+$ that $\mathcal{B}$ includes an $\boldsymbol{A} \notin \mathbb{S}^n_+$. Then $\boldsymbol{B} \succeq_* \boldsymbol{A}$ and $\boldsymbol{A} \not\succeq_* \boldsymbol{B}$ for any $\boldsymbol{B} \in \mathbb{S}^n_+$. Hence $\mathcal{B}$ cannot contain any $\boldsymbol{B} \in \mathbb{S}^n_+$, and condition (A-4) is satisfied. □

6 Examples

In [2, Section 4.1], several examples satisfying condition (B) with finite $\mathcal{B}$ were provided. We present five examples that are not covered by those cases in this section. We recall that if $\mathcal{B}$ satisfies condition (B), then so does $\{\boldsymbol{L}^T\boldsymbol{B}\boldsymbol{L} : \boldsymbol{B} \in \mathcal{B}\}$ for every $n \times n'$ matrix L with $1 \leq n'$ by Theorem 3.7. This is illustrated in Example 6.3.

Example 6.1. This example shows that the reduction of $\text{COP}(\boldsymbol{\Gamma}^{n'} \cap \mathbb{J}')$ to $\text{COP}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}))$ with $\mathcal{B}$ satisfying conditions (A-3), (A-4) and (A-5), which has been described in Section 5, is effective. Let $n' = 4$ and $\mathcal{B}' = \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$. Then $\mathbb{J}' = \mathbb{J}_+(\mathcal{B}')$, where

$$\boldsymbol{A'} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ \boldsymbol{B'} = \begin{pmatrix} -1 & -2 & 0 & -1 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & -1 \end{pmatrix}, \ \boldsymbol{C'} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 1 & 0 & 2 & -1 \end{pmatrix}.$$

Let $\widetilde{\boldsymbol{X}} = \mathrm{diag}(0,0,1,1)$ (the 4×4 diagonal matrix with elements $0,0,1,1$). Then $\widetilde{\boldsymbol{X}} \in \mathbb{J}_0(\boldsymbol{B'})$ but $\widetilde{\boldsymbol{X}} \notin \mathbb{J}_+(\boldsymbol{A'})$. Hence $\mathcal{B}'$ does not satisfy condition (B). Also, (8) in Theorem 3.4 dose not hold. In fact, assume on the contrary that (8) holds, which implies that the diagonal of $\alpha \boldsymbol{A'} + \beta \boldsymbol{B'} \in \mathbb{S}^4_+$ is nonnegative. Hence

$$2\alpha - \beta \geq 0, \ \alpha - \beta \geq 0, \ -\alpha + \beta \geq 0, \ -\alpha - \beta \geq 0. \tag{21}$$

Obviously, only $(\alpha, \beta) = \boldsymbol{0}$ satisfies the above inequalities as shown in Figure 3 which illustrates the region of (α, β) determined by the first, third and fourth inequalities in (21). Therefore, (8) in Theorem 3.4 dose not hold.

Now, we apply the reduction stated in Section 5 to $\mathrm{COP}(\boldsymbol{\Gamma}^4 \cap \mathbb{J}_+(\mathcal{B}'))$. We first observe that

$$\begin{aligned} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{J}_+(\mathcal{B}') \quad & \subseteq \quad \left\{ \boldsymbol{X} \in \mathbb{S}^4_+ : \langle \boldsymbol{B'} + \boldsymbol{C'}, \, \boldsymbol{X} \rangle \geq 0 \right\} \\ & = \quad \left\{ \boldsymbol{X} \in \mathbb{S}^4_+ : \langle \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}, \, \boldsymbol{X} \rangle \geq 0 \right\} \\ & = \quad \mathbb{F}, \text{ where } \mathbb{F} = \left\{ \begin{pmatrix} \boldsymbol{U} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} \in \mathbb{S}^4_+ : \boldsymbol{U} \in \mathbb{S}^2_+ \right\}. \end{aligned}$$

Since the left 4×4 positive semidefinite matrix with rank 2 is contained in $\mathbb{F} \cap \mathbb{J}_+(\mathcal{B}')$, $\mathbb{F}$ forms the minimal face of $\mathbb{S}^4_+$ that contains $\mathbb{J}_+(\mathcal{B}')$. Thus, identifying $\mathbb{F}$ with $\mathbb{S}^2_+$, we obtain $\mathrm{COP}(\boldsymbol{\Gamma}^2 \cap \mathbb{J})$ equivalent to $\mathrm{COP}(\boldsymbol{\Gamma}^4 \cap \mathbb{J}_+(\mathcal{B}'))$, where $\mathbb{J} = \mathbb{F} \cap \mathbb{J}_+(\mathcal{B}')$. Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be projections of $\boldsymbol{A'}$, $\boldsymbol{B'}$ and $\boldsymbol{C'}$ onto $\mathbb{F}$, respectively, such that

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \ \boldsymbol{B} = \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}, \ \boldsymbol{C} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Then $\mathbb{S}^2_{++} \ni \boldsymbol{A} \succeq_* \boldsymbol{B}$. Hence $\boldsymbol{A}$ is redundant. Letting $\mathcal{B} = \{\boldsymbol{B}, \boldsymbol{C}\}$, we obtain $\mathcal{B}$ satisfying conditions (A-1) through (A-5), and $\mathrm{COP}(\boldsymbol{\Gamma}^4 \cap \mathbb{J}_+(\mathcal{B}'))$ has been reduced to $\mathrm{COP}(\boldsymbol{\Gamma}^2 \cap \mathbb{J}_+(\mathcal{B}))$. Obviously, $\boldsymbol{B} + \boldsymbol{C} = \boldsymbol{O} \in \mathbb{S}^n_+$. Hence $\langle \boldsymbol{B}, \, \boldsymbol{X} \rangle = 0$ if and only if $\langle \boldsymbol{C}, \, \boldsymbol{X} \rangle = 0$, which implies that condition (B) holds. Therefore $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^2)$ and $\mathbb{J}_+(\mathcal{B}') \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^4)$. We also see that $\mathbb{J}_+(\mathcal{B}) = \mathbb{J}_0(\boldsymbol{B}) = \mathbb{J}_0(\boldsymbol{C})$ holds. Thus case (a) in Theorem 3.6 occurs.

Example 6.2. This example provides an infinite $\mathcal{B} \subseteq \mathbb{S}^n$ satisfying conditions (B)' and (C)'. Let

$$\boldsymbol{B}(\boldsymbol{t}) = \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{t} \\ -\boldsymbol{t}^T & \boldsymbol{t}^T\boldsymbol{t} - r^2 \end{pmatrix} \in \mathbb{S}^n \ (\boldsymbol{t} \in T), \ \mathcal{B} = \{\boldsymbol{B}(\boldsymbol{t}) : \boldsymbol{t} \in T\},$$

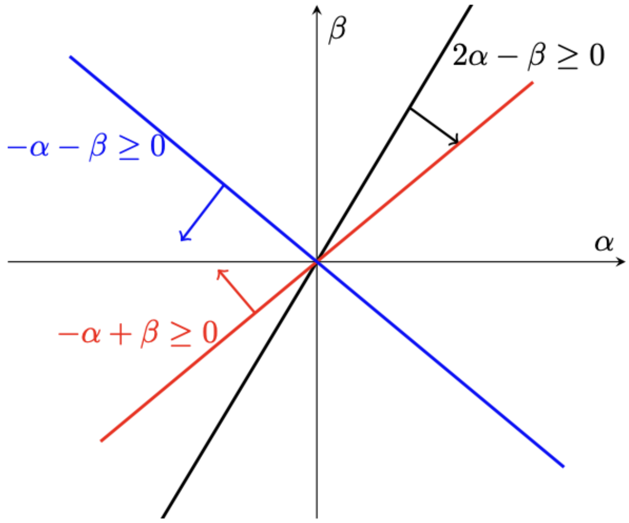

Figure 3: The region of (α, β) determined by the first, third and fourth inequalities in (21).

where $0 < r \leq 1/2$, $T \subseteq \mathbb{Z}^n$ (the set of integer vectors in $\mathbb{R}^n$) and $\boldsymbol{I}$ denotes the $(n-1) \times (n-1)$ identity matrix. Then,

$$
\begin{aligned}
& q(\boldsymbol{u}, z, \boldsymbol{B}(\boldsymbol{t})) = \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix}^T \boldsymbol{B}(\boldsymbol{t}) \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix} = \| \boldsymbol{u} - \boldsymbol{t}z \|^2 - r^2 z^2, \\
& \not\!f_+(1, \boldsymbol{B}(\boldsymbol{t})),\ \not\!f_0(1, \boldsymbol{B}(\boldsymbol{t})) \text{ or } \not\!f_-(1, \boldsymbol{B}(\boldsymbol{t})) \\
& \qquad\qquad = \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : \| \boldsymbol{u} - \boldsymbol{t} \|^2 - r^2 \geq 0,\ = 0 \text{ or } \leq 0, \text{ respectively} \right\}
\end{aligned}
$$

for every $\boldsymbol{t} \in T$ and $(\boldsymbol{u}, z) \in \mathbb{R}^n$. See Figure 4. It is easily seen that conditions (B)' and (C)' are satisfied. Therefore, by Theorem 1.3, we obtain $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$.

As a generalization, it is straightforward to construct an ellipsoid-based constraint by replacing each $\boldsymbol{B}(\boldsymbol{t})$ with $\boldsymbol{L}^T \boldsymbol{B}(\boldsymbol{t}) \boldsymbol{L}$ ($\boldsymbol{t} \in T$), where $\boldsymbol{L}$ denotes an $n \times n$ nonsingular matrix of the form $\boldsymbol{L} = \begin{pmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix}$. We also note that the equivalence relation (1) between COP($\mathbb{J} \cap \boldsymbol{\Gamma}^n, \boldsymbol{Q}, \boldsymbol{H}$) and its SDP relaxation COP($\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}$) with $\mathbb{J} = \mathbb{J}_+(\mathcal{B})$ (or $\mathbb{J} = \mathbb{J}_+(\{\boldsymbol{L}^T \boldsymbol{B} \boldsymbol{L} : \boldsymbol{B} \in \mathcal{B}\})$) holds for any choice of $\boldsymbol{Q} \in \mathbb{S}^n$ and $\boldsymbol{H} \in \mathbb{S}^n$ by Theorem 1.1. For example, we can take $\boldsymbol{H} = \delta \boldsymbol{I}$ for some $\delta > 0$ where $\boldsymbol{I}$ is the $n \times n$ identity matrix. In this case, COP($\mathbb{J} \cap \boldsymbol{\Gamma}^n, \boldsymbol{Q}, \boldsymbol{H}$) turns out to be

$$
\eta(\mathbb{J} \cap \boldsymbol{\Gamma}^n, \boldsymbol{Q}, \boldsymbol{H}) \quad = \quad \inf \left\{ \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{z} \end{pmatrix}^T \boldsymbol{Q} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{z} \end{pmatrix} : \begin{array}{l} \| \boldsymbol{u} - \boldsymbol{t}z \|^2 - r^2 z^2 \geq 0 \ (\boldsymbol{t} \in T), \\ (\text{or } \| \boldsymbol{M}\boldsymbol{u} - \boldsymbol{t}z \|^2 - r^2 z^2 \geq 0 \ (\boldsymbol{t} \in T)), \\ u_1^2 + \cdots + u_{n-1}^2 + z^2 = 1/\delta \end{array} \right\}.
$$

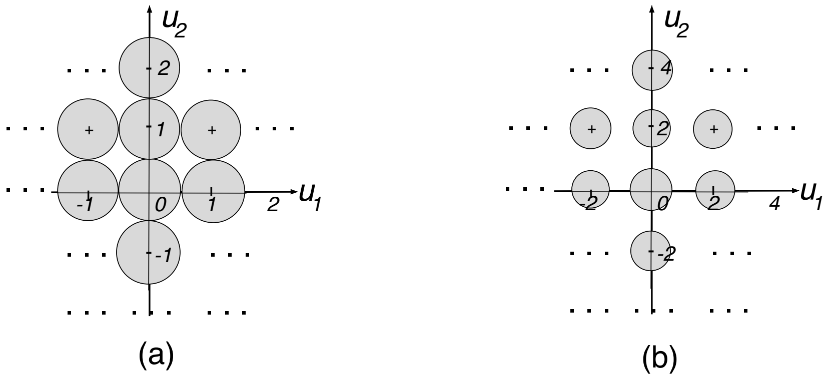

Figure 4: Illustration of $\not\!\!f_-(1,\boldsymbol{B}(\boldsymbol{t}))$. Each gray disk region corresponds to $\not\!\!f_-(1,\boldsymbol{B}(\boldsymbol{t}))$ for some $\boldsymbol{t} \in T$. (a) $T = \mathbb{Z}^{n-1}$ and $r = 1/2$. (b) $T = 2\mathbb{Z}^{n-1}$ and $r = 1/3$.

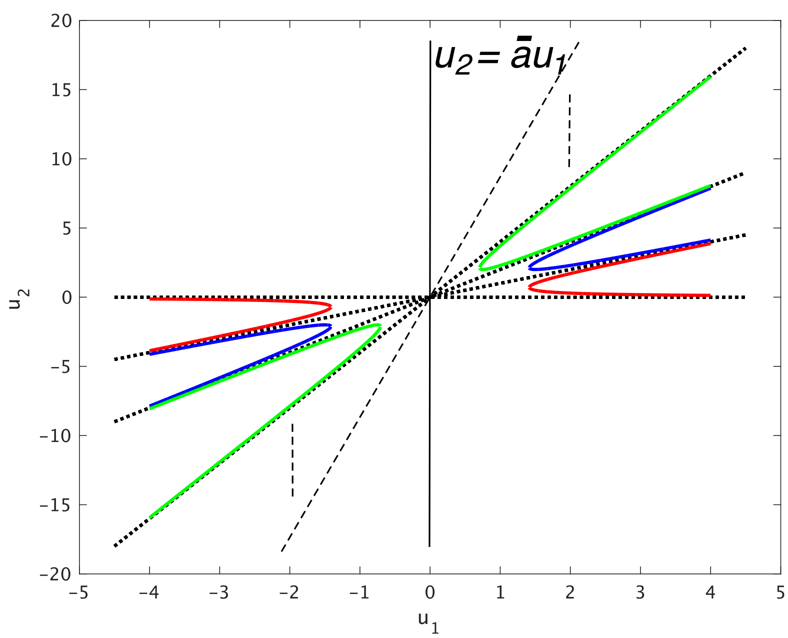

Figure 5: Example 6.3. We take $a_0 = 0,\ a_1 = 1,\ a_2 = 2,\ a_3 = 4$. The red, blue and green curves represent the hyperbolas defined by $(u_2 - a_{k-1}u_1)(a_k u_1 - u_2) - 0.5 = 0$ $(k = 1, 2, 3)$, respectively.

Example 6.3. This example presents another infinite $\mathcal{B}$ satisfying conditions (B)' and (C)'. Let

$$\boldsymbol{C}_k = \begin{pmatrix} a_{k-1}a_k & -\frac{a_{k-1}+a_k}{2} \\ -\frac{a_{k-1}+a_k}{2} & 1 \end{pmatrix} \in \mathbb{S}^2, \boldsymbol{B}_k = \begin{pmatrix} \boldsymbol{C}_k & \boldsymbol{0} \\ \boldsymbol{0}^T & r^2 \end{pmatrix} \in \mathbb{S}^3 \ (1 \leq k < \infty),$$
$$\mathcal{B}_m = \{\boldsymbol{B}_k : k = 1, \ldots, m\} \ (m = 1, 2, \ldots), \ \mathcal{B} = \{\boldsymbol{B}_k : k = 1, 2, \ldots\},$$

where $0 < r^2$ and $\{a_k : k = 0, 1, \ldots\}$ denotes an infinite sequence of nonnegative real numbers such that $a_{k-1} < a_k$ $(k = 1, 2, \ldots)$ and $a_k \to \bar{a}$ as $k \to \infty$ for some $\bar{a} > 0$. Then,

$$q(\boldsymbol{u}, y, \boldsymbol{B}_k) = \begin{pmatrix} \boldsymbol{u} \\ y \end{pmatrix}^T \boldsymbol{B}_k \begin{pmatrix} \boldsymbol{u} \\ y \end{pmatrix} = (u_2 - a_{k-1}u_1)(u_2 - a_k u_1) + r^2 y^2,$$
$$\mathcal{f}_+(1, \boldsymbol{B}_k), \ \mathcal{f}_0(1, \boldsymbol{B}_k) \text{ or } \mathcal{f}_-(1, \boldsymbol{B}_k) =$$
$$\left\{\boldsymbol{u} \in \mathbb{R}^{n-1} : (u_2 - a_{k-1}u_1)(u_2 - a_k u_1) + r^2 \geq 0, \ = 0, \text{ or } \leq 0, \text{ respectively}\right\}$$

for every $(\boldsymbol{u}, y) \in \mathbb{R}^3$ $(k = 1, 2, \ldots)$. We note that $\mathcal{f}_0(1, \boldsymbol{B}_k)$ forms a hyperbola with the asymptote $\{\boldsymbol{u} \in \mathbb{R}^2 : u_2 - a_{k-1}u_1 = 0\}$ and $\{\boldsymbol{u} \in \mathbb{R}^2 : a_k u_1 - u_2 = 0\}$ $(1 \leq k < \infty)$. See Figure 5. From the figure, we see that $\mathcal{B}$ satisfies conditions (B)' and (C)'. Therefore, by Theorem 1.3, we obtain $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$.

Each $\mathcal{B}_m$ $(1 \leq m < \infty)$ is finite so that it is obviously closed, but $\mathcal{B}$ is not. In fact, $\boldsymbol{B}_k \in \mathcal{B}$ converges $\overline{\boldsymbol{B}} = \begin{pmatrix} \overline{\boldsymbol{C}} & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \notin \mathcal{B}$ as $k \to \infty$, where $\overline{\boldsymbol{C}} = \begin{pmatrix} \bar{a}^2 & -\bar{a} \\ -\bar{a} & 1 \end{pmatrix}$. Hence

$$q(\boldsymbol{u}, y, \overline{\boldsymbol{B}}) = (u_2 - \bar{a}u_1)^2 + r^2 y^2, \ \mathcal{f}_+(y, \overline{\boldsymbol{B}}) = \mathbb{R}^2, \ \mathcal{f}_{--}(y, \overline{\boldsymbol{B}}) = \emptyset,$$
$$\mathcal{f}_-(y, \overline{\boldsymbol{B}}) = \begin{cases} \emptyset & \text{if } y \in (0, 1], \\ \{\boldsymbol{u} \in \mathbb{R}^2 : u_2 - \bar{a}u_1 = 0\}, & \text{if } y = 0 \end{cases}$$

for every $(\boldsymbol{u}, y) \in \mathbb{R}^2 \times [0, 1]$. We also see $\overline{\boldsymbol{B}} \in \mathbb{S}^3_+$. Hence $\mathbb{J}_+(\boldsymbol{B}_k) \subseteq \mathbb{S}^3_+ = \mathbb{J}_+(\overline{\boldsymbol{B}})$ $(k = 1, 2, \ldots,)$. Therefore, cl$\mathcal{B}$ does not satisfy condition (A-4), (A-5) and (C)' although $\mathbb{J}_+(\text{cl}\mathcal{B}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^3)$ remains true.

As an application of Theorem 3.7, we can extend the above 3-dimensional QCQP to a general n'-dimensional QCQP where $n' \geq 3$. Let

$$\boldsymbol{L} \ = \ \begin{pmatrix} \boldsymbol{b}^T & 0 \\ \boldsymbol{c}^T & 0 \\ \boldsymbol{0}^T & 1 \end{pmatrix} \in \mathbb{R}^{3 \times n'},$$

where $\boldsymbol{b}, \ \boldsymbol{c} \in \mathbb{R}^{n'-1}$ are linearly independent, and apply the linear transformation $\begin{pmatrix} \boldsymbol{u} \\ y \end{pmatrix} = \boldsymbol{L} \begin{pmatrix} \boldsymbol{v} \\ z \end{pmatrix}$ to the feasible region of COP($\boldsymbol{\Gamma}^3 \cap \mathbb{J}_+(\mathcal{B})$) above. Let $\boldsymbol{B}'_k = \boldsymbol{L}^T \boldsymbol{B}_k \boldsymbol{L}$ $(k = 1, 2, \ldots)$ and $\mathcal{B}' = \{\boldsymbol{B}'_k : k = 1, 2, \ldots\}$. Then we know that $\mathcal{B}'$ satisfies condition (B) by Theorem 3.7. In this case, we see that

$$q(\boldsymbol{v}, z, \boldsymbol{B}'_k) = \begin{pmatrix} \boldsymbol{v} \\ z \end{pmatrix}^T \boldsymbol{B}'_k \begin{pmatrix} \boldsymbol{v} \\ z \end{pmatrix} = (\boldsymbol{b}^T\boldsymbol{v} - a_{k-1}\boldsymbol{c}^T\boldsymbol{v})(\boldsymbol{b}^T\boldsymbol{v} - a_k\boldsymbol{c}^T\boldsymbol{v}) + r^2 z^2,$$
$$\mathcal{f}_+(1, \boldsymbol{B}'_k), \ \mathcal{f}_0(1, \boldsymbol{B}'_k) \text{ or } \mathcal{f}_-(1, \boldsymbol{B}'_k) =$$
$$\left\{\boldsymbol{v} \in \mathbb{R}^{n'-1} : (\boldsymbol{b}^T\boldsymbol{v} - a_{k-1}\boldsymbol{c}^T\boldsymbol{v})(\boldsymbol{b}^T\boldsymbol{v} - a_k\boldsymbol{c}^T\boldsymbol{v}) + r^2 \geq 0, \ = 0 \text{ or } \leq 0, \text{ respectively}\right\}$$

for every $(\boldsymbol{v}, z) \in \mathbb{R}^{n'-1} \times [0,1]$ $(k = 1, 2, \ldots)$. Since $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly independent, so are $\boldsymbol{b} - a_{k-1}\boldsymbol{c}$ and $\boldsymbol{b} - a_k\boldsymbol{c}$ $(k = 1, 2, \ldots)$. Hence, for every $k = 1, 2, \ldots$, the linear equation

$$(\boldsymbol{b} - a_{k-1}\boldsymbol{c})^T\boldsymbol{v} = 2r, \ (\boldsymbol{b} - a_k\boldsymbol{c})^T\boldsymbol{v} = -r$$

has a solution $\boldsymbol{v}_k$, which satisfies $q(\boldsymbol{v}_k, 1, \boldsymbol{B}_k) = -r^2 < 0$; hence $\boldsymbol{v}_k \in ⨏_{--}(1, \boldsymbol{B}_k)$ and $\mathcal{B}'$ satisfies condition (C)'. We also see that

$$\begin{aligned} \boldsymbol{v} \in ⨏_{-}(1, \boldsymbol{B}'_k) \ \Rightarrow \ & 0 < \boldsymbol{c}^T\boldsymbol{v} \text{ and } a_{k-1}\boldsymbol{c}^T\boldsymbol{v} \le \boldsymbol{b}^T\boldsymbol{v} \le a_k\boldsymbol{c}^T\boldsymbol{v}, \text{ or} \\ & \boldsymbol{c}^T\boldsymbol{v} < 0 \text{ and } a_k\boldsymbol{c}^T\boldsymbol{v} \le \boldsymbol{b}^T\boldsymbol{v} \le a_{k-1}\boldsymbol{c}^T\boldsymbol{v} \end{aligned}$$

$(k = 1, 2, \ldots)$, which implies $⨏_{-}(1, \boldsymbol{B}'_k) \cap ⨏_{--}(1, \boldsymbol{B}'_\ell) = \emptyset$ if $k \neq \ell$. Thus $\mathcal{B}'$ satisfies condition (B)' and $\mathbb{J}_+(\mathcal{B}') \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$ by Theorem 1.3.

Example 6.4. We generalize the hyperbola-based constraint in $\mathbb{R}^2$ given in Example 6.3. Let $n \ge 3$ and $1 \le \ell \le n-2$ be fixed. For every $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) > \boldsymbol{0}$ and $\sigma \in \mathbb{R}$, we consider a homogeneous quadratic function (quadratic form) in $(\boldsymbol{u}, z) \in \mathbb{R}^n$:

$$q(\boldsymbol{u}, z) = -\sum_{i=1}^{\ell} \lambda_i u_i^2 + \sum_{j=\ell+1}^{n-1} \sum_{i=1}^{\ell} \lambda_j (u_j - \sigma u_i)^2 + \lambda_n z^2. \tag{22}$$

We can take $\boldsymbol{B}(\boldsymbol{\lambda}, \sigma) \in \mathbb{S}^n$ such that $q(\boldsymbol{u}, z) = \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix}^T \boldsymbol{B}(\boldsymbol{\lambda}, \sigma) \begin{pmatrix} \boldsymbol{u} \\ z \end{pmatrix}$ for every $(\boldsymbol{u}, z) \in \mathbb{R}^n$, but the precise description of $\boldsymbol{B}(\boldsymbol{\lambda}, \sigma)$ is not relevant in the subsequent discussion. When $n = 3$, $\ell = 1$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, and $\sigma = 0$, $⨏_0(1, \boldsymbol{B}(\boldsymbol{\lambda}, \sigma))$ forms a simple 2-dimensional hyperbola illustrated in Figure 1 (b). In general, we have

$$\begin{aligned} & ⨏_+(1, \boldsymbol{B}(\boldsymbol{\lambda}, \sigma)) \text{ or } ⨏_-(1, \boldsymbol{B}(\boldsymbol{\lambda}, \sigma)) \\ & \quad = \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : -\sum_{i=1}^{\ell} \lambda_i u_i^2 + \sum_{j=\ell+1}^{n-1} \sum_{i=1}^{\ell} \lambda_j (u_j - \sigma u_i)^2 + \lambda_n \ge 0 \text{ or } \le 0, \text{ respectively} \right\}. \end{aligned}$$

Let $\boldsymbol{\lambda} > \boldsymbol{0}$ and $\sigma \in \mathbb{R}$ be fixed. We will show that

$$⨏_-(1, \boldsymbol{B}(\boldsymbol{\lambda}, \tau)) \cap ⨏_-(1, \boldsymbol{B}(\boldsymbol{\lambda}, \sigma)) = \emptyset \text{ for every sufficiently large } \tau > \sigma. \tag{23}$$

Assume on the contrary that for every $\tau > \sigma$, there exists a $\boldsymbol{u} = \boldsymbol{u}(\tau) \in ⨏_-(1, \boldsymbol{B}(\boldsymbol{\lambda}, \sigma)) \cap ⨏_-(1, \boldsymbol{B}(\boldsymbol{\lambda}, \tau))$. Let $\max\{\lambda_i u_i^2 : i = 1, \ldots, \ell\} = \lambda_k u_k^2$. Then $\lambda_n \le \sum_{i=1}^{\ell} \lambda_i u_i^2 \le \ell \lambda_k u_k^2$. Here, k may depend on $\boldsymbol{u}(\tau)$ but we may assume without loss of generality that a common k can be taken along a sequence $\{\boldsymbol{u}(\tau^p) : p = 1, 2, \ldots\}$ with $\tau^p \to \infty$ as $p \to \infty$. Let $j' \in \{\ell+1, \ldots, n-1\}$ be fixed arbitrary. We observe that

$$\begin{aligned} \boldsymbol{u} \in ⨏_-(1, \boldsymbol{B}(\boldsymbol{\lambda}, \sigma)) \ \Rightarrow \ & \sum_{j=\ell+1}^{n-1} \sum_{i=1}^{\ell} \lambda_j (u_j - \sigma u_i)^2 + \lambda_n \le \sum_{i=1}^{\ell} \lambda_i u_i^2 \\ \Rightarrow \ & \lambda_{j'} (u_{j'} - \sigma u_k)^2 \le \sum_{i=1}^{\ell} \lambda_i u_i^2 \le \ell \lambda_k u_k^2 && (24) \\ \Rightarrow \ & |u_{j'} - \sigma u_k| \le \sqrt{\ell \lambda_k / \lambda_{j'}} |u_k|, && (25) \end{aligned}$$

Similarly,

$$\boldsymbol{u} \in \mathbb{f}_-(1, \boldsymbol{B}(\boldsymbol{\lambda}, \tau)) \;\Rightarrow\; \lambda_{j'}(u_{j'} - \tau u_k)^2 \leq \sum_{i=1}^{\ell} \lambda_i u_i^2 \leq \ell \lambda_k u_k^2. \tag{26}$$

Hence,

$$\begin{array}{rcl} \lambda_{j'}(u_{j'} - \tau u_k)^2 & = & \lambda_{j'}\big(- (\tau - \sigma)u_k + u_{j'} - \sigma u_k\big)^2 \\ & \geq & \lambda_{j'}\big((\tau-\sigma)^2 u_k^2 - 2(\tau-\sigma)|u_{j'} - \tau u_k||u_k| - (u_{j'} - \sigma u_k)^2\big) \\ & \geq & \lambda_{j'}\big((\tau-\sigma)^2 u_k^2 - 2(\tau-\sigma)\sqrt{\ell\lambda_k/\lambda_{j'}}u_k^2 - (\ell\lambda_k/\lambda_{j'})u_k^2\big) \\ & & \text{(by (24) and (25))} \\ & = & \lambda_{j'}\big((\tau-\sigma)^2 - 2(\tau-\sigma)\sqrt{\ell\lambda_k/\lambda_{j'}} - (\ell\lambda_k/\lambda_{j'})\big)u_k^2 \\ & > & \ell\lambda_k u_k^2 \text{ for every sufficiently large } \tau > 0. \end{array}$$

(Choose a τ such that $\lambda_{j'}\big((\tau-\sigma)^2 - 2(\tau-\sigma)\sqrt{\ell\lambda_k/\lambda_{j'}} - (\ell\lambda_k/\lambda_{j'})\big) > \ell\lambda_k$.) The last inequality contradicts (26). Thus we have shown (23). As a consequence, we can take a finite or infinite monotone increasing sequence $\Sigma = \{\tau^p \geq \sigma : p = 1, 2, \ldots\}$ such that

$$\mathbb{f}_-(1, \boldsymbol{B}(\boldsymbol{\lambda}, \sigma)) \cap \mathbb{f}_-(1, \boldsymbol{B}(\boldsymbol{\lambda}, \tau)) = \emptyset \text{ for every distinct } \sigma,\ \tau \in \Sigma.$$

Thus $\mathcal{B} = \{\boldsymbol{B}(\boldsymbol{\lambda}, \tau) : \tau \in \Sigma\}$ satisfies condition (B)'. Since $\boldsymbol{0} \in \mathbb{f}_{--}(1, \boldsymbol{B}(\boldsymbol{\lambda}, \tau))$ for every $\tau \in \Sigma$, $\mathcal{B}$ also satisfies condition (C)'.

For simplicity of notation, we have taken a common $\sigma \in \mathbb{R}$ for all $j = \ell+1, \ldots, n-1$ in (22). Replacing σ with $\boldsymbol{\sigma} = (\sigma_{\ell+1}, \ldots, \sigma_{n-1})$ in (22), we have

$$q(\boldsymbol{u}, z) = -\sum_{i=1}^{\ell} \lambda_i u_i^2 + \sum_{j=\ell+1}^{n-1} \sum_{i=1}^{\ell} \lambda_j (u_j - \sigma_j u_i)^2 + \lambda_n z^2.$$

In this case, we can prove in a similar manner that

$$\begin{array}{l} \mathbb{f}_-(1, \boldsymbol{B}(\boldsymbol{\lambda}, \boldsymbol{\sigma})) \cap \mathbb{f}_-(1, \boldsymbol{B}(\boldsymbol{\lambda}, \boldsymbol{\tau})) = \emptyset \\ \qquad\qquad \text{if } |\tau_{j'} - \sigma_{j'}| \text{ is sufficiently large for some } j' \in \{\ell+1, \ldots, n-1\}. \end{array}$$

Example 6.5. We consider a parabola-based constraint. Let $n \geq 3$ and

$$B_{ij} = \begin{cases} \lambda_i > 0 & \text{if } 2 \leq i = j \leq n, \\ -0.5 & \text{if } (i,j) = (1,n) \text{ or } (i,j) = (n,1), \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{array}{l} q(\boldsymbol{u}, z, \boldsymbol{B}) = -u_1 z + \displaystyle\sum_{i=2}^{n-1} \lambda_i u_i^2 + \lambda_n z \text{ for every } (\boldsymbol{u}, z) \in \mathbb{R}^{n-1} \times [0,1], \\ \mathbb{f}_+(1, \boldsymbol{B}),\ \mathbb{f}_0(1, \boldsymbol{B}) \text{ or } \mathbb{f}_-(1, \boldsymbol{B}) \\ \qquad = \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : -u_1 + \displaystyle\sum_{i=2}^{n-1} \lambda_i u_i^2 + \lambda_n \geq 0,\ = 0 \text{ or } \leq 0, \text{ respectively} \right\}, \qquad (27) \\ \mathbb{f}_-(1, \boldsymbol{B}) \subseteq \{\boldsymbol{u} \in \mathbb{R}^n : \lambda_i u_i^2 + \lambda_n \leq u_1 \ (i = 2, \ldots, n-1)\} \\ \qquad\qquad \subseteq \mathbb{K}_-(\boldsymbol{B}) \equiv \{\boldsymbol{u} \in \mathbb{R}^{n-1} : 0 \leq u_1,\ -u_1 \leq 2\sqrt{\lambda_i \lambda_n} u_i \leq u_1 \ (i = 2, \ldots, n-1)\}. \end{array}$$

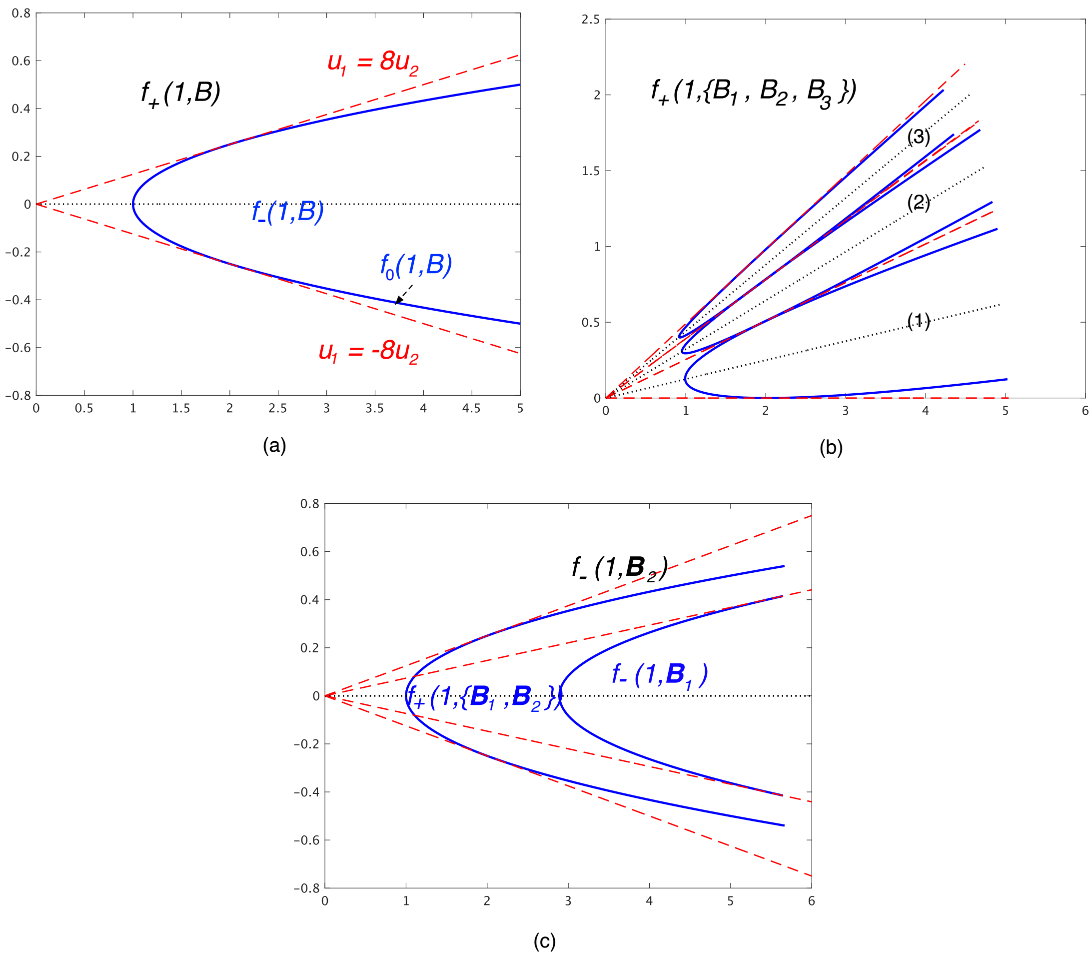

Figure 6: Parabola-based constraints with $n-1=2$. (a) Parabola $\not{f}_-(1,\boldsymbol{B})$ defined by (27) where $\lambda_2 = 16$ and $\lambda_3 = 1$. (b) (1): $\not{f}_{--}(1,\boldsymbol{B}_1)$, (2): $\not{f}_{--}(1,\boldsymbol{B}_2)$ and (3): $\not{f}_{--}(1,\boldsymbol{B}_3)$. $\not{f}_+(1,\{\boldsymbol{B}_1,\boldsymbol{B}_2,\boldsymbol{B}_3\}) = \mathbb{R}^2 \backslash \big((1)\cup(2)\cup(3)\big)$. (c) $\not{f}_+(1,\{\boldsymbol{B}_1,\boldsymbol{B}_2\}) = \not{f}_+(1,\boldsymbol{B}_2)\backslash \not{f}_{--}(1,\boldsymbol{B}_1)$, where $\not{f}_-(1,\boldsymbol{B}_1) = \{\boldsymbol{u}\in\mathbb{R}^2 : -u_1 + 16u_2^2 + 3 \le 0\}$ and $\not{f}_+(1,\boldsymbol{B}_2) = \{\boldsymbol{u}\in\mathbb{R}^2 : -(-u_1 + 16u_2^2 + 1) \ge 0\}$.

See Figure 6 (a). We note that $\mathbb{K}_-(\boldsymbol{B})$ forms a polyhedral cone in $\mathbb{R}^{n-1}$, which converges to the half line $\{\boldsymbol{u}\in\mathbb{R}^{n-1} : u_1 \ge 0,\ u_i = 0\ (i=2,\ldots,n-1)\}$ as all λ_i $(i=2,\ldots,n-1)$ tend to ∞. We see that each $\not{f}_0(1,\boldsymbol{B}) \cap \{\boldsymbol{u}\in\mathbb{R}^{n-1} : u_j = 0\ (2 \le j \ne i \le n-1)\}$ forms a 2-dimensional parabola $(i=2,\ldots,n-1)$. By applying a linear transformation $\boldsymbol{B} \to \boldsymbol{L}^T\boldsymbol{B}\boldsymbol{L} \in \mathbb{S}^n$ with a nonsingular $\boldsymbol{L}$ to $\not{f}_0(z,\boldsymbol{B})$ with different $\lambda_i > 0$ $(i=2,\ldots,n)$, we can create various parabolas. Furthermore, we can arrange some of those parabolas such that the assoicated $\mathcal{B}$ satisfies conditions (B)' and (C)'. See Figure 6 (b) and (c).

7 Concluding remarks

We have presented two sufficient conditions for semi-infinite QCQPs to be equivalent to their SDP relaxation. The first condition, condition (B), extends the result from [2] for QCQPs with finitely many inequality constraints to those with infinitely many inequality constraints. The effectiveness of this condition becomes particularly evident when it is combined with the reduction of a given QCQP to a QCQP satisfying Slater's constraint qualification and some additional conditions (conditions (A-3), (A-4) and (A-5)) as illustrated in Example 6.1. As a result, a wider class of QCQPs can be shown to be equivalent to their SDP relaxations.

The second condition, denoted as condition (B)', is a special case of condition (B) adapted for the standard inequality form (semi-infinite) of QCQP (3). As shown in the examples in Sections 1 and 6, condition (B)' geometrically characterizes the feasible region of a QCQP that can be reformulated as its SDP relaxation. Specifically, some examples in Sections 1 and 6 can be viewed as the feasible regions of ball-, parabola-, and hyperbola-based constrained quadratic programs. It will be interesting to find practical applications of condition (B)'.

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