

Exact SDP relaxations for a class of quadratic programs with finite and infinite quadratic constraints

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Abstract

We investigate exact semidefinite programming (SDP) relaxations for the problem of minimizing a nonconvex quadratic objective function over a feasible region defined by both finitely and infinitely many nonconvex quadratic inequality constraints (semi-infinite QCQPs). Sufficient conditions for the exactness of SDP relaxations for QCQPs with finitely many constraints have been extensively studied, notably by Argue et al. (MOR, 48:100-126, 2023), Arima et al. (SIOPT, 34:3194-3211, 2024), and Joyse and Yang (MP, 205:539-558, 2024). In this work, we present three new sufficient conditions that generalize the existing conditions in these works for both finite and semi-infinite QCQPs. Specifically, we establish relationships among the proposed and existing conditions, and prove that one of the proposed conditions is the weakest among them. Illustrative examples are also provided to demonstrate the effectiveness of the proposed conditions in comparison to the existing ones.

Key words. Finite and semi-infinite quadratically constrained quadratic program, exact semidefinite programming relaxations, rank-one generated cones, non-intersecting quadratic constraints, ball-, parabola- and hyperbola-based constraints.

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1 Introduction

We begin by considering a general conic optimization problem (COP). Let \mathbb{V} be a finite-dimensional vector space equipped with an inner product $\langle \mathbf{A}, \mathbf{B} \rangle$ for every $\mathbf{A}, \mathbf{B} \in \mathbb{V}$. For every closed cone $\mathbb{C} \subset \mathbb{V}$, $\mathbf{Q} \in \mathbb{V}$ and $\mathbf{H} \in \mathbb{V}$, $\text{COP}(\mathbb{C}, \mathbf{Q}, \mathbf{H})$ denotes the problem of minimizing $\langle \mathbf{Q}, \mathbf{X} \rangle$ subject to $\mathbf{X} \in \mathbb{C}$ and $\langle \mathbf{H}, \mathbf{X} \rangle = 1$, *i.e.*,

$$\eta(\mathbb{C}, \mathbf{Q}, \mathbf{H}) = \inf \{ \langle \mathbf{Q}, \mathbf{X} \rangle : \mathbf{X} \in \mathbb{C}, \langle \mathbf{H}, \mathbf{X} \rangle = 1 \},$$

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where $\mathbb{C} \subset \mathbb{V}$ is a cone if $\lambda \mathbf{X} \in \mathbb{C}$ holds for every $\mathbf{X} \in \mathbb{C}$ and $\lambda \geq 0$. We note that a cone \mathbb{C} is not necessarily convex. When $\mathbf{Q}, \mathbf{H} \in \mathbb{V}$ are unspecified and arbitrary, we often denote $\text{COP}(\mathbb{C}, \mathbf{Q}, \mathbf{H})$ and $\eta(\mathbb{C}, \mathbf{Q}, \mathbf{H})$ as $\text{COP}(\mathbb{C})$ and $\eta(\mathbb{C})$, respectively. If $\text{COP}(\mathbb{C})$ is infeasible, we assume that $\eta(\mathbb{C}) = +\infty$.

Let \mathbb{K} be a closed nonconvex cone in \mathbb{V} . Let $\text{co}\mathbb{K}$ and $\overline{\text{co}}\mathbb{K}$ denote the convex hull of \mathbb{K} and its closure, respectively. For every closed convex cone $\mathbb{J} \subseteq \text{co}\mathbb{K}$, $\mathbf{Q} \in \mathbb{V}$ and $\mathbf{H} \in \mathbb{V}$, we consider a nonconvex conic optimization problem $\text{COP}(\mathbb{K} \cap \mathbb{J})$:

$$\eta(\mathbb{K} \cap \mathbb{J}) = \inf \{ \langle \mathbf{Q}, \mathbf{X} \rangle : \mathbf{X} \in \mathbb{K} \cap \mathbb{J}, \langle \mathbf{H}, \mathbf{X} \rangle = 1 \}$$

and its convex relaxation $\text{COP}(\mathbb{J})$:

$$\eta(\mathbb{J}) = \inf \{ \langle \mathbf{Q}, \mathbf{X} \rangle : \mathbf{X} \in \mathbb{J}, \langle \mathbf{H}, \mathbf{X} \rangle = 1 \}$$

Obviously, $\eta(\mathbb{J}) \leq \eta(\mathbb{K} \cap \mathbb{J})$ holds. If $\eta(\mathbb{J}) = \eta(\mathbb{K} \cap \mathbb{J})$, we say that $\text{COP}(\mathbb{K} \cap \mathbb{J})$ and its convex relaxation $\text{COP}(\mathbb{J})$ are *equivalent*, or that $\text{COP}(\mathbb{J})$ is *an exact convex relaxation of* $\text{COP}(\mathbb{K} \cap \mathbb{J})$.

The above framework of a nonconvex conic optimization problem $\text{COP}(\mathbb{K} \cap \mathbb{J})$ and its convex relaxation $\text{COP}(\mathbb{J})$ was originally introduced in [22] for a unified geometrical analysis on the completely positive programming (CPP) reformulation of quadratically constrained quadratic problems (QCQPs) and their extension to polynomial optimization problems (POPs). They introduced

$$\widehat{\mathcal{F}}(\mathbb{K}) = \text{the family of closed convex cones } \mathbb{J} \subset \text{co}\mathbb{K} \text{ such that } \text{co}(\mathbb{K} \cap \mathbb{J}) = \mathbb{J},$$

and established the following results.

Theorem 1.1. *Let $\mathbf{Q}, \mathbf{H} \in \mathbb{V}$, and $\mathbb{K} \subseteq \mathbb{V}$ be a closed cone.*

(i) *Assume that $\mathbb{J} \in \widehat{\mathcal{F}}(\mathbb{K})$. Then*

$$-\infty < \eta(\mathbb{J}, \mathbf{Q}, \mathbf{H}) \text{ if and only if } -\infty < \eta(\mathbb{J}, \mathbf{Q}, \mathbf{H}) = \eta(\mathbb{K} \cap \mathbb{J}, \mathbf{Q}, \mathbf{H}). \quad (1)$$

(ii) *If \mathbb{J} is a face of $\text{co}\mathbb{K}$, then $\mathbb{J} \in \widehat{\mathcal{F}}(\mathbb{K})$.*

(iii) *$\mathbb{J}' \in \widehat{\mathcal{F}}(\mathbb{K})$ for every face \mathbb{J}' of $\mathbb{J} \in \widehat{\mathcal{F}}(\mathbb{K})$.*

(iv) *Assume that $\mathbf{H} \in \text{int } \text{co}\mathbb{K}$ (the interior of $\text{co}\mathbb{K}$.) Then $\mathbb{J} \in \widehat{\mathcal{F}}(\mathbb{K})$ if and only if $\eta(\mathbb{J} \cap \mathbb{K}, \mathbf{Q}, \mathbf{H}) = \eta(\mathbb{J}, \mathbf{Q}, \mathbf{H})$ for every $\mathbf{Q} \in \mathbb{S}^n$.*

Proof. For assertion (i), we refer to [22, Theorem 3.1] (see also [2, Corollary 2.2]); for (ii) and (iii), to [22, Lemma 2.1]; for (iv) to [2, Theorem 1.2].

In particular, Kim et al. [22] applied Theorem 1.1 (i) and (ii) to equivalent completely positive programming (CPP) relaxation of a class of QCQPs in binary and nonnegative variables, which includes Burer's class of QCQPs [10], and its extension to POPs.

1.1 A quadratically constrained quadratic program (QCQP) and its equivalent semidefinite programming (SDP) relaxation

This paper deals with a special case: $\mathbb{V} = \mathbb{S}^n$ (the linear space of $n \times n$ symmetric matrices). and $\mathbb{K} = \mathbf{\Gamma}^n \equiv \{\mathbf{x}\mathbf{x}^T : \mathbf{x} \in \mathbb{R}^n\}$, where \mathbb{R}^n is the n -dimensional Euclidean space of column vectors $\mathbf{x} = (x_1, \dots, x_n)$ and \mathbf{x}^T denotes the row vector obtained by transposing $\mathbf{x} \in \mathbb{R}^n$. In this case,

- $\text{co}\mathbb{K} = \text{co}\mathbf{\Gamma}^n = \mathbb{S}_+^n$ (the convex cone of $n \times n$ positive semidefinite matrices),
- $\text{COP}(\mathbf{\Gamma}^n \cap \mathbb{J})$ and its convex relaxation $\text{COP}(\mathbb{J})$ correspond to a (geometric form of) QCQP and its semidefinite programming (SDP) relaxation,
- each closed convex cone $\mathbb{J} \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$ is characterized as $\mathbb{J} = \text{co}\{\mathbf{x}\mathbf{x}^T : \mathbf{x} \in \mathbb{R}^n, \mathbf{x}\mathbf{x}^T \in \mathbb{J}\}$, and is called *rank-one-generated (ROG)* in the literature [1, 6, 15].

Argue et al. [1] demonstrated independently from [22], that the ROG property is crucial for ensuring the equivalence relation (1) in case $\mathbb{K} = \mathbf{\Gamma}^n$ [1, Lemma 19]. They provided a thorough study of characterizations of ROG cones and established several necessary and sufficient conditions. In particular, condition (II) below characterizes when a closed convex cone $\mathbb{J} \subseteq \mathbb{S}_+^n$, defined by linear matrix inequalities, has the ROG property. In [2], Arima et al. also presented a sufficient condition, condition (I) below for $\mathbb{J} \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$.

A closed convex cone $\mathbb{J} \subseteq \mathbb{S}_+^n$ is often represented using linear matrix inequalities. For every $\mathbf{B} \in \mathbb{S}^n$, let

$$\mathbb{J}_+(\mathbf{B}), \mathbb{J}_0(\mathbf{B}) \text{ or } \mathbb{J}_-(\mathbf{B}) = \{\mathbf{X} \in \mathbb{S}_+^n : \langle \mathbf{B}, \mathbf{X} \rangle \geq, = \text{ or } \leq 0, \text{ respectively}\},$$

and $\mathbb{J}_+(\mathcal{B}) = \{\mathbf{X} \in \mathbb{S}_+^n : \langle \mathbf{B}, \mathbf{X} \rangle \geq 0 \text{ } (\mathbf{B} \in \mathcal{B})\}$ for every $\mathcal{B} \subseteq \mathbb{S}^n$. Since $\mathbb{J} \subseteq \mathbb{S}_+^n$ is a closed convex cone, \mathbb{J} can be represented as the intersection of (possibly infinitely many) half spaces and \mathbb{S}_+^n such that $\mathbb{J} = \mathbb{J}_+(\mathcal{B})$ for some $\mathcal{B} \subseteq \mathbb{S}^n$. We should mention that there are many choices for such a \mathcal{B} . For example, we can take $\mathcal{B} = \mathbb{J}^*$, where $\mathbb{J}^* = \{\mathbf{Y} \in \mathbb{S}^n : \langle \mathbf{X}, \mathbf{Y} \rangle \geq 0 \text{ for every } \mathbf{X} \in \mathbb{J}\}$ (the dual of \mathbb{J}). This trivial choice of \mathcal{B} , however, involves many redundant matrices to represent \mathbb{J} .

Now we represent a QCQP and its SDP relaxation as $\text{COP}(\mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}))$

$$\eta(\mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})) = \inf \left\{ \langle \mathbf{Q}, \mathbf{x}\mathbf{x}^T \rangle : \begin{array}{l} \mathbf{x} \in \mathbb{R}^n, \langle \mathbf{B}, \mathbf{x}\mathbf{x}^T \rangle \geq 0 \text{ } (\mathbf{B} \in \mathcal{B}), \\ \langle \mathbf{H}, \mathbf{x}\mathbf{x}^T \rangle = 1 \end{array} \right\} \quad (2)$$

$$= \inf \left\{ \langle \mathbf{Q}, \mathbf{X} \rangle : \begin{array}{l} \mathbf{X} \in \mathbb{S}^n, \mathbf{X} \in \mathbf{\Gamma}^n, \\ \mathbf{X} \in \mathbb{J}_+(\mathcal{B}) \text{ } (\mathbf{B} \in \mathcal{B}), \langle \mathbf{H}, \mathbf{X} \rangle = 1 \end{array} \right\}, \quad (3)$$

and $\text{COP}(\mathbb{J}_+(\mathcal{B}))$

$$\eta(\mathbb{J}_+(\mathcal{B})) = \inf \left\{ \langle \mathbf{Q}, \mathbf{X} \rangle : \begin{array}{l} \mathbf{X} \in \mathbb{S}_+^n, \\ \mathbf{X} \in \mathbb{J}_+(\mathcal{B}) \text{ } (\mathbf{B} \in \mathcal{B}), \langle \mathbf{H}, \mathbf{X} \rangle = 1 \end{array} \right\}, \quad (4)$$

respectively. We note that $\mathbf{x} \in \mathbb{R}^n$ is variable in (2) while $\mathbf{X} \in \mathbb{S}^n$ is variable in (3). The process of embedding QCQP (2), defined in $\mathbf{x} \in \mathbb{R}^n$, into QCQP (3) is often called *a lifting into the matrix space \mathbb{S}^n* . The following results are known:

Theorem 1.2. *Let $\mathcal{B} \subseteq \mathbb{S}^n$. If one of the following conditions (I) and (II) holds, then $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\Gamma^n)$. In particular, $\mathbb{J}_+(\mathbf{B}) \in \widehat{\mathcal{F}}(\Gamma^n)$, $\mathbb{J}_-(\mathbf{B}) = \mathbb{J}_+(-\mathbf{B}) \in \widehat{\mathcal{F}}(\Gamma^n)$ and $\mathbb{J}_0(\mathbf{B}) = \mathbb{J}_+(\{\mathbf{B}, -\mathbf{B}\}) \in \widehat{\mathcal{F}}(\Gamma^n)$ for every $\mathbf{B} \in \mathbb{S}^n$.*

- (I) \mathcal{B} is finite. $\mathbb{J}_0(\mathbf{B}) \subseteq \mathbb{J}_+(\mathbf{A})$ for every $\mathbf{A}, \mathbf{B} \in \mathcal{B}$. [2, Theorem 4.1].
- (II) \mathcal{B} is finite. For every distinct $\mathbf{A}, \mathbf{B} \in \mathcal{B}$, there exists a nonzero $(\alpha, \beta) \in \mathbb{R}^2$ such that $\alpha\mathbf{A} + \beta\mathbf{B} \in \mathbb{S}_+^n$ [1, Proposition 1].

For QCQP examples that satisfy condition (I) in [18, 27, 33, 36], we refer the reader to [2]. For fundamental properties of ROG cones and their applications to equivalent SDP relaxations of QCQPs, see [1, 20].

1.2 Non-intersecting quadratic constraint conditions

Conditions (I) and (II) are defined in \mathbb{S}^n , the space of the variable matrix \mathbf{X} of $\text{COP}(\mathbb{J}_+(\mathcal{B}))$. For practical applications, however, it is more convenient to provide a direct characterization of QCQP (2), which involves \mathcal{B} satisfying condition (B), in the space \mathbb{R}^n of its variable vector \mathbf{x} . We examine the special case where $\mathbf{H} = \mathbf{H}^1 \equiv \text{diag}(0, \dots, 0, 1) \in \mathbb{S}_+^n$ (the $n \times n$ diagonal matrix with diagonal elements $0, \dots, 0, 1$). Letting

$$\mathbb{H}^1 = \{\mathbf{X} \in \mathbb{S}^n : \langle \mathbf{H}^1, \mathbf{X} \rangle = 1\} = \{\mathbf{X} \in \mathbb{S}^n : X_{nn} = 1\},$$

we can rewrite $\text{COP}(\Gamma^n \cap \mathbb{J}_+(\mathcal{B}), \mathbf{Q}, \mathbf{H}^1)$ and $\text{COP}(\mathbb{J}_+(\mathcal{B}), \mathbf{Q}, \mathbf{H}^1)$ as

$$\begin{aligned} \eta(\Gamma^n \cap \mathbb{J}_+(\mathcal{B}), \mathbf{Q}, \mathbf{H}^1) &= \inf \{ \langle \mathbf{Q}, \mathbf{X} \rangle : \mathbf{X} \in \Gamma^n \cap \mathbb{J}_+(\mathcal{B}) \cap \mathbb{H}^1 \} \\ &= \inf \{ \langle \mathbf{Q}, \mathbf{X} \rangle : \mathbf{X} \in \overline{\text{co}}(\Gamma^n \cap \mathbb{J}_+(\mathcal{B}) \cap \mathbb{H}^1) \} \\ &\quad (\text{since the objective function } \langle \mathbf{Q}, \mathbf{X} \rangle \text{ is linear in } \mathbf{X} \in \mathbb{S}^n) \end{aligned}$$

and

$$\eta(\mathbb{J}_+(\mathcal{B}), \mathbf{Q}, \mathbf{H}^1) = \inf \{ \langle \mathbf{Q}, \mathbf{X} \rangle : \mathbf{X} \in \mathbb{J}_+(\mathcal{B}) \cap \mathbb{H}^1 \},$$

respectively. Therefore, $\overline{\text{co}}(\Gamma^n \cap \mathbb{J}_+(\mathcal{B}) \cap \mathbb{H}^1) = \mathbb{J}_+(\mathcal{B}) \cap \mathbb{H}^1$ serves as a sufficient condition for the equivalence of $\text{COP}(\Gamma^n \cap \mathbb{J}_+(\mathcal{B}), \mathbf{Q}, \mathbf{H}^1)$ and $\text{COP}(\mathbb{J}_+(\mathcal{B}), \mathbf{Q}, \mathbf{H}^1)$, which will be utilized in Theorem 1.3 below.

For the subsequent discussion, $\text{COP}(\Gamma^n \cap \mathbb{J}_+(\mathcal{B}), \mathbf{Q}, \mathbf{H}^1)$ will be expressed in an alternative form. Define

$$q(\mathbf{u}, \mathbf{B}) = \langle \mathbf{B}, \begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix}^T \rangle \text{ for every } (\mathbf{u}, \mathbf{B}) \in \mathbb{R}^{n-1} \times \mathbb{S}^n.$$

If $\mathbf{B} = \begin{pmatrix} \mathbf{C} & \mathbf{c}^T \\ \mathbf{c} & \gamma \end{pmatrix}$, where $\mathbf{C} \in \mathbb{S}^{n-1}$, $\mathbf{c} \in \mathbb{R}^{n-1}$ and $\gamma \in \mathbb{R}$, then $q(\mathbf{u}, \mathbf{B})$ is a quadratic function of the form $\mathbf{u}^T \mathbf{C} \mathbf{u} + 2\mathbf{c}^T \mathbf{u} + \gamma$ in $\mathbf{u} \in \mathbb{R}^{n-1}$. For every $\mathbf{B} \in \mathbb{S}^n$ and $\mathcal{B} \subseteq \mathbb{S}^n$, define

$$\begin{aligned} \mathbf{B}_{\geq}, \mathbf{B}_= \text{ or } \mathbf{B}_{\leq} &= \{ \mathbf{u} \in \mathbb{R}^{n-1} : q(\mathbf{u}, \mathbf{B}) \geq 0, = \text{ or } \leq 0, \text{ respectively} \}, \\ \mathcal{B}_{\geq} &= \bigcap_{\mathbf{B} \in \mathcal{B}} \mathbf{B}_{\geq} = \{ \mathbf{u} \in \mathbb{R}^{n-1} : q(\mathbf{u}, \mathbf{B}) \geq 0 \text{ } (\mathbf{B} \in \mathcal{B}) \}. \end{aligned}$$

Now, we rewrite QCQP (2) with $\mathbf{H} = \mathbf{H}^1 = \text{diag}(0, \dots, 0, 1) \in \mathbb{S}_+^n$ as

$$\begin{aligned}\eta(\Gamma^n \cap \mathbb{J}_+(\mathcal{B}), \mathbf{Q}, \mathbf{H}^1) &= \inf \{q(\mathbf{u}, \mathbf{Q}) : \mathbf{u} \in \mathbb{R}^{n-1}, q(\mathbf{u}, \mathbf{B}) \geq 0 \ (\mathbf{B} \in \mathcal{B})\} \\ &= \inf \{q(\mathbf{u}, \mathbf{Q}) : \mathbf{u} \in \mathcal{B}_\geq\}.\end{aligned}\tag{5}$$

The following result is obtained from [19, Corollary 2] as a special case, adapted to $\text{COP}(\Gamma^n \cap \mathbb{J}_+(\mathcal{B}), \mathbf{Q}, \mathbf{H}^1)$ and $\text{COP}(\mathbb{J}_+(\mathcal{B}), \mathbf{Q}, \mathbf{H}^1)$.

Theorem 1.3. *Let $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathbb{S}^n$. Assume that condition (III_g) below is satisfied. Then*

$$\mathbb{J}_+(\mathcal{B}) \cap \mathbb{H}^1 = \overline{\text{co}}(\Gamma^n \cap \mathbb{J}_+(\mathcal{B}) \cap \mathbb{H}^1),\tag{6}$$

$$\eta(\mathbb{J}_+(\mathcal{B}), \mathbf{Q}, \mathbf{H}^1) = \eta(\Gamma^n \cap \mathbb{J}_+(\mathcal{B}), \mathbf{Q}, \mathbf{H}^1).\tag{7}$$

(Recall that (6) implies (7)).

(III_g) \mathcal{B} is finite. $q(\cdot, \mathbf{B}) : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ is not affine (i.e., if $q(\cdot, \mathbf{B})$ is described as $q(\mathbf{u}, \mathbf{B}) = \mathbf{u}^T \mathbf{D} \mathbf{u} + 2\mathbf{d}^T \mathbf{u} + \delta$ for every $\mathbf{u} \in \mathbb{R}^{n-1}$, then $\mathbf{D} \neq \mathbf{O}$) ($\mathbf{B} \in \mathcal{B} \setminus \mathcal{A}$) [19, Assumption 3], and

$$\left. \begin{aligned} \mathbb{J}_+(\mathcal{A}) \cap \mathbb{H}^1 &= \overline{\text{co}}(\Gamma^n \cap \mathbb{J}_+(\mathcal{A}) \cap \mathbb{H}^1), \\ \mathbf{B}_= &\subseteq \mathcal{B}_\geq \text{ for every } \mathbf{B} \in \mathcal{B} \setminus \mathcal{A} \end{aligned} \right\} \quad [19, \text{Assumption 4}].\tag{8}$$

It was shown in [19, Example 1] that the condition “ $q(\cdot, \mathbf{B}) : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ ($\mathbf{B} \in \mathcal{B}$) is not affine” is necessary for (6) to hold in Theorem 1.3.

Suppose that $\mathcal{A} = \emptyset$ in condition (III_g). Then the first identity in (8) holds automatically (see Theorem 1.2). Hence, (III_g) is simplified to

(III) \mathcal{B} is finite. $q(\cdot, \mathbf{B}) : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ ($\mathbf{B} \in \mathcal{B}$) is not affine and

$$\mathbf{B}_= \subseteq \mathcal{B}_\geq \text{ for every } \mathbf{B} \in \mathcal{B}\tag{9}$$

The condition (9) is frequently called *the non-intersecting quadratic constraint condition* (NIQCC) [11, 19, 35]. Early studies considered special cases of NIQCC for simple QCQPs, particularly those arising from the generalized trust region subproblem (TRS) [5, 11, 13, 18, 28]. As an extension of the generalized TRS, the works in [27, 36] investigated a QCQP of the form $\inf \{q_0(\mathbf{u}) : -1 \leq q_1(\mathbf{u}) \leq 1\}$, where q_0, q_1 are quadratic functions in $\mathbf{u} \in \mathbb{R}^{n-1}$. This problem can be reformulated as a QCQP satisfying NIQCC. Similarly, a quadratic program with non-intersecting ellipsoidal hollows [35] provides another extension of the generalized TRS. Thus, condition (9) may be regarded as a unified representation or generalization of NIQCC-type assumptions that appear in these QCQPs. When compared with condition (I), (9) can be viewed as a *non-homogenized form of NIQCC*, while condition (I) represents a *homogenized form of NIQCC*. As a corollary of Theorem 1.3, we obtain:

Theorem 1.4. *Let $\mathcal{B} \subseteq \mathbb{S}^n$. Assume that condition (III) is satisfied. Then (6) and (7) hold.*

1.3 Summary of main results

The main purpose of the paper is to propose new conditions that generalize previously mentioned conditions (I), (II) and (III) and to investigate their relationships in detail. To describe the new conditions, we introduce a face \mathbb{F} of \mathbb{S}_+^n that contains $\mathbb{J}_+(\mathcal{B})$. Clearly, $\mathbb{J}_+(\mathcal{B}) = \mathbb{J}_+(\mathcal{B}) \cap \mathbb{F}$. To ensure that the conditions are nontrivial and meaningful, we implicitly assume that \mathbb{F} is a proper face of \mathbb{S}_+^n . If a linear equality $\mathbf{G} \begin{pmatrix} u \\ 1 \end{pmatrix} = 0$ is given for some $r \times n$ full row-rank matrix \mathbf{G} , it can be incorporated into QCQP (5) as a quadratic constraint $q(\mathbf{u}, \mathbf{B}) \geq 0$, where $\mathbf{B} = -\mathbf{G}^T \mathbf{G}$. In this case, $\mathbb{J}_+(\mathcal{B})$ is contained in the face $\mathbb{F} = \mathbb{J}_+(-\mathbf{G}^T \mathbf{G}) = \mathbb{J}_0(\mathbf{G}^T \mathbf{G})$. (Note that $\mathbf{G}^T \mathbf{G} \in \mathbb{S}_+^n$). Moreover, even if none of $\mathbb{J}_+(\mathbf{B})$ ($\mathbf{B} \in \mathcal{B}$) is a face of \mathbb{S}_+^n , $\mathbb{J}_+(\mathcal{B})$ may still be contained in a proper face of \mathbb{S}_+^n , as demonstrated in Section 2. Therefore, studying such cases is both theoretically and practically important.

As a weaker variant of condition (I), we propose the following condition:

(B) $\mathbb{J}_0(\mathcal{B}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\mathcal{A}) \cap \mathbb{F}$ or $\mathbb{J}_+(\mathcal{A}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\mathcal{B}) \cap \mathbb{F}$ holds for every $\mathcal{A}, \mathcal{B} \in \mathcal{B}$.

Theorem 1.5. *Let $\mathcal{B} \subseteq \mathbb{S}^n$ and \mathbb{F} be a face of \mathbb{S}_+^n that includes $\mathbb{J}_+(\mathcal{B})$. Assume that condition (B) holds. Then $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\Gamma^n)$.*

We emphasize that an infinite set \mathcal{B} is allowed in condition (B); hence QCQP (3) may be a semi-infinite program [14, 29]. The inclusion $\mathbb{J}_+(\mathcal{A}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\mathcal{B}) \cap \mathbb{F}$ in condition (B) for distinct $\mathcal{A}, \mathcal{B} \in \mathcal{B}$ implies that \mathcal{B} is redundant to describe $\mathbb{J}_+(\mathcal{B}) \cap \mathbb{F}$, i.e., $\mathbb{J}_+(\mathcal{B}) \cap \mathbb{F} = \mathbb{J}_+(\mathcal{B} \setminus \{\mathcal{B}\}) \cap \mathbb{F}$, but $\mathbb{J}_+(\mathcal{B})$ can be a proper subset of $\mathbb{J}_+(\mathcal{B} \setminus \{\mathcal{B}\})$, as shown in Section 2. Theorem 1.5 strengthens [2, Theorem 3.1] where the authors assume that \mathcal{B} is finite and fix $\mathbb{F} = \mathbb{S}_+^n$ as in condition (I).

It is well-known that every face of \mathbb{S}_+^n is isomorphic to \mathbb{S}_+^r for some $r \in \{0, \dots, n\}$ [25, 26]. For proofs of Theorems 1.5, 1.6 and 1.7, we apply facial reduction from \mathbb{F} onto \mathbb{S}_+^r . (See [7, 8, 34] for numerical methods for facial reduction). Let Φ be a linear isomorphism from $\mathbb{F} \supseteq \mathbb{J}_+(\mathcal{B})$ onto \mathbb{S}_+^r , and $\Psi : \mathbb{S}^n \rightarrow \mathbb{S}^r$ the adjoint map with respect to Φ . Then we can prove that $\Phi(\mathbb{J}_0(\mathcal{B}) \cap \mathbb{F}) = \mathbb{J}_0(\Psi(\mathcal{B}))$ and $\Phi(\mathbb{J}_+(\mathcal{B}) \cap \mathbb{F}) = \mathbb{J}_+(\Psi(\mathcal{B}))$ for every $\mathcal{B} \in \mathbb{S}^n$, and that $\Phi(\mathbb{J}_+(\mathcal{B})) = \Phi(\mathbb{J}_+(\mathcal{B}) \cap \mathbb{F}) = \mathbb{J}_+(\Psi(\mathcal{B}))$. Thus condition (B) is equivalent to

(\tilde{B}) $\mathbb{J}_0(\tilde{\mathcal{B}}) \subseteq \mathbb{J}_+(\tilde{\mathcal{A}})$ or $\mathbb{J}_+(\tilde{\mathcal{A}}) \subseteq \mathbb{J}_+(\tilde{\mathcal{B}})$ holds for every $\tilde{\mathcal{A}}, \tilde{\mathcal{B}} \in \Psi(\mathcal{B})$.

Also, $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\Gamma^n)$ holds if and only if $\mathbb{J}_+(\Psi(\mathcal{B})) = \Phi(\mathbb{J}_+(\mathcal{B})) \in \Phi(\widehat{\mathcal{F}}(\Gamma^n)) = \widehat{\mathcal{F}}(\Gamma^r)$. As a result, when \mathcal{B} is finite, we can reduce Theorem 1.5 to Theorem 1.2. We note that the isomorphism Φ and the adjoint Ψ do not appear explicitly in the descriptions of the proposed conditions (B), (C) and (D), although they play an essential role in the proofs of our main results, including Theorems 1.5, 1.6 and 1.7. It should be emphasized that conditions (B), (C) and (D) are invariant under the isomorphism Φ , while conditions (I), (II) and (III) are not. Further details will be provided in Section 3. The equivalence of $\mathbb{J}_0(\mathcal{B}) \in \widehat{\mathcal{F}}(\Gamma^n)$ and $\mathbb{J}_0(\Psi(\mathcal{B})) \in \widehat{\mathcal{F}}(\Gamma^r)$, which was obtained similarly in [1, Observation 1], was utilized for the proof of [1, Proposition 1] in a different context, where $\mathbb{J}_0(\mathcal{B}) = \{\mathbf{X} \in \mathbb{S}_+^n : \langle \mathbf{B}, \mathbf{X} \rangle = 0 \text{ } (\mathbf{B} \in \mathcal{B})\}$. But the condition (II) derived there is not invariant under Φ .

We also propose non-homogenized NIQCCs (C) and (D) below. Let

$$L(\mathbb{F}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}\mathbf{x}^T \in \mathbb{F}\}, \quad L_1(\mathbb{F}) = \{\mathbf{u} \in \mathbb{R}^{n-1} : \begin{pmatrix} u \\ 1 \end{pmatrix} \in L(\mathbb{F})\}.$$

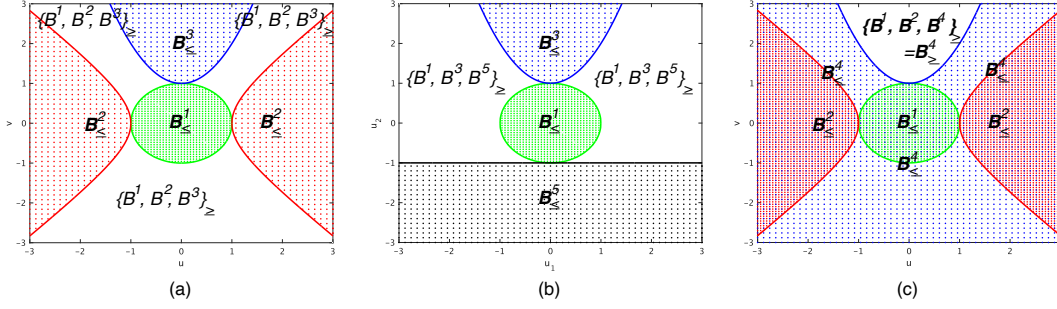


Figure 1: Illustration for conditions (C) and (D). $n = 3$, $\mathbb{F} = \mathbb{S}_+^3$ and $L_1(\mathbb{F}) = \mathbb{R}^2$. (a) $\mathcal{B}^1 = \{\mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^3\}$, (b) $\mathcal{B}^2 = \{\mathbf{B}^1, \mathbf{B}^3, \mathbf{B}^5\}$ and (c) $\mathcal{B}^3 = \{\mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^4\}$, where $q(\mathbf{u}, \mathbf{B}^1) = u_1^2 + u_2^2 - 1$, $q(\mathbf{u}, \mathbf{B}^2) = -u_1^2 + u_2^2 + 1$, $q(\mathbf{u}, \mathbf{B}^3) = u_1^2 - u_2 + 1$, $q(\mathbf{u}, \mathbf{B}^4) = -q(\mathbf{u}, \mathbf{B}^3)$ and $q(\mathbf{u}, \mathbf{B}^5) = u_2 + 1$. \mathcal{B}_{\geq}^k : the unshaded region ($k = 1, 2, 3$).

Since \mathbb{F} is a face of \mathbb{S}_+^n , there exists an $\mathbf{F} \in \mathbb{S}_+^n$ such that $\mathbb{F} = \{\mathbf{X} \in \mathbb{S}_+^n : \langle \mathbf{F}, \mathbf{X} \rangle = 0\}$. Hence $L(\mathbb{F}) = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{F}, \mathbf{x}\mathbf{x}^T \rangle = 0\} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{F}\mathbf{x} = \mathbf{0}\}$, which implies that $L(\mathbb{F})$ is a linear subspace of \mathbb{R}^n , and $L_1(\mathbb{F}) = \{\mathbf{u} \in \mathbb{R}^{n-1} : \mathbf{F} \begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix} = \mathbf{0}\}$ an affine subspace of \mathbb{R}^{n-1} . It follows from $\mathbb{J}_+(\mathcal{B}) \subseteq \mathbb{F}$ that $\mathcal{B}_{\geq} \subseteq L_1(\mathbb{F})$; hence $\mathcal{B}_{\geq} = \mathcal{B}_{\geq} \cap L_1(\mathbb{F})$. Specifically, if $\mathbb{F} = \mathbb{S}_+^n$ then $L(\mathbb{F}) = \mathbb{R}^n$ and $L_1(\mathbb{F}) = \mathbb{R}^{n-1}$.

Theorem 1.6. *Let $\mathcal{B} \subseteq \mathbb{S}^n$ and \mathbb{F} be a face of \mathbb{S}_+^n that includes $\mathbb{J}_+(\mathcal{B})$. Assume that the following condition (C) is satisfied. Then condition (B) holds; hence $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\Gamma^n)$.*

(C) $\mathcal{B}_{\geq} \cap L_1(\mathbb{F}) \neq \emptyset$, and

$$\emptyset \neq \mathcal{B}_{\leq} \cap L_1(\mathbb{F}) \subseteq \mathcal{A}_{\geq} \cap L_1(\mathbb{F}) \text{ or } \mathcal{A}_{\geq} \cap L_1(\mathbb{F}) \subseteq \mathcal{B}_{\geq} \cap L_1(\mathbb{F}) \text{ for every } \mathcal{A}, \mathcal{B} \in \mathcal{B}.$$

Theorem 1.7. *Let $\mathcal{B} \subseteq \mathbb{S}^n$ and \mathbb{F} be a face of \mathbb{S}_+^n that includes $\mathbb{J}_+(\mathcal{B})$. Assume that the following condition (D) is satisfied. Then (6) and (7) hold.*

(D) \mathcal{B} is finite, $q(\cdot, \mathbf{B}) : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ ($\mathbf{B} \in \mathcal{B}$) is not affine on $L_1(\mathbb{F})$, and

$$\mathcal{B}_{\leq} \cap L_1(\mathbb{F}) \subseteq \mathcal{A}_{\geq} \cap L_1(\mathbb{F}) \text{ for every } \mathcal{A}, \mathcal{B} \in \mathcal{B}. \quad (10)$$

If we take $\mathbb{F} = \mathbb{S}_+^n$ then condition (D) coincides with condition (III). Thus Theorem 1.7 is a generalization of Theorem 1.4. Neither of conditions (C) and (D) implies the other; the inclusion relation $\mathcal{B}_{\leq} \cap L_1(\mathbb{F}) \subseteq \mathcal{A}_{\geq} \cap L_1(\mathbb{F})$ in (C) implies the inclusion relation $\mathcal{B}_{\leq} \cap L_1(\mathbb{F}) \subseteq \mathcal{A}_{\geq} \cap L_1(\mathbb{F})$ in (D), while (C) allows $q(\cdot, \mathbf{B})$ for some $\mathbf{B} \in \mathcal{B}$ to be affine on $L_1(\mathbb{F})$. Figure 1 shows three cases (a), (b) and (c) for conditions (C) and (D), where condition (C) is satisfied in all cases but condition (D) is satisfied only in cases (a). Case (b) involves an affine function $q(\cdot, \mathbf{B}^5) = u_2 + 1$. In case (c), $\mathcal{B}_{\leq}^1 \not\subseteq \mathcal{B}_{\geq}^4$, $\mathcal{B}_{\leq}^2 \not\subseteq \mathcal{B}_{\geq}^4$ and $\{\mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^4\}_{\geq} = \{\mathbf{B}^4\}_{\geq}$ hold. This case illustrates that any redundant \mathbf{B} should be removed from \mathcal{B} for condition (D) to be more effective. In conditions (C), ‘or $\mathcal{A}_{\geq} \cap L_1(\mathbb{F}) \subseteq \mathcal{B}_{\geq} \cap L_1(\mathbb{F})$ ’ is added to adapt such cases.

Condition (B) (also (C) and (D)) depends on the choice of a face \mathbb{F} of \mathbb{S}_+^n that includes $\mathbb{J}_+(\mathcal{B})$. If \mathbb{F}^1 and \mathbb{F}^2 are faces of \mathbb{S}_+^n such that $\mathbb{F}^1 \supseteq \mathbb{F}^2 \supseteq \mathbb{J}_+(\mathcal{B})$, then the condition with $\mathbb{F} = \mathbb{F}^1$ implies the condition with $\mathbb{F} = \mathbb{F}^2$. Thus, by choosing the minimal face \mathbb{F}_{\min} of \mathbb{S}_+^n that includes $\mathbb{J}_+(\mathcal{B})$ for \mathbb{F} , we obtain the weakest condition in the sense that the condition is satisfied if it holds with any face \mathbb{F} of \mathbb{S}_+^n containing $\mathbb{J}_+(\mathcal{B})$. In this case, $\mathbb{J}_+(\Psi(\mathcal{B}))$ satisfies Slater's constraint qualification $\mathbb{J}_+(\Psi(\mathcal{B})) \cap \mathbb{S}_{++}^r \neq \emptyset$, where \mathbb{S}_{++}^r denotes the interior of \mathbb{S}_+^r , *i.e.*, the set of $r \times r$ positive definite matrices.

We now outline possible applications. QCQPs of the form (5) arise in a variety of domains, including robotics and autonomous systems for avoiding multiple exclusion zones [14, 30, 31], sensor placement problems [9, 12] where optimal locations must lie outside risky regions, and data classification tasks involving the placement of test points outside known clusters [24]. The infeasible region of QCQP (5), given by the interior of \mathbf{B}_{\leq} ($\mathbf{B} \in \mathcal{B}$), corresponds to the multiple exclusion zones, risky regions, and known clusters, respectively (see Figures 1, 3, 4 and 5). When the formulated QCQPs satisfy a NIQCC, such as conditions (I) and (III), they can be solved exactly through their SDP relaxations. Furthermore, the proposed conditions (B), (C) and (D) are expected to substantially broaden the scope of such applications; for example, by allowing the free addition of linear equality constraints and by effectively handling degenerate cases in which the feasible region fails to satisfy Slater's constraint qualification. Additionally, in Section 6, we illustrate some geometric examples for the reader interested in possible applications.

1.4 Outline of the paper

In Section 2, a simple example is provided to demonstrate the effectiveness of the proposed conditions (B), (C) and (D) in comparison to the existing conditions (I), (II) and (III). In Section 3.1, we introduce an isomorphism Φ from a face \mathbb{F} of \mathbb{S}_+^n onto \mathbb{S}_+^r and some basic theoretical issues related on Φ that play an essential role in the subsequent sections. Based on them, we present a proof of Theorem 1.5 in Section 3.2, and proofs of Theorem 1.6 and 1.7 in Section 4. In Section 5, we investigate the relationships of the conditions presented above, (I), (II), (III), (B), (C), and (D) in detail. It is shown that (B) is the weakest condition among them, and that all four conditions (I), (II), (B), and (C) are equivalent under Slater's constraint qualification and the absence of redundant constraints. Under these assumptions, condition (II) may be regarded as the dual of condition (B) (see Remark 5.4). In Section 6, we provide three geometric QCQP examples that satisfy condition (C). Finally, Section 7 contains our concluding remarks.

2 An example illustrating the effectiveness of Theorems 1.5, 1.6 and 1.7 in comparison to Theorems 1.2 and 1.4

In this section, we compare our main results stated in Theorems 1.5, 1.6 and 1.7 with the known results in Theorem 1.2 and 1.4 through an example to demonstrate the effectiveness

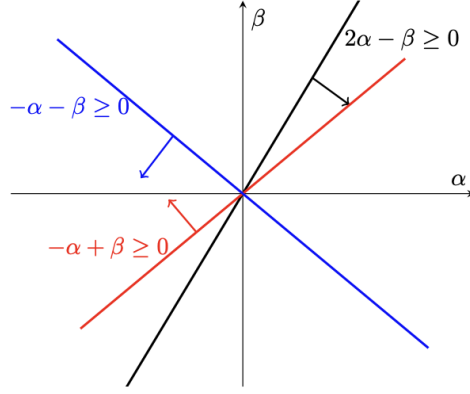


Figure 2: The region of (α, β) determined by the inequalities in (11).

of our main results. Let $n = 4$, $\mathcal{B} = \{\mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^3\}$ and $\mathbb{J} = \mathbb{J}_+(\mathcal{B})$, where

$$\mathbf{B}^1 = \begin{pmatrix} -1 & -2 & 0 & 1 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 0 & 2 & 2 \end{pmatrix}, \quad \mathbf{B}^2 = \begin{pmatrix} 1 & 1 & 0 & -2 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ -2 & 0 & -1 & -1 \end{pmatrix}, \quad \mathbf{B}^3 = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 \end{pmatrix}.$$

Let

$$\mathbf{X}^1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{X}^2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{X}^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{X}^1 &\in \mathbb{J}_+(\{\mathbf{B}^2, \mathbf{B}^3\}) \setminus \mathbb{J}_+(\{\mathbf{B}^1\}), \quad \mathbf{X}^2 \in \mathbb{J}_+(\{\mathbf{B}^1, \mathbf{B}^3\}) \setminus \mathbb{J}_+(\{\mathbf{B}^2\}), \\ \mathbf{X}^3 &\in \mathbb{J}_+(\{\mathbf{B}^1, \mathbf{B}^2\}) \setminus \mathbb{J}_+(\{\mathbf{B}^3\}). \end{aligned}$$

This implies that none of \mathbf{B}^1 , \mathbf{B}^2 and \mathbf{B}^3 is redundant to describe $\mathbb{J}_+(\{\mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^3\})$.

Let $\widehat{\mathbf{X}} = \text{diag}(1, 1, 0, 0)$ (the 4×4 diagonal matrix with elements 1, 1, 0, 0). Then $\widehat{\mathbf{X}} \in \mathbb{J}_0(\mathbf{B}^2)$ but $\widehat{\mathbf{X}} \notin \mathbb{J}_+(\mathbf{B}^1)$. Hence \mathcal{B} does not satisfy condition (I) assumed in Theorem 1.2. Also, condition (II) assumed in Theorem 1.2 does not hold. In fact, assume on the contrary that (II) holds, which implies that the diagonal of $\alpha \mathbf{B}^1 + \beta \mathbf{B}^2 \in \mathbb{S}_+^4$ is nonnegative for some nonzero $(\alpha, \beta) \in \mathbb{R}^2$. Hence

$$-\alpha + \beta \geq 0, \quad -\alpha - \beta \geq 0, \quad 2\alpha - \beta \geq 0, \quad 2\alpha + \beta \geq 0. \quad (11)$$

Clearly, only $(\alpha, \beta) = \mathbf{0}$ satisfies the above inequalities, as shown in Figure 2, which illustrates the region of (α, β) determined by the inequalities in (11). Therefore, condition (II) does not hold. It is also easily verified that if $\bar{\mathbf{u}} = \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^4$ then $\bar{\mathbf{u}} \in \mathbf{B}^1$ but $\bar{\mathbf{u}} \notin \mathbf{B}^2$. Therefore, (9) in condition (III) is not satisfied.

To see whether condition (B) is satisfied for some face \mathbb{F} that contains \mathcal{B} , we observe that

$$\begin{aligned}\mathbb{J}_+(\mathcal{B}) &\subseteq \{\mathbf{X} \in \mathbb{S}_+^4 : \langle \mathbf{B}^1 + \mathbf{B}^2 + \mathbf{B}^3, \mathbf{X} \rangle \geq 0\} \\ &= \left\{ \mathbf{X} \in \mathbb{S}_+^4 : \left\langle \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{X} \right\rangle \geq 0 \right\} \\ &= \left\{ \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{U} \end{pmatrix} \in \mathbb{S}^4 : \mathbf{U} \in \mathbb{S}_+^2 \right\}.\end{aligned}$$

Hence, by letting

$$\begin{aligned}\mathbb{F}^1 &= \left\{ \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{U} \end{pmatrix} \in \mathbb{S}^4 : \mathbf{U} \in \mathbb{S}_+^2 \right\}, \\ \widetilde{\mathbf{B}}^1 &= \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad \widetilde{\mathbf{B}}^2 = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}, \quad \widetilde{\mathbf{B}}^3 = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}, \quad \widetilde{\mathcal{B}} = \{\widetilde{\mathbf{B}}^1, \widetilde{\mathbf{B}}^2, \widetilde{\mathbf{B}}^3\},\end{aligned}$$

we see that $\mathbb{J}_+(\mathcal{B}) \subseteq \mathbb{F}^1$, which indicates that $X_{ij} = 0$ ($1 \leq i \leq 2$ or $1 \leq j \leq 2$) if $\mathbf{X} \in \mathbb{J}_+(\mathcal{B})$, and condition (B) is equivalent to

$$\mathbb{J}_0(\widetilde{\mathcal{B}}) \subseteq \mathbb{J}_+(\widetilde{\mathcal{A}}) \text{ or } \mathbb{J}_+(\widetilde{\mathcal{A}}) \subseteq \mathbb{J}_+(\widetilde{\mathcal{B}}) \text{ for every distinct } \widetilde{\mathcal{A}}, \widetilde{\mathcal{B}} \in \widetilde{\mathcal{B}},$$

which obviously holds since

$$\begin{aligned}\mathbb{J}_0(\widetilde{\mathcal{B}}^1) &= \mathbb{J}_0(\widetilde{\mathcal{B}}^2) = \mathbb{J}_0(\widetilde{\mathcal{B}}^3) = \mathbb{J}_+(\widetilde{\mathcal{B}}^2) = \mathbb{J}_+(\widetilde{\mathcal{B}}^3) \\ &= \left\{ \mathbf{U} \in \mathbb{S}_+^2 : \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \mathbf{U} \right\rangle = 0 \right\} \subseteq \mathbb{S}_+^2 = \mathbb{J}_+(\widetilde{\mathcal{B}}^1).\end{aligned}$$

Now, we verify whether condition (C) is satisfied with $\mathbb{F} = \mathbb{F}^1$. By definition,

$$\begin{aligned}L(\mathbb{F}^1) &= \{\mathbf{x} \in \mathbb{R}^4 : \mathbf{x}\mathbf{x}^T \in \mathbb{F}^1\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : x_3, x_4 \in \mathbb{R} \right\}, \\ L_1(\mathbb{F}^1) &= \left\{ \mathbf{u} \in \mathbb{R}^3 : \begin{pmatrix} u \\ 1 \end{pmatrix} \in L(\mathbb{F}^1) \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix} \in \mathbb{R}^3 : u_3 \in \mathbb{R} \right\}.\end{aligned}$$

For every $\mathbf{u} \in L_1(\mathbb{F}^1)$,

$$q(\mathbf{u}, \mathbf{B}^k) = \langle \widetilde{\mathbf{B}}^k, \begin{pmatrix} u_3 \\ 1 \end{pmatrix} \rangle = \begin{cases} 2(u_3 + 1)^2 & (k = 1) \\ -(u_3 + 1)^2 & (k = 2, 3). \end{cases}$$

holds. Hence

$$\begin{aligned}\mathbf{B}_{\leq}^1 \cap L_1(\mathbb{F}^1) &= \mathbf{B}_{\geq}^2 \cap L_1(\mathbb{F}^1) = \mathbf{B}_{\geq}^3 \cap L_1(\mathbb{F}^1) = \left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}, \\ \mathbf{B}_{\geq}^2 \cap L_1(\mathbb{F}^1) &= \mathbf{B}_{\geq}^3 \cap L_1(\mathbb{F}^1) = \left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\} \subseteq \left\{ \begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix} : u_3 \in \mathbb{R} \right\} = \mathbf{B}_{\geq}^1 \cap L_1(\mathbb{F}^1).\end{aligned}$$

Therefore condition (C) with $\mathbb{F} = \mathbb{F}^1$ is satisfied. It is also easy to see that condition (D) with $\mathbb{F} = \mathbb{F}^1$ is satisfied.

We note that \mathbb{F}^1 is not the minimal face of \mathbb{S}_+^4 that contains $\mathbb{J}_+(\mathcal{B})$. In fact, if we let

$$\mathbb{F}^2 = \left\{ \mathbf{X} \in \mathbb{S}_+^4 : \left\langle \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \mathbf{X} \right\rangle = 0 \right\},$$

then $\mathbb{F} = \mathbb{F}^1 \cap \mathbb{F}^2$ forms the minimal face. In this case

$$\begin{aligned} L(\mathbb{F}) &= \left\{ \begin{pmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} : x_3 = -x_4 \in \mathbb{R} \right\}, \quad L_1(\mathbb{F}) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}, \\ \mathbb{J}_+(\mathcal{B}) \cap \mathbb{F} &= \mathbb{F} = \left\{ \begin{pmatrix} O & O \\ O & U \end{pmatrix} \in \mathbb{S}^4 : \left\langle \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, U \right\rangle = 0, U \in \mathbb{S}_+^2 \right\}, \\ \mathcal{B}_{\geq} \cap L_1(\mathbb{F}) &= L_1(\mathbb{F}) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\} \end{aligned}$$

hold.

3 On Theorem 1.5

Throughout Sections 3 and 4, let $\mathcal{B} \subseteq \mathbb{S}^n$ and \mathbb{F} be a face of \mathbb{S}_+^n that contains $\mathbb{J}_+(\mathcal{B})$.

3.1 Facial reduction of $\mathbb{J}_+(\mathcal{B})$ into \mathbb{S}_+^r

We first represent \mathbb{F} as $\mathbb{F} = \{ \mathbf{X} \in \mathbb{S}_+^n : \langle \mathbf{F}, \mathbf{X} \rangle = 0 \}$ for some $\mathbf{F} \in \mathbb{S}_+^n$. Suppose that $\text{rank} \mathbf{F} = n - r$ for some $r \in \{0, 1, \dots, n\}$. We can take an $n \times n$ orthogonal matrix \mathbf{P} that diagonalizes \mathbf{F} such that $\mathbf{P}^T \mathbf{F} \mathbf{P} = \text{diag}(0, \dots, 0, \lambda_{r+1}, \dots, \lambda_n) \in \mathbb{S}_+^n$ for some positive numbers $\lambda_{r+1}, \dots, \lambda_n$, where $0, \dots, 0, \lambda_{r+1}, \dots, \lambda_n$ denote the eigenvalues of \mathbf{F} with $\lambda_i > 0$ ($r+1 \leq i \leq n$). Let $\mathbf{P}_{1,r}$ denote the $n \times r$ matrix consisting of the first r columns of \mathbf{P} ; each j th columns of $\mathbf{P}_{1,r}$ is an eigenvector of \mathbf{F} associated with the zero eigenvalue. Let \mathbf{M} be an arbitrary $r \times r$ nonsingular matrix. Define

$$\begin{aligned} \Phi(\mathbf{X}) &= \mathbf{M}^T \mathbf{P}_{1,r}^T \mathbf{X} \mathbf{P}_{1,r} \mathbf{M} \in \mathbb{S}_+^r \text{ for every } \mathbf{X} \in \mathbb{F}, \\ \Psi(\mathbf{Y}) &= \mathbf{M}^{-1} \mathbf{P}_{1,r}^T \mathbf{Y} \mathbf{P}_{1,r} \mathbf{M}^{-T} \text{ for every } \mathbf{Y} \in \mathbb{S}^n, \\ \theta(\mathbf{x}) &= \mathbf{M}^T \mathbf{P}_{1,r}^T \mathbf{x} \text{ for every } \mathbf{x} \in L(\mathbb{F}). \end{aligned}$$

Then $\Phi : \mathbb{F} \rightarrow \mathbb{S}_+^r$ forms a linear isomorphism from \mathbb{F} onto \mathbb{S}_+^r and $\Psi : \mathbb{S}^n \rightarrow \mathbb{S}^r$ the adjoint map with respect to Φ (see, for example, [7, 25, 26]). Here $\mathbf{V} \in \mathbb{S}_+^r \rightarrow \mathbf{M}^T \mathbf{V} \mathbf{M} \in \mathbb{S}_+^r$ serves as an automorphism on \mathbb{S}_+^r . The choice of \mathbf{M} is not relevant here, but it becomes relevant in Section 4 where QCQP (5) is considered, *i.e.*, $\text{COP}(\Gamma^n \cap \mathbb{J}_+(\mathcal{B}) \cap \mathbb{F}, \mathbf{Q}, \mathbf{H})$ with $\mathbf{H} = \text{diag}(0, \dots, 0, 1) \in \mathbb{S}_+^n$. So we may assume that \mathbf{M} is the $r \times r$ identity matrix in this section.

Lemma 3.1.

- (i) The map $\mathbf{V} \in \mathbb{S}_+^r \rightarrow \mathbf{P}_{1,r} \mathbf{M}^{-T} \mathbf{V} \mathbf{M}^{-1} \mathbf{P}_{1,r}^T \in \mathbb{F}$ serves as the inverse of $\Phi : \mathbb{F} \rightarrow \mathbb{S}_+^r$.
- (ii) $\theta : L(\mathbb{F}) \rightarrow \mathbb{R}^r$ is linear, one-to-one and onto.
- (iii) $\langle \mathbf{A}, \mathbf{X} \rangle = \langle \Psi(\mathbf{A}), \Phi(\mathbf{X}) \rangle$ for every $\mathbf{A} \in \mathbb{S}^n$ and $\mathbf{X} \in \mathbb{F}$.

- (iv) $\langle \mathbf{A}, \mathbf{x}\mathbf{x}^T \rangle = \langle \Psi(\mathbf{A}), \theta(\mathbf{x})\theta(\mathbf{x})^T \rangle$ for every $\mathbf{A} \in \mathbb{S}^n$ and $\mathbf{x} \in L(\mathbb{F})$.
- (v) $\Phi(\mathbb{J}_-(\mathbf{B})) \cap \mathbb{F} = \mathbb{J}_-(\Psi(\mathbf{B}))$, $\Phi(\mathbb{J}_0(\mathbf{B})) \cap \mathbb{F} = \mathbb{J}_0(\Psi(\mathbf{B}))$ and $\Phi(\mathbb{J}_+(\mathbf{B})) \cap \mathbb{F} = \mathbb{J}_+(\Psi(\mathbf{B}))$ for every $\mathbf{B} \in \mathcal{B}$.
- (vi) $\Phi(\mathbb{J}_+(\mathcal{B})) = \Phi(\mathbb{J}_+(\mathcal{B})) \cap \mathbb{F} = \mathbb{J}_+(\Psi(\mathcal{B}))$.

Proof. (i) For every $\mathbf{V} \in \mathbb{S}_+^r$, we see

$$\langle \mathbf{F}, \mathbf{P}_{1,r} \mathbf{M}^{-T} \mathbf{V} \mathbf{M}^{-1} \mathbf{P}_{1,r}^T \rangle = \langle \mathbf{P} \text{diag}(0, \dots, 0, \lambda_{r+1}, \dots, \lambda_n) \mathbf{P}^T, \mathbf{P}_{1,r} \mathbf{M}^{-T} \mathbf{V} \mathbf{M}^{-1} \mathbf{P}_{1,r}^T \rangle = 0.$$

Hence $\mathbf{P}_{1,r} \mathbf{M}^{-T} \mathbf{V} \mathbf{M}^{-1} \mathbf{P}_{1,r}^T \in \mathbb{F}$. Also $\Phi(\mathbf{P}_{1,r} \mathbf{M}^{-T} \mathbf{V} \mathbf{M}^{-1} \mathbf{P}_{1,r}^T) = \mathbf{V}$ for every $\mathbf{V} \in \mathbb{S}_+^r$. Therefore, the desired result follows.

(ii) By definition, $\theta : L(\mathbb{F}) \rightarrow \mathbb{R}^r$ is linear and $\theta(L(\mathbb{F})) \subseteq \mathbb{R}^r$. Let $\mathbf{v} \in \mathbb{R}^r$. Define $\mathbf{x} = \mathbf{P}_{1,r} \mathbf{M}^{-T} \mathbf{v}$ and $\mathbf{V} = \mathbf{v}\mathbf{v}^T \in \mathbb{S}_+^r$. Then $\mathbf{x}\mathbf{x}^T = \mathbf{P}_{1,r} \mathbf{M}^{-T} \mathbf{V} \mathbf{M}^{-1} \mathbf{P}_{1,r}^T \in \mathbb{F}$ as we have seen in the proof (i) above. Hence $\mathbf{x} \in L(\mathbb{F})$ by the definition of $L(\mathbb{F})$. We also see that $\theta(\mathbf{x}) = \mathbf{M}^T \mathbf{P}_{1,r}^T \mathbf{x} = \mathbf{M}^T \mathbf{P}_{1,r}^T \mathbf{P}_{1,r} \mathbf{M}^{-T} \mathbf{v} = \mathbf{v}$. Hence $\theta(L(\mathbb{F})) = \mathbb{R}^r$, and we have shown $\theta(L(\mathbb{F})) = \mathbb{R}^r$. To see that $\theta : L(\mathbb{F}) \rightarrow \mathbb{R}^r$ is one-to-one, assume that $\theta(\mathbf{x}^1) = \theta(\mathbf{x}^2)$ for some $\mathbf{x}^1, \mathbf{x}^2 \in L(\mathbb{F})$. Then $\theta(\mathbf{x}^2 - \mathbf{x}^1) = \mathbf{0}$. Hence $\mathbb{S}_+^r \ni \mathbf{O} = \theta(\mathbf{x}^2 - \mathbf{x}^1)\theta(\mathbf{x}^2 - \mathbf{x}^1)^T = \Phi((\mathbf{x}^2 - \mathbf{x}^1)(\mathbf{x}^2 - \mathbf{x}^1)^T)$, which implies $(\mathbf{x}^2 - \mathbf{x}^1)(\mathbf{x}^2 - \mathbf{x}^1)^T = \mathbf{O}$ and $\mathbf{x}^2 - \mathbf{x}^1 = \mathbf{0}$.

(iii) Let $\mathbf{A} \in \mathbb{S}^n$, $\mathbf{X} \in \mathbb{F}$ and $\mathbf{V} = \Phi(\mathbf{X}) \in \mathbb{S}_+^r$. By (i), $\mathbf{X} = \mathbf{P}_{1,r} \mathbf{M}^{-T} \mathbf{V} \mathbf{M}^{-1} \mathbf{P}_{1,r}^T \in \mathbb{F}$. Thus, it follows that

$$\langle \mathbf{A}, \mathbf{X} \rangle = \langle \mathbf{A}, \mathbf{P}_{1,r} \mathbf{M}^{-T} \mathbf{V} \mathbf{M}^{-1} \mathbf{P}_{1,r}^T \rangle = \langle \mathbf{M}^{-1} \mathbf{P}_{1,r}^T \mathbf{A} \mathbf{P}_{1,r} \mathbf{M}^{-T}, \mathbf{V} \rangle = \langle \Psi(\mathbf{A}), \Phi(\mathbf{X}) \rangle.$$

(iv) Let $\mathbf{A} \in \mathbb{S}^n$ and $\mathbf{x} \in L(\mathbb{F})$. Then $\mathbf{x}\mathbf{x}^T \in \mathbb{F}$. Consequently, $\langle \mathbf{A}, \mathbf{x}\mathbf{x}^T \rangle = \langle \Psi(\mathbf{A}), \Phi(\mathbf{x}\mathbf{x}^T) \rangle = \langle \Psi(\mathbf{A}), \theta(\mathbf{x})\theta(\mathbf{x})^T \rangle$ follows from (iii).

Assertion (v) can be proved easily by assertions (i) and (iii). Assertion (vi) follows from assertion (v). \square

The mapping $\mathbb{J}_+(\mathcal{B}) \subseteq \mathbb{F} \rightarrow \Phi(\mathbb{J}_+(\mathcal{B})) \cap \mathbb{F} = \mathbb{J}_+(\Psi(\mathcal{B})) \subseteq \mathbb{S}^r$ in (vi) is interpreted as a *facial reduction* [7] of $\mathbb{J}_+(\mathcal{B}) \subseteq \mathbb{F}$ into \mathbb{S}^r . For simplicity of notation, we denote $\Psi(\mathbf{B})$ by $\tilde{\mathbf{B}}$ for every $\mathbf{B} \in \mathbb{S}^n$, and $\Psi(\mathcal{B}) = \{\Psi(\mathbf{B}) : \mathbf{B} \in \mathcal{B}\}$ by $\tilde{\mathcal{B}}$. Then, we can simplify QCQP (2), its SDP relaxation (4), and condition (B) by the mappings Φ from \mathbb{F} onto \mathbb{S}_+^r and θ from $L(\mathbb{F})$ onto \mathbb{R}^r as follows.

$\text{COP}(\Gamma^r \cap \mathbb{J}_+(\tilde{\mathcal{B}}), \tilde{\mathbf{Q}}, \tilde{\mathbf{H}})$:

$$\eta(\Gamma^r \cap \mathbb{J}_+(\tilde{\mathcal{B}}), \tilde{\mathbf{Q}}, \tilde{\mathbf{H}}) = \inf \left\{ \langle \tilde{\mathbf{Q}}, \mathbf{v}\mathbf{v}^T \rangle : \begin{array}{l} \mathbf{v} \in \mathbb{R}^r, \mathbf{v}\mathbf{v}^T \in \mathbb{J}_+(\tilde{\mathcal{B}}), \\ \langle \tilde{\mathbf{H}}, \mathbf{v}\mathbf{v}^T \rangle = 1 \end{array} \right\}. \quad (12)$$

$\text{COP}(\mathbb{J}_+(\tilde{\mathcal{B}}), \tilde{\mathbf{Q}}, \tilde{\mathbf{H}})$:

$$\eta(\mathbb{J}_+(\tilde{\mathcal{B}}), \tilde{\mathbf{Q}}, \tilde{\mathbf{H}}) = \inf \left\{ \langle \tilde{\mathbf{Q}}, \mathbf{V} \rangle : \begin{array}{l} \mathbf{V} \in \mathbb{S}_+^r, \mathbf{V} \in \mathbb{J}_+(\tilde{\mathcal{B}}), \\ \langle \tilde{\mathbf{H}}, \mathbf{V} \rangle = 1 \end{array} \right\}. \quad (13)$$

($\tilde{\mathbf{B}}$) For every distinct $\tilde{\mathbf{A}}, \tilde{\mathbf{B}} \in \tilde{\mathcal{B}}$, either $\mathbb{J}_0(\tilde{\mathbf{B}}) \subseteq \mathbb{J}_+(\tilde{\mathbf{A}})$ or $\mathbb{J}_+(\tilde{\mathbf{A}}) \subseteq \mathbb{J}_+(\tilde{\mathbf{B}})$ holds.

More precisely, Lemma 3.1 ensures:

- [a] $\mathbf{x} \in \mathbb{R}^n$ is a feasible solution of (2) with the objective value $\langle \mathbf{Q}, \mathbf{x}\mathbf{x}^T \rangle$ if and only if $\mathbf{v} = \theta(\mathbf{x})$ is a feasible solution of (12) with the objective value $\langle \tilde{\mathbf{Q}}, \mathbf{v}\mathbf{v}^T \rangle = \langle \mathbf{Q}, \mathbf{x}\mathbf{x}^T \rangle$.
 - [b] $\mathbf{X} \in \mathbb{S}_+^n$ is a feasible solution of (4) with the objective value $\langle \mathbf{Q}, \mathbf{X} \rangle$ if and only if $\mathbf{V} = \Phi(\mathbf{X})$ is a feasible solution of (13) with the objective value $\langle \tilde{\mathbf{Q}}, \mathbf{V} \rangle = \langle \mathbf{Q}, \mathbf{X} \rangle$.
 - [c] Condition (B) holds if and only if condition ($\tilde{\text{B}}$) does.
- In addition, we know that when $\mathbf{X} \in \mathbb{F}$ and $\mathbf{V} = \Phi(\mathbf{X}) = \mathbf{M}^T \mathbf{P}_{1,r}^T \mathbf{X} \mathbf{P}_{1,r} \mathbf{M}$, which imply $\mathbf{X} = \mathbf{P}_{1,r} \mathbf{M}^{-T} \mathbf{V} \mathbf{M}^{-1} \mathbf{P}_{1,r}^T$, \mathbf{X} is rank-1 if and only if so is \mathbf{V} . It follows that
- [d] $\mathbb{J}_+(\mathcal{B}) = \mathbb{J}_+(\mathcal{B}) \cap \mathbb{F} \in \widehat{\mathcal{F}}(\Gamma^n)$ if and only if $\mathbb{J}_+(\tilde{\mathcal{B}}) = \mathbb{J}_+(\Psi(\mathcal{B})) = \Phi(\mathbb{J}_+(\mathcal{B}) \cap \mathbb{F}) \in \widehat{\mathcal{F}}(\Gamma^r)$.

3.2 Proof of Theorem 1.5

In view of the discussion of Section 3.1, it suffices to prove $\mathbb{J}_+(\tilde{\mathcal{B}}) \in \widehat{\mathcal{F}}(\Gamma^r)$ under condition ($\tilde{\text{B}}$) (see [c] and [d]). For simplicity of notation, we omit \sim from $\tilde{\mathcal{B}}$ ($\mathcal{B} \in \mathcal{B}$), $\tilde{\mathcal{B}}$, $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{H}}$ for the proof, or equivalently, we prove $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\Gamma^n)$ under condition (B) with $\mathbb{F} = \mathbb{S}_+^n$.

3.2.1 The finite case of \mathcal{B}

This case can be derived easily from Theorem 1.2. Assume that \mathcal{B} is finite and condition (B) holds with $\mathbb{F} = \mathbb{S}_+^n$. If $\mathbb{J}_+(\mathbf{A}) \subseteq \mathbb{J}_+(\mathbf{B})$ for distinct $\mathbf{A}, \mathbf{B} \in \mathcal{B}$, then \mathbf{B} is redundant to describe $\mathbb{J}_+(\mathcal{B})$, i.e., $\mathbb{J}_+(\mathcal{B} \setminus \{\mathbf{B}\}) = \mathbb{J}_+(\mathcal{B})$. Hence, we can remove such elements \mathbf{B} from \mathcal{B} one by one, recursively, to construct a reduced set \mathcal{B} . Then, the resulting \mathcal{B} eventually satisfies condition (I). □

3.2.2 The infinite case of \mathcal{B}

We present three lemmas for the proof.

Lemma 3.2. *Let $\{\mathcal{B}^k \subseteq \mathbb{S}_+^n : k = 1, 2, \dots\}$ be a sequence such that $\bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}^k) = \mathbb{J}$ for some closed convex cone $\mathbb{J} \subseteq \mathbb{S}_+^n$. Then $\bigcap_{m=1}^{\infty} \text{co}(\Gamma^n \cap (\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k))) = \text{co}(\Gamma^n \cap \mathbb{J})$.*

Proof. Since $\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k) \supseteq \mathbb{J}$ ($m = 1, 2, \dots$), $\bigcap_{m=1}^{\infty} \text{co}(\Gamma^n \cap (\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k))) \supseteq \text{co}(\Gamma^n \cap \mathbb{J})$ follows. To prove the converse inclusion, let $\bar{\mathbf{X}} \in \text{co}(\Gamma^n \cap (\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k)))$ ($m = 1, 2, \dots$). Then, for each $m = 1, 2, \dots$, there exist $\mathbf{X}_m^p \in \Gamma^n \cap (\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k))$ ($p = 1, 2, \dots, \ell$) for some $\ell \leq \dim \mathbb{S}^n = n(n-1)/2$ such that $\bar{\mathbf{X}} = \sum_{p=1}^{\ell} \mathbf{X}_m^p$. Let $q \in \{1, 2, \dots, \ell\}$ be fixed arbitrarily. Consider the sequence $\{\mathbf{X}_m^q : m = 1, 2, \dots\}$. The sequence is bounded since $\bar{\mathbf{X}} \in \mathbb{S}_+^n$, $\mathbf{X}_m^p \in \mathbb{S}_+^n$ ($p = 1, \dots, \ell$) and $\langle \mathbf{I}, \bar{\mathbf{X}} \rangle = \langle \mathbf{I}, \sum_{p=1}^{\ell} \mathbf{X}_m^p \rangle \geq \langle \mathbf{I}, \mathbf{X}_m^q \rangle$. Hence, we may assume without loss of generality that the sequence converges to some $\bar{\mathbf{X}}^q$ as $m \rightarrow \infty$. Then, $\bar{\mathbf{X}}^q \in \text{cl}(\bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}^k)) = \mathbb{J}$. Since Γ^n is closed, we also see $\bar{\mathbf{X}}^q \in \Gamma^n$. Therefore, taking the limit of the identity $\bar{\mathbf{X}} = \sum_{p=1}^{\ell} \mathbf{X}_m^p$ as $m \rightarrow \infty$, we obtain that $\bar{\mathbf{X}} = \sum_{p=1}^{\ell} \bar{\mathbf{X}}^p$ and $\bar{\mathbf{X}}^p \in \Gamma^n \cap \mathbb{J}$ ($p = 1, \dots, \ell$). Therefore, we have shown that $\bar{\mathbf{X}} \in \text{co}(\Gamma^n \cap \mathbb{J})$ and $\bigcap_{m=1}^{\infty} \text{co}(\Gamma^n \cap (\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k))) \subseteq \text{co}(\Gamma^n \cap \mathbb{J})$. □

We may assume without loss of generality that \mathcal{B} is bounded since we can replace \mathcal{B} with $\mathcal{B}' = \{\mathbf{B} / \|\mathbf{B}\| : \mathbf{O} \neq \mathbf{B} \in \mathcal{B}\}$ if \mathcal{B} is unbounded, where $\|\mathbf{B}\|$ denotes the Frobenius norm of $\mathbf{B} \in \mathbb{S}^n$. For each $\epsilon > 0$, define an open neighborhood $U(\mathbf{B}, \epsilon) = \{\mathbf{A} \in \mathbb{S}^n : \|\mathbf{A} - \mathbf{B}\| < \epsilon\}$ of each $\mathbf{B} \in \text{cl}\mathcal{B}$. Let $\{\epsilon_k\}$ be a sequence of positive numbers that converges to 0. Let k be fixed. Since \mathcal{B} is bounded, its closure $\text{cl}\mathcal{B}$ is a compact subset of \mathbb{S}^n . Hence, we can choose a finite subset \mathcal{A}_k of $\text{cl}\mathcal{B}$ such that $\bigcup_{\mathbf{A} \in \mathcal{A}_k} U(\mathbf{A}, \epsilon_k/2) \supseteq \text{cl}\mathcal{B}$ and $U(\mathbf{A}, \epsilon_k/2)$ contains a $\mathbf{B} \in \mathcal{B}$ ($\mathbf{A} \in \mathcal{A}_k$). Let \mathcal{B}^k be the set of such \mathbf{B} 's. Then, the sequence $\{\mathcal{B}^k \ (k = 1, 2, \dots)\}$ satisfies that

$$\mathcal{B}^k \subseteq \mathcal{B}, \mathbb{J}_+(\mathcal{B}^k) \supseteq \mathbb{J}_+(\mathcal{B}) \ (k = 1, 2, \dots), \bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}^k) \supseteq \mathbb{J}_+(\mathcal{B}), \quad (14)$$

$$\forall \mathbf{B} \in \mathcal{B}, \exists \mathbf{B}' \in \mathcal{B}^k; \|\mathbf{B}' - \mathbf{B}\| < \epsilon_k \ (k = 1, 2, \dots). \quad (15)$$

Lemma 3.3. $\bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}^k) = \mathbb{J}_+(\mathcal{B})$.

Proof. By (14), it suffices to show that $\bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}^k) \subseteq \mathbb{J}_+(\mathcal{B})$. Let $\overline{\mathbf{X}} \in \bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}^k)$. To prove $\overline{\mathbf{X}} \in \mathbb{J}_+(\mathcal{B})$, we show that $\langle \overline{\mathbf{B}}, \overline{\mathbf{X}} \rangle \geq 0$ for an arbitrarily chosen $\overline{\mathbf{B}} \in \mathcal{B}$. By (15), there exists a sequence $\{\mathbf{B}_k \in \mathcal{B}^k\}$ that converges $\overline{\mathbf{B}} \in \mathcal{B}$ as $k \rightarrow \infty$. Since $\overline{\mathbf{X}} \in \mathbb{J}_+(\mathcal{B}^k)$, we see that $\langle \mathbf{B}_k, \overline{\mathbf{X}} \rangle \geq 0 \ (k = 1, 2, \dots)$. Hence, we obtain $\langle \overline{\mathbf{B}}, \overline{\mathbf{X}} \rangle \geq 0$ by taking the limit as $k \rightarrow \infty$. \square

Lemma 3.4. Assume that \mathcal{B} satisfies condition (B). Then $\text{co}(\Gamma^n \cap (\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k))) = \bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k) \ (m = 1, 2, \dots)$.

Proof. Let $m \in \{1, 2, \dots\}$ be fixed. We note that $\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k) = \mathbb{J}_+(\bigcup_{k=1}^m \mathcal{B}^k)$. Each $\bigcup_{k=1}^m \mathcal{B}^k$ satisfies condition (B) since it is a subset of \mathcal{B} . Since $\bigcup_{k=1}^m \mathcal{B}^k$ is finite, $\mathbb{J}_+(\bigcup_{k=1}^m \mathcal{B}^k) \in \widehat{\mathcal{F}}(\Gamma^n)$ as shown in Section 3.2.1. Therefore, $\text{co}(\Gamma^n \cap (\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k))) = \bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k)$. \square

Now, we show $\text{co}(\Gamma^n \cap \mathbb{J}_+(\mathcal{B})) = \mathbb{J}_+(\mathcal{B})$, which is equivalent to $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\Gamma^n)$ by definition. By Lemmas 3.3 and 3.4, we see that

$$\text{co}(\Gamma^n \cap (\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k))) = \bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k) \supseteq \bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}^k) = \mathbb{J}_+(\mathcal{B}) \ (m = 1, 2, \dots),$$

which implies that $\bigcap_{m=1}^{\infty} \text{co}(\Gamma^n \cap (\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k))) \supseteq \mathbb{J}_+(\mathcal{B})$. By Lemma 3.2, $\bigcap_{m=1}^{\infty} \text{co}(\Gamma^n \cap (\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k))) = \text{co}(\Gamma^n \cap \mathbb{J}_+(\mathcal{B}))$. Therefore, we have shown that $\text{co}(\Gamma^n \cap \mathbb{J}_+(\mathcal{B})) \supseteq \mathbb{J}_+(\mathcal{B})$. The converse inclusion $\text{co}(\Gamma^n \cap \mathbb{J}_+(\mathcal{B})) \subseteq \mathbb{J}_+(\mathcal{B})$ is straightforward since $\mathbb{J}_+(\mathcal{B})$ is convex. \square

4 On Theorems 1.6 and 1.7

4.1 Facial reduction of $\mathcal{B}_{\geq} \cap L_1(\mathbb{F})$ into \mathbb{R}^{r-1}

To adapt the argument in Section 3.1 to QCQP (5) (i.e., $\text{COP}(\Gamma^n \cap \mathbb{J}_+(\mathcal{B}), \mathbf{Q}, \mathbf{H}^1)$ with $\mathbf{H}^1 = \text{diag}(0, \dots, 0, 1) \in \mathbb{S}_+^n$), its SDP relaxation, conditions (C) and (D), we need some additional arguments. First, we assume that the feasible region $\mathcal{B}_{\geq} \cap L_1(\mathbb{F})$ of QCQP (5) is nonempty, as required in condition (C). Consequently, the feasible region of $\text{COP}(\Gamma^n \cap \mathbb{J}_+(\widetilde{\mathcal{B}}), \widetilde{\mathbf{Q}}, \widetilde{\mathbf{H}}^1)$ (12) is also nonempty. For any choice of a nonsingular \mathbf{M} , the rank of $\widetilde{\mathbf{H}}^1 = \Psi(\mathbf{H}^1) = \mathbf{M}^{-1} \mathbf{P}_{1,r}^T \mathbf{H}^1 \mathbf{P}_{1,r} \mathbf{M}^{-T} \in \mathbb{S}_+^r$ is 1, otherwise the rank is 0 or $\widetilde{\mathbf{H}}^1 = \mathbf{O} \in \mathbb{S}_+^r$;

hence $\text{COP}(\Gamma^r \cap \mathbb{J}_+(\widetilde{\mathcal{B}}), \widetilde{\mathcal{Q}}, \widetilde{\mathbf{H}^1})$ (12) is infeasible. Specifically, we can take a nonsingular matrix \mathbf{M} such that $\widetilde{\mathbf{H}^1} = \mathbf{M}^{-1} \mathbf{P}_{1,r}^T \mathbf{H}^1 \mathbf{P}_{1,r} \mathbf{M}^{-T} = \text{diag}(0, \dots, 0, 1) \in \mathbb{S}_+^r$. In this case, if $\mathbf{u} \in L_1(\mathbb{F})$, then $\begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix} \in L(\mathbb{F})$ and $\langle \widetilde{\mathbf{H}^1}, \theta\left(\begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix}\right) \theta\left(\begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix}\right)^T \rangle = \langle \mathbf{H}^1, \begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix}^T \rangle = 1$ holds by Lemma 3.1 (iv). Since $\theta : L(\mathbb{F}) \rightarrow \mathbb{R}^{r-1}$ is linear, we have either $\theta\left(\begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix}\right)_r = 1$ for every $\mathbf{u} \in L(\mathbb{F})$ or $\theta\left(\begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix}\right)_r = -1$ for every $\mathbf{u} \in L(\mathbb{F})$. Define $\theta_1 : L_1(\mathbb{F}) \rightarrow \mathbb{R}^{r-1}$ by

$$\theta_1(\mathbf{u}) = \begin{cases} \left(\theta\left(\begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix}\right)_1, \dots, \theta\left(\begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix}\right)_{r-1} \right)^T & \text{in the former case,} \\ \left(-\theta\left(\begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix}\right)_1, \dots, -\theta\left(\begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix}\right)_{r-1} \right)^T & \text{in the latter case.} \end{cases}$$

Then,

$$q(\mathbf{u}, \mathbf{B}) = \langle \mathbf{B}, \begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix}^T \rangle = \langle \widetilde{\mathbf{B}}, \theta\left(\begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix}\right) \theta\left(\begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix}\right)^T \rangle = \langle \widetilde{\mathbf{B}}, \begin{pmatrix} \theta_1(\mathbf{u}) \\ 1 \end{pmatrix} \begin{pmatrix} \theta_1(\mathbf{u}) \\ 1 \end{pmatrix}^T \rangle$$

for every $\mathbf{u} \in L_1(\mathbb{F})$ and $\mathbf{B} \in \mathbb{S}^n$. Now, for every $\mathbf{w} \in \mathbb{R}^{r-1}$ and $\widetilde{\mathbf{B}} \in \mathbb{S}^r$, we define

$$\begin{aligned} q(\mathbf{w}, \widetilde{\mathbf{B}}) &= \begin{pmatrix} \mathbf{w} \\ 1 \end{pmatrix}^T \widetilde{\mathbf{B}} \begin{pmatrix} \mathbf{w} \\ 1 \end{pmatrix}, \\ \widetilde{\mathbf{B}}_{\geq}, \widetilde{\mathbf{B}}_{=} \text{ or } \widetilde{\mathbf{B}}_{\leq} &= \left\{ \mathbf{w} \in \mathbb{R}^{r-1} : q(\mathbf{w}, \widetilde{\mathbf{B}}) \geq 0, = \text{ or } \leq 0, \text{ resp.} \right\}, \\ \widetilde{\mathcal{B}}_{\geq} &= \bigcap_{\widetilde{\mathbf{B}} \in \widetilde{\mathcal{B}}} \widetilde{\mathbf{B}}_{\geq} = \left\{ \mathbf{w} \in \mathbb{R}^{r-1} : q(\mathbf{w}, \widetilde{\mathbf{B}}) \geq 0 \ (\widetilde{\mathbf{B}} \in \widetilde{\mathcal{B}}) \right\}. \end{aligned}$$

Using the notation above, QCQP (5) is transformed into $\text{COP}(\Gamma^r \cap \mathbb{J}_+(\widetilde{\mathcal{B}}), \widetilde{\mathcal{Q}}, \widetilde{\mathbf{H}^1})$ with $\widetilde{\mathbf{H}^1} = \text{diag}(0, \dots, 0, 1) \in \mathbb{S}_+^r$:

$$\begin{aligned} \eta(\Gamma^r \cap \mathbb{J}_+(\widetilde{\mathcal{B}}), \widetilde{\mathcal{Q}}, \widetilde{\mathbf{H}^1}) &= \inf \{ q(\mathbf{w}, \widetilde{\mathcal{Q}}) : q(\mathbf{w}, \widetilde{\mathbf{B}}) \geq 0 \ (\widetilde{\mathbf{B}} \in \widetilde{\mathcal{B}}) \} \\ &= \inf \{ q(\mathbf{w}, \widetilde{\mathcal{Q}}) : \mathbf{w} \in \widetilde{\mathcal{B}}_{\geq} \}, \end{aligned} \tag{16}$$

and conditions (C) and (D) into

- (C) $\widetilde{\mathcal{B}}_{\geq} \neq \emptyset$, and $\emptyset \neq \widetilde{\mathbf{B}}_{\leq} \subseteq \widetilde{\mathbf{A}}_{\geq}$ or $\widetilde{\mathbf{A}}_{\geq} \subseteq \widetilde{\mathbf{B}}_{\geq}$ holds for every $\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}} \in \widetilde{\mathcal{B}}$.
- (D) $\widetilde{\mathcal{B}}$ is finite. $q(\cdot, \widetilde{\mathbf{B}}) : \mathbb{R}^{r-1} \rightarrow \mathbb{R}$ ($\widetilde{\mathbf{B}} \in \widetilde{\mathcal{B}}$) is not affine and $\widetilde{\mathbf{B}}_{=} \subseteq \widetilde{\mathbf{A}}_{\geq}$ for every $\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}} \in \widetilde{\mathcal{B}}$.

Lemma 4.1.

- (i) $\theta_1 : L_1(\mathbb{F}) \rightarrow \mathbb{R}^{r-1}$ is one-to-one and onto.
- (ii) $q(\mathbf{u}, \mathbf{A}) = q(\theta_1(\mathbf{u}), \widetilde{\mathbf{A}})$ for every $\mathbf{u} \in L_1(\mathbb{F})$ and $\mathbf{A} \in \mathbb{S}^n$.
- (iii) $\theta_1(\mathbf{B}_{\geq} \cap L_1(\mathbb{F})) = \widetilde{\mathbf{B}}_{\geq}$, $\theta_1(\mathbf{B}_{=} \cap L_1(\mathbb{F})) = \widetilde{\mathbf{B}}_{=}$ and $\theta_1(\mathbf{B}_{\leq} \cap L_1(\mathbb{F})) = \widetilde{\mathbf{B}}_{\leq}$.

Proof. (i) This assertion follows from Lemma 3.1 (ii) and the definition of $\theta_1 : L_1(\mathbb{F}) \rightarrow \mathbb{R}^{r-1}$.

(ii) Let $\mathbf{u} \in L_1(\mathbb{F})$ and $\mathbf{A} \in \mathbb{S}^n$. Then,

$$\begin{aligned} q(\mathbf{u}, \mathbf{A}) &= \langle \mathbf{A}, \begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix}^T \rangle \\ &= \langle \tilde{\mathbf{A}}, \theta\left(\begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix}\right) \theta\left(\begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix}\right)^T \rangle \text{ (by Theorem 3.1 (iv))} \\ &= \langle \tilde{\mathbf{A}}, \begin{pmatrix} \theta_1(\mathbf{u}) \\ 1 \end{pmatrix} \begin{pmatrix} \theta_1(\mathbf{u}) \\ 1 \end{pmatrix}^T \rangle = q(\tilde{\mathbf{A}}, \theta_1(\mathbf{u})). \end{aligned}$$

(iii) By (i) and (ii), we see that

$$\begin{aligned} \theta_1(\mathbf{B}_{\geq} \cap L_1(\mathbb{F})) &= \{\theta_1(\mathbf{u}) : q(\mathbf{u}, \mathbf{B}) \geq 0, \mathbf{u} \in L_1(\mathbb{F})\} \\ &= \left\{ \theta_1(\mathbf{u}) : q(\theta_1(\mathbf{u}), \tilde{\mathbf{B}}) \geq 0, \theta_1(\mathbf{u}) \in \theta_1(L_1(\mathbb{F})) \right\} \\ &= \left\{ \mathbf{w} \in \mathbb{R}^{r-1} : q(\mathbf{w}, \tilde{\mathbf{B}}) \geq 0 \right\} = \tilde{\mathbf{B}}_{\geq}. \end{aligned}$$

The second and third identity can be proved similarly. □

As a result of Lemma 4.1, we obtain that:

- [e] $\mathbf{u} \in \mathbb{R}^{n-1}$ is a feasible solution of QCQP (5) with the objective value $q(\mathbf{u}, \mathbf{Q})$ if and only if $\mathbf{w} = \theta_1(\mathbf{u})$ is a feasible solution of QCQP (16) with the objective value $q(\mathbf{w}, \tilde{\mathbf{Q}}) = q(\mathbf{u}, \mathbf{Q})$.
- [f] Condition (C) holds if and only if condition ($\tilde{\mathbf{C}}$) holds.
- [g] Condition (D) holds if and only if condition ($\tilde{\mathbf{D}}$) holds.

4.2 Proof of Theorem 1.6

We have shown in [c] of Section 3.1 and [f] above that conditions (B) and (C) are equivalent to conditions ($\tilde{\mathbf{B}}$) and ($\tilde{\mathbf{C}}$), respectively. Hence, for Theorem 1.6, it is sufficient to prove that condition ($\tilde{\mathbf{C}}$) implies condition ($\tilde{\mathbf{B}}$). For simplicity of notation, we omit \sim from $\tilde{\mathbf{B}}$ ($\mathbf{B} \in \mathcal{B}$), $\tilde{\mathbf{B}}$, $\tilde{\mathbf{Q}}$ and \mathbf{H}^1 , or equivalently, we prove that condition (C) implies condition (B) under the assumption that $\mathbb{F} = \mathbb{S}_+^n$, $L(\mathbb{F}) = \mathbb{R}^r$ and $L_1(\mathbb{F}) = \mathbb{R}^{r-1}$.

Lemma 4.2. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$. Then*

- (i) $\mathbb{J}_-(\mathbf{B}) \subseteq \mathbb{J}_+(\mathbf{A}) \Rightarrow \mathbb{J}_0(\mathbf{B}) \subseteq \mathbb{J}_+(\mathbf{A})$.
 - (ii) $\mathbb{J}_-(\mathbf{B}) \subseteq \mathbb{J}_+(\mathbf{A}) \Leftrightarrow \mathbf{\Gamma}^n \cap \mathbb{J}_-(\mathbf{B}) \subseteq \mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathbf{A})$
- $$\Updownarrow$$
- $$\{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{B}, \mathbf{x}\mathbf{x}^T \rangle \leq 0\} \subseteq \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{A}, \mathbf{x}\mathbf{x}^T \rangle \geq 0\} \quad (17)$$

Proof. (i) is obvious since $\mathbb{J}_0(\mathbf{B}) \subseteq \mathbb{J}_-(\mathbf{B})$. (ii) For the first \Leftrightarrow , \Rightarrow is straightforward. Conversely, if $\mathbf{\Gamma}^n \cap \mathbb{J}_-(\mathbf{B}) \subseteq \mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathbf{A})$, then

$$\mathbb{J}_-(\mathbf{B}) = \text{co}(\mathbf{\Gamma}^n \cap \mathbb{J}_-(\mathbf{B})) \subseteq \text{co}(\mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathbf{A})) = \mathbb{J}_+(\mathbf{A})$$

follows by Theorem 1.2. Hence \Rightarrow follows. The second \Updownarrow is straightforward. □

Lemma 4.3. Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$. Assume that $\emptyset \neq \mathbf{B}_\leq \subseteq \mathbf{A}_\geq$. Then $\mathbb{J}_-(\mathbf{B}) \subseteq \mathbb{J}_+(\mathbf{A})$.

Proof. By Lemma 4.2 (ii), it suffices to show that (17). We represent $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ as $\mathbf{A} = \begin{pmatrix} \mathbf{C} & \mathbf{c} \\ \mathbf{c}^T & \gamma \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \mathbf{D} & \mathbf{d} \\ \mathbf{d}^T & \delta \end{pmatrix}$, where $\mathbf{C}, \mathbf{D} \in \mathbb{S}^{n-1}$, $\mathbf{c}, \mathbf{d} \in \mathbb{R}^{n-1}$, $\gamma, \delta \in \mathbb{R}$. By assumption, there exists a $\hat{\mathbf{u}} \in \mathbb{R}^{n-1}$ such that

$$\hat{\mathbf{u}} \in \mathbf{B}_\leq, \text{ i.e., } \hat{\mathbf{u}}^T \mathbf{D} \hat{\mathbf{u}} + 2\mathbf{d}^T \hat{\mathbf{u}} + \delta = \langle \mathbf{B}, \begin{pmatrix} \hat{\mathbf{u}} \\ 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{u}} \\ 1 \end{pmatrix}^T \rangle \leq 0$$

Let $\bar{\mathbf{x}} = \begin{pmatrix} \bar{\mathbf{u}} \\ \bar{\xi} \end{pmatrix} \in \mathbb{R}^n$. By assuming

$$\begin{aligned} \emptyset \neq \{ \mathbf{u} \in \mathbb{R}^{n-1} : \mathbf{u}^T \mathbf{D} \mathbf{u} + 2\mathbf{d}^T \mathbf{u} + \delta \leq 0 \} &= \mathbf{B}_\leq \\ &\subseteq \mathbf{A}_\geq = \{ \mathbf{u} \in \mathbb{R}^{n-1} : \mathbf{u}^T \mathbf{C} \mathbf{u} + 2\mathbf{c}^T \mathbf{u} + \gamma \geq 0 \}, \\ \langle \mathbf{B}, \bar{\mathbf{x}} \bar{\mathbf{x}}^T \rangle &= \langle \mathbf{B}, \begin{pmatrix} \bar{\mathbf{u}} \\ \bar{\xi} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{u}} \\ \bar{\xi} \end{pmatrix}^T \rangle = \bar{\mathbf{u}}^T \mathbf{D} \bar{\mathbf{u}} + 2\mathbf{d}^T \bar{\mathbf{u}} \bar{\xi} + \delta \bar{\xi}^2 \leq 0, \end{aligned}$$

we will show that

$$\langle \mathbf{A}, \bar{\mathbf{x}} \bar{\mathbf{x}}^T \rangle = \langle \mathbf{A}, \begin{pmatrix} \bar{\mathbf{u}} \\ \bar{\xi} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{u}} \\ \bar{\xi} \end{pmatrix}^T \rangle = \bar{\mathbf{u}}^T \mathbf{C} \bar{\mathbf{u}} + 2\mathbf{c}^T \bar{\mathbf{u}} \bar{\xi} + \gamma \bar{\xi}^2 \geq 0. \quad (18)$$

One of the following for four cases (a), (b), (c) and (d) occurs.

- (a) $\bar{\xi} \neq 0$: In this case, $\begin{pmatrix} \bar{\mathbf{u}}/\bar{\xi} \\ 1 \end{pmatrix} \in \mathbf{B}_\leq \subseteq \mathbf{A}_\geq$, which implies (18).
- (b) $\bar{\xi} = 0$, $\bar{\mathbf{u}}^T \mathbf{D} \bar{\mathbf{u}} < 0$: We have

$$(\alpha \bar{\mathbf{u}})^T \mathbf{D} (\alpha \bar{\mathbf{u}}) + 2\mathbf{d}^T (\alpha \bar{\mathbf{u}}) + \delta < 0; \text{ hence } \alpha \bar{\mathbf{u}} \in \mathbf{B}_\leq \subseteq \mathbf{A}_\geq$$

for every sufficiently large $\alpha > 0$. Hence,

$$\bar{\mathbf{u}}^T \mathbf{C} \bar{\mathbf{u}} + 2\mathbf{c}^T \bar{\mathbf{u}} (1/\alpha) + \gamma (1/\alpha)^2 \geq 0 \text{ for every sufficiently large } \alpha > 0.$$

Taking the limit as $\alpha \rightarrow \infty$, we obtain $\bar{\mathbf{u}}^T \mathbf{C} \bar{\mathbf{u}} \geq 0$, i.e., (18) with $\bar{\xi} = 0$ holds.

- (c) $\bar{\xi} = 0$, $\bar{\mathbf{u}}^T \mathbf{D} \bar{\mathbf{u}} = 0$, $\mathbf{D} \in \mathbb{S}_+^{n-1}$: We may assume without loss of generality that $(\bar{\mathbf{u}}^T \mathbf{D} + \mathbf{d}^T) \bar{\mathbf{u}} \leq 0$ since $\langle \mathbf{B}, \begin{pmatrix} \bar{\mathbf{u}} \\ \bar{\xi} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{u}} \\ \bar{\xi} \end{pmatrix}^T \rangle = \langle \mathbf{B}, \begin{pmatrix} -\bar{\mathbf{u}} \\ -\bar{\xi} \end{pmatrix} \begin{pmatrix} -\bar{\mathbf{u}} \\ -\bar{\xi} \end{pmatrix}^T \rangle$ and $\langle \mathbf{A}, \begin{pmatrix} \bar{\mathbf{u}} \\ \bar{\xi} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{u}} \\ \bar{\xi} \end{pmatrix}^T \rangle = \langle \mathbf{A}, \begin{pmatrix} -\bar{\mathbf{u}} \\ -\bar{\xi} \end{pmatrix} \begin{pmatrix} -\bar{\mathbf{u}} \\ -\bar{\xi} \end{pmatrix}^T \rangle$. Since $\mathbf{D} \bar{\mathbf{u}} = \mathbf{0}$ and $\hat{\mathbf{u}} \in \mathbf{B}_\leq$, we see

$$\begin{aligned} &(\alpha \bar{\mathbf{u}} + \hat{\mathbf{u}})^T \mathbf{D} (\alpha \bar{\mathbf{u}} + \hat{\mathbf{u}}) + 2\mathbf{d}^T (\alpha \bar{\mathbf{u}} + \hat{\mathbf{u}}) + \delta \\ &= 2\alpha (\bar{\mathbf{u}}^T \mathbf{D} + \mathbf{d}^T) \hat{\mathbf{u}} + \hat{\mathbf{u}}^T \mathbf{D} \hat{\mathbf{u}} + 2\mathbf{d}^T \hat{\mathbf{u}} + \delta \\ &\leq \hat{\mathbf{u}}^T \mathbf{D} \hat{\mathbf{u}} + 2\mathbf{d}^T \hat{\mathbf{u}} + \delta \leq 0 \end{aligned}$$

for every $\alpha > 0$. Hence, $\alpha \bar{\mathbf{u}} + \hat{\mathbf{u}} \in \mathbf{B}_\leq \subseteq \mathbf{A}_\geq$ for every $\alpha > 0$. Therefore, for every $\alpha > 0$,

$$((\alpha \bar{\mathbf{u}} + \hat{\mathbf{u}})^T \mathbf{C} (\alpha \bar{\mathbf{u}} + \hat{\mathbf{u}}) + 2\mathbf{c}^T (\alpha \bar{\mathbf{u}} + \hat{\mathbf{u}}) + \gamma) / \alpha^2 \geq 0.$$

Taking the limit as $\alpha \rightarrow \infty$, we obtain (18) with $\bar{\xi} = 0$.

- (d) $\bar{\xi} = 0$, $\bar{\mathbf{u}}^T \mathbf{D} \bar{\mathbf{u}} = 0$, $\mathbf{D} \notin \mathbb{S}_+^{n-1}$. In this case, there exists $\mathbf{v} \in \mathbb{R}^{n-1}$ such that $\mathbf{v}^T \mathbf{D} \mathbf{v} = -1$. We may assume that $\bar{\mathbf{u}}^T \mathbf{D} \mathbf{v} \leq 0$; if $\bar{\mathbf{u}}^T \mathbf{D} \mathbf{v} > 0$, then replace \mathbf{v} by $-\mathbf{v}$. For every $\epsilon > 0$,

$$\begin{aligned} (\bar{\mathbf{u}} + \epsilon \mathbf{v})^T \mathbf{D} (\bar{\mathbf{u}} + \epsilon \mathbf{v}) &= \bar{\mathbf{u}}^T \mathbf{D} \bar{\mathbf{u}} + 2\epsilon \bar{\mathbf{u}}^T \mathbf{D} \mathbf{v} + \epsilon^2 \mathbf{v}^T \mathbf{D} \mathbf{v} \\ &= 2\epsilon \bar{\mathbf{u}}^T \mathbf{D} \mathbf{v} + \epsilon^2 \mathbf{v}^T \mathbf{D} \mathbf{v} \\ &\leq -\epsilon^2 < 0. \end{aligned}$$

As shown in case (b), we see that $(\bar{\mathbf{u}} + \epsilon \mathbf{v})^T \mathbf{C} (\bar{\mathbf{u}} + \epsilon \mathbf{v}) \geq 0$ for every $\epsilon > 0$. Therefore, taking the limit as $\epsilon \rightarrow 0$, we obtain $\bar{\mathbf{u}}^T \mathbf{C} \bar{\mathbf{u}} \geq 0$, *i.e.*, (18) with $\bar{\xi} = 0$ holds. \square

Remark 4.4. The following example shows the necessity of the assumption $\emptyset \neq \mathbf{B}_{\leq}$ in Lemma 4.3. Let $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $\mathbf{B}_{\leq} = \{(u_1, u_2) : u_1^2 + 1 \leq 0\} = \emptyset \subseteq \mathbf{A}_{\geq} = \{(u_1, u_2) : -u_2^2 \geq 0\}$. But $\mathbb{J}_-(\mathbf{B}) \ni \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \notin \mathbb{J}_+(\mathbf{A})$; hence $\mathbb{J}_-(\mathbf{B}) \not\subseteq \mathbb{J}_+(\mathbf{A})$.

Now we are ready to prove Theorem 1.6. By Lemmas 4.3, we see that for every $\mathbf{A}, \mathbf{B} \in \mathbb{S}_+^n$

$$\begin{aligned} \emptyset \neq \mathbf{B}_{\leq} \subseteq \mathbf{A}_{\geq} &\Rightarrow \mathbb{J}_0(\mathbf{B}) \subseteq \mathbb{J}_-(\mathbf{B}) \subseteq \mathbb{J}_+(\mathbf{A}), \\ \emptyset \neq \mathbf{A}_{\geq} = (-\mathbf{A})_{\leq} \subseteq \mathbf{B}_{\geq} &\Rightarrow \mathbb{J}_+(\mathbf{A}) = \mathbb{J}_-(-\mathbf{A}) \subseteq \mathbb{J}_+(\mathbf{B}). \end{aligned}$$

(Note that the assumption $\mathcal{B}_{\geq} \neq \emptyset$ of condition (C) with $\mathbb{F} = \mathbb{S}_+^n$ implies $\mathbf{A}_{\geq} \neq \emptyset$ for every $\mathbf{A} \in \mathcal{B}$.) Therefore, condition (C) implies condition (B). \square

4.3 Proof of Theorem 1.7

We have shown in [g] that condition (D) is equivalent to condition ($\tilde{\text{D}}$). By Theorem 1.4, we obtain

$$\begin{aligned} \mathbb{J}_+(\tilde{\mathcal{B}}) \cap \widetilde{\mathbb{H}^1} &= \overline{\text{co}}(\Gamma^r \cap \mathbb{J}_+(\tilde{\mathcal{B}}) \cap \widetilde{\mathbb{H}^1}), \\ \eta(\mathbb{J}_+(\tilde{\mathcal{B}}), \tilde{\mathcal{Q}}, \widetilde{\mathbf{H}^1}) &= \eta(\Gamma^r \cap \mathbb{J}_+(\tilde{\mathcal{B}}), \tilde{\mathcal{Q}}, \widetilde{\mathbf{H}^1}), \end{aligned}$$

which are equivalent to (6) and (7), where $\widetilde{\mathbb{H}^1} = \{\mathbf{V} \in \mathbb{S}^r : \langle \widetilde{\mathbf{H}^1}, \mathbf{V} \rangle = 1\}$. \square

5 Relationships among conditions (I), (II), (III), (B), (C), and (D)

Throughout this section, we assume $\mathbb{F} = \mathbb{F}_{\min}$ in conditions (B), (C) and (D). We have already observed that (I) \Rightarrow (B), (III) \Rightarrow (D) and (C) \Rightarrow (B) (Theorem 1.6). To prove that (B) is the weakest among the conditions, we show (II) \Rightarrow (B) in Section 5.1, and (D) \Rightarrow (B) in Section 5.2. Under appropriate assumptions, equivalence of conditions (I), (II), (B) and (C) are shown in Section 5.3.

5.1 Condition (II) \Rightarrow condition (B)

Assume that condition (II) is satisfied. Let $\mathbf{A}, \mathbf{B} \in \mathcal{B}$ and $\mathbf{A} \neq \mathbf{B}$. Then, there exists an $(\alpha, \beta) \neq \mathbf{0}$ such that $\alpha\mathbf{A} + \beta\mathbf{B} \in \mathbb{S}_+^n$, which implies

$$\langle \alpha\mathbf{A} + \beta\mathbf{B}, \mathbf{X} \rangle \geq 0 \text{ for every } \mathbf{X} \in \mathbb{S}_+^n. \quad (19)$$

We will show that $\mathbb{J}_0(\mathbf{B}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\mathbf{A}) \cap \mathbb{F}$ or $\mathbb{J}_+(\mathbf{A}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\mathbf{B}) \cap \mathbb{F}$ holds. One of the following cases (a),(b),...,(f) occurs.

- (a) $\alpha > 0$: If $\mathbf{X} \in \mathbb{J}_0(\mathbf{B}) \cap \mathbb{F}$, then we see from (19) that $\mathbf{X} \in \mathbb{J}_+(\mathbf{A}) \cap \mathbb{F}$. Hence, $\mathbb{J}_0(\mathbf{B}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\mathbf{A}) \cap \mathbb{F}$ holds.
- (b) $\alpha = 0$ and $\beta > 0$: By (19), $\mathbb{J}_+(\mathbf{A}) \cap \mathbb{F} \subseteq \mathbb{F} = \mathbb{J}_+(\mathbf{B}) \cap \mathbb{F}$ holds.
- (c) $\alpha = 0$ and $\beta < 0$: By (19), $\langle -\mathbf{B}, \mathbf{X} \rangle \geq 0$ holds for every $\mathbf{X} \in \mathbb{S}_+^n$, which implies $-\mathbf{B} \in \mathbb{S}_+^n$ and $\mathbb{J}_+(\mathbf{B})$ forms a face of \mathbb{S}_+^n . If $\mathbb{J}_+(\mathbf{B}) \cap \mathbb{F} = \mathbb{F}$, then $\mathbb{J}_+(\mathbf{A}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\mathbf{B}) \cap \mathbb{F}$ holds. Otherwise, $\mathbb{J}_+(\mathbf{B}) \cap \mathbb{F}$ is contained in the proper face $\mathbb{J}_+(\mathbf{B}) \cap \mathbb{F}$ of $\mathbb{F} = \mathbb{F}_{\min}$, which contradicts the definition of \mathbb{F}_{\min} .
- (d) $\alpha < 0$ and $\beta > 0$: Then $\mathbf{B} = (-\alpha/\beta)\mathbf{A} + \mathbf{Y}$ for some $\mathbf{Y} \in \mathbb{S}_+^n$, which implies $\mathbb{J}_+(\mathbf{A}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\mathbf{B}) \cap \mathbb{F}$.
- (e) $\alpha < 0, \beta = 0$. This case can be treated similarly to case (c) to show that $\mathbb{J}_+(\mathbf{A}) \cap \mathbb{F}$ forms a face of $\mathbb{F} = \mathbb{F}_{\min}$.
- (f) $\alpha < 0, \beta < 0$. In this case, we observe that

$$\begin{aligned} \mathbb{J}_+(\mathcal{B}) &\subseteq \{\mathbf{X} \in \mathbb{F} : \langle \mathbf{A}, \mathbf{X} \rangle \geq 0, \langle \mathbf{B}, \mathbf{X} \rangle \geq 0\} \\ &= \{\mathbf{X} \in \mathbb{F} : \langle \mathbf{A}, \mathbf{X} \rangle \geq 0, \langle \mathbf{B}, \mathbf{X} \rangle \geq 0, -\langle \alpha\mathbf{A} + \beta\mathbf{B}, \mathbf{X} \rangle \geq 0\} \\ &\quad (\text{since } \alpha < 0 \text{ and } \beta < 0) \\ &\subseteq \{\mathbf{X} \in \mathbb{F} : \langle -\alpha\mathbf{A} - \beta\mathbf{B}, \mathbf{X} \rangle = 0\} \quad (\text{since } \alpha\mathbf{A} + \beta\mathbf{B} \in \mathbb{S}_+^n). \end{aligned}$$

Hence, $\mathbb{J}_+(\mathcal{B})$ is included in a face $\mathbb{F}' \equiv \{\mathbf{X} \in \mathbb{F} : \langle -\alpha\mathbf{A} - \beta\mathbf{B}, \mathbf{X} \rangle = 0\}$ of \mathbb{F} . If $\mathbb{F}' = \mathbb{F}$ then

$$\mathbb{J}_0(\mathbf{B}) \cap \mathbb{F} = \mathbb{J}_0(\mathbf{B}) \cap \mathbb{F}' = \mathbb{J}_0(\mathbf{A}) \cap \mathbb{F}' = \mathbb{J}_0(\mathbf{A}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\mathbf{A}) \cap \mathbb{F}.$$

Otherwise, \mathbb{F}' is a proper face of $\mathbb{F} = \mathbb{F}_{\min}$ that contains $\mathbb{J}_+(\mathcal{B})$, a contradiction to the definition of \mathbb{F}_{\min} . □

5.2 Condition (D) \Rightarrow condition (B)

We have shown the equivalence of conditions (B) and $(\tilde{\mathbf{B}})$ in Section 3.1 [c] and the equivalence of (D) and $(\tilde{\mathbf{D}})$ in Section 4.1 [g]. Therefore, it suffices to show $(\tilde{\mathbf{B}}) \Rightarrow (\tilde{\mathbf{D}})$ for (B) \Rightarrow (D). Since we take $\mathbb{F} = \mathbb{F}_{\min}$, $\mathbb{J}_+(\mathcal{B}) \cap \mathbb{S}_{++}^n \neq \emptyset$ (Slater's constraint qualification for SDP (13)). For simplicity of notation, we omit \sim and assume that $\mathbb{J}_+(\mathcal{B})$ itself satisfies condition

$$(A-1) \quad \mathbb{J}_+(\mathcal{B}) \cap \mathbb{S}_{++}^n \neq \emptyset.$$

throughout this and the next sections.

The following lemma can be proved similarly as Lemma 4.2 (ii), and the proof is omitted.

Lemma 5.1. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$. Then*

$$\begin{aligned} \mathbb{J}_0(\mathbf{B}) \subseteq \mathbb{J}_+(\mathbf{A}) &\Leftrightarrow \mathbf{\Gamma}^n \cap \mathbb{J}_0(\mathbf{B}) \subseteq \mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathbf{A}) \\ &\Updownarrow \\ \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}\mathbf{x}^T \in \mathbb{J}_0(\mathbf{B})\} &\subseteq \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}\mathbf{x}^T \in \mathbb{J}_+(\mathbf{A})\} \end{aligned} \quad (20)$$

Lemma 5.2. *Let $\mathbf{B} = \begin{pmatrix} \mathbf{D} & \mathbf{d} \\ \mathbf{d}^T & \delta \end{pmatrix} \in \mathbb{S}^n$ and $\delta \neq 0$. Assume that $\mathbf{B} \notin \mathbb{S}_+^n$ and $\mathbb{J}_+(\mathbf{B}) \cap \mathbb{S}_{++} \neq \emptyset$ (condition (A-1)). Then, there exists a $\mathbf{v} \in \mathbb{R}^{n-1}$ such that $-\delta \mathbf{v}^T (\mathbf{D} - \mathbf{d}\mathbf{d}^T/\delta) \mathbf{v} > 0$.*

Proof. Assume that $\delta > 0$. Since $\mathbf{D} - \mathbf{d}\mathbf{d}^T/\delta$ is the Schur complement of $\mathbf{B} \notin \mathbb{S}_+^n$ with $\delta > 0$, $(\mathbf{D} - \mathbf{d}\mathbf{d}^T/\delta) \notin \mathbb{S}_+^{n-1}$. Thus, there exists $\mathbf{v} \in \mathbb{R}^{n-1}$ such that $\mathbf{v}^T (\mathbf{D} - \mathbf{d}\mathbf{d}^T/\delta) \mathbf{v} < 0$, and the desired result follows. Now assume that $\delta < 0$. If $\mathbf{O} \neq \mathbf{B}$ was negative semidefinite, then $\mathbb{J}_+(\mathbf{B})$ would be a proper face of \mathbb{S}_+^n , which is a contradiction to the assumption $\mathbb{J}_+(\mathbf{B}) \cap \mathbb{S}_{++} \neq \emptyset$. Consequently, $-\mathbf{B} = \begin{pmatrix} -\mathbf{D} & -\mathbf{d} \\ -\mathbf{d}^T & -\delta \end{pmatrix}$ with $-\delta > 0$ is not positive semidefinite. Hence its Schur complement $(-\mathbf{D} + \mathbf{d}\mathbf{d}^T/\delta)$ is not positive semidefinite or $(\mathbf{D} - \mathbf{d}\mathbf{d}^T/\delta)$ is not negative semidefinite, and there exists a $\mathbf{v} \in \mathbb{R}^{n-1}$ such that $\mathbf{v}^T (\mathbf{D} - \mathbf{d}\mathbf{d}^T/\delta) \mathbf{v} > 0$. Therefore, the desired result follows. \square

Now we are ready to show condition (D) \Rightarrow condition (B) under the assumption $\mathbb{F} = \mathbb{S}_+^n$ and $\mathbb{J}_+(\mathcal{B}) \cap \mathbb{S}_{++}^n \neq \emptyset$ (condition (A-1)). By Lemma 5.1, it suffices to show that (20) or $\mathbb{J}_+(\mathbf{A}) \subseteq \mathbb{J}_+(\mathbf{B})$ holds for every distinct $\mathbf{A}, \mathbf{B} \in \mathcal{B}$. Let $\mathbf{A}, \mathbf{B} \in \mathcal{B}$ be distinct. We represent $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ as $\mathbf{A} = \begin{pmatrix} \mathbf{C} & \mathbf{c} \\ \mathbf{c}^T & \gamma \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \mathbf{D} & \mathbf{d} \\ \mathbf{d}^T & \delta \end{pmatrix}$, where $\mathbf{C}, \mathbf{D} \in \mathbb{S}^{n-1}$, $\mathbf{c}, \mathbf{d} \in \mathbb{R}^{n-1}$, $\gamma, \delta \in \mathbb{R}$. Let $\bar{\mathbf{x}} = \begin{pmatrix} \bar{\mathbf{u}} \\ \bar{\xi} \end{pmatrix} \in \mathbb{R}^n$. We may exclude the case $\mathbf{B} \in \mathbb{S}_+^n$ because it implies $\mathbb{J}_+(\mathbf{A}) \subseteq \mathbb{S}_+^n = \mathbb{J}_+(\mathbf{B})$. Assuming

$$\begin{aligned} \mathbf{D} \neq \mathbf{O}, \mathbb{J}_+(\mathbf{B}) \cap \mathbb{S}_{++}^n \neq \emptyset, \mathbf{B} \notin \mathbb{S}_+^n \\ \{\mathbf{u} \in \mathbb{R}^{n-1} : \mathbf{u}^T \mathbf{D} \mathbf{u} + 2\mathbf{d}^T \mathbf{u} + \delta = 0\} = \mathbf{B}_= \\ \subseteq \mathbf{A}_\geq = \{\mathbf{u} \in \mathbb{R}^{n-1} : \mathbf{u}^T \mathbf{C} \mathbf{u} + 2\mathbf{c}^T \mathbf{u} + \gamma \geq 0\}, \\ \langle \mathbf{B}, \bar{\mathbf{x}}\bar{\mathbf{x}}^T \rangle = \langle \mathbf{B}, \begin{pmatrix} \bar{\mathbf{u}} \\ \bar{\xi} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{u}} \\ \bar{\xi} \end{pmatrix}^T \rangle = \bar{\mathbf{u}}^T \mathbf{D} \bar{\mathbf{u}} + 2\mathbf{d}^T \bar{\mathbf{u}} \bar{\xi} + \delta \bar{\xi}^2 = 0, \end{aligned} \quad (21)$$

we show that

$$\langle \mathbf{A}, \bar{\mathbf{x}}\bar{\mathbf{x}}^T \rangle = \langle \mathbf{A}, \begin{pmatrix} \bar{\mathbf{u}} \\ \bar{\xi} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{u}} \\ \bar{\xi} \end{pmatrix}^T \rangle = \bar{\mathbf{u}}^T \mathbf{C} \bar{\mathbf{u}} + 2\mathbf{c}^T \bar{\mathbf{u}} \bar{\xi} + \gamma \bar{\xi}^2 \geq 0. \quad (22)$$

If $\bar{\mathbf{x}} = \mathbf{0}$, then (22) obviously holds. If $\bar{\xi} \neq 0$, then $(\bar{\mathbf{u}}/\bar{\xi}) \in \mathbf{B}_= \subseteq \mathbf{A}_\geq$, which implies (22). Therefore, we assume in the rest of the proof that $\bar{\mathbf{u}} \neq \mathbf{0}$ and $\bar{\xi} = 0$, which together with (21) implies $\bar{\mathbf{u}}^T \mathbf{D} \bar{\mathbf{u}} = 0$.

Let $\mathbf{0} \neq \mathbf{v} \in \mathbb{R}^{n-1}$. We consider the following quadratic equation in $\alpha \in \mathbb{R}$ with a parameter $\xi \in \mathbb{R}$.

$$\begin{aligned} 0 &= (\bar{\mathbf{u}} + \alpha \mathbf{v})^T \mathbf{D} (\bar{\mathbf{u}} + \alpha \mathbf{v}) + 2\mathbf{d}^T (\bar{\mathbf{u}} + \alpha \mathbf{v}) \xi + \delta \xi^2 \\ &= \alpha^2 \mathbf{v}^T \mathbf{D} \mathbf{v} + 2\alpha (\bar{\mathbf{u}}^T \mathbf{D} \mathbf{v} + \mathbf{d}^T \mathbf{v} \xi) + (2\mathbf{d}^T \bar{\mathbf{u}} + \delta \xi) \xi. \end{aligned} \quad (23)$$

(Note that $\bar{\mathbf{u}}^T \mathbf{D} \bar{\mathbf{u}} = 0$). We will specify $\mathbf{v} \in \mathbb{R}^{n-1}$ depending on $\mathbf{d}^T \bar{\mathbf{u}}$, $\mathbf{D} \bar{\mathbf{u}}$ and δ such that $\mathbf{v}^T \mathbf{D} \mathbf{v} \neq 0$ and the equation (23) has a real solution $\alpha(\xi)$ for every $\xi \in (-\epsilon, 0)$ or $\xi \in (0, \epsilon)$ and some $\epsilon > 0$ such that $\alpha(\xi) \rightarrow 0$ as $\xi \rightarrow 0$. Then $\langle \mathbf{B}, \left(\bar{\mathbf{u}} + \frac{\alpha(\xi) \mathbf{v}}{\xi} \right) \left(\bar{\mathbf{u}} + \frac{\alpha(\xi) \mathbf{v}}{\xi} \right)^T \rangle = 0$ with every $\xi \in (-\epsilon, 0)$ or $(0, \epsilon)$. As shown above, $(\bar{\mathbf{u}} + \alpha(\xi) \mathbf{v})/\xi \in \mathbf{B}_= \subseteq \mathbf{A}_\geq$ for every $\xi \in (-\epsilon, 0)$ or $(0, \epsilon)$; hence

$$(\bar{\mathbf{u}} + \alpha(\xi) \mathbf{v})^T \mathbf{C} (\bar{\mathbf{u}} + \alpha(\xi) \mathbf{v}) + 2\mathbf{c}^T (\bar{\mathbf{u}} + \alpha(\xi) \mathbf{v}) \xi + \delta \xi^2 \geq 0$$

for every $\xi \in (-\epsilon, 0)$ or $(0, \epsilon)$.

Therefore, by taking the limit as $\xi \rightarrow 0$, we obtain (22) with $\bar{\xi} = 0$. The solutions of the equation (23) are given by

$$\begin{aligned} \alpha(\xi) &= \frac{-(\bar{\mathbf{u}}^T \mathbf{D} \mathbf{v} + \mathbf{d}^T \mathbf{v} \xi) \pm \sqrt{f(\xi)}}{\mathbf{v}^T \mathbf{D} \mathbf{v}}, \text{ where} \\ f(\xi) &= (\bar{\mathbf{u}}^T \mathbf{D} \mathbf{v} + \mathbf{d}^T \mathbf{v} \xi)^2 - \mathbf{v}^T \mathbf{D} \mathbf{v} (2\mathbf{d}^T \bar{\mathbf{u}} \xi + \delta \xi) \xi. \end{aligned}$$

One of the four cases (a), (b), (c), and (d) below occurs.

- (a) $\mathbf{D} \bar{\mathbf{u}} \neq \mathbf{0}$: It follows from $\bar{\mathbf{u}}^T \mathbf{D} \bar{\mathbf{u}} = 0$ and $\mathbf{D} \bar{\mathbf{u}} \neq \mathbf{0}$ that $\mathbf{D} \in \mathbb{S}^{n-1}$ is neither positive semidefinite nor negative semidefinite. Hence, there exist $\mathbf{v}_+, \mathbf{v}_- \in \mathbb{R}^{n-1}$ such that $\mathbf{v}_+^T \mathbf{D} \mathbf{v}_+ = 1$ and $\mathbf{v}_-^T \mathbf{D} \mathbf{v}_- = -1$. Let $\mathbf{v} = \mathbf{v}_+$ if $(2\mathbf{d}^T \bar{\mathbf{u}} + \delta \xi) \leq 0$ for every sufficiently small $\xi > 0$, and $\mathbf{v} = \mathbf{v}_-$ otherwise. Then

$$\begin{aligned} f(\xi) &= (\bar{\mathbf{u}}^T \mathbf{D} \mathbf{v} + \mathbf{d}^T \mathbf{v} \xi)^2 - \mathbf{v}^T \mathbf{D} \mathbf{v} (2\mathbf{d}^T \bar{\mathbf{u}} \xi + \delta \xi) \xi \geq 0 \\ &\quad \text{for every sufficiently small } \xi > 0, \\ &\rightarrow (\bar{\mathbf{u}}^T \mathbf{D} \mathbf{v})^2 \text{ as } 0 < \xi \rightarrow 0, \\ \alpha(\xi) &= \frac{-(\bar{\mathbf{u}}^T \mathbf{D} \mathbf{v} + \mathbf{d}^T \mathbf{v} \xi) \pm \sqrt{f(\xi)}}{\mathbf{v}^T \mathbf{D} \mathbf{v}} \rightarrow \frac{-\bar{\mathbf{u}}^T \mathbf{D} \mathbf{v} \pm |\bar{\mathbf{u}}^T \mathbf{D} \mathbf{v}|}{\mathbf{v}^T \mathbf{D} \mathbf{v}} \text{ as } 0 < \xi \rightarrow 0. \end{aligned}$$

Therefore, at least one of the solutions of $\alpha(\xi)$ converges to zero.

- (b) $\mathbf{D} \bar{\mathbf{u}} = \mathbf{0}$, $\mathbf{d}^T \bar{\mathbf{u}} \neq 0$: In this case, we have that

$$\begin{aligned} f(\xi) &= (\mathbf{d}^T \mathbf{v})^2 \xi^2 - \mathbf{v}^T \mathbf{D} \mathbf{v} (2\mathbf{d}^T \bar{\mathbf{u}} \xi + \delta \xi) \xi \\ &= -2\mathbf{v}^T \mathbf{D} \mathbf{v} \mathbf{d}^T \bar{\mathbf{u}} \xi + ((\mathbf{d}^T \mathbf{v})^2 - \mathbf{v}^T \mathbf{D} \mathbf{v} \delta) \xi^2, \\ \alpha(\xi) &= \frac{-\mathbf{d}^T \mathbf{v} \xi \pm \sqrt{f(\xi)}}{\mathbf{v}^T \mathbf{D} \mathbf{v}}. \end{aligned}$$

Choose $\mathbf{v} \in \mathbb{R}^{n-1}$ such that $\mathbf{v}^T \mathbf{D} \mathbf{v} \neq 0$. Then, if $\mathbf{v}^T \mathbf{D} \mathbf{v} \mathbf{d}^T \bar{\mathbf{u}} < 0$, then $0 < f(\xi) \rightarrow 0$ as $0 < \xi \rightarrow 0$, otherwise $0 < f(\xi) \rightarrow 0$ as $0 > \xi \rightarrow 0$. In both case $\mathbb{R} \ni \alpha(\xi) \rightarrow 0$ as $\xi \rightarrow 0$.

- (c) $\mathbf{D} \bar{\mathbf{u}} = \mathbf{0}$, $\mathbf{d}^T \bar{\mathbf{u}} = 0$, $\delta = 0$: Choose $\mathbf{v} \in \mathbb{R}^{n-1}$ such that $\mathbf{v}^T \mathbf{D} \mathbf{v} \neq 0$. Then,

$$\alpha(\xi) = \frac{-\mathbf{d}^T \mathbf{v} \xi \pm \sqrt{f(\xi)}}{\mathbf{v}^T \mathbf{D} \mathbf{v}}, \quad f(\xi) = (\mathbf{d}^T \mathbf{v})^2 \xi^2.$$

Hence, $0 \leq f(\xi) \rightarrow 0$ and $\mathbb{R} \ni \alpha(\xi) \rightarrow 0$ as $0 \neq \xi \rightarrow 0$.

- (d) $D\bar{\mathbf{u}} = \mathbf{0}$, $d^T\bar{\mathbf{u}} = 0$, $\delta \neq 0$: Since $\mathbf{B} \notin \mathbb{S}_+^n$ and $\mathbb{J}_+(\mathbf{B}) \cap \mathbb{S}_{++}^n \neq \emptyset$, we can apply Lemma 5.2 for the existence of a $\mathbf{v} \in \mathbb{R}^{n-1}$ such that $-\delta\mathbf{v}^T(\mathbf{D} - d d^T/\delta)\mathbf{v} > 0$ and $\mathbf{v}^T \mathbf{D} \mathbf{v} \neq 0$. Then,

$$\begin{aligned} f(\xi) &= ((d^T \mathbf{v})^2 - \mathbf{v}^T \mathbf{D} \mathbf{v} \delta) \xi^2 = -\delta \mathbf{v}^T (\mathbf{D} - d d^T / \delta) \mathbf{v} \xi^2 > 0 \text{ for every } \xi \neq 0, \\ \alpha(\xi) &= \frac{-d^T \mathbf{v} \xi \pm \sqrt{f(\xi)}}{\mathbf{v}^T \mathbf{D} \mathbf{v}} \rightarrow 0 \text{ as } \xi \rightarrow 0. \end{aligned}$$

□

5.3 Equivalence of conditions (I), (II), (B), and (C)

Throughout this section, we allow $\mathcal{B} \subseteq \mathbb{S}^n$ to be infinite in conditions (I) and (II) as in condition (B) even though conditions (I) and (II) are originally stated for finite $\mathcal{B} \subseteq \mathbb{S}^n$. In addition to $\mathbb{F} = \mathbb{S}_+^n$ and condition (A-1) assumed in the previous section, we assume conditions

(A-2) $\mathbb{J}_+(\mathbf{B}) \not\subseteq \mathbb{J}_+(\mathbf{A})$ for every distinct $\mathbf{A}, \mathbf{B} \in \mathcal{B}$.

(A-3) $\mathbf{B}_{\leq} \neq \emptyset$ for every $\mathbf{B} \in \mathcal{B}$.

Under these conditions, we show the equivalence of conditions (I), (II), (B), and (C). In the case where \mathcal{B} is finite, if $\mathbb{J}_+(\mathbf{A}) \subseteq \mathbb{J}_+(\mathbf{B})$ for distinct $\mathbf{A}, \mathbf{B} \in \mathcal{B}$, we can remove $\mathbf{B} \in \mathcal{B}$ from \mathcal{B} one by one, recursively, so that $\mathbb{J}_+(\mathcal{B}) = \mathbb{J}_+(\mathcal{B} \setminus \{\mathbf{B}\})$. Therefore, condition (A-2) is attained. When \mathcal{B} is infinite, we need Zorn's lemma (see, for example, [16, 17]) to remove such \mathbf{B} 's consistently from \mathcal{B} so that the resulting \mathcal{B} satisfies condition (A-2). The details are omitted here. Condition (A-3) is also reasonable since if $\mathbf{B}_{\leq} = \emptyset$ then $\mathbf{B}_{\geq} = \mathbb{R}^{n-1}$ and $\mathcal{B}_{\geq} = \{\mathcal{B} \setminus \{\mathbf{B}\}\}_{\geq}$.

The equivalence of (I) and (B) is obvious under (A-2). We have already seen that (II) \Rightarrow (B) in Section 5.1 and that (C) \Rightarrow (B) in Theorem 1.6. To prove (B) \Rightarrow (II) and (B) \Rightarrow (C), it suffices to prove the following lemma, which also shows that (B) implies (10) included in (D).

Lemma 5.3. *Assume that $\mathcal{B} \subseteq \mathbb{S}^n$ satisfies conditions (A-1) and (A-2). Let $\mathbf{A}, \mathbf{B} \in \mathcal{B}$ and $\mathbf{A} \neq \mathbf{B}$. Then*

$$\alpha \mathbf{A} + \beta \mathbf{B} \in \mathbb{S}_+^n \text{ for some } (\alpha, \beta) \neq \mathbf{0}. \quad (24)$$

$$\begin{aligned} &\uparrow \\ \mathbb{J}_0(\mathbf{B}) \subseteq \mathbb{J}_+(\mathbf{A}) &\Leftrightarrow \mathbb{J}_-(\mathbf{B}) \subseteq \mathbb{J}_+(\mathbf{A}) \\ &\downarrow \qquad \qquad \downarrow \\ \mathbf{B}_{=} \subseteq \mathbf{A}_{\geq} &\qquad \mathbf{B}_{\leq} \subseteq \mathbf{A}_{\geq}. \end{aligned} \quad (25)$$

Proof. (i) In (25), \Leftarrow is straightforward since $\mathbb{J}_0(\mathbf{B}) \subseteq \mathbb{J}_-(\mathbf{B})$. To prove \Rightarrow , assume on the contrary that $\mathbb{J}_0(\mathbf{B}) \subseteq \mathbb{J}_+(\mathbf{A})$ but $\mathbb{J}_-(\mathbf{B}) \not\subseteq \mathbb{J}_+(\mathbf{A})$ or equivalently that $\langle \mathbf{B}, \bar{\mathbf{X}} \rangle < 0$ and $\langle \mathbf{A}, \bar{\mathbf{X}} \rangle < 0$ for some $\bar{\mathbf{X}} \in \mathbb{S}_+^n$. By condition (A-2), $\langle \mathbf{B}, \widetilde{\mathbf{X}} \rangle > 0$ and $\langle \mathbf{A}, \widetilde{\mathbf{X}} \rangle < 0$ for some $\widetilde{\mathbf{X}} \in \mathbb{S}_+^n$. Hence there exists $\lambda \in (0, 1)$ such that

$$\langle \mathbf{B}, \lambda \bar{\mathbf{X}} + (1 - \lambda) \widetilde{\mathbf{X}} \rangle = 0, \quad \langle \mathbf{A}, \lambda \bar{\mathbf{X}} + (1 - \lambda) \widetilde{\mathbf{X}} \rangle < 0, \quad \lambda \bar{\mathbf{X}} + (1 - \lambda) \widetilde{\mathbf{X}} \in \mathbb{S}_+^n.$$

This contradicts $\mathbb{J}_0(\mathbf{B}) \subseteq \mathbb{J}_+(\mathbf{A})$.

(ii) $\mathbb{J}_0(\mathbf{B}) \subseteq \mathbb{J}_+(\mathbf{A}) \Rightarrow (24)$: Consider the primal-dual pair of SDPs

$$\begin{aligned}\zeta_p &= \inf\{\langle \mathbf{A}, \mathbf{X} \rangle : \mathbf{X} \in \mathbb{S}_+^n, \langle \mathbf{B}, \mathbf{X} \rangle = 0\} = \inf\{\langle \mathbf{A}, \mathbf{X} \rangle : \mathbf{X} \in \mathbb{J}_0(\mathbf{B})\}, \\ \zeta_d &= \sup\{0 : \mathbf{A} + \tau \mathbf{B} \in \mathbb{S}_+^n, \tau \in \mathbb{R}\}.\end{aligned}\quad (26)$$

Obviously, $\zeta_p = 0$ if and only if $\mathbb{J}_0(\mathbf{B}) \subseteq \mathbb{J}_+(\mathbf{A})$. By (A-1), $-\mathbf{B} \notin \mathbb{S}_+^n$, which implies $\langle \mathbf{B}, \mathbf{X}^1 \rangle > 0$ for some $\mathbf{X}^1 \in \mathbb{S}_{++}^n$. By (A-2), $\mathbf{B} \notin \mathbb{S}_+^n$, which implies $\langle \mathbf{B}, \mathbf{X}^2 \rangle < 0$ for some $\mathbf{X}^2 \in \mathbb{S}_{++}^n$. Hence a convex combination $\bar{\mathbf{X}} \in \mathbb{S}_{++}^n$ of \mathbf{X}^1 and \mathbf{X}^2 satisfies $\langle \mathbf{B}, \bar{\mathbf{X}} \rangle = 0$, *i.e.*, $\bar{\mathbf{X}}$ is an interior feasible solution of primal SDP (26). By the duality theorem, $\zeta_p = \zeta_d = 0$ if and only if the dual SDP is feasible, *i.e.*, $\mathbf{A} + \tau \mathbf{B} \in \mathbb{S}_+^n$ for some $\tau \in \mathbb{R}$. Therefore $\mathbb{J}_0(\mathbf{B}) \subseteq \mathbb{J}_+(\mathbf{A}) \Rightarrow (24)$ has been shown.

(iii) $\mathbb{J}_-(\mathbf{B}) \subseteq \mathbb{J}_+(\mathbf{A}) \Rightarrow \mathbf{B}_\leq \subseteq \mathbf{A}_\geq$: We observe that

$$\begin{aligned}\mathbb{J}_-(\mathbf{B}) &\subseteq \mathbb{J}_+(\mathbf{A}) \\ &\Updownarrow \text{ (by Lemma 4.2 (ii))} \\ \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{B}, \mathbf{x}\mathbf{x}^T \rangle \leq 0\} &\subseteq \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{A}, \mathbf{x}\mathbf{x}^T \rangle \geq 0\} \\ &\Downarrow \\ \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{B}, \mathbf{x}\mathbf{x}^T \rangle \leq 0, x_n = 1\} &\subseteq \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{A}, \mathbf{x}\mathbf{x}^T \rangle \geq 0, x_n = 1\} \\ &\Updownarrow \\ \mathbf{B}_\leq &\subseteq \mathbf{A}_\geq.\end{aligned}$$

(iv) $\mathbb{J}_0(\mathbf{B}) \subseteq \mathbb{J}_+(\mathbf{A}) \Rightarrow \mathbf{B}_= \subseteq \mathbf{A}_\geq$: This assertion can be proved similarly as in (iii) by applying Lemma 5.1 instead of Lemma 4.2 (ii). \square

Remark 5.4. From (ii) of the proof above, we see that condition (II) can be regarded as the dual of condition (B) under assumptions (A-1) and (A-2).

6 Examples

In [2, Section 4.1], several examples satisfying condition (B) with finite \mathcal{B} and $\mathbb{F} = \mathbb{S}_+^n$ were provided. We present three examples that are not covered by those examples in this section.

Example 6.1. This example provides an infinite $\mathcal{B} \subseteq \mathbb{S}^n$ satisfying condition (C). Let

$$\mathbf{B}(\mathbf{t}) = \begin{pmatrix} \mathbf{I} & -\mathbf{t} \\ -\mathbf{t}^T & \mathbf{t}^T \mathbf{t} - r^2 \end{pmatrix} \in \mathbb{S}^n \ (\mathbf{t} \in T), \ \mathcal{B} = \{\mathbf{B}(\mathbf{t}) : \mathbf{t} \in T\},$$

where $0 < r \leq 1/2$, $T \subseteq \mathbb{Z}^n$ (the set of integer vectors in \mathbb{R}^n) and \mathbf{I} denotes the $(n-1) \times (n-1)$ identity matrix. Then,

$$\begin{aligned}q(\mathbf{u}, \mathbf{B}(\mathbf{t})) &= \begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix}^T \mathbf{B}(\mathbf{t}) \begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix} = \|\mathbf{u} - \mathbf{t}\|^2 - r^2, \\ \mathbf{B}(\mathbf{t})_\geq \text{ or } \mathbf{B}(\mathbf{t})_\leq &= \{\mathbf{u} \in \mathbb{R}^{n-1} : \|\mathbf{u} - \mathbf{t}\|^2 - r^2 \geq 0 \text{ or } \leq 0, \text{ respectively}\}\end{aligned}$$

for every $\mathbf{t} \in T$ and $\mathbf{u} \in \mathbb{R}^{n-1}$. See Figure 3. It is easily seen that condition (C) is satisfied. Therefore, by Theorem 1.6, we obtain $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\mathbb{I}^n)$. When T is finite, this example is a special case of quadratic programs with hollows [35].

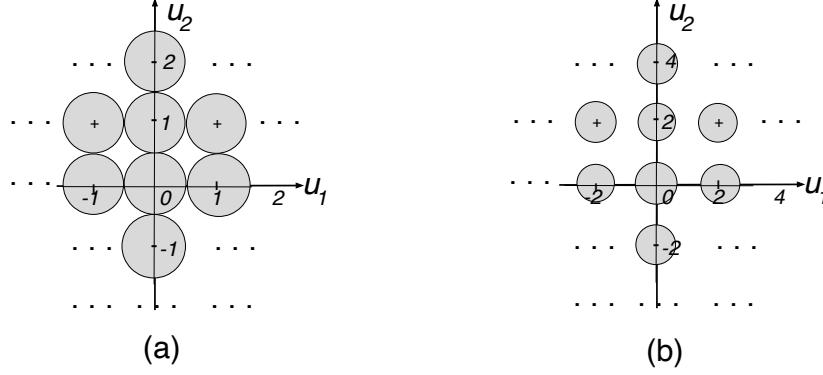


Figure 3: Illustration of $\mathbf{B}(\mathbf{t})_{\leq}$. Each gray disk region corresponds to $\mathbf{B}(\mathbf{t})_{\leq}$ for some $\mathbf{t} \in T$. (a) $T = \mathbb{Z}^{n-1}$ and $r = 1/2$. (b) $T = 2\mathbb{Z}^{n-1}$ and $r = 1/3$.

As a generalization, it is straightforward to construct an ellipsoid-based constraint by replacing each $\mathbf{B}(\mathbf{t})$ with $\mathbf{L}^T \mathbf{B}(\mathbf{t}) \mathbf{L}$ ($\mathbf{t} \in T$), where \mathbf{L} denotes an $n \times n$ nonsingular matrix of the form $\mathbf{L} = \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}$. We also note that the equivalence relation (1) between $\text{COP}(\mathbb{J} \cap \Gamma^n, \mathbf{Q}, \mathbf{H})$ and its SDP relaxation $\text{COP}(\mathbb{J}, \mathbf{Q}, \mathbf{H})$ with $\mathbb{J} = \mathbb{J}_+(\mathcal{B})$ (or $\mathbb{J} = \mathbb{J}_+(\{\mathbf{L}^T \mathbf{B} \mathbf{L} : \mathbf{B} \in \mathcal{B}\})$) holds for any choice of $\mathbf{Q} \in \mathbb{S}^n$ and $\mathbf{H} \in \mathbb{S}^n$ by Theorem 1.1. For example, we can take $\mathbf{H} = \delta \mathbf{I}$ for some $\delta > 0$ where \mathbf{I} is the $n \times n$ identity matrix. In this case, $\text{COP}(\mathbb{J} \cap \Gamma^n, \mathbf{Q}, \mathbf{H})$ turns out to be

$$\eta(\mathbb{J} \cap \Gamma^n, \mathbf{Q}, \mathbf{H}) = \inf \left\{ \begin{pmatrix} \mathbf{u} \\ \mathbf{z} \end{pmatrix}^T \mathbf{Q} \begin{pmatrix} \mathbf{u} \\ \mathbf{z} \end{pmatrix} : \begin{array}{l} \|\mathbf{u} - \mathbf{t}\mathbf{z}\|^2 - r^2 \mathbf{z}^2 \geq 0 \ (\mathbf{t} \in T), \\ \text{(or } \|\mathbf{M}\mathbf{u} - \mathbf{t}\mathbf{z}\|^2 - r^2 \mathbf{z}^2 \geq 0 \ (\mathbf{t} \in T)), \\ u_1^2 + \dots + u_{n-1}^2 + \mathbf{z}^2 = 1/\delta \end{array} \right\}.$$

Example 6.2. This example presents another infinite \mathcal{B} satisfying condition (C). For a finite or infinite sequence $\{(\gamma_p, \mu_p, \lambda_p, \sigma_p) \in \mathbb{R}^4 : p = 1, 2, \dots\}$ satisfying

$$\gamma_p > 0, \mu_p \geq 0, \lambda_p > 0 \text{ for every } p, \text{ and } |\sigma_q - \sigma_p| \geq \sqrt{\lambda_p} + \sqrt{\lambda_q} \text{ if } p < q, \quad (27)$$

we consider a sequence of quadratic functions

$$q_p(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \sigma_p \mathbf{u}\|^2 - \lambda_p \|\mathbf{u}\|^2 + \mu_p \|\mathbf{w}\|^2 + \gamma_p \quad \text{for every } \mathbf{u} \in \mathbb{R}^\ell, \mathbf{v} \in \mathbb{R}^\ell \text{ and } \mathbf{w} \in \mathbb{R}^m \quad (28)$$

($p = 1, 2, \dots$). We can take a matrix $\mathbf{B}^p \in \mathbb{S}^{2\ell+m+1}$ representing the quadratic function $q^\sigma(\mathbf{u}, \mathbf{v}, \mathbf{w})$ such that

$$q^\sigma(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \\ 1 \end{pmatrix}^T \mathbf{B}^p \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \\ 1 \end{pmatrix} \text{ for every } \mathbf{u} \in \mathbb{R}^\ell, \mathbf{v} \in \mathbb{R}^\ell \text{ and } \mathbf{w} \in \mathbb{R}^m$$

($p = 1, 2, \dots$).

We show that $\mathcal{B} = \{\mathbf{B}^p : p = 1, 2, \dots\}$ satisfies condition (C) with $\mathbb{F} = \mathbb{S}_+^{2\ell+m+1}$. Let $p < q$. Assume on the contrary that $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbf{B}_\leq^p \cap \mathbf{B}_\leq^q$ for some $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{R}^{2\ell+m}$.

Then

$$\begin{aligned}\| \mathbf{v} - \sigma_p \mathbf{u} \|^2 + \mu_p \| \mathbf{w} \|^2 + \gamma_p &\leq \lambda_p \| \mathbf{u} \|^2, \\ \| \mathbf{v} - \sigma_q \mathbf{u} \|^2 + \mu_q \| \mathbf{w} \|^2 + \gamma_q &\leq \lambda_q \| \mathbf{u} \|^2.\end{aligned}$$

Hence,

$$0 < \| \mathbf{u} \|, \quad \| \mathbf{v} - \sigma_p \mathbf{u} \| < \sqrt{\lambda_p} \| \mathbf{u} \|, \quad \| \mathbf{v} - \sigma_q \mathbf{u} \| < \sqrt{\lambda_q} \| \mathbf{u} \|.$$

Therefore,

$$\begin{aligned}|\sigma_q - \sigma_p| \| \mathbf{u} \| &= \| (\sigma_q \mathbf{u} - \mathbf{v}) + (\mathbf{v} - \sigma_p \mathbf{u}) \| \\ &\leq \| \sigma_q \mathbf{u} - \mathbf{v} \| + \| \mathbf{v} - \sigma_p \mathbf{u} \| \\ &< (\sqrt{\lambda_p} + \sqrt{\lambda_q}) \| \mathbf{u} \|,\end{aligned}$$

which implies $\sigma_q - \sigma_p < \sqrt{\lambda_p} + \sqrt{\lambda_q}$, a contradiction to the last inequality of (27). Thus we have shown $\mathbf{B}_{\leq}^p \cap \mathbf{B}_{\leq}^q = \emptyset$, which implies $\mathbf{B}_{\leq}^p \subseteq \mathbf{B}_{\geq}^q$ and $\mathbf{B}_{\leq}^q \subseteq \mathbf{B}_{\geq}^p$.

If we take $\gamma_p = \lambda_p = \mu_p = 1$ and $\sigma_p = 2p$ ($p = 1, 2, \dots$), then $\{(\gamma_p, \mu_p, \lambda_p, \sigma_p) : p = 1, 2, \dots\}$ satisfies (27). As another case, we consider

$$\gamma_p > 0, \quad \lambda_p = (a_p - a_{p-1})^2/4, \quad \mu_p = 0, \quad \sigma_p = -(a_{p-1} + a_p)/2 \quad (p = 1, 2, \dots), \quad (29)$$

where $\{a_p : p = 0, 1, \dots\}$ denotes an infinite sequence of positive numbers such that $a_{p-1} < a_p$. In this case, if $p < q$ then

$$\begin{aligned}|\sigma_q - \sigma_p| - (\sqrt{\lambda_q} + \sqrt{\lambda_p}) &= \frac{(a_q + a_{q-1}) - (a_p + a_{p-1})}{2} - \frac{(a_q - a_{q-1}) + (a_p - a_{p-1})}{2} \\ &= a_{q-1} - a_p \geq 0.\end{aligned}$$

Therefore, the sequence $\{(\gamma_p, \lambda_p, \mu_p, \sigma_p) : p = 1, 2, \dots\}$ defined by (29) satisfies (27). Now suppose that $\ell = 1$ and $m = 0$. Then, the quadratic functions (28) turns out to be

$$\begin{aligned}q_p(u, v) &= (v - \sigma_p u)^2 - \lambda_p u^2 + \gamma_p \\ &= (v - (a_{p-1} + a_p)u/2)^2 - (a_p - a_{p-1})^2 u^2/4 + \gamma_p \\ &= a_{p-1} a_p u^2 - (a_{p-1} + a_p)uv + v^2 + \gamma_p \\ &= \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}^T \mathbf{B}^p \begin{pmatrix} u \\ v \\ 1 \end{pmatrix},\end{aligned}$$

where

$$\mathbf{B}^p = \begin{pmatrix} a_{p-1} a_p & -(a_{p-1} + a_p)/2 & 0 \\ -(a_{p-1} + a_p)/2 & 1 & 0 \\ 0 & 0 & \gamma_p \end{pmatrix}$$

$\mathcal{B} = \{\mathbf{B}^p : p = 1, 2, \dots\}$ satisfies condition (C) with $\mathbb{F} = \mathbb{S}_+^3$. See Figure 4.

Now assume that $a_p \rightarrow \bar{a}$ as $p \rightarrow \infty$. Then $\mathbf{B}^p \rightarrow \bar{\mathbf{B}} = \begin{pmatrix} \bar{a}^2 & -\bar{a} & 0 \\ -\bar{a} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \notin \mathcal{B}$ as $p \rightarrow \infty$. Therefore, \mathcal{B} is not closed. We also see $\bar{\mathbf{B}} \in \mathbb{S}_+^3$. Hence, $\mathbb{J}_+(\mathbf{B}^p) \subseteq \mathbb{S}_+^3 = \mathbb{J}_+(\bar{\mathbf{B}})$ ($p = 1, 2, \dots$). Therefore, $\text{cl}\mathcal{B}$ (the closure of \mathcal{B}) does not satisfy condition (A-2) although $\mathbb{J}_+(\text{cl}\mathcal{B}) \in \widehat{\mathcal{F}}(\mathbf{T}^3)$ remains true.

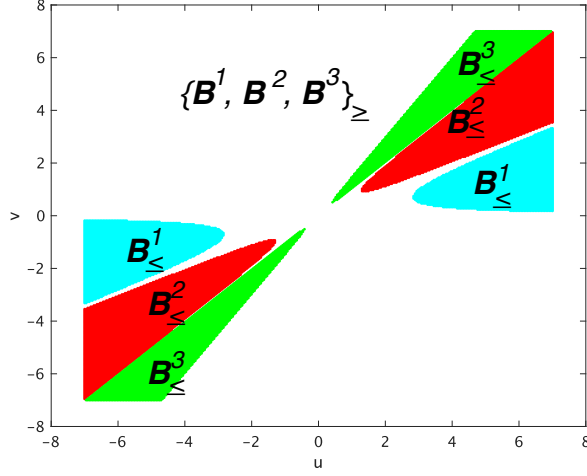


Figure 4: Example 6.2. We take $a_0 = 0$, $a_1 = 0.5$, $a_2 = 1$, $a_3 = 1.5$, $\gamma_1 = 0.5$, $\gamma_2 = 0.1$, $\gamma_2 = 0.01$.

Example 6.3. We consider a parabola-based constraint. Let $n \geq 3$ and

$$B_{ij} = \begin{cases} \lambda_i > 0 & \text{if } 2 \leq i = j \leq n, \\ -0.5 & \text{if } (i, j) = (1, n) \text{ or } (i, j) = (n, 1), \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} q(\mathbf{u}, \mathbf{B}) &= -u_1 + \sum_{i=2}^{n-1} \lambda_i u_i^2 + \lambda_n \text{ for every } \mathbf{u} \in \mathbb{R}^{n-1}, \\ \mathbf{B}_{\geq}, \mathbf{B}_{=} \text{ or } \mathbf{B}_{\leq} &= \left\{ \mathbf{u} \in \mathbb{R}^{n-1} : -u_1 + \sum_{i=2}^{n-1} \lambda_i u_i^2 + \lambda_n \geq 0, = 0 \text{ or } \leq 0, \text{ respectively} \right\} (30) \\ \mathbf{B}_{\leq} &\subseteq \{ \mathbf{u} \in \mathbb{R}^n : \lambda_i u_i^2 + \lambda_n \leq u_1 \ (i = 2, \dots, n-1) \} \\ &\subseteq \mathbb{K}_-(\mathbf{B}) \equiv \{ \mathbf{u} \in \mathbb{R}^{n-1} : 0 \leq u_1, -u_1 \leq 2\sqrt{\lambda_i \lambda_n} u_i \leq u_1 \ (i = 2, \dots, n-1) \}. \end{aligned}$$

See Figure 5 (a). We note that $\mathbb{K}_-(\mathbf{B})$ forms a polyhedral cone in \mathbb{R}^{n-1} , which converges to the half line $\{ \mathbf{u} \in \mathbb{R}^{n-1} : u_1 \geq 0, u_i = 0 \ (i = 2, \dots, n-1) \}$ as all $\lambda_i \ (i = 2, \dots, n-1)$ tend to ∞ . We see that each $\mathbf{B}_{=} \cap \{ \mathbf{u} \in \mathbb{R}^{n-1} : u_j = 0 \ (2 \leq j \neq i \leq n-1) \}$ forms a 2-dimensional parabola $(i = 2, \dots, n-1)$. By applying a linear transformation $\mathbf{B} \rightarrow \mathbf{L}^T \mathbf{B} \mathbf{L} \in \mathbb{S}^n$ with a nonsingular \mathbf{L} to $\mathbf{B}_{=}$ with different $\lambda_i > 0 \ (i = 2, \dots, n)$, we can create various parabolas. Furthermore, we can arrange some of those parabolas such that the associated \mathbf{B} satisfies condition (C). See Figure 5 (b) and (c).

7 Concluding remarks

We have presented two types of sufficient conditions under which a semi-infinite QCQP is equivalent to its SDP relaxation. The first type, condition (B), extends the assumptions in

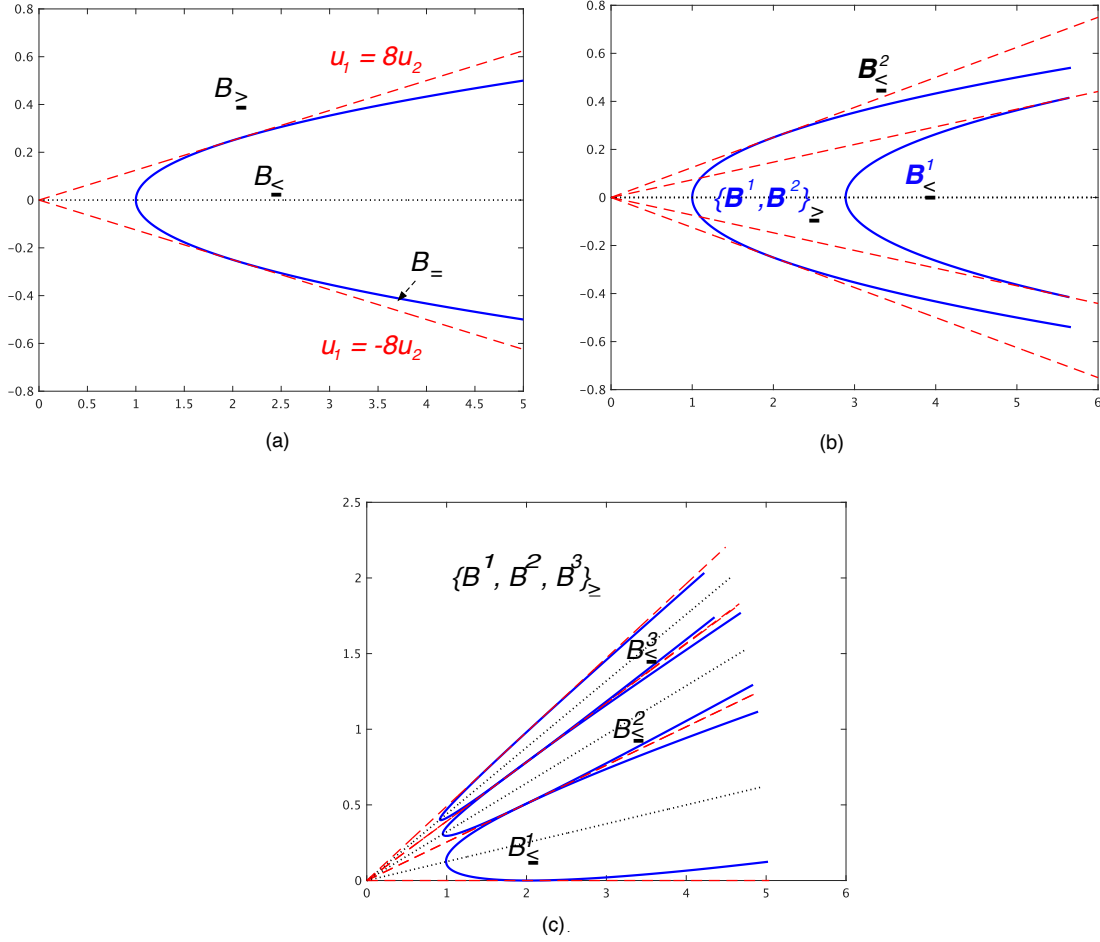


Figure 5: Parabola-based constraints with $n = 3$. (a) Parabola B_{\leq} defined by (30) where $\lambda_2 = 16$ and $\lambda_3 = 1$. (b) $B_{\geq}^1 = \{u \in \mathbb{R}^2 : -u_1 + 16u_2^2 + 3 \geq 0\}$, $B_{\geq}^2 = \{u \in \mathbb{R}^2 : -(-u_1 + 16u_2^2 + 1) \geq 0\}$ and $\{B^1, B^2\}_{\geq} = B_{\geq}^1 \cap B_{\geq}^2$.

[2, Theorem 4.1] and [1, Proposition 1] for QCQPs with finitely many inequality constraints to the case of infinitely many inequality constraints. Even for QCQPs with finitely many constraints, condition (B) is weaker than the corresponding conditions in those works. The second type, condition (C), is a new variant of NIQCC. We have shown that (C) implies (B) in general, and that the two are equivalent under appropriate additional assumptions. Furthermore, we have generalized condition (III) (the NIQCC from [19, Corollary 2]) to condition (D).

In addition to the equivalence results for semi-infinite QCQPs and their SDP relaxations, which have been studied here based on ROG cones and NIQCCs, there exist several other well-studied classes of QCQPs whose SDP relaxations are exact. Notable examples include convex QCQPs, where both the objective and all quadratic constraints are convex, and QCQPs defined by specific sign pattern conditions on the objective and constraint matrices [3, 4, 21, 32]. These QCQPs have a fundamentally different nature, and most existing studies on them are largely independent of the ROG and NIQCC properties investigated in this paper. In our recent paper [23], we have discussed how homogenized and non-homogenized

NIQCC properties can be incorporated into such QCQPs.

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