Exact SDP relaxations for a class of quadratic programs with finite and infinite quadratic constraints

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Abstract

We investigate exact semidefinite programming (SDP) relaxations for the problem of minimizing a nonconvex quadratic objective function over a feasible region defined by both finitely and infinitely many nonconvex quadratic inequality constraints (semi-infinite QCQPs). Sufficient conditions for the exactness of SDP relaxations for QCQPs with finitely many constraints have been extensively studied, notably by Argue et al. (MOR, 48:100-126, 2023), Arima et al. (SIOPT, 34:3194-3211, 2024), and Joyce and Yang (MP, 205:539-558, 2024). In this work, we present three new sufficient conditions that generalize the existing conditions in these works for both finite and semi-infinite QCQPs. Specifically, we establish relationships among the proposed and existing conditions, and prove that one of the proposed conditions is the weakest among them, since it is implied by all the others. Illustrative examples are also provided to demonstrate the effectiveness of the proposed conditions in comparison to the existing ones.

Key words. Finite and semi-infinite quadratically constrained quadratic program, exact semidefinite programming relaxations, rank-one generated cones, non-intersecting quadratic constraints, ball-, parabola- and hyperbola-based constraints.

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1 Introduction

We begin by considering a general conic optimization problem (COP). Let $\mathbb{V}$ be a finite-dimensional vector space equipped with an inner product $\langle \boldsymbol{A}, \boldsymbol{B} \rangle$ for every $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{V}$. For every closed cone $\mathbb{C} \subseteq \mathbb{V}$, $\boldsymbol{Q} \in \mathbb{V}$ and $\boldsymbol{H} \in \mathbb{V}$, COP$(\mathbb{C}, \boldsymbol{Q}, \boldsymbol{H})$ denotes the problem of minimizing $\langle \boldsymbol{Q}, \boldsymbol{X} \rangle$ subject to $\boldsymbol{X} \in \mathbb{C}$ and $\langle \boldsymbol{H}, \boldsymbol{X} \rangle = 1$, *i.e.*,

$$\eta(\mathbb{C}, \boldsymbol{Q}, \boldsymbol{H}) \;=\; \inf \{ \langle \boldsymbol{Q}, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{C}, \; \langle \boldsymbol{H}, \boldsymbol{X} \rangle = 1 \},$$

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where $\mathbb{C} \subseteq \mathbb{V}$ is a cone if $\lambda \boldsymbol{X} \in \mathbb{C}$ holds for every $\boldsymbol{X} \in \mathbb{C}$ and $\lambda \geq 0$. We note that a cone $\mathbb{C}$ is not necessarily convex. When $\boldsymbol{Q}, \boldsymbol{H} \in \mathbb{V}$ are unspecified and arbitrary, we often denote COP$(\mathbb{C}, \boldsymbol{Q}, \boldsymbol{H})$ and $\eta(\mathbb{C}, \boldsymbol{Q}, \boldsymbol{H})$ as COP$(\mathbb{C})$ and $\eta(\mathbb{C})$, respectively. If COP$(\mathbb{C})$ is infeasible, we assume that $\eta(\mathbb{C}) = +\infty$.

Let $\mathbb{K}$ be a closed nonconvex cone in $\mathbb{V}$. Let co$\mathbb{K}$ and $\overline{\text{co}}\mathbb{K}$ denote the convex hull of $\mathbb{K}$ and its closure, respectively. For every closed convex cone $\mathbb{J} \subseteq \text{co}\mathbb{K}$, $\boldsymbol{Q} \in \mathbb{V}$ and $\boldsymbol{H} \in \mathbb{V}$, we consider a nonconvex conic optimization problem COP$(\mathbb{K} \cap \mathbb{J})$:

$$\eta(\mathbb{K} \cap \mathbb{J}) \;\; = \;\; \inf \left\{ \langle \boldsymbol{Q}, \, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{K} \cap \mathbb{J}, \; \langle \boldsymbol{H}, \, \boldsymbol{X} \rangle = 1 \right\}$$

and its convex relaxation COP$(\mathbb{J})$:

$$\eta(\mathbb{J}) \;\; = \;\; \inf \left\{ \langle \boldsymbol{Q}, \, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{J}, \; \langle \boldsymbol{H}, \, \boldsymbol{X} \rangle = 1 \right\}.$$

Obviously, $\eta(\mathbb{J}) \leq \eta(\mathbb{K} \cap \mathbb{J})$ holds. If $\eta(\mathbb{J}) = \eta(\mathbb{K} \cap \mathbb{J})$, we say that COP$(\mathbb{K} \cap \mathbb{J})$ and its convex relaxation COP$(\mathbb{J})$ are *equivalent*, or that COP$(\mathbb{J})$ is *an exact convex relaxation of* COP$(\mathbb{K} \cap \mathbb{J})$.

The above framework of a nonconvex conic optimization problem COP$(\mathbb{K} \cap \mathbb{J})$ and its convex relaxation COP$(\mathbb{J})$ was originally introduced in [22] for a unified geometrical analysis on the completely positive programming (CPP) reformulation of quadratically constrained quadratic problems (QCQPs) and their extension to polynomial optimization problems (POPs). They introduced

$$\widehat{\mathcal{F}}(\mathbb{K}) \;\; = \;\; \text{the family of closed convex cones } \mathbb{J} \subseteq \text{co}\mathbb{K} \text{ such that } \text{co}(\mathbb{K} \cap \mathbb{J}) = \mathbb{J},$$

and established the following results.

Theorem 1.1. *Let* $\boldsymbol{Q}, \; \boldsymbol{H} \in \mathbb{V}$*, and* $\mathbb{K} \subseteq \mathbb{V}$ *be a closed cone.*

(i) Assume that $\mathbb{J} \in \widehat{\mathcal{F}}(\mathbb{K})$*. Then*

$$-\infty < \eta(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}) \text{ if and only if } -\infty < \eta(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}) = \eta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}). \quad (1)$$

(ii) If $\mathbb{J}$ *is a face of* co$\mathbb{K}$*, then* $\mathbb{J} \in \widehat{\mathcal{F}}(\mathbb{K})$*.*

(iii) $\mathbb{J}' \in \widehat{\mathcal{F}}(\mathbb{K})$ *for every face* $\mathbb{J}'$ *of* $\mathbb{J} \in \widehat{\mathcal{F}}(\mathbb{K})$*.*

(iv) Assume that $\boldsymbol{H} \in$ *int* co$\mathbb{K}$ *(the interior of* co$\mathbb{K}$*). Then* $\mathbb{J} \in \widehat{\mathcal{F}}(\mathbb{K})$ *if and only if* $\eta(\mathbb{J} \cap \mathbb{K}, \boldsymbol{Q}, \boldsymbol{H}) = \eta(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H})$ *for every* $\boldsymbol{Q} \in \mathbb{S}^n$*.*

Proof. For assertion (i), we refer to [22, Theorem 3.1] (see also [2, Corollary 2.2]); for (ii) and (iii), to [22, Lemma 2.1]; for (iv) to [2, Theorem 1.2].

In particular, Kim et al. [22] applied Theorem 1.1 (i) and (ii) to equivalent CPP relaxation of a class of QCQPs in binary and nonnegative variables, which includes Burer's class of QCQPs [10], and its extension to POPs.

1.1 A quadratically constrained quadratic program (QCQP) and its equivalent semidefinite programming (SDP) relaxation

In this paper, we focus on the case $\mathbb{V} = \mathbb{S}^n$, the linear space of $n \times n$ symmetric matrices, and $\mathbb{K} = \mathbf{\Gamma}^n \equiv \{\boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} \in \mathbb{R}^n\}$, where $\mathbb{R}^n$ is the n-dimensional Euclidean space of column vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{x}^T$ denotes the row vector obtained by transposing $\boldsymbol{x} \in \mathbb{R}^n$. In this case,

- $\text{co}\mathbb{K} = \text{co}\mathbf{\Gamma}^n = \mathbb{S}^n_+$ (the convex cone of $n \times n$ positive semidefinite matrices),
- $\text{COP}(\mathbf{\Gamma}^n \cap \mathbb{J})$ and its convex relaxation $\text{COP}(\mathbb{J})$ correspond to a (geometric form of) QCQP and its semidefinite programming (SDP) relaxation,
- each closed convex cone $\mathbb{J} \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$ is characterized as $\mathbb{J} = \text{co}\left\{\boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{x}\boldsymbol{x}^T \in \mathbb{J}\right\}$, and is called *rank-one-generated (ROG)* in the literature [1, 6, 15].

Argue et al. [1] demonstrated independently from [22], that the ROG property is crucial for ensuring the equivalence relation (1) in case $\mathbb{K} = \mathbf{\Gamma}^n$ [1, Lemma 19]. They provided a thorough study of characterizations of ROG cones and established several necessary and sufficient conditions. In particular, condition (II) below characterizes when a closed convex cone $\mathbb{J} \subseteq \mathbb{S}^n_+$, defined by linear matrix inequalities, has the ROG property. In [2], Arima et al. also presented a sufficient condition, condition (I) below for $\mathbb{J} \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$.

A closed convex cone $\mathbb{J} \subseteq \mathbb{S}^n_+$ is often represented using linear matrix inequalities. For every $\boldsymbol{B} \in \mathbb{S}^n$, let

$$\mathbb{J}_+(\boldsymbol{B}),\ \mathbb{J}_0(\boldsymbol{B}) \text{ or } \mathbb{J}_-(\boldsymbol{B}) = \left\{\boldsymbol{X} \in \mathbb{S}^n_+ : \langle \boldsymbol{B},\, \boldsymbol{X}\rangle \geq,\ = \text{or}\ \leq 0,\ \text{respectively}\right\},$$

and $\mathbb{J}_+(\mathcal{B}) = \left\{\boldsymbol{X} \in \mathbb{S}^n_+ : \langle \boldsymbol{B},\, \boldsymbol{X}\rangle \geq 0\ (\boldsymbol{B} \in \mathcal{B})\right\}$ for every $\mathcal{B} \subseteq \mathbb{S}^n$. Since $\mathbb{J} \subseteq \mathbb{S}^n_+$ is a closed convex cone, $\mathbb{J}$ can be represented as the intersection of (possibly infinitely many) half spaces and $\mathbb{S}^n_+$ such that $\mathbb{J} = \mathbb{J}_+(\mathcal{B})$ for some $\mathcal{B} \subseteq \mathbb{S}^n$. We should mention that there are many choices for such a $\mathcal{B}$. For example, we can take $\mathcal{B} = \mathbb{J}^*$, where $\mathbb{J}^* = \{\boldsymbol{Y} \in \mathbb{S}^n : \langle \boldsymbol{X},\, \boldsymbol{Y}\rangle \geq 0 \text{ for every } \boldsymbol{X} \in \mathbb{J}\}$ (the dual of $\mathbb{J}$). This trivial choice of $\mathcal{B}$, however, involves many redundant matrices to represent $\mathbb{J}$.

Now we represent a QCQP and its SDP relaxation as $\text{COP}(\mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}))$

$$\eta(\mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})) = \inf\left\{\langle \boldsymbol{Q},\, \boldsymbol{x}\boldsymbol{x}^T\rangle : \begin{array}{l} \boldsymbol{x} \in \mathbb{R}^n,\ \langle \boldsymbol{B},\, \boldsymbol{x}\boldsymbol{x}^T\rangle \geq 0\ (\boldsymbol{B} \in \mathcal{B}), \\ \langle \boldsymbol{H},\, \boldsymbol{x}\boldsymbol{x}^T\rangle = 1 \end{array}\right\} \tag{2}$$

$$= \inf\left\{\langle \boldsymbol{Q},\, \boldsymbol{X}\rangle : \begin{array}{l} \boldsymbol{X} \in \mathbb{S}^n, \boldsymbol{X} \in \mathbf{\Gamma}^n, \\ \boldsymbol{X} \in \mathbb{J}_+(\boldsymbol{B})\ (\boldsymbol{B} \in \mathcal{B}),\ \langle \boldsymbol{H},\, \boldsymbol{X}\rangle = 1 \end{array}\right\}, \tag{3}$$

and $\text{COP}(\mathbb{J}_+(\mathcal{B}))$

$$\eta(\mathbb{J}_+(\mathcal{B})) = \inf\left\{\langle \boldsymbol{Q},\, \boldsymbol{X}\rangle : \begin{array}{l} \boldsymbol{X} \in \mathbb{S}^n_+, \\ \boldsymbol{X} \in \mathbb{J}_+(\boldsymbol{B})\ (\boldsymbol{B} \in \mathcal{B}),\ \langle \boldsymbol{H},\, \boldsymbol{X}\rangle = 1 \end{array}\right\}, \tag{4}$$

respectively. We note that $\boldsymbol{x} \in \mathbb{R}^n$ is the variable in (2) while $\boldsymbol{X} \in \mathbb{S}^n$ is the variable in (3). The process of embedding QCQP (2), defined in $\boldsymbol{x} \in \mathbb{R}^n$, into QCQP (3) is often called *a lifting into the matrix space* $\mathbb{S}^n$. The following results are known:

Theorem 1.2. *Let $\mathcal{B} \subseteq \mathbb{S}^n$. If one of the following conditions (I) and (II) holds, then $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$. In particular, $\mathbb{J}_+(\boldsymbol{B}) \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$, $\mathbb{J}_-(\boldsymbol{B}) = \mathbb{J}_+(-\boldsymbol{B}) \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$ and $\mathbb{J}_0(\boldsymbol{B}) = \mathbb{J}_+(\{\boldsymbol{B}, -\boldsymbol{B}\}) \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$ for every $\boldsymbol{B} \in \mathbb{S}^n$.*

(I) $\mathcal{B}$ is finite. $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$ for every $\boldsymbol{A}, \ \boldsymbol{B} \in \mathcal{B}$. [2, Theorem 4.1].

(II) $\mathcal{B}$ is finite. For every distinct $\boldsymbol{A}, \ \boldsymbol{B} \in \mathcal{B}$, there exists a nonzero $(\alpha, \beta) \in \mathbb{R}^2$ such that $\alpha\boldsymbol{A} + \beta\boldsymbol{B} \in \mathbb{S}^n_+$ [1, Proposition 1].

For QCQP examples that satisfy condition (I) in [18, 28, 34, 39], we refer the reader to [2]. For fundamental properties of ROG cones and their applications to equivalent SDP relaxations of QCQPs, see [1, 20].

1.2 Non-intersecting quadratic constraint conditions

Conditions (I) and (II) are defined in $\mathbb{S}^n$, the space of the variable matrix $\boldsymbol{X}$ of COP($\mathbb{J}_+(\mathcal{B})$). For practical applications, however, it is more convenient to provide a direct characterization of QCQP (2), which involves $\mathcal{B}$ satisfying condition (B), in the space $\mathbb{R}^n$ of its variable vector $\boldsymbol{x}$. We examine the special case where $\boldsymbol{H} = \boldsymbol{H}^1 \equiv \mathrm{diag}(0, \ldots, 0, 1) \in \mathbb{S}^n_+$ (the $n \times n$ diagonal matrix with diagonal elements $0, \ldots, 0, 1$). Letting

$$\mathbb{H}^1 = \{\boldsymbol{X} \in \mathbb{S}^n : \langle \boldsymbol{H}^1, \ \boldsymbol{X} \rangle = 1\} = \{\boldsymbol{X} \in \mathbb{S}^n : X_{nn} = 1\},$$

we can rewrite COP($\mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1$) and COP($\mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1$) as

$$\begin{aligned} \eta(\mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1) &= \inf\left\{\langle \boldsymbol{Q}, \ \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}) \cap \mathbb{H}^1\right\} \\ &= \inf\left\{\langle \boldsymbol{Q}, \ \boldsymbol{X} \rangle : \boldsymbol{X} \in \overline{\mathrm{co}}(\mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}) \cap \mathbb{H}^1)\right\} \\ &\quad \text{(since the objective function } \langle \boldsymbol{Q}, \ \boldsymbol{X} \rangle \text{ is linear in } \boldsymbol{X} \in \mathbb{S}^n\text{)} \end{aligned}$$

and

$$\eta(\mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1) = \inf\left\{\langle \boldsymbol{Q}, \ \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{J}_+(\mathcal{B}) \cap \mathbb{H}^1\right\},$$

respectively. Therefore, $\overline{\mathrm{co}}(\mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}) \cap \mathbb{H}^1) = \mathbb{J}_+(\mathcal{B}) \cap \mathbb{H}^1$ serves as a sufficient condition for the equivalence of COP($\mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1$) and COP($\mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1$), which will be utilized in Theorem 1.3 below.

For the subsequent discussion, COP($\mathbf{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1$) will be expressed in an alternative form. Define

$$q(\boldsymbol{u}, \boldsymbol{B}) = \left\langle \boldsymbol{B}, \ \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix}\begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix}^T \right\rangle \text{ for every } (\boldsymbol{u}, \boldsymbol{B}) \in \mathbb{R}^{n-1} \times \mathbb{S}^n.$$

If $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{c}^T \\ \boldsymbol{c} & \gamma \end{pmatrix}$, where $\boldsymbol{C} \in \mathbb{S}^{n-1}$, $\boldsymbol{c} \in \mathbb{R}^{n-1}$ and $\gamma \in \mathbb{R}$, then $q(\boldsymbol{u}, \boldsymbol{Q})$ is a quadratic function of the form $\boldsymbol{u}^T\boldsymbol{C}\boldsymbol{u} + 2\boldsymbol{c}^T\boldsymbol{u} + \gamma$ in $\boldsymbol{u} \in \mathbb{R}^{n-1}$. For every $\boldsymbol{B} \in \mathbb{S}^n$ and $\mathcal{B} \subseteq \mathbb{S}^n$, define

$$\begin{aligned} &\boldsymbol{B}_\geq, \ \boldsymbol{B}_= \text{ or } \boldsymbol{B}_\leq = \left\{\boldsymbol{u} \in \mathbb{R}^{n-1} : q(\boldsymbol{u}, \boldsymbol{B}) \geq 0, \ = \ \text{or} \ \leq 0, \ \text{respectively}\right\}, \\ &\mathcal{B}_\geq = \bigcap_{\boldsymbol{B} \in \mathcal{B}} \boldsymbol{B}_\geq = \left\{\boldsymbol{u} \in \mathbb{R}^{n-1} : q(\boldsymbol{u}, \boldsymbol{B}) \geq 0 \ (\boldsymbol{B} \in \mathcal{B})\right\}. \end{aligned}$$

Now, we rewrite QCQP (2) with $\boldsymbol{H} = \boldsymbol{H}^1 = \mathrm{diag}(0,\ldots,0,1) \in \mathbb{S}^n_+$ as

$$\begin{aligned}\eta(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1) &= \inf\left\{q(\boldsymbol{u},\boldsymbol{Q}) : \boldsymbol{u} \in \mathbb{R}^{n-1},\ q(\boldsymbol{u},\boldsymbol{B}) \geq 0\ (\boldsymbol{B} \in \mathcal{B})\right\} \\ &= \inf\{q(\boldsymbol{u},\boldsymbol{Q}) : \boldsymbol{u} \in \mathcal{B}_{\geq}\}. \end{aligned} \tag{5}$$

The following result follows from [19, Corollary 3], specialized to $\mathrm{COP}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1)$ and $\mathrm{COP}(\mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1)$.

Theorem 1.3. *Let $\mathcal{B} \subseteq \mathbb{S}^n$. Assume that condition (III) holds. Then*

$$\mathbb{J}_+(\mathcal{B}) \cap \mathbb{H}^1 = \overline{co}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}) \cap \mathbb{H}^1), \tag{6}$$

$$\eta(\mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1) = \eta(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1). \tag{7}$$

(Moreover, (6) *immediately implies* (7)*.)*

(III) $\mathcal{B}$ is finite. $q(\cdot,\boldsymbol{B}) : \mathbb{R}^{n-1} \to \mathbb{R}$ is not affine (*i.e.*, if $q(\cdot,\boldsymbol{B})$ is described as $q(\boldsymbol{u},\boldsymbol{B}) = \boldsymbol{u}^T\boldsymbol{D}\boldsymbol{u} + 2\boldsymbol{d}^T\boldsymbol{u} + \delta$ for every $\boldsymbol{u} \in \mathbb{R}^{n-1}$, then $\boldsymbol{D} \neq \boldsymbol{O}$) ($\boldsymbol{B} \in \mathcal{B}$) and

$$\boldsymbol{B}_{=} \subseteq \mathcal{B}_{\geq} \quad \text{for every } \boldsymbol{B} \in \mathcal{B}. \tag{8}$$

The condition (8) is commonly referred to as *the non-intersecting quadratic constraint condition* (NIQCC) [11, 19, 38]. Early works examined special instances of NIQCC for relatively simple QCQPs, most notably those arising from the generalized trust region subproblem (TRS) [5, 11, 13, 18, 29]. Extending the generalized TRS framework, the authors in [28, 39] studied QCQPs of the form $\inf\{q_0(\boldsymbol{u}) : -1 \leq q_1(\boldsymbol{u}) \leq 1\}$, where q_0, q_1 are quadratic functions in $\boldsymbol{u} \in \mathbb{R}^{n-1}$. This problem can be reformulated as a QCQP satisfying NIQCC. Similarly, a quadratic program with non-intersecting ellipsoidal hollows [38] provides another extension of the generalized TRS. Thus, condition (8) may be viewed as a unified formulation of NIQCC-type assumptions that arise in these classes of QCQPs. In comparison with condition (I), (8) represents a *non-homogenized form of NIQCC, whereas condition (I) corresponds to* a *homogenized form of NIQCC*.

1.3 Summary of main results

The main purpose of the paper is to propose new conditions that generalize previously mentioned conditions (I), (II) and (III) and to investigate their relationships in detail. To describe the new conditions, we introduce a face $\mathbb{F}$ of $\mathbb{S}^n_+$ that contains $\mathbb{J}_+(\mathcal{B})$. Clearly, $\mathbb{J}_+(\mathcal{B}) = \mathbb{J}_+(\mathcal{B}) \cap \mathbb{F}$. To ensure that the conditions are nontrivial and meaningful, we implicitly assume that $\mathbb{F}$ is a proper face of $\mathbb{S}^n_+$. If a linear equality $\boldsymbol{G}\binom{\boldsymbol{u}}{1} = 0$ is given for some $r \times n$ full row-rank matrix $\boldsymbol{G}$, it can be incorporated into QCQP (5) as a quadratic constraint $q(\boldsymbol{u},\boldsymbol{B}) \geq 0$, where $\boldsymbol{B} = -\boldsymbol{G}^T\boldsymbol{G}$. In this case, $\mathbb{J}_+(\mathcal{B})$ is contained in the face $\mathbb{F} = \mathbb{J}_+(-\boldsymbol{G}^T\boldsymbol{G}) = \mathbb{J}_0(\boldsymbol{G}^T\boldsymbol{G})$. (Note that $\boldsymbol{G}^T\boldsymbol{G} \in \mathbb{S}^n_+$). Moreover, even if none of $\mathbb{J}_+(\boldsymbol{B})$ ($\boldsymbol{B} \in \mathcal{B}$) is a face of $\mathbb{S}^n_+$, $\mathbb{J}_+(\mathcal{B})$ may still be contained in a proper face of $\mathbb{S}^n_+$, as demonstrated in Section 2. Therefore, studying such cases is both theoretically and practically important.

As a weaker variant of condition (I), we propose the following condition:

(B) $\mathbb{J}_0(\boldsymbol{B}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\boldsymbol{A}) \cap \mathbb{F}$ or $\mathbb{J}_+(\boldsymbol{A}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\boldsymbol{B}) \cap \mathbb{F}$ holds for every $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}$.

Theorem 1.4. *Let $\mathcal{B} \subseteq \mathbb{S}^n$ and $\mathbb{F}$ be a face of $\mathbb{S}^n_+$ that includes $\mathbb{J}_+(\mathcal{B})$. Assume that condition (B) holds. Then $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$.*

We emphasize that an infinite set $\mathcal{B}$ is allowed in condition (B); hence QCQP (3) may be a semi-infinite program [14, 30]. The inclusion $\mathbb{J}_+(\boldsymbol{A}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\boldsymbol{B}) \cap \mathbb{F}$ in condition (B) for distinct $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}$ implies that $\boldsymbol{B}$ is redundant to describe $\mathbb{J}_+(\mathcal{B}) \cap \mathbb{F}$, *i.e.*, $\mathbb{J}_+(\mathcal{B}) \cap \mathbb{F} = \mathbb{J}_+(\mathcal{B}\backslash\{\boldsymbol{B}\}) \cap \mathbb{F}$, but $\mathbb{J}_+(\mathcal{B})$ can be a proper subset of $\mathbb{J}_+(\mathcal{B}\backslash\boldsymbol{B}))$, as shown in Section 2. Theorem 1.4 strengthens [2, Theorem 3.1] where the authors assume that $\mathcal{B}$ is finite and fix $\mathbb{F} = \mathbb{S}^n_+$ as in condition (I).

It is well-known that every face of $\mathbb{S}^n_+$ is isomorphic to $\mathbb{S}^r_+$ for some $r \in \{0, \ldots, n\}$ [26, 27]. For proofs of Theorems 1.4, 1.5 and 1.6, we apply facial reduction from $\mathbb{F}$ onto $\mathbb{S}^r_+$. (See [7, 8, 36] for numerical methods for facial reduction). Let Φ be a linear isomorphism from $\mathbb{F} \supseteq \mathbb{J}_+(\mathcal{B})$ onto $\mathbb{S}^r_+$, and $\Phi^* : \mathbb{S}^n \to \mathbb{S}^r$ the adjoint map with respect to Φ. Then we can prove that $\Phi(\mathbb{J}_0(\boldsymbol{B}) \cap \mathbb{F}) = \mathbb{J}_0(\Phi^*(\boldsymbol{B}))$ and $\Phi(\mathbb{J}_+(\boldsymbol{B}) \cap \mathbb{F}) = \mathbb{J}_+(\Phi^*(\boldsymbol{B}))$ for every $\boldsymbol{B} \in \mathbb{S}^n$, and that $\Phi(\mathbb{J}_+(\mathcal{B})) = \Phi(\mathbb{J}_+(\mathcal{B}) \cap \mathbb{F}) = \mathbb{J}_+(\Phi^*(\mathcal{B}))$. Thus condition (B) is equivalent to

($\widetilde{B}$) $\mathbb{J}_0(\widetilde{\boldsymbol{B}}) \subseteq \mathbb{J}_+(\widetilde{\boldsymbol{A}})$ or $\mathbb{J}_+(\widetilde{\boldsymbol{A}}) \subseteq \mathbb{J}_+(\widetilde{\boldsymbol{B}})$ holds for every $\widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{B}} \in \Phi^*(\mathcal{B})$.

Also, $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$ holds if and only if $\mathbb{J}_+(\Phi^*(\mathcal{B})) = \Phi(\mathbb{J}_+(\mathcal{B})) \in \Phi(\widehat{\mathcal{F}}(\mathbf{\Gamma}^n)) = \widehat{\mathcal{F}}(\mathbf{\Gamma}^r)$. As a result, when $\mathcal{B}$ is finite, we can reduce Theorem 1.4 to Theorem 1.2 as will be shown in Section 3.2.1. We note that the isomorphism Φ and the adjoint Φ^* do not appear explicitly in the descriptions of the proposed conditions (B), (C) and (D), although they play an essential role in the proofs of our main results, including Theorems 1.4, 1.5 and 1.6. It should be emphasized that conditions (B), (C) and (D) are invariant under the isomorphism Φ, while conditions (I), (II) and (III) are not. Further details will be provided in Section 3. The equivalence of $\mathbb{J}_0(\mathcal{B}) \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$ and $\mathbb{J}_0(\Phi^*(\mathcal{B})) \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^r)$, which was obtained similarly in [1, Observation 1], was utilized for the proof of [1, Proposition 1] in a different context, where $\mathbb{J}_0(\mathcal{B}) = \{\boldsymbol{X} \in \mathbb{S}^n_+ : \langle \boldsymbol{B}, \boldsymbol{X} \rangle = 0 \ (\boldsymbol{B} \in \mathcal{B})\}$. But the condition (II) derived there is not invariant under Φ.

We also propose non-homogenized NIQCCs (C) and (D) below. Let

$$L(\mathbb{F}) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}\boldsymbol{x}^T \in \mathbb{F}\}, \ L_1(\mathbb{F}) = \{\boldsymbol{u} \in \mathbb{R}^{n-1} : \binom{\boldsymbol{u}}{1} \in L(\mathbb{F})\}.$$

Since $\mathbb{F}$ is a face of $\mathbb{S}^n_+$, there exists an $\boldsymbol{F} \in \mathbb{S}^n_+$ such that $\mathbb{F} = \{\boldsymbol{X} \in \mathbb{S}^n_+ : \langle \boldsymbol{F}, \boldsymbol{X} \rangle = 0\}$. Hence $L(\mathbb{F}) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{F}, \boldsymbol{x}\boldsymbol{x}^T \rangle = 0\} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{F}\boldsymbol{x} = \boldsymbol{0}\}$, which implies that $L(\mathbb{F})$ is a linear subspace of $\mathbb{R}^n$, and $L_1(\mathbb{F}) = \{\boldsymbol{u} \in \mathbb{R}^{n-1} : \boldsymbol{F}\binom{\boldsymbol{u}}{1} = \boldsymbol{0}\}$ an affine subspace of $\mathbb{R}^{n-1}$. It follows from $\mathbb{J}_+(\mathcal{B}) \subseteq \mathbb{F}$ that $\mathcal{B}_{\geq} \subseteq L_1(\mathbb{F})$; hence $\mathcal{B}_{\geq} = \mathcal{B}_{\geq} \cap L_1(\mathbb{F})$. Specifically, if $\mathbb{F} = \mathbb{S}^n_+$ then $L(\mathbb{F}) = \mathbb{R}^n$ and $L_1(\mathbb{F}) = \mathbb{R}^{n-1}$.

Theorem 1.5. *Let $\mathcal{B} \subseteq \mathbb{S}^n$ and $\mathbb{F}$ be a face of $\mathbb{S}^n_+$ that includes $\mathbb{J}_+(\mathcal{B})$. Assume that the following condition (C) is satisfied. Then condition (B) holds; hence $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$.*

(C) $\mathcal{B}_{\geq} \cap L_1(\mathbb{F}) \neq \emptyset$, and

$$\emptyset \neq \boldsymbol{B}_{\leq} \cap L_1(\mathbb{F}) \subseteq \boldsymbol{A}_{\geq} \cap L_1(\mathbb{F}) \text{ or } \boldsymbol{A}_{\geq} \cap L_1(\mathbb{F}) \subseteq \boldsymbol{B}_{\geq} \cap L_1(\mathbb{F}) \text{ for every } \boldsymbol{A}, \ \boldsymbol{B} \in \mathcal{B}.$$

Theorem 1.6. *Let $\mathcal{B} \subseteq \mathbb{S}^n$ and $\mathbb{F}$ be a face of $\mathbb{S}^n_+$ that includes $\mathbb{J}_+(\mathcal{B})$. Assume that the following condition (D) is satisfied. Then (6) and (7) hold.*

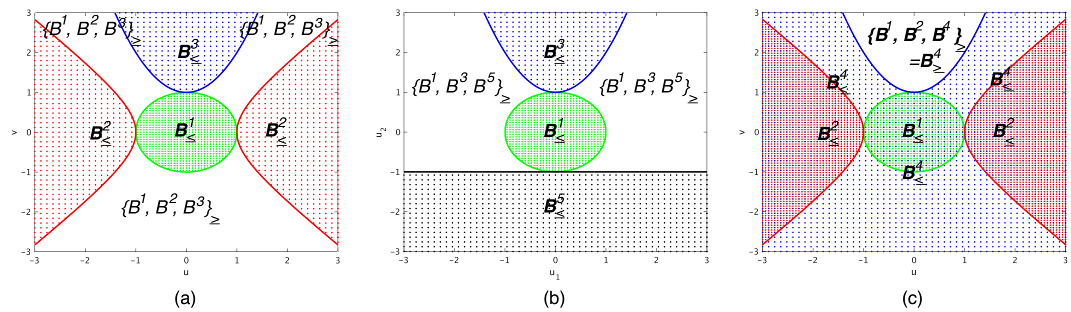

Figure 1: Illustration for conditions (C) and (D). $n = 3$, $\mathbb{F} = \mathbb{S}^3_+$ and $L_1(\mathbb{F}) = \mathbb{R}^2$. (a) $\mathcal{B}^1 = \{\boldsymbol{B}^1, \boldsymbol{B}^2, \boldsymbol{B}^3\}$, (b) $\mathcal{B}^2 = \{\boldsymbol{B}^1, \boldsymbol{B}^3, \boldsymbol{B}^5\}$ and (c) $\mathcal{B}^3 = \{\boldsymbol{B}^1, \boldsymbol{B}^2, \boldsymbol{B}^4\}$, where $q(\boldsymbol{u}, \boldsymbol{B}^1) = u_1^2 + u_2^2 - 1$, $q(\boldsymbol{u}, \boldsymbol{B}^2) = -u_1^2 + u_2^2 + 1$, $q(\boldsymbol{u}, \boldsymbol{B}^3) = u_1^2 - u_2 + 1$, $q(\boldsymbol{u}, \boldsymbol{B}^4) = -q(\boldsymbol{u}, \boldsymbol{B}^3)$, and $q(\boldsymbol{u}, \boldsymbol{B}^5) = u_2 + 1$. $\mathcal{B}^k_{\geq}$: the unshaded region ($k = 1, 2, 3$).

(D) $\mathcal{B}$ is finite, $q(\cdot, \boldsymbol{B}) : \mathbb{R}^{n-1} \to \mathbb{R}$ ($\boldsymbol{B} \in \mathcal{B}$) is not affine on $L_1(\mathbb{F})$, and

$$\boldsymbol{B}_= \cap L_1(\mathbb{F}) \subseteq \boldsymbol{A}_\geq \cap L_1(\mathbb{F}) \text{ for every } \boldsymbol{A}, \ \boldsymbol{B} \in \mathcal{B}. \tag{9}$$

If we take $\mathbb{F} = \mathbb{S}^n_+$ then condition (D) coincides with condition (III). Thus Theorem 1.6 is a generalization of Theorem 1.3. Neither of conditions (C) and (D) implies the other; the inclusion relation $\boldsymbol{B}_\leq \cap L_1(\mathbb{F}) \subseteq \boldsymbol{A}_\geq \cap L_1(\mathbb{F})$ in (C) implies the inclusion relation $\boldsymbol{B}_= \cap L_1(\mathbb{F}) \subseteq \boldsymbol{A}_\geq \cap L_1(\mathbb{F})$ in (D), while (C) allows $q(\cdot, \boldsymbol{B})$ for some $\boldsymbol{B} \in \mathcal{B}$ to be affine on $L_1(\mathbb{F})$. Figure 1 shows three cases (a), (b) and (c) for conditions (C) and (D), where condition (C) is satisfied in all cases but condition (D) is satisfied only in cases (a). Case (b) involves an affine function $q(\cdot, \boldsymbol{B}^5) = u_2 + 1$. In case (c), $\boldsymbol{B}^1_= \not\subseteq \boldsymbol{B}^4_\geq$, $\boldsymbol{B}^2_= \not\subseteq \boldsymbol{B}^4_\geq$ and $\{\boldsymbol{B}^1, \boldsymbol{B}^2, \boldsymbol{B}^4\}_\geq = \{\boldsymbol{B}^4\}_\geq$ hold. This case illustrates that any redundant $\boldsymbol{B}$ should be removed from $\mathcal{B}$ for condition (D) to be more effective. In conditions (C), 'or $\boldsymbol{A}_\geq \cap L_1(\mathbb{F}) \subseteq \boldsymbol{B}_\geq \cap L_1(\mathbb{F})$' is added to adapt such cases.

Condition (B) (also (C) and (D)) depends on the choice of a face $\mathbb{F}$ of $\mathbb{S}^n_+$ that includes $\mathbb{J}_+(\mathcal{B})$. If $\mathbb{F}^1$ and $\mathbb{F}^2$ are faces of $\mathbb{S}^n_+$ such that $\mathbb{F}^1 \supseteq \mathbb{F}^2 \supseteq \mathbb{J}_+(\mathcal{B})$, then the condition with $\mathbb{F} = \mathbb{F}^1$ implies the condition with $\mathbb{F} = \mathbb{F}^2$. Thus, by choosing the minimal face $\mathbb{F}_{\min}$ of $\mathbb{S}^n_+$ that contains $\mathbb{J}_+(\mathcal{B})$, we obtain the weakest condition, since the condition with $\mathbb{F}_{\min}$ is implied by the condition with any larger face $\mathbb{F}$. In this case, $\mathbb{J}_+(\Phi^*(\mathcal{B}))$ satisfies Slater's constraint qualification $\mathbb{J}_+(\Phi^*(\mathcal{B})) \cap \mathbb{S}^r_{++} \neq \emptyset$, where $\mathbb{S}^r_{++}$ denotes the interior of $\mathbb{S}^r_+$, *i.e.*, the set of $r \times r$ positive definite matrices.

We now outline possible applications. QCQPs of the form (5) arise in a variety of domains, including robotics and autonomous systems for avoiding multiple exclusion zones [14, 31, 32], sensor placement problems [9, 12] where optimal locations must lie outside risky regions, and data classification tasks involving the placement of test points outside known clusters [25]. The infeasible region of QCQP (5), given by the interior of $\boldsymbol{B}_\leq$ ($\boldsymbol{B} \in \mathcal{B}$), corresponds to the multiple exclusion zones, risky regions, and known clusters, respectively (see Figures 1, 4, 5 and 6). When the formulated QCQPs satisfy a NIQCC, such as conditions (I) and (III), they can be solved exactly through their SDP relaxations. Furthermore,

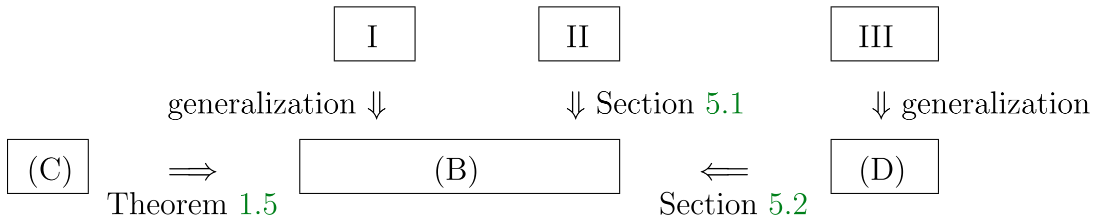

Figure 2: Relationships among conditions (I), (II), (III), (B), (C) and (D). All conditions are equivalent under additional assumptions including Slater's constraint qualification ($\mathbb{J}_+(\mathcal{B})$ intersects with the interior of $\mathbb{S}^n_+$ or $\mathbb{F}_{\min} = \mathbb{S}^n_+$) and no redundancy on $\mathcal{B}$ to represent $\mathbb{J}_+(\mathcal{B})$. See Section 5.3.

the proposed conditions (B), (C) and (D) are expected to substantially broaden the scope of such applications; for example, by allowing the free addition of linear equality constraints and by effectively handling degenerate cases in which the feasible region fails to satisfy Slater's constraint qualification. Additionally, in Section 6, we illustrate some geometric examples for the reader interested in possible applications.

1.4 Outline of the paper

In Section 2, a simple example is provided to demonstrate the effectiveness of the proposed conditions (B), (C) and (D) in comparison to the existing conditions (I), (II) and (III). In Section 3.1, we introduce an isomorphism Φ from a face $\mathbb{F}$ of $\mathbb{S}^n_+$ onto $\mathbb{S}^r_+$ and some basic theoretical issues related on Φ that play an essential role in the subsequent sections. Based on them, we present a proof of Theorem 1.4 in Section 3.2, and proofs of Theorem 1.5 and 1.6 in Section 4. In Section 5, we investigate the relationships of the conditions presented above, (I), (II), (III), (B), (C), and (D) in detail. It is shown that condition (B) is the weakest among them, since it is implied by all the others, and that all four conditions (I), (II), (B), and (C) are equivalent under Slater's constraint qualification and the absence of redundant constraints. See Figure 2. Under these assumptions, condition (II) may be regarded as the dual of condition (B) (see Remark 5.3). In Section 6, we provide three geometric QCQP examples that satisfy condition (C). Finally, Section 7 contains our concluding remarks.

2 An example illustrating the effectiveness of Theorems 1.4, 1.5 and 1.6 in comparison to Theorems 1.2 and 1.3

In this section, we compare our main results stated in Theorems 1.4, 1.5 and 1.6 with the known results in Theorems 1.2 and 1.3 through an example to demonstrate the effectiveness of our main results. Let $n = 4$, $\mathcal{B} = \{\boldsymbol{B}^1, \boldsymbol{B}^2, \boldsymbol{B}^3\}$ and $\mathbb{J} = \mathbb{J}_+(\mathcal{B})$, where

$$\boldsymbol{B}^1 = \begin{pmatrix} -1 & -2 & 0 & 1 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 0 & 2 & 2 \end{pmatrix},\ \boldsymbol{B}^2 = \begin{pmatrix} 1 & 1 & 0 & -2 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ -2 & 0 & -1 & -1 \end{pmatrix},\ \boldsymbol{B}^3 = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 \end{pmatrix}.$$

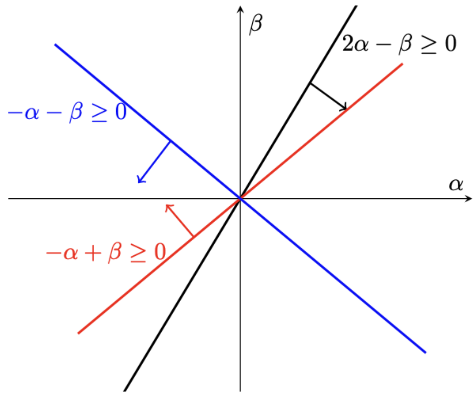

Figure 3: The region of (α, β) determined by the inequalities in (10).

Let

$$\boldsymbol{X}^1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \boldsymbol{X}^2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},\ \boldsymbol{X}^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} &\boldsymbol{X}^1 \in \mathbb{J}_+(\{\boldsymbol{B}^2, \boldsymbol{B}^3\}) \backslash \mathbb{J}_+(\{\boldsymbol{B}^1\}),\ \boldsymbol{X}^2 \in \mathbb{J}_+(\{\boldsymbol{B}^1, \boldsymbol{B}^3\}) \backslash \mathbb{J}_+(\{\boldsymbol{B}^2\}), \\ &\boldsymbol{X}^3 \in \mathbb{J}_+(\{\boldsymbol{B}^1, \boldsymbol{B}^2\}) \backslash \mathbb{J}_+(\{\boldsymbol{B}^3\}). \end{aligned}$$

This implies that none of $\boldsymbol{B}^1$, $\boldsymbol{B}^2$ and $\boldsymbol{B}^3$ is redundant to describe $\mathbb{J}_+(\{\boldsymbol{B}^1, \boldsymbol{B}^2, \boldsymbol{B}^3\})$.

Let $\widehat{\boldsymbol{X}} = \mathrm{diag}(1, 1, 0, 0)$ (the 4×4 diagonal matrix with elements $1, 1, 0, 0$). Then $\widehat{\boldsymbol{X}} \in \mathbb{J}_0(\boldsymbol{B}^2)$ but $\widehat{\boldsymbol{X}} \notin \mathbb{J}_+(\boldsymbol{B}^1)$. Hence $\mathcal{B}$ does not satisfy condition (I) assumed in Theorem 1.2. Also, condition (II) assumed in Theorem 1.2 dose not hold. In fact, assume on the contrary that (II) holds, which implies that the diagonal of $\alpha\boldsymbol{B}^1 + \beta\boldsymbol{B}^2 \in \mathbb{S}^4_+$ is nonnegative for some nonzero $(\alpha, \beta) \in \mathbb{R}^2$. Hence

$$-\alpha + \beta \geq 0,\ -\alpha - \beta \geq 0,\ 2\alpha - \beta \geq 0,\ 2\alpha - \beta \geq 0. \tag{10}$$

Clearly, only $(\alpha, \beta) = \boldsymbol{0}$ satisfies the above inequalities, as shown in Figure 3, which illustrates the region of (α, β) determined by the inequalities in (10). Therefore, condition (II) dose not hold. It is also easily verified that if $\bar{\boldsymbol{u}} = \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} \in \mathbb{R}^3$ then $\bar{\boldsymbol{u}} \in \boldsymbol{B}^1_=$ but $\bar{\boldsymbol{u}} \notin \boldsymbol{B}^2_\geq$. Therefore, (8) in condition (III) is not satisfied.

To see whether condition (B) is satisfied for some face $\mathbb{F}$ that contains $\mathcal{B}$, we observe that

$$\begin{aligned} \mathbb{J}_+(\mathcal{B}) &\subseteq \left\{ \boldsymbol{X} \in \mathbb{S}^4_+ : \langle \boldsymbol{B}^1 + \boldsymbol{B}^2 + \boldsymbol{B}^3,\ \boldsymbol{X} \rangle \geq 0 \right\} \\ &= \left\{ \boldsymbol{X} \in \mathbb{S}^4_+ : \langle \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \boldsymbol{X} \rangle \geq 0 \right\} \\ &= \left\{ \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{U} \end{pmatrix} \in \mathbb{S}^4 : \boldsymbol{U} \in \mathbb{S}^2_+ \right\}. \end{aligned}$$

Hence, by letting

$$\mathbb{F}^1 = \left\{ \begin{pmatrix} O & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{U} \end{pmatrix} \in \mathbb{S}^4 : \boldsymbol{U} \in \mathbb{S}^2_+ \right\},$$

$$\widetilde{\boldsymbol{B}^1} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},\ \widetilde{\boldsymbol{B}^2} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix},\ \widetilde{\boldsymbol{B}^3} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix},\ \widetilde{\mathcal{B}} = \{\widetilde{\boldsymbol{B}^1}, \widetilde{\boldsymbol{B}^2}, \widetilde{\boldsymbol{B}^3}\},$$

we see that $\mathbb{J}_+(\mathcal{B}) \subseteq \mathbb{F}^1$, which indicates that $X_{ij} = 0$ $(1 \leq i \leq 2$ or $1 \leq j \leq 2)$ if $\boldsymbol{X} \in \mathbb{J}_+(\mathcal{B})$, and condition (B) is equivalent to

$$\mathbb{J}_0(\widetilde{\boldsymbol{B}}) \subseteq \mathbb{J}_+(\widetilde{\boldsymbol{A}}) \text{ or } \mathbb{J}_+(\widetilde{\boldsymbol{A}}) \subseteq \mathbb{J}_+(\widetilde{\boldsymbol{B}}) \text{ for every } \widetilde{\boldsymbol{A}},\ \widetilde{\boldsymbol{B}} \in \widetilde{\mathcal{B}},$$

which obviously holds since

$$\begin{aligned} \mathbb{J}_0(\widetilde{\boldsymbol{B}^1}) &= \mathbb{J}_0(\widetilde{\boldsymbol{B}^2}) = \mathbb{J}_0(\widetilde{\boldsymbol{B}^3}) = \mathbb{J}_+(\widetilde{\boldsymbol{B}^2}) = \mathbb{J}_+(\widetilde{\boldsymbol{B}^3}) \\ &= \left\{ \boldsymbol{U} \in \mathbb{S}^2_+ : \langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \boldsymbol{U} \rangle = 0 \right\} \subseteq \mathbb{S}^2_+ = \mathbb{J}_+(\widetilde{\boldsymbol{B}^1}). \end{aligned}$$

Now, we verify whether condition (C) is satisfied with $\mathbb{F} = \mathbb{F}^1$. By definition,

$$\begin{aligned} L(\mathbb{F}^1) &= \left\{ \boldsymbol{x} \in \mathbb{R}^4 : \boldsymbol{x}\boldsymbol{x}^T \in \mathbb{F}^1 \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : x_3,\ x_4 \in \mathbb{R} \right\}, \\ L_1(\mathbb{F}^1) &= \left\{ \boldsymbol{u} \in \mathbb{R}^3 : \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix} \in L(\mathbb{F}^1) \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix} \in \mathbb{R}^3 : u_3 \in \mathbb{R} \right\}. \end{aligned}$$

For every $\boldsymbol{u} \in L_1(\mathbb{F}^1)$,

$$q(\boldsymbol{u}, \boldsymbol{B}^k) = \langle \widetilde{\boldsymbol{B}^k}, \begin{pmatrix} u_3 \\ 1 \end{pmatrix} \rangle = \begin{cases} 2(u_3+1)^2 & (k=1) \\ -(u_3+1)^2 & (k=2,3). \end{cases}$$

holds. Hence

$$\begin{aligned} &\boldsymbol{B}^1_{\leq} \cap L_1(\mathbb{F}^1) = \boldsymbol{B}^2_{\geq} \cap L_1(\mathbb{F}^1) = \boldsymbol{B}^3_{\geq} \cap L(\mathbb{F}^1) = \left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}, \\ &\boldsymbol{B}^1_{\geq} \cap L(\mathbb{F}^1) = \boldsymbol{B}^2_{\leq} \cap L_1(\mathbb{F}^1) = \boldsymbol{B}^3_{\leq} \cap L_1(\mathbb{F}^1) = \left\{ \begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix} : u_3 \in \mathbb{R} \right\}. \end{aligned}$$

Therefore condition (C) with $\mathbb{F} = \mathbb{F}^1$ is satisfied. It is also easy to see that condition (D) with $\mathbb{F} = \mathbb{F}^1$ is satisfied.

We note that $\mathbb{F}^1$ is not the minimal face of $\mathbb{S}^4_+$ that contains $\mathbb{J}_+(\mathcal{B})$. In fact, if we let

$$\mathbb{F}^2 = \left\{ \boldsymbol{X} \in \mathbb{S}^4_+ : \langle \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \boldsymbol{X} \rangle = 0 \right\},$$

then $\mathbb{F} = \mathbb{F}^1 \cap \mathbb{F}^2$ forms the minimal face. In this case

$$\begin{aligned} &L(\mathbb{F}) = \left\{ \begin{pmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} : x_3 + x_4 = 0 \right\},\ L_1(\mathbb{F}) = \left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}, \\ &\mathbb{J}_+(\mathcal{B}) \cap \mathbb{F} = \mathbb{F} = \left\{ \begin{pmatrix} O & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{U} \end{pmatrix} \in \mathbb{S}^4 : \langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \boldsymbol{U} \rangle = 0,\ \boldsymbol{U} \in \mathbb{S}^2_+ \right\}, \\ &\mathcal{B}_{\geq} \cap L_1(\mathbb{F}) = L_1(\mathbb{F}) = \left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\} \end{aligned}$$

hold.

3 On Theorem 1.4

Throughout Sections 3 and 4, let $\mathcal{B} \subseteq \mathbb{S}^n$ and $\mathbb{F}$ be a face of $\mathbb{S}^n_+$ that contains $\mathbb{J}_+(\mathcal{B})$.

3.1 Facial reduction of $\mathbb{J}_+(\mathcal{B})$ into $\mathbb{S}^r_+$

We first represent $\mathbb{F}$ as $\mathbb{F} = \left\{ \boldsymbol{X} \in \mathbb{S}^n_+ : \langle \boldsymbol{F}, \boldsymbol{X} \rangle = 0 \right\}$ for some $\boldsymbol{F} \in \mathbb{S}^n_+$. Suppose that rank$\boldsymbol{F} = n - r$ for some $r \in \{0, 1, \ldots, n\}$. We can take an $n \times n$ orthogonal matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{F}$ such that $\boldsymbol{P}^T \boldsymbol{F} \boldsymbol{P} = \mathrm{diag}(0, \ldots, 0, \lambda_{r+1}, \ldots, \lambda_n) \in \mathbb{S}^n_+$ for some positive numbers $\lambda_{r+1}, \ldots, \lambda_n$, where $0, \ldots, 0, \lambda_{r+1}, \ldots, \lambda_n$ denote the eigenvalues of $\boldsymbol{F}$ with $\lambda_i > 0$ $(r+1 \leq i \leq n)$. Let $\boldsymbol{P}_{1\_r}$ denote the $n \times r$ matrix consisting of the first r columns of $\boldsymbol{P}$; each j th columns of $\boldsymbol{P}_{1\_r}$ is an eigenvector of $\boldsymbol{F}$ associated with the zero eigenvalue. Let $\boldsymbol{M}$ be an arbitrary $r \times r$ nonsingular matrix. Define

$$
\begin{aligned}
\Phi(\boldsymbol{X}) &= \boldsymbol{M}^T \boldsymbol{P}^T_{1\_r} \boldsymbol{X} \boldsymbol{P}_{1\_r} \boldsymbol{M} \in \mathbb{S}^r_+ \text{ for every } \boldsymbol{X} \in \mathbb{F}, \\
\Phi^*(\boldsymbol{Y}) &= \boldsymbol{M}^{-1} \boldsymbol{P}^T_{1\_r} \boldsymbol{Y} \boldsymbol{P}_{1\_r} \boldsymbol{M}^{-T} \text{ for every } \boldsymbol{Y} \in \mathbb{S}^n, \\
\theta(\boldsymbol{x}) &= \boldsymbol{M}^T \boldsymbol{P}^T_{1\_r} \boldsymbol{x} \text{ for every } \boldsymbol{x} \in L(\mathbb{F}).
\end{aligned}
$$

Then $\Phi : \mathbb{F} \to \mathbb{S}^r_+$ forms a linear isomorphism from $\mathbb{F}$ onto $\mathbb{S}^r_+$ and $\Phi^* : \mathbb{S}^n \to \mathbb{S}^r$ the adjoint map with respect to Φ (see, for example, [7, 26, 27]). Here $\boldsymbol{V} \in \mathbb{S}^r_+ \to \boldsymbol{M}^T \boldsymbol{V} \boldsymbol{M} \in \mathbb{S}^r_+$ serves as an automorphism on $\mathbb{S}^r_+$. The choice of $\boldsymbol{M}$ is not relevant here, but it becomes relevant in Section 4 where QCQP (5) is considered, *i.e.*, COP$(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}) \cap \mathbb{F}, \boldsymbol{Q}, \boldsymbol{H})$ with $\boldsymbol{H} = \mathrm{diag}(0, \ldots, 0, 1) \ \in \mathbb{S}^n_+$. So we may assume that $\boldsymbol{M}$ is the $r \times r$ identity matrix in this section.

Lemma 3.1.

(i) *The map* $\boldsymbol{V} \in \mathbb{S}^r_+ \to \boldsymbol{P}_{1\_r} \boldsymbol{M}^{-T} \boldsymbol{V} \boldsymbol{M}^{-1} \boldsymbol{P}^T_{1\_r} \in \mathbb{F}$ *serves as the inverse of* $\Phi : \mathbb{F} \to \mathbb{S}^r_+$.

(ii) $\theta : L(\mathbb{F}) \to \mathbb{R}^r$ *is linear, one-to-one and onto.*

(iii) $\langle \boldsymbol{A}, \boldsymbol{X} \rangle = \langle \Phi^*(\boldsymbol{A}), \Phi(\boldsymbol{X}) \rangle$ *for every* $\boldsymbol{A} \in \mathbb{S}^n$ *and* $\boldsymbol{X} \in \mathbb{F}$.

(iv) $\langle \boldsymbol{A}, \boldsymbol{x}\boldsymbol{x}^T \rangle = \langle \Phi^*(\boldsymbol{A}), \theta(\boldsymbol{x})\theta(\boldsymbol{x})^T \rangle$ *for every* $\boldsymbol{A} \in \mathbb{S}^n$ *and* $\boldsymbol{x} \in L(\mathbb{F})$.

(v) $\Phi(\mathbb{J}_-(\boldsymbol{B})) \cap \mathbb{F}) = \mathbb{J}_-(\Phi^*(\boldsymbol{B}))$, $\Phi(\mathbb{J}_0(\boldsymbol{B})) \cap \mathbb{F}) = \mathbb{J}_0(\Phi^*(\boldsymbol{B}))$ *and* $\Phi(\mathbb{J}_+(\boldsymbol{B})) \cap \mathbb{F}) = \mathbb{J}_+(\Phi^*(\boldsymbol{B}))$ *for every* $\boldsymbol{B} \in \mathcal{B}$.

(vi) $\Phi(\mathbb{J}_+(\mathcal{B}))) = \Phi(\mathbb{J}_+(\mathcal{B})) \cap \mathbb{F}) = \mathbb{J}_+(\Phi^*(\mathcal{B}))$.

Proof. (i) For every $\boldsymbol{V} \in \mathbb{S}^r_+$, we see

$$
\begin{aligned}
&\langle \boldsymbol{F}, \boldsymbol{P}_{1\_r} \boldsymbol{M}^{-T} \boldsymbol{V} \boldsymbol{M}^{-1} \boldsymbol{P}^T_{1\_r} \rangle \\
&\quad = \langle \boldsymbol{P}^T_{1\_r} \boldsymbol{P} \mathrm{diag}(0, \ldots, 0, \lambda_{r+1}, \ldots, \lambda_n) \boldsymbol{P}^T \boldsymbol{P}_{1\_r}, \boldsymbol{M}^{-T} \boldsymbol{V} \boldsymbol{M}^{-1} \rangle \\
&\quad = \langle \boldsymbol{O}, \boldsymbol{M}^{-T} \boldsymbol{V} \boldsymbol{M}^{-1} \rangle = 0.
\end{aligned}
$$

Hence $\boldsymbol{P}_{1\_r} \boldsymbol{M}^{-T} \boldsymbol{V} \boldsymbol{M}^{-1} \boldsymbol{P}^T_{1\_r} \in \mathbb{F}$. Also $\Phi(\boldsymbol{P}_{1\_r} \boldsymbol{M}^{-T} \boldsymbol{V} \boldsymbol{M}^{-1} \boldsymbol{P}^T_{1\_r}) = \boldsymbol{V}$ for every $\boldsymbol{V} \in \mathbb{S}^r_+$. Therefore, the desired result follows.

(ii) By definition, $\theta : L(\mathbb{F}) \to \mathbb{R}^r$ is linear and $\theta(L(\mathbb{F})) \subseteq \mathbb{R}^r$. Let $\boldsymbol{v} \in \mathbb{R}^r$. Define $\boldsymbol{x} = \boldsymbol{P}_{1\_r} \boldsymbol{M}^{-T} \boldsymbol{v}$ and $\boldsymbol{V} = \boldsymbol{v}\boldsymbol{v}^T \in \mathbb{S}^r_+$. Then $\boldsymbol{x}\boldsymbol{x}^T = \boldsymbol{P}_{1\_r} \boldsymbol{M}^{-T} \boldsymbol{V} \boldsymbol{M}^{-1} \boldsymbol{P}^T_{1\_r} \in \mathbb{F}$ as we have seen in the proof (i) above. Hence $\boldsymbol{x} \in L(\mathbb{F})$ by the definition of $L(\mathbb{F})$. We also see that

$\theta(\boldsymbol{x}) = \boldsymbol{M}^T\boldsymbol{P}_{1\_r}^T\boldsymbol{x} = \boldsymbol{M}^T\boldsymbol{P}_{1\_r}^T\boldsymbol{P}_{1\_r}\boldsymbol{M}^{-T}\boldsymbol{v} = \boldsymbol{v}$. Hence $\theta(L(\mathbb{F})) = \mathbb{R}^r$, and we have shown $\theta(L(\mathbb{F})) = \mathbb{R}^r$. To see that $\theta : L(\mathbb{F}) \to \mathbb{R}^r$ is one-to-one, assume that $\theta(\boldsymbol{x}^1) = \theta(\boldsymbol{x}^2)$ for some $\boldsymbol{x}^1, \boldsymbol{x}^2 \in L(\mathbb{F})$. Then $\theta(\boldsymbol{x}^2 - \boldsymbol{x}^1) = \boldsymbol{0}$. Hence $\mathbb{S}^r_+ \ni \boldsymbol{O} = \theta(\boldsymbol{x}^2 - \boldsymbol{x}^1)\theta(\boldsymbol{x}^2 - \boldsymbol{x}^1)^T = \Phi((\boldsymbol{x}^2 - \boldsymbol{x}^1)(\boldsymbol{x}^2 - \boldsymbol{x}^1)^T)$, which implies $(\boldsymbol{x}^2 - \boldsymbol{x}^1)(\boldsymbol{x}^2 - \boldsymbol{x}^1)^T = \boldsymbol{O}$ and $\boldsymbol{x}^2 - \boldsymbol{x}^1 = \boldsymbol{0}$.

(iii) Let $\boldsymbol{A} \in \mathbb{S}^n$, $\boldsymbol{X} \in \mathbb{F}$ and $\boldsymbol{V} = \Phi(\boldsymbol{X}) \in \mathbb{S}^r_+$. By (i), $\boldsymbol{X} = \boldsymbol{P}_{1\_r}\boldsymbol{M}^{-T}\boldsymbol{V}\boldsymbol{M}^{-1}\boldsymbol{P}_{1\_r}^T \in \mathbb{F}$. Thus, it follows that

$$\langle \boldsymbol{A},\, \boldsymbol{X}\rangle = \langle \boldsymbol{A},\, \boldsymbol{P}_{1\_r}\boldsymbol{M}^{-T}\boldsymbol{V}\boldsymbol{M}^{-1}\boldsymbol{P}_{1\_r}^T\rangle = \langle \boldsymbol{M}^{-1}\boldsymbol{P}_{1\_r}^T\boldsymbol{A}\boldsymbol{P}_{1\_r}\boldsymbol{M}^{-T},\, \boldsymbol{V}\rangle = \langle \Phi^*(\boldsymbol{A}),\, \Phi(\boldsymbol{X})\rangle.$$

(iv) Let $\boldsymbol{A} \in \mathbb{S}^n$ and $\boldsymbol{x} \in L(\mathbb{F})$. Then $\boldsymbol{x}\boldsymbol{x}^T \in \mathbb{F}$. Consequently, $\langle \boldsymbol{A},\, \boldsymbol{x}\boldsymbol{x}^T\rangle = \langle \Phi^*(\boldsymbol{A}),\, \Phi(\boldsymbol{x}\boldsymbol{x}^T)\rangle = \langle \Phi^*(\boldsymbol{A}),\, \theta(\boldsymbol{x})\theta(\boldsymbol{x})^T\rangle$ follows from (iii).

Assertion (v) can be proved easily by assertions (i) and (iii). Assertion (vi) follows from assertion (v). □

The mapping $\mathbb{J}_+(\mathcal{B}) \subseteq \mathbb{F} \to \Phi(\mathbb{J}_+(\mathcal{B})) \cap \mathbb{F}) = \mathbb{J}_+(\Phi^*(\mathcal{B})) \subseteq \mathbb{S}^r$ in (vi) is interpreted as *a facial reduction* [7] of $\mathbb{J}_+(\mathcal{B})) \subseteq \mathbb{F}$ into $\mathbb{S}^r$. For simplicity of notation, we denote $\Phi^*(\boldsymbol{B})$ by $\widetilde{\boldsymbol{B}}$ for every $\boldsymbol{B} \in \mathbb{S}^n$, and $\Phi^*(\mathcal{B}) = \{\Phi^*(\boldsymbol{B}) : \boldsymbol{B} \in \mathcal{B}\}$ by $\widetilde{\mathcal{B}}$. Then, we can simplify QCQP (2), its SDP relaxation (4), and condition (B) by the mappings Φ from $\mathbb{F}$ onto $\mathbb{S}^r_+$ and θ from $L(\mathbb{F})$ onto $\mathbb{R}^r$ as follows.

COP($\boldsymbol{\Gamma}^r \cap \mathbb{J}_+(\widetilde{\mathcal{B}}), \widetilde{\boldsymbol{Q}}, \widetilde{\boldsymbol{H}}$):

$$\eta(\boldsymbol{\Gamma}^r \cap \mathbb{J}_+(\widetilde{\mathcal{B}}), \widetilde{\boldsymbol{Q}}, \widetilde{\boldsymbol{H}}) \;=\; \inf\left\{ \langle \widetilde{\boldsymbol{Q}},\, \boldsymbol{v}\boldsymbol{v}^T\rangle : \begin{array}{l} \boldsymbol{v} \in \mathbb{R}^r,\ \boldsymbol{v}\boldsymbol{v}^T \in \mathbb{J}_+(\widetilde{\mathcal{B}}), \\ \langle \widetilde{\boldsymbol{H}},\, \boldsymbol{v}\boldsymbol{v}^T\rangle = 1 \end{array} \right\}. \tag{11}$$

COP($\mathbb{J}_+(\widetilde{\mathcal{B}}), \widetilde{\boldsymbol{Q}}, \widetilde{\boldsymbol{H}}$):

$$\eta(\mathbb{J}_+(\widetilde{\mathcal{B}}), \widetilde{\boldsymbol{Q}}, \widetilde{\boldsymbol{H}}) \;=\; \inf\left\{ \langle \widetilde{\boldsymbol{Q}},\, \boldsymbol{V}\rangle : \begin{array}{l} \boldsymbol{V} \in \mathbb{S}^r_+,\ \boldsymbol{V} \in \mathbb{J}_+(\widetilde{\mathcal{B}}), \\ \langle \widetilde{\boldsymbol{H}},\, \boldsymbol{V}\rangle = 1 \end{array} \right\}. \tag{12}$$

($\widetilde{\boldsymbol{B}}$) For every $\widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{B}} \in \widetilde{\mathcal{B}}$, either $\mathbb{J}_0(\widetilde{\boldsymbol{B}}) \subseteq \mathbb{J}_+(\widetilde{\boldsymbol{A}})$ or $\mathbb{J}_+(\widetilde{\boldsymbol{A}}) \subseteq \mathbb{J}_+(\widetilde{\boldsymbol{B}})$ holds.

More precisely, Lemma 3.1 ensures:

[a] $\boldsymbol{x} \in \mathbb{R}^n$ is a feasible solution of (2) with the objective value $\langle \boldsymbol{Q},\, \boldsymbol{x}\boldsymbol{x}^T\rangle$ if and only if $\boldsymbol{v} = \theta(\boldsymbol{x})$ is a feasible solution of (11) with the objective value $\langle \widetilde{\boldsymbol{Q}},\, \boldsymbol{v}\boldsymbol{v}^T\rangle = \langle \boldsymbol{Q},\, \boldsymbol{x}\boldsymbol{x}^T\rangle$.

[b] $\boldsymbol{X} \in \mathbb{S}^n_+$ is a feasible solution of (4) with the objective value $\langle \boldsymbol{Q},\, \boldsymbol{X}\rangle$ if and only if $\boldsymbol{V} = \Phi(\boldsymbol{X})$ is a feasible solution of (12) with the objective value $\langle \widetilde{\boldsymbol{Q}},\, \boldsymbol{V}\rangle = \langle \boldsymbol{Q},\, \boldsymbol{X}\rangle$.

[c] Condition (B) holds if and only if condition ($\widetilde{\mathrm{B}}$) does.

In addition, we know that when $\boldsymbol{X} \in \mathbb{F}$ and $\boldsymbol{V} = \Phi(\boldsymbol{X}) = \boldsymbol{M}^T\boldsymbol{P}_{1\_r}^T\boldsymbol{X}\boldsymbol{P}_{1\_r}\boldsymbol{M}$, which imply $\boldsymbol{X} = \boldsymbol{P}_{1\_r}\boldsymbol{M}^{-T}\boldsymbol{V}\boldsymbol{M}^{-1}\boldsymbol{P}_{1\_r}^T$, $\boldsymbol{X}$ is rank-1 if and only if so is $\boldsymbol{V}$. It follows that

[d] $\mathbb{J}_+(\mathcal{B}) = \mathbb{J}_+(\mathcal{B}) \cap \mathbb{F} \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$ if and only if $\mathbb{J}_+(\widetilde{\mathcal{B}}) = \mathbb{J}_+(\Phi^*(\mathcal{B})) = \Phi(\mathbb{J}_+(\mathcal{B}) \cap \mathbb{F}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^r)$.

3.2 Proof of Theorem 1.4

In view of the discussion of Section 3.1, it suffices to prove $\mathbb{J}_+(\widetilde{\mathcal{B}}) \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^r)$ under condition $(\widetilde{\mathrm{B}})$ (see [c] and [d]). For simplicity of notation, we omit $\widetilde{}$ from $\widetilde{\boldsymbol{B}}$ ($\boldsymbol{B} \in \mathcal{B}$), $\widetilde{\mathcal{B}}$, $\widetilde{\boldsymbol{Q}}$ and $\widetilde{\boldsymbol{H}}$ for the proof, or equivalently, we prove $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$ under condition (B) with $\mathbb{F} = \mathbb{S}^n_+$.

3.2.1 The finite case of $\mathcal{B}$

This case can be derived easily from Theorem 1.2. Assume that $\mathcal{B}$ is finite and condition (B) holds with $\mathbb{F} = \mathbb{S}^n_+$. If $\mathbb{J}_+(\boldsymbol{A}) \subseteq \mathbb{J}_+(\boldsymbol{B})$ for distinct $\boldsymbol{A}$, $\boldsymbol{B} \in \mathcal{B}$, then $\boldsymbol{B}$ is redundant to describe $\mathbb{J}_+(\mathcal{B})$, *i.e.*, $\mathbb{J}_+(\mathcal{B}\backslash\{\boldsymbol{B}\}) = \mathbb{J}_+(\mathcal{B})$. Hence, we can remove such elements $\boldsymbol{B}$ from $\mathcal{B}$ one by one, recursively, to construct a reduced set $\mathcal{B}$. Then, the resulting $\mathcal{B}$ eventually satisfies condition (I). □

3.2.2 The infinite case of $\mathcal{B}$

We present three lemmas for the proof.

Lemma 3.2. *Let $\{\mathcal{B}^k \subseteq \mathbb{S}^n : k = 1, 2, \ldots\}$ be a sequence such that $\bigcap_{k=1}^\infty \mathbb{J}_+(\mathcal{B}^k) = \mathbb{J}$ for some closed convex cone $\mathbb{J} \subseteq \mathbb{S}^n_+$. Then $\bigcap_{m=1}^\infty co\big(\mathbf{\Gamma}^n \cap \big(\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k)\big)\big) = co\big(\mathbf{\Gamma}^n \cap \mathbb{J}\big)$.*

Proof. Since $\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k) \supseteq \mathbb{J}$ $(m = 1, 2, \ldots)$, $\bigcap_{m=1}^\infty \mathrm{co}\big(\mathbf{\Gamma}^n \cap \big(\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k)\big)\big) \supseteq \mathrm{co}\big(\mathbf{\Gamma}^n \cap \mathbb{J}\big)$ follows. To prove the converse inclusion, let $\overline{\boldsymbol{X}} \in \mathrm{co}\big(\mathbf{\Gamma}^n \cap \big(\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k)\big)\big)$ $(m = 1, 2, \ldots\ )$. Then, for each $m = 1, 2, \ldots$, there exist $\boldsymbol{X}^p_m \in \mathbf{\Gamma}^n \cap \big(\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k)\big) \subseteq \mathbb{S}^n_+$ $(p = 1, 2, \ldots, \ell)$ for some $\ell \leq \mathrm{dim}\mathbb{S}^n = n(n-1)/2$ such that $\overline{\boldsymbol{X}} = \sum_{p=1}^\ell \boldsymbol{X}^p_m$. Let $q \in \{1, 2, \ldots, \ell\}$ be fixed arbitrarily. Consider the sequence $\{\boldsymbol{X}^q_m \in \mathbb{S}^n_+ : m = 1, 2, \ldots, \}$. The sequence is bounded since $\overline{\boldsymbol{X}} \in \mathbb{S}^n_+$, $\boldsymbol{X}^p_m \in \mathbb{S}^n_+$ $(p = 1, \ldots, \ell)$ and $\langle \boldsymbol{I}, \overline{\boldsymbol{X}}\rangle = \langle \boldsymbol{I}, \sum_{p=1}^\ell \boldsymbol{X}^p_m\rangle \geq \langle \boldsymbol{I}, \boldsymbol{X}^q_m\rangle$. Hence the sequence admits a subsequence converging to some $\overline{\boldsymbol{X}}^q \in \mathbb{S}^n_+$. For notational simplicity, we relabel this subsequence as the sequence itself. Then, $\overline{\boldsymbol{X}}^q \in \mathrm{cl}\big(\bigcap_{k=1}^\infty \mathbb{J}_+(\mathcal{B}^k)\big) = \mathbb{J}$. Since $\mathbf{\Gamma}^n$ is closed, we also see $\overline{\boldsymbol{X}}^q \in \mathbf{\Gamma}^n$. Therefore, taking the limit of the identity $\overline{\boldsymbol{X}} = \sum_{p=1}^\ell \boldsymbol{X}^p_m$ as $m \to \infty$, we obtain that $\overline{\boldsymbol{X}} = \sum_{p=1}^\ell \overline{\boldsymbol{X}}^p$ and $\overline{\boldsymbol{X}}^p \in \mathbf{\Gamma}^n \cap \mathbb{J}$ $(p = 1, \ldots, \ell)$. Therefore, we have shown that $\overline{\boldsymbol{X}} \in \mathrm{co}\big(\mathbf{\Gamma}^n \cap \mathbb{J}\big)$ and $\bigcap_{m=1}^\infty \mathrm{co}\big(\mathbf{\Gamma}^n \cap \big(\bigcap_{k=1}^m \mathbb{J}_+(\mathcal{B}^k)\big)\big) \subseteq \mathrm{co}\big(\mathbf{\Gamma}^n \cap \mathbb{J}\big)$. □

We may assume without loss of generality that $\mathcal{B}$ is bounded since we can replace $\mathcal{B}$ with $\mathcal{B}' = \{\boldsymbol{B}/\parallel \boldsymbol{B} \parallel : \boldsymbol{O} \neq \boldsymbol{B} \in \mathcal{B}\}$ if $\mathcal{B}$ is unbounded, where $\parallel \boldsymbol{B} \parallel$ denotes the Frobenius norm of $\boldsymbol{B} \in \mathbb{S}^n$. For each $\epsilon > 0$, define an open neighborhood $U(\boldsymbol{B}, \epsilon) = \{\boldsymbol{A} \in \mathbb{S}^n : \parallel \boldsymbol{A} - \boldsymbol{B} \parallel < \epsilon\}$ of each $\boldsymbol{B} \in \mathrm{cl}\mathcal{B}$. Let $\{\epsilon_k\}$ be a sequence of positive numbers that converges to 0. Let k be fixed. Since $\mathcal{B}$ is bounded, $\mathrm{cl}\mathcal{B}$ is a compact in $\mathbb{S}^n$. Hence we can take a finite subset $\mathcal{B}^k \subseteq \mathcal{B}$ such that the union of all $U(\boldsymbol{B}, \epsilon_k)$ $(\boldsymbol{B} \in \mathcal{B}^k)$ covers $\mathcal{B}$. Then, the sequence $\{\mathcal{B}^k\ (k = 1, 2, \ldots)\}$ satisfies that

$$\mathcal{B}^k \subseteq \mathcal{B},\ \mathbb{J}_+(\mathcal{B}^k) \supseteq \mathbb{J}_+(\mathcal{B})\ (k = 1, 2, \ldots),\ \bigcap_{k=1}^\infty \mathbb{J}_+(\mathcal{B}^k) \supseteq \mathbb{J}_+(\mathcal{B}), \tag{13}$$

$$\forall \boldsymbol{B} \in \mathcal{B},\ \exists \boldsymbol{B}' \in \mathcal{B}^k ; \parallel \boldsymbol{B}' - \boldsymbol{B} \parallel < \epsilon_k\ (k = 1, 2, \ldots). \tag{14}$$

Lemma 3.3. *$\bigcap_{k=1}^\infty \mathbb{J}_+(\mathcal{B}^k) = \mathbb{J}_+(\mathcal{B})$.*

Proof. By (13), it suffices to show that $\bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}^k) \subseteq \mathbb{J}_+(\mathcal{B})$. Let $\overline{\boldsymbol{X}} \in \bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}^k)$. To prove $\overline{\boldsymbol{X}} \in \mathbb{J}_+(\mathcal{B})$, we show that $\langle \overline{\boldsymbol{B}}, \overline{\boldsymbol{X}} \rangle \geq 0$ for an arbitrarily chosen $\overline{\boldsymbol{B}} \in \mathcal{B}$. By (14), there exists a sequence $\{\boldsymbol{B}_k \in \mathcal{B}^k\}$ that converges $\overline{\boldsymbol{B}} \in \mathcal{B}$ as $k \to \infty$. Since $\overline{\boldsymbol{X}} \in \mathbb{J}_+(\mathcal{B}^k)$, we see that $\langle \boldsymbol{B}_k, \overline{\boldsymbol{X}} \rangle \geq 0$ $(k = 1, 2, \dots)$. Hence, we obtain $\langle \overline{\boldsymbol{B}}, \overline{\boldsymbol{X}} \rangle \geq 0$ by taking the limit as $k \to \infty$. □

Lemma 3.4. *Assume that $\mathcal{B}$ satisfies condition (B). Then* $co\big(\boldsymbol{\Gamma}^n \cap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}^k)\big)\big) = \bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}^k)$ $(m = 1, 2, \dots)$.

Proof. Let $m \in \{1, 2, \dots\}$ be fixed. We note that $\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}^k) = \mathbb{J}_+(\bigcup_{k=1}^{m} \mathcal{B}^k)$. Each $\bigcup_{k=1}^{m} \mathcal{B}^k$ satisfies condition (B) since it is a subset of $\mathcal{B}$. Since $\bigcup_{k=1}^{m} \mathcal{B}^k$ is finite, $\mathbb{J}_+(\bigcup_{k=1}^{m} \mathcal{B}^k) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$ as shown in Section 3.2.1. Therefore, $\mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}^k)\big)\big) = \bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}^k)$. □

Now, we show $\mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})\big) = \mathbb{J}_+(\mathcal{B})$, which is equivalent to $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$ by definition. By Lemmas 3.3 and 3.4, we see that

$$\mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}^k)\big)\big) = \bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}^k) \supseteq \bigcap_{k=1}^{\infty} \mathbb{J}_+(\mathcal{B}^k) = \mathbb{J}_+(\mathcal{B}) \ (m = 1, 2, \dots),$$

which implies that $\bigcap_{m=1}^{\infty} \mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}^k)\big)\big) \supseteq \mathbb{J}_+(\mathcal{B})$. By Lemma 3.2, $\bigcap_{m=1}^{\infty} \mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \big(\bigcap_{k=1}^{m} \mathbb{J}_+(\mathcal{B}^k)\big)\big) = \mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})\big)$. Therefore, we have shown that $\mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})\big) \supseteq \mathbb{J}_+(\mathcal{B})$. The converse inclusion $\mathrm{co}\big(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B})\big) \subseteq \mathbb{J}_+(\mathcal{B})$ is straightforward since $\mathbb{J}_+(\mathcal{B})$ is convex. □

4 On Theorems 1.5 and 1.6

4.1 Facial reduction of $\mathcal{B}_{\geq} \cap \boldsymbol{L}_1(\mathbb{F})$ into $\mathbb{R}^{r-1}$

To adapt the argument in Section 3.1 to QCQP (5) (*i.e.*, $\mathrm{COP}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1)$ with $\boldsymbol{H}^1 = \mathrm{diag}(0, \dots, 0, 1) \in \mathbb{S}^n_+$), its SDP relaxation, conditions (C) and (D), we need some additional arguments. First, we assume that the feasible region $\mathcal{B}_{\geq} \cap L_1(\mathbb{F})$ of QCQP (5) is nonempty, as required in condition (C). Consequently, the feasible region of $\mathrm{COP}(\boldsymbol{\Gamma}^r \cap \mathbb{J}_+(\widetilde{\mathcal{B}}), \widetilde{\boldsymbol{Q}}, \widetilde{\boldsymbol{H}^1})$, QCQP (11) with $\widetilde{\boldsymbol{H}} = \widetilde{\boldsymbol{H}^1}$ is also nonempty. For any choice of a nonsingular $\boldsymbol{M}$, the rank of $\widetilde{\boldsymbol{H}^1} = \Phi^*(\boldsymbol{H}^1) = \boldsymbol{M}^{-1}\boldsymbol{P}^T_{1.r}\boldsymbol{H}^1\boldsymbol{P}_{1.r}\boldsymbol{M}^{-T} \in \mathbb{S}^r_+$ is 1, otherwise the rank is 0 or $\widetilde{\boldsymbol{H}^1} = \boldsymbol{O} \in \mathbb{S}^r_+$; hence $\mathrm{COP}(\boldsymbol{\Gamma}^r \cap \mathbb{J}_+(\widetilde{\mathcal{B}}), \widetilde{\boldsymbol{Q}}, \widetilde{\boldsymbol{H}^1})$ (11) is infeasible. Specifically, we can take a nonsingular matrix $\boldsymbol{M}$ such that $\widetilde{\boldsymbol{H}^1} = \boldsymbol{M}^{-1}\boldsymbol{P}^T_{1.r}\boldsymbol{H}^1\boldsymbol{P}_{1.r}\boldsymbol{M}^{-T} = \mathrm{diag}(0, \dots, 0, 1) \in \mathbb{S}^r_+$. In this case, if $\boldsymbol{u} \in L_1(\mathbb{F})$, then $\binom{\boldsymbol{u}}{1} \in L(\mathbb{F})$ and $\langle \widetilde{\boldsymbol{H}^1}, \theta\big(\binom{\boldsymbol{u}}{1}\big)\theta\big(\binom{\boldsymbol{u}}{1}\big)^T \rangle = \langle \boldsymbol{H}^1, \binom{\boldsymbol{u}}{1}\binom{\boldsymbol{u}}{1}^T \rangle = 1$ hold by Lemma 3.1 (iv). Since $\theta : L(\mathbb{F}) \to \mathbb{R}^{r-1}$ is linear, we have either $\theta\big(\binom{\boldsymbol{u}}{1}\big)_r = 1$ for every $\boldsymbol{u} \in L(\mathbb{F})$ or $\theta\big(\binom{\boldsymbol{u}}{1}\big)_r = -1$ for every $\boldsymbol{u} \in L(\mathbb{F})$. Define $\theta_1 : L_1(\mathbb{F}) \to \mathbb{R}^{r-1}$ by

$$\theta_1(\boldsymbol{u}) = \begin{cases} \Big(\theta\big(\binom{\boldsymbol{u}}{1}\big)_1, \dots, \theta\big(\binom{\boldsymbol{u}}{1}\big)_{r-1}\Big)^T & \text{in the former case,} \\ \Big(-\theta\big(\binom{\boldsymbol{u}}{1}\big)_1, \dots, -\theta\big(\binom{\boldsymbol{u}}{1}\big)_{r-1}\Big)^T & \text{in the latter case.} \end{cases}$$

Then,

$$q(\boldsymbol{u},\boldsymbol{B}) = \langle \boldsymbol{B},\, \binom{\boldsymbol{u}}{1}\binom{\boldsymbol{u}}{1}^T \rangle = \langle \widetilde{\boldsymbol{B}},\, \theta\Big(\binom{\boldsymbol{u}}{1}\Big)\theta\Big(\binom{\boldsymbol{u}}{1}\Big)^T \rangle = \langle \widetilde{\boldsymbol{B}},\, \binom{\theta_1(\boldsymbol{u})}{1}\binom{\theta_1(\boldsymbol{u})}{1}^T \rangle$$

for every $\boldsymbol{u} \in L_1(\mathbb{F})$ and $\boldsymbol{B} \in \mathbb{S}^n$. Now, for every $\boldsymbol{w} \in \mathbb{R}^{r-1}$ and $\widetilde{\boldsymbol{B}} \in \mathbb{S}^r$, we define

$$\begin{aligned}
& q(\boldsymbol{w},\widetilde{\boldsymbol{B}}) = \binom{\boldsymbol{w}}{1}^T \widetilde{\boldsymbol{B}} \binom{\boldsymbol{w}}{1}, \\
& \widetilde{\boldsymbol{B}}_{\geq},\ \widetilde{\boldsymbol{B}}_{=} \text{ or } \widetilde{\boldsymbol{B}}_{\leq} = \left\{ \boldsymbol{w} \in \mathbb{R}^{r-1} : q(\boldsymbol{w},\widetilde{\boldsymbol{B}}) \geq 0,\ =\ \text{ or } \ \leq 0,\ \text{resp.} \right\}, \\
& \widetilde{\mathcal{B}}_{\geq} = \textstyle\bigcap_{\widetilde{\boldsymbol{B}} \in \widetilde{\mathcal{B}}} \widetilde{\boldsymbol{B}}_{\geq} = \left\{ \boldsymbol{w} \in \mathbb{R}^{r-1} : q(\boldsymbol{w},\widetilde{\boldsymbol{B}}) \geq 0 \ (\widetilde{\boldsymbol{B}} \in \widetilde{\mathcal{B}}) \right\}.
\end{aligned}$$

Using the notation above, QCQP (5) is transformed into COP$(\boldsymbol{\Gamma}^r \cap \mathbb{J}_+(\widetilde{\mathcal{B}}), \widetilde{\boldsymbol{Q}}, \widetilde{\boldsymbol{H}^1})$ with $\widetilde{\boldsymbol{H}^1} = \text{diag}(0,\ldots,0,1) \in \mathbb{S}^r_+$:

$$\begin{aligned}
\eta(\boldsymbol{\Gamma}^r \cap \mathbb{J}_+(\widetilde{\mathcal{B}}), \widetilde{\boldsymbol{Q}}, \widetilde{\boldsymbol{H}^1}) &= \inf\{q(\boldsymbol{w},\widetilde{\boldsymbol{Q}}) : q(\boldsymbol{w},\widetilde{\boldsymbol{B}}) \geq 0 \ (\widetilde{\boldsymbol{B}} \in \widetilde{\mathcal{B}})\} \\
&= \inf\{q(\boldsymbol{w},\widetilde{\boldsymbol{Q}}) : \boldsymbol{w} \in \widetilde{\mathcal{B}}_{\geq}\}, \qquad (15)
\end{aligned}$$

and conditions (C) and (D) into

- ($\widetilde{\text{C}}$) $\widetilde{\mathcal{B}}_{\geq} \neq \emptyset$, and $\emptyset \neq \widetilde{\boldsymbol{B}}_{\leq} \subseteq \widetilde{\boldsymbol{A}}_{\geq}$ or $\widetilde{\boldsymbol{A}}_{\geq} \subseteq \widetilde{\boldsymbol{B}}_{\geq}$ holds for every $\widetilde{\boldsymbol{A}},\ \widetilde{\boldsymbol{B}} \in \widetilde{\mathcal{B}}$.
- ($\widetilde{\text{D}}$) $\mathcal{B}$ is finite. $q(\cdot,\widetilde{\boldsymbol{B}}) : \mathbb{R}^{r-1} \to \mathbb{R}$ $(\widetilde{\boldsymbol{B}} \in \widetilde{\mathcal{B}})$ is not affine and $\widetilde{\boldsymbol{B}}_{=} \subseteq \widetilde{\boldsymbol{A}}_{\geq}$ for every $\widetilde{\boldsymbol{A}},\ \widetilde{\boldsymbol{B}} \in \widetilde{\mathcal{B}}$.

Lemma 4.1.

(i) $\theta_1 : L_1(\mathbb{F}) \to \mathbb{R}^{r-1}$ *is one-to-one and onto.*

(ii) $q(\boldsymbol{u},\boldsymbol{A}) = q(\theta_1(\boldsymbol{u}),\widetilde{\boldsymbol{A}})$ *for every* $\boldsymbol{u} \in L_1(\mathbb{F})$ *and* $\boldsymbol{A} \in \mathbb{S}^n$.

(iii) $\theta_1(\boldsymbol{B}_{\geq} \cap L_1(\mathbb{F})) = \widetilde{\boldsymbol{B}}_{\geq}$, $\theta_1(\boldsymbol{B}_{=} \cap L_1(\mathbb{F})) = \widetilde{\boldsymbol{B}}_{=}$ *and* $\theta_1(\boldsymbol{B}_{\leq} \cap L_1(\mathbb{F})) = \widetilde{\boldsymbol{B}}_{\leq}$.

Proof. (i) This assertion follows from Lemma 3.1 (ii) and the definition of $\theta_1 : L_1(\mathbb{F}) \to \mathbb{R}^{r-1}$.

(ii) Let $\boldsymbol{u} \in L_1(\mathbb{F})$ and $\boldsymbol{A} \in \mathbb{S}^n$ Then,

$$\begin{aligned}
q(\boldsymbol{u},\boldsymbol{A}) &= \langle \boldsymbol{A},\, \binom{\boldsymbol{u}}{1}\binom{\boldsymbol{u}}{1}^T \rangle \\
&= \langle \widetilde{\boldsymbol{A}},\, \theta\Big(\binom{\boldsymbol{u}}{1}\Big)\theta\Big(\binom{\boldsymbol{u}}{1}\Big)^T \rangle \text{ (by Theorem 3.1 (iv))} \\
&= \langle \widetilde{\boldsymbol{A}},\, \binom{\theta_1(\boldsymbol{u})}{1}\binom{\theta_1(\boldsymbol{u})}{1}^T \rangle \ = \ q(\widetilde{\boldsymbol{A}},\theta_1(\boldsymbol{u})).
\end{aligned}$$

(iii) By (i) and (ii), we see that

$$\begin{aligned}
\theta_1(\boldsymbol{B}_{\geq} \cap L_1(\mathbb{F})) &= \{\theta_1(\boldsymbol{u}) : q(\boldsymbol{u},\boldsymbol{B}) \geq 0,\ \boldsymbol{u} \in L_1(\mathbb{F})\} \\
&= \left\{\theta_1(\boldsymbol{u}) : q(\theta_1(\boldsymbol{u}),\widetilde{\boldsymbol{B}}) \geq 0,\ \theta_1(\boldsymbol{u}) \in \theta_1(L_1(\mathbb{F}))\right\} \\
&= \left\{\boldsymbol{w} \in \mathbb{R}^{r-1} : q(\boldsymbol{w},\widetilde{\boldsymbol{B}}) \geq 0\right\} = \widetilde{\boldsymbol{B}}_{\geq}.
\end{aligned}$$

The second and third identity can be proved similarly. □

As a result of Lemma 4.1, we obtain that:

[e] $\boldsymbol{u} \in \mathbb{R}^{n-1}$ is a feasible solution of QCQP (5) with the objective value $q(\boldsymbol{u}, \boldsymbol{Q})$ if and only if $\boldsymbol{w} = \theta_1(\boldsymbol{u})$ is a feasible solution of QCQP (15) with the objective value $q(\boldsymbol{w}, \widetilde{\boldsymbol{Q}}) = q(\boldsymbol{u}, \boldsymbol{Q})$.

[f] Condition (C) holds if and only if condition ($\widetilde{\text{C}}$) holds.

[g] Condition (D) holds if and only if condition ($\widetilde{\text{D}}$) holds.

4.2 Proof of Theorem 1.5

We have shown in [c] of Section 3.1 and [f] above that conditions (B) and (C) are equivalent to conditions ($\widetilde{\text{B}}$) and ($\widetilde{\text{C}}$), respectively. Hence, for Theorem 1.5, it is sufficient to prove that condition ($\widetilde{\text{C}}$) implies condition ($\widetilde{\text{B}}$). For simplicity of notation, we omit $\tilde{}$ from $\widetilde{\boldsymbol{B}}$ ($\boldsymbol{B} \in \mathcal{B}$), $\widetilde{\mathcal{B}}$, $\widetilde{\boldsymbol{Q}}$ and $\widetilde{\boldsymbol{H}^1}$, or equivalently, we prove that condition (C) implies condition (B) under the assumption that $\mathbb{F} = \mathbb{S}^n_+$, $L(\mathbb{F}) = \mathbb{R}^r$ and $L_1(\mathbb{F}) = \mathbb{R}^{r-1}$.

Lemma 4.2. *Let $\boldsymbol{A}$, $\boldsymbol{B} \in \mathbb{S}^n$. Then*

(i) $\mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}) \Rightarrow \mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$.

(ii) $\mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}) \Leftrightarrow \boldsymbol{\Gamma}^n \cap \mathbb{J}_-(\boldsymbol{B}) \subseteq \boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\boldsymbol{A})$

$$\Updownarrow$$

$$\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{B}, \boldsymbol{x}\boldsymbol{x}^T \rangle \leq 0\} \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{A}, \boldsymbol{x}\boldsymbol{x}^T \rangle \geq 0\} \tag{16}$$

Proof. (i) is obvious since $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_-(\boldsymbol{B})$. (ii) For the first $\Leftrightarrow$, $\Rightarrow$ is straightforward. Conversely, if $\boldsymbol{\Gamma}^n \cap \mathbb{J}_-(\boldsymbol{B}) \subseteq \boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\boldsymbol{A})$, then

$$\mathbb{J}_-(\boldsymbol{B}) = \text{co}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_-(\boldsymbol{B})) \subseteq \text{co}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\boldsymbol{A})) = \mathbb{J}_+(\boldsymbol{A})$$

follows by Theorem 1.2. Hence $\Rightarrow$ follows. The second $\Updownarrow$ is straightforward. □

Lemma 4.3. *Let $\boldsymbol{A}$, $\boldsymbol{B} \in \mathbb{S}^n$. Assume that $\emptyset \neq \boldsymbol{B}_{\leq} \subseteq \boldsymbol{A}_{\geq}$. Then $\mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$.*

Proof. By Lemma 4.2 (ii), it suffices to prove (16) under the assumption $\emptyset \neq \boldsymbol{B}_{\leq} \subseteq \boldsymbol{A}_{\geq}$. For every subset $D \subseteq \mathbb{R}^{n-1}$, define

$$\mathbb{H}(D) = \left\{ \begin{pmatrix} u \\ \xi \end{pmatrix} \in \mathbb{R}^n : \ u \in \mathbb{R}^{n-1}, \ \xi > 0, \ u/\xi \in D \right\}.$$

Since $\boldsymbol{B}_{\leq} \neq \emptyset$ and $\boldsymbol{A}_{\geq} \neq \emptyset$, we may apply the homogenization identity in [35, Lemma 2] to obtain

$$\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{B}, \boldsymbol{x}\boldsymbol{x}^T \rangle \leq 0\} = \mathbb{H}(\boldsymbol{B}_{\leq}) \cup \mathbb{H}(-\boldsymbol{B}_{\leq}),$$
$$\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{A}, \boldsymbol{x}\boldsymbol{x}^T \rangle \geq 0\} = \mathbb{H}(\boldsymbol{A}_{\geq}) \cup \mathbb{H}(-\boldsymbol{A}_{\geq}).$$

Since $\boldsymbol{B}_{\leq} \subseteq \boldsymbol{A}_{\geq}$, we obtain $\mathbb{H}(\boldsymbol{B}_{\leq}) \subseteq \mathbb{H}(\boldsymbol{A}_{\geq})$ and $\mathbb{H}(-\boldsymbol{B}_{\leq}) \subseteq \mathbb{H}(-\boldsymbol{A}_{\geq})$. Therefore,

$$\mathbb{H}(\boldsymbol{B}_{\leq}) \cup \mathbb{H}(-\boldsymbol{B}_{\leq}) \subseteq \mathbb{H}(\boldsymbol{A}_{\geq}) \cup \mathbb{H}(-\boldsymbol{A}_{\geq}),$$

which implies (16). □

Remark 4.4. The following example shows the necessity of the assumption $\emptyset \neq \boldsymbol{B}_{\leq}$ in Lemma 4.3. Let $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $\boldsymbol{B}_{\leq} = \{(u_1, u_2) : u_1^2 + 1 \leq 0\} = \emptyset \subseteq \boldsymbol{A}_{\geq} = \{(u_1, u_2) : -u_2^2 \geq 0\} = \{(u_1, 0)\}$. But $\mathbb{J}_-(\boldsymbol{B}) \ni \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \not\in \mathbb{J}_+(\boldsymbol{A})$; hence $\mathbb{J}_-(\boldsymbol{B}) \not\subseteq \mathbb{J}_+(\boldsymbol{A})$.

Now we are ready to prove Theorem 1.5. By Lemmas 4.3, we see that for every $\boldsymbol{A}$, $\boldsymbol{B} \in \mathbb{S}^n_+$

$$\begin{aligned} \emptyset \neq \boldsymbol{B}_{\leq} \subseteq \boldsymbol{A}_{\geq} \quad &\Rightarrow \quad \mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}), \\ \emptyset \neq \boldsymbol{A}_{\geq} = (-\boldsymbol{A})_{\leq} \subseteq \boldsymbol{B}_{\geq} \quad &\Rightarrow \quad \mathbb{J}_+(\boldsymbol{A}) = \mathbb{J}_-(-\boldsymbol{A}) \subseteq \mathbb{J}_+(\boldsymbol{B}). \end{aligned}$$

(Note that the assumption $\mathcal{B}_{\geq} \neq \emptyset$ of condition (C) with $\mathbb{F} = \mathbb{S}^n_+$ implies $\boldsymbol{A}_{\geq} \neq \emptyset$ for every $\boldsymbol{A} \in \mathcal{B}$.) Therefore, condition (C) implies condition (B). □

4.3 Proof of Theorem 1.6

We have shown in [g] that condition (D) is equivalent to condition ($\widetilde{\mathrm{D}}$). By Theorem 1.3, we obtain

$$\begin{aligned} &\mathbb{J}_+(\widetilde{\mathcal{B}}) \cap \widetilde{\mathbb{H}^1} = \overline{\mathrm{co}}(\boldsymbol{\Gamma}^r \cap \mathbb{J}_+(\widetilde{\mathcal{B}}) \cap \widetilde{\mathbb{H}^1}), \\ &\eta(\mathbb{J}_+(\widetilde{\mathcal{B}}), \widetilde{\boldsymbol{Q}}, \widetilde{\boldsymbol{H}^1}) = \eta(\boldsymbol{\Gamma}^r \cap \mathbb{J}_+(\widetilde{\mathcal{B}}), \widetilde{\boldsymbol{Q}}, \widetilde{\boldsymbol{H}^1}), \end{aligned}$$

which are equivalent to (6) and (7), where $\widetilde{\mathbb{H}^1} = \{\boldsymbol{V} \in \mathbb{S}^r : \langle \widetilde{\boldsymbol{H}^1}, \boldsymbol{V} \rangle = 1\}$. □

5 Relationships among conditions (I), (II), (III), (B), (C), and (D)

Throughout this section, we assume $\mathbb{F} = \mathbb{F}_{\min}$ in conditions (B), (C) and (D). We have already observed that (I) ⇒ (B), (III) ⇒ (D) and (C) ⇒ (B) (Theorem 1.5). To prove that condition (B) is the weakest of these conditions, in the sense it is implied by all the other conditions, we show (II) ⇒ (B) in Section 5.1, and (D) ⇒ (B) in Section 5.2. Under appropriate assumptions, equivalence of conditions (I), (II), (B) and (C) are shown in Section 5.3.

5.1 Condition (II) ⇒ condition (B)

Assume that condition (II) is satisfied. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}$ and $\boldsymbol{A} \neq \boldsymbol{B}$. Then, there exists an $(\alpha, \beta) \neq \boldsymbol{0}$ such that $\alpha\boldsymbol{A} + \beta\boldsymbol{B} \in \mathbb{S}^n_+$, which implies

$$\langle \alpha\boldsymbol{A} + \beta\boldsymbol{B}, \boldsymbol{X} \rangle \geq 0 \text{ for every } \boldsymbol{X} \in \mathbb{S}^n_+. \tag{17}$$

We will show that $\mathbb{J}_0(\boldsymbol{B}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\boldsymbol{A}) \cap \mathbb{F}$ or $\mathbb{J}_+(\boldsymbol{A}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\boldsymbol{B}) \cap \mathbb{F}$ holds. One of the following cases (a),(b),...,(f) occurs.

(a) $\alpha > 0$: If $\boldsymbol{X} \in \mathbb{J}_0(\boldsymbol{B}) \cap \mathbb{F}$, then we see from (17) that $\boldsymbol{X} \in \mathbb{J}_+(\boldsymbol{A}) \cap \mathbb{F}$. Hence, $\mathbb{J}_0(\boldsymbol{B}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\boldsymbol{A}) \cap \mathbb{F}$ holds.

(b) $\alpha = 0$ and $\beta > 0$: By (17), $\mathbb{J}_+(\boldsymbol{A}) \cap \mathbb{F} \subseteq \mathbb{F} = \mathbb{J}_+(\boldsymbol{B}) \cap \mathbb{F}$ holds.

(c) $\alpha = 0$ and $\beta < 0$: By (17), $\langle -\boldsymbol{B}, \boldsymbol{X} \rangle \geq 0$ holds for every $\boldsymbol{X} \in \mathbb{S}^n_+$, which implies $-\boldsymbol{B} \in \mathbb{S}^n_+$ and $\mathbb{J}_+(\boldsymbol{B})$ forms a face of $\mathbb{S}^n_+$. If $\mathbb{J}_+(\boldsymbol{B}) \cap \mathbb{F} = \mathbb{F}$, then $\mathbb{J}_+(\boldsymbol{A}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\boldsymbol{B}) \cap \mathbb{F}$ holds. Otherwise, $\mathbb{J}_+(\mathcal{B}) \cap \mathbb{F}$ is contained in the proper face $\mathbb{J}_+(\boldsymbol{B}) \cap \mathbb{F}$ of $\mathbb{F} = \mathbb{F}_{\min}$, which contradicts the definition of $\mathbb{F}_{\min}$.

(d) $\alpha < 0$ and $\beta > 0$: Then $\boldsymbol{B} = (-\alpha/\beta)\boldsymbol{A} + \boldsymbol{Y}$ for some $\boldsymbol{Y} \in \mathbb{S}^n_+$, which implies $\mathbb{J}_+(\boldsymbol{A}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\boldsymbol{B}) \cap \mathbb{F}$.

(e) $\alpha < 0$, $\beta = 0$. This case can be treated similarly to case (c) to show that $\mathbb{J}_+(\boldsymbol{A}) \cap \mathbb{F}$ forms a face of $\mathbb{F} = \mathbb{F}_{\min}$.

(f) $\alpha < 0$, $\beta < 0$. In this case, we observe that

$$\begin{aligned} \mathbb{J}_+(\mathcal{B}) &\subseteq \{\boldsymbol{X} \in \mathbb{F} : \langle \boldsymbol{A}, \boldsymbol{X} \rangle \geq 0, \ \langle \boldsymbol{B}, \boldsymbol{X} \rangle \geq 0\} \\ &= \{\boldsymbol{X} \in \mathbb{F} : \langle \boldsymbol{A}, \boldsymbol{X} \rangle \geq 0, \ \langle \boldsymbol{B}, \boldsymbol{X} \rangle \geq 0, \ -\langle \alpha\boldsymbol{A} + \beta\boldsymbol{B}, \boldsymbol{X} \rangle \geq 0\} \\ &\quad \text{(since } \alpha < 0 \text{ and } \beta < 0\text{)} \\ &\subseteq \{\boldsymbol{X} \in \mathbb{F} : \langle -\alpha\boldsymbol{A} - \beta\boldsymbol{B}, \boldsymbol{X} \rangle = 0\} \ \text{(since } \alpha\boldsymbol{A} + \beta\boldsymbol{B} \in \mathbb{S}^n_+\text{)}. \end{aligned}$$

Hence, $\mathbb{J}_+(\mathcal{B})$ is included in a face $\mathbb{F}' \equiv \{\boldsymbol{X} \in \mathbb{F} : \langle -\alpha\boldsymbol{A} - \beta\boldsymbol{B}, \boldsymbol{X} \rangle = 0\}$ of $\mathbb{F}$. If $\mathbb{F}' = \mathbb{F}$ then

$$\mathbb{J}_0(\boldsymbol{B}) \cap \mathbb{F} = \mathbb{J}_0(\boldsymbol{B}) \cap \mathbb{F}' = \mathbb{J}_0(\boldsymbol{A}) \cap \mathbb{F}' = \mathbb{J}_0(\boldsymbol{A}) \cap \mathbb{F} \subseteq \mathbb{J}_+(\boldsymbol{A}) \cap \mathbb{F}.$$

Otherwise, $\mathbb{F}'$ is a proper face of $\mathbb{F} = \mathbb{F}_{\min}$ that contains $\mathbb{J}_+(\mathcal{B})$, a contradiction to the definition of $\mathbb{F}_{\min}$.

□

5.2 Condition (D) ⇒ condition (B)

We have shown the equivalence of conditions (B) and ($\widetilde{\text{B}}$) in Section 3.1 [c] and the equivalence of (D) and ($\widetilde{\text{D}}$) in Section 4.1 [g]. Therefore, it suffices to show ($\widetilde{\text{D}}$) ⇒ ($\widetilde{\text{B}}$) for (D) ⇒ (B). Since we take $\mathbb{F} = \mathbb{F}_{\min}$, $\mathbb{J}_+(\widetilde{\mathcal{B}}) \cap \mathbb{S}^r_{++} \neq \emptyset$ (Slater's constraint qualification for SDP (12)). For simplicity of notation, we omit $\widetilde{\ }$ and assume that $\mathbb{J}_+(\mathcal{B})$ itself satisfies condition

(A-1) $\mathbb{J}_+(\mathcal{B}) \cap \mathbb{S}^n_{++} \neq \emptyset$.

throughout this and the next sections.

For the proof of (D) ⇒ (B), we fix arbitrary $\boldsymbol{A}, \ \boldsymbol{B} \in \mathcal{B}$. The case $\boldsymbol{B} \in \mathbb{S}^n_+$ is immediate. Indeed, if $\boldsymbol{B} \in \mathbb{S}^n_+$, then $\langle \boldsymbol{B}, \boldsymbol{X} \rangle \geq 0$ for every $\boldsymbol{X} \in \mathbb{S}^n_+$, and hence $\mathbb{J}_+(\boldsymbol{B}) = \mathbb{S}^n_+$. Therefore $\mathbb{J}_+(\boldsymbol{A}) \subseteq \mathbb{J}_+(\boldsymbol{B})$, and condition (B) holds. Thus we assume $\boldsymbol{B} \notin \mathbb{S}^n_+$. Let us write $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{c} \\ \boldsymbol{c}^T & \gamma \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{D} & \boldsymbol{d} \\ \boldsymbol{d}^T & \delta \end{pmatrix}$, where $\boldsymbol{C}, \boldsymbol{D} \in \mathbb{S}^{n-1}$, $\boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^{n-1}$, and $\gamma, \delta \in \mathbb{R}$. We show that $\boldsymbol{B}_= \subseteq \boldsymbol{A}_\geq \Rightarrow \mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$ under the assumptions that

$$\mathbb{J}_+(\boldsymbol{B}) \cap \mathbb{S}^n_{++} \neq \emptyset, \ \boldsymbol{B} \notin \mathbb{S}^n_+, \ \text{and } \boldsymbol{D} \neq \boldsymbol{O}. \tag{18}$$

Define the quadratic functions h and f on $\mathbb{R}^{n-1}$ as

$$h(\boldsymbol{u}) = \boldsymbol{u}^T\boldsymbol{D}\boldsymbol{u} + 2\boldsymbol{d}^T\boldsymbol{u} + \delta, \ f(\boldsymbol{u}) = \boldsymbol{u}^T\boldsymbol{C}\boldsymbol{u} + 2\boldsymbol{c}^T\boldsymbol{u} + \gamma \text{ for every } \boldsymbol{u} \in \mathbb{R}^{n-1}.$$

Lemma 5.1.

(i) The quadratic function $h : \mathbb{R}^{n-1} \to \mathbb{R}$ takes both negative and positive values; $h(\boldsymbol{u}^1) < 0 < h(\boldsymbol{u}^2)$ for some $\boldsymbol{u}^1,\ \boldsymbol{u}^2 \in \mathbb{R}^{n-1}$.

(ii) The following two conditions are equivalent.

E_1: $h(\boldsymbol{u}) = 0 \Rightarrow f(\boldsymbol{u}) \geq 0$.

E_2: There exists $\tau \in \mathbb{R}$ such that $f(\boldsymbol{u}) + \tau h(\boldsymbol{u})$ for every $\boldsymbol{u} \in \mathbb{R}^{n-1}$.

Proof. (i) The existence of $\boldsymbol{u}^1$ follows from $\boldsymbol{B} \notin \mathbb{S}^n_+$, and that of $\boldsymbol{u}^2$ follows from $\mathbb{J}_+(\boldsymbol{B}) \cap \mathbb{S}^n_{++} \neq \emptyset$. (ii) Under the assumptions (i) and $\boldsymbol{D} \neq \boldsymbol{O}$, the equivalence of E_1 and E_2 follows from [37], which established the so-called S-Lemma with equality, and provided the equivalence of E_1 and E_2 under appropriate assumptions. In particular, Theorem 3 of [37] stated a necessary and sufficient condition for the equivalence when (i) holds, where $\boldsymbol{D} \neq \boldsymbol{O}$ was shown to be sufficient for the equivalence. □

Obviously, $\boldsymbol{B}_= \subseteq \boldsymbol{A}_\geq$ is equivalent to E_1. Hence, by Lemma 5.1, there exists a $\tau \in \mathbb{R}$ such that $f(\boldsymbol{u}) + \tau h(\boldsymbol{u})$ for every $\boldsymbol{u} \in \mathbb{R}^{n-1}$. By [24, Lemma 3.3], this is equivalent to $\boldsymbol{A} + \tau\boldsymbol{B} \in \mathbb{S}^n_+$, that is, $\langle \boldsymbol{A} + \tau\boldsymbol{B},\, \boldsymbol{X}\rangle \geq 0$ for every $\boldsymbol{X} \in \mathbb{S}^n_+$. Consequently, if $\langle \boldsymbol{B},\, \boldsymbol{X}\rangle = 0$ with $\boldsymbol{X} \in \mathbb{S}^n_+$, then $\langle \boldsymbol{A},\, \boldsymbol{X}\rangle \geq 0$. This shows that $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$. □

5.3 Equivalence of conditions (I), (II), (B), and (C)

Throughout this section, we allow $\mathcal{B} \subseteq \mathbb{S}^n$ to be infinite in conditions (I) and (II) as in condition (B) even though conditions (I) and (II) are originally stated for finite $\mathcal{B} \subseteq \mathbb{S}^n$. In addition to $\mathbb{F} = \mathbb{S}^n_+$ and condition (A-1) assumed in the previous section, we assume conditions

(A-2) $\mathbb{J}_+(\boldsymbol{B}) \not\subseteq \mathbb{J}_+(\boldsymbol{A})$ for every distinct $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}$.

(A-3) $\boldsymbol{B}_\leq \neq \emptyset$ for every $\boldsymbol{B} \in \mathcal{B}$.

Under these conditions, we show the equivalence of conditions (I), (II), (B), and (C). In the case where $\mathcal{B}$ is finite, if $\mathbb{J}_+(\boldsymbol{A}) \subseteq \mathbb{J}_+(\boldsymbol{B})$ for distinct $\boldsymbol{A},\ \boldsymbol{B} \in \mathcal{B}$, we can remove $\boldsymbol{B} \in \mathcal{B}$ from $\mathcal{B}$ one by one, recursively, so that $\mathbb{J}_+(\mathcal{B}) = \mathbb{J}_+(\mathcal{B}\backslash\{\boldsymbol{B}\})$. Therefore, condition (A-2) is attained. When $\mathcal{B}$ is infinite, we need Zorn's lemma (see, for example, [16, 17]) to remove such $\boldsymbol{B}$'s consistently from $\mathcal{B}$ so that the resulting $\mathcal{B}$ satisfies condition (A-2). The details are omitted here. Condition (A-3) is also reasonable since if $\boldsymbol{B}_\leq = \emptyset$ then $\boldsymbol{B}_\geq = \mathbb{R}^{n-1}$ and $\mathcal{B}_\geq = \{\mathcal{B}\backslash\{\boldsymbol{B}\}\}_\geq$.

The equivalence of (I) and (B) is obvious under (A-2). We have already seen that (II) $\Rightarrow$ (B) in Section 5.1 and that (C) $\Rightarrow$ (B) in Theorem 1.5. To prove (B) $\Rightarrow$ (II) and (B) $\Rightarrow$ (C), it suffices to prove the following lemma, which also shows that (B) implies (9) included in (D).

Lemma 5.2. *Assume that $\mathcal{B} \subseteq \mathbb{S}^n$ satisfies conditions (A-1) and (A-2). Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}$ and $\boldsymbol{A} \neq \boldsymbol{B}$. Then*

$$\alpha\boldsymbol{A} + \beta\boldsymbol{B} \in \mathbb{S}^n_+ \text{ for some } (\alpha, \beta) \neq \boldsymbol{0}. \tag{19}$$

$$\Uparrow$$

$$\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}) \ \Leftrightarrow\ \mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}) \tag{20}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\boldsymbol{B}_= \subseteq \boldsymbol{A}_\geq \qquad\qquad \boldsymbol{B}_\leq \subseteq \boldsymbol{A}_\geq.$$

Proof. (i) In (20), $\Leftarrow$ is straightforward since $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_-(\boldsymbol{B})$. To prove $\Rightarrow$, assume on the contrary that $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$ but $\mathbb{J}_-(\boldsymbol{B}) \not\subseteq \mathbb{J}_+(\boldsymbol{A})$ or equivalently that $\langle \boldsymbol{B}, \overline{\boldsymbol{X}} \rangle < 0$ and $\langle \boldsymbol{A}, \overline{\boldsymbol{X}} \rangle < 0$ for some $\overline{\boldsymbol{X}} \in \mathbb{S}^n_+$. By condition (A-2), $\mathbb{J}_+(\boldsymbol{B}) \not\subseteq \mathbb{J}_+(\boldsymbol{A})$, which together with $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$ implies $\langle \boldsymbol{B}, \widetilde{\boldsymbol{X}} \rangle > 0$ and $\langle \boldsymbol{A}, \widetilde{\boldsymbol{X}} \rangle < 0$ for some $\widetilde{\boldsymbol{X}} \in \mathbb{S}^n_+$. Hence there exists $\lambda \in (0, 1)$ such that

$$\langle \boldsymbol{B}, \lambda\overline{\boldsymbol{X}} + (1-\lambda)\widetilde{\boldsymbol{X}} \rangle = 0, \ \langle \boldsymbol{A}, \lambda\overline{\boldsymbol{X}} + (1-\lambda)\widetilde{\boldsymbol{X}} \rangle < 0, \ \lambda\overline{\boldsymbol{X}} + (1-\lambda)\widetilde{\boldsymbol{X}} \in \mathbb{S}^n_+.$$

This contradicts $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$.

(ii) $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}) \ \Rightarrow \ $(19): Consider the primal-dual pair of SDPs

$$\begin{aligned} \zeta_\mathrm{p} &= \inf\{\langle \boldsymbol{A}, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{S}^n_+, \ \langle \boldsymbol{B}, \boldsymbol{X} \rangle = 0\} = \inf\{\langle \boldsymbol{A}, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{J}_0(\boldsymbol{B})\}, \\ \zeta_\mathrm{d} &= \sup\{0 : \boldsymbol{A} + \tau\boldsymbol{B} \in \mathbb{S}^n_+, \ \tau \in \mathbb{R}\}. \end{aligned} \quad (21)$$

Obviously, $\zeta_\mathrm{p} = 0$ if and only if $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A})$. By (A-1), $-\boldsymbol{B} \notin \mathbb{S}^n_+$, which implies $\langle \boldsymbol{B}, \boldsymbol{X}^1 \rangle > 0$ for some $\boldsymbol{X}^1 \in \mathbb{S}^n_{++}$. By (A-2), $\boldsymbol{B} \notin \mathbb{S}^n_+$, which implies $\langle \boldsymbol{B}, \boldsymbol{X}^2 \rangle < 0$ for some $\boldsymbol{X}^2 \in \mathbb{S}^n_{++}$. Hence a convex combination $\overline{\boldsymbol{X}} \in \mathbb{S}^n_{++}$ of $\boldsymbol{X}^1$ and $\boldsymbol{X}^2$ satisfies $\langle \boldsymbol{B}, \overline{\boldsymbol{X}} \rangle = 0$, *i.e.*, $\overline{\boldsymbol{X}}$ is an interior feasible solution of primal SDP (21). By the duality theorem, $\zeta_p = \zeta_d = 0$ if and only if the dual SDP is feasible, *i.e.*, $\boldsymbol{A} + \tau\boldsymbol{B} \in \mathbb{S}^n_+$ for some $\tau \in \mathbb{R}$. Therefore, we have shown that $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}) \ \Rightarrow \ $(19).

(iii) $\mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}) \Rightarrow \boldsymbol{B}_\leq \subseteq \boldsymbol{A}_\geq$: We observe that

$$\begin{array}{c} \mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}) \\ \Updownarrow \ \text{(by Lemma 4.2 (ii))} \\ \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{B}, \boldsymbol{x}\boldsymbol{x}^T \rangle \leq 0\} \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{A}, \boldsymbol{x}\boldsymbol{x}^T \rangle \geq 0\} \\ \Downarrow \\ \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{B}, \boldsymbol{x}\boldsymbol{x}^T \rangle \leq 0, \ x_n = 1\} \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{A}, \boldsymbol{x}\boldsymbol{x}^T \rangle \geq 0, \ x_n = 1\} \\ \Updownarrow \\ \boldsymbol{B}_\leq \subseteq \boldsymbol{A}_\geq. \end{array}$$

(iv) $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{J}_+(\boldsymbol{A}) \Rightarrow \boldsymbol{B}_= \subseteq \boldsymbol{A}_\geq$: This assertion can be proved similarly as in (iii). □

Remark 5.3. From (ii) of the proof above, we see that condition (II) can be regarded as the dual of condition (B) under assumptions (A-1) and (A-2).

6 Examples

In [2, Section 4.1], several examples satisfying condition (B) with finite $\mathcal{B}$ and $\mathbb{F} = \mathbb{S}^n_+$ were provided. We present three examples that are not covered by those examples in this section.

Example 6.1. This example provides an infinite $\mathcal{B} \subseteq \mathbb{S}^n$ satisfying condition (C). Let

$$\boldsymbol{B}(\boldsymbol{t}) = \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{t} \\ -\boldsymbol{t}^T & \boldsymbol{t}^T\boldsymbol{t} - r^2 \end{pmatrix} \in \mathbb{S}^n \ (\boldsymbol{t} \in T), \ \mathcal{B} = \{\boldsymbol{B}(\boldsymbol{t}) : \boldsymbol{t} \in T\},$$

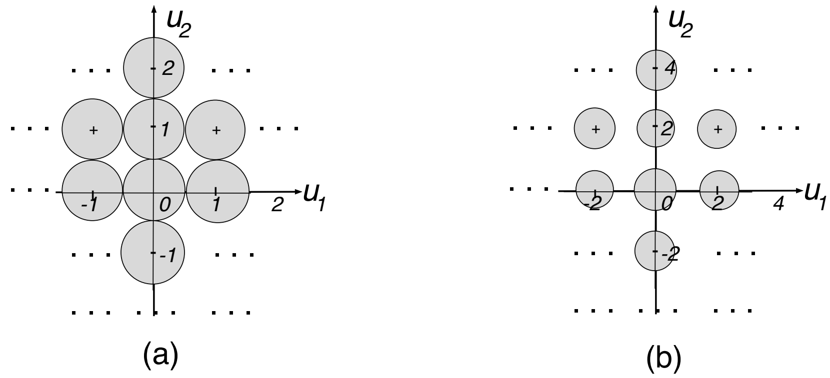

Figure 4: Illustration of $\boldsymbol{B}(\boldsymbol{t})_{\leq}$. Each gray disk region corresponds to $\boldsymbol{B}(\boldsymbol{t})_{\leq}$ for some $\boldsymbol{t} \in T$. (a) $T = \mathbb{Z}^{n-1}$ and $r = 1/2$. (b) $T = 2\mathbb{Z}^{n-1}$ and $r = 1/3$.

where $0 < r \leq 1/2$, $T \subseteq \mathbb{Z}^n$ (the set of integer vectors in $\mathbb{R}^n$) and $\boldsymbol{I}$ denotes the $(n-1) \times (n-1)$ identity matrix. Then,

$$q(\boldsymbol{u}, \boldsymbol{B}(\boldsymbol{t})) = \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix}^T \boldsymbol{B}(\boldsymbol{t}) \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix} = \parallel \boldsymbol{u} - \boldsymbol{t} \parallel^2 - r^2,$$
$$\boldsymbol{B}(\boldsymbol{t})_{\geq} \text{ or } \boldsymbol{B}(\boldsymbol{t})_{\leq} = \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : \parallel \boldsymbol{u} - \boldsymbol{t} \parallel^2 - r^2 \geq 0 \text{ or } \leq 0, \text{ respectively} \right\}$$

for every $\boldsymbol{t} \in T$ and $\boldsymbol{u} \in \mathbb{R}^{n-1}$. See Figure 3. It is easily seen that condition (C) with $\mathbb{F} = \mathbb{S}^n_+$ is satisfied. Therefore, by Theorem 1.5, we obtain $\mathbb{J}_+(\mathcal{B}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^n)$. When T is finite, this example is a special case of quadratic programs with hollows [38].

As a generalization, it is straightforward to construct an ellipsoid-based constraint by replacing each $\boldsymbol{B}(\boldsymbol{t})$ with $\boldsymbol{L}^T\boldsymbol{B}(\boldsymbol{t})\boldsymbol{L}$ ($\boldsymbol{t} \in T$), where $\boldsymbol{L}$ denotes an $n \times n$ nonsingular matrix of the form $\boldsymbol{L} = \begin{pmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix}$. We also note that the equivalence relation (1) between COP($\mathbb{J} \cap \boldsymbol{\Gamma}^n, \boldsymbol{Q}, \boldsymbol{H}$) and its SDP relaxation COP($\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}$) with $\mathbb{J} = \mathbb{J}_+(\mathcal{B})$ (or $\mathbb{J} = \mathbb{J}_+(\{\boldsymbol{L}^T\boldsymbol{B}\boldsymbol{L} : \boldsymbol{B} \in \mathcal{B}\})$) holds for any choice of $\boldsymbol{Q} \in \mathbb{S}^n$ and $\boldsymbol{H} \in \mathbb{S}^n$ by Theorem 1.1. For example, we can take $\boldsymbol{H} = \delta\boldsymbol{I}$ for some $\delta > 0$ where $\boldsymbol{I}$ is the $n \times n$ identity matrix. In this case, COP($\mathbb{J} \cap \boldsymbol{\Gamma}^n, \boldsymbol{Q}, \boldsymbol{H}$) turns out to be

$$\eta(\mathbb{J} \cap \boldsymbol{\Gamma}^n, \boldsymbol{Q}, \boldsymbol{H}) \;=\; \inf \left\{ \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{z} \end{pmatrix}^T \boldsymbol{Q} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{z} \end{pmatrix} : \begin{array}{l} \parallel \boldsymbol{u} - \boldsymbol{t}z \parallel^2 - r^2z^2 \geq 0 \ (\boldsymbol{t} \in T), \\ (\text{or } \parallel \boldsymbol{M}\boldsymbol{u} - \boldsymbol{t}z \parallel^2 - r^2z^2 \geq 0 \ (\boldsymbol{t} \in T)), \\ u_1^2 + \cdots + u_{n-1}^2 + z^2 = 1/\delta \end{array} \right\}.$$

Example 6.2. This example presents another infinite $\mathcal{B}$ satisfying condition (C). For a finite or infinite sequence $\{(\gamma_p, \mu_p, \lambda_p, \sigma_p) \in \mathbb{R}^4 : p = 1, 2, \ldots\}$ satisfying

$$\gamma_p > 0,\ \mu_p \geq 0, \lambda_p > 0 \text{ for every } p, \text{ and } |\sigma_q - \sigma_p| \geq \sqrt{\lambda_p} + \sqrt{\lambda_q} \text{ if } p < q, \tag{22}$$

we consider a sequence of quadratic functions

$$\begin{aligned} q_p(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \;&=\; \parallel \boldsymbol{v} - \sigma_p\boldsymbol{u} \parallel^2 - \lambda_p \parallel \boldsymbol{u} \parallel^2 + \mu_p \parallel \boldsymbol{w} \parallel^2 + \gamma_p \\ &\quad \text{for every } \boldsymbol{u} \in \mathbb{R}^\ell,\ \boldsymbol{v} \in \mathbb{R}^\ell \text{ and } \boldsymbol{w} \in \mathbb{R}^m \end{aligned} \tag{23}$$

$(p = 1, 2, \ldots)$. We can take a matrix $\boldsymbol{B}^p \in \mathbb{S}^{2\ell+m+1}$ representing the quadratic function $q^\sigma(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ such that

$$q^\sigma(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \;=\; \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{w} \\ 1 \end{pmatrix}^T \boldsymbol{B}^p \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{w} \\ 1 \end{pmatrix} \text{ for every } \boldsymbol{u} \in \mathbb{R}^\ell,\ \boldsymbol{v} \in \mathbb{R}^\ell \text{ and } \boldsymbol{w} \in \mathbb{R}^m$$

$(p = 1, 2, \ldots)$.

We show that $\mathcal{B} = \{\boldsymbol{B}^p : p = 1, 2, \ldots\}$ satisfies condition (C) with $\mathbb{F} = \mathbb{S}^{2\ell+m+1}_+$. Let $p < q$. Assume on the contrary that $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{B}^p_\leq \cap \boldsymbol{B}^q_\leq$ for some $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \mathbb{R}^{2\ell+m}$. Then

$$\begin{aligned} &\| \boldsymbol{v} - \sigma_p \boldsymbol{u} \|^2 + \mu_p \| \boldsymbol{w} \|^2 + \gamma_p \leq \lambda_p \| \boldsymbol{u} \|^2, \\ &\| \boldsymbol{v} - \sigma_q \boldsymbol{u} \|^2 + \mu_q \| \boldsymbol{w} \|^2 + \gamma_q \leq \lambda_q \| \boldsymbol{u} \|^2 . \end{aligned}$$

Hence,

$$0 < \| \boldsymbol{u} \|,\ \| \boldsymbol{v} - \sigma_p \boldsymbol{u} \| < \sqrt{\lambda_p} \| \boldsymbol{u} \|,\ \| \boldsymbol{v} - \sigma_q \boldsymbol{u} \| < \sqrt{\lambda_q} \| \boldsymbol{u} \| .$$

Therefore,

$$\begin{aligned} |\sigma_q - \sigma_p| \, \| \boldsymbol{u} \| &= \| (\sigma_q \boldsymbol{u} - \boldsymbol{v}) + (\boldsymbol{v} - \sigma_p \boldsymbol{u}) \| \\ &\leq \| \sigma_q \boldsymbol{u} - \boldsymbol{v} \| + \| \boldsymbol{v} - \sigma_p \boldsymbol{u} \| \\ &< (\sqrt{\lambda_p} + \sqrt{\lambda_q}) \| \boldsymbol{u} \|, \end{aligned}$$

which implies $\sigma_q - \sigma_p < \sqrt{\lambda_p} + \sqrt{\lambda_q}$, a contradiction to the last inequality of (22). Thus we have shown $\boldsymbol{B}^p_\leq \cap \boldsymbol{B}^q_\leq = \emptyset$, which implies $\boldsymbol{B}^p_\leq \subseteq \boldsymbol{B}^q_\geq$ and $\boldsymbol{B}^q_\leq \subseteq \boldsymbol{B}^p_\geq$.

If we take $\gamma_p = \lambda_p = \mu_p = 1$ and $\sigma_p = 2p$ $(p = 1, 2, \ldots)$, then $\{(\gamma_p, \mu_p, \lambda_p, \sigma_p) : p = 1, 2, \ldots\}$ satisfies (22). As another case, we consider

$$\gamma_p > 0,\ \lambda_p = (a_p - a_{p-1})^2/4,\ \mu_p = 0,\ \sigma_p = -(a_{p-1} + a_p)/2 \ (p = 1, 2, \ldots), \tag{24}$$

where $\{a_p : p = 0, 1, \ldots\}$ denotes an infinite sequence of positive numbers such that $a_{p-1} < a_p$ In this case, if $p < q$ then

$$\begin{aligned} |\sigma_q - \sigma_p| - (\sqrt{\lambda_q} + \sqrt{\lambda_p}) &= \frac{(a_q + a_{q-1}) - (a_p + a_{p-1})}{2} - \frac{(a_q - a_{q-1}) + (a_p - a_{p-1})}{2} \\ &= a_{q-1} - a_p \geq 0. \end{aligned}$$

Therefore, the sequence $\{(\gamma_p, \lambda_p, \mu_p, \sigma_p) : p = 1, 2, \ldots\}$ defined by (24) satisfies (22). Now suppose that $\ell = 1$ and $m = 0$. Then, the quadratic functions (23) turns out to be

$$\begin{aligned} q_p(u, v) &= (v - \sigma_p u)^2 - \lambda_p u^2 + \gamma_p \\ &= (v - (a_{p-1} + a_p)u/2)^2 - (a_p - a_{p-1})^2 u^2/4 + \gamma_p \\ &= a_{p-1} a_p u^2 - (a_{p-1} + a_p) uv + v^2 + \gamma_p \\ &= \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}^T \boldsymbol{B}^p \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}, \end{aligned}$$

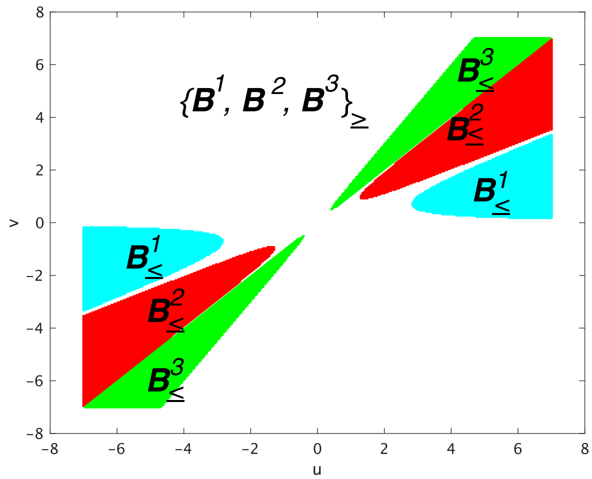

Figure 5: Example 6.2. We take $a_0 = 0,\ a_1 = 0.5,\ a_2 = 1,\ a_3 = 1.5,\ \gamma_1 = 0.5,\ \gamma_2 = 0.1,\ \gamma_2 = 0.01$.

where

$$\boldsymbol{B}^p = \begin{pmatrix} a_{p-1}a_p & -(a_{p-1}+a_p)/2 & 0 \\ -(a_{p-1}+a_p)/2 & 1 & 0 \\ 0 & 0 & \gamma_p \end{pmatrix}$$

$\mathcal{B} = \{\boldsymbol{B}^p : p = 1, 2, \ldots\}$ satisfies condition (C) with $\mathbb{F} = \mathbb{S}^3_+$. See Figure 4.

Now assume that $a_p \to \bar{a}$ as $p \to \infty$. Then $\boldsymbol{B}^p \to \overline{\boldsymbol{B}} = \begin{pmatrix} \bar{a}^2 & -\bar{a} & \boldsymbol{0} \\ -\bar{a} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \notin \mathcal{B}$ as $p \to \infty$. Therefore, $\mathcal{B}$ is not closed. We also see $\overline{\boldsymbol{B}} \in \mathbb{S}^3_+$. Hence, $\mathbb{J}_+(\boldsymbol{B}^p) \subsetneq \mathbb{S}^3_+ = \mathbb{J}_+(\overline{\boldsymbol{B}})$ $(p = 1, 2, \ldots,)$. Therefore, cl$\mathcal{B}$ (the closure of $\mathcal{B}$) does not satisfy condition (A-2) although $\mathbb{J}_+(\mathrm{cl}\mathcal{B}) \in \widehat{\mathcal{F}}(\boldsymbol{\Gamma}^3)$ remains true.

Example 6.3. We consider a parabola-based constraint. Let $n \geq 3$ and

$$B_{ij} = \begin{cases} \lambda_i > 0 & \text{if } 2 \leq i = j \leq n, \\ -0.5 & \text{if } (i,j) = (1,n) \text{ or } (i,j) = (n,1), \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} & q(\boldsymbol{u}, \boldsymbol{B}) = -u_1 + \sum_{i=2}^{n-1} \lambda_i u_i^2 + \lambda_n \text{ for every } \boldsymbol{u} \in \mathbb{R}^{n-1}, \\ & \boldsymbol{B}_{\geq}, \boldsymbol{B}_{=} \text{ or } \boldsymbol{B}_{\leq} = \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : \begin{array}{l} -u_1 + \sum_{i=2}^{n-1} \lambda_i u_i^2 + \lambda_n \geq 0, = 0 \text{ or } \leq 0, \\ \text{respectively} \end{array} \right\}, \qquad (25) \\ & \boldsymbol{B}_{\leq} \subseteq \{ \boldsymbol{u} \in \mathbb{R}^n : \lambda_i u_i^2 + \lambda_n \leq u_1 \ (i = 2, \ldots, n-1) \} \\ & \quad \subseteq \mathbb{K}_-(\boldsymbol{B}) \equiv \{ \boldsymbol{u} \in \mathbb{R}^{n-1} : 0 \leq u_1,\ -u_1 \leq 2\sqrt{\lambda_i \lambda_n} u_i \leq u_1 \ (i = 2, \ldots, n-1) \}. \end{aligned}$$

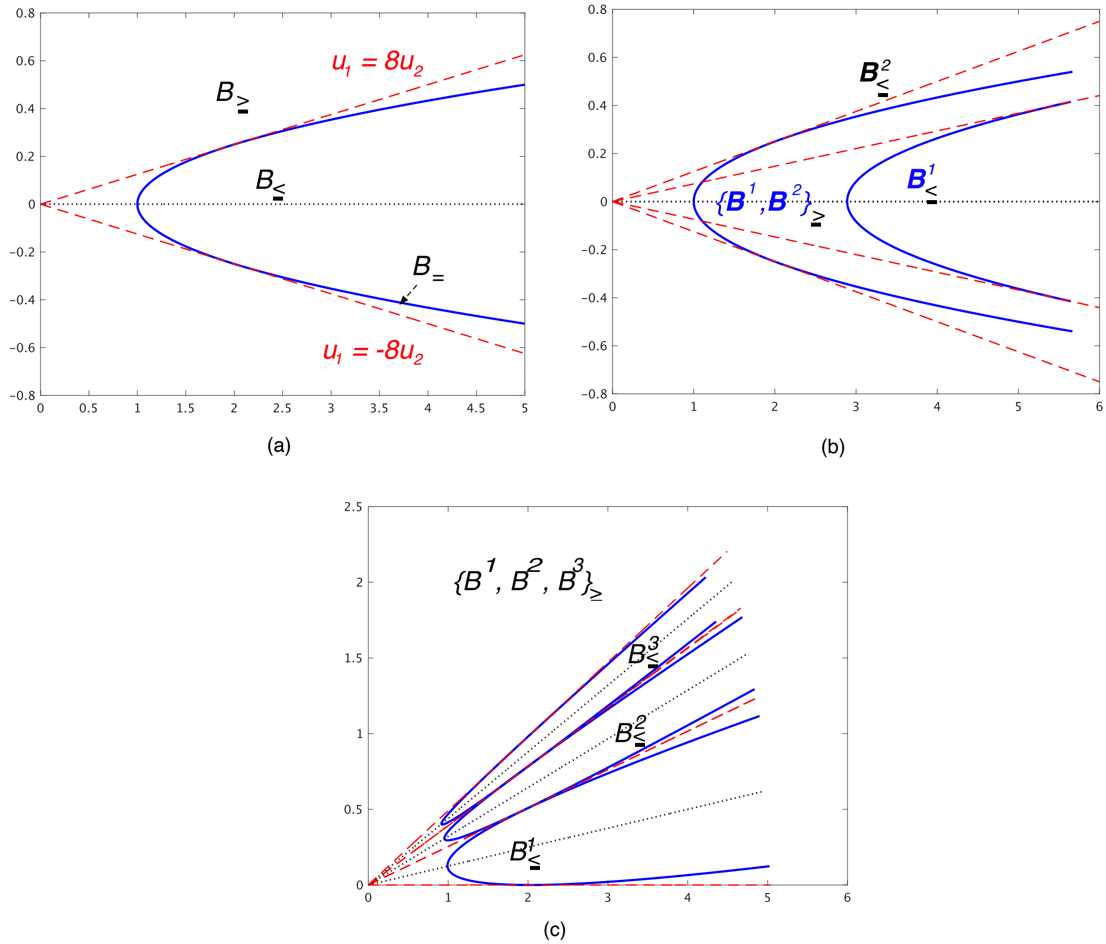

Figure 6: Parabola-based constraints with $n = 3$. (a) Parabola $\boldsymbol{B}_{\leq}$ defined by (25) where $\lambda_2 = 16$ and $\lambda_3 = 1$. (b) $\boldsymbol{B}^1_{\geq} = \{\boldsymbol{u} \in \mathbb{R}^2 : -u_1 + 16u_2^2 + 3 \geq 0\}$, $\boldsymbol{B}^2_{\geq} = \{\boldsymbol{u} \in \mathbb{R}^2 : -(-u_1 + 16u_2^2 + 1) \geq 0\}$ and $\{\boldsymbol{B}^1, \boldsymbol{B}^2\}_{\geq} = \boldsymbol{B}^1_{\geq} \cap \boldsymbol{B}^2_{\geq}$.

See Figure 6 (a). We note that $\mathbb{K}_{-}(\boldsymbol{B})$ forms a polyhedral cone in $\mathbb{R}^{n-1}$, which converges to the half line $\{\boldsymbol{u} \in \mathbb{R}^{n-1} : u_1 \geq 0,\ u_i = 0\ (i = 2, \ldots, n-1)\}$ as all λ_i $(i = 2, \ldots, n-1)$ tend to ∞. We see that each $\boldsymbol{B}_{=} \cap \{\boldsymbol{u} \in \mathbb{R}^{n-1} : u_j = 0\ (2 \leq j \neq i \leq n-1)\}$ forms a 2-dimensional parabola $(i = 2, \ldots, n-1)$. By applying a linear transformation $\boldsymbol{B} \to \boldsymbol{L}^T\boldsymbol{B}\boldsymbol{L} \in \mathbb{S}^n$ with a nonsingular $\boldsymbol{L}$ to $\boldsymbol{B}_{=}$ with different $\lambda_i > 0$ $(i = 2, \ldots, n)$, we can create various parabolas. Furthermore, we can arrange some of those parabolas such that the assoicated $\mathcal{B}$ satisfies condition (C). See Figure 6 (b) and (c).

7 Concluding remarks

Finally, we comment on the role of the representation of the closed convex cone $\mathbb{J} \subseteq \mathbb{S}^n_+$ in the geometric QCQP, COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}$) and its SDP relaxation COP($\mathbb{J}$). Our sufficient conditions for the equivalence of COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}$) and COP($\mathbb{J}$) are formulated in terms of a set $\mathcal{B}$ with $\mathbb{J} = \mathbb{J}_+(\mathcal{B})$, and are therefore representation-dependent: different choices of $\mathcal{B}$ may affect whether conditions (I), (II), (III), (B), (C), or (D) hold, even though the cone $\mathbb{J}$ itself is unchanged. In contrast, some properties studied here are intrinsic to $\mathbb{J}$. For example, Slater's condition $\mathbb{J} \cap \mathbb{S}^n_{++} \neq \emptyset$ and the minimal face of $\mathbb{S}^n_+$ containing $\mathbb{J}$ depend only on $\mathbb{J}$ and not on a particular representation. Redundant constraints, however, are features of a chosen $\mathcal{B}$ rather than structural properties of $\mathbb{J}$. Thus, our results provide verifiable sufficient conditions, expressed in terms of a representation $\mathcal{B}$, that guarantee the intrinsic geometric property $\mathbb{J} \in \widehat{\mathcal{F}}(\mathbf{\Gamma}^n)$, *i.e.*, that $\mathbb{J}$ is ROG.

We have presented two types of sufficient conditions under which a semi-infinite QCQP is equivalent to its SDP relaxation. The first type, condition (B), extends the assumptions in [2, Theorem 4.1] and [1, Proposition 1] for QCQPs with finitely many inequality constraints to the case of infinitely many inequality constraints. Even for QCQPs with finitely many constraints, condition (B) is weaker than the corresponding conditions in those works. Condition (B) may be viewed as a sufficient structural condition on a representation of $\mathbb{J}$ ensuring that $\mathbb{J}$ (or equivalently, its minimal face) is ROG. The second type, condition (C), is a new variant of NIQCC. We have shown that (C) implies (B) in general, and that the two are equivalent under appropriate additional assumptions. Furthermore, we have generalized condition (III) (the NIQCC from [19, Corollary 2]) to condition (D).

In addition to the equivalence results for semi-infinite QCQPs and their SDP relaxations, which have been studied here based on ROG and NIQCC, there exist several other well-studied classes of QCQPs whose SDP relaxations are exact. Notable examples include convex QCQPs, where both the objective and all quadratic constraints are convex, and QCQPs defined by specific sign pattern conditions on the objective and constraint matrices [3, 4, 21, 33]. These QCQPs have a fundamentally different nature, and most existing studies on them are largely independent of the ROG and NIQCC properties investigated in this paper. In our recent paper [23], we have discussed how homogenized and non-homogenized NIQCC properties can be incorporated into such QCQPs.

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References

[1] C. J. Argue, F. Kilinç-Karzan, and A.L. Wang. Necessary and sufficient conditions for rank-one-generated cones. *Math. Oper. Res.*, 48(1):100–126, 2023.

[2] N. Arima, S. Kim, and M. Kojima. Further development in convex conic reformulation of geometric nonconvex conic optimization problems. *SIAM J. Optim.*, 34(4):3194–3211, 2024.

[3] G. Azuma, Fukuda M., S. Kim, and M. Yamashita. Exact SDP relaxations of quadratically constrained quadratic programs with forest structures. *J. of Global Optim.*, 82:243–262, 2022.

[4] G. Azuma, Fukuda M., S. Kim, and M. Yamashita. Exact SDP relaxations for quadratic programs with bipartite graph structures. *J. of Global Optim.*, 86:671–691, 2023.

[5] A. Beck and Y.C. Eldar. Strong duality in nonconvex quadratic optimization with two quadratic constraints. *SIAM J. Optim.*, 17:844–860, 2006.

[6] G. Blekherman, R. Sinn, and M. Velasco. Do sums of squares dream of free resolutions? *SIAM J. Appl. Algebra Geom.*, 1:175–199, 2017.

[7] J. M. Borwein and H. Wolkowicz. Facial reduction for a cone-convex programming problem. *J. Aust. Math. Soc.*, 30:369–380, 1981.

[8] J.M. Borwein and H. Wolkowicz. Regularizing the abstract convex program. *J. Math. Anal. Appl.*, 83:495–530, 1981.

[9] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[10] S. Burer. On the copositive representation of binary and continuous non-convex quadratic programs. *Math. Program.*, 120:479–495, 2009.

[11] S. Burer and Yang B. The trust region subproblem with non-intersecting linear constraints. *Math. Program.*, 149:253–264, 2015.

[12] C.-Y. Chong and S. P. Kumar. Sensor networks: Evolution, opportunities, and challenges. *Proceedings of the IEEE*, 91(8):1247–1256, 2003.

[13] A.R. Conn, N.I.M. Gould, and P.L.: Toint. *Trust-Region Methods. MPS-SIAM Series in Optimization, SIAM, Philadelphia, PA (2000)Conn, A.R., Gould, N.I.M., Toint, P.L.:*, volume Trust-Region Methods of *MPS-SIAM Series in Optimization*. SIAM, Philadelphia, 2000.

[14] R. Hettich and K. O. Kortanek. Semi-infinite programming: theory, methods and applications. *SIAM Review*, 35(3):380–429, 1993.

[15] R. Hildebrand. Spectrahedral cones generated by rank 1 matrices. *J. Global Optim*, 64:349–397, 2016.

[16] K. Hrbacek and T. Jech. *Introduction to Set Theory, Revised and Expanded*. Taylor & Francis, 1999.

[17] T. Jeck. *Set Theory*. Springer, 2006.

[18] V. Jeyakumar and Li. G. Y. Trust-region problems with linear inequality constraints: exact SDP relaxation, global optimality and robust optimization. *Math. Program.*, 147:171–206, 2014.

[19] A. Joyce and B. Yang. Convex hull results on quadratic programs with non-intersecting constraints. *Math. Program.*, 205:539–558, 2024.

[20] F. Kilinç-Karzan and A.L. Wang. Exactness in sdp relaxations of qcqps: Theory and applications. Technical Report arXiv:2107.06885, 2021.

[21] S. Kim and M. Kojima. Exact solutions of some nonconvex quadratic optimization problems via SDP and SOCP relaxations. *Comput. Optim. Appl.*, 26(2):143–154, 2003.

[22] S. Kim, M. Kojima, and K. C. Toh. A geometrical analysis of a class of nonconvex conic programs for convex conic reformulations of quadratic and polynomial optimization problems. *SIAM J. Optim.*, 30:1251–1273, 2020.

[23] M. Kojima, S. Kim, and N. Arima. Extending exact convex relaxations of quadratically constrained quadratic programs. Technical Report arXiv:2504.03204, 2025.

[24] M. Laurent. Sums of squares, moment matrices and optimization over polynomials sums of squares, moment matrices and optimization over polynomials. In M. Putinar and Sullivant, editors, *Emerging Applications of Algebraic Geometry*, volume 149 of *The IMA Volumes in Mathematics and its Applications*, pages 157–270, 2008.

[25] G. Pang, C. Shen, L. Cao, and van den H. Anton. Deep learning for anomaly detection: A review. *ACM Computing Surveys*, 54(2):1–38, 2021.

[26] G. Pataki. *The geometry of semidefinite programming*, volume Handbook of semidefinite programming: Theory, Algorithms, and Applications, chapter 1. Springer, 2000.

[27] G. Pataki. Strong duality in conic linear programming: facial reduction and extended duals. In et al. Bailey, D., editor, *Computational and Analytical Mathematics*, volume 50 of *Springer Proceedings in Mathematics and Statistics*, New York, NY., 2013. Springer.

[28] B. T. Polyak. Convexity of quadratic transformations and its use in control and optimization. *J. Optim. Theory Appl.*, 99(3):553–583, 1998.

[29] T. Pong and H. Wolkowicz. The generalized trust region subproblem. *Comput. Optim. Appl.*, 58:273–322, 2014.

[30] R. Reemtsen and J. J. Rückmann. *Semi-infinite programming*. Springer, 2013.

[31] A. Richards and J. How. Aircraft trajectory planning with collision avoidance using mixed integer linear programming. In *Proceedings of the American Control Conference*, volume 3, pages 1936–1941, 2002.

[32] J. Schulman, J. Ho, A. Lee, I. Awwal, H. Bradlow, and P. Abbeel. Finding locally optimal, collision-free trajectories with sequential convex optimization. In *Robotics: Science and Systems (RSS)*, 2013.

[33] S. Sojoudi and J. Lavaei. Exactness of semidefinite relaxations for nonlinear optimization problems with underlying graph structure. *SIAM J. Optim.*, 24(4):1746–1778, 2014.

[34] R. Stern and H. Wolkowicz. Indefinite trust region subproblems and nonsymmetric eigenvalue perturbations. *SIAM J. Optim.*, 5(2):286–313, 1995.

[35] J. F. Sturm and S. Zhang. On cones of nonnegative quadratic functions. *Math. Oper. Res.*, 28(2):246–267, 2003.

[36] H. Waki and M. Muramatsu. Facial reduction algorithms for conic optimization problems. *J. Optim. Theory Appl.*, 158(1):188–215, 2013.

[37] Y. Xia, S. Wang, and R. L. Sheu. S-lemma with equality and its applications. *Math. Program.*, 156:513–547, 2016.

[38] B. Yang, K. Anstreiher, and S. Burer. Quadratic programs with hollows. *Math. Program.*, 170:541–553, 2018.

[39] Y. Ye and S. Zhang. New results on quadratic minimization. *SIAM J. Optim.*, 14:245–267, 2003.