How Many Policies Do We Need in K-Adaptability for Two-stage Robust Integer Optimization?

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Abstract

In the realm of robust optimization the k-adaptability approach is one promising method to derive approximate solutions for two-stage robust optimization problems. Instead of allowing all possible second-stage decisions, the k-adaptability approach aims at calculating a limited set of k such decisions already in the first-stage before the uncertainty reveals. The parameter k can be adjusted to control the quality of the approximation. However, not much is known on how many solutions k are needed to achieve an optimal solution for the two-stage robust problem. In this work we derive bounds on k which guarantee optimality for general non-linear problems with integer decisions where the uncertainty appears in the objective function or in the constraints. We show that for objective uncertainty the bound is the same as for the linear case and depends linearly on the dimension of the uncertainty, while for constraint uncertainty the dependence can be exponential, still providing the first generic bound for a wide class of problems. The results give new insights on how many solutions are needed for problems as the decision dependent information discovery problem or the capital budgeting problem with constraint uncertainty.

1 Introduction

Two-stage robust optimization problems appear in a variety of applications where decisions are influenced by uncertain parameters, e.g., the demands of a customer, the travel time or the population density of a certain district; see [GYDH15, YGdH19]. As common in robust optimization, it is assumed that the uncertain parameters lie in an uncertainty set which is pre-constructed by the user. In the two-stage robust setting some of the decisions have to be taken *here-and-now* while some decisions can be taken after the uncertain parameters of the problem are known (*wait-and-see decisions*). The goal is to find a here-and-now decision which optimizes the worst possible objective value over all scenarios in the uncertainty set.

While a large amount of works concentrate on the case where the decision variables are continuous, many real-world applications and combinatorial problem structures require integer decisions; [BK18b]. Unfortunately, two-stage robust optimization problems with integer wait-and-see decisions are computationally extremely challenging while at the same time the variety of solution methods is still limited. For the case where the uncertain parameters only appear in the objective function promising algorithms based on column-generation or branch & bound methods were developed ([KK20, AD22, DLMM24]). At the same time the constraint uncertainty case is still insufficiently investigated. Here, classical column-and-constraint generation (CCG) approaches were adapted for the general mixed-integer case [ZZ12] or for interdiction-type problems [LST23]. Recently, a neural network supported CCG was developed which can calculate heuristic solutions of high quality much faster than state-of-the-art approaches [DJKK24].

One promising method to approximate two-stage robust optimization problems is the k-adaptability approach, where, instead of considering all wait-and-see solutions, a limited set of k such solutions is calculated in the first-stage such that the best of it can be chosen after the uncertain parameters are known. This approach was first studied in [BC10] and gained more attention later in [HKW15, SGW20,

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Kur24, RP21]. A related special case of the problem, where no first-stage solutions are considered, sometimes called *min-max-min robust optimization*, was studied first in [BK17] and later in several other works [CGKP19, BK18a, GKP20, CG21, APS22].

One important research question is: How many second-stage solutions k do we need such that the k-adaptability approach returns an optimal solution for the two-stage robust optimization problem? If we know such a number k we can use the k-adaptability approach to solve the two-stage robust problem exactly. Furthermore, it provides insights on the complexity of the uncertainty set in connection with the second-stage problem, since larger values for k indicate a more diverse set of scenarios and required second-stage reactions. However, insights on the number of wait-and-see solutions needed for optimality are sparse. In [HKW15, BK17] it was shown that $k = n + 1$ solutions are enough for linear problems with objective uncertainty (where n is the minimum of the dimension of the problem and the dimension of the uncertainty). For the constraint uncertainty case the authors in [HKW15] present an example where all second-stage solutions are needed to guarantee optimality. To the best of our knowledge there are no better bounds known for the constraint uncertainty case with integer recourse.

Contributions

- We show that in the objective uncertainty case the bound on k which is known for the linear case holds even if we consider general non-linear objective functions which are concave in the uncertain parameters. As a consequence we can show for the first time that for robust optimization with decision dependent information discovery at most $k = n_\xi + 1$ solutions are needed to guarantee optimality, where n_ξ is the dimension of the uncertainty.
- Based on the latter bounds on k, we derive bounds on the approximation guarantee of the k-adaptability approach for arbitrary values of k in the objective uncertainty case.
- We derive bounds on k to guarantee optimality for the constraint uncertainty case. To this end we introduce a new concept called *recourse-stability* which leads to a bound on k which depends on the uncertainty dimension and the number of recourse-stable regions needed to cover the uncertainty set.
- We show that for certain problem structures the bound on k for the constraint uncertainty case can significantly reduce the value k which is needed for optimality.

2 Preliminaries

2.1 Notation and Preliminaries

For any given positive integer n we denote $[n] = \{1, 2, \ldots, n\}$, we denote all n-dimensional vectors of non-negative real numbers as $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \geq 0\}$ and all n-dimensional vectors of non-negative integers as $\mathbb{Z}^n_+ := \{x \in \mathbb{Z}^n : x \geq 0\}$. The euclidean norm is denoted as $\|\cdot\|$, i.e., $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ for any $x \in \mathbb{R}^n$. For a given set $S \subseteq \mathbb{R}^n$ we define the diameter of the set as $\operatorname{diam}(\mathcal{S}) = \max_{x,y\in\mathcal{S}} \|x - y\|$, the closure of the set as $\operatorname{cl}(\mathcal{S})$, where a point $x \in \mathbb{R}^n$ is contained in the closure of $\mathcal{S}$ if and only if for every radius $\varepsilon > 0$ there exists a point $s \in \mathcal{S}$ with $\|x - s\| < \varepsilon$.

For any $\mathcal{X} \subseteq \mathbb{R}^n$ we call a function $f : \mathcal{X} \to \mathbb{R}$ *convex* if and only if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

holds for all $x, y \in \mathcal{X}$ and $0 \leq \lambda \leq 1$. A function f is *concave* if $-f$ is convex. The function f is *Lipschitz continuous* with Lipschitz constant $L > 0$ if and only if

$$|f(x) - f(y)| \leq L\|x - y\|$$

holds for all $x, y \in \mathcal{X}$.

One preliminary result we will use in Section 3 and 4 was derived in [CC05]. In this work the authors study convex optimization problems of the form

$$\begin{aligned} \mathcal{P} : \min_{x\in\mathbb{R}^n} \ & c^\top x \\ s.t. \ & x \in \mathcal{X}_i \quad i \in [m] \end{aligned}$$

where $m \in \mathbb{Z}_+$, $c \in \mathbb{R}^n$ and $\mathcal{X}_i$ is a closed and convex set for every $i \in [m]$. The authors define the constraint X_k to be a *support constraint* if removing it from the problem leads to a strictly better optimal value compared to the original problem $\mathcal{P}$. They prove the following theorem.

Theorem 1 ([CC05]). *The number of support constraints for Problem $\mathcal{P}$ is at most n.*

Note that assuming a linear objective function in $\mathcal{P}$ is without loss of generality since we can always move a convex objective function into the constraints by using the epigraph reformulation.

2.2 Problem Definition

In this work we consider the general class of (non-linear) two-stage robust optimization problems of the form

$$\inf_{x\in\mathcal{X}} \sup_{\xi\in\mathcal{U}} \inf_{y\in\mathcal{Y}(x)} f(x,y,\xi) \tag{2RO}$$

where $\mathcal{X} \subseteq \mathbb{R}^{n_x}$ is an arbitrary compact set containing all possible first-stage decisions, $\mathcal{Y}(x) \subseteq \mathcal{Y} \subset \mathbb{Z}^{n_y}$ is the set of feasible second-stage decisions y which can depend on the chosen first-stage decision x and $\mathcal{U} \subset \mathbb{R}^{n_\xi}$ is a convex and compact uncertainty set containing all possible scenarios ξ. We assume that $\mathcal{Y}$ is bounded, i.e., it contains a finite number of solutions. Furthermore, $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{U} \to \mathbb{R}$ is an arbitrary function, if not stated otherwise.

While the uncertainty parameters ξ seem to appear only in the objective function in (2RO), the problem definition also covers the case of constraint uncertainty due to the generality of the objective function f. Indeed, we will consider the case of constraint uncertainty in Section 4, by considering the function

$$f(x,y,\xi) := \begin{cases} g(x,y,\xi) & \text{if } A(\xi)x + B(\xi)y \geq h(\xi) \\ \infty & \text{otherwise,} \end{cases}$$

where $g : \mathcal{X} \times \mathcal{Y} \times \mathcal{U} \to \mathbb{R}$ is a given continuous objective function, $A(\xi) \in \mathbb{R}^{m\times n_x}$, $B(\xi) \in \mathbb{R}^{m\times n_y}$ and $h(\xi) \in \mathbb{R}^m$ are constraint parameters which are given as functions of the uncertain parameters. The latter function f ensures that for an optimal $x \in \mathcal{X}$ and every $\xi \in \mathcal{U}$ a feasible second-stage decision $y \in \mathcal{Y}(x)$ is available, which minimizes $g(x,y,\xi)$, since otherwise the chosen x has objective value ∞ in Problem (2RO).

The k-adaptability approach aims at finding approximate solutions $x \in \mathcal{X}$ for (2RO). The idea is, for a fixed parameter $k \in \mathbb{N}$, to calculate a set of k second-stage policies $y^1, \ldots, y^k$ already in the first stage, and choose the best of it in the second-stage after the scenario is known. This leads to the problem

$$\inf_{\substack{x\in\mathcal{X} \\ y^1,\ldots,y^k\in\mathcal{Y}(x)}} \sup_{\xi\in\mathcal{U}} \inf_{i=1,\ldots,k} f(x,y^i,\xi). \tag{k-ARO}$$

However, using this idea we cannot guarantee that the calculated solution $x \in \mathcal{X}$ is optimal for (2RO). In fact the quality of the optimal k-adaptable solution depends on the parameter k. The larger k, the better is the approximation for the original two-stage problem (2RO). On the other hand, the larger we choose k, the more complex Problem (k-ARO) becomes, since we have to introduce more second-stage decision variables. Hence, an interesting research question is: How many second-stage policies k do we need, such that the optimal solution of (k-ARO) is also optimal for (2RO)?

This question was studied before for several special cases: In [HKW15] the authors show that if the uncertainty only appears in the objective function and if this objective function is linear, at most $k = n + 1$ second-stage policies are needed, where n is the minimum of the problem dimension and the uncertainty dimension. This coincides with the result observed in [BK17] for min-max-min robust combinatorial optimization problems, which is a special case of the k-adaptability problem. In [Kur24] it was shown that under objective uncertainty, if we want to approximate (2RO) by a factor of $1 + \alpha(n_y)$, then it is enough to use $k = qn_y$ policies where $q = \frac{M(n_y)}{M(n_y)+a(n_y)}$ and $M(n_y)$ is a value depending on the problem parameters and the dimension n_y. In this work we will generalize the latter results to the case of non-linear objective functions.

In case the uncertain parameters appear in the constraints, the best known bounds on k are not very promising. In [BC10] the authors argue that under a given continuity assumption and for continuous second-stage decisions the k-adaptability approach converges to an optimal solution of (2RO) for $k \to \infty$. Unfortunately, this result is not correct as it was shown later in [KSMM23]. The authors provide counterexamples where the k-adaptability approach does not lead to an optimal solution of (2RO) for any $k \in \mathbb{N}$. However, the authors show that the continuity assumption can be adjusted such that original convergence result holds. Again, for the case of continuous second-stage decisions, the authors in [EHG18] derive approximation guarantees which (k-ARO) provides for (2RO) and show that, if the number of policies k is bounded by a polynomial in the problem parameters, (k-ARO) cannot approximate (2RO) better than by a factor of $m^{1-\varepsilon}$.

In the setting which is studied in this work, namely $\mathcal{Y}$ is bounded and only contains integer solutions, the number of possible second-stage policies is finite. Hence, the convergence discussion above is not necessary, since trivially for $k = |\mathcal{Y}|$ the k-adaptability problem will return the optimal solution of (2RO). It is shown in [HKW15] that indeed there are problem instances where all $k = |\mathcal{Y}|$ policies are needed, hence finding a better bound is impossible in the general setting. However, in this work we will derive better bounds on k for certain problem structures.

3 Objective Uncertainty

In case of objective uncertainty and a linear objective function the k-adaptability problem provides an optimal solution of (2RO) if $k \geq \min\{n_y, n_\xi\} + 1$; see [HKW15]. The following theorem shows that a similar result holds for arbitrary non-linear objective functions which are concave in the uncertain parameters.

Theorem 2. *Let $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{U} \to \mathbb{R}$ be a continuous function such that $f(x, y, \xi)$ is concave in ξ for every $x \in \mathcal{X}, y \in \mathcal{Y}$ and let $k \geq n_\xi + 1$. Then, a solution $x \in \mathcal{X}$ is optimal for (k-ARO) if and only if it is optimal for (2RO).*

Proof. First, note that since all sets $\mathcal{X}, \mathcal{U}, \mathcal{Y}$ are compact and f is continuous, all the maxima and minima in the problem definition (2RO) exist and are finite.

Since $\mathcal{Y}$ is bounded and contains only integer decisions, we know that for $k = |\mathcal{Y}|$ the problems (k-ARO) and (2RO) are equivalent. Fix any first-stage decision $x \in \mathcal{X}$. By using an epigraph reformulation we can rewrite the inner max-min problem of (k-ARO) with $k = |\mathcal{Y}(x)|$ as

$$\begin{aligned} \max_{z,\xi} \; & z \\ s.t. \quad & f(x, y, \xi) - z \geq 0 \quad \forall y \in \mathcal{Y}(x) \\ & \xi \in \mathcal{U} \\ & z \in \mathbb{R}. \end{aligned}$$

Since f is concave and continuous in ξ the function $f(x, y, \xi) - z$ is concave and continuous in (ξ, z). Hence, the latter problem is convex, where for every $y \in \mathcal{Y}(x)$ the feasible set corresponding to the constraint is closed and convex. Additionally, $\xi \in \mathcal{U}$ and $z \in \mathbb{R}$ are convex constraints with closed and convex region. From Theorem 1 it follows, that the number of support constraints is at most the dimension of the problem, i.e., $n_\xi + 1$. Hence, we can remove all constraints except $n_\xi + 1$ from the problem without changing the optimal solution. We can conclude that at most $n_\xi + 1$ of the second-stage solutions $y \in \mathcal{Y}(x)$ are needed. This holds for any $x \in \mathcal{X}$ which proves the result. □

The latter result is interesting since the bound on k does only depend on the dimension of the uncertainty set and not on the dimension of the decision variables x and y. Furthermore, we do not make any assumptions on the function f regarding x and y; especially no convexity in x or y is required.

The following example shows that we can apply Theorem 2 to the robust optimization problem with decision-dependent information discovery (DDID).

Example 3 (Robust Optimization with Decision-Dependent Information Discovery)**.** *Consider the DDID which was introduced in [VGY20] and later studied in [PGDT22, OP23]. In both of the works [VGY20, PGDT22] the k-adaptability version of the problem is studied which is given as*

$$\min_{\substack{w \in \mathcal{W} \\ y^1,\ldots,y^k \in \mathcal{Y}}} \max_{\bar{\xi} \in \mathcal{U}} \min_{i=1,\ldots,k} \max_{\xi \in \mathcal{U}(w,\bar{\xi})} \xi^\top C w + \xi^\top P y^i$$

for matrices C, P of appropriate size, where $\mathcal{W} \subseteq \{0,1\}^{n_w}$, $\mathcal{Y} \subseteq \{0,1\}^{n_y}$, $\mathcal{U} \subset \mathbb{R}^{n_\xi}$ is a polyhedral uncertainty set and $U(w, \bar{\xi}) = \{\xi \in \mathcal{U} : w_i \xi_i = w_i \bar{\xi}_i, i = 1, \ldots, n_\xi\}$. We can rewrite the problem into the form (k-ARO) where

$$f(x, y, \bar{\xi}) := \max_{\xi \in \mathcal{U}(w, \bar{\xi})} \xi^\top C w + \xi^\top P y.$$

To apply Theorem 2 we have to show that f is concave in $\bar{\xi}$. We can reformulate f as

$$\begin{aligned} \max_{\xi} \ & \xi^\top C w + \xi^\top P y \\ s.t. \ & w_i \xi = w_i \bar{\xi}_i \quad i = 1, \ldots, n_\xi \\ & \xi \in \mathcal{U}. \end{aligned}$$

Taking the dual the problem can be transformed into the minimum of linear functions in $\bar{\xi}$, which is concave and continuous in $\bar{\xi}$. Hence, from Theorem 2 it follows, that at most $k = n_\xi + 1$ second-stage policies are needed to get an optimal solution for DDID.

The next example shows that for the capital budgeting problem the number of policies needed to guarantee optimality can be very small, namely the number of risk factors plus one.

Example 4 (Capital Budgeting)**.** *The k-adaptable version of the two-stage robust capital budgeting problem (CB) with objective uncertainty was studied in [SGW20]. The problem is given as*

$$\max_{\substack{x \in \mathcal{X} \\ y^1, \ldots, y^k \in \mathcal{Y}(x)}} \min_{\xi \in \mathcal{U}} \max_{i=1,\ldots,k} r(\xi)^\top (x + \kappa y^i)$$

where $\mathcal{X} = \{0,1\}^n$ and $\mathcal{Y}(x) = \{y \in \{0,1\}^n : c^\top (x + y) \leq B, \ x + y \leq e\}$. Furthermore, $\mathcal{U} = [-1,1]^\rho$ is an uncertainty set of all realizations of ρ different risk factors and e is the all-one vector. The risk of project i is given as $r_i(\xi) = (1 + \frac{1}{2}\Psi_i^\top \xi) r_i^0$ where Ψ_i is the i-th row of a given factor loading matrix Ψ. Note that the number of risk factors ρ is usually a small number which is independent of the other dimensions of the problem. Clearly, the objective function $f(x, y, \xi) = r(\xi)^\top (x + \kappa y)$ is linear (and therefore concave and continuous) in ξ and we can apply Theorem 2 to show that at most $k = \rho + 1$ second-stage policies are needed.

Next, we derive approximation bounds which the k-adaptability problem provides for (2RO). In the following we denote by $\text{opt}(k)$ the optimal value of the k-adaptability problem.

Theorem 5. *Let $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{U} \to \mathbb{R}$ be a continuous function such that $f(x, y, \xi)$ is concave in ξ for every $x \in \mathcal{X}, y \in \mathcal{Y}$. Furthermore, assume f is Lipschitz continuous in y, i.e., there exists a constant $L > 0$ such that*

$$|f(x, y, \xi) - f(x, y', \xi)| \leq L \|y - y'\| \quad \forall x \in \mathcal{X}, \xi \in \mathcal{U}, y, y' \in \mathcal{Y}.$$

Then, for any $s, k \in \mathbb{N}$ with $s \leq k$ it holds

$$opt(s) - opt(k) \leq L diam(\mathcal{Y}) \frac{k - s}{s + 1}.$$

Proof. First, we reformulate (k-ARO) as

$$\min_{\substack{x \in \mathcal{X} \\ y^1, \ldots, y^k \in \mathcal{Y}(x)}} \max_{\xi \in \mathcal{U}} \min_{\substack{\lambda \in \mathbb{R}^k_+ \\ \sum_{i=1}^k \lambda_i = 1}} \sum_{i=1}^k \lambda_i f(x, y^i, \xi).$$

Since f is concave in ξ and $\lambda \geq 0$, also the function $\sum_{i=1}^k \lambda_i f(x, y^i, \xi)$ is concave in ξ. We can apply the classical minimax theorem and swap the inner maximum and minimum operator which leads to the reformulation

$$\min_{\substack{x \in \mathcal{X} \\ y^1, \ldots, y^k \in \mathcal{Y}(x) \\ \lambda \in \mathbb{R}^k_+ \\ \sum_{i=1}^k \lambda_i = 1}} \max_{\xi \in \mathcal{U}} \sum_{i=1}^k \lambda_i f(x, y^i, \xi).$$

Let $(x^*, y^{1^*}, \dots, y^{k^*}, \lambda^*)$ be an optimal solution of the latter problem and assume w.l.o.g. that $\lambda_1^* \geq \dots \geq \lambda_k^*$. We define a feasible solution for the s-adaptability problem as

$$x(s) = x^*, \quad y^1(s) = y^{1^*}, \dots, y^s(s) = y^{s^*},$$

and

$$\lambda(s)_1 = \lambda_1^*, \dots, \lambda(s)_{s-1} = \lambda_{s-1}^*, \ \lambda(s)_s = \sum_{i=s}^{k} \lambda_i^*.$$

Then we have

$$\text{opt}(s) - \text{opt}(k) \leq \max_{\xi \in \mathcal{U}} \sum_{i=1}^{s} \lambda(s)_i f(x(s), y^i(s), \xi) - \max_{\xi \in \mathcal{U}} \sum_{i=1}^{k} \lambda_i^* f(x^*, y^{i^*}, \xi).$$

Let $\xi^*(s)$ be a scenario which maximizes the first maximum of the latter expression. Then we can further bound

$$\begin{aligned}
&\max_{\xi \in \mathcal{U}} \sum_{i=1}^{s} \lambda(s)_i f(x(s), y^i(s), \xi) - \max_{\xi \in \mathcal{U}} \sum_{i=1}^{k} \lambda_i^* f(x^*, y^{i^*}, \xi) \\
&\leq \sum_{i=1}^{s} \lambda(s)_i f(x(s), y^i(s), \xi^*(s)) - \sum_{i=1}^{k} \lambda_i^* f(x^*, y^{i^*}, \xi^*(s)) \\
&= \sum_{i=s+1}^{k} \lambda_i^* \left(f(x^*, y^{s^*}, \xi^*(s)) - f(x^*, y^{i^*}, \xi^*(s)) \right) \\
&\leq L \sum_{i=s+1}^{k} \lambda_i^* \| y^{s^*} - y^{i^*} \| \\
&\leq L \text{diam}(\mathcal{Y}) \sum_{i=s+1}^{k} \lambda_i^*,
\end{aligned}$$

where the first inequality follows since $\xi^*(s)$ is optimal for the first maximum and feasible for the second maximum, the first equality follows from the definition of $x(s)$, $y^i(s)$ and $\lambda(s)$, the second inequality follows from the Lipschitz continuity of f, and the last inequality follows from the definition of the diameter. From the sorting $\lambda_1^* \geq \dots \geq \lambda_k^*$ and since $\sum_{i=1}^{k} \lambda_i^* = 1$ it follows $\lambda_i \leq \frac{1}{i}$. Hence we can further bound

$$L\text{diam}(\mathcal{Y}) \sum_{i=s+1}^{k} \lambda_i^* \leq L\text{diam}(\mathcal{Y}) \sum_{i=s+1}^{k} \frac{1}{i} \leq L\text{diam}(\mathcal{Y}) \frac{k-s}{s+1},$$

where the last inequality follows from $\frac{1}{i} \leq \frac{1}{s+1}$ for all $i \geq s+1$. This proves the result. □

The bound in Theorem 5 can depend on the dimension n_y, since $\text{diam}(\mathcal{Y})$ can depend on n_y. However, it goes to zero if $s \to k$. A similar bound was derived in [Kur24] for the linear case and it was shown that the bound leads to interesting conclusions. Similarly, for the non-linear case studied in this work, we can apply Theorem 5 with $k = n_\xi + 1$ to obtain bounds on the quality of the k-adaptable approximation for (2RO); see Figure 1. Furthermore, we can use Theorem 5 to provide bounds on the number of policies k which lead to a certain additive approximation guarantee α.

Corollary 6. *Assume that* $k \geq n_\xi + 1 - \max\{0, \frac{\alpha n_\xi}{L diam(\mathcal{Y}) + \alpha}\}$ *and* $\alpha > 0$. *Then it holds*

$$opt(k) \leq opt(2RO) + \alpha,$$

where $opt(2RO)$ *is the optimal value of* (2RO).

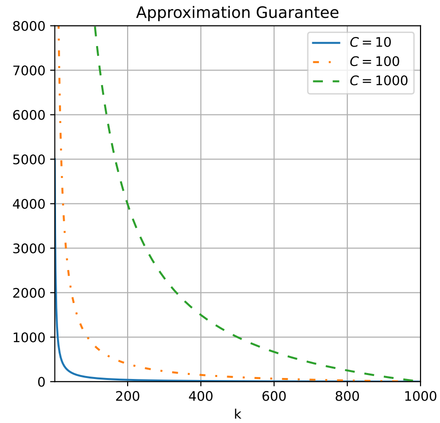

Figure 1: Plot of the additive approximation guarantee $\alpha = C\frac{n_\xi+1-k}{k+1}$ which (k-ARO) provides for (2RO) depending on k for different constants C and $n_\xi = 1000$.

Proof. From Theorem 2 we know that $\mathrm{opt}(2\mathrm{RO}) = \mathrm{opt}(n_\xi + 1)$. Hence, if $\max\{0, \frac{\alpha n_\xi}{L\mathrm{diam}(\mathcal{Y})+\alpha}\} = 0$, the result holds trivially even for $\alpha = 0$. Assume now that $\frac{\alpha n_\xi}{L\mathrm{diam}(\mathcal{Y})+\alpha} > 0$. Then, for $l := \max\{0, \frac{\alpha n_\xi}{L\mathrm{diam}(\mathcal{Y})+\alpha}\}$ from Theorem 5 we obtain

$$\begin{aligned}
\mathrm{opt}(k) - \mathrm{opt}(2\mathrm{RO}) &\le \mathrm{opt}(n_\xi + 1 - l) - \mathrm{opt}(n_\xi + 1) \\
&\le L\mathrm{diam}(\mathcal{Y})\frac{l}{n_\xi + 2 - l} \\
&= \frac{\frac{L\mathrm{diam}(\mathcal{Y})\alpha n_\xi}{L\mathrm{diam}(\mathcal{Y})+\alpha}}{\frac{(n_\xi+2)(L\mathrm{diam}(\mathcal{Y})+\alpha)-\alpha n_\xi}{L\mathrm{diam}(\mathcal{Y})+\alpha}} \\
&= \frac{L\mathrm{diam}(\mathcal{Y})\alpha n_\xi}{n_\xi L\mathrm{diam}(\mathcal{Y}) + 2L\mathrm{diam}(\mathcal{Y}) + 2\alpha} \\
&\le \frac{L\mathrm{diam}(\mathcal{Y})\alpha n_\xi}{n_\xi L\mathrm{diam}(\mathcal{Y})} \\
&= \alpha,
\end{aligned}$$

where the first inequality follows from $k \ge n_\xi + 1 - l$, the second inequality follows from Theorem 5, the first equality follows from the definition of l, and the last inequality follows since $2L\mathrm{diam}(\mathcal{Y}) + 2\alpha \ge 0$. □

4 Constraint Uncertainty

In this section we study the connection between Problems (2RO) and (k-ARO) when the uncertainty appears in the constraints. More precisely, we consider functions

$$f(x, y, \xi) := \begin{cases} g(x, y, \xi) & \text{if } A(\xi)x + B(\xi)y \ge h(\xi) \\ \infty & \text{otherwise,} \end{cases}$$

where $g : \mathcal{X} \times \mathcal{Y} \times \mathcal{U} \to \mathbb{R}$ is a given continuous objective function which is concave in ξ and $A(\xi) \in \mathbb{R}^{m\times n_x}$, $B(\xi) \in \mathbb{R}^{m\times n_y}$ and $h(\xi) \in \mathbb{R}^m$ are the constraint parameters which are given as affine-linear functions of the uncertain parameters. The two-stage robust problem is then given as

$$\inf_{x\in\mathcal{X}} \sup_{\xi\in\mathcal{U}} \inf_{y\in\mathcal{Y}(x)} f(x, y, \xi) \tag{2RO-C}$$

and the k-adaptability problem is given as

$$\inf_{\substack{x\in\mathcal{X}\\ y^1,\dots,y^k\in\mathcal{Y}(x)}} \sup_{\xi\in\mathcal{U}} \inf_{i=1,\dots,k} f(x,y^i,\xi). \tag{k-ARO-C}$$

Note that in contrast to the objective uncertainty case we have to use the infimum and supremum operators since for discontinuous functions f we cannot guarantee that the maximum or minimum is always attained; see [HKW15] for an example. Since $\mathcal{Y}$ is finite at least the inner infimum could be replaced by the minimum operator, but for comprehensibility we will use the infimum operator instead.

Note that for any $x\in\mathcal{X}$ for which a $\xi\in\mathcal{U}$ exists such that there exists no $y\in\mathcal{Y}(x)$ which is feasible for $A(\xi)x+B(\xi)y\geq h(\xi)$, the objective value is ∞. We call such a solution *infeasible*. Furthermore, we assume that (k-ARO-C) with $k=1$ always has at least one feasible solution x, i.e., the problems (2RO-C) and (k-ARO-C) are feasible for any k. Since g is continuous and all sets $\mathcal{X},\mathcal{U},\mathcal{Y}$ are compact, it follows that the optimal value of all the latter problems is finite.

Unfortunately, the bounds on k derived in the previous section are not valid in the constraint uncertainty case. In [HKW15] the authors provide an example where in (k-ARO-C) all $k=|\mathcal{Y}|$ second-stage policies are needed to obtain an optimal solution to (2RO-C). Hence, there is no hope to obtain a better bound in the general setting. However, we will derive better bounds in this section for certain problem structures.

The main idea for the results is presented in the following. Consider any fixed first-stage solution $x\in\mathcal{X}$ which is feasible. Following the reformulation of the proof of Theorem 2 we can reformulate the inner sup-inf problem of (k-ARO-C) for $k=|\mathcal{Y}(x)|$ as

$$\begin{aligned} \sup_{z,\xi}\ & z \\ s.t.\ \ & f(x,y,\xi)-z\geq 0 \quad \forall y\in\mathcal{Y}(x) \\ & \xi\in\mathcal{U},\ z\in\mathbb{R}. \end{aligned} \tag{1}$$

Unfortunately, we cannot apply the same argumentation as in the proof of Theorem 2 since now the function f is not concave in ξ. In fact, the latter problem is a problem with up to $|\mathcal{Y}(x)|$ non-convex constraints and we cannot use Theorem 2 in [CC05] to bound the number of support constraints. However, assume we know a convex region $\mathcal{D}\subset\mathcal{U}$ for which the following holds: for every $y\in\mathcal{Y}(x)$, the solution y is feasible for the constraint system $B(\xi)y\geq h(\xi)-A(\xi)x$ either for all $\xi\in\mathcal{D}$ or for no $\xi\in\mathcal{D}$. We call such a region *recourse-stable* and we denote by $\mathcal{Y}_{\mathcal{D}}(x)$ the set of second-stage solutions in $\mathcal{Y}(x)$ which are feasible for all $\xi\in\mathcal{D}$. Note that $\mathcal{D}$ can be an open set.

If we consider Problem (1) only on a convex recourse-stable region $\mathcal{D}$ (instead of $\mathcal{U}$) then we can remove all constraints for which the corresponding second-stage solution y is infeasible on $\mathcal{D}$ since the left-hand-side constraint value is infinity. For all others, we can replace the function f by the function g, leading to

$$\begin{aligned} \sup_{z,\xi}\ & z \\ s.t.\ \ & g(x,y,\xi)-z\geq 0 \quad \forall y\in\mathcal{Y}_{\mathcal{D}}(x) \\ & \xi\in\operatorname{cl}(\mathcal{D}),\ z\in\mathbb{R}, \end{aligned} \tag{2}$$

where we additionally replaced $\mathcal{D}$ by its closure. This can be done since h,A,B are affine linear functions in ξ and hence the set of ξ which fulfill the constraints $B(\xi)y\geq h(\xi)-A(\xi)x$ for a given $y\in\mathcal{Y}_{\mathcal{D}}(x)$ is closed and contains the set $\mathcal{D}$. It follows that all solutions in $\mathcal{Y}_{\mathcal{D}}(x)$ are also feasible for all $\xi\in\operatorname{cl}(\mathcal{D})$ and using function g instead of f is valid.

Since g is concave in ξ, Problem (2) is a convex problem and since g is continuous in ξ every constraint describes a closed convex set and the supremum can be replaced by the maximum. We can apply Theorem 1 to show that at most $n_\xi+1$ support constraints exists, i.e., we can remove all but $n_\xi+1$ of the second-stage policies without changing the optimal solution. Assume now we have R convex recourse-stable regions $\mathcal{D}_1,\dots,\mathcal{D}_R\subseteq\mathcal{U}$ such that $\operatorname{cl}(\mathcal{D}_1)\cup\ldots\cup\operatorname{cl}(\mathcal{D}_R)=\mathcal{U}$. We can now apply the latter idea to every of the recourse-stable regions, which indicates that we need at most $R(n_\xi+1)$ second-stage policies in total. Note that the recourse-stability of a region depends on the solution $x\in\mathcal{X}$. However, if such a cover of at most R convex recourse-stable regions exists for every x the previous derivation motivates the following Theorem.

Theorem 7. *Let $g : \mathcal{X} \times \mathcal{Y} \times \mathcal{U} \to \mathbb{R}$ be a continuous function such that $g(x,y,\xi)$ is concave in ξ for every $x \in \mathcal{X}, y \in \mathcal{Y}$. Furthermore, assume that for every $x \in \mathcal{X}$ there exist R convex recourse-stable regions $\mathcal{D}_1, \ldots, \mathcal{D}_R \subseteq \mathcal{U}$ such that $\mathrm{cl}(\mathcal{D}_1) \cup \ldots \cup \mathrm{cl}(\mathcal{D}_R) = \mathcal{U}$. Then, if*

$$k \geq \min \{R(n_\xi + 1), |\mathcal{Y}|\},$$

a solution $x \in \mathcal{X}$ is optimal for (k-ARO-C) if and only if it is optimal for (2RO-C).

Proof. Consider any fixed $x \in \mathcal{X}$ and R convex recourse-stable regions $\mathcal{D}_1, \ldots, \mathcal{D}_R \subseteq \mathcal{U}$ such that $\mathrm{cl}(\mathcal{D}_1) \cup \ldots \cup \mathrm{cl}(\mathcal{D}_R) = \mathcal{U}$. Since g is concave in ξ, for every $i \in [R]$ Problem (2) with $\mathcal{D} = \mathcal{D}_i$ is convex and since g is continuous in ξ every constraint corresponds to a convex closed set. Hence, we can apply Theorem 1 which shows that we can remove all constraints except $n_\xi + 1$ support constraints without changing the optimal value of the problem. For every $i \in [R]$ let $y^{i1}, \ldots, y^{i(n_\xi+1)} \in \mathcal{Y}_{\mathcal{D}_i}(x)$ be the solutions related to the support constraints. We define now the problem

$$\begin{aligned} \sup_{z,\xi} \ & z \\ s.t. \quad & f(x, y^{ij}, \xi) - z \geq 0 \quad \forall i \in [R], j \in [n_\xi + 1] \\ & \xi \in \mathcal{U}, \ z \in \mathbb{R}. \end{aligned} \tag{3}$$

which uses at most $R(n_\xi + 1)$ second-stage solutions. To prove the lemma we show that the optimal value of (3) is equal to the optimal value of (1).

First, consider the case where x is an infeasible solution for (2RO-C), i.e., there exists a $\xi \in \mathcal{U}$ such that no $y \in \mathcal{Y}(x)$ is feasible for the constraint system $A(\xi)x + B(\xi)y \geq h(\xi)$. Clearly the optimal value of (1) and (3) are both ∞ in this case.

Now, consider the case where x is a feasible solution, i.e., for every $\xi \in \mathcal{U}$ there exists a feasible second-stage solution. In the following we denote the optimal value of Problem (1) and (3) as opt(1) and opt(3). Since $y^{ij} \in \mathcal{Y}(x)$ for all $i \in [R]$ and $j \in [n_\xi + 1]$, it follows that opt(1) $\leq$ opt(3).

To show the reverse inequality let (ξ^*, z^*) be an optimal solution of (3). Then there exists an $i^* \in [R]$ such that $\xi^* \in \mathrm{cl}(\mathcal{D}_{i^*})$. Hence, we obtain

$$\mathrm{opt}(3) \ = \ \begin{aligned} \sup_{z,\xi} \ & z \\ s.t. \quad & f(x, y^{ij}, \xi) - z \geq 0 \quad i \in [R], j \in [n_\xi + 1] \\ & \xi \in \mathrm{cl}(\mathcal{D}_{i^*}), \ z \in \mathbb{R}, \end{aligned}$$

which is smaller or equal to

$$\begin{aligned} \sup_{z,\xi} \ & z \\ s.t. \quad & f(x, y^{i^*j}, \xi) - z \geq 0 \quad j \in [n_\xi + 1] \\ & \xi \in \mathrm{cl}(\mathcal{D}_{i^*}), \ z \in \mathbb{R}. \end{aligned}$$

The optimal value of the last problem is equal to the optimal value of

$$\begin{aligned} \sup_{z,\xi} \ & z \\ s.t. \quad & g(x, y^{i^*j}, \xi) - z \geq 0 \quad j \in [n_\xi + 1] \\ & \xi \in \mathrm{cl}(\mathcal{D}_{i^*}), \ z \in \mathbb{R}, \end{aligned} \tag{4}$$

since for every $\xi \in \mathcal{D}_{i^*}$ it holds $f(x, y^{i^*j}, \xi) = g(x, y^{i^*j}, \xi)$ since $y^{i^*j} \in \mathcal{Y}_{\mathcal{D}_{i^*}}(x)$. The same can be shown for $\xi \in \mathrm{cl}(\mathcal{D}_{i^*}) \setminus \mathcal{D}_{i^*}$ since h, A, B are affine linear functions in ξ and hence the set of ξ which fulfill the constraints $B(\xi)y \geq h(\xi) - A(\xi)x$ for a given $y \in \mathcal{Y}_{\mathcal{D}_{i^*}}(x)$ is closed and contains the set $\mathcal{D}_{i^*}$. Hence, it must contain $\mathrm{cl}(\mathcal{D}_{i^*})$ and we can conclude that all solutions in $\mathcal{Y}_{\mathcal{D}_{i^*}}(x)$ are also feasible for all $\xi \in \mathrm{cl}(\mathcal{D}_{i^*})$ and we can use function g instead of f on the whole closure.

By the definition of the support constraints the optimal value of (4) is equal to

$$\begin{aligned} \sup_{z,\xi} \ & z \\ s.t. \quad & g(x, y, \xi) - z \geq 0 \quad y \in \mathcal{Y}_{D_{i^*}}(x) \\ & \xi \in \mathrm{cl}(\mathcal{D}_{i^*}), \ z \in \mathbb{R}. \end{aligned} \tag{5}$$

If we can show that any optimal solution of (5) is feasible for (1) then we proved opt(3) $\le$ opt(1). Consider any optimal solution $(\bar{\xi}, \bar{z})$ for (5) with $\bar{\xi} \in \mathcal{D}_{i^*}$. Then it holds that $f(x, y, \bar{\xi}) = g(x, y, \bar{\xi})$ for every $y \in \mathcal{Y}_{\mathcal{D}_{i^*}}(x)$ and $f(x, y, \bar{\xi}) = \infty$ otherwise. From feasibility for (5) it follows that the corresponding solution must be feasible for (1). Now consider the remaining case where $\bar{\xi} \in \mathrm{cl}(\mathcal{D}_{i^*}) \setminus \mathcal{D}_{i^*}$. Then there exists an infinite sequence $\{\bar{\xi}_t\}_{t\in\mathbb{N}}$ with $\bar{\xi}_t \in \mathcal{D}_{i^*}$ and $\lim_{t\to\infty} \bar{\xi}_t = \bar{\xi}$. Set $\bar{z}_t = \min_{y\in\mathcal{Y}_{\mathcal{D}_{i^*}}(x)} g(x, y, \bar{\xi}_t)$. Then $(\bar{\xi}_t, \bar{z}_t)$ is feasible for (5) for every t and $\lim_{t\to\infty}(\bar{\xi}_t, \bar{z}_t) = (\bar{\xi}, \bar{z})$. Since $\bar{\xi}_t \in \mathcal{D}_{i^*}$ for every t, by the discussion above every $(\bar{\xi}_t, \bar{z}_t)$ is feasible for (1) and hence, the optimal value of (1) must be at least $\bar{z}$ which is the optimal value of (5). This shows opt(1) $\ge$ opt(3) and we proved opt(1) = opt(3).

In summary we showed that for every $x \in \mathcal{X}$ there exist at most $R(n_\xi + 1)$ second-stage solutions such that Problems (1) and (3) have the same optimal value which proves the result. □

As for the bound derived in Theorem 2, the result in Theorem 7 is interesting since the bound on k only depends on the dimension of the uncertain parameters. However, the dimension n_y may be hidden in the number R as we will see in the following section. Note, again no convexity is required for g regarding the variables x and y.

Remark 8. *If the objective function g does not depend on ξ, i.e., $g(x, y, \xi) = \bar{g}(x, y)$, then for each recourse-stable region $\mathcal{D}_i$ the Problem (2) is equivalent to*

$$\max_{\xi\in\mathrm{cl}(\mathcal{D}_i)} \min_{y\in\mathcal{Y}_{\mathcal{D}_i}} \bar{g}(x, y)$$

and hence there exists a single solution in $\mathcal{Y}_{\mathcal{D}_i}$ which minimizes $\bar{g}(x, y)$ for every $\xi \in \mathrm{cl}(\mathcal{D}_i)$. It follows that in this case the bound on k from Theorem 7 can be improved to

$$k \ge \min\{R, |\mathcal{Y}|\}.$$

Finally, we can show that for constraint-wise uncertainty and fixed-recourse we only need $k = 1$ second-stage solutions for optimality. A similar result was shown for continuous decisions in [MDH18].

Corollary 9. *Assume g does not depend on ξ, $B(\xi) = B$ for all $\xi \in \mathcal{U}$ and the uncertainty appears constraint-wise, i.e., we consider the problem*

$$\inf_{x\in\mathcal{X}} \sup_{\xi^1,\dots,\xi^m\in\mathcal{U}} \inf_{y\in\mathcal{Y}(x)} f(x, y, \xi)$$

where the constraints are given as

$$a_i(\xi^i)^\top x + b_i^\top y \ge h_i(\xi^i) \quad i \in [m].$$

Then, for $k = 1$ a solution for (k-ARO-C) is optimal if and only if it is optimal for (2RO-C).

Proof. Since for every $x \in \mathcal{X}$ the function $h_i(\xi^i) - a_i(\xi^i)^\top x$ is continuous in ξ^i and $\mathcal{U}$ compact, for every $i \in [m]$ there exists $\bar{\xi}^i \in \mathcal{U}$ which maximizes the latter function over $\mathcal{U}$. This scenario leads to the smallest number of feasible second-stage solutions and is hence a maximizing scenario. The problem reduces to

$$\begin{aligned} &\inf_{x,y} g(x, y) \\ s.t. \quad & a_i(\bar{\xi}^i)^\top x + b_i^\top y \ge h_i(\bar{\xi}^i) \quad i \in [m] \\ & x \in \mathcal{X}, y \in \mathcal{Y}(x) \end{aligned}$$

which shows that only $k = 1$ solutions are needed. □

The main task in the following is to bound the number of recourse-stable regions for certain problem structures to obtain good values for R.

4.1 Bounds on the Number of Recourse-Stable Regions

In this section we derive bounds on the number of recourse-stable regions which are needed to cover the uncertainty set $\mathcal{U}$. By Theorem 7 we obtain then a bound on the number of policies k which are needed to get an optimal solution for (2RO-C).

We assumed that all constraint parameters are given as affine-linear functions of the uncertain parameters ξ, i.e., we have

$$h(\xi) = h + H\xi, \quad A(\xi) = A + \sum_{i=1}^{n_\xi} A^i \xi_i, \quad B(\xi) = B + \sum_{i=1}^{n_\xi} B^i \xi_i$$

where $h \in \mathbb{Z}^m$, $H \in \mathbb{Z}^{m\times n_\xi}$, $A, A^i \in \mathbb{Z}^{m\times n_x}$, $B, B^i \in \mathbb{Z}^{m\times n_y}$ for all $i \in [n_\xi]$ are given parameters. We can reformulate the constraint system $A(\xi)x + B(\xi)y \geq h(\xi)$ as

$$\sum_{i=1}^{n_\xi} (A^i x + B^i y - H_i)\xi_i \geq h - Ax - By \tag{6}$$

where H_i is the i-th column of H. The latter inequality system describes a polyhedron in the ξ-space defined by m halfspaces.

The main idea to derive the results of this section is the following: if we can bound the number of hyperplanes in the ξ-space which can appear (over all different second-stage solutions y) in (6) then we can bound the number of regions which are enclosed by hyperplanes and which are not intersected by any other hyperplane. We will show that the interior of each of these regions is a convex recourse-stable region and taking the union of the closures of all these regions defines a cover for $\mathcal{U}$. Then we can apply Theorem 7 to get a bound on k.

Fix any $x \in \mathcal{X}$ and define the set of all possible hyperplanes appearing in the constraints in (6) over all $y \in \mathcal{Y}(x)$ which intersect with $\mathcal{U}$ as

$$\mathcal{H}(x) := \left\{ H = \{\xi : a_i(x,y)^\top \xi = h_i(x,y)\} : H \cap \mathcal{U} \neq \emptyset,\ y \in \mathcal{Y}(x), i = 1, \dots, m \right\}$$

where $a_i(x,y)$ is the i-th row of the matrix

$$A(x,y) := \left(A^1 x + B^1 y - H_1, \dots, A^{n_\xi} x + B^{n_\xi} y - H_{n_\xi}\right)$$

and $h_i(x,y)$ is the i-th entry of the vector $h(x,y) := h - Ax - By$. Define the maximum number of hyperplanes over all feasible x as

$$\eta := \max_{\substack{x\in\mathcal{X} \\ x \text{ feasible}}} |\mathcal{H}(x)|.$$

Note that η can be significantly smaller than $|\mathcal{Y}|$, e.g. if B, B^i are matrices with integer values. Then η can be bounded by terms in the size of the numbers in B, B^i which we will discuss later in more detail. Consider Example 10 to motivate the results of this section.

Example 10. *Consider the problem without first-stage solutions where the second-stage feasible region is given as*

$$\mathcal{Y} = \left\{ y \in \{0,1\}^2 : y_1 + \xi_2 y_2 \geq \xi_1,\ y_1 + 3y_2 \geq \xi_2 \right\}$$

and uncertainty set $\mathcal{U} = [\frac{1}{2}, \frac{7}{2}] \times [\frac{1}{2}, \frac{5}{2}]$. *We can go through all possible second-stage solutions in* $\{0,1\}^2$ *and draw the corresponding hyperplanes of the two constraints in* $\mathcal{Y}$*; see Figure 2. The interior of the resulting regions are all recourse-stable. Only the ones which intersect with* $\mathcal{U}$ *are relevant. The feasible second-stage solutions for each region are given as*

$$\mathcal{Y}_{\mathcal{D}_1} = \left\{ \begin{pmatrix}1\\1\end{pmatrix} \right\}, \quad \mathcal{Y}_{\mathcal{D}_2} = \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}, \quad \mathcal{Y}_{\mathcal{D}_3} = \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}.$$

We can prove now the following lemma.

Lemma 11. *For every feasible* $x \in \mathcal{X}$ *there exist at most* $R \in \mathcal{O}\left(\eta^{n_\xi}\right)$ *convex recourse-stable regions such that the union of its closures covers* $\mathcal{U}$.

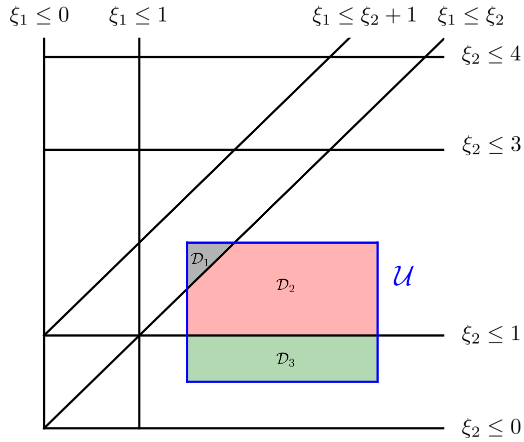

Figure 2: All hyperplanes in $\mathcal{H}(x)$ and corresponding recourse stable regions for Example 10.

Proof. Let $x \in \mathcal{X}$ be an arbitrary feasible solution. Consider $\mathcal{H}(x)$ which contains at most η hyperplanes. This set of hyperplanes induces a set of full-dimensional regions, where each region is enclosed by a subset of these hyperplanes and no other hyperplane in $\mathcal{H}(x)$ is intersecting the region. It was shown in Lemma 4 in [STR18] that this number of regions can be bounded by

$$\sum_{i=0}^{\operatorname{rank}(V)} \binom{\eta}{i} \in \mathcal{O}\left(\eta^{\operatorname{rank}(V)}\right),$$

where V is the matrix which results from the concatenation of all row vectors $a_i(x, y)$ appearing in $\mathcal{H}(x)$. We can bound the rank of V by n_ξ which yields the number of regions stated in the lemma.

Next, we have to show that the interior of each of the induced regions is in fact a convex recourse-stable region. Convexity follows since each region is polyhedral. To show recourse-stability consider any full-dimensional region $\bar{\mathcal{D}}$ induced by the set of hyperplanes in $\mathcal{H}(x)$. For any ξ in the interior of this region consider any $y \in \mathcal{Y}(x)$ which is feasible for this ξ. Then ξ lies in the polyhedron (6) for this y. Since none of the hyperplanes which defines (6) intersects the interior of $\bar{\mathcal{D}}$, the whole interior of the region must be contained in (6) and hence y is feasible for all ξ in the interior of this region. Hence, we proved that the interior of each region is recourse-stable.

Finally, we have to show that the union of the closures of these regions covers $\mathcal{U}$. This is trivially the case since we actually bounded the number of regions to cover the whole space $\mathbb{R}^{n_\xi}$. □

The bound on the size of the cover can be improved if we consider fixed recourse, i.e., the recourse matrix $B(\xi)$ does not depend on ξ.

Lemma 12. *Assume fixed recourse, i.e., $B(\xi) =: B$ for all $\xi \in \mathcal{U}$. For every feasible $x \in \mathcal{X}$ there exists a cover of at most $R \in \mathcal{O}\left(\eta^{\min\{m, n_\xi\}}\right)$ convex recourse-stable regions for $\mathcal{U}$.*

Proof. In the case of fixed recourse, we have $B^i = 0$ for all $i \in [n_\xi]$. Hence, $A(x, y)$ is the same matrix for every $y \in \mathcal{Y}(x)$. Following the proof of Lemma 11 the number of regions for the cover can be bounded by $\mathcal{O}\left(\eta^{\operatorname{rank}(V)}\right)$ where V is the matrix derived from concatenating the same matrix $A(x, y)$ multiple times. Hence, V has rank at most $\min\{m, n_\xi\}$. Following the rest of the proof of Lemma 11 proves the result. □

We can summarize the latter results now in the following theorem.

Theorem 13. *Let $g : \mathcal{X} \times \mathcal{Y} \times \mathcal{U} \to \mathbb{R}$ be a continuous function such that $g(x, y, \xi)$ is concave in ξ for every $x \in \mathcal{X}, y \in \mathcal{Y}$. Then, the number of second-stage policies needed in (k-ARO-C) to ensure an optimal solution for (2RO-C) is*

- *$k \in O\left(\eta^{\min\{m, n_\xi\}}(n_\xi + 1)\right)$ if fixed recourse holds*

- $k \in \mathcal{O}\left(\eta^{n_\xi}(n_\xi+1)\right)$ *if random recourse holds.*

Proof. The result directly follows from Lemma 11 and 12 together with Theorem 7. □

By Remark 8 it follows that, if g does not depend on ξ, we can improve the bounds to $k \in O\left(\eta^{\min\{m,n_\xi\}}\right)$ if fixed recourse holds and $k \in \mathcal{O}\left(\eta^{n_\xi}\right)$ if random recourse holds.

Calculating the value η which is required for the bounds in Theorem 13 is not always easy. In the following remarks we provide bounds which are easier to calculate.

Remark 14. *Assume $\mathcal{Y} \subseteq \mathbb{Z}_+^{n_y} \cap [0,u]$ where $u \in \mathbb{Z}_+^{n_y}$ are given upper bounds on the second-stage decision values. Furthermore, assume fixed recourse and $B(\xi) = B$ with $B \in \mathbb{Z}^{n_y}$. Since B is integer for each of the m rows b_i, the term $b_i^\top y$ can attain at most*

$$\beta = \max_{i=1,\dots,m} 2|b_i|^\top u$$

different values, where $|b_i|$ denotes the vector containing the absolute values of all entries in b_i. We can conclude that there are at most β values for $h_i(x,y)$. Since all $B^i = 0$ there is only one matrix $A(x,y)$. For each of the m constraints we therefore have at most β many right-hand-side values and we can conclude that $\mathcal{H}(x)$ contains at most $\eta \leq m\beta$ hyperplanes. Note that this value only depends on the number of constaints and the values in B and not on the dimensions of the problem.

Remark 15. *The number of right-hand-side values in Remark 14 can be improved in some situations. Note that for every right-hand-side value in (6) which is larger than all left-hand-side values, the corresponding regions do not intersect with $\mathcal{U}$. On the other hand if the right-hand-side value is smaller than the smallest left-hand-side value the regions contain the full set $\mathcal{U}$. Hence, we do not need to consider these values in $\mathcal{H}(x)$.*

Remark 16. *Assume $\mathcal{Y} \subseteq \mathbb{Z}_+^{n_y} \cap [0,u]$ where $u \in \mathbb{Z}_+^{n_y}$ are given upper bounds on the second-stage decision values. Furthermore, assume random recourse and $B, B^i \in \mathbb{Z}^{n_y}$ for all i. Define similar to the previous remark the number of possible values which can appear in row i of matrix B^j, i.e., the values $(b_i^j)^\top y$ over all $y \in \mathcal{Y}(x)$ as*

$$\tilde{\beta} = \max_{i=1,\dots,m} \max_{j=1,\dots,n_\xi} 2|b_i^j|^\top u,$$

where $|b_i^j|$ denotes the vector containing the absolute values of all entries in b_i^j. Fix one constraint $i \in [m]$ and consider all hyperplanes $a_i(x,y)^\top \xi = h_i(x,y)$ in $\mathcal{H}(x)$. For $a_i(x,y)$ we have at most $\tilde{\beta}^{n_\xi}$ vectors and for $h_i(x,y)$ at most β values, as derived in Remark 14. Hence we have at most $\tilde{\beta}^{n_\xi}\beta$ hyperplanes for constraint i which leads to $\eta \leq m\tilde{\beta}^{n_\xi}\beta$.

We now present applications studied in other works and apply the previous results.

Example 17 (Capital Budgeting with Constraint Uncertainty)**.** *Consider the capital budgeting problem from Example 4 where now also the budget-constraint contains uncertain parameters, i.e., we have $\mathcal{Y}(x) = \{y \in \{0,1\}^n : c(\xi)^\top(x+y) \leq B,\ x+y \leq e\}$ where again $\mathcal{U} = [-1,1]^\rho$ is an uncertainty set of all realizations of ρ different risk factors and e is the all-one vector. The costs of project i are given as $c_i(\xi) = (1 + \frac{1}{2}\Phi_i^\top\xi)c_i^0$ where Φ_i is the i-th row of a given factor loading matrix Φ. We assume that all entries of Φ and c^0 are integer, which can be obtained after scaling.*

We can reformulate the budget constraint as

$$\sum_{j=1}^{\rho}\left(\sum_{i=1}^{n} c_i^0(x_i+y_i)\Phi_{ij}\right)\xi_j + \sum_{i=1}^{n} c_i^0(x_i+y_i) \leq B. \tag{7}$$

To apply Theorem 13 we have to calculate η, which is the maximum number of hyperplanes which can appear in (7) over all y for a given x. The maximum number of options appears clearly if $x=0$. In this case the maximum number of values over all y for each coefficient $\sum_{i=1}^n c_i^0(x_i+y_i)\Phi_{ij}$ is

$$\phi := 2\max_{j=1,\dots,\rho}\sum_{i=1}^{n}|\Phi_{ij}c_i^0|.$$

The constant term can have at most $\bar{c} = 2\sum_{i=1}^{n} |c_i^0|$ *values over all* y*. Hence the number of possible hyperplanes is* $\eta \leq \phi^\rho \bar{c}$*. From Theorem 13 it follows that we need at most* $k \in \mathcal{O}\left((\phi^\rho \bar{c})^\rho (\rho + 1)\right)$ *second-stage policies to ensure optimality. From Remark 8 it follows, that if the objective parameters are not uncertain this bound improves to* $k \in \mathcal{O}\left((\phi^\rho \bar{c})^\rho\right)$*. Note again that* ρ *is usually a fixed and small number and the actual number of possible second-stage solutions can be* $|\mathcal{Y}| = 2^{n_y}$*.*

Example 18. *Consider a capacitated facility location problem with uncertain transportation costs and uncertain capacities. We have a set of customers* $\mathcal{J}$ *and a set of locations* $\mathcal{I}$ *and transportation costs* $t_{ij}(\xi)$ *between each* $i \in \mathcal{I}$ *and* $j \in \mathcal{J}$ *which depend on the uncertain scenario* $\xi \in \mathcal{U}$*. Furthermore, each location* $i \in \mathcal{I}$ *has a capacity* $C_i(\xi) = c_i^\top \xi$ *which is uncertain as well. Each customer* $j \in \mathcal{J}$ *has a known integer demand* $d_j > 0$*. We are allowed to open at most* p *facilities in the first-stage and for every scenario* ξ *we afterwards have to assign each customer to an opened facilities such that the sum of assigned demands for each facility does not exceed the capacity. The problem can be formulated in the form (2RO-C) where* $\mathcal{X} = \left\{x \in \{0,1\}^{\mathcal{I}} : \sum_{i\in\mathcal{I}} x_i \leq p\right\}$ *and*

$$\mathcal{Y}(x) = \left\{ y \in \{0,1\}^{\mathcal{I}\times\mathcal{J}} : \sum_{i\in\mathcal{I}} y_{ij} = 1 \ \forall j \in \mathcal{J}, \ \sum_{j\in\mathcal{J}} d_j y_{ij} \leq c_i^\top \xi x_i \ \forall i \in \mathcal{I} \right\}.$$

The objective function is $g(y,\xi) = \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} t_{ij}(\xi) y_{ij}$*. To calculate* η *first observe that, since at most* p *facilities can be openend, at most* p *of the capacity constraints are non-zero. Since all demands are integer, the left-hand-side of the capacity constraints can only take the values* $\{0, 1, \ldots, D\}$*, where* $D = \sum_{j\in\mathcal{J}} d_j$*. For each opened facility the coefficient vector* c_i *is fixed which leads to* $D + 1$ *different hyperplanes for each opened facility. In total we have at most* $\eta \leq p(D+1)$ *hyperplanes. Applying Theorem 13 show that we need at most* $k \in \mathcal{O}\left((p(D+1))^p (n_\xi + 1)\right)$ *second-stage policies. From Remark 8 it follows, that if the travel costs are not uncertain this bound improves to* $k \in \mathcal{O}\left((p(D+1))^p\right)$*. In contrast, the number of second-stage solutions can be* $|\mathcal{Y}| = 2^{|\mathcal{I}||\mathcal{J}|}$*.*

5 Conclusion

In this work we derived bounds on the number k of second-stage solutions which are needed such that the k-adaptability approach returns an optimal solution for the original two-stage robust problem. We distinguished the two cases of objective uncertainty and constraint uncertainty. Interestingly, for objective uncertainty the number of solutions needed is $k = n_\xi + 1$, i.e., it depends only on the dimension of the uncertainty. This results hold for a very general class of (non-linear) objective functions. We used the latter result to derive approximation guarantees the k-adaptability problem provides for all values of k smaller than the bound above.

For constraint uncertainty we developed a new concept called recourse-stability. A recourse-stable region is a subset of the uncertainty set such that each second-stage solution is either feasible or infeasible on the whole set. We could show that to guarantee optimality we need at most $k = R(n_\xi + 1)$ second-stage solutions, where R is the number of recourse-stable regions needed to cover the uncertainty set. We show that we can determine a value for R by considering all possible hyperplanes which can appear in the uncertain constraints by plugging in all possible second-stage solutions. Examples show that the derived value for R provides good bounds on k for many problem structures.

There remain several open questions to be tackled in the future. First, it would be interesting if we can achieve better values for R by focusing on certain applications and the corresponding problem structures. Furthermore, it would be interesting if approximation bounds as for objective uncertainty could also be derived for the constraint uncertainty case. Finally, an important question is if the methodology derived in this work can contribute to the development of efficient solution methods.

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