FACETS FROM SOLITARY ITEMS FOR THE 0/1 KNAPSACK POLYTOPE

SHREYAS BHOWMIK* AND SACHIN JAYASWAL*

Abstract. We present a new class of valid inequalities, called *Solitary item inequality*, which is facet-defining for any 0/1 knapsack polytope. We prove that any facet-defining inequality of 0/1 knapsack polytope with nonnegative integral coefficients and right hand side 1 belongs to this class, and hence, the set of facet-defining inequalities corresponding to strong covers of cardinality 2 is its special case. Next, we provide a counterexample to show that Theorem 6.2 in [13], which claims to provide the complete characterization of the convex hull of a special type of 0/1 knapsack set, called a *graphic* knapsack set, is invalid. Furthermore, we define a subset of the *graphic* knapsack set, for which the above theorem becomes valid. We also show that the convex hull of another subset of the *graphic* knapsack set for which Theorem 6.2 in [13] is invalid, is completely characterized by *Solitary item inequalities*, along with the trivial nonnegative inequalities.

Key words. 0/1 Knapsack, Knapsack polytope, Solitary item inequalities, Facets, Convex hull

MSC codes. 90C10

1. Introduction. A 0/1 knapsack problem is defined as follows. Given a set of items $N = \{1, 2, \cdots, n\}$ with weights a_i and profits p_i $\forall i \in N$, select a subset of N to pack in a knapsack of limited capacity b that maximizes the total profit. Corresponding to a 0/1 knapsack problem, the knapsack set X is defined as $X = \{x \in \mathbb{B}^n : \sum_{i \in N} a_i x_i \leqslant b\}$ and the knapsack polytope is represented by its convex hull $K = \text{conv}(X)$. Since a 0/1 knapsack problem is known to be $\mathcal{NP}$-hard [6], a partial characterization of K using a subset of its facets is of significant interest and has been extensively studied in the literature. Cover inequalities are among the initial categories of valid inequalities for K, which under certain conditions become facet-defining [1]. For cover inequalities that are not facet-defining for a given K, there are several procedures to strengthen them, such as (sequential) up-lifting [1, 2, 8, 13], (sequential) down-lifting [16] and simultaneous lifting [5, 14]. Weismantel proposed another class of inequalities, called weight inequalities. He also provided a way to strengthen weight inequalities using a reduction parameter and conditions under which the strengthened inequalities, called weight reduction inequalities, become facet-defining [12]. Parberg proposed another class of inequalities based the idea of (1,k) configurations [9], which was generalized in [4].

Despite the difficulty of completely characterizing knapsack polytopes in general, there is a stream of literature that provides complete characterization of special classes of the knapsack polytope. A complete linear description of *graphic* knapsack polytope [13], *weakly super-increasing* knapsack polytope [7], special-weight knapsack polytope (where the weights of the items belong to a set containing only two elements) [11], and *sequential* knapsack polytope [10] are well known. For a comprehensive review on the knapsack polytope, we refer the reader to [6].

The rest of the paper is organized as follows: In section 2, we introduce a new class of valid inequalities (VIs) for 0/1 knapsack polytopes and demonstrate that they are always facet-defining. We also show that this class includes any facet-defining inequality of a 0/1 knapsack polytope with nonnegative integral coefficients and a right-hand side of 1. This means that the set of facet-defining inequalities that correspond to

*Operations & Decision Sciences, Indian Institute of Management Ahmedabad, Vastrapur, Ahmedabad, 380015, Gujarat, India (phd22shreyasb@iima.ac.in, sachin@iima.ac.in).

strong covers of cardinality 2 is a special case of this class. In section 3, we introduce a special class of 0/1 knapsack sets, referred to as *graphic* knapsack set, for which [13] claims to provide the complete characterization of its convex hull. Specifically, Theorem 6.2 in [13] states that the convex hull of a *graphic* knapsack set is given by the set of all inequalities corresponding to strong covers of cardinality 2, along with the trivial nonnegative inequalities. We provide a counterexample to show that this set of inequalities may be insufficient in some cases. Further, we define a subset of the *graphic* knapsack set, called *1-graphic* knapsack set, for which the above theorem becomes valid. In subsection 3.1, we show that *Solitary item inequalities*, along with the trivial nonnegative inequalities, are sufficient to completely characterize the convex hull of another subset of *graphic* knapsack set (which we refer to as *2-graphic* knapsack set), for which Theorem 6.2 in [13] fails.

2. Solitary item inequality. For the rest of the paper, we consider the weights of the items and the knapsack capacity to be a positive integer as otherwise the set can be represented in similar fashion [6]. To ensure the full dimensionality of the convex hull of the knapsack set, we assume that the weights of all the items are at most equal to the knapsack capacity. We also assume that the items are sorted in the order of non-decreasing weights. Under these assumptions, a 0/1 knapsack set is defined as $X = \{x \in \mathbb{B}^n : \sum_{i\in N} a_i x_i \leqslant b,\ 0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant b, a_i \in \mathbb{Z}_> \ \forall i \in N, b \in \mathbb{Z}_>\}$ and its polytope $\mathrm{K} = \mathrm{conv}(X)$.

Definition 2.1 (Type 1 solitary item). *Given a 0/1 knapsack set X, we define $k \in N$ as a Type 1 solitary item if $2a_k \leqslant b+1$.*

Definition 2.2 (Type 2 solitary item). *Given a 0/1 knapsack set X and $a_0 = 0$, we define $k \in N$ as a Type 2 solitary item if: (i) $2a_k > b+1$; (ii) $a_{k-1} + a_k \leqslant b$.*

Proposition 2.3. *If $k \in N$ is a Type 1 solitary item, then $m \in N : m \leqslant k-1$ is also a Type 1 solitary item.*

Proof. Since $k \in N$ is a *Type 1 solitary item*, $2a_k \leqslant b+1$, which implies $2a_m \leqslant b+1$ (since $a_m \leqslant a_k\ \forall m \in N : m \leqslant k-1$). ∎

Proposition 2.4. *If $k \in N$ is not a Type 1 solitary item, then there is no Type 1 solitary item $m \in N : m \geqslant k+1$.*

Proof. Since $k \in N$ is not a *Type 1 solitary item*, $2a_k > b+1$, which implies $2a_m > b+1$ (since $a_m \geqslant a_k\ \forall m \in N : m \geqslant k+1$). ∎

Proposition 2.5. *If $k \in N$ is a Type 2 solitary item, then $m \in N : m \leqslant k-1$ is a Type 1 solitary item.*

Proof. Since $k \in N$ is a *Type 2 solitary item*, $a_k + a_{k-1} \leqslant b$, which implies $2a_{k-1} \leqslant a_k + a_{k-1} \leqslant b < b+1$. Therefore, item $k-1$ is a *Type 1 solitary item*. Further, using Proposition 2.3, item $k-1$ is a *Type 1 solitary item* implies $m \in N : m \leqslant k-2$ is also a *Type 1 solitary item*. ∎

Proposition 2.6. *$\exists$ at most one Type 2 solitary item.*

Proof. Let item $k \in N$ be a *Type 2 solitary item*. From Proposition 2.5, $m \in N : m \leqslant k-1$ is a *Type 1 solitary item*, which implies that they cannot be *Type 2 solitary item*. Furthermore, since $k \in N$ is a *Type 2 solitary item*, $2a_k > b+1$, which implies $a_m + a_{m-1} > b+1\ \forall m \in N : m \geqslant k+1$. Therefore, $m \in N : m \geqslant k+1$ cannot be a *Type 2 solitary item*. ∎

Proposition 2.7. *If $k \in N$ is a Type 2 solitary item, then there is no Type 1 solitary item $m \in N : m \geqslant k+1$.*

Proof. The proof follows directly from Proposition 2.4. ∎

Definition 2.8 (Solitary item inequality). *Given a Type 1 solitary item or a Type 2 solitary item $k \in N$, we define a Solitary item inequality as:*

$$x_k + \sum_{j \in N \setminus \{k\}: a_j \geqslant b - a_k + 1} x_j \leqslant 1 \tag{2.1}$$

Proposition 2.9. *A Solitary item inequality* (2.1) *is valid for* $\mathrm{K} = \mathrm{conv}(X)$.

Proof. We show that (2.1) is facet-defining for $\mathrm{K} = \mathrm{conv}(X)$ when k in (2.1) is: (i) a *Type 1 solitary item*; (ii) a *Type 2 solitary item*.

(i) Let $k \in N$ is a *Type 1 solitary item*. Consider the set $N' := \{j \in N \setminus \{k\} : a_j \geqslant b - a_k + 1\}$. Then, it is easy to see the following:
  (a) $x_k + x_j \leqslant 1 \; \forall j \in N'$ since $a_j \geqslant b - a_k + 1$, which implies $a_k + a_j > b$.
  (b) $x_i + x_j \leqslant 1 \; \forall i, j \in N' : i \neq j$ since $a_i + a_j \geqslant 2(b - a_k + 1) = 2b + 2 - 2a_k \geqslant b + 1 > b$.

  (a) and (b) above together imply (2.1).

(ii) Let $k \in N$ is a *Type 2 solitary item*. Consider the set $\widetilde{N} := \{j \in N \setminus \{k\} : a_j \geqslant b - a_k + 1\}$. Then, it is easy to see the following:
  (a) $x_k + x_j \leqslant 1 \; \forall j \in \widetilde{N}$ since $a_j \geqslant b - a_k + 1$, which implies $a_k + a_j > b$.
  (b) Since item k is a *Type 2 solitary item*, $a_k + a_{k-1} \leqslant b$, which implies $k - 1 \notin \widetilde{N}$. This also implies that $m \in N : m \leqslant k - 2 \notin \widetilde{N}$. From Proposition 2.6, we know that $a_k + a_{k+1} > b$, which implies $k + 1 \in \widetilde{N}$, which in turn implies $m \in N : m \geqslant k + 2 \in \widetilde{N}$. Therefore, $\widetilde{N} = \{k+1, k+2, \cdots, n\}$, which implies $x_i + x_j \leqslant 1 \; \forall i, j \in \widetilde{N} : i \neq j$ since $a_i + a_j \geqslant 2a_k > b + 1$.

  (a) and (b) above together imply (2.1). ∎

Example 2.10. *Consider a 0/1 knapsack set $X = \{x \in \mathbb{B}^5 : 4x_1 + 6x_2 + 8x_3 + 9x_4 + 10x_5 \leqslant 14\}$. Following are the Solitary item inequalities for* $\mathrm{K} = \mathrm{conv}(X)$.

- *Corresponding to the Type 1 solitary item* 1 *(since* $2a_1 = 8 < b + 1 = 14 + 1 = 15$*), the Solitary item inequality is* $x_1 \leqslant 1$.
- *Corresponding to the Type 1 solitary item* 2 *(since* $2a_2 = 12 < b + 1 = 15$*), the Solitary item inequality is* $x_2 + x_4 + x_5 \leqslant 1$.
- *Corresponding to the Type 2 solitary item* 3 *(since* $2a_3 = 16 > b + 1 = 15$*, and* $a_2 + a_3 = 14 = b$*), the Solitary item inequality is* $x_3 + x_4 + x_5 \leqslant 1$.

Proposition 2.11. *A Solitary item inequality* (2.1) *is facet-defining for* $\mathrm{K} = \mathrm{conv}(X)$.

Proof. We show that (2.1) is facet-defining for $\mathrm{K} = \mathrm{conv}(X)$ when k in (2.1) is: (i) a *Type 1 solitary item*; (ii) a *Type 2 solitary item*.

(i) Let $k \in N$ is a *Type 1 solitary item*. Consider the set $N' := \{j \in N \setminus \{k\} : a_j \geqslant b - a_k + 1\}$. We have already shown in Proposition 2.9 that a *Solitary item inequality* (2.1) is valid for $\mathrm{K} = \mathrm{conv}(X)$. Let e_i denote the ith unit vector in $\mathbb{R}^n$. Now, consider the following points:
  - $e_i \; \forall \; i \in \{k\} \cup N'$: clearly, these points are feasible for X and satisfy (2.1) at equality.
  - $e_k + e_j \; \forall j \in N \setminus (\{k\} \cup N')$: these points are also feasible for X (as $a_k + a_j \leqslant b \; \forall j \in N \setminus (\{k\} \cup N')$) and satisfy (2.1) at equality.

Hence, the above $|N|$ points lie on the face $F := \{x \in \text{conv}(X) : x_k + \sum_{j\in N\setminus\{k\}: a_j \geqslant b-a_k+1} x_j = 1\}$. It is easy to show that all these points are affinely independent; hence, F is of dimension $|N|-1$. So, a *Solitary item inequality* (2.1) corresponding to a *Type 1 solitary item* $k \in N$ is facet-defining for $\mathrm{K} = \text{conv}(X)$.

(ii) Let $k \in N$ is a *Type 2 solitary item*. Consider the set $\widetilde{N} := \{j \in N \setminus \{k\} : a_j \geqslant b - a_k + 1\}$, which is equal to $\{k+1, k+2, \cdots, n\}$ as shown in the proof of Proposition 2.9. We have already shown that in Proposition 2.9, a *Solitary item inequality* (2.1) is valid for $\mathrm{K} = \text{conv}(X)$. Let e_i denote the ith unit vector in $\mathbb{R}^n$. Now, consider the following points:
- $e_i \ \forall\, i \in \{k\} \cup \widetilde{N}$: clearly, these points are feasible for X and satisfy (2.1) at equality.
- $e_k + e_j \ \forall j \in N \setminus (\{k\} \cup \widetilde{N})$: these points are also feasible for X (as $a_k + a_j \leqslant a_k + a_{k-1} \leqslant b \ \forall j \in N \setminus (\{k\} \cup \widetilde{N})$) and satisfy (2.1) at equality.

Hence, the above $|N|$ points lie on the face $F := \{x \in \text{conv}(X) : x_k + \sum_{j\in N\setminus\{k\}: a_j \geqslant b-a_k+1} x_j = 1\}$. It is easy to show that all these points are affinely independent; hence, F is of dimension $|N|-1$. So, a *Solitary item inequality* (2.1) corresponding to a *Type 2 solitary item* $k \in N$ is facet-defining for $\mathrm{K} = \text{conv}(X)$. ∎

EXAMPLE 2.12 (Continued). *For Example 2.10, all the Solitary item inequalities identified above are facet-defining.*

PROPOSITION 2.13. *Given $\pi_i \in \mathbb{Z}_\geqslant \ \forall i \in N \cup \{0\}$, if $\sum_{i\in N} \pi_i x_i \leqslant \pi_0$ is a facet-defining inequality for $\mathrm{K} = \text{conv}(X)$, then $\pi_i \leqslant \pi_0 \ \forall i \in N$.*

Proof. We prove this by contradiction. For this, let $\exists j \in N : \pi_j > \pi_0$ such that

$$\sum_{i\in N\setminus\{j\}} \pi_i x_i + \pi_j x_j \leqslant \pi_0 \tag{2.2}$$

is facet-defining for $\mathrm{K} = \text{conv}(X)$. Clearly, $x_j = 0$ for (2.2) to be valid for $\text{conv}(X)$. Hence, (2.2) can define a face of $\text{conv}(X)$ of dimension at most $|N|-2$ and, therefore, cannot be facet-defining. This proves that given $\pi_i \in \mathbb{Z}_\geqslant \ \forall i \in N \cup \{0\}$, if $\sum_{i\in N} \pi_i x_i \leqslant \pi_0$ is a facet-defining inequality for $\mathrm{K} = \text{conv}(X)$, then $\pi_i \leqslant \pi_0 \ \forall i \in N$. ∎

COROLLARY 2.14. *Given $\pi_i \in \mathbb{Z}_\geqslant \ \forall i \in N$, if $\sum_{i\in N} \pi_i x_i \leqslant 1$ is a facet-defining inequality for $\mathrm{K} = \text{conv}(X)$, then $\pi_i \in \{0,1\} \ \forall i \in N$.*

Proof. The proof immediately follows from Proposition 2.13. ∎

THEOREM 2.15. *Given $\pi_i \in \mathbb{Z}_\geqslant \ \forall i \in N$, any facet-defining inequality for K that is of the form $\sum_{i\in N} \pi_i x_i \leqslant 1$ is a Solitary item inequality.*

Proof. From Corollary 2.14, we know that any facet-defining inequality of K given by $\sum_{i\in N} \pi_i x_i \leqslant 1$, where $\pi_i \in \mathbb{Z}_\geqslant \ \forall i \in N$, can be expressed as $\sum_{i\in S\subseteq N} x_i \leqslant 1$ (since $\pi_i \in \{0,1\} \ \forall i \in N$). We need to show that the facet-defining inequality $\sum_{i\in S\subseteq N} x_i \leqslant 1$ must be a *Solitary item inequality*, which we prove by contradiction. For this, assume that $\exists$ a facet-defining inequality $\sum_{i\in S\subseteq N} x_i \leqslant 1$, which is not a *Solitary item inequality*. Since the inequality is associated neither with a *Type 1 solitary item* nor with a *Type 2 solitary item*, the following conditions hold $\forall i \in S$:

(i) $2a_i > b + 1$

(ii) $a_i + a_{i-1} > b$

Now, let $l := \min\{\text{argmin}_{j\in S} a_j\}$, i.e., it is the lowest index among the items with the lowest weight in the set S. For l, we make the following claims:

CLAIM 2.16. $l \neq 1$.

Proof. This is true since condition (ii) above gets violated for $l = 1$ as $a_0 = 0$ by assumption. ☐

CLAIM 2.17. $l \neq i\ \forall i \in N \setminus \{1\}$.

Proof. We prove this by contradiction. For this, suppose $l \in N \setminus \{1\}$. Then, $a_{l-1} + a_i \geqslant a_{l-1} + a_l > b\ \forall i \in S$ (from condition (ii) above). This implies $x_{l-1} + \sum_{i\in S} x_i \leqslant 1$, which contradicts the fact that $l = \min\{\text{argmin}_{j\in S}\{a_j\}\}$. ☐

From Claims 2.16 and 2.17, we conclude that $S = \emptyset$, which implies that $\sum_{i\in S\subseteq N} x_i \leqslant 1$ reduces to $0 \leqslant 1$, which cannot be a facet-defining inequality. This contradicts our assumption that the inequality $\sum_{i\in S\subseteq N} x_i \leqslant 1$, which is not a *Solitary item inequality*, is facet-defining. Hence, any facet defining inequality of the form $\sum_{i\in N} \pi_i x_i \leqslant 1$ where $\pi_i \in \mathbb{Z}_{\geqslant}\ \forall i \in N$ for K must be a *Solitary item inequality.* ☐

COROLLARY 2.18. *Any inequality corresponding to a strong cover (or possibly an extension of strong cover) of cardinality 2 that is facet-defining for* K *is a Solitary item inequality.*

Proof. This follows immediately from Theorem 2.15. ☐

In the next section, we introduce the *graphic* knapsack set, for which Theorem 6.2 in [13] claims to provide the complete characterization of its convex hull. We provide a counterexample to show that the set of inequalities described in the Theorem 6.2 in [13] may be insufficient in some cases. We also define *1-graphic* knapsack set, for which the above theorem becomes valid. Furthermore, we show that *Solitary item inequalities*, along with the trivial nonnegative inequalities, are sufficient to completely characterize the convex hull of the *2-graphic* knapsack set, for which Theorem 6.2 in [13] fails.

3. Graphic knapsack set.

DEFINITION 3.1 (*Graphic* Knapsack). *X is called graphic*[1] *[13] if there exists* $t \in \{1, 2, \cdots, n-1\}$ *for which the following conditions hold:*
(i) $a_t + a_{t+1} > b$,
(ii) $\sum_{i=1}^{t} a_i \leqslant b$

Theorem 6.2 in [13] claims that the set of all inequalities corresponding to (extended) strong covers of cardinality 2, along with the trivial nonnegative inequalities, is sufficient to define the convex hull of the *graphic* knapsack set. We provide a counterexample to show that this set might be insufficient.

EXAMPLE 3.2. *Consider a 0/1 knapsack set* $X = \{x \in \mathbb{B}^5 : 1x_1 + 1x_2 + 2x_3 + 4x_4 + 4x_5 \leqslant 5\}$.
Here, X is graphic since for $t = 3$*: (i)* $a_t + a_{t+1} = 2 + 4 = 6 > 5 = b$*; (ii)* $\sum_{i=1}^{t} a_i = 1 + 1 + 2 = 4 < 5 = b$. *The complete convex hull of X, as obtained from PORTA [3], is shown in Table 1.*

[1]Please note that our notation assumes that the items are sorted in a nondecreasing order of their weights, which is consistent with the recent literature on knapsack problems [for example, see 6]. This is in contrast to the notation used in [13], which assumes that the items are sorted in a noninecreasing order of their weights.

TABLE 1
Facets of Example 3.2 generated from PORTA

Sl. No.	Facets of Example 3.2	Remark
1	$x_1 \geqslant 0$	Trivial
2	$x_2 \geqslant 0$	Trivial
3	$x_3 \geqslant 0$	Trivial
4	$x_4 \geqslant 0$	Trivial
5	$x_5 \geqslant 0$	Trivial
6	$x_1 \leqslant 1$	Not a cover inequality
7	$x_2 \leqslant 1$	Not a cover inequality
8	$x_3 + x_4 + x_5 \leqslant 1$	Cardinality 2 strong cover ({3,4}) inequality
9	$x_1 + x_2 + x_4 + x_5 \leqslant 2$	Not a cardinality 2 strong cover inequality

Clearly, the facet-defining inequalities $6, 7,$ *and* 9 *do not correspond to any cover (strong or otherwise) of cardinality 2. This example highlights the invalidity of the claim in Theorem 6.2 in [13]. The flaw in its proof lies in the following claims:*

- *Any strong cover of a graphic knapsack set is of cardinality 2.*
- $\forall i \in \{1, 2, \cdots, t\}, \exists j \in \{t+1, t+2, \cdots, n\}$ *such that* (i, j) *forms a strong cover*[2].

Clearly, the above claims are not valid in our example since {1,2,4} is a strong cover of cardinality 3, and there exists no item that can form a strong cover either with item 1 or with item 2.

Next, we provide a subset of *graphic* knapsack set, which we refer to as *1-graphic*, for which the theorem becomes valid.

DEFINITION 3.3 (*1-graphic* Knapsack). *X is called 1-graphic if there exists* $t \in \{1, 2, \cdots, n-1\}$ *for which the following conditions hold:*

(i) $a_1 + a_{t+1} > b,$
(ii) $\sum_{i=1}^{t} a_i \leqslant b$

PROPOSITION 3.4. *The set of all inequalities corresponding to strong covers of cardinality 2, along with the trivial nonnegative inequalities, is sufficient to characterize the convex hull of 1-graphic knapsack set completely.*

Proof. Consider the following partition of the set of items N corresponding to a *1-graphic* knapsack set: $N_1 := \{1, \cdots, t\}; N_2 := \{t+1, \cdots, n\}$. Any *minimal* cover of a *1-graphic* knapsack set is of cardinality 2 since:

(a) $a_{j_1} + a_{j_2} > b \; \forall j_1 \in N_2, j_2 \in N_2, j_1 \neq j_2$ (follows directly from condition (i) of Definition 3.3)
(b) $a_i + a_j > b \; \forall i \in N_1, j \in N_2$ (condition (i) of Definition 3.3)
(c) $\sum_{i \in N_1} a_i \leqslant b$ (condition (ii) of Definition 3.3)

Hence, any strong cover of a *1-graphic* knapsack set is of cardinality 2 (since a strong cover is necessarily a minimal cover). The rest of the proof is the same as for Theorem 6.2 in [13]. □

Remark 3.5. *Solitary item inequalities*, along with the trivial nonnegative inequalities, are sufficient to characterize the convex hull of *1-graphic* knapsack set (due to Corollary 2.18).

In the next subsection, we introduce the *2-graphic* knapsack set and show that its con-

[2]Please note that we have rephrased the statement to make it consistent with our notation.

vex hull is also completely characterized by *Solitary item inequalities*, along with the trivial nonnegative inequalities. Further, we also show that any *1-graphic* knapsack set is also a *2-graphic* knapsack set. Hence, our complete characterization of a *2-graphic* knapsack set also characterizes the convex hull of a *1-graphic* knapsack set.

3.1. Convex hull of a *2-graphic* knapsack set.

DEFINITION 3.6 (*2-graphic* Knapsack). *X is called 2-graphic if there exists $t \in \{1, 2, \cdots, n-1\}$ for which the following conditions hold:*
(i) $a_2 + a_{t+1} > b$,
(ii) $\sum_{i=1}^{t} a_i \leqslant b$

In the rest of the paper, we denote a *2-graphic* knapsack set X and its convex hull as $\widetilde{X}$ and $\widetilde{\mathrm{K}}$, respectively.

Remark 3.7. Any *1-graphic* knapsack set is also a *2-graphic* knapsack set since $a_1 + a_{t+1} > b \implies a_2 + a_{t+1} > b$.

THEOREM 3.8. *The system of inequalities*

$$x_i \geqslant 0, \quad \forall\, i \in N, \tag{3.1}$$

$$x_i + \sum_{j \in N \setminus \{i\}: a_j \geqslant b - a_i + 1} x_j \leqslant 1 \ \forall i = 1, 2, \cdots, t \tag{3.2}$$

is sufficient to describe the convex hull of a 2-graphic knapsack set.

Proof. The proof is based on the following Claims 3.9 and 3.10:

CLAIM 3.9. *All the strong covers for $\widetilde{X}$ are of cardinality 2.*

Proof. The proof is similar to the proof of Proposition 3.4 if we consider the two disjoint subsets $N_1 := \{2, \cdots, t\}; N_2 := \{t+1, \cdots, n\}$. □

This claim suggests that all the nontrivial facet-defining inequalities should be of the form $\sum_{i \in N} \pi_i x_i \leqslant 1$, where $\pi_i \in \{0, 1\}\ \forall i \in N$ [13]. Hence, all the *Solitary item inequalities* along with the trivial non-negative inequalities, are sufficient to characterize the complete convex hull of a *2-graphic* knapsack set. Next we show that only the set of *Solitary item inequalities* described in Theorem 3.8, along with the trivial nonnegative inequalities, are sufficient to describe the convex hull of a *2-graphic* knapsack set depending on the next claim.

CLAIM 3.10. *Item t satisfying conditions (i) and (ii) in Definition 3.6 is either a Type 1 solitary item or a Type 2 solitary item.*

Proof. We prove this by contradiction. For this, assume that item t is neither a *Type 1 solitary item* nor a *Type 2 solitary item*, which implies the following conditions:
(a) $2a_t > b + 1$
(b) $a_t + a_{t-1} > b$

Condition (b) above contradicts condition (ii) in Definition 3.6. This completes the proof of the claim. □

Next, we identify all the *Solitary item inequalities* separately for the two mutually exclusive and exhaustive Cases 3.11 and 3.12.

Case 3.11. Item t is a *Type 2 solitary item.*
If t is a *Type 2 solitary item*, then:

- $i \in N : i \leqslant t - 1$ is a *Type 1 solitary item* (from Proposition 2.5)
- $i \in N : i \geqslant t + 1$ is neither a *Type 1 solitary item* nor a *Type 2 solitary item* (from Propositions 2.6 and 2.7)

Hence, all the facet-defining *Solitary item inequalities* for $\widetilde{\mathrm{K}}$ are given by

$$x_i + \sum_{j \in N \setminus \{i\}: a_j \geqslant b - a_i + 1} x_j \leqslant 1 \;\forall i = 1, 2, \cdots, t.$$

Case 3.12. Item t is a *Type 1 solitary item.*
Below, we first identify all the items that can be either *Type 1 solitary item* or *Type 2 solitary item*, followed by the *Solitary item inequalities* defined by them. If t is a *Type 2 solitary item*, then:

- $i \in N : i \leqslant t-1$ is a *Type 1 solitary item* (from Proposition 2.3).
- $i \in N : i \geqslant t+1$ can never be a *Type 2 solitary item* since $a_t + a_{t+1} > b \implies a_{i-1} + a_i > b$. Now we consider the following two subcases under which $i \in N : i \geqslant t+1$: (i) can never be a *Type 1 solitary item*; (ii) may be a *Type 1 solitary item.*
  - (i) If $a_{t+1} > a_t$, then $i \in N : i \geqslant t+1$ can never be a *Type 1 solitary item* since for any *graphic* knapsack set, we have $a_{t+1} + a_t > b$, which implies $2a_{t+1} > b+1 \implies 2a_i > b+1$ (since $i \geqslant t+1$). Hence, there are no *Solitary item inequalities* defined by items $i \in N : i \geqslant t+1$ when $a_{t+1} > a_t$. Therefore, all the facet-defining *Solitary item inequalities* for $\widetilde{\mathrm{K}}$ are given by

$$x_i + \sum_{j \in N \setminus \{i\}: a_j \geqslant b - a_i + 1} x_j \leqslant 1 \;\forall i = 1, 2, \cdots, t$$

  - (ii) If $a_{t+1} = a_t$, then let us assume that item $t+1$ is a *Type 1 solitary item* (otherwise, all the facet-defining *Solitary item inequalities* for $\widetilde{\mathrm{K}}$ are given by $x_i + \sum_{j \in N \setminus \{i\}: a_j \geqslant b - a_i + 1} x_j \leqslant 1 \;\forall i = 1, 2, \cdots, t$, as shown in (i) above). In that case, the following conditions hold:
    (a) $2a_t \leqslant b+1$ (since t is a *Type 1 solitary item*)
    (b) $2a_{t+1} \leqslant b+1$ (since $t+1$ is a *Type 1 solitary item*)
    (a) and (b) together imply $a_t + a_{t+1} \leqslant b+1$. However, $a_t + a_{t+1} \geqslant b+1$ (true for any *graphic* knapsack set). Also, $a_t = a_{t+1}$ (by assumption). These three conditions can simultaneously hold only if $a_t = a_{t+1} = \lceil b/2 \rceil$ and b is odd. Further, since $\widetilde{X}$ is *2-graphic*, $a_2 + a_{t+1} \geqslant b+1$ (condition (i) in Definition 3.6) and $\sum_{i=1}^{t} a_i \leqslant b$ (condition (ii) in Definition 3.6), which together imply $t = 2$. Hence, $i = 3$ is also a *Type 1 solitary item* (since by assumption, $a_{t+1} = a_t$). Therefore, the *Solitary item inequalities* defined by $i = 2, 3$ are given by

$$x_2 + \sum_{j \in N \setminus \{2\}: a_j \geqslant b - a_2 + 1} x_j \leqslant 1$$
$$x_3 + \sum_{j \in N \setminus \{3\}: a_j \geqslant b - a_3 + 1} x_j \leqslant 1$$

    Clearly, these two inequalities are identical since $a_2 = a_3$.
    Similarly, it can be easily shown that if $a_2 = a_{2+m} = \lceil b/2 \rceil$, where $1 \leqslant m \leqslant n-2$ and b is odd, then $i \in N : 2 \leqslant i \leqslant 2+m$ is a *Type 1 solitary item.* Therefore,

$$x_i + \sum_{j \in N \setminus \{i\}: a_j \geqslant b - a_i + 1} x_j \leqslant 1 \;\forall i \in \{2, 3, \cdots, 2+m\}$$

Clearly, all the above inequalities corresponding to $i \in \{2, 3, \cdots, 2+m\}$ are identical since $a_2 = a_3 = \cdots = a_{2+m}$. Hence, all the facet-defining *Solitary item inequalities* for $\widetilde{\mathrm{K}}$ are given by

$$x_i + \sum_{j \in N \setminus \{i\} : a_j \geqslant b - a_i + 1} x_j \leqslant 1 \; \forall i = 1, 2 = t$$

($i = 1$ is also included since $i = 1$ is already shown above to be a *Type 1 solitary item*).

The system of *Solitary item inequalities* can be rewritten in the following form:

$$\begin{aligned} & x_1 + \sum_{j \in S_1} x_j \leqslant 1 \\ & x_2 + \sum_{j \in S_2} x_j \leqslant 1 \\ & x_3 + \sum_{j \in S_3} x_j \leqslant 1 \\ & \cdots \\ & x_{t-2} + \sum_{j \in S_{t-2}} x_j \leqslant 1 \\ & x_{t-1} + \sum_{j \in S_{t-1}} x_j \leqslant 1 \\ & x_t + \sum_{j \in S_t} x_j \leqslant 1 \end{aligned}$$

where $S_i := \{j \in N \setminus \{i\} : a_j \geqslant b - a_i + 1\}$. It is obvious that $S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots \subseteq S_{t-1} \subseteq S_t = \{t+1, t+2, \cdots, n\}$.

Clearly, the above set of inequalities can be expressed as $(\mathbf{I}, \mathbf{A})x \leqslant e$, where $e \in \mathbb{R}^t$ is the vector of all ones. After subtracting row l of $\mathbf{A}$ from row $l+1$ of $\mathbf{A}$, $\forall\ l = 1, 2, \cdots, t-1$, we can easily show that $\mathbf{A}$ is totally unimodular (since after the row operations, $\mathbf{A}$ contains at most one 1 in every column). Hence $(\mathbf{I}, \mathbf{A})$ is also totally unimodular [15], which completes the proof. ∎

TABLE 2

Facets of Example 3.13 generated from PORTA

Sl. No.	Facets of Example 3.13	Type
1	$x_1 \geqslant 0$	(3.1) in Theorem 3.8 for $i = \{1\}$
2	$x_2 \geqslant 0$	(3.1) in Theorem 3.8 for $i = \{2\}$
3	$x_3 \geqslant 0$	(3.1) in Theorem 3.8 for $i = \{3\}$
4	$x_4 \geqslant 0$	(3.1) in Theorem 3.8 for $i = \{4\}$
5	$x_5 \geqslant 0$	(3.1) in Theorem 3.8 for $i = \{5\}$
6	$x_1 \leqslant 1$	(3.2) in Theorem 3.8 for $i = \{1\}$
7	$x_2 + x_4 + x_5 \leqslant 1$	(3.2) in Theorem 3.8 for $i = \{2\}$
8	$x_3 + x_4 + x_5 \leqslant 1$	(3.2) in Theorem 3.8 for $i = \{3\}$

EXAMPLE 3.13. *Consider a 0/1 knapsack set* $X = \{x \in \mathbb{B}^5 : 8x_1 + 15x_2 + 18x_3 + 37x_4 + 40x_5 \leqslant 48\}$.

Here, X is 2-graphic since for $t = 3$*: (i)* $a_2 + a_{t+1} = 15 + 37 = 52 > b = 48$*; (ii)* $\sum_{i=1}^{t} a_i = 8 + 15 + 18 = 41 < b = 48$*. The complete convex hull of X is shown in Table 2.*

Remark 3.14. Theorem 6.2 in [13] is insufficient to completely characterize the convex hull of *2-graphic* knapsack set in Example 3.13 since (1,i) is not a strong cover for any $i \in N \setminus \{1\}$.

Remark 3.15. *Solitary item inequalities* with the nonnegative trivial inequalities (i.e., the set of inequalities described in Theorem 3.8) are also sufficient to define the convex hull of a *1-graphic* knapsack set (follows from Remark 3.7).

References.

[1] E. Balas, *Facets of the knapsack polytope*, Mathematical programming, 8 (1975), pp. 146–164.

[2] E. Balas and E. Zemel, *Facets of the knapsack polytope from minimal covers*, SIAM Journal on Applied Mathematics, 34 (1978), pp. 119–148.

[3] T. Christof, A. Löbel, and M. Stoer, *Porta-polyhedron representation transformation algorithm*, Software package, available for download at http://www. zib. de/Optimization/Software/Porta, (1997).

[4] A. Del Pia, J. Linderoth, and H. Zhu, *On the complexity of separating cutting planes for the knapsack polytope*, Mathematical Programming, (2023), pp. 1–27.

[5] Z. Gu, G. L. Nemhauser, and M. W. Savelsbergh, *Sequence independent lifting in mixed integer programming*, Journal of Combinatorial Optimization, 4 (2000), pp. 109–129.

[6] C. Hojny, T. Gally, O. Habeck, H. Lüthen, F. Matter, M. E. Pfetsch, and A. Schmitt, *Knapsack polytopes: a survey*, Annals of Operations Research, 292 (2020), pp. 469–517.

[7] M. Laurent and A. Sassano, *A characterization of knapsacks with the max-flow-—min-cut property*, Operations Research Letters, 11 (1992), pp. 105–110.

[8] M. W. Padberg, *A note on zero-one programming*, Operations Research, 23 (1975), pp. 833–837.

[9] M. W. Padberg, *(1, k)-configurations and facets for packing problems*, Mathematical Programming, 18 (1980), pp. 94–99.

[10] Y. Pochet and R. Weismantel, *The sequential knapsack polytope*, SIAM Journal on Optimization, 8 (1998), pp. 248–264.

[11] R. Weismantel, *Hilbert bases and the facets of special knapsack polytopes*, Mathematics of operations research, 21 (1996), pp. 886–904.

[12] R. Weismantel, *On the 0/1 knapsack polytope*, Mathematical Programming, 77 (1997), pp. 49–68.

[13] L. A. Wolsey, *Faces for a linear inequality in 0–1 variables*, Mathematical Programming, 8 (1975), pp. 165–178.

[14] L. A. Wolsey, *Valid inequalities and superadditivity for 0–1 integer programs*, Mathematics of Operations Research, 2 (1977), pp. 66–77.

[15] L. A. Wolsey, *Integer programming*, John Wiley & Sons, New York, 2020.

[16] L. A. Wolsey and G. L. Nemhauser, *Integer and combinatorial optimization*, John Wiley & Sons, New York, 2014.