Time-dependent Stackelberg Protection Location Games

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Abstract

We study a Stackelberg game in which a government positions rapid response teams and thereafter a terrorist attacks a location on a line segment. We assume the damage associated to such an attack to be time dependent. We show that there exists a subgame perfect Nash equilibrium that balances the possible damage on all intervals of the line segment that result from positioning the rapid response teams. We discuss implications for an instance of the model.

Keywords: Counter-terrorism, Stackelberg Game, Resource Allocation

1. Introduction

In the last years, several games have been studied that optimize the location of defensive resources against terrorism attacks (see, e.g., Berman and Gavious [2007],Bier et al. [2007], Zhuang and Bier [2007], Powell [2009], Hausken and Zhuang [2011], van Aken et al. [2024]). One of those papers, van Aken et al. [2024], introduce and analyze a so-called Stackelberg protection location setting in which a government positions heavily-armed and highly-trained response teams on a line segment. This line segment could, for instance, represent a long boulevard. After the government positions response teams, a terrorist selects an attack location on the line segment. The associated damage of the attack is determined by the product of two components: (1) the time it takes the closest response team to arrive at the attack location, and (2) the damage of an attack per time unit. An underlying situation is formally defined by a line segment $[0, 1]$, a number of response teams $n \in \mathbb{N}$, and a continuous damage rate function $f : [0, 1] \to \mathbb{R}_{\geq 0}$. The associated game focuses on the strategic positioning of the response teams, which is represented by $d = (d_1, \ldots, d_n) \in D^n$, where $D^n = \{d \in [0, 1]^n \mid d_1 \leq d_2 \leq \ldots \leq d_n\}$, and the location of the attack, given by $a \in A$, where $A = [0, 1]$. Formally, for a given $d \in D$ and $a \in A$, damage reads:

$$D(d, a) = \min_{i \in \{1,..,n\}} \frac{|d_i - a|}{v} \cdot f(a)$$

with $v \in \mathbb{R}_{>0}$ being the constant speed of the response teams. Van Aken et al. [2024] consider the terrorist to be a maximizer of this damage and the government to be a minimizer of this

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damage. Hence, the government is interested in solving the following problem:

$$\mathscr{P} := \quad \min_{d \in D} \max_{a \in A} D(d, a). \tag{1}$$

Van Aken et al. [2024] refer to a strategy $d \in D$ leading to an optimal solution of $\mathscr{P}$ as an optimal strategy, which we will continue to do in the remainder of this paper. Note that an optimal strategy is not affected by v. For that reason, van Aken et al. [2024] assume without loss of generality that $v = 1$. Below, we present an illustrative example of such a game.

Example 1. *Let $n = 2$ and $f(x) = 1$ for all $x \in [0, 1]$. Suppose the government positions its response teams at $d = (0.2, 0.8)$ and the attacker attacks at $a = 0.4$. Then, $D((0.2, 0.8), 0.4) = \min\{|0.2 - 0.4|, |0.8 - 0.4|\} \cdot 1 = 0.2$. This situation is visualized in Figure 1.*

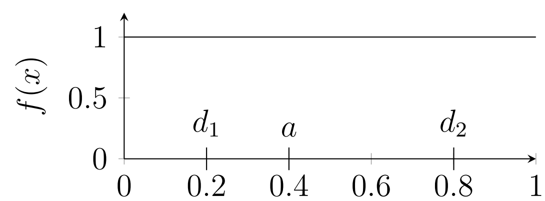
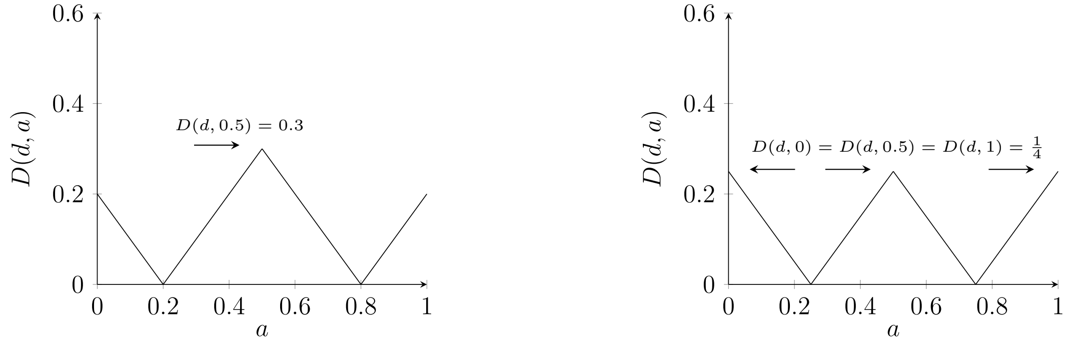


Figure 1: Visualization of a situation with $n = 2$, $f(x) = 1$ for all $x \in [0, 1]$, $d = (0.2, 0.8)$, and $a = 0.4$.

◇

To identify an optimal strategy $d \in D$, van Aken et al. [2024] introduce local damage problems. A local damage problem identifies the maximal damage in between two consecutive response teams, to the left of the first response team, or to the right of the last response team. Van Aken et al. [2024] show that there exists an optimal strategy for which the maximal damage of all these local damage problems coincide. They refer to such a strategy as a *balanced strategy*. Below, this is illustrated by an example.

Example 2. *Reconsider Example 1. As we have $n = 2$ teams, there exist three local damage problems: one to the left of the first response team, one in between the two response teams and one to the right of the second response team. In Figure 2(a) we demonstrate the damage for all $a \in [0, 1]$ given that $d = (0.2, 0.8)$. For this figure, we learn that the maximal damage to the left of the first as well as to the right of the second response team equals 0.2, while the maximal damage in between the two response teams equals 0.3. In Figure 2(b) we demonstrate the damage for all $a \in [0, 1]$ given strategy $d = (0.25, 0.75)$. This time, all three maximal local damages coincide (0.25). This implies that $d = (0.25, 0.75)$ is a balanced strategy, and, hence, optimal.*



(a) $d = (0.2, 0.8)$.

(b) $d = (0.25, 0.75)$.

Figure 2: The damage of an attack for a setting with $n - 2$, $f(x) = 1$ for all $x \in [0, 1]$ and a given d.

◇

Van Aken et al. [2024] assume that the damage per time unit remains constant until the closest response team arrives at the attack location. In our opinion, this is a restrictive assumption. For instance, think of a terrorist who decides to start a shooting somewhere at a market place. Then, the damage rate function could represent the expected number of victims per time unit, but this number will typically decrease over time as people run away once a shooting starts. Moreover, it is also likely that some local police teams are close by. In case of a terrorist attack, such teams will also respond and, although they are most likely unable to neutralize the threat, they can hinder it and consequently lower the damage per time unit. Oppositely, it is also possible that the damage rate function increases for a certain amount of time. For instance, think of a terrorist who after attacking people on the street decides to enter a shop, restaurant or museum where (still) many people may be around.

In summary, we believe that the implicit assumption of a constant damage rate function over time by van Aken et al. [2024] is restrictive and for that reason we will investigate and study a generalized version of the Stackelberg protection location game in this paper. More precisely, we formulate a generalized time-dependent damage rate function and show that, even in this case, a balanced strategy is optimal. In doing so, we make use of the continuity and the non-increasing/non-decreasing behavior of local damage problems defined for our generalized version of the Stackelberg protection location game. After that, we discuss implications for a specific type of time-dependent damage rate function and discuss the impact of time-dependency in the damage rate function on the results.

2. Model

As in a Stackelberg protection location setting, we consider an underlying situation and the associated game. However, this time, we consider a time-dependent damage rate function $h : [0, 1]^2 \to \mathbb{R}_{\geq 0}$ where the first argument resembles the location of the attack and the second argument the time. Similar to van Aken et al. [2024] we set the speed v equal to 1, implying that we can restrict ourselves to a time range of 1. If speed is not restricted to 1, the time domain needs to be scaled accordingly. The damage of an attack is given by

$$D(d, a) = \int_{t=0}^{t=\min_{i \in \{1,\ldots,n\}} |d_i - a|} h(a, t) dt \text{ for all } a \in A \text{ and all } d \in D^n.$$

As before, the attacker tries to maximize this damage, while the government tries to minimize this damage. Hence, the government is interested in solving optimization problem:

$$\mathscr{P}^* := \min_{d \in D} \max_{a \in A} D(d, a). \tag{2}$$

We will refer to a strategy $d \in D$ leading to an optimal solution of $\mathscr{P}^*$ as an optimal strategy in the remaining of the paper. We denote a specific time-dependent protection location situation (TPL situation) by $\theta = (n, h)$ with n the number of response teams and h the time-dependent damage rate function, and we focus on the analysis of the associated time-dependent protection location Stackelberg game which we call a TPLS game. We limit ourselves to the set of TPL situations for which D as well as optimization problem (2) is well-defined and denote this set by Θ. We now identify three relevant subclasses of Θ. The first subclass that we identify is the class for which the damage rate function is constant over time and continuous on a closed interval, and thus, uniformly continuous. We denote this class of TPL situations by Θ^C. An example of a situation in this class is given below.

Example 3. *Let $\theta \in \Theta^C$ with $n = 2$ and $h(a, t) = 1$ for all $(a, t) \in [0, 1]^2$. Suppose the government positions its response teams at $d = (0.2, 0.8)$ and the attacker attacks at $a = 0.4$. Then, $D((0.2, 0.8), 0.4) = \int_{t=0}^{t=\min\{|0.2-0.4|,|0.8-0.4|\}} 1dt = 0.2$. The situation is visualized in Figure 3. Note that damage function D coincides with the damage function given in Example 1. Moreover, as the damage rate function is constant over time, the visual representation of Figure 1 can be recognized as an alternative representation of θ.*

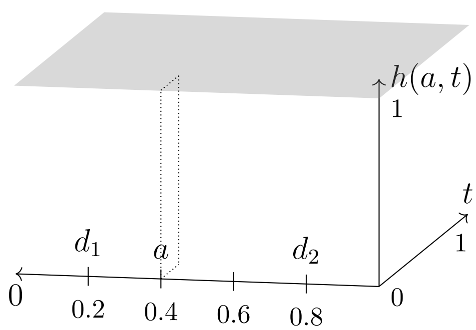


Figure 3: Visual representation of θ of Example 3 with $d = (0.2, 0.8)$ and $a = 0.4$.

◇

Note that there exists a bijection between Θ^C and the set of Stackelberg protection location situations in van Aken et al. [2024]. Therefore, whenever we consider class Θ^C, we have in mind a Stackelberg protection location situation and its associated game. Consequently, with a slight abuse of notation, we will denote the set of situations studied in van Aken et al. [2024] by Θ^C.

The second subclass of TPL situations that we identify is the class for which the damage rate function is uniformly continuous. We denote this class of TPL situations by Θ^U. An example of such a situation is given below.

Example 4. *Let $\theta \in \Theta^U$ with $n = 2$ and $h(a, t) = 1 - t$ for all $(a, t) \in [0, 1]^2$. Suppose the government positions its response teams at $d = (0.2, 0.8)$ and the attacker attacks at $a = 0.4$.*

Then, $D((0.2,0.8),0.4) = \int_{t=0}^{t=\min\{|0.2-0.4|,|0.8-0.4|\}} 1 - t dt = \frac{9}{50}$*. This situation is visualized in Figure 4. The damage of the attack is given by the area within the dotted quadrangle.*

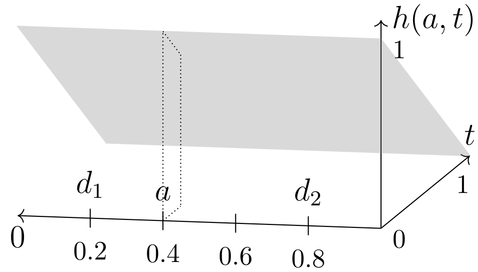


Figure 4: Visual representation of θ of Example 4 with $d = (0.2, 0.8)$ and $a = 0.4$. ◇

The third subclass of TPL situations that we identify is the class for which there exists a possible "jump" in the damage rate function. Such a jump could, for instance, represent the effect of police arriving at the scene. Formally, for a TPL situation in this class, we introduce a uniformly continuous function $f : [0,1]^2 \to \mathbb{R}_{\geq 0}$, which represents a time-dependent damage rate function before the jump. Additionally, we introduce a uniformly continuous function $l : [0,1]^2 \to \mathbb{R}_{\geq 0}$, which represents a time dependent damage rate function after the jump. Moreover, continuously differentiable function $p : [0,1] \to [0,1]$ indicates the time it takes the police team to be present at the location of the attack. Then, function h reads:

$$h(a,t) = \begin{cases} f(a,t) & \text{if } t \leq p(a) \\ l(a,t) & \text{otherwise} \end{cases}$$

for all $(a,t) \in [0,1]^2$. We denote this class of specific TPL situations by Θ^J. An example of such a situation is given below.

Example 5. *Let* $\theta \in \Theta^J$ *with* $n = 2$ *and* $h(a,t) = \begin{cases} 1 - t & \textit{if } t \leq 0.3 \\ 0 & \textit{otherwise} \end{cases}$ *for all* $(a,t) \in [0,1]^2$. *Suppose the government positions the response teams at* $d = (0, 0.8)$ *and the attacker attacks at* $a = 0.4$*. Then,* $D((0,0.8),0.4) = \int_{t=0}^{t=\min\{0.4-0,0.8-0.4\}} h(0.4,t)dt = 0.255$*. The situation is visualized in Figure 5.*

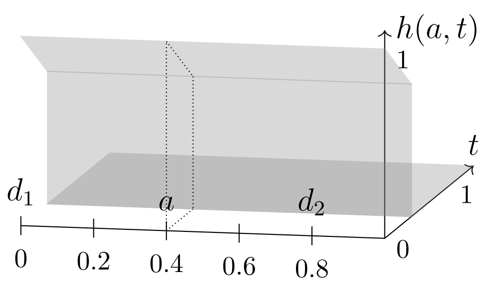


Figure 5: Visual representation of θ of Example 5 with $d = (0, 0.8)$ and $a = 0.4$. ◇

Note that the introduced subclasses can be categorized as subsets of each other, more specifically as $\Theta^C \subseteq \Theta^U \subseteq \Theta^J \subseteq \Theta$. In the next sections we will focus on class Θ^J.

3. Analysis of TPL situations and games

In this section, we will show that for all TPL situations that belong to class Θ^J a balanced strategy is optimal in the TPLS game. In doing so, we first need to introduce some new definitions and notation. Similar to the paper of van Aken et al. [2024], we start by separating the maximization problem of the terrorist, i.e., the inner maximization problem in equation (2), into $n+1$ local maximization problems. We refer to them as *local damage problems* and call their optimal values the *local damages*. The first local damage problem, which identifies the maximal damage to the left of the first response team, is denoted by $\mathscr{L} : [0,1] \to \mathbb{R}$ with

$$\mathscr{L}(d_1) = \max_{a\in[0,d_1]} \int_{t=0}^{t=d_1-a} h(a,t)dt \text{ for all } d_1 \in [0,1].$$

Then, we introduce $n-1$ local damage problems that each identify the maximal damage between two adjacent response teams. Let $i \in \{1,2,...,n-1\}$ and denote $D^{i,i+1} = \{(d_i,d_{i+1}) \in [0,1]^2 \mid d_i \leq d_{i+1}\}$, which is the set of feasible locations of response teams i and $i+1$. Then, the local damage problem between response team i and $i+1$ is given by $\mathscr{I}^i : D^{i,i+1} \to \mathbb{R}$ with

$$\mathscr{I}^i(d_i,d_{i+1}) = \max_{a\in[d_i,d_{i+1}]} \int_{t=0}^{t=\min\{a-d_i,d_{i+1}-a\}} h(a,t)dt \text{ for all } (d_i,d_{i+1}) \in D^{i,i+1}.$$

The last local damage problem, which identifies the maximal damage to the right of the last response team, is denoted by $\mathscr{R} : [0,1] \to \mathbb{R}$ with

$$\mathscr{R}(d_n) = \max_{a\in[d_n,1]} \int_{t=0}^{t=a-d_n} h(a,t)dt \text{ for all } d_n \in [0,1].$$

Van Aken et al. [2024] show that if the functions $\mathscr{L}$, $\mathscr{I}^i$ for all $i \in \{1,\dots,n-1\}$ and $\mathscr{R}$ satisfy some desirable properties, then there exists an optimal balanced strategy. That is, there exists a balanced strategy d^* that minimizes $\mathscr{P}$ and for which:

$$\mathscr{L}(d_1^*) = \mathscr{I}^1(d_1^*,d_2^*) = \mathscr{I}^2(d_2^*,d_3^*) = \ldots = \mathscr{R}(d_n^*),$$

and hence, is optimal. The properties that van Aken et al. [2024] need are as follows:

(i) $\mathscr{L}$, $\mathscr{I}^i$ for all $i \in \{1,\dots,n-1\}$ and $\mathscr{R}$ are continuous

(ii) $\mathscr{L}$ is non-decreasing

(iii) $\mathscr{R}$ is non-increasing

(iv) For all $i \in \{1,\dots,n-1\}$, $\mathscr{I}^i(\cdot,d_{i+1})$ is non-increasing for all $d_{i+1} \in [0,1]$ and $\mathscr{I}^i(d_i,\cdot)$ is non-decreasing for all $d_i \in [0,1]$.

It turns out that these properties are also sufficient in Θ^J to prove the existence of an optimal balanced strategy. That is, although van Aken et al. [2024] formulate its proof for Θ^C, their arguments can still be used to prove the existence of an optimal balanced strategy in Θ^J. This is captured in the following lemma.

Lemma 3.1. *Let $\theta = (n, h) \in \Theta^J$. If $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{I}^i$ for all $i \in \{1, \ldots, n-1\}$ are continuous and $\mathscr{L}$ is non-decreasing, $\mathscr{R}$ is non-increasing and for all $i \in \{1, \ldots, n-1\}$ the following holds:*
(i) $\mathscr{I}^i(\cdot, d_{i+1})$ is non-increasing for all $d_{i+1} \in [0, 1]$
(ii) $\mathscr{I}^i(d_i, \cdot)$ is non-decreasing for all $d_i \in [0, 1]$,
then there exists a balanced strategy and any balanced strategy is optimal.

Proof. By applying Lemma 4.3 until Lemma 4.8 of van Aken et al. [2024] to θ and the associated TPLS game, we are able to conclude that there exists a balanced strategy and any balanced strategy is optimal. Note that we can directly apply Lemma 4.3 until Lemma 4.8 of van Aken et al. [2024] as the proofs of these Lemmas only use the continuity of $\mathscr{L}$, $\mathscr{I}^i$ for all $i \in \{1, \ldots, n-1\}$ and $\mathscr{R}$ and the non-increasing/non-decreasing behavior of these damage rate functions. □

In what follows, we will show that $\mathscr{L}$, $\mathscr{I}^i$ for all $i \in \{1, \ldots, n-1\}$ and $\mathscr{R}$ satisfy the sufficient conditions of Lemma 3.1. We start by showing continuity. Note that the functions $\mathscr{L}$, $\mathscr{I}^i$ for all $i \in \{1, \ldots, n-1\}$ and $\mathscr{R}$ all include a maximization term. In order to prove the continuity of such maximization functions, we will make use of Berge's maximum theorem.

Theorem 3.2 (Berge's maximum theorem (Herings [1996]))**.** *Let $S \subseteq \mathbb{R}^m$, let $T \subseteq \mathbb{R}^n$, and let $\varphi : S \to T$ be a continuous, compact-valued correspondence. Let $h : S \times T \to \mathbb{R}$ be a continuous function and let the relation $g : S \to \mathbb{R}$ be defined by $g(x) = \max_{y \in \varphi(x)} h(x, y)$, for all $x \in S$. Then g is a continuous function.*

To apply Berge's maximum theorem, we first show continuity of the argument of the maximization functions. The results are presented in Lemma 3.3, Lemma 3.4 and Lemma 3.5.

Lemma 3.3. *Let $\theta = (n, h) \in \Theta^J$. Let $D = \{(d_1, a) \in [0, 1]^2 \mid a \leq d_1\}$ and $g : D \to \mathbb{R}$ be a function with $g(d_1, a) = \int_{t=0}^{t=d_1-a} h(a, t)dt$ for all $(d_1, a) \in D$. Function g is continuous.*

Proof. Let $(d_1, a) \in D$ and let $\epsilon > 0$. Take a $\delta' > 0$ such that $|f(a, t) - f(a', t')| < \frac{1}{6}\epsilon$ for all $(a, t), (a', t') \in [0, 1]^2$ with $||(a, t) - (a', t')|| < \delta'$. Note that such a δ' exists as f is uniformly continuous. Take a $\delta'' > 0$ such that $|l(a, t) - l(a', t')| < \frac{1}{6}\epsilon$ for all $(a, t), (a', t') \in [0, 1]^2$ with $||(a, t) - (a', t')|| < \delta''$. Note that such a δ'' exists as l is uniformly continuous. Let $M = \max\{\max_{(a,t) \in [0,1]^2} f(a, t), \max_{(a,t) \in [0,1]^2} l(a, t), 1\}$. Note that M is well-defined as f and l are continuous and thus the maximum on a closed rectangle for both functions exists. Choose $\delta = \min\{\frac{1}{6 \cdot M \cdot (\max_{a \in [0,1]} |p'(a)|+1)} \cdot \epsilon, \delta', \delta''\}$. Note that δ is well defined as $M > 0$ by definition and $\max_{a \in [0,1]} |p'(a)| + 1 > 0$.

Choose a $(d_1', a') \in D$ such that $||(d_1', a') - (d_1, a)|| < \delta$.
The following holds:

$$|g(d_1', a') - g(d_1, a)| = \left| \int_{t=0}^{t=d_1'-a'} h(a', t)dt - \int_{t=0}^{t=d_1-a} h(a, t)dt \right|$$

$$
\begin{aligned}
&= \left| \int_{t=0}^{t=d_1-a} h(a',t) - h(a,t)dt + \int_{t=d_1-a}^{t=d_1'-a'} h(a',t)dt \right| \\
&\leq \left| \int_{t=0}^{t=d_1-a} h(a',t) - h(a,t)dt \right| + \left| \int_{t=d_1-a}^{t=d_1'-a'} h(a',t)dt \right| \\
&\leq \int_{t=0}^{t=d_1-a} |h(a',t) - h(a,t)|\, dt + \left| \int_{t=d_1-a}^{t=d_1'-a'} h(a',t)dt \right| \\
&\leq \int_{t=0}^{t=1} |h(a',t) - h(a,t)|\, dt + \left| \int_{t=d_1-a}^{t=d_1'-a'} h(a',t)dt \right| \\
&\leq \int_{t=0}^{t=\min\{p(a'),p(a)\}} \frac{1}{6}\epsilon dt + \int_{t=\min\{p(a'),p(a)\}}^{t=\max\{p(a'),p(a)\}} M dt + \\
&\quad \int_{t=\max\{p(a'),p(a)\}}^{t=1} \frac{1}{6}\epsilon dt + \left| \int_{t=d_1-a}^{t=d_1'-a'} h(a',t)dt \right| \\
&\leq \frac{1}{6}\epsilon + M \cdot |a'-a| \cdot \max_{a\in[0,1]} |p'(a)| + \frac{1}{6}\epsilon + \left| \int_{t=d_1-a}^{t=d_1'-a'} h(a',t)dt \right| \\
&\leq \frac{1}{3}\epsilon + M \cdot \max_{a\in[0,1]} |p'(a)| \cdot \delta + \left| \int_{t=d_1-a}^{t=d_1'-a'} h(a',t)dt \right| \\
&= \frac{1}{3}\epsilon + M \cdot \max_{a\in[0,1]} |p'(a)| \cdot \delta + \int_{\min\{d_1-a,d_1'-a'\}}^{\max\{d_1-a,d_1'-a'\}} h(a',t)dt \\
&\leq \frac{1}{3}\epsilon + M \cdot \max_{a\in[0,1]} |p'(a)| \cdot \delta + M \cdot |(d_1'-a') - (d_1-a)| \\
&\leq \frac{1}{3}\epsilon + M \cdot \max_{a\in[0,1]} |p'(a)| \cdot \delta + M \cdot 2 \cdot \delta \\
&\leq \frac{1}{3}\epsilon + \frac{1}{6}\epsilon + \frac{1}{3}\epsilon \\
&< \epsilon.
\end{aligned}
$$

Where the second equality holds as $\int_{t=0}^{t=d_1'-a'} h(a',t)dt = \int_{t=0}^{t=d_1-a} h(a',t)dt + \int_{t=d_1-a}^{t=d_1'-a'} h(a',t)dt$ and as $-\int_{t=0}^{t=d_1-a} h(a,t)dt = \int_{t=0}^{t=d_1-a} -h(a,t)dt$. The first inequality holds by the triangle inequality. The second inequality holds by the continuous version of the triangle inequality (Dragomir [2007]). The third inequality holds as $d_1 - a \leq 1$. The fourth inequality holds as for $t \in [0,1]$ we can distinguish between three cases: (1) if $t \in [0, \min\{p(a'), p(a)\}]$ it holds that $|h(a,t) - h(a',t)| < \frac{1}{6}\epsilon$ (2) if $t \in [\min\{p(a'), p(a)\}, \max\{p(a'), p(a)\}]$ it holds that $|h(a,t) - h(a',t)| \leq M$ and (3) if $t \in [\max\{p(a'), p(a)\}, 1]$ it holds that $|h(a,t) - h(a',t)| < \frac{1}{6}\epsilon$. The fifth inequality holds as for the first integral it is true that $\min\{p(a'), p(a)\} \leq 1$ and $\int_{t=0}^{t=1} \frac{1}{6}\epsilon dt = \frac{1}{6}\epsilon$. For the second integral, note that $|p(a') - p(a)| \leq |a' - a| \cdot \max_{a\in[0,1]} |p'(a)|$ holds, consequently it is true that $\int_{t=\min\{p(a'),p(a)\}}^{t=\max\{p(a'),p(a)\}} M dt \leq M \cdot |a' - a| \cdot \max_{a\in[0,1]} |p'(a)|$. For the third integral note that $\max\{p(a'), p(a)\} \geq 0$ holds, and we have $\int_{t=0}^{t=1} \frac{1}{6}\epsilon dt = \frac{1}{6}\epsilon$. The sixth inequality holds as $|a' - a| \leq ||(d_1', a') - (d_1, a)|| \leq \delta$. The third equality holds as $h(a',t) \geq 0$ for all $t \in [0,1]$ and $\max\{d_1 - a, d_1' - a'\} \geq \min\{d_1 - a, d_1' - a'\}$. The seventh inequality holds as $M \geq h(a',t)$ for all $t \in [0,1]$. The eighth inequality holds as we have chosen (d_1', a') in such a way that $||(d_1', a') - (d_1, a)|| < \delta$. Because of this, $|d_1' - d_1| < \delta$ and $|a - a'| < \delta$, and consequently $|d'_1 - d_1 + a - a'| < 2\delta$. The ninth inequality holds as $\delta \leq \frac{1}{6 \cdot M \cdot (\max_{a \in [0,1]} |p'(a)|+1)}\epsilon$, and thus $\delta \leq \frac{1}{6 \cdot M}\epsilon$. The tenth inequality holds as $\epsilon > 0$. □

Lemma 3.4. *Let $\theta = (n, h) \in \Theta^J$. Let $D = \{(d_n, a) \in [0, 1]^2 \mid d_n \leq a\}$ and $g : D \to \mathbb{R}$ be a function with $g(d_n, a) = \int_{t=0}^{t=a-d_n} h(a, t)dt$ for all $(d_n, a) \in D$. Function g is continuous.*

Proof. Let $h^* : [0, 1]^2 \to \mathbb{R}$ be described by

$$h^*(a, t) = \begin{cases} f(1 - a, t) & \text{if } t \leq p(1 - a) \\ l(1 - a, t) & \text{otherwise.} \end{cases}$$

Note that $h^*(1-a, t) = h(a, t)$ for all $(a, t) \in [0, 1]^2$. Let $g^*(1-d_n, 1-a) = \int_{t=0}^{t=1-d_n-(1-a)} h^*(1-a, t)dt$ for all $(1 - d_n, 1 - a) \in D^*$ where $D^* = \{(1 - d_n, 1 - a) \in [0, 1]^2 \mid 1 - a \leq 1 - d_n\}$. Take a $(d_n, a) \in D$. We derive the following:

$$g(d_n, a) = \int_{t=0}^{t=a-d_n} h(a, t)dt = \int_{t=0}^{t=1-d_n-(1-a)} h^*(1 - a, t)dt = g^*(1 - d_n, 1 - a).$$

As $g(d_n, a) = g^*(1 - d_n, 1 - a)$ for all $(d_n, a) \in D$ and g^* is continuous according to Lemma 3.3, it holds that g is continuous. □

Lemma 3.5. *Let $\theta = (n, h) \in \Theta^J$. Let $D = \{(d_i, d_{i+1}, a) \in [0, 1]^3 \mid d_i \leq a \leq d_{i+1}\}$ and $g : D \to \mathbb{R}$ be a function with $g(d_i, d_{i+1}, a) = \int_{t=0}^{t=\min\{a-d_i, d_{i+1}-a\}} h(a, t)dt$ for all $(d_i, d_{i+1}, a) \in D$ and for all $i \in \{1, \ldots, n - 1\}$. Function g is continuous.*

Proof. Note that the following holds for all $i \in \{1, \ldots, n - 1\}$:

$$g(d_i, d_{i+1}, a) = \int_{t=0}^{t=\min\{a-d_i, d_{i+1}-a\}} h(a, t)dt = \min\left\{\int_{t=0}^{t=a-d_i} h(a, t)dt, \int_{t=0}^{t=d_{i+1}-a} h(a, t)dt\right\}.$$

Here the second equality holds as h is a non-negative function. Additionally, note that according to Lemma 3.4 $g(d_i, a)$ for all $(d_i, a) \in D$ with $D = \{(d_i, a) \in [0, 1]^2 \mid d_i \leq a\}$ is continuous. Thus, $\int_{t=0}^{t=a-d_i} h(a, t)dt$ is continuous in d_i and a. Additionally, note that according to Lemma 3.3 $g(d_{i+1}, a)$ for all $(d_{i+1}, a) \in D$ with $D = \{(d_{i+1}, a) \in [0, 1]^2 \mid a \leq d_{i+1}\}$ is continuous. Thus, $\int_{t=0}^{t=d_{i+1}-a} h(a, t)dt$ is continuous in d_{i+1} and a. As the minimum of two continuous functions is continuous, we have proven that function g is continuous. □

As it is now proven that the arguments of $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{I}^i$ for all $i \in \{1, \ldots, n - 1\}$ are continuous, we can directly apply Berge's maximum theorem to show that $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{I}^i$ for all $i \in \{1, \ldots, n - 1\}$ are continuous. This is shown in Lemma 3.6.

Lemma 3.6. *Let $\theta = (n, h) \in \Theta^J$. $\mathscr{L}$, $\mathscr{I}^i$ for all $i \in \{1, 2, \ldots, n-1\}$ and $\mathscr{R}$ are continuous.*

Proof. First, we focus on $\mathscr{L}$, then on $\mathscr{I}^i$ for all $i \in \{1, 2, \ldots, n-1\}$ and finally on $\mathscr{R}$.

Let $\varphi : [0,1] \to [0,1]$ be a correspondence with $\varphi(d_1) = [0, d_1]$ for all $d_1 \in [0,1]$. Let $r : [0,1] \to \mathbb{R}$ be a function with $r(d_1) = 0$ for all $d_1 \in [0,1]$ and let $q : [0,1] \to \mathbb{R}$ be a function with $q(d_1) = d_1$ for all $d_1 \in [0,1]$. Let $g : [0,1]^2 \to \mathbb{R}$ be a function with $g(d_1, a) = \int_{t=0}^{t=d_1-a} h(a,t)dt$ for all $(d_1, a) \in [0,1]^2$ with $a \leq d_1$. Lemma 3.3 states that g is continuous. Since r and q are continuous and bounded functions and $r(d_1) \leq q(d_1)$ for all $d_1 \in [0,1]$, by Lemma 2.3 of van Aken et al. [2024] it is true that φ is continuous and compact-valued. Moreover, g is continuous and thus by Theorem 3.2 it is true that $\mathscr{L}(d_1) = \max_{a \in [0,d_1]} g(d_1, a)$ is continuous for all $d_1 \in [0,1]$.

If $n \geq 2$, let $\varphi : D^{1,2} \to [0,1]$ be a correspondence with $\varphi(d_1, d_2) = [d_1, d_2]$ for all $(d_1, d_2) \in D^{1,2}$. Let $r : D^{1,2} \to \mathbb{R}$ be a function with $r(d_1, d_2) = d_1$ for all $(d_1, d_2) \in D^{1,2}$ and let $q : D^{1,2} \to \mathbb{R}$ be a function with $q(d_1, d_2) = d_2$ for all $(d_1, d_2) \in D^{1,2}$. Let $g : D^{1,2} \to \mathbb{R}$ be a function defined by $g(d_1, d_2) = \max_{a \in [d_1, d_2]} \int_{t=0}^{t=\min\{a-d_1, d_2-a\}} h(a,t)dt$ for all $(d_1, d_2) \in D^{1,2}$. Lemma 3.5 states that g is a continuous function. Since r and q are continuous and bounded functions and $r(d_1, d_2) \leq q(d_1, d_2)$ for all $(d_1, d_2) \in D^{1,2}$, by Lemma 2.3 of van Aken et al. [2024] is is true that φ is continuous and compact-valued. Moreover, g is continuous and thus by Theorem 3.2 it is true that $\mathscr{I}^1(d_1, d_2) = \max_{a \in [d_1, d_2]} g(d_1, d_2)$ is continuous for all $(d_1, d_2) \in D^{1,2}$. As $\mathscr{I}^{i+1}(d_i, d_{i+1}) = \mathscr{I}^i(d_i, d_{i+1})$ for all $(d_i, d_{i+1}) \in D^{i,i+1}$ for all $i \in \{1, 2, \ldots, n-2\}$, we conclude that $\mathscr{I}^i$ is continuous for all $i \in \{1, .., n-1\}$.

Let $\varphi : [0,1] \to [0,1]$ be a correspondence with $\varphi(d_n) = [d_n, 1]$ for all $d_n \in [0,1]$. Let $r : [0,1] \to \mathbb{R}$ be a function with $r(d_n) = d_n$ for all $d_n \in [0,1]$ and let $q : [0,1] \to \mathbb{R}$ be a function with $q(d_n) = 1$ for all $d_n \in [0,1]$. Let $g : [0,1]^2 \to \mathbb{R}$ be a function with $g(d_n, a) = \int_{t=0}^{t=a-d_n} h(a,t)dt$ for all $(d_n, a) \in [0,1]^2$ with $d_n \leq a$. Lemma 3.4 states that g is continuous. Since r and q are continuous and bounded functions and $r(d_n) \leq q(d_n)$ for all $d_n \in [0,1]$, by Lemma 2.3 of van Aken et al. [2024] it is true that φ is continuous and compact-valued. Moreover, g is continuous and thus by Theorem 3.2 it is true that $\mathscr{R}(d_n) = \max_{a \in [d_n, 1]} g(d_n, a)$ is continuous for all $d_n \in [0,1]$. □

Next, in order to prove that all conditions in Lemma 3.1 are satisfied, we show the non-increasing/non-decreasing behavior of $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{I}^i$ for all $i \in \{1, \ldots, n-1\}$. This is shown in Lemma 3.7.

Lemma 3.7. *Let $\theta = (n, h) \in \Theta^J$. $\mathscr{L}$ is non-decreasing, $\mathscr{R}$ is non-increasing, and for all $i \in \{1, ..., n-1\}$ the following holds*

(i) $\mathscr{I}^i(\cdot, d_{i+1})$ is non-increasing for all $d_{i+1} \in [0,1]$

(ii) $\mathscr{I}^i(d_i, \cdot)$ is non-decreasing for all $d_i \in [0,1]$.

Proof. First we focus on $\mathscr{L}$, then on $\mathscr{R}$ and finally on $\mathscr{I}^i$ for all $i \in \{1, 2, \ldots, n-1\}$.
Let $d_1 \in [0,1]$ and $d_1' \in [0,1]$ such that $d_1' > d_1$. Let $a^* \in \arg\max_{a \in [0,d_1]} \int_{t=0}^{t=d_1-a} h(a,t)$. We derive the following:

$$\mathscr{L}(d_1) = \int_{t=0}^{t=d_1-a^*} h(a^*, t)dt \leq \int_{t=0}^{t=d_1'-a^*} h(a^*, t)dt \leq \max_{a \in [0,d_1']} \int_{t=0}^{t=d_1'-a} h(a,t)dt = \mathscr{L}(d_1').$$

The first inequality holds as $d_1' > d_1$ and h is a non-negative function. The second inequality holds as we know that $a^* \in [0, d_1]$. Consequently, a^* is a feasible solution to the optimization problem $\max_{a \in [0,d_1']} \int_{t=0}^{t=d_1'-a} h(a,t)$. Hence, $\mathscr{L}$ is non-decreasing.

Let $d_n \in [0,1]$ and $d_n' \in [0,1]$ such that $d_n' < d_n$. Let $a^* \in \arg\max_{a \in [d_n,1]} \int_{t=0}^{t=a-d_n} h(a,t)dt$. We derive the following:

$$\mathscr{R}(d_n) = \int_{t=0}^{t=a^*-d_n} h(a^*,t)dt \leq \int_{t=0}^{t=a^*-d_n'} h(a^*,t)dt \leq \max_{a \in [d_n',1]} \int_{t=0}^{t=a-d_n'} h(a,t)dt = \mathscr{R}(d_n').$$

The first inequality holds as $d_n' < d_n$ and h is a non-negative function. The second inequality holds as we know that $a^* \in [d_n, 1]$. Consequently, a^* is a feasible solution to the optimization problem $\max_{a \in [d_n',1]} \int_{t=0}^{t=a-d_n} h(a,t)dt$. Hence, $\mathscr{R}$ is non-increasing.

Let $i \in \{1, \dots, n-1\}$ and $d_{i+1} \in [0,1]$. First, we prove that $\mathscr{I}^1(\cdot, d_{i+1})$ is non-increasing. Let $d_i \in [0, d_{i+1}]$ and $d_i' \in [0, d_{i+1}]$ such that $d_i < d_i'$.
Let $a^* \in \arg\max_{a \in [d_i', d_{i+1}]} \int_{t=0}^{t=\min\{d_{i+1}-a, a-d_i'\}} h(a,t)dt$. We derive the following:

$$\begin{aligned}
\mathscr{I}^1(d_i', d_{i+1}) &= \max_{a \in [d_i', d_{i+1}]} \int_{t=0}^{t=\min\{d_{i+1}-a, a-d_i'\}} h(a,t)dt \\
&= \int_{t=0}^{t=\min\{d_{i+1}-a^*, a^*-d_i'\}} h(a^*,t)dt \\
&\leq \int_{t=0}^{t=\min\{d_{i+1}-a^*, a^*-d_i\}} h(a^*,t)dt \\
&\leq \max_{a \in [d_i, d_{i+1}]} \int_{t=0}^{t=\min\{d_{i+1}-a, a-d_i\}} h(a,t)dt \\
&= \mathscr{I}^1(d_i, d_{i+1}).
\end{aligned}$$

The first inequality holds as $d_i < d_i'$. The second inequality holds as we know that $a^* \in [d_i', d_{i+1}]$. Consequently, a^* is a feasible solution to the optimization problem $\max_{a \in [d_i, d_{i+1}]} \int_{t=0}^{t=\min\{d_{i+1}-a, a-d_i\}} h(a,t)dt$. Hence, $\mathscr{I}^1(\cdot, d_{i+1})$ is non-increasing. As by definition it holds that $\mathscr{I}^i(d_i, d_{i+1}) = \mathscr{I}^1(d_i, d_{i+1})$ for all $i \in \{1, \dots, n-1\}$, it follows that $\mathscr{I}^i(\cdot, d_{i+1})$ is non-increasing for all $i \in \{1, \dots, n-1\}$.
Let $i \in \{1, \dots, n-1\}$ and $d_i \in [0,1]$. First, we prove that $\mathscr{I}^1(d_i, \cdot)$ is non-decreasing. Let $d_{i+1} \in [d_i, 1]$ and $d_{i+1}' \in [d_i, 1]$ such that $d_{i+1} < d_{i+1}'$. Let $a^* \in \arg\max_{a \in [d_i, d_{i+1}]} \int_{t=0}^{t=\min\{d_{i+1}-a, a-d_i\}} h(a,t)dt$. We derive the following:

$$\begin{aligned}
\mathscr{I}^1(d_i, d_{i+1}) &= \max_{a \in [d_i, d_{i+1}]} \int_{t=0}^{t=\min\{d_{i+1}-a, a-d_i\}} h(a,t)dt \\
&= \int_{t=0}^{t=\min\{d_{i+1}-a^*, a^*-d_i\}} h(a^*,t)dt \\
&\leq \int_{t=0}^{t=\min\{d_{i+1}'-a^*, a^*-d_i\}} h(a^*,t)dt
\end{aligned}$$

$$\leq \max_{a \in [d_i, d'_{i+1}]} \int_{t=0}^{t=\min\{d'_{i+1}-a, a-d_i\}} h(a,t)dt$$
$$= \mathscr{I}^1(d_i, d'_{i+1}).$$

The first inequality holds as $d_{i+1} < d'_{i+1}$. The second inequality holds as we know that $a^* \in [d_i, d_{i+1}]$. Consequently, a^* is a feasible solution to the optimization problem $\max_{a \in [d_i, d'_{i+1}]} \int_{t=0}^{t=\min\{d'_{i+1}-a, a-d_i\}} h(a,t)dt$. Hence, $\mathscr{I}^1(d_i, \cdot)$ is non-decreasing. As by definition it holds that $\mathscr{I}^i(d_i, d_{i+1}) = \mathscr{I}^1(d_i, d_{i+1})$ for all $i \in \{1, \ldots, n-1\}$, it follows that $\mathscr{I}^i(d_i, \cdot)$ is non-decreasing for all $i \in \{1, \ldots, n-1\}$. □

We have proven that $\mathscr{L}$, $\mathscr{I}^i$ for all $i \in \{1, \ldots, n-1\}$ and $\mathscr{R}$ satisfy the properties given in the beginning of this section. By applying Lemma 3.1, in combination with Lemma 3.6 and Lemma 3.7, it follows that there exists a balanced strategy and any balanced strategy is optimal. This is formalized in Theorem 3.8.

Theorem 3.8. *Let $\theta = (n, h) \in \Theta^J$. There exists a balanced strategy and any balanced strategy is optimal.*

Proof. By Lemma 3.6 it holds that $\mathscr{L}$, $\mathscr{I}^i$ for all $i \in \{1, 2, \ldots, n-1\}$ and $\mathscr{R}$ are continuous. Additionally, by Lemma 3.7 it holds that $\mathscr{L}$ is non-decreasing, $\mathscr{R}$ is non-increasing, and for all $i \in \{1, \ldots, n-1\}$:
(i) $\mathscr{I}^i(\cdot, d_{i+1})$ is non-increasing for all $d_{i+1} \in [0, 1]$
(ii) $\mathscr{I}^i(d_i, \cdot)$ is non-decreasing for all $d_i \in [0, 1]$.

As all the sufficient conditions in Lemma 3.1 are met, we are able to apply Lemma 3.1 and conclude that there exists a balanced strategy and any balanced strategy is optimal. □

4. A police team scenario

In this section, we consider a specific instance of the model. We study a stylized busy shopping avenue with a market square at the end of the street. The beginning of the shopping avenue is still relatively quiet, but the more we move towards the market square, the busier it gets. We can represent this setting by a linearly increasing damage rate function. Next to the still to be located response teams, there is also a police office located at the busiest location of the district, i.e., at $a = 1$. The police is capable to react to a terrorist attack and hinder the attack. When the police arrives at the location of the attack, the damage per time unit from that moment onwards will be cut in half. A damage rate function $h : [0, 1]^2 \to \mathbb{R}_{\geq 0}$ representing this setting is described by:

$$h(a,t) = \begin{cases} a & \text{if } t \leq 1-a \\ \frac{1}{2} \cdot a & \text{otherwise} \end{cases} \tag{3}$$

for all $(a, t) \in [0, 1]^2$ and a visual representation of function h is illustrated by Figure 7.

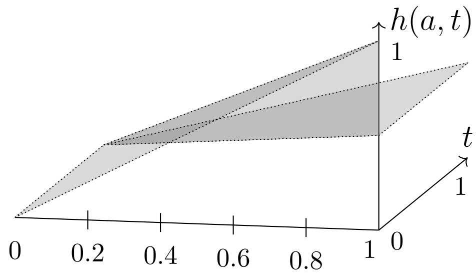


Figure 6: Visual representation of the damage rate function h.

In Theorem 4.1, we describe how to optimally locate the response teams.

Theorem 4.1. *Let* $\theta = (n, h) \in \Theta$ *with* $h(a,t) = \begin{cases} a & \text{if } t \leq 1-a \\ \frac{1}{2}a & \text{otherwise} \end{cases}$ *for all* $(a,t) \in [0,1]^2$. *An optimal location of the response teams is given by* $d_i^* = \sqrt{(2+n)\cdot i} - \sqrt{n \cdot i}$ *for all* $i \in \{1, \ldots, n\}$, *with associated damage* $\frac{1}{2}(1 - \sqrt{2n+n^2} + n)$.

Proof. Let $d_i^* = \sqrt{(2+n)\cdot i} - \sqrt{n \cdot i}$ for all $i \in \{1, \ldots, n\}$. We show that $\mathscr{L}(d_1^*) = \mathscr{I}^1(d_1^*, d_2^*) = \mathscr{I}^2(d_2^*, d_3^*) = \ldots = \mathscr{R}(d_n^*)$.

We derive the following:

$$\begin{aligned} \mathscr{L}(d_1^*) &= \max_{a \in [0, d_1^*]} \int_{t=0}^{t=d_1^*-a} h(a,t)dt = \max_{a \in [0, d_1^*]} \int_{t=0}^{t=d_1^*-a} adt \\ &= \max_{a \in [0, d_1^*]} a \cdot d_1^* - a^2 dt = \frac{1}{4} d_1^{*2} = \frac{1}{2} \cdot (1 + n - \sqrt{n}\sqrt{2+n}). \end{aligned}$$

The first equality follows by definition. The second equality follows by noting that $h(a,t) = a$ in the relevant domain as $d_1^* - a \leq 1 - a$ for all $a \in [0,1]$ and consequently also for all $a \in [0, d_1^*]$. The third equality follows by calculating the integral. The fourth equality follows by noting that the maximum is attained at $a^* = \frac{d_1^*}{2}$. The fifth equality results from substituting the optimal location for the response team into the equation.
We derive the following for $\mathscr{I}^i(d_i^*, d_{i+1}^*)$ for all $i \in \{1, \ldots, n-1\}$:

$$\begin{aligned} \mathscr{I}^i(d_i^*, d_{i+1}^*) &= \max_{a \in [d_i^*, d_{i+1}^*]} \int_{t=0}^{t=\min\{a-d_i^*, d_{i+1}^*-a\}} h(a,t)dt \\ &= \max_{a \in [d_i^*, d_{i+1}^*]} \int_{t=0}^{t=\min\{a-d_i^*, d_{i+1}^*-a\}} adt \\ &= \max_{a \in [d_i^*, d_{i+1}^*]} a \cdot \min\{a - d_i^*, d_{i+1}^* - a\}dt \\ &= \frac{d_{i+1}^{*2} - d_i^{*2}}{4} \\ &= \frac{1}{4}(2 + 2n - 2\sqrt{(2+n)(i+1)}\sqrt{n(i+1)} + 2\sqrt{(2+n)i}\sqrt{ni}) \\ &= \frac{1}{2} \cdot (1 + n - \sqrt{n}\sqrt{2+n}). \end{aligned}$$

The first equality follows by definition. The second equality follows by noting that $h(a,t) = a$ in the relevant domain as $\min\{a - d_i^*, d_{i+1}^* - a\} \leq d_{i+1}^* - a \leq 1 - a$ for all $a \in [0,1]$ and consequently also for all $a \in [d_i^*, d_{i+1}^*]$. The third equality follows by calculating the integral. The fourth equality follows by noting that the maximum is attained at $a^* = \frac{1}{2}(d_{i+1} + d_i)$. Note that the objective function is equal to $a \cdot (a - d_i^*)$ if $a \in [d_i^*, \frac{d_{i+1}+d_i}{2}]$ and is equal to $a \cdot (d_{i+1}^* - a)$ if $a \in [\frac{d_{i+1}+d_i}{2}, d_{i+1}^*]$. Consequently $a^* = \frac{1}{2}(d_{i+1} + d_i)$ follows since the objective function is increasing in a on the interval $a \in [d_i^*, \frac{d_{i+1}+d_i}{2}]$ and the objective function is decreasing in a on the interval $a \in [\frac{d_{i+1}+d_i}{2}, d_{i+1}^*]$. The fifth equality follows from substituting the optimal location of the response team in to the equation and expanding the squares. The sixth equality results from simplifying the expression.

We derive the following for $\mathscr{R}(d_n^*)$:

$$
\begin{aligned}
\mathscr{R}(d_n^*) &= \max_{a \in [d_n^*, 1]} \int_{t=0}^{t=a-d_n^*} h(a,t)dt \\
&= \max\left\{ \max_{a \in [d_n^*, \frac{1}{2}+\frac{1}{2}d_n^*]} \int_{t=0}^{t=a-d_n^*} h(a,t)dt,\ \max_{a \in [\frac{1}{2}+\frac{1}{2}d_n^*, 1]} \int_{t=0}^{t=a-d_n^*} h(a,t)dt \right\} \\
&= \max\left\{ \max_{a \in [d_n^*, \frac{1}{2}+\frac{1}{2}d_n^*]} \int_{t=0}^{t=a-d_n^*} a dt,\ \max_{a \in [\frac{1}{2}+\frac{1}{2}d_n^*, 1]} \int_{t=0}^{t=a-d_n^*} h(a,t)dt \right\} \\
&= \max\left\{ \max_{a \in [d_n^*, \frac{1}{2}+\frac{1}{2}d_n^*]} a^2 - a \cdot d_n^*,\ \max_{a \in [\frac{1}{2}+\frac{1}{2}d_n^*, 1]} \int_{t=0}^{t=a-d_n^*} h(a,t)dt \right\} \\
&= \max\left\{ \frac{1}{4} - \frac{1}{4}d_n^{*2},\ \max_{a \in [\frac{1}{2}+\frac{1}{2}d_n^*, 1]} \int_{t=0}^{t=a-d_n^*} h(a,t)dt \right\} \\
&= \max\left\{ \frac{1}{4} - \frac{1}{4}d_n^{*2},\ \max_{a \in [\frac{1}{2}+\frac{1}{2}d_n^*, 1]} \left\{ \int_{t=0}^{t=1-a} a dt + \int_{t=1-a}^{t=a-d_n^*} \frac{1}{2} a dt \right\} \right\} \\
&= \max\left\{ \frac{1}{4} - \frac{1}{4}d_n^{*2},\ \max_{a \in [\frac{1}{2}+\frac{1}{2}d_n^*, 1]} \frac{1}{2}a - \frac{1}{2}a \cdot d_n^* \right\} \\
&= \max\left\{ \frac{1}{4} - \frac{1}{4}d_n^{*2}, \frac{1}{2} - \frac{1}{2}d_n^* \right\} \\
&= \frac{1}{2} - \frac{1}{2}d_n^* \\
&= \frac{1}{2} \cdot (1 + n - \sqrt{n}\sqrt{2+n}).
\end{aligned}
$$

The first equality follows by definition. The second equality follows by noting that it is possible to split the maximization problem into separate maximization problems. The third equality follows by noting that $h(a,t) = a$ in the relevant domain for the integral in the first maximization problem as $a - d_n^* \leq 1 - a$ for all $a \in [d_n^*, \frac{1}{2} + \frac{1}{2}d_n^*]$. The fourth equality follows from calculating the integral of the first maximization problem. The fifth equality follows by noting that the maximum is attained at $a^* = \frac{1}{2} + \frac{1}{2}d_n^*$ in the first maximization problem and consequently that $a^2 - a \cdot d_n^* = (\frac{1}{2}+\frac{1}{2}d_n^*)^2 - (\frac{1}{2}+\frac{1}{2}d_n^*)\cdot d_n^* = \frac{1}{4}+\frac{1}{2}d_n^*+\frac{1}{4}d_n^{*2}-\frac{1}{2}d_n^*-\frac{1}{2}d_n^{*2} = \frac{1}{4}-\frac{1}{4}d_n^{*2}$. The sixth equality follows by noting that it is possible to split the integral into two separate parts. In the relevant domain of the first integral $h(a,t) = a$ as $1-a \leq 1-a$ for all $a \in [0,1]$ and consequently also for all $a \in [\frac{1}{2}+\frac{1}{2}d_n^*, 1]$. In the relevant domain of the second integral $h(a,t) = \frac{1}{2}a$ as $t > 1-a$ for all $t \in (1-a, a-d_n^*]$ and for all $a \in [\frac{1}{2}+\frac{1}{2}d_n^*, 1]$. The seventh equality follows from calculating the integrals, noting that $\int_{t=0}^{t=1-a} a dt = a - a^2$ and $\int_{t=1-a}^{t=a-d_n^*} \frac{1}{2}a dt = a^2 - \frac{1}{2}ad_n^* - \frac{1}{2}a$ and adding the outcomes of these two integrals. The eighth equality follows by noting that the maximum is attained at $a^* = 1$. The ninth equality follows by noting that $\frac{1}{2}-\frac{1}{2}d_n^* > \frac{1}{4}-\frac{1}{4}d_n^*$ for all $d_n^* \in [0,1]$. The tenth equality results from substituting the optimal location for the response team into the equation.

With these equalities we are able to conclude that the following holds:

$$\mathscr{L}(d_1^*) = \mathscr{I}(d_1^*, d_2^*) = \ldots = \mathscr{I}(d_{n-1}^*, d_n^*) = \mathscr{R}(d_n^*)$$

and consequently this strategy is balanced. By theorem 3.8 we conclude that the location of response teams is given by $d_i^* = \sqrt{(2+n)\cdot i} - \sqrt{n \cdot i}$ for all $i \in \{1, \ldots, n\}$ and the associated damage is $\frac{1}{2}(1 - \sqrt{2n+n^2} + n)$. □

By means of an example, we now investigate the impact on damage when the diminishing effect of the police team on the damage function is neglected. That is, for a given $\theta \in \Theta^J$, we first calculate an optimal location of the response teams and associated damage. Subsequently, using the model of van Aken et al. [2024], i.e., the model that ignores the diminishing effect, we determine an optimal location of the response teams as well. We stress that these locations are optimal for the incorrect model in which the effect of the police team is neglected. Finally, we evaluate the damage when using these locations, i.e., the ones derived from the model of van Aken et al. [2024], and study the relative increase in damage compared to the setting where an optimal location (for the correct model, incorporating the effect of the police team) of the teams is applied.

Example 6. *Let us first consider the scenario in which we take the police office into account in determining the location of the response teams. Let* $\theta = (n, h) \in \Theta^J$ *with* $n = 4$ *and* $h(a,t) = \begin{cases} a & \text{if } t \leq 1-a \\ \frac{1}{2}a & \text{otherwise} \end{cases}$ *for all* $(a,t) \in [0,1]^2$. *According to Theorem 4.1, an optimal positioning of the response teams is* $d \approx (0.45, 0.64, 0.78, 0.90)$[3]*, with associated damage being equal to* 0.0505*, rounded to four decimal places. The location of the response teams is depicted in Figure 7 by the lower four arrows.*

Let us now consider the scenario in which we neglect the diminishing effect of the police on the damage per time unit when determining an optimal location of the response teams. Thus, to determine an optimal location of the response teams, we consider $\theta = (n, \hat{h}) \in \Theta^C$

[3]These expressions are rounded to two decimal places. The exact expressions of the location of the response teams are given by $d = (\sqrt{6}-\sqrt{4}, \sqrt{12}-\sqrt{8}, \sqrt{18}-\sqrt{12}, \sqrt{24}-\sqrt{16})$. The exact associated damage is equal to $\frac{1}{2}(5-\sqrt{24})$.

with $n = 4$ *and* $\hat{h}(a) = a$ *for all* $(a,t) \in [0,1]^2$*. According to Theorem 5.3 of van Aken et al. [2024], an optimal positioning of the response teams is given by* $d' \approx (0.47, 0.67, 0.82, 0.94)$[4] *The location of the response teams is depicted in Figure 7 by the upper four arrows. We evaluate this positioning in our original setting* $\theta = (n, h)$*. This leads to a damage of* 0.0557*, rounded to four decimal places.*

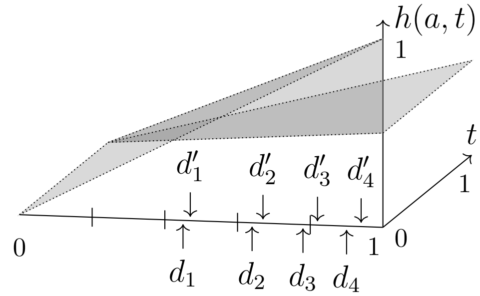


Figure 7: Visual representation of the situation with damage rate function $h(a,t) = \begin{cases} a & \text{if } t \leq 1-a \\ \frac{1}{2}a & \text{otherwise} \end{cases}$ for all $(a,t) \in [0,1]^2$ and the location of the four response teams in both scenarios of the example.

It can be noted that when the diminishing effect of the police on the damage per time unit is ignored, the response teams are located slightly more towards the end of the shopping avenue compared to when this effect is taken into account. As a result, the actual damage increases with more than 10% when, in positioning the response teams, one ignores the diminishing effect of the police on the damage per time unit. ◇

Example 6 shows that there is a significant impact by incorporating time-dependent effects such as the presence of a police office. Therefore, it is worthwhile to consider these time-dependent effects in the damage rate function and position the response teams according to this time-dependent damage rate function.

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[4]These expressions are rounded to two decimal places. The exact expressions of the location of the response teams are given by $d' = (2(\sqrt{5}-\sqrt{4}), 2\sqrt{2}(\sqrt{5}-\sqrt{4}), 2\sqrt{3}(\sqrt{5}-\sqrt{4}), 4(\sqrt{5}-\sqrt{4}))$. The exact associated damage is equal to $\frac{1}{(\sqrt{5}+\sqrt{4})^2}$.

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