Advances in Polyhedral Relaxations of the Quadratic Linear Ordering Problem

Sven Mallach[1,2]

[1]Chair of Management Science, University of Siegen, Germany
[2]University of Bonn, Germany

September 16, 2024

Abstract

We report on results concerning the polyhedral structure of the quadratic linear ordering problem and its associated integer linear programming formulations. Specifically, we provide a deeper analysis of the characteristic equation system that partly describes the corresponding polytope, i.e., the convex hull of the feasible solutions to the quadratic linear ordering problem, and determine an accessible description of a restricted and inextensible subset of the odd-cycle inequalities that induces facets of it. Further, we present an extended formulation that provides a replacement for the commonly used linearization applied to products of linear ordering variables that share an index.

1 Introduction

The quadratic linear ordering problem (QLOP) asks for a permutation of the elements of a finite set S that maximizes an objective that depends on the relative order of element pairs and combinations of these pairs. More formally, assuming without loss of generality that $S = [n] := \{1, \ldots, n\}$, let Π_n be the set of all permutations of n elements. Given matrices $C \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{n^2 \times n^2}$, define the objective function

$$f(\pi) := \sum_{i,j \in [n]: \pi(i) < \pi(j)} c_{ij} + \sum_{i,j,k,\ell \in [n]: \pi(i) < \pi(j); \pi(k) < \pi(\ell)} q_{ijk\ell}. \tag{1}$$

The QLOP is then to determine a permutation $\pi^* \in \Pi_n$ such that $f(\pi^*) \geq f(\pi)$ for all $\pi \in \Pi_n$. In case $Q = 0$, the QLOP reduces to the classical linear ordering problem which is well-known to be $\mathcal{NP}$-hard [6].

The QLOP has several applications, for instance the single-row facility layout problem modeling several challenging problems in operations research [8, 10], crossing minimization in hierarchical (layered) graph drawings [2], and cutwidth minimization [5].

In this paper, we extend the seminal polyhedral results and revisit the integer linear programming formulation for the QLOP presented by Buchheim, Wiegele and Zheng in [2]. Particularly, we amend their polyhedral study by an analysis of the continuous relaxation of this formulation, and provide a deeper analysis of the impact and implications of its characteristic minimal equation system. We then address the open question from [2] which instances of the so-called odd-cycle inequalities, predominantly studied in the context of the strongly related Boolean Quadric Polytope, remain facet-inducing for the convex hull of the feasible solutions to the QLOP, and identify a restricted subset of triangle inequalities for which this holds true. By taking up some of the findings by DeVries in her PhD thesis [3], we describe how to determine more compact and irredundant integer linear programming formulations for the QLOP, and provide a perspective on further known facet-defining inequalities that facilitates to assess their combinatorial structure. Finally, we suggest an extended formulation that implicitly linearizes quadratic terms referring to the relative order of exactly three distinct elements $i, j, k \in [n]$.

The paper is organized as follows. In Sect. 2, we review the prevalent results on the QLOP that are extended in this work and state the basic definitions and terminology needed for this purpose. Sect. 3 is devoted to the analysis of the minimum equation system that is valid for each solution to the QLOP, revealing also results concerning triangle inequalities, before addressing more general odd-cycle inequalities in Sect. 4. The extended formulation is presented in Sect. 5. A brief conclusion is given in Sect. 6. In the appendix, Sect. A provides an explicit list of the facet-defining triangle inequalities and Sect. B lists the further facet-defining inequalities from [3].

2 Preparations and Review of Results

Concerning the objective function (1), we will assume w.l.o.g. that the matrix C is strictly upper triangular and that the matrix Q is replaced by an $\binom{n}{2} \times \binom{n}{2}$ matrix that is strictly upper triangular as well. These new matrices provide coefficients for pairs (i, j) where $i, j \in [n]$, $i < j$, and their combinations with pairs (k, ℓ) where $k, \ell \in [n]$, $k < \ell$ and either $i < k$ or $i = k$ and $j < \ell$, respectively. As in [2], the resulting index set for the quadratic terms of interest is thus $I = \{(i, j, k, \ell) : i, j, k, \ell \in [n],\ i < j \text{ and } k < \ell \text{ and } (i < k \text{ or } (i = k \text{ and } j < \ell))\}$, and the one concerning single pairs is $I_2 := \{(i, j) : i, j \in [n],\ i < j\}$. Upon the inclusion of a constant term $\tilde{C}$ and an update of the coefficients (see e.g. [3]), we may then rewrite (1) as

$$f(\pi) = \tilde{C} + \sum_{(i,j) \in I_2, \pi(i) < \pi(j)} c_{ij} + \sum_{(i,j,k,\ell) \in I : \pi(i) < \pi(j); \pi(k) < \pi(\ell)} q_{ijk\ell}. \qquad (2)$$

For a permutation $\pi \in \Pi_n$, define its associated incidence vector $\xi(\pi) \in \mathbb{R}^{\binom{n}{2}}$ such that, for all $(i, j) \in I_2$, $\xi(\pi)_{ij} = 1$ if $\pi(i) < \pi(j)$ and $\xi(\pi)_{ij} = 0$ otherwise. Moreover, let Ξ_n be the collection of the vectors $\xi(\pi)$ for all $\pi \in \Pi_n$. Then the quadratic linear ordering polytope of order n can be defined as

$$P^n_{\text{QLO}} := \operatorname{conv}\left\{(x, y)^\mathsf{T} \in \{0,1\}^{\binom{n}{2} \times |I|} : x \in \Xi_n \text{ and } y_{ijk\ell} = x_{ij} x_{k\ell} \text{ for all } (i, j, k, \ell) \in I\right\}.$$

In [2], Buchheim, Wiegele and Zheng show that the integer linear program

$$\text{(QLOP-ILP)}$$

$$\begin{aligned}
\max \quad & \sum_{(i,j)\in I_2} c_{ij}x_{ij} + \sum_{(i,j,k,\ell)\in I} q_{ijkl}y_{ijk\ell} && \\
\text{s.t.} \quad & y_{ijik} + y_{ikjk} - y_{ijjk} = x_{ik} && \text{for all } i,j,k \in [n] : i<j<k \quad (3)\\
& y_{ijk\ell} - x_{ij} \le 0 && \text{for all } (i,j,k,\ell) \in I \quad (4)\\
& y_{ijk\ell} - x_{k\ell} \le 0 && \text{for all } (i,j,k,\ell) \in I \quad (5)\\
& x_{ij} + x_{k\ell} - y_{ijk\ell} \le 1 && \text{for all } (i,j,k,\ell) \in I \quad (6)\\
& y_{ijk\ell} \ge 0 && \text{for all } (i,j,k,\ell) \in I \quad (7)\\
& x_{ij} \in \{0,1\} && \text{for all } (i,j) \in I_2
\end{aligned}$$

is an exact formulation for the QLOP, by verifying that the equations (3) ensure $x \in \Xi_n$ if $x \in \{0,1\}^{\binom{n}{2}}$. Even more, they proved that (3) constitute a minimal equation system for P^n_{QLO}. Besides these equations, the inequalities (4)–(7) implement the "standard" linearization (see [4, 7]) for the products $y_{ijk\ell} = x_{ij}x_{k\ell}$. We refer to the program that results from QLOP-ILP by removing the final integrality restrictions on x as its *continuous* or, more precisely, *linear programming* (LP) *relaxation*, and remark that (4)–(7) imply $0 \le x \le 1$ and $0 \le y \le 1$.

For the discussion to follow, it is convenient to define the further index sets $I_3 := \{(i,j,i,k),(i,j,j,k),(i,k,j,k) : i,j,k \in [n],\ i<j<k\} \subseteq I$ and $I_4 := I \setminus I_3$ with respect to the (linearized) quadratic terms. I_3 corresponds to the only index combinations $(i,j,k,\ell) \in I$ such that $|\{i,j,k,\ell\}| = 3$ while I_4 could also be written as $I_4 = \{(i,j,k,\ell) : i,j,k,\ell \in [n], |\{i,j,k,\ell\}| = 4,\ i<j \text{ and } k<\ell \text{ and } i<k\}$.

As DeVries [3] observed, for each triple $i,j,k \in [n] : i<j<k$, only half of the twelve inequalities appearing as (4)–(7) for $(i,j,i,k),(i,j,j,k),(i,k,j,k) \in I_3$ are truly required. More precisely, one may choose e.g.

$$\begin{aligned}
& y_{ijik} - x_{ij} \le 0 && \text{for all } i,j,k \in [n] : i<j<k \quad (8)\\
& y_{ijik} - x_{ik} \le 0 && \text{for all } i,j,k \in [n] : i<j<k \quad (9)\\
& y_{ikjk} - x_{ik} \le 0 && \text{for all } i,j,k \in [n] : i<j<k \quad (10)\\
& y_{ikjk} - x_{jk} \le 0 && \text{for all } i,j,k \in [n] : i<j<k \quad (11)\\
& x_{ij} + x_{jk} - y_{ijjk} \le 1 && \text{for all } i,j,k \in [n] : i<j<k\\
& -y_{ijjk} \le 0 && \text{for all } i,j,k \in [n] : i<j<k
\end{aligned}$$

while the other instances of (4)–(7) are then implied by combining two of these inequalities for the respective other two I_3-variables, e.g. the neglected inequality (4) for y_{ijjk} is implied by the two selected inequalities (4) for y_{ijik} and (5) for y_{ikjk}. For ease of reference, we will call any choice of such a proper restriction of (4)–(7) an *I_3-linearization reduction*. Moreover, as opposed to the variables in I_3 which are necessary to formulate (3) and thus to model feasible permutations, the variables $(i,j,k,\ell) \in I_4$ only need to be introduced if $q_{ijk\ell} \ne 0$, while (4) and (5) can be omitted if $q_{ijk\ell} < 0$, and (6) and (7) can be omitted if $q_{ijk\ell} > 0$. For the polyhedral results concerning P^n_{QLO} presented in the paper, we assume that all I_4-variables are present.

Noteworthy, the equations (3) may be straightforwardly employed to eliminate variables and then also themselves, as has been briefly considered by DeVries [3] as well. For instance, one could substitute for the identity $y_{ijjk} = y_{ijik} + y_{ikjk} - x_{ik}$ for all $i, j, k \in [n] : i < j < k$. Any such elimination will be called an I_3-*equation reduction* in the following. Choosing the exemplified one, and combining it with the above I_3-linearization reduction, one arrives at the following formulation.

(QLOP-ILP-RED)

$$\max \sum_{(i,j)\in I_2} c_{ij}x_{ij} + \sum_{i<j<k} \big(q_{ijjk}(y_{ijik} + y_{ikjk} - x_{ik}) + q_{ijik}y_{ijik} + q_{ikjk}y_{ikjk}\big) + \sum_{(i,j,k,\ell)\in I_4} q_{ijkl}y_{ijk\ell}$$

$$\begin{aligned}
\text{s.t.}\quad & (8), (9), (10), (11), \\
-y_{ijik} - y_{ikjk} + x_{ik} &\leq 0 && \text{for all } i,j,k \in [n] : i < j < k \\
x_{ij} + x_{jk} + x_{ik} - y_{ijik} - y_{ikjk} &\leq 1 && \text{for all } i,j,k \in [n] : i < j < k \\
y_{ijk\ell} - x_{ij} &\leq 0 && \text{for all } (i,j,k,\ell) \in I_4 \\
y_{ijk\ell} - x_{k\ell} &\leq 0 && \text{for all } (i,j,k,\ell) \in I_4 \\
x_{ij} + x_{k\ell} - y_{ijk\ell} &\leq 1 && \text{for all } (i,j,k,\ell) \in I_4 \\
y_{ijk\ell} &\geq 0 && \text{for all } (i,j,k,\ell) \in I_4 \\
x_{ij} &\in \{0,1\} && \text{for all } (i,j) \in I_2
\end{aligned}$$

3 A Deeper Analysis of the Minimal Equation System and the Related Triangle Inequalities

Apparently, the equations (3) play a central role for P^n_{QLO} and integer linear programming formulations derived by linearizing the products of linear ordering variables. The following extended analysis unveils new aspects regarding their impact and implications, so as to explain their role for the strength of the formulations from Sect. 2, respectively of their continuous relaxations in characterizing the solution space. This leads to a discussion of triangle inequalities for P^n_{QLO} in a natural way.

We begin this analysis by extending the result by Buchheim, Wiegele and Zheng in [2], showing that the equations (3) imply the well-known three-di-cycle inequalities

$$0 \leq x_{ij} + x_{jk} - x_{ik} \leq 1 \text{ for all } i,j,k \in [n] : i < j < k \tag{12}$$

if $x \in \{0,1\}^{\binom{n}{2}}$ (and thus transitivity in the sense that $x_{ik} = 1$ if $x_{ij} = x_{jk} = 1$ and $x_{ik} = 0$ if $x_{ij} = x_{jk} = 0$), to the continuous case. In other words, we show that equations (3) imply inequalities (12) already for the LP relaxation of QLOP-ILP. Moreover, we show that the converse is not true, i.e., that the relaxation with (3) is strictly stronger.

Theorem 1. *Let $(\bar{x}, \bar{y})$ be a feasible solution to the continuous relaxation of QLOP-ILP. Then, for all $i, j, k \in [n]$, $i < j < k$, $0 \leq \bar{x}_{ij} + \bar{x}_{jk} - \bar{x}_{ik} \leq 1$.*

Proof. Fix some arbitrary triple $i, j, k \in [n]$, $i < j < k$. By equations (3), a feasible solution $(\bar{x}, \bar{y})$ to the relaxation of QLOP-ILP satisfies

$$-\bar{y}_{ijik} - \bar{y}_{ikjk} + \bar{y}_{ijjk} + \bar{x}_{ik} \leq 0 \tag{$*$}$$

When linearly combining $(*)$ as follows with (6) for $\bar{y}_{ijjk}$, (4) for $\bar{y}_{ikjk}$, and (5) for $\bar{y}_{ijik}$, we obtain:

$$
\begin{array}{rrrrrl}
-\bar{y}_{ijik} - \bar{y}_{ikjk} + \bar{y}_{ijjk} & & & & +\bar{x}_{ik} & \leq 0 \\
 & -\bar{y}_{ijjk} & +\bar{x}_{ij} & +\bar{x}_{jk} & & \leq 1 \\
+\bar{y}_{ikjk} & & & & -\bar{x}_{ik} & \leq 0 \\
+\bar{y}_{ijik} & & & & -\bar{x}_{ik} & \leq 0 \\
\hline
 & & +\bar{x}_{ij} & +\bar{x}_{jk} & -\bar{x}_{ik} & \leq 1
\end{array}
$$

Analogously, when combining $(*)$ with (7) for $\bar{y}_{ijjk}$, (5) for $\bar{y}_{ikjk}$, and (4) for $\bar{y}_{ijik}$, one obtains $-\bar{x}_{ij} - \bar{x}_{jk} + \bar{x}_{ik} \leq 0$. □

Theorem 2. *For any fixed $n \in \mathbb{N}$, $n \geq 3$, let $\tilde{P}$ be the polytope given by the feasible region of the continuous relaxation of QLOP-ILP, and let $\tilde{Q}$ be the respective polytope given when replacing (3) by (12) in QLOP-ILP. Then, $\tilde{P} \subsetneq \tilde{Q}$.*

Proof. The part $\tilde{P} \subseteq \tilde{Q}$ follows directly from Theorem 1. To show $\tilde{Q} \not\subseteq \tilde{P}$, consider the vector $(\hat{x}, \hat{y}) = (\frac{1}{2}, 0)$. It is easy to verify that $(\hat{x}, \hat{y})$ satisfies $0 \leq \bar{x}_{ij} + \bar{x}_{jk} - \bar{x}_{ik} \leq 1$ for all $i, j, k \in [n]$, $i < j < k$, as well as (4)–(7), and thus is feasible for $\tilde{Q}$. However, $(\hat{x}, \hat{y})$ strictly satisfies (i.e. is not binding for) *any* instance of (13) and thus violates *each* instance of (3) for $n \geq 3$, i.e., $(\hat{x}, \hat{y}) \notin \tilde{P}$. □

Theorem 2 theoretically underpins and partially explains the experimental observation in [2] that for specific instances the solution times with an ILP solver may significantly improve with QLOP-ILP compared to when replacing (3) by (12).

The following proposition subsumes a simple observation that promotes insights about a certain role of the variables x_{ik}, $(i,k) \in I_2$, $k - i > 1$, in QLOP-ILP.

Proposition 3. *Let $(\bar{x}, \bar{y})$ be a feasible solution to the continuous relaxation of QLOP-ILP. Then, for all $i, j, k \in [n]$, $i < j < k$, we have $\bar{y}_{ijjk} \leq \bar{x}_{ik}$*

Proof. By equations (3), as well as (4) for $\bar{y}_{ikjk}$ and (5) for $\bar{y}_{ijik}$, we have $\bar{y}_{ijjk} = \underbrace{\bar{y}_{ijik}}_{\leq \bar{x}_{ik}} + \underbrace{\bar{y}_{ikjk}}_{\leq \bar{x}_{ik}} - \bar{x}_{ik}$. □

This result emphasizes in particular that if $x_{ik} = 0$ for *some* $(i,k) \in I_2$, $i+1 < k$, then $y_{ijik} = y_{ikjk} = y_{ijjk} = 0$ for *all* $j \in [n]$, $i < j < k$, which may be exploited for instance in the implementation of branch-and-bound algorithms for QLOP.

Remark 4. *By the derivations in Sect. 2, the above results naturally extend to an I_3-linearization reduction of QLOP-ILP and to QLOP-ILP-RED.*

We now direct our attention to the well-observed fact that each of the equations (3) describes a facet of the Boolean Quadric Polytope $P_{\text{BQP}}^{\binom{n}{2}}$ [11] that is induced by the corresponding triangle inequality:

$$y_{ijik} + y_{ikjk} - y_{ijjk} - x_{ik} \leq 0 \qquad \text{for all } i, j, k \in [n] : i < j < k \tag{13}$$

Besides (13), there are three further triangle inequalities defined for $P_{\mathrm{BQP}}^{\binom{n}{2}}$ and the same I_3-index respectively y-variable triples:

$$y_{ijik} - y_{ikjk} + y_{ijjk} - x_{ij} \leq 0 \qquad \text{for all } i,j,k \in [n] : i < j < k \qquad (14)$$

$$-y_{ijik} + y_{ikjk} + y_{ijjk} - x_{jk} \leq 0 \qquad \text{for all } i,j,k \in [n] : i < j < k \qquad (15)$$

$$-y_{ijik} - y_{ikjk} - y_{ijjk} + x_{ij} + x_{jk} + x_{ik} \leq 1 \qquad \text{for all } i,j,k \in [n] : i < j < k \qquad (16)$$

As has been observed also in [5] in matrix notation for the linear constraints (3)–(7) of QLOP-ILP, these triangle inequalities are implied by its continuous relaxation as well. More precisely, the proof of the following theorem shows that any point on the affine space defined by one instance of (3) that additionally satisfies (a weakening of) the "standard" linearization inequalities (4)–(7) lies in the feasible halfspaces of the according instances of (14), (15), and (16).

Theorem 5. *Let $(\bar{x},\bar{y})$ be a feasible solution to the continuous relaxation of QLOP-ILP. Then, for all $i,j,k \in [n]$, $i < j < k$, $(\bar{x},\bar{y})$ satisfies (14), (15), and (16).*

Proof. Substituting for $\bar{y}_{ijjk}$ by $\bar{y}_{ijik} + \bar{y}_{ikjk} - \bar{x}_{ik}$ in (14) gives $2\bar{y}_{ikjk} - \bar{x}_{ik} - \bar{x}_{jk} \leq 0$ which is implied by (4) and (5) for $\bar{y}_{ikjk}$. Analogously, substituting for $\bar{y}_{ijjk}$ by $\bar{y}_{ijik} + \bar{y}_{ikjk} - \bar{x}_{ik}$ in (15) gives $2\bar{y}_{ijik} - \bar{x}_{ij} - \bar{x}_{ik} \leq 0$ which is implied by (4) and (5) for $\bar{y}_{ijik}$. Finally, substituting for $\bar{x}_{ik}$ by $\bar{y}_{ijik} + \bar{y}_{ikjk} - \bar{y}_{ijjk}$ in (16) gives $\bar{x}_{ij} + \bar{x}_{jk} - 2\bar{y}_{ijjk} \leq 1$ which is implied by (6) and (7) for $\bar{y}_{ijjk}$. □

For $n \geq 4$, there are up to four classes of further triangle inequalities defined besides (14)–(16) that are not implied by (3), and so may be violated by a feasible solution to the continuous relaxation of QLOP-ILP. We now explore them one after one regarding the question whether they define facets of P_{QLO}^{n}, $n \geq 4$.

The first additional class stems from triangles combining three I_3-variables that do not all belong to the same triple $i,j,k \in [n]$, $i < j < k$, but to different ones. An example configuration $(\bar{x},\bar{y})$ that is feasible for the LP relaxation of QLOP-ILP for $n = 4$ and that violates the triangle inequality

$$y_{1213} - y_{1214} + y_{1314} - x_{13} \leq 0$$

belonging to this class is shown in Figure 1. The inequalities stemming from such triangles however do *not* define facets of P_{QLO}^{n}, $n \geq 4$. In particular, for $n = 4$, their induced faces F have $\dim F < 16$ while the dimension of P_{QLO}^{4} is $\dim P_{\mathrm{QLO}}^{4} = \binom{4}{2} + \binom{\binom{4}{2}}{2} - \binom{4}{3} = 17$ [2].

The second additional class combines two I_3-variables and one I_4-variable. Exactly one of the four inequalities associated with each such triangle defines a facet of P_{QLO}^{4}, $n \geq 4$. A proof of this statement and an explicit list of these inequalities is given in the appendix section A. The result has also been equivalently derived by DeVries w.r.t. a variable space that results from an I_3-equation reduction (Table 3.4 in [3]). As an asset, the more natural representation considered in the appendix permits to identify the general (sign) pattern (in terms of the $\{-1,1\}$-coefficients) that distinguishes the facet-defining inequalities from the other three ones per triangle. It turns out that the number of y-variables of the form y_{ijjk} for some $i,j,k \in [n] : i < j < k$, is characteristic in this respect. We therefore subdivide the index set I_3 into the set $I_3^* := \{(i,j,j,k) \in$

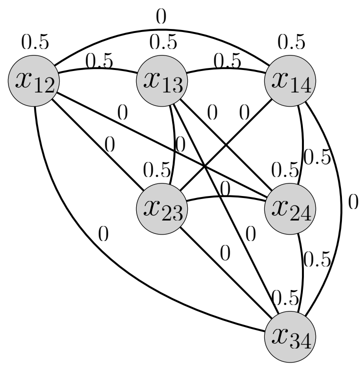

Figure 1: Graph-based illustration where the vertices (edges) correspond to the x-(y-)variables for P^4_{QLO}. The attached values reflect a feasible solution to the continuous relaxation of QLOP-ILP that violates the (non-facet-inducing) triangle inequality $y_{1213} + y_{1314} - y_{1214} - x_{13} \leq 0$.

$I_3 : i < j < k\}$ collecting these variable subscripts and the set $I_3^- = I_3 \setminus I_3^*$. The subset of the second class of triangle inequalities that induce facets for P^4_{QLO}, $n \geq 4$, can then be compactly written as follows where we use elements from I interchangeably with their corresponding y-indices (i.e., with or without parenthesis) and the notation $e \cap f$ for two y-variable indices to denote their common index pair from I_2 (e.g. $(i,j,j,k) \cap (i,j,k,\ell) = (i,j)$).

$$-y_e + y_f + y_g - x_{f\cap g} \leq 0 \qquad \text{for all } e \in I_4, f \in I_3^-, g \in I_3^-, f \neq g$$
$$y_e + y_f - y_g - x_{e\cap f} \leq 0 \qquad \text{for all } e \in I_4, f \in I_3^-, g \in I_3^*$$
$$-y_f - y_g - y_e + x_{e\cap f} + x_{f\cap g} + x_{e\cap g} \leq 1 \qquad \text{for all } e \in I_4, f \in I_3^*, g \in I_3^*, f \neq g$$

This representation is thus to be interpreted as follows. If no index from I_3^* is involved in the triangle then the single I_4-variable is the only one with a negative coefficient (right-hand side zero). If exactly one I_3^*-index is involved, then the respective variable is the only one with a negative coefficient (right-hand side zero). Finally, if two I_3^*-variables are involved, then all y-variables obtain a negative coefficient and the right-hand side is one. The x-variables receive their unique coefficients (signs) according to the inherent pattern that is given in (14)–(16) or, more generally, described for (17) in Sect. 4.

For $n \geq 5$, there is a third additional class that combines one I_3-variable and two I_4-variables. Here, there are two facet-defining inequalities per triangle. Again, an explicit list of these and a proof is given in the appendix section A. Using the same notation as before, these can be compactly expressed as follows.

$$-y_e + y_f + y_g - x_{f\cap g} \leq 0 \qquad \text{for all } e \in I_4, f \in I_4, g \in I_3^-, e \neq f$$
$$y_e - y_f + y_g - x_{e\cap g} \leq 0 \qquad \text{for all } e \in I_4, f \in I_4, g \in I_3^-, e \neq f$$
$$y_e + y_f - y_g - x_{e\cap f} \leq 0 \qquad \text{for all } e \in I_4, f \in I_4, g \in I_3^*, e \neq f$$
$$-y_f - y_g - y_e + x_{e\cap f} + x_{f\cap g} + x_{e\cap g} \leq 1 \qquad \text{for all } e \in I_4, f \in I_4, g \in I_3^*, e \neq f$$

The interpretation is now again based on the number of involved I_3^*-variables. If there are no such variables, then both I_4-variables obtain once a negative coefficient, giving the first two inequality types. If there is a variable from I_3^*, then there is one facet-defining inequality where only this variable has a negative coefficient, and one where all y-variables have negative coefficients (right-hand side one).

Finally, for $n \geq 6$, there is a fourth additional class that combines three I_4-variables. Here, all four inequalities per triangle are facet-defining which are listed together with a proof in the appendix section A. Even though thus no distinction among the inequalities associated with a single triangle is necessary, we continue for consistency the notation used for the previous classes.

$$
\begin{aligned}
&-y_e + y_f + y_g - x_{f\cap g} \leq 0 && \text{for all } e,f,g \in I_4, e \neq f \neq g \neq e \\
&y_e - y_f + y_g - x_{e\cap g} \leq 0 && \text{for all } e,f,g \in I_4, e \neq f \neq g \neq e \\
&y_e + y_f - y_g - x_{e\cap f} \leq 0 && \text{for all } e,f,g \in I_4, e \neq f \neq g \neq e \\
&-y_f - y_g - y_e + x_{e\cap f} + x_{f\cap g} + x_{e\cap g} \leq 1 && \text{for all } e,f,g \in I_4, e \neq f \neq g \neq e
\end{aligned}
$$

We conclude that the facial structure for P^n_{QLO}, $n \geq 4$, includes triangle inequalities, but in a more complicated pattern than in case of the Boolean Quadric Polytope. From a practical perspective, the results suggest a tailored separation procedure for the triangle inequalities. When using QLOP-ILP-RED instead of QLOP-RED, one further needs to take into account that the edges corresponding to the I_3-equation reduction do not anymore appear in the graph G if it were straightforwardly constructed from the variables present in the formulation.

The triangle inequalities constitute a subset of the more general odd-cycle inequalities that we address in Section 4, thereby further confirming the conjecture by Buchheim, Wiegele, and Zheng in [2] that it is likely to obtain inequalities from a usual separation procedure which are not facet-inducing.

4 Odd-Cycle Inequalities for the Quadratic Linear Ordering Polytope

To extend the discussion of odd-cycle inequalities for the QLOP beyond the special case of triangles addressed in Sect. 3, it is helpful to associate an instance of the QLOP with a graph. To this end, let $G = (V,E)$ be the undirected graph such that $v_{ij} \in V$ corresponds to $(i,j) \in I_2$ and the corresponding variable x_{ij}, and there is an edge $\{v_{ij}, v_{k\ell}\} \in E$ corresponding to the variable $y_{ijk\ell}$, respectively $(i,j,k,\ell) \in I$. For $n = 4$, G is precisely the graph shown in Figure 1.

Given $G = (V,E)$, consider now an arbitrary cycle consisting of the edges $C \subseteq E$. Then, for P^n_{QLO} and $P^{\binom{n}{2}}_{\text{BQP}}$, $n \geq 4$, the following odd-cycle inequality is defined for any partition of C into $C_A \cup C_B$ such that $|C_A|$ is odd.

$$
\sum_{v_{ij} \in V_A} x_{ij} - \sum_{v_{ij} \in V_B} x_{ij} - \sum_{\{v_{ij}, v_{jk}\} \in C_A} y_{ijk\ell} + \sum_{\{v_{ij}, v_{k\ell}\} \in C_B} y_{ijk\ell} \leq \left\lfloor \frac{|C_A|}{2} \right\rfloor \tag{17}
$$

Here, V_A (V_B) are the vertices incident to exactly two edges in C_A (C_B).

In view of the results for $|C|=3$ presented in Sect. 3, we will focus on cycles of length at least four, and thereby make use of the fact that it is a necessary condition for an odd-cycle inequality to be facet-defining that the respective cycle is chordless in G.

Lemma 1. *Let C, $|C|>3$, be a cycle in the graph G that has a chord. Then, the odd-cycle inequalities (17) defined for C do not induce facets of P^n_{QLO}, $n \geq 4$.*

Proof. If $c \in E \setminus C$ is a chord of C, then (17) can be written as a linear combination of two other valid odd-cycle inequalities [1, 11, 9] obtained from splitting C along c, and thus cannot be a facet of P^n_{QLO}. □

In case of the Boolean Quadric Polytope $P_{\mathrm{BQP}}^{\binom{n}{2}}$, chordlessness is also a sufficient condition for an odd-cycle inequality to define a facet [1, 11]. This is not true for P^n_{QLO}, as we already see from the fact that not all triangle inequalities are facet-inducing for it. Conversely, by taking a closer look at the structure of the graph G, it turns out that Lemma 1 already serves as a sufficient basis to prove the following result.

Theorem 6. *For $n \geq 4$, no odd-cycle inequality (17) other than the triangle inequalities described in Sect. 3, respectively (22)–(153) in the appendix, induces a facet of P^n_{QLO}.*

Proof. Since every possible I_4-edge is present, G is a complete graph. It follows that each cycle of length larger than three has a chord, and so, by Lemma 1, the corresponding odd-cycle inequalities cannot induce a facet of P^n_{QLO}. □

5 An Extended Formulation

In this section, we take on an alternative perspective on the I_3-variables that concern the relative positions of entity triples $i,j,k \in [n]$, $i<j<k$, only. Replacing these variables by (asymmetric) betweenness variables, in a reverse fashion compared to the approach in [10], permits to derive an extended formulation for the QLOP whose continuous relaxation implies the "standard" linearization with respect to the (replaced) I_3-part.

To build this extended formulation, we first replace, for all $i,j,k \in [n]$, $i<j<k$, the variables $y_{ijik}, y_{ikjk} \in I_3$ (while keeping y_{ijjk}) by the variables y_{kiij} and y_{ikkj} so that the common index can be dropped. Switching further from y to b for further umambiguity, we obtain the three variables b_{ijk}, b_{kij} and b_{ikj}, and further introduce the "reverse" betweenness variables b_{jik}, b_{kji} and b_{jki} for all $i,j,k \in [n]$, $i<j<k$, supposed to be equal to one if and only if the order specified by the respective index is respected by the permutation. The objective coefficients q_{ijjk}, q_{ikjk}, and q_{ijik} are accordingly reassigned to b_{ijk}, $x_{ik} - b_{ikj}$, and $x_{ij} - b_{kij}$.

The next step is to replace (3) by the following equation, which can be obtained from it by simple term substitutions [10] as well:

$$b_{kij} + b_{ikj} + b_{ijk} - x_{ij} = 0 \text{ for all } i,j,k \in [n] : i<j<k \tag{18}$$

Finally, we append three more pendants of (18) from [10], for instance:

$$\begin{aligned}
b_{jik} + b_{ikj} + b_{ijk} - x_{ik} &= 0 && \text{for all } i,j,k \in [n] : i<j<k && (19)\\
b_{jik} + b_{jki} + b_{ijk} - x_{jk} &= 0 && \text{for all } i,j,k \in [n] : i<j<k && (20)\\
b_{kji} + b_{jki} + b_{jik} + x_{ij} &= 1 && \text{for all } i,j,k \in [n] : i<j<k && (21)
\end{aligned}$$

The resulting formulation is subsequently displayed as EXT-QLOP-ILP, thereby emphasizing how to redistribute the objective coefficients of QLOP-ILP, while further objective coefficients for b-variables could be inserted in a straightforward way.

(EXT-QLOP-ILP)

$$\max \sum_{(i,j)\in I_2} c_{ij}x_{ij} + \sum_{i<j<k} \big(q_{ijjk}b_{ijk} + q_{ijik}(x_{ij} - b_{kij}) + q_{ikjk}(x_{ik} - b_{ikj})\big) + \sum_{(i,j,k,\ell)\in I_4} q_{ijkl}y_{ijk\ell}$$

$$\begin{aligned}
\text{s.t.}\quad b_{kij} + b_{ikj} + b_{ijk} - x_{ij} &= 0 && \text{for all } i,j,k \in [n] : i < j < k\\
b_{jik} + b_{ikj} + b_{ijk} - x_{ik} &= 0 && \text{for all } i,j,k \in [n] : i < j < k\\
b_{jik} + b_{jki} + b_{ijk} - x_{jk} &= 0 && \text{for all } i,j,k \in [n] : i < j < k\\
b_{kji} + b_{jki} + b_{jik} + x_{ij} &= 1 && \text{for all } i,j,k \in [n] : i < j < k\\
y_{ijk\ell} - x_{ij} &\le 0 && \text{for all } (i,j,k,\ell) \in I_4\\
y_{ijk\ell} - x_{k\ell} &\le 0 && \text{for all } (i,j,k,\ell) \in I_4\\
x_{ij} + x_{k\ell} - y_{ijk\ell} &\le 1 && \text{for all } (i,j,k,\ell) \in I_4\\
y_{ijk\ell} &\ge 0 && \text{for all } (i,j,k,\ell) \in I_4\\
x_{ij} &\in \{0,1\} && \text{for all } (i,j) \in I_2
\end{aligned}$$

As shown in [10], the equation system has full rank, and it implies the three-dicycle inequalities (12) as well as the inequalities of the "standard" linearization (for all b-variables) already for the continuous relaxation of EXT-QLOP-ILP. Thus, the explicit linearization is only kept for the I_4-variables.

Accounting for the program size, the extended formulation replaces the $3\binom{n}{3}$ I_3-variables of QLOP-ILP by $6\binom{n}{3}$ b-variables, and in turn its $6\binom{n}{3}$ inequalities (assuming an I_3-linearization reduction and counting potential inequalities of the form $y_e \ge 0$, $e \in I_3$) by $4\binom{n}{3}$ equations. Importantly, it has no equivalent in the variable space of QLOP-ILP which is evident from the fact that equations (3) constitute a minimal equation system for P^n_{QLO}. Indeed, when re-applying the variable-term substitutions, then any of (18)–(21) turns into (3).

At the same time, the equation system in EXT-QLOP-ILP may be used to eliminate variables in the same fashion as carried out for QLOP-ILP in Sect. 2, e.g. using the identities:

$$\begin{aligned}
b_{ijk} &= -b_{kij} - b_{ikj} + x_{ij} && \text{for all } i,j,k \in [n] : i < j < k\\
b_{jik} &= b_{kij} - x_{ij} + x_{ik} && \text{for all } i,j,k \in [n] : i < j < k\\
b_{jki} &= b_{ikj} - x_{ik} + x_{jk} && \text{for all } i,j,k \in [n] : i < j < k\\
b_{kji} &= 1 - x_{jk} - b_{kij} - b_{ikj} && \text{for all } i,j,k \in [n] : i < j < k
\end{aligned}$$

The resulting reduced version of EXT-QLOP-ILP however does not imply the non-negativity of the right-hand sides of these equations (whereas, for the single equation used for elimination to obtain QLOP-ILP-RED, the inequality $-y_{ijik} - y_{ikjk} + x_{ik} \le 0$ stemming from the standard linearization establishes it). On the other hand, it is easy to verify that the right-hand sides of these equations must be less or equal to one if they are enforced to be non-negative. Indeed, when studying the polytope spanned by the six vertices corresponding to the permutations of three distinct i, j, k, in the variables x_{ij}, x_{jk}, x_{ik}, b_{kij}, and b_{ikj}, it turns out that the corresponding inequalities for non-negativity of the above right-hand sides, $b_{kij} \geq 0$, and $b_{ikj} \geq 0$ induce a complete facet description. Therefore, in terms of formulation size, a net reduction of $4\binom{n}{3}$ variables is achieved compared to EXT-QLOP-ILP while $4\binom{n}{3}$ equations (with support size four) are replaced by equally many inequalities (with support size three).

$$
\begin{aligned}
&&& \text{(EXT-QLOP-ILP-RED)}\\
\max \; & \sum_{(i,j)\in I_2} c_{ij}x_{ij} + \sum_{i<j<k} \big(q_{ijjk}(-b_{kij}-b_{ikj}+x_{ij}) + q_{ijik}(x_{ij}-b_{kij}) + q_{ikjk}(x_{ik}-b_{ikj})\big) + \sum_{(i,j,k,\ell)\in I_4} q_{ijkl}y_{ijk\ell} \\
\text{s.t.} \; & b_{kij}+b_{ikj}-x_{ij} \leq 0 && \text{for all } i,j,k \in [n] : i<j<k \\
& -b_{kij}+x_{ij}-x_{ik} \leq 0 && \text{for all } i,j,k \in [n] : i<j<k \\
& -b_{ikj}+x_{ik}-x_{jk} \leq 0 && \text{for all } i,j,k \in [n] : i<j<k \\
& b_{kij}+b_{ikj}+x_{jk} \leq 1 && \text{for all } i,j,k \in [n] : i<j<k \\
& b_{kij},\, b_{ikj} \geq 0 && \text{for all } i,j,k \in [n] : i<j<k \\
& y_{ijk\ell}-x_{ij} \leq 0 && \text{for all } (i,j,k,\ell) \in I_4 \\
& y_{ijk\ell}-x_{k\ell} \leq 0 && \text{for all } (i,j,k,\ell) \in I_4 \\
& x_{ij}+x_{k\ell}-y_{ijk\ell} \leq 1 && \text{for all } (i,j,k,\ell) \in I_4 \\
& y_{ijk\ell} \geq 0 && \text{for all } (i,j,k,\ell) \in I_4 \\
& x_{ij} \in \{0,1\} && \text{for all } (i,j) \in I_2
\end{aligned}
$$

When now comparing EXT-QLOP-ILP-RED with QLOP-ILP-RED, it is verified by substituting for $b_{kij} = x_{ij} - y_{ijik}$ and $b_{ikj} = x_{ik} - y_{ikjk}$ that they are equivalent.

We close this section with the remark that the results for QLOP-ILP concerning the triangle inequalities defined for I_3-variables in Sect. 3 carry over to EXT-QLOP-ILP, i.e., $b_{kij}+b_{ikj}+b_{ijk}-x_{ij} \leq 0$, $-b_{kij}+b_{ikj}+b_{ijk}-x_{ik} \leq 0$, $x_{jk}+b_{kij}-b_{ijk}+b_{ikj} \leq 1$, and $b_{kij}+b_{ijk}-b_{ikj}+x_{ik}-x_{ij}-x_{jk} \leq 0$ are implied by solutions satisfying (18).

6 Conclusion

We have shown that the characteristic minimal equation system partly describing the (convex hull of the) feasible solutions to the quadratic linear ordering problem implies the three-di-cycle inequalities known from the linear ordering problem already when solving the continuous relaxation of the corresponding integer linear program. We further illustrate their impact on the related triangle inequalities and reveal a pattern to identify the restricted subset of these inequalities that define facets of the quadratic linear ordering polytope. We further proved that no odd-cycle inequalities other than these particular triangle inequalities induce facets of this polytope, and demonstrated that there is an extended formulation that implicitly linearizes the quadratic terms that refer to three distinct elements.

The contributed results suggest tailored algorithms to separate triangle inequalities in the context of the quadratic linear ordering problem. Moreover, they suggest that a further structural analysis of the polytope P^n_{QLO}, as considered by DeVries [3] for $n = 3$ and $n = 4$, is worthwhile. Even more since, as DeVries further points out, no facet of the linear ordering polytope is a facet of the quadratic linear ordering polytope.

For completeness and convenience, we list the remaining 54 facet-defining inequalities found by DeVries and contributing to a complete description of P^4_{QLO} in the original QLOP variable space in the appendix section B.

A The facet-defining triangle inequalities for P^n_{QLO}, $n \geq 4$

We start with the facet-defining triangle inequalities involving exactly one I_4-variable for P^n_{QLO}, $n \geq 4$:

$$y_{ijik} - y_{ijj\ell} + y_{ikj\ell} - x_{ik} \leq 0 \quad \text{for all } i,j,k,\ell \in [n] : i < j < k < \ell \tag{22}$$
$$y_{iji\ell} - y_{ijjk} + y_{i\ell jk} - x_{i\ell} \leq 0 \quad \text{for all } i,j,k,\ell \in [n] : i < j < k < \ell \tag{23}$$
$$y_{iji\ell} - y_{ijk\ell} + y_{i\ell k\ell} - x_{i\ell} \leq 0 \quad \text{for all } i,j,k,\ell \in [n] : i < j < k < \ell \tag{24}$$
$$y_{iki\ell} - y_{ikj\ell} + y_{i\ell j\ell} - x_{i\ell} \leq 0 \quad \text{for all } i,j,k,\ell \in [n] : i < j < k < \ell \tag{25}$$
$$y_{ikjk} - y_{ikj\ell} + y_{jkj\ell} - x_{jk} \leq 0 \quad \text{for all } i,j,k,\ell \in [n] : i < j < k < \ell \tag{26}$$
$$y_{ikj\ell} - y_{ikk\ell} + y_{j\ell k\ell} - x_{j\ell} \leq 0 \quad \text{for all } i,j,k,\ell \in [n] : i < j < k < \ell \tag{27}$$
$$y_{ijik} + y_{ijk\ell} - y_{ikk\ell} - x_{ij} \leq 0 \quad \text{for all } i,j,k,\ell \in [n] : i < j < k < \ell \tag{28}$$
$$y_{iki\ell} + y_{ikjk} - y_{i\ell jk} - x_{ik} \leq 0 \quad \text{for all } i,j,k,\ell \in [n] : i < j < k < \ell \tag{29}$$
$$y_{i\ell jk} + y_{i\ell k\ell} - y_{jkk\ell} - x_{i\ell} \leq 0 \quad \text{for all } i,j,k,\ell \in [n] : i < j < k < \ell \tag{30}$$
$$-y_{ijj\ell} + y_{ijk\ell} + y_{j\ell k\ell} - x_{k\ell} \leq 0 \quad \text{for all } i,j,k,\ell \in [n] : i < j < k < \ell \tag{31}$$
$$-y_{i\ell jk} + y_{i\ell j\ell} + y_{jkj\ell} - x_{j\ell} \leq 0 \quad \text{for all } i,j,k,\ell \in [n] : i < j < k < \ell \tag{32}$$
$$-y_{ijjk} - y_{ijk\ell} - y_{jkk\ell} + x_{ij} + x_{jk} + x_{k\ell} \leq 1 \quad \text{for all } i,j,k,\ell \in [n] : i < j < k < \ell \tag{33}$$

The following result has been equivalently derived by DeVries w.r.t. a variable space that results from an I_3-equation reduction (Table 3.4 in [3]).

Theorem 7. *The triangle inequalities (22)–(33) define facets of P^n_{QLO} for $n \geq 4$.*

Proof. Let $F = \{(x,y)^\mathsf{T} \in \mathbb{R}^{\binom{n}{2} \times |I|} : a^\mathsf{T}(x,y)^\mathsf{T} = b\}$ be the affine space induced by any of the triangle inequalities (22)–(33), and $F^{\leq}$ the corresponding halfspace of interest. Clearly, we have $P^n_{\text{QLO}} \subseteq F^{\leq}$ as follows directly from the validity of the triangle inequalities for $P^{\binom{n}{2}}_{\text{BQP}}$. Any $(x,y)^\mathsf{T} \in F$ is a convex (linear) combination of these vertices, so the dimension $\dim F$ of F is equal to the maximal number of linear independent vectors spanning F (minus one in case $(0,0)^\mathsf{T} \notin F$).

Consider now the case $n = 4$ giving $\dim P^4_{\text{QLO}} = 17$ and exactly 12 instances of (22)–(33) in total. By insertion, one verifies that 22 of the 24 vertices of P^4_{QLO} lie on F. It follows that $F \cap P^4_{\text{QLO}} \neq \emptyset$, and so F is a face of P^4_{QLO}. Moreover, it is spanned by the respective 22 integral vertices since P^4_{QLO} is an integral polytope.

We have $(0,0)^\mathsf{T} \notin F$ only for the face F induced by (33) which contains 17 linear independent vertices of P^4_{QLO}, so $\dim F = 16$ and F is a facet of P^4_{QLO}. All the other induced faces F have a basis of 16 linear independent vertices of P^4_{QLO} and $(0,0)^\mathsf{T} \in F$, so $\dim F = 16$ as well.

Finally, as has been shown in [3], any facet for P^k_{QLO}, $k \in \mathbb{N}$, is a facet of P^{k+1}_{QLO}, so the statement follows. □

For $n \geq 5$, the additional inequalities with two I_4 variables lead to 60 different shapes, each to be read for all $i, j, k, \ell, m \in [n]$, $i < j < k < \ell < m$, whence we divide them into two parts.

$$y_{ijik} + y_{ij\ell m} - y_{ik\ell m} - x_{ij} \leq 0 \tag{34}$$
$$y_{ijik} - y_{ij\ell m} + y_{ik\ell m} - x_{ik} \leq 0 \tag{35}$$
$$y_{iji\ell} + y_{ijkm} - y_{i\ell km} - x_{ij} \leq 0 \tag{36}$$
$$y_{iji\ell} - y_{ijkm} + y_{i\ell km} - x_{i\ell} \leq 0 \tag{37}$$
$$y_{ijim} + y_{ijk\ell} - y_{imk\ell} - x_{ij} \leq 0 \tag{38}$$
$$y_{ijim} - y_{ijk\ell} + y_{imk\ell} - x_{im} \leq 0 \tag{39}$$
$$-y_{ijjk} + y_{ij\ell m} + y_{jk\ell m} - x_{\ell m} \leq 0 \tag{40}$$
$$-y_{ijjk} - y_{ij\ell m} - y_{jk\ell m} + x_{ij} + x_{jk} + x_{\ell m} \leq 1 \tag{41}$$
$$-y_{ijj\ell} + y_{ijkm} + y_{j\ell km} - x_{km} \leq 0 \tag{42}$$
$$-y_{ijj\ell} - y_{ijkm} - y_{j\ell km} + x_{ij} + x_{j\ell} + x_{km} \leq 1 \tag{43}$$
$$-y_{ijjm} + y_{ijk\ell} + y_{jmk\ell} - x_{k\ell} \leq 0 \tag{44}$$
$$-y_{ijjm} - y_{ijk\ell} - y_{jmk\ell} + x_{ij} + x_{jm} + x_{k\ell} \leq 1 \tag{45}$$
$$y_{ijk\ell} - y_{ijkm} + y_{k\ell km} - x_{k\ell} \leq 0 \tag{46}$$
$$-y_{ijk\ell} + y_{ijkm} + y_{k\ell km} - x_{km} \leq 0 \tag{47}$$
$$y_{ijk\ell} + y_{ij\ell m} - y_{k\ell\ell m} - x_{ij} \leq 0 \tag{48}$$
$$-y_{ijk\ell} - y_{ij\ell m} - y_{k\ell\ell m} + x_{ij} + x_{k\ell} + x_{\ell m} \leq 1 \tag{49}$$
$$y_{ijkm} - y_{ij\ell m} + y_{km\ell m} - x_{km} \leq 0 \tag{50}$$
$$-y_{ijkm} + y_{ij\ell m} + y_{km\ell m} - x_{\ell m} \leq 0 \tag{51}$$
$$y_{iki\ell} + y_{ikjm} - y_{i\ell jm} - x_{ik} \leq 0 \tag{52}$$
$$y_{iki\ell} - y_{ikjm} + y_{i\ell jm} - x_{i\ell} \leq 0 \tag{53}$$
$$y_{ikim} + y_{ikj\ell} - y_{imj\ell} - x_{ik} \leq 0 \tag{54}$$
$$y_{ikim} - y_{ikj\ell} + y_{imj\ell} - x_{im} \leq 0 \tag{55}$$
$$y_{ikjk} + y_{ik\ell m} - y_{jk\ell m} - x_{ik} \leq 0 \tag{56}$$
$$y_{ikjk} - y_{ik\ell m} + y_{jk\ell m} - x_{jk} \leq 0 \tag{57}$$
$$y_{ikj\ell} - y_{ikjm} + y_{j\ell jm} - x_{j\ell} \leq 0 \tag{58}$$
$$-y_{ikj\ell} + y_{ikjm} + y_{j\ell jm} - x_{jm} \leq 0 \tag{59}$$
$$y_{ikj\ell} - y_{ikkm} + y_{j\ell km} - x_{j\ell} \leq 0 \tag{60}$$
$$-y_{ikj\ell} - y_{ikkm} - y_{j\ell km} + x_{ik} + x_{j\ell} + x_{km} \leq 1 \tag{61}$$
$$y_{ikj\ell} + y_{ik\ell m} - y_{j\ell\ell m} - x_{ik} \leq 0 \tag{62}$$
$$-y_{ikj\ell} - y_{ik\ell m} - y_{j\ell\ell m} + x_{ik} + x_{j\ell} + x_{\ell m} \leq 1 \tag{63}$$

$$y_{ikjm} - y_{ikk\ell} + y_{jmk\ell} - x_{jm} \leq 0 \tag{64}$$
$$-y_{ikjm} - y_{ikk\ell} - y_{jmk\ell} + x_{ik} + x_{jm} + x_{k\ell} \leq 1 \tag{65}$$
$$y_{ikjm} - y_{ik\ell m} + y_{jm\ell m} - x_{jm} \leq 0 \tag{66}$$
$$-y_{ikjm} + y_{ik\ell m} + y_{jm\ell m} - x_{\ell m} \leq 0 \tag{67}$$
$$y_{i\ell im} + y_{i\ell jk} - y_{imjk} - x_{i\ell} \leq 0 \tag{68}$$
$$y_{i\ell im} - y_{i\ell jk} + y_{imjk} - x_{im} \leq 0 \tag{69}$$
$$y_{i\ell jk} - y_{i\ell jm} + y_{jkjm} - x_{jk} \leq 0 \tag{70}$$
$$-y_{i\ell jk} + y_{i\ell jm} + y_{jkjm} - x_{jm} \leq 0 \tag{71}$$
$$y_{i\ell jk} + y_{i\ell km} - y_{jkkm} - x_{i\ell} \leq 0 \tag{72}$$
$$-y_{i\ell jk} - y_{i\ell km} - y_{jkkm} + x_{i\ell} + x_{jk} + x_{km} \leq 1 \tag{73}$$
$$y_{i\ell jk} - y_{i\ell\ell m} + y_{jk\ell m} - x_{jk} \leq 0 \tag{74}$$
$$-y_{i\ell jk} - y_{i\ell\ell m} - y_{jk\ell m} + x_{i\ell} + x_{jk} + x_{\ell m} \leq 1 \tag{75}$$
$$y_{i\ell j\ell} + y_{i\ell km} - y_{j\ell km} - x_{i\ell} \leq 0 \tag{76}$$
$$y_{i\ell j\ell} - y_{i\ell km} + y_{j\ell km} - x_{j\ell} \leq 0 \tag{77}$$
$$y_{i\ell jm} + y_{i\ell k\ell} - y_{jmk\ell} - x_{i\ell} \leq 0 \tag{78}$$
$$-y_{i\ell jm} + y_{i\ell k\ell} + y_{jmk\ell} - x_{k\ell} \leq 0 \tag{79}$$
$$y_{i\ell jm} - y_{i\ell km} + y_{jmkm} - x_{jm} \leq 0 \tag{80}$$
$$-y_{i\ell jm} + y_{i\ell km} + y_{jmkm} - x_{km} \leq 0 \tag{81}$$
$$y_{imjk} - y_{imj\ell} + y_{jkj\ell} - x_{jk} \leq 0 \tag{82}$$
$$-y_{imjk} + y_{imj\ell} + y_{jkj\ell} - x_{j\ell} \leq 0 \tag{83}$$
$$y_{imjk} + y_{imk\ell} - y_{jkk\ell} - x_{im} \leq 0 \tag{84}$$
$$-y_{imjk} - y_{imk\ell} - y_{jkk\ell} + x_{im} + x_{jk} + x_{k\ell} \leq 1 \tag{85}$$
$$y_{imjk} + y_{im\ell m} - y_{jk\ell m} - x_{im} \leq 0 \tag{86}$$
$$-y_{imjk} + y_{im\ell m} + y_{jk\ell m} - x_{\ell m} \leq 0 \tag{87}$$
$$y_{imj\ell} - y_{imk\ell} + y_{j\ell k\ell} - x_{j\ell} \leq 0 \tag{88}$$
$$-y_{imj\ell} + y_{imk\ell} + y_{j\ell k\ell} - x_{k\ell} \leq 0 \tag{89}$$
$$y_{imj\ell} + y_{imkm} - y_{j\ell km} - x_{im} \leq 0 \tag{90}$$
$$-y_{imj\ell} + y_{imkm} + y_{j\ell km} - x_{km} \leq 0 \tag{91}$$
$$y_{imjm} + y_{imk\ell} - y_{jmk\ell} - x_{im} \leq 0 \tag{92}$$
$$y_{imjm} - y_{imk\ell} + y_{jmk\ell} - x_{jm} \leq 0 \tag{93}$$

Theorem 8. *The triangle inequalities (34)–(93) define facets of P^n_{QLO} for $n \geq 5$.*

Proof. The proof is analogous to the one for the previous case, except that is carried out for $n = 5$. Each of these inequalities has 100 of the 120 vertices of P^5_{QLO} binding, while the dimension of their associated faces is 44 and $\dim P^5_{\mathrm{QLO}} = 45$. □

For $n \geq 6$, the additional inequalities with three I_4 variables lead to 60 different instances per index configuration as well which are again divided into two parts. They are to be read for all $i, j, k, \ell, m, o \in [n]$, $i < j < k < \ell < m < o$.

$$y_{ijk\ell}+y_{ijmo}-y_{k\ell mo}-x_{ij}\leq 0 \tag{94}$$
$$y_{ijk\ell}-y_{ijmo}+y_{k\ell mo}-x_{k\ell}\leq 0 \tag{95}$$
$$-y_{ijk\ell}+y_{ijmo}+y_{k\ell mo}-x_{mo}\leq 0 \tag{96}$$
$$-y_{ijk\ell}-y_{ijmo}-y_{k\ell mo}+x_{ij}+x_{k\ell}+x_{mo}\leq 1 \tag{97}$$
$$y_{ijkm}+y_{ij\ell o}-y_{km\ell o}-x_{ij}\leq 0 \tag{98}$$
$$y_{ijkm}-y_{ij\ell o}+y_{km\ell o}-x_{km}\leq 0 \tag{99}$$
$$-y_{ijkm}+y_{ij\ell o}+y_{km\ell o}-x_{\ell o}\leq 0 \tag{100}$$
$$-y_{ijkm}-y_{ij\ell o}-y_{km\ell o}+x_{ij}+x_{km}+x_{\ell o}\leq 1 \tag{101}$$
$$y_{ijko}+y_{ij\ell m}-y_{ko\ell m}-x_{ij}\leq 0 \tag{102}$$
$$y_{ijko}-y_{ij\ell m}+y_{ko\ell m}-x_{ko}\leq 0 \tag{103}$$
$$-y_{ijko}+y_{ij\ell m}+y_{ko\ell m}-x_{\ell m}\leq 0 \tag{104}$$
$$-y_{ijko}-y_{ij\ell m}-y_{ko\ell m}+x_{ij}+x_{ko}+x_{\ell m}\leq 1 \tag{105}$$
$$y_{ikj\ell}+y_{ikmo}-y_{j\ell mo}-x_{ik}\leq 0 \tag{106}$$
$$y_{ikj\ell}-y_{ikmo}+y_{j\ell mo}-x_{j\ell}\leq 0 \tag{107}$$
$$-y_{ikj\ell}+y_{ikmo}+y_{j\ell mo}-x_{mo}\leq 0 \tag{108}$$
$$-y_{ikj\ell}-y_{ikmo}-y_{j\ell mo}+x_{ik}+x_{j\ell}+x_{mo}\leq 1 \tag{109}$$
$$y_{ikjm}+y_{ik\ell o}-y_{jm\ell o}-x_{ik}\leq 0 \tag{110}$$
$$y_{ikjm}-y_{ik\ell o}+y_{jm\ell o}-x_{jm}\leq 0 \tag{111}$$
$$-y_{ikjm}+y_{ik\ell o}+y_{jm\ell o}-x_{\ell o}\leq 0 \tag{112}$$
$$-y_{ikjm}-y_{ik\ell o}-y_{jm\ell o}+x_{ik}+x_{jm}+x_{\ell o}\leq 1 \tag{113}$$
$$y_{ikjo}+y_{ik\ell m}-y_{jo\ell m}-x_{ik}\leq 0 \tag{114}$$
$$y_{ikjo}-y_{ik\ell m}+y_{jo\ell m}-x_{jo}\leq 0 \tag{115}$$
$$-y_{ikjo}+y_{ik\ell m}+y_{jo\ell m}-x_{\ell m}\leq 0 \tag{116}$$
$$-y_{ikjo}-y_{ik\ell m}-y_{jo\ell m}+x_{ik}+x_{jo}+x_{\ell m}\leq 1 \tag{117}$$
$$y_{i\ell jk}+y_{i\ell mo}-y_{jkmo}-x_{i\ell}\leq 0 \tag{118}$$
$$y_{i\ell jk}-y_{i\ell mo}+y_{jkmo}-x_{jk}\leq 0 \tag{119}$$
$$-y_{i\ell jk}+y_{i\ell mo}+y_{jkmo}-x_{mo}\leq 0 \tag{120}$$
$$-y_{i\ell jk}-y_{i\ell mo}-y_{jkmo}+x_{i\ell}+x_{jk}+x_{mo}\leq 1 \tag{121}$$
$$y_{i\ell jm}+y_{i\ell ko}-y_{jmko}-x_{i\ell}\leq 0 \tag{122}$$
$$y_{i\ell jm}-y_{i\ell ko}+y_{jmko}-x_{jm}\leq 0 \tag{123}$$

$$-y_{i\ell jm}+y_{i\ell ko}+y_{jmko}-x_{ko}\leq 0 \tag{124}$$
$$-y_{i\ell jm}-y_{i\ell ko}-y_{jmko}+x_{i\ell}+x_{jm}+x_{ko}\leq 1 \tag{125}$$
$$y_{i\ell jo}+y_{i\ell km}-y_{jokm}-x_{i\ell}\leq 0 \tag{126}$$
$$y_{i\ell jo}-y_{i\ell km}+y_{jokm}-x_{jo}\leq 0 \tag{127}$$
$$-y_{i\ell jo}+y_{i\ell km}+y_{jokm}-x_{km}\leq 0 \tag{128}$$
$$-y_{i\ell jo}-y_{i\ell km}-y_{jokm}+x_{i\ell}+x_{jo}+x_{km}\leq 1 \tag{129}$$
$$y_{imjk}+y_{im\ell o}-y_{jk\ell o}-x_{im}\leq 0 \tag{130}$$
$$y_{imjk}-y_{im\ell o}+y_{jk\ell o}-x_{jk}\leq 0 \tag{131}$$
$$-y_{imjk}+y_{im\ell o}+y_{jk\ell o}-x_{\ell o}\leq 0 \tag{132}$$
$$-y_{imjk}-y_{im\ell o}-y_{jk\ell o}+x_{im}+x_{jk}+x_{\ell o}\leq 1 \tag{133}$$
$$y_{imj\ell}+y_{imko}-y_{j\ell ko}-x_{im}\leq 0 \tag{134}$$
$$y_{imj\ell}-y_{imko}+y_{j\ell ko}-x_{j\ell}\leq 0 \tag{135}$$
$$-y_{imj\ell}+y_{imko}+y_{j\ell ko}-x_{ko}\leq 0 \tag{136}$$
$$-y_{imj\ell}-y_{imko}-y_{j\ell ko}+x_{im}+x_{j\ell}+x_{ko}\leq 1 \tag{137}$$
$$y_{imjo}+y_{imk\ell}-y_{jok\ell}-x_{im}\leq 0 \tag{138}$$
$$y_{imjo}-y_{imk\ell}+y_{jok\ell}-x_{jo}\leq 0 \tag{139}$$
$$-y_{imjo}+y_{imk\ell}+y_{jok\ell}-x_{k\ell}\leq 0 \tag{140}$$
$$-y_{imjo}-y_{imk\ell}-y_{jok\ell}+x_{im}+x_{jo}+x_{k\ell}\leq 1 \tag{141}$$
$$y_{iojk}+y_{io\ell m}-y_{jk\ell m}-x_{io}\leq 0 \tag{142}$$
$$y_{iojk}-y_{io\ell m}+y_{jk\ell m}-x_{jk}\leq 0 \tag{143}$$
$$-y_{iojk}+y_{io\ell m}+y_{jk\ell m}-x_{\ell m}\leq 0 \tag{144}$$
$$-y_{iojk}-y_{io\ell m}-y_{jk\ell m}+x_{io}+x_{jk}+x_{\ell m}\leq 1 \tag{145}$$
$$y_{ioj\ell}+y_{iokm}-y_{j\ell km}-x_{io}\leq 0 \tag{146}$$
$$y_{ioj\ell}-y_{iokm}+y_{j\ell km}-x_{j\ell}\leq 0 \tag{147}$$
$$-y_{ioj\ell}+y_{iokm}+y_{j\ell km}-x_{km}\leq 0 \tag{148}$$
$$-y_{ioj\ell}-y_{iokm}-y_{j\ell km}+x_{io}+x_{j\ell}+x_{km}\leq 1 \tag{149}$$
$$y_{iojm}+y_{iok\ell}-y_{jmk\ell}-x_{io}\leq 0 \tag{150}$$
$$y_{iojm}-y_{iok\ell}+y_{jmk\ell}-x_{jm}\leq 0 \tag{151}$$
$$-y_{iojm}+y_{iok\ell}+y_{jmk\ell}-x_{k\ell}\leq 0 \tag{152}$$
$$-y_{iojm}-y_{iok\ell}-y_{jmk\ell}+x_{io}+x_{jm}+x_{k\ell}\leq 1 \tag{153}$$

Theorem 9. *The triangle inequalities (94)–(153) define facets of P^n_{QLO} for $n\geq 6$.*

Proof. The proof is analogous to the previous ones, now carried out for $n=6$. Each of these inequalities has 540 of the 720 vertices of P^6_{QLO} binding, while the dimension of their associated faces is 99 and $\dim P^6_{\text{QLO}}=100$. □

B Remaining facet-defining inequalities for P^4_{QLO} from [3]

Up to $n=4$, complete linear descriptions of P^n_{QLO} are known as derived by DeVries [3]. We list the remaining facet-defining inequalities for P^4_{QLO} that have not been part of the discussion in this paper and display them based on the full variable space of QLOP-ILP (and in general notation for $n \geq 4$ to be read for all $i,j,k,\ell \in [n]$, $i<j<k<\ell$) which increases the accessibility of their combinatorial relations or logical implications.

The first class of inequalities (Table 3.3 in [3]) are arranged in blocks of four, and stem from multiplication of a three-di-cycle expression (of the form $x_{ij}+x_{jk}-x_{ik}$) with (the complement of) an I_2-variable that contains one different index. Each of these blocks relates to three distinct edges having one common endpoint in the graph G defined in Sect. 4. Assuming these have the indices (i,j,i,ℓ), (i,k,i,ℓ), and (j,k,i,ℓ), the first $\geq$-inequality states that if i is placed before k and ℓ ($y_{iki\ell}=1$), then j must be placed either after i ($y_{iji\ell}=1$) or before k ($y_{jki\ell}=1$), or both. The first $\leq$-inequality enforces that if $y_{iji\ell}=y_{jki\ell}=1$ (which is only possible if $x_{i\ell}=1$) then $y_{iki\ell}=1$ as well. The other two inequalities of a block consider complemented factors.

$$
\begin{aligned}
0 &\leq y_{iji\ell}-y_{iki\ell}+y_{i\ell jk} \leq x_{i\ell}\\
0 &\leq x_{ij}-x_{ik}+x_{jk}-y_{iji\ell}+y_{iki\ell}-y_{i\ell jk} \leq 1-x_{i\ell}\\
0 &\leq y_{ijj\ell}-y_{ikj\ell}+y_{jkj\ell} \leq x_{j\ell}\\
0 &\leq x_{ij}-x_{ik}+x_{jk}-y_{ijj\ell}+y_{ikj\ell}-y_{jkj\ell} \leq 1-x_{j\ell}\\
0 &\leq y_{ijk\ell}-y_{ikk\ell}+y_{jkk\ell} \leq x_{k\ell}\\
0 &\leq x_{ij}-x_{ik}+x_{jk}-y_{ijk\ell}+y_{ikk\ell}-y_{jkk\ell} \leq 1-x_{k\ell}\\
0 &\leq y_{ijik}-y_{iki\ell}+y_{ikj\ell} \leq x_{ik}\\
0 &\leq x_{ij}-x_{i\ell}+x_{j\ell}-y_{ijik}+y_{iki\ell}-y_{ikj\ell} \leq 1-x_{ik}\\
0 &\leq y_{ijjk}-y_{i\ell jk}+y_{jkj\ell} \leq x_{jk}\\
0 &\leq x_{ij}-x_{i\ell}+x_{j\ell}-y_{ijjk}+y_{i\ell jk}-y_{jkj\ell} \leq 1-x_{jk}\\
0 &\leq y_{ijk\ell}-y_{i\ell k\ell}+y_{j\ell k\ell} \leq x_{k\ell}\\
0 &\leq x_{ij}-y_{ijk\ell}-x_{i\ell}+y_{i\ell k\ell}+x_{j\ell}-y_{j\ell k\ell} \leq 1-x_{k\ell}\\
0 &\leq y_{ijik}-y_{iji\ell}+y_{ijk\ell} \leq x_{ij}\\
0 &\leq x_{ik}-x_{i\ell}+x_{k\ell}-y_{ijik}+y_{iji\ell}-y_{ijk\ell} \leq 1-x_{ij}\\
0 &\leq y_{ikjk}-y_{i\ell jk}+y_{jkk\ell} \leq x_{jk}\\
0 &\leq -x_{i\ell}+x_{k\ell}+x_{ik}-y_{ikjk}+y_{i\ell jk}-y_{jkk\ell} \leq 1-x_{jk}\\
0 &\leq -y_{i\ell j\ell}+y_{ikj\ell}+y_{j\ell k\ell} \leq x_{j\ell}\\
0 &\leq x_{ik}+x_{k\ell}-x_{i\ell}+y_{i\ell j\ell}-y_{ikj\ell}-y_{j\ell k\ell} \leq 1-x_{j\ell}\\
0 &\leq y_{ijjk}-y_{ijj\ell}+y_{ijk\ell} \leq x_{ij}\\
0 &\leq x_{jk}-x_{j\ell}+x_{k\ell}-y_{ijjk}+y_{ijj\ell}-y_{ijk\ell} \leq 1-x_{ij}\\
0 &\leq y_{ikjk}-y_{ikj\ell}+y_{ikk\ell} \leq x_{ik}\\
0 &\leq x_{jk}-x_{j\ell}+x_{k\ell}-y_{ikjk}+y_{ikj\ell}-y_{ikk\ell} \leq 1-x_{ik}\\
0 &\leq -y_{i\ell j\ell}+y_{i\ell k\ell}+y_{i\ell jk} \leq x_{i\ell}\\
0 &\leq x_{jk}-x_{j\ell}+x_{k\ell}+y_{i\ell j\ell}-y_{i\ell k\ell}-y_{i\ell jk} \leq 1-x_{i\ell}
\end{aligned}
$$

The second class of inequalities (Tables 3.5 in [3]) relates to 4-cliques in G. Each member stems from the multiplication of two distinct three-di-cycle expressions as well.

$$-x_{jk} - x_{i\ell} + y_{ikjk} + y_{i\ell j\ell} + y_{iki\ell} - y_{ikj\ell} - y_{i\ell jk} + y_{jkj\ell} \leq 0$$
$$-x_{j\ell} - x_{ik} + y_{ikjk} + y_{i\ell j\ell} + y_{iki\ell} - y_{ikj\ell} - y_{i\ell jk} + y_{jkj\ell} \leq 0$$
$$-x_{i\ell} + y_{iji\ell} - y_{ijjk} - y_{ijk\ell} + y_{i\ell k\ell} + y_{i\ell jk} - y_{jkk\ell} \leq 0$$
$$x_{ij} - x_{i\ell} + x_{jk} + x_{k\ell} + y_{iji\ell} - y_{ijjk} - y_{ijk\ell} + y_{i\ell k\ell} + y_{i\ell jk} - y_{jkk\ell} \leq 1$$
$$-x_{ij} - x_{j\ell} + y_{ijik} - y_{ijj\ell} + y_{ijk\ell} + y_{ikj\ell} - y_{ikk\ell} + y_{j\ell k\ell} \leq 0$$
$$-x_{ik} - x_{k\ell} + y_{ijik} - y_{ijj\ell} + y_{ijk\ell} + y_{ikj\ell} - y_{ikk\ell} + y_{j\ell k\ell} \leq 0$$

The third class of inequalities (Table 3.7 in [3]) considers paths in G of the form $(i,j),(j,k),(k,\ell)$ together with the edge (i,ℓ,j,k) and the "(j,k)-disjoint" edge (i,k,j,ℓ), for all permutations of four distinct indices. The associated logical implication is that if the variables on the path are both zero, i.e., $y_{ijjk} = 0$ and $y_{jkk\ell} = 0$, but $y_{i\ell jk} = 1$, then this uniquely determines the ordering $i - k - j - \ell$ on this quadruple, and so $y_{ikj\ell}$ must be equal to 1 as well. The x-variables stem from complements to occur when a pair (quadruple) as mentioned above does not comply with the definition of I_2 (I).

$$y_{i\ell jk} - y_{ijjk} - y_{jkk\ell} - y_{ikj\ell} \leq 0$$
$$y_{ikj\ell} - y_{ijj\ell} - x_{j\ell} + y_{j\ell k\ell} - y_{i\ell jk} \leq 0$$
$$x_{i\ell} - y_{i\ell jk} - x_{ik} + y_{ikjk} - x_{j\ell} + y_{jkj\ell} - y_{ijk\ell} \leq 0$$
$$y_{ijk\ell} - y_{ikk\ell} - x_{k\ell} + y_{j\ell k\ell} - x_{i\ell} + y_{i\ell jk} \leq 0$$
$$x_{ik} - y_{ikj\ell} - x_{i\ell} + y_{i\ell j\ell} - x_{jk} + y_{jkj\ell} - x_{ij} + y_{ijk\ell} \leq 0$$
$$x_{ij} - y_{ijk\ell} - x_{i\ell} + y_{i\ell k\ell} + x_{jk} + x_{k\ell} - y_{jkk\ell} - x_{ik} + y_{ikj\ell} \leq 1$$
$$y_{ikj\ell} - x_{ik} + y_{ijik} - y_{ikk\ell} - y_{i\ell jk} \leq 0$$
$$y_{i\ell jk} - jx_{i\ell} + y_{iji\ell} + y_{i\ell k\ell} - y_{ikj\ell} \leq 0$$
$$x_{j\ell} - y_{j\ell ik} - x_{jk} + y_{jkik} - x_{i\ell} + y_{iki\ell} - x_{k\ell} + y_{ijk\ell} \leq 0$$
$$-y_{ijk\ell} - y_{jkk\ell} + y_{i\ell k\ell} - x_{j\ell} + y_{j\ell ik} \leq 0$$
$$x_{jk} - y_{i\ell jk} - x_{j\ell} + y_{i\ell j\ell} - x_{ik} + y_{iki\ell} + x_{ij} + x_{k\ell} - y_{ijk\ell} \leq 1$$
$$-x_{ij} - x_{k\ell} + y_{ijk\ell} - x_{j\ell} + y_{j\ell k\ell} + x_{ik} + x_{k\ell} - y_{ikk\ell} - x_{jk} + y_{i\ell jk} \leq 0$$
$$y_{ijk\ell} - x_{ij} + y_{ijik} - y_{ijj\ell} - x_{i\ell} + y_{i\ell jk} \leq 0$$
$$-y_{i\ell jk} + y_{iki\ell} - x_{i\ell} + y_{i\ell j\ell} - y_{ijk\ell} \leq 0$$
$$-x_{k\ell} + y_{ijk\ell} + x_{jk} - y_{ijjk} - x_{i\ell} + y_{iji\ell} - x_{j\ell} + y_{ikj\ell} \leq 0$$
$$-y_{ikj\ell} + y_{jkj\ell} - x_{j\ell} + y_{i\ell j\ell} + x_{ij} + x_{k\ell} - y_{ijk\ell} \leq 1$$
$$-x_{i\ell} - x_{jk} + y_{i\ell jk} - x_{k\ell} + y_{i\ell k\ell} - x_{ij} + y_{iji\ell} + x_{ik} + x_{j\ell} - y_{ikj\ell} \leq 0$$
$$-x_{ik} + y_{ikj\ell} - x_{k\ell} + y_{j\ell k\ell} + x_{ij} - y_{ijj\ell} + x_{i\ell} + x_{jk} - y_{i\ell jk} \leq 1$$
$$-y_{ijk\ell} + y_{iji\ell} - y_{ijjk} - x_{ik} + y_{ikj\ell} \leq 0$$
$$-y_{ikj\ell} + y_{iki\ell} - x_{ik} + y_{ikjk} - x_{ij} + y_{ijk\ell} \leq 0$$
$$-x_{k\ell} + y_{ijk\ell} + x_{j\ell} - y_{ijj\ell} - x_{ik} + y_{ijik} - x_{jk} + y_{i\ell jk} \leq 0$$
$$-y_{i\ell jk} + y_{jkj\ell} - x_{jk} + y_{jkik} + x_{ij} + x_{k\ell} - y_{ijk\ell} \leq 1$$
$$-x_{j\ell} + y_{ikj\ell} + x_{k\ell} - y_{ikk\ell} - x_{ij} + y_{ijik} + x_{i\ell} + x_{jk} - y_{i\ell jk} \leq 1$$
$$-x_{i\ell} + y_{i\ell jk} + x_{k\ell} - y_{jkk\ell} + x_{ij} + x_{jk} - y_{ijjk} + x_{ik} + x_{j\ell} - y_{ikj\ell} \leq 2$$

References

[1] F. Barahona and A. R. Mahjoub. On the cut polytope. *Mathematical Programming*, 36(2):157–173, 1986.

[2] C. Buchheim, A. Wiegele, and L. Zheng. Exact algorithms for the quadratic linear ordering problem. *INFORMS Journal on Computing*, 22(1):168–177, 2010.

[3] A. N. DeVries. *Tight Representations of Specially-Structured 0-1 Linear, Quadratic, and Polynomial Programs*. PhD thesis, Clemson University, 2018.

[4] R. Fortet. L'algèbre de boole et ses applications en recherche opérationnelle. *Cahiers du Centre d'Etudes de Recherche Opérationnelle*, 1:5–36, 1959.

[5] E. Gaar, D. Puges, and A. Wiegele. Strong SDP based bounds on the cutwidth of a graph. *Computers & Operations Research*, 161:1–11, 2024.

[6] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., New York, NY, USA, 1979.

[7] F. Glover and E. Woolsey. Further reduction of zero-one polynomial programming problems to zero-one linear programming problems. *Oper. Res.*, 21(1):156–161, 1973.

[8] P. Hungerländer and F. Rendl. A computational study and survey of methods for the single-row facility layout problem. *Computational Optimization and Applications*, 55(1):1–20, May 2013.

[9] M. Jünger and S. Mallach. Exact facetial odd-cycle separation for maximum cut and binary quadratic optimization. *INFORMS J. on Computing*, 33(4):1419–1430, 2021.

[10] S. Mallach. Binary programs for asymmetric betweenness problems and relations to the quadratic linear ordering problem. *EURO Journal on Computational Optimization*, 11:1–21, 2023.

[11] M. Padberg. The boolean quadric polytope: Some characteristics, facets and relatives. *Mathematical Programming*, 45(1):139–172, Aug 1989.