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Convergence of subgradient extragradient methods with novel stepsizes for equilibrium problems in Hilbert spaces

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Abstract In this paper, by combining the inertial technique and subgradient extragradient method with a new strategy of stepsize selection, we propose a novel extragradient method to solve pseudomonotone equilibrium problems in real Hilbert spaces. Our method is designed such that the stepsize sequence is increasing after a finite number of iterations. This distinguishes our method from most other extragradient-type methods for equilibrium problems. The weak and strong convergence of new algorithms under standard assumptions are established. We examine the performance of our methods on the Nash-Cournot oligopolistic equilibrium models of electricity markets. The reported numerical results demonstrate the efficiency of the proposed method.

Keywords Inertial method · extragradient algorithm · Variational inequality · Equilibrium problem · Pseudomonotone · Lipschitz type inequality

Mathematics Subject Classification (2010) 47H10

1 Introduction

Let C be a non-empty, closed and convex subset of a real Hilbert space $\mathcal{H}$, $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a bi-function such that $C \subset \text{int}(\text{dom} f(x,\cdot))$ for every $x \in \mathcal{H}$. An equilibrium problem (in short, EP) for f on C is stated as follows:

$$\text{Determine an element } x^* \in C \text{ such that } f(x^*,y) \geq 0 \quad \forall y \in C. \tag{1}$$

We denote the solutions set of EP (1) by $\text{Sol}(f,C)$. The normal cone N_C to C at a point $x \in C$ is defined by $N_C(x) = \{w \in \mathcal{H} : \langle w, y - x \rangle \leq 0,\ \forall y \in C\}$. This paper investigates the algorithms and numerical results of the equilibrium problem under the following conditions:
(C1) f is pseudomonotone on C;

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(C2) $f(\cdot,y)$ is weakly upper semicontinuous on C;
(C3) $f(x,\cdot)$ is convex and lower semicontinuous;
(C4) There exist positive numbers c_1 and c_2 such that the Lipschitz-type condition

$$f(x,y)+f(y,z) \geq f(x,z)-c_1\|x-y\|^2-c_2\|y-z\|^2 \tag{2}$$

holds for all $x,y,z \in H$.
(C5) The solution set $\mathrm{Sol}(f,C) \neq \emptyset$.
(C6) Either $f(x,.)$ is continuous at some point of C, or there is an interior point of C where $f(x,.)$ is finite for every $x \in C$.

It easy to see that conditions (C1) and (C4) imply that $f(x,x)=0$ for all $x \in C$. The phrase "equilibrium problem" was used in the research article by Muu and Oettli in 1992 [28], and it was further explored by Blum and Oettli [4]. The equilibrium problem is of interest to researchers because it unifies many nonlinear problems, including fixed-point problems, variational inequalities, Nash equilibrium problems in non-cooperative games, vector and scalar minimization problems, complementarity problems, saddle point problems, etc (see [3, 4, 28] for more details). The equilibrium problem (shortly, EP) is also known as the Ky Fan inequality since Fan [9] gave the first existence result of solutions of (EP). Because of its applications, several authors have established and generalized many results concerning the existence of solutions for equilibrium problems (e.g., see [5,20] and the references therein). An important direction in the equilibrium problem theory is the study of efficient iterative algorithms for finding approximate solutions and their convergence analysis. Several methods have been proposed to solve equilibrium problems in finite dimensional spaces (e.g., see [1,7,17,33]) and infinite dimensional spaces (e.g., see [6,15,21,36]).

Recently, the authors in [8,11–13] introduced various methods for solving strongly pseudomonotone and Lipschitz-type bifunctions equilibrium problems. It is worth mentioning that the step size of the methods often depends on c_1 and c_2. In general equilibrium problems, finding the constants c_1 and c_2 is not an easy task, a fact which might affect the efficiency of the involved methods. To overcome this difficulty, Yang and Liu [46] introduced the following method:

Algorithm 1.1 (Subgradient extragradient type method with self-adaptive stepsizes for EPs)

(Step 0) Take $\lambda_0 > 0, x_0 \in \mathcal{H}, \mu \in (0,1)$.
(Step 1) Given the current iterate x_n, compute

$$y_n = \operatorname{argmin}\left\{\lambda_n f(x_n,y)+\frac{1}{2}\|x_n-y\|^2, y \in C\right\} = \operatorname{prox}_{\lambda_n f(x_n,.)}(x_n).$$

(Step 2) Choose $w_n \in \partial_2 f(x_n,y_n)$ such that $x_n-\lambda_n w_n - y_n \in N_C(y_n)$, compute

$$z_n = \operatorname{argmin}\left\{\lambda_n f(y_n,y)+\frac{1}{2}\|x_n-y\|^2, y \in T_n\right\} = \operatorname{prox}_{\lambda_n f(y_n,.)}(x_n),$$

where $T_n = \{v \in H \mid \langle x_n - \lambda_n w_n - y_n, v - y_n\rangle \leq 0\}$.
(Step 3) Compute $t_n = \alpha_n x_0 + (1-\alpha_n) z_n, x_{n+1} = \beta_n z_n + (1-\beta_n) S t_n$, where $S:\mathcal{H}\to\mathcal{H}$ is a mapping and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\left(\|x_n-y_n\|^2+\|z_n-y_n\|^2\right)}{2(f(x_n,z_n)-f(x_n,y_n)-f(y_n,z_n))}, \lambda_n\right\}, & \text{if } f(x_n,z_n)-f(x_n,y_n)-f(y_n,z_n)>0, \\ \lambda_n, & \text{otherwise.}\end{cases}$$

Set $n := n+1$ and return to Step 1.

Recently, Ngamkhum et al. [30] introduced the following algorithm with non-monotonic step sizes and proved its strong convergence:

Algorithm 1.2 (Modified inertial extragradient algorithm with non-monotonic step sizes)

Initialization. Choose parameters $\lambda_1 > 0, \tau \in [0,1), \mu \in (0,1), \sigma \in \left(0, \frac{1}{2\mu}\right), \eta \in \left[\sigma, \frac{1}{\mu}\right), \{\gamma_k\} \subset [0,1]$ such that $\lim_{k\to\infty} \gamma_k = 1, \{\alpha_k\} \subset (0,1)$ with $0 < \inf \alpha_k \leq \sup \alpha_k < 1$, $\{\xi_k\} \subset [1,\infty)$ with $\sum_{k=0}^{\infty} (\xi_k - 1) < \infty, \{\rho_k\} \subset [0,\infty)$ with $\sum_{k=0}^{\infty} \rho_k < \infty$, and $\{\epsilon_k\} \subset [0,\infty)$, $\{\beta_k\} \subset (0,1)$ such that $\sum_{k=0}^{\infty} \beta_k = \infty, \lim_{k\to\infty} \beta_k = 0$, and $\lim_{k\to\infty} \frac{\epsilon_k}{\beta_k} = 0$. Pick $x_0, x_1 \in \mathcal{H}$ and set $k = 1$.

Step 1. Choose θ_k such that $0 \leq \theta_k \leq \bar{\theta}_k$, where

$$\bar{\theta}_k = \begin{cases} \min\left\{\tau, \frac{\epsilon_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1} \\ \tau, & \text{otherwise} \end{cases}$$

and compute

$$w_k = (1-\beta_k)\left(x_k + \theta_k\left(x_k - x_{k-1}\right)\right).$$

Step 2. Solve the strongly convex programs

$$y_k = \arg\min\left\{\eta\lambda_k f\left(w_k, y\right) + \frac{1}{2}\left\|y - w_k\right\|^2 : y \in C\right\}$$
$$z_k = \arg\min\left\{\sigma\lambda_k f\left(y_k, y\right) + \frac{1}{2}\left\|y - w_k\right\|^2 : y \in C\right\}.$$

Step 3. Compute

$$\lambda_{k+1} = \begin{cases} \min\left\{\frac{\mu\left(\|w_k - y_k\|^2 + \|z_k - y_k\|^2\right)}{2\left(f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k)\right)}, \xi_k\lambda_k + \rho_k\right\}, & \\ & \text{if } f\left(w_k, z_k\right) - f\left(w_k, y_k\right) - f\left(y_k, z_k\right) > 0, \\ \xi_k\lambda_k + \rho_k, & \text{otherwise.} \end{cases}$$

Step 4. Compute

$$v_k = \gamma_k w_k + (1-\gamma_k) T w_k,$$
$$x_{k+1} = \alpha_k v_k + (1-\alpha_k) T z_k,$$

where $T : \mathcal{H} \to \mathcal{H}$ is a mapping.
Step 5. Put $k := k+1$ and go to Step 1.

Other results related to this kind of stepsize, please consult [22, 34, 35, 39, 40, 45] and the references therein. However, the sequence $\{\lambda_n\}$ generated by Algorithm 1.1 is a monotonically decreasing sequence. If the stepsize is small then the computations could be expensive and time demanding. Moreover, the line-search type projection method [7, 44] often decreases to zeros that can make the convergence of the algorithm quite slowly at large iterations. So, these observations lead us to phrase the following research question:

> **Question.** *Can we design algorithms such that the sequence of our stepsizes is increasing from some fixed iteration?*

The goal of this paper is to answer the above question. Motivated and inspired by the works of Yang and Liu [45], Hoai [15], Hoai et al. [16] and mentioned methods, we introduce two subgradient extragradient type algorithms for solving problem (1). The main advantage of our method is the increasing property of the sequence of adaptive stepsizes after a finite number of iterations. This gives a new method which is able to overcome drawbacks of some algorithms mentioned above.

The paper is organized as follows. We first recall some basic definitions and results in Section 2. Our new iterative methods are proposed and analyzed in Section 3 and Section 4. In Section 5 we present an application to variational inequalities. Then, in Section 6 we illustrate the performances of our schemes with related methods for solving the Nash-Cournot oligopolistic equilibrium model.

2 Preliminaries

In this section, we present some preliminary results that we will use in our upcoming results. From now on, we assume that $\mathcal{H}$ is a real Hilbert space endowed with the inner product $\langle .,. \rangle$ and the associated norm $\|.\|$. It is easy to see that

$$\|\alpha\bar{x}+\beta\bar{y}+\gamma\bar{z}\|^2 = \alpha\|\bar{x}\|^2+\beta\|\bar{y}\|^2+\gamma\|\bar{z}\|^2-\alpha\beta\|\bar{x}-\bar{y}\|^2 - \beta\gamma\|\bar{y}-\bar{z}\|^2-\alpha\gamma\|\bar{x}-\bar{z}\|^2,$$

for all $\bar{x},\bar{y},\bar{z} \in \mathcal{H}$ and for all $\alpha,\beta,\gamma \in \mathbb{R}$ with $\alpha+\beta+\gamma = 1$.

When $\{x^k\}$ is a sequence in $\mathcal{H}$, we denote strong convergence of $\{x^k\}$ to $x \in \mathcal{H}$ by $x^k \to x$ and weak convergence by $x^k \rightharpoonup x$. For a given sequence $\{x^k\} \subset \mathcal{H}$, $\omega_w(x^k)$ denotes the weak ω-limit set of $\{x^k\}$, i.e.,

$$\omega_w(x^k) := \{x \in \mathcal{H} : x^{k_l} \rightharpoonup x \ \text{ for some subsequence } \{k_l\} \text{ of } \{k\}\}.$$

Let C be a nonempty closed convex subset of $\mathcal{H}$. For every element $x \in \mathcal{H}$, there exists a unique nearest point in C, denoted by $P_C x$, that is

$$||x-P_C x|| = \min\{||x-y|| : \ y \in C\}.$$

The operator P_C is called the *metric projection* of $\mathcal{H}$ onto C and some of its properties are summarized in the next lemma.

Lemma 2.1 *Let $C \subseteq \mathcal{H}$ be a closed convex set, P_C fulfils the following:*

(1) $\langle x-P_C x, y-P_C x\rangle \le 0$ *for all* $x \in \mathcal{H}$ *and* $y \in C$*;*

(2) $\|P_C x-y\|^2 \le \|x-y\|^2-\|x-P_C x\|^2$ *for all* $x \in \mathcal{H}$, $y \in C$*;*

Let $g : \mathcal{H} \to (-\infty,\infty]$ be a proper, convex, and lower semicontinuous function and $\gamma > 0$.

$$\begin{aligned} \operatorname{prox}_g(x) : \mathcal{H} &\to \mathcal{H} \\ x &\mapsto \underset{y\in C}{\operatorname{argmin}}\left\{g(y)+\frac{1}{2\gamma}\|y-x\|^2\right\} \end{aligned}$$

is well-defined and is said to be the proximity operator of g.

We also recall that the subdifferential of $g : \mathcal{H} \to (-\infty, \infty]$ at $x \in \mathcal{H}$ is defined as the set of all subgradient of g at x:

$$\partial g(x) := \{w \in \mathcal{H} : g(y) - g(x) \geq \langle w, y - x\rangle \quad \forall y \in \mathcal{H}\}.$$

The function g is called subdifferentiable at x if $\partial g(x) \neq \emptyset$, g is said to be subdifferentiable on a subset $C \subset \mathcal{H}$ if it is subdifferentiable at each point $x \in C$, and it is said to be subdifferentiable, if it is subdifferentiable at each point $x \in \mathcal{H}$, i.e., if $D(\partial g) = \mathcal{H}$.

We now recall some classical concepts of monotonicity for nonlinear operators.

Definition 2.1 An operator $A : \mathcal{H} \to \mathcal{H}$ with $\mathrm{dom}A \supseteq C$ is said to be

(1) (see Minty [26]) monotone on C if

$$\langle Ax - Ay, x - y\rangle \geq 0 \quad \forall x, y \in C.$$

(2) (see Karamardian [18]) pseudomonotone on C if

$$\langle Ax, y - x\rangle \geq 0 \Longrightarrow \langle Ay, x - y\rangle \leq 0 \quad \forall x, y \in C.$$

Analogous to Definition 2.1, we have the following concepts for bifunctions.

Definition 2.2 Let C be a nonempty closed convex subset of $\mathcal{H}$ and $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ such that $C \times C$ is contained in the domain of f. Then f is said to be

(1) (see Blum and Oettli [4]) monotone on C if

$$f(x, y) + f(y, x) \leq 0 \quad \forall x, y \in C.$$

(2) (see Bianchi and Schaible [2]) pseudomonotone on C if

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq 0 \quad \forall x, y \in C.$$

Remark 2.1 It is obvious that if $A : \mathcal{H} \to \mathcal{H}$ is monotone (pseudomonotone) on C in the sense of Definition 2.1 then the corresponding bifunction defined by $f(x, y) = \langle Ax, y - x\rangle$ is monotone (pseudomonotone) on C in the sense of Definition 2.2.

In the proof of the strong convergence theorem, we will use the subdifferential inequality:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \tag{3}$$

for all $x, y \in \mathcal{H}$.

The following lemmas will be useful for proving the convergence results of this paper.

Lemma 2.2 (Proposition 3.61 [31]) *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let $g : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be proper, lower-semicontinuous, and convex. Assume either that g is continuous at some point of C, or that there is an interior point of C where g is finite. Then, x^* is a solution of the convex optimization problem*

$$\min\{g(x) : x \in C\}$$

if and only if

$$0 \in \partial g(x^*) + N_C(x^*).$$

Lemma 2.3 (Opial [29]) *Let $\mathcal{H}$ be a real Hilbert space and $\{x^k\}$ a sequence in $\mathcal{H}$ such that there exists a nonempty closed set $S \subset \mathcal{H}$ satisfying*

(1) *For every $z \in S$, $\lim\limits_{k\to\infty} \|x^k - z\|$ exists;*

(2) *Any weak cluster point of $\{x^k\}$ belongs to S.*

Then, there exists $\bar{x} \in S$ such that $\{x^k\}$ converges weakly to $\bar{x}$.

Lemma 2.4 (See [37]) *Assume that $\{a_k\}$ and $\{b_k\}$ are two sequences of nonnegative numbers such that $a_{k+1} \le a_k + b_k$ $\forall k \in \mathbb{N}$. If $\sum_{k=1}^{\infty} b_k < \infty$ then $\lim_{k\to\infty} a_k$ exists.*

Lemma 2.5 ([23, 42])*Let $\{a_k\}_{k=0}^{\infty}$ and $\{c_k\}_{k=0}^{\infty}$ are sequences of nonnegative real numbers such that*

$$a_{k+1} \le (1-\delta_k)a_k + b_k + c_k, \quad k \ge 0,$$

where $\{\delta_k\}_{k=0}^{\infty}$ is a sequence in $(0,1)$ and $\{b_k\}_{k=0}^{\infty}$ is a sequence in $\mathbb{R}$. Assume $\sum_{k=0}^{\infty} c_k < \infty$. Then the following results hold:

(1) *If $b_k \le \delta_k M$ for some $M \ge 0$, then $\{a_k\}_{k=0}^{\infty}$ is a bounded sequence.*

(2) *If $\sum_{k=0}^{\infty} \delta_k = \infty$ and $\limsup_{k\to\infty} b_k/\delta_k \le 0$, then $\lim_{k\to\infty} a_k = 0$.*

Lemma 2.6 ([24, Lemma 2.2]) *Let the sequences $\{\phi_k\}_{k=0}^{\infty} \subset [0,+\infty)$ and $\{\delta_k\}_{k=0}^{\infty} \subset [0,+\infty)$ which satisfy:*

(1) $\phi_{k+1} - \phi_k \le \theta_k(\phi_k - \phi_{k-1}) + \delta_k$,

(2) $\sum\limits_{k=1}^{\infty} \delta_k < \infty$,

(3) $\{\theta_k\}_{k=0}^{\infty} \subset [0,\theta]$, *where $\theta \in [0,1)$.*

Then $\{\phi_k\}_{k=0}^{\infty}$ is a converging sequence and $\sum\limits_{k=1}^{\infty} [\phi_{k+1} - \phi_k]_+ < \infty$, where $[t]_+ := \max\{t, 0\}$ (for any $t \in \mathbb{R}$).

Lemma 2.7 ([24]) *Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{a_{k_j}\}_{j=0}^{\infty}$ of $\{a_k\}_{k=0}^{\infty}$ such that $a_{k_j} < a_{k_j+1}$ for all $j \ge 0$. Also consider the sequence of integers $\{\tau(k)\}_{k\ge k_0}$ defined by*

$$\tau(k) = \max\{i \le k : a_i < a_{i+1}\}.$$

Then $\{\tau(k)\}_{k\ge k_0}$ is a nondecreasing sequence verifying $\lim\limits_{k\to\infty} \tau(k) = \infty$ and, for all $k \ge k_0$,

$$\max\{a_{\tau(k)}, a_k\} \le a_{\tau(k)+1}.$$

3 Algorithm and its convergence

Inspired by the algorithms in [14, 15, 43, 45], we propose the following algorithm for pseudomonotone equilibrium problems:

Algorithm 3.1 (Subgradient extragradient method with novel stepsizes for EPs)

Step 0 (Initialization). Select $\lambda_0 > 0$, $\mu_1 < \mu_0 < \sigma < 1/2$, $\theta \in [0,1)$. Take positive sequences $\{\epsilon_k\}, \{\xi_k\} \subset [0,\infty)$ satisfying

$$\sum_{k=0}^{\infty} \epsilon_k < \infty, \ \sum_{k=0}^{\infty} \xi_k < \infty.$$

Choose initial iterates $x^0, x^1 \in C$ and set $k = 1$.

Step 1. Given the current iterates x^{k-1} and x^k $(k \geq 1)$, choose α_k such that $0 \leq \alpha_k \leq \bar{\alpha}_k$, where

$$\bar{\alpha}_k = \begin{cases} \min\left\{\theta, \dfrac{\xi_k}{\|x^k - x^{k-1}\|}\right\} & \text{if } x^k \neq x^{k-1}, \\ \theta & \text{otherwise.} \end{cases}$$

Compute

$$\begin{cases} w^k = x^k + \alpha_k (x^k - x^{k-1}), \\ y^k = \underset{y \in C}{\operatorname{argmin}} \left\{ \lambda_k f(w^k, y) + \dfrac{1}{2}\|y - w^k\|^2 \right\}. \end{cases}$$

If $y^k = w^k$ then terminate: w^k is a solution. Otherwise, go to **Step 2**.

Step 2. Take $\xi^k \in \partial_2 f(w^k, y^k)$ such that $w^k - \lambda_k \xi^k - y^k \in N_C(y^k)$. Construct the half-space

$$T_k = \{x \in \mathcal{H} : \langle w^k - \lambda_k \xi^k - y^k, x - y^k \rangle \leq 0\}.$$

Calculate

$$\begin{aligned} x^{k+1} &= \underset{y \in T_k}{\operatorname{argmin}} \left\{ \lambda_k f(y^k, y) + \frac{1}{2}\|y - w^k\|^2 \right\}, \\ \eta_k &= f(w^k, x^{k+1}) - f(w^k, y^k) - f(y^k, x^{k+1}), \\ \delta_k &= \|w^k - y^k\|^2 + \|y^k - x^{k+1}\|^2 \end{aligned}$$

and update λ_k by

$$\lambda_{k+1} = \begin{cases} \mu_1 \dfrac{\delta_k}{\eta_k}, & \text{if } \eta_k > \frac{\mu_0}{\lambda_k}\delta_k, \\ (1+\epsilon_k)\lambda_k, & \text{otherwise.} \end{cases} \tag{4}$$

Set $k := k + 1$, and return to **Step 1**.

The following lemma shows that if the algorithm terminates at iteration k, then w^k is a solution.

Lemma 3.1 *If $y^k = w^k$ then $y^k \in \mathrm{Sol}(f, C)$.*

Proof We have

$$y^k = \underset{y \in C}{\operatorname{argmin}} \left\{ \lambda_k f(w^k, y) + \frac{1}{2}\|y - w^k\|^2 \right\}.$$

Therefore, from Lemma 2.2 we have

$$0 \in \partial_2\left[\lambda_k f(w^k,.) + \frac{1}{2}\|. - w^k\|^2\right](y^k) + N_C(y^k),$$

i.e.,

$$0 \in \partial_2(\lambda_k f(w^k,.))(y^k) + y^k - w^k + N_C(y^k).$$

Hence, there exists $\xi^k \in \partial_2(f(w^k,.))(y^k)$ such that

$$w^k - \lambda_k \xi^k - y^k \in N_C(y^k),$$

which implies that

$$\langle w^k - \lambda_k \xi^k - y^k, y - y^k\rangle \le 0 \quad \forall y \in C.$$

If $y^k = w^k$ then

$$\langle \xi^k, y - y^k\rangle \ge 0 \quad \forall y \in C.$$

By the assumption (C3), we get

$$f(y^k,y) = f(y^k,y) - f(y^k,y^k) \ge \langle \xi^k, y - y^k\rangle \ge 0 \quad \forall y \in C,$$

which means that $y^k \in \mathrm{Sol}(f,C)$. □

Remark 3.1 From the proof of Lemma 3.1 we have $C \subset T_k$.

Lemma 3.2 *Let $\{\lambda_k\}$ be the stepsize sequence generated by Algorithm 3.1. Then*

(i) *for all $k \ge 1$ we have $\lambda_k \ge \lambda_{min} := \min\left\{\frac{\mu_1}{\max\{c_1,c_2\}}, \lambda_0\right\} > 0$;*

(ii) *$\{\lambda_k\}$ is convergent;*

(iii) *there exists a positive integer k_0 such that $\lambda_{k+1} \ge \lambda_k$ for all $k \ge k_0$.*

Proof (i) Indeed, since f satisfies the Lipschitz type inequality (2), we obtain

$$\begin{aligned}\mu_1 \frac{\|w^k - y^k\|^2 + \|y^k - x^{k+1}\|^2}{\eta_k} &\ge \mu_1 \frac{\|w^k - y^k\|^2 + \|y^k - x^{k+1}\|^2}{c_1\|w^k - y^k\|^2 + c_2\|y^k - x^{k+1}\|^2} \\ &\ge \frac{\mu_1}{\max\{c_1,c_2\}}\end{aligned}$$

for all n.

Hence

$$\lambda_1 \ge \left\{\frac{\mu_1}{\max\{c_1,c_2\}}, \lambda_0\right\}.$$

By induction, we obtain that the sequence $\{\lambda_k\}$ is bounded below by $\min\left\{\frac{\mu_1}{\max\{c_1,c_2\}}, \lambda_0\right\}$.
(ii) The proof is similar to that of [14,15]. We provide it here for completeness. Let

$$u_k = \ln \lambda_{k+1} - \ln \lambda_k \quad \forall k \ge 0,$$

we have $u_k = u_k^+ - u_k^-$, where

$$u_k^+ = \max\{0, u_k\} \ge 0 \text{ and } u_k^- = -\min\{0, u_k\} \ge 0 \text{ for all } k \ge 0.$$

From the definition of λ_k in Algorithm 3.1, we derive that

$$u_k = \ln \frac{\lambda_{k+1}}{\lambda_k} \le \ln(1+\epsilon_k) \le \epsilon_k \quad \forall k \ge 0,$$

which implies $u_k^+ \le \epsilon_k$. Since $\sum\limits_{k=0}^{+\infty} \epsilon_k$ is convergent, we obtain $\sum\limits_{k=0}^{+\infty} u_k^+ < +\infty$. Observing that $\sum\limits_{k=0}^{+\infty} u_k^-$ is a nonnegative series and using the following relation

$$\ln \lambda_{k+1} - \ln \lambda_0 = \sum_{i=0}^{k} u_i = \sum_{i=0}^{k} (u_i^+ - u_i^-) = \sum_{i=0}^{k} u_i^+ - \sum_{i=0}^{k} u_i^-, \tag{5}$$

we see that if $\lim\limits_{k\to+\infty} \sum\limits_{i=0}^{k} u_i^- = +\infty$ then

$$\lim_{k\to+\infty} (\ln \lambda_{k+1}) = -\infty,$$

i.e.,

$$\lim_{k\to+\infty} \lambda_k = 0.$$

This contradicts (i) and hence $\sum\limits_{k=0}^{+\infty} u_k^-$ must be convergent. Finally, from (5) we get the desired conclusion that $\lim\limits_{k\to+\infty} \lambda_k = \lambda^*$, where $\lambda_{min} \le \lambda^* < +\infty$.

(iii) We show that there exists k_0 such that

$$\eta_k \le \frac{\mu_0}{\lambda_k} \delta_k \quad \forall k \ge k_0.$$

Then

$$\lambda_{k+1} = (1+\epsilon_k)\lambda_k \ge \lambda_k.$$

Suppose by contradiction that there exists $\{k_l\}, k_l \to +\infty$ such that

$$\eta_{k_l} > \frac{\mu_0}{\lambda_{k_l}} \delta_{k_l}.$$

For this case

$$\lambda_{k_l+1} = \mu_1 \frac{\delta_{k_l}}{\eta_{k_l}}$$

Therefore

$$\frac{\mu_1 \delta_{k_l}}{\lambda_{k_l+1}} = \eta_{k_l} > \frac{\mu_0}{\lambda_{k_l}} \delta_{k_l},$$

i.e.,

$$\frac{\lambda_{k_l+1}}{\lambda_{k_l}} < \frac{\mu_1}{\mu_0} \quad \forall k_l.$$

From (ii), we have

$$\lim_{l\to+\infty} \lambda_{k_l} = \lim_{l\to+\infty} \lambda_{k_l+1} = \lim_{k\to+\infty} \lambda_k = \lambda^*.$$

Hence we deduce that

$$\frac{\lambda^*}{\lambda^*} \le \frac{\mu_1}{\mu_0} < 1.$$

It is a contradiction and we finish the proof.

Remark 3.2 The stepsize sequence generated by Algorithm 3.1 is increasing after a finite number of iterations. Therefore, our method is different from other extragradient-type methods for equilibrium problems.

The next statement plays a crucial role in proving the convergence result.

Lemma 3.3 *Let $\{x^k\}$ and $\{y^k\}$ be the sequences generated by Algorithm 3.1 and $z \in C$. Then the following inequality holds.*

$$\|x^{k+1}-z\|^2 \le \|w^k - z\|^2 - \left(1-2\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\|w^k-y^k\|^2 - \left(1-2\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\|x^{k+1}-y^k\|^2 + 2\lambda_k f(y^k,z),$$

for all $k \ge n_1$.

Proof It follows from $x^{k+1} = \underset{y\in T_k}{\operatorname{argmin}}\left\{\lambda_k f(y^k,y) + \frac{1}{2}\|y-w^k\|^2\right\}$ and Lemma 2.2 that

$$0 = \lambda_k g^k + x^{k+1} - w^k + q^k,$$

where $g^k \in \partial f(y^k,.)(x^{k+1})$ and $q^k \in N_{T_k}(x^{k+1})$.

From the definition

$$N_{T_k}(x^{k+1}) = \{q \in \mathcal{H} : \langle q, y - x^{k+1}\rangle \le 0,\ \forall y \in T_k\}$$

and the fact that $C \subset T_k$, we have

$$\langle w^k - x^{k+1} - \lambda_k g^k, z - x^{k+1}\rangle \le 0.$$

Consequently,

$$\langle w^k - x^{k+1}, z - x^{k+1}\rangle \le \lambda_k\langle g^k, z-x^{k+1}\rangle \le \lambda_k(f(y^k,z) - f(y^k,x^{k+1})).$$

We have

$$\begin{aligned}
\|x^{k+1}-z\|^2 &= \|w^k-z\|^2 + \|x^{k+1}-w^k\|^2 + 2\langle x^{k+1}-w^k, w^k-z\rangle \\
&= \|w^k-z\|^2 - \|x^{k+1}-w^k\|^2 + 2\langle x^{k+1}-w^k, x^{k+1}-z\rangle \\
&\le \|w^k-z\|^2 - \|x^{k+1}-w^k\|^2 + 2\lambda_k(f(y^k,z) - f(y^k,x^{k+1})) \\
&= \|w^k-z\|^2 - \|x^{k+1}-w^k\|^2 + 2\lambda_k[f(w^k,y^k) - f(w^k,x^{k+1})] + \\
&\quad + 2\lambda_k[f(w^k,x^{k+1}) - f(w^k,y^k) - f(y^k,x^{k+1})] + 2\lambda_k f(y^k,z) \\
&= \|w^k-z\|^2 - \|x^{k+1}-w^k\|^2 + A + B + 2\lambda_k f(y^k,z), \qquad (6)
\end{aligned}$$

where

$$\begin{aligned}
A &= 2\lambda_k[f(w^k,y^k) - f(w^k,x^{k+1})], \\
B &= 2\lambda_k[f(w^k,x^{k+1}) - f(w^k,y^k) - f(y^k,x^{k+1})].
\end{aligned}$$

From the definition of T_k and $x^{k+1} \in T_k$, we obtain

$$\langle w^k - \lambda_k \xi^k - y^k, x^{k+1} - y^k\rangle \le 0,$$

where $\xi^k \in \partial f(w^k, .)(y^k)$. Using the definition of the subdifferential we arrive at

$$f(w^k, y) - f(w^k, y^k) \geq \langle \xi^k, y - y^k \rangle, \quad \forall y \in \mathcal{H}.$$

Therefore,

$$\begin{aligned} 2\lambda_k[f(w^k, x^{k+1}) - f(w^k, y^k)] &\geq 2\lambda_k \langle \xi^k, x^{k+1} - y^k \rangle \\ &\geq 2\langle w^k - y^k, x^{k+1} - y^k \rangle. \end{aligned} \tag{7}$$

It follows that

$$\begin{aligned} A &\leq 2\langle y^k - w^k, x^{k+1} - y^k \rangle \\ &= \|x^{k+1} - w^k\|^2 - \|w^k - y^k\|^2 - \|x^{k+1} - y^k\|^2. \end{aligned} \tag{8}$$

Now, since the positive series $\sum_{k=0}^{+\infty} \epsilon_k$ converges and $0 < \mu_0 < \sigma$ then one can choose $n_1 \in \mathbb{N}$ such that

$$\epsilon_k < \frac{\sigma}{\mu_0} - 1 \ \forall k \geq n_1.$$

The way of choosing n_1 like that helps us to show the correctness of the following inequality

$$B \leq 2\sigma \frac{\lambda_k}{\lambda_{k+1}} [\|w^k - y^k\|^2 + \|y^k - x^{k+1}\|^2], \ \forall k \geq n_1. \tag{9}$$

Indeed, if $\eta_k > \frac{\mu_0}{\lambda_k}\delta_k$ then by (4) we get

$$\begin{aligned} B &= 2\lambda_k[f(w^k, x^{k+1}) - f(w^k, y^k) - f(y^k, x^{k+1})] \\ &= 2\mu_1 \frac{\lambda_k}{\lambda_{k+1}} [\|w^k - y^k\|^2 + \|y^k - x^{k+1}\|^2] \\ &< 2\sigma \frac{\lambda_k}{\lambda_{k+1}} [\|w^k - y^k\|^2 + \|y^k - x^{k+1}\|^2] \end{aligned} \tag{10}$$

Otherwise, we have $\eta_k \leq \frac{\mu_0}{\lambda_k}\delta_k$ and

$$\begin{aligned} B &= 2\lambda_k[f(w^k, x^{k+1}) - f(w^k, y^k) - f(y^k, x^{k+1})] \\ &\leq 2\mu_0[\|w^k - y^k\|^2 + \|y^k - x^{k+1}\|^2] \\ &< 2\frac{\sigma}{1+\epsilon_k} [\|w^k - y^k\|^2 + \|y^k - x^{k+1}\|^2] \\ &= 2\sigma \frac{\lambda_k}{\lambda_{k+1}} [\|w^k - y^k\|^2 + \|y^k - x^{k+1}\|^2]. \end{aligned} \tag{11}$$

From (10) and (11) we obtain (9). It follows from (6), (8) and (9) that

$$\begin{aligned} \|x^{k+1} - z\|^2 \leq \|w^k - z\|^2 - \left(1 - 2\sigma \frac{\lambda_k}{\lambda_{k+1}}\right)\|w^k - y^k\|^2 - \left(1 - 2\sigma \frac{\lambda_k}{\lambda_{k+1}}\right)\|x^{k+1} - y^k\|^2 + \\ + 2\lambda_k f(y^k, z), \end{aligned} \tag{12}$$

for all $k \geq n_1$.

If $z \in \mathrm{Sol}(f, C)$ then it follows from the pseudomonotonicity of f that $f(y^k, z) \leq 0$. Then the inequality (12) implies

$$\|x^{k+1} - z\|^2 \leq \|w^k - z\|^2 - \left(1 - 2\sigma \frac{\lambda_k}{\lambda_{k+1}}\right)\|w^k - y^k\|^2 - \left(1 - 2\sigma \frac{\lambda_k}{\lambda_{k+1}}\right)\|x^{k+1} - y^k\|^2, \tag{13}$$

for all $k \geq n_1$.

The proof is complete. □

Using the above lemmas we can state and prove the following convergence result for the algorithm.

Theorem 3.1 *Let C be a nonempty closed convex subset in a real Hilbert space, $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a bifunction satisfying (C1)-(C6). Then the sequence $\{x^k\}$ generated by Algorithm 3.1 converges weakly to a solution of equilibrium problem $EP(f,C)$.*

Proof Let $z \in \mathrm{Sol}(f,C)$. We first show that $\lim_{k\to\infty} \|x^k - z\|$ exists. Indeed, we first consider the limit

$$\lim_{k\to\infty} \left(1 - 2\sigma \frac{\lambda_k}{\lambda_{k+1}}\right) = 1 - 2\sigma > 0, \tag{14}$$

therefore, there exists $n_2 > n_1$ such that $\left(1 - 2\sigma \frac{\lambda_k}{\lambda_{k+1}}\right) > 0$ for all $k \geq n_2$. It follows from (13) that

$$\|x^{k+1} - z\| \leq \|w^k - z\|$$

for all $k \geq n_2$.

Hence, we have

$$\|x^{k+1} - z\| \leq \|x^k - z\| + \alpha_k \|x^k - x^{k-1}\|.$$

for all $n \geq n_2$.

Applying Lemma 2.4 with the data $a_k := \|x^k - z\|,\ b_k := \alpha_k \|x^k - x^{k-1}\|$ we deduce that $\lim_{k\to\infty} \|x^k - z\|$ exists. Hence $\{x_k\}$ is bounded.

On the other hand, by a simple calculation, we get

$$\begin{aligned} \|w^k - z\|^2 &= \|(1+\alpha_k)(x^k - z) - \alpha_k(x^{k-1} - z)\|^2 \\ &= (1+\alpha_k)\|x^k - z\|^2 - \alpha_k \|x^{k-1} - z\|^2 + \alpha_k(1+\alpha_k)\|x^k - x^{k-1}\|^2 \\ &\leq (1+\alpha_k)\|x^k - z\|^2 - \alpha_k \|x^{k-1} - z\|^2 + 2\alpha_k \|x^k - x^{k-1}\|^2. \end{aligned} \tag{15}$$

Moreover, from (13), we obtain

$$\begin{aligned} &0 \leq \left(1 - 2\sigma \frac{\lambda_k}{\lambda_{k+1}}\right) \|w^k - y^k\|^2 + \left(1 - 2\sigma \frac{\lambda_k}{\lambda_{k+1}}\right) \|x^{k+1} - y^k\|^2 \\ &\leq \|w^k - z\|^2 - \|x^{k+1} - z\|^2 \\ &\leq \|x^k - z\|^2 - \|x^{k+1} - z\|^2 + \alpha_k [\|x^k - z\|^2 - \|x^{k-1} - z\|^2] + 2\alpha_k \|x^k - x^{k-1}\|^2. \end{aligned} \tag{16}$$

Passing to the limit in (16) and taking into account that $\{\|x^k - z\|^2\}$ is convergent, we obtain

$$\lim_{k\to+\infty} \|w^k - y^k\| = \lim_{k\to+\infty} \|x^{k+1} - y^k\| = 0. \tag{17}$$

hence, we infer that

$$\lim_{k\to\infty} \|w^k - x^{k+1}\| = 0. \tag{18}$$

By Lemma 2.3, it remains to show that any weak cluster point of the sequence $\{x^k\}$ belongs to the solution set $\mathrm{Sol}(f,C)$. Let $\bar{x}$ be an arbitrary weakly cluster point of $\{x^k\}$. Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_l}\}$ of $\{x^k\}$ such that $x^{k_l} \rightharpoonup \bar{x}$. It follows from (18) that $y^{k_l} \rightharpoonup \bar{x} \in C$.

It follows from

$$x^{k+1} = \underset{y \in T_k}{\text{argmin}} \left\{ \lambda_k f(y^k, y) + \frac{1}{2}\|y - w^k\|^2 \right\}$$

and Lemma 2.2 that there exist $u^k \in \partial f(y^k, .)(x^{k+1})$ and $q^k \in N_{T_k}(x^{k+1})$ such that

$$0 = \lambda_k u^k + x^{k+1} - w^k + q^k.$$

From the definition of $N_{T_k}(x^{k+1})$, we deduce that

$$\langle w^k - x^{k+1} - \lambda_k u^k, y - x^{k+1} \rangle \leq 0 \quad \forall y \in T_k.$$

Since $C \subset T_k$, we have

$$\langle w^k - x^{k+1} - \lambda_k u^k, y - x^{k+1} \rangle \leq 0 \quad \forall y \in C,$$

$$\langle w^k - x^{k+1}, y - x^{k+1} \rangle \leq \langle \lambda_k u^k, y - x^{k+1} \rangle \quad \forall y \in C.$$

On the other hand, since $u^k \in \partial f(y^k, .)(x^{k+1})$, we get

$$\langle u^k, y - x^{k+1} \rangle \leq f(y^k, y) - f(y^k, x^{k+1}) \ \forall y \in C.$$

Hence, we arrive at

$$\frac{\langle w^k - x^{k+1}, y - x^{k+1} \rangle}{\lambda_k} \leq f(y^k, y) - f(y^k, x^{k+1}) \quad \forall y \in C. \tag{19}$$

From (9) and (7) we have

$$\begin{aligned} 2\lambda_k f(y^k, x^{k+1}) &\geq 2\lambda_k [f(w^k, x^{k+1}) - f(w^k, y^k)] - 2\sigma \frac{\lambda_k}{\lambda_{k+1}} [\|w^k - y^k\|^2 + \|y^k - x^{k+1}\|^2] \\ &\geq 2\langle w^k - y^k, x^{k+1} - y^k \rangle - 2\sigma \frac{\lambda_k}{\lambda_{k+1}} [\|w^k - y^k\|^2 + \|y^k - x^{k+1}\|^2]. \end{aligned} \tag{20}$$

Combining (19) and (20) we get

$$\begin{aligned} \langle w^k - x^{k+1}, y - x^{k+1} \rangle &\leq \lambda_k f(y^k, y) - \langle w^k - y^k, x^{k+1} - y^k \rangle \\ &\quad + 2\sigma \frac{\lambda_k}{\lambda_{k+1}} [\|w^k - y^k\|^2 + \|y^k - x^{k+1}\|^2] \quad \forall y \in C. \end{aligned} \tag{21}$$

Replacing k in (21) by k_l we get

$$\begin{aligned} \langle w^{k_l} - x^{k_l+1}, y - x^{k_l+1} \rangle &\leq \lambda_{k_l} f(y^{k_l}, y) - \langle w^{k_l} - y^{k_l}, x^{k_l+1} - y^{k_l} \rangle \\ &\quad + \sigma \frac{\lambda_{k_l}}{\lambda_{k_l+1}} [\|w^{k_l} - y^{k_l}\|^2 + \|y^{k_l} - x^{k_l+1}\|^2] \quad \forall y \in C. \end{aligned} \tag{22}$$

Passing to the limit when l tending to ∞ in (22) and using weak lower semicontinuity of the function $f(., y)$ together with (17) we obtain

$$f(\bar{x}, y) \geq 0 \quad \forall y \in C.$$

It means that $\bar{x} \in \text{Sol}(f, C)$. Since $\bar{x}$ is an arbitrary weak cluster point we can conclude that the set of all weak cluster points belongs to the solution set $\text{Sol}(f, C)$. Hence, it follows from Lemma 2.3 that the sequence $\{x^k\}$ converges weakly to a solution of EP (1). The proof thus is complete. □

Now, motivated by Alogorithm 2.1 of [40], in Algorithm 3.1, we can choose the parameter θ in another way as follows:

Algorithm 3.2 (Novel inertial subgradient extragradient method for equilibrium problems)

Step 0 (Initialization). Select $\lambda_0 > 0$, $\mu_1 < \mu_0 < \sigma < 1/2$, $\theta \in \left[0, \frac{1}{\sqrt{\tau}}\right)$, $\tau \geq 1$, $\alpha \in (0,1)$. Take a positive sequence $\{\epsilon_k\}$ satisfying $\sum_{k=0}^{\infty} \epsilon_k < \infty$. Choose initial iterates $x^0, x^1 \in C$ and set $k = 1$.

Step 1. Given the current iterates x^{k-1} and x^k $(k \geq 1)$, compute

$$\begin{cases} w^k = x^k + \theta(x^k - x^{k-1}), \\ y^k = \operatorname*{argmin}_{y \in C} \left\{ \lambda_k f(w^k, y) + \dfrac{1}{2}\|y - w^k\|^2 \right\}. \end{cases}$$

If $y^k = w^k$ then terminate: w^k is a solution. Otherwise, go to Step 2.

Step 2. Take $\xi^k \in \partial_2 f(w^k, y^k)$ such that $w^k - \lambda_k \xi^k - y^k \in N_C(y^k)$. Construct the half-space

$$T_k = \{x \in \mathcal{H} : \langle w^k - \lambda_k \xi^k - y^k, x - y^k \rangle \leq 0\}.$$

Calculate

$$\begin{aligned} z^k &= \operatorname*{argmin}_{y \in T_k} \left\{ \lambda_k f(y^k, y) + \frac{1}{2}\|y - w^k\|^2 \right\}, \\ x^{k+1} &= (1-\alpha) w^k + \alpha z^k, && (23) \\ \eta_k &= f(w^k, x^{k+1}) - f(w^k, y^k) - f(y^k, x^{k+1}), && (24) \\ \delta_k &= \|w^k - y^k\|^2 + \|y^k - x^{k+1}\|^2 \end{aligned}$$

and update λ_k by

$$\lambda_{k+1} = \begin{cases} \mu_1 \dfrac{\delta_k}{\eta_k}, & \text{if } \eta_k > \frac{\mu_0}{\lambda_k}\delta_k, \\ (1+\epsilon_k)\lambda_k, & \text{otherwise.} \end{cases}$$

Set $k := k+1$, and return to **Step 1**.

Theorem 3.2 *Let C be a nonempty closed convex subset in a real Hilbert space, $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a bifunction satisfying (C1)-(C6). If*

$$\alpha \in \left(0, \frac{\sqrt{\tau}\theta^2 - (\tau+1)\theta + \sqrt{\tau}}{\sqrt{\tau}\theta^2 - (\tau+1)\theta + \sqrt{\tau} + \theta(1+\theta)}\right)$$

then the sequence $\{x^k\}$ generated by Algorithm 3.2 converges weakly to a solution of equilibrium problem $EP(f,C)$.

Proof Arguing as in Lemma 3.3 we obtain

$$\|z^k - z\|^2 \leq \|w^k - z\|^2 - \left(1 - 2\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\|w^k - y^k\|^2 - \left(1 - 2\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\|z^k - y^k\|^2, \quad (25)$$

for all $k \geq k_1$.

We follow the line of the proof in [40, Theorem 3.1]. By (14), there exists $k_2 > k_1$ such that

$$1 - 2\sigma\frac{\lambda_k}{\lambda_{k+1}} > 0$$

for all $k \geq k_2$.

From (23) we have

$$\begin{aligned}
\left\|x^{k+1} - z\right\|^2 &= \left\|(1-\alpha)w^k + \alpha z^k - z\right\|^2 \\
&= \left\|(1-\alpha)\left(w^k - z\right) + \alpha\left(z^k - z\right)\right\|^2 \\
&= (1-\alpha)\left\|w^k - z\right\|^2 + \alpha\left\|z^k - z\right\|^2 - (1-\alpha)\alpha\left\|z^k - w^k\right\|^2. \qquad (26)
\end{aligned}$$

Combining (25) and (26) we get

$$\begin{aligned}
\left\|x^{k+1} - z\right\|^2 \leq \left\|w^k - z\right\|^2 &- \left(1 - 2\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\alpha\left\|w^k - y^k\right\|^2 \\
&- \left(1 - 2\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\alpha\left\|z^k - y^k\right\|^2 - (1-\alpha)\alpha\left\|z^k - w^k\right\|^2. \qquad (27)
\end{aligned}$$

We find from (23) that $z^k - w^k = \frac{1}{\alpha}\left(x^{k+1} - w^k\right)$, which together with (27) implies that

$$\begin{aligned}
\left\|x^{k+1} - z\right\|^2 \leq \left\|w^k - z\right\|^2 &- \left(1 - 2\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\alpha\left\|w^k - y^k\right\|^2 \\
&- \left(1 - 2\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\alpha\left\|z^k - y^k\right\|^2 - \frac{1-\alpha}{\alpha}\left\|x^{k+1} - w^k\right\|^2 \\
&\leq \left\|w^k - z\right\|^2 - \frac{1-\alpha}{\alpha}\left\|x^{k+1} - w^k\right\|^2 \\
&= \left\|w^k - z\right\|^2 - \rho\left\|x^{k+1} - w^k\right\|^2 \ \forall k \geq k_2, \qquad (28)
\end{aligned}$$

where $\rho := \frac{1-\alpha}{\alpha}$.

Moreover, we obtain

$$\begin{aligned}
\left\|w^k - z\right\|^2 &= \left\|x^k + \theta\left(x^k - x^{k-1}\right) - z\right\|^2 \\
&= \left\|(1+\theta)\left(x^k - z\right) - \theta\left(x^{k-1} - z\right)\right\|^2 \\
&= (1+\theta)\left\|x^k - z\right\|^2 - \theta\left\|x^{k-1} - z\right\|^2 + (1+\theta)\theta\left\|x^k - x^{k-1}\right\|^2. \qquad (29)
\end{aligned}$$

Besides, we have

$$\begin{aligned}
\left\|x^{k+1} - w^k\right\|^2 &= \left\|x^{k+1} - x^k - \theta\left(x^k - x^{k-1}\right)\right\|^2 \\
&= \left\|x^{k+1} - x^k\right\|^2 + \theta^2\left\|x^k - x^{k-1}\right\|^2 - 2\theta\left\langle x^{k+1} - x^k, x^k - x^{k-1}\right\rangle \\
&\geq \left\|x^{k+1} - x^k\right\|^2 + \theta^2\left\|x^k - x^{k-1}\right\|^2 - 2\theta\left\|x^{k+1} - x^k\right\|\left\|x^k - x^{k-1}\right\| \\
&\geq (1 - \theta\sqrt{\tau})\left\|x^{k+1} - x^k\right\|^2 + \left(\theta^2 - \frac{\theta}{\sqrt{\tau}}\right)\left\|x^k - x^{k-1}\right\|^2. \qquad (30)
\end{aligned}$$

Combining (28), (29) and (30) we arrive at

$$\begin{aligned}\left\|x^{k+1}-z\right\|^2 &\leq (1+\theta)\left\|x^k-z\right\|^2-\theta\left\|x^{k-1}-z\right\|^2+(1+\theta)\theta\left\|x^k-x^{k-1}\right\|^2\\ &\quad -\rho(1-\theta\sqrt{\tau})\left\|x^{k+1}-x^k\right\|^2-\rho\left(\theta^2-\frac{\theta}{\sqrt{\tau}}\right)\left\|x^k-x^{k-1}\right\|^2\\ &= (1+\theta)\left\|x^k-z\right\|^2-\theta\left\|x^{k-1}-z\right\|^2-\rho(1-\theta\sqrt{\tau})\left\|x^{k+1}-x^k\right\|^2\\ &\quad +\left[(1+\theta)\theta-\rho\left(\theta^2-\frac{\theta}{\sqrt{\tau}}\right)\right]\left\|x^k-x^{k-1}\right\|^2 \quad \forall k\geq k_2. \end{aligned} \tag{31}$$

This leads to the following estimation:

$$\begin{aligned}&\left\|x^{k+1}-z\right\|^2-\theta\left\|x^k-z\right\|^2+\left[(1+\theta)\theta-\rho\left(\theta^2-\frac{\theta}{\sqrt{\tau}}\right)\right]\left\|x^{k+1}-x^k\right\|^2\\ &\leq \left\|x^k-z\right\|^2-\theta\left\|x^{k-1}-z\right\|^2+\left[(1+\theta)\theta-\rho\left(\theta^2-\frac{\theta}{\sqrt{\tau}}\right)\right]\left\|x^k-x^{k-1}\right\|^2\\ &\quad -\left[\rho(1-\theta\sqrt{\tau})-(1+\theta)\theta+\rho\left(\theta^2-\frac{\theta}{\sqrt{\tau}}\right)\right]\left\|x^{k+1}-x^k\right\|^2 \quad \forall k\geq k_2. \end{aligned} \tag{32}$$

Setting

$$\Gamma_k=\left\|x^k-z\right\|^2-\theta\left\|x^{k-1}-z\right\|^2+\left[(1+\theta)\theta-\rho\left(\theta^2-\frac{\theta}{\sqrt{\tau}}\right)\right]\left\|x^k-x^{k-1}\right\|^2$$

and

$$\begin{aligned}\gamma &= \rho(1-\theta\sqrt{\tau})-(1+\theta)\theta+\rho\left(\theta^2-\frac{\theta}{\sqrt{\tau}}\right)\\ &= \rho\Big(\frac{\sqrt{\tau}\theta^2-(\tau+1)\theta+\sqrt{\tau}}{\sqrt{\tau}}\Big)-\theta(1+\theta).\end{aligned}$$

In view of $\theta\in\left[0,\frac{1}{\sqrt{\tau}}\right)$ and

$$\alpha\in\left(0,\frac{\sqrt{\tau}\theta^2-(\tau+1)\theta+\sqrt{\tau}}{\sqrt{\tau}\theta^2-(\tau+1)\theta+\sqrt{\tau}+\theta(1+\theta)}\right),$$

it is easy to see that

$$\gamma=\rho\Big(\frac{\sqrt{\tau}\theta^2-(\tau+1)\theta+\sqrt{\tau}}{\sqrt{\tau}}\Big)-\theta(1+\theta)>0. \tag{33}$$

Hence (32) becomes

$$\Gamma_{k+1}-\Gamma_k\leq-\gamma\left\|x^{k+1}-x^k\right\|^2 \;\forall k\geq k_2. \tag{34}$$

This means that the sequence $\{\Gamma_k\}$ is nonincreasing for all $k\geq k_2$. We also note that

$$\begin{aligned}\Gamma_k &= \left\|x^k-z\right\|^2-\theta\left\|x^{k-1}-z\right\|^2+\underbrace{\left[(1+\theta)\theta-\rho\left(\theta^2-\frac{\theta}{\sqrt{\tau}}\right)\right]}_{>0}\left\|x^k-x^{k-1}\right\|^2\\ &\geq \left\|x^k-z\right\|^2-\theta\left\|x^{k-1}-z\right\|^2.\end{aligned}$$

Therefore, we get for all $k \geq k_2$ that

$$
\begin{aligned}
\left\|x^k - z\right\|^2 &\leq \theta \left\|x^{k-1} - z\right\|^2 + \Gamma_k \\
&\leq \theta \left\|x^{k-1} - z\right\|^2 + \Gamma_{k_2} \\
&\leq \cdots \leq \theta^{k-k_2} \left\|x^{k_2} - z\right\|^2 + \Gamma_{k_2}\left(\theta^{k-k_2-1} + \cdots + 1\right) \\
&\leq \theta^{k-k_2} \left\|x^{k_2} - z\right\|^2 + \frac{\Gamma_{k_2}}{1-\theta}.
\end{aligned} \tag{35}
$$

On the other hand, we have

$$
\begin{aligned}
\Gamma_{k+1} &= \left\|x^{k+1} - z\right\|^2 - \theta \left\|x^k - z\right\|^2 + \left[(1+\theta)\theta - \rho\left(\theta^2 - \frac{\theta}{\sqrt{2}}\right)\right] \left\|x^k - x^{k-1}\right\|^2 \\
&\geq -\theta \left\|x^k - z\right\|^2.
\end{aligned} \tag{36}
$$

It follows from (35) and (36) that

$$
-\Gamma_{k+1} \leq \theta \left\|x^k - z\right\|^2 \leq \theta^{k-k_2+1} \left\|x^{k_2} - z\right\|^2 + \frac{\theta \Gamma_{k_2}}{1-\theta}.
$$

Moreover, from (34) we have

$$
\begin{aligned}
\gamma \sum_{j=k_2}^{k} \left\|x^{k+1} - x^k\right\|^2 \leq \Gamma_{k_2} - \Gamma_{k+1} &\leq \theta^{k-k_2+1} \left\|x^{k_2} - z\right\|^2 + \frac{\Gamma_{k_2}}{1-\theta} \ \forall k \geq k_2 \\
&\leq \left\|x^{k_2} - z\right\|^2 + \frac{\Gamma_{k_2}}{1-\theta} \quad \forall k \geq k_2.
\end{aligned}
$$

We infer that $\sum_{k=1}^{\infty} \left\|x^{k+1} - x^k\right\|^2 < +\infty$. Hence, we get $\left\|x^{k+1} - x^k\right\| \to 0$. Further,

$$
\left\|x^{k+1} - w^k\right\|^2 = \left\|x^{k+1} - x^k\right\|^2 + \theta^2 \left\|x^k - x^{k-1}\right\|^2 - 2\theta \left\langle x^{k+1} - x^k, x^k - x^{k-1}\right\rangle.
$$

This implies that $\left\|x^{k+1} - w^k\right\| \to 0$ and

$$
\lim_{k\to\infty} \left\|z^k - w^k\right\| = \frac{1}{\alpha} \lim_{k\to\infty} \left\|x^{k+1} - w^k\right\| = 0, \tag{37}
$$

$$
\left\|w^k - x^k\right\|^2 = \theta^2 \left\|x^k - x^{k-1}\right\|^2 \to 0 \text{ as } k \to \infty.
$$

From (31), we get

$$
\begin{aligned}
\left\|x^{k+1} - z\right\|^2 \leq (1+\theta) \left\|x^k - z\right\|^2 - \theta \left\|x^{k-1} - z\right\|^2 \\
+ \left[(1+\theta)\theta - \rho\left(\theta^2 - \frac{\theta}{\sqrt{\tau}}\right)\right] \left\|x^k - x^{k-1}\right\|^2 \quad \forall k \geq k_2.
\end{aligned} \tag{38}
$$

Applying Lemma 2.6 to (38) with data

$$
\phi_k := \|x^k - z\|^2, \ \delta_k := \left[(1+\theta)\theta - \rho\left(\theta^2 - \frac{\theta}{\sqrt{\tau}}\right)\right] \left\|x^k - x^{k-1}\right\|^2, \ \theta_k := \theta
$$

we deduce that $\lim_{k\to\infty}\left\|x^k-z\right\|^2$ exists. We also get from (29) that $\lim_{k\to\infty}\left\|w^k-z\right\|^2$ exists. From this and (37) we have

$$\begin{aligned}\left\|w^k-z\right\|^2-\left\|z^k-z\right\|^2&=\left(\left\|w^k-z\right\|+\left\|z^k-z\right\|\right)\left(\| w^k-z\|-\|z^k-z\|\right)\\&\le\left(\| w^k-z\|+\|z^k-z\|\right)\left\|w^k-z^k\right\|\to 0 \text{ as } k\to\infty,\end{aligned}\tag{39}$$

i.e.,

$$\lim_{k\to\infty}\left\|w^k-z\right\|^2-\left\|z^k-z\right\|^2=0.$$

Therefore, from (25), we obtain

$$\left(1-\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\left\|y^k-w^k\right\|^2+\left(1-\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\left\|z^k-y^k\right\|^2\le\left\|w^k-z\right\|^2-\left\|z^k-z\right\|^2\to 0 \text{ as } k\to\infty,$$

i.e.,

$$\lim_{k\to\infty}\left[\left(1-\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\left\|y^k-w^k\right\|^2+\left(1-\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\left\|z^k-y^k\right\|^2\right]=0.$$

This yields that

$$\lim_{k\to\infty}\left\|y^k-w^k\right\|=0 \text{ and } \lim_{k\to\infty}\left\|z^k-y^k\right\|=0.$$

Similarly to (21), we have

$$\begin{aligned}\langle w^k-z^k,y-z^k\rangle\le\lambda_k f(y^k,y)-\langle w^k-y^k,z^k-y^k\rangle\\+2\sigma\frac{\lambda_k}{\lambda_{k+1}}[\|w^k-y^k\|^2+\|y^k-z^k\|^2]\quad\forall y\in C.\end{aligned}$$

The rest of the proof is similar to that of Theorem 3.1. Thus, we come to the conclusion of Theorem 3.2.

Remark 3.3 From Algorithm 3.2 and Theorem 3.2 we see that

1. In some cases, the selection of $\theta=\theta(\tau)$ and the suitable parameter τ in our method allows us to increase the parameter α in (23). So we can say that our choice of α is more flexible than that of Algorithm 2.1 in [40]. Please see the comparision of numerical results with different parameters in Example 6.1.

2. Arguing as in the proof of Theorem 3.2 of [40], we can establish the linear convergence rate of Algorithm 3.2 under a strong pseudomonotonicity assumption of f.

4 Strong convergence result

In this section, we give a viscosity version of Algorithm 3.1 to obtain a new strong convergence result for equilibrium problems.

Algorithm 4.1 (Viscosity-type subgradient extragradient method with novel stepsizes)

Step 0 (Initialization). Let $\varphi : \mathcal{H} \to \mathcal{H}$ be a contraction (i.e., there exists a constant $\alpha \in [0,1)$ such that $\|\varphi(x) - \varphi(y)\| \leq \alpha \|x - y\|$ for all $x, y \in \mathcal{H}$). Select $\lambda_0 > 0$, $\mu_1 < \mu_0 < \sigma < 1/2$, $\theta \in [0,1)$. Take positive sequences $\{\epsilon_k\}, \{\xi_k\}, \{\alpha_k\} \subset [0,\infty)$ satisfying

$$\sum_{k=0}^{\infty} \epsilon_k < \infty, \ \sum_{k=0}^{\infty} \xi_k < \infty, \ \sum_{k=0}^{\infty} \alpha_k = \infty, \ \lim_{k\to\infty} \alpha_k = 0, \ \lim_{k\to\infty} \frac{\xi_k}{\alpha_k} = 0.$$

Choose initial iterates $x^0, x^1 \in C$ and set $k = 1$.

Step 1. Given the current iterates x^{k-1} and x^k $(k \geq 1)$, choose θ_k such that $0 \leq \theta_k \leq \bar{\theta}_k$, where

$$\bar{\theta}_k = \begin{cases} \min\left\{\theta, \dfrac{\xi_k}{\|x^k - x^{k-1}\|}\right\} & \text{if } x^k \neq x^{k-1}, \\ \theta & \text{otherwise.} \end{cases}$$

Compute

$$\begin{cases} w^k = x^k + \theta_k (x^k - x^{k-1}), \\ y^k = \underset{y\in C}{\operatorname{argmin}} \left\{ \lambda_k f(w^k, y) + \dfrac{1}{2}\|y - w^k\|^2 \right\}. \end{cases}$$

If $y^k = w^k$ then terminate: w^k is a solution. Otherwise, go to Step 2.

Step 2. Take $\xi^k \in \partial_2 f(w^k, y^k)$ such that $w^k - \lambda_k \xi^k - y^k \in N_C(y^k)$. Construct the half-space

$$T_k = \{x \in \mathcal{H} : \langle w^k - \lambda_k \xi^k - y^k, x - y^k \rangle \leq 0\}.$$

Calculate

$$\begin{aligned} z^k &= \underset{y\in T_k}{\operatorname{argmin}} \left\{ \lambda_k f(y^k, y) + \frac{1}{2}\|y - w^k\|^2 \right\}, \\ x^{k+1} &= \alpha_k \varphi(w^k) + (1 - \alpha_k) z^k, \\ \eta_k &= f(w^k, z^k) - f(w^k, y^k) - f(y^k, z^k), \ \delta_k = \|w^k - y^k\|^2 + \|y^k - z^k\|^2 \end{aligned}$$

and update λ_k by

$$\lambda_{k+1} = \begin{cases} \mu_1 \dfrac{\delta_k}{\eta_k}, & \text{if } \eta_k > \frac{\mu_0}{\lambda_k}\delta_k, \\ (1 + \epsilon_k)\lambda_k, & \text{otherwise.} \end{cases}$$

Set $k := k + 1$, and return to **Step 1**.

Theorem 4.1 *Under the conditions (C1)-(C6), the sequence $\{x^k\}$ generated by Algorithm 4.1 converges strongly to an element of* $\mathrm{Sol}(f,C)$.

Proof First, from Proposition 2 of [10], we have that the solution set $\mathrm{Sol}(f,C)$ is closed and convex. We now show that $\{x^k\}$ is bounded. Let $z = P_{\mathrm{Sol}(f,C)}(\varphi(z))$. From (13) we obtain

$$\|z^k - z\|^2 \leq \|w^k - z\|^2 - \left(1 - 2\sigma \frac{\lambda_k}{\lambda_{k+1}}\right)\|w^k - y^k\|^2 - \left(1 - 2\sigma \frac{\lambda_k}{\lambda_{k+1}}\right)\|z^k - y^k\|^2.$$

Hence, for all $k \in \mathbb{N}$,

$$\|z^k - z\| \leq \|w^k - z\|.$$

We note that

$$\begin{aligned}\|x^{k+1} - z\| &\leq (1-\alpha_k)\|z^k - z\| + \alpha_k\|\varphi(w^k) - z\| \\ &\leq (1-\alpha_k)\|w^k - z\| + \alpha_k(\|\varphi(w^k) - \varphi(z)\| + \|\varphi(z) - z\|) \\ &\leq (1-\alpha_k)\|w^k - z\| + \alpha_k(\alpha\|w^k - z\| + \|\varphi(z) - z\|) \\ &= (1-(1-\alpha)\alpha_k)\|w^k - z\| + \alpha_k\|\varphi(z) - z\|.\end{aligned}$$

Hence,

$$\begin{aligned}\|x^{k+1} - z\| &\leq (1-(1-\alpha)\alpha_k)\|w^k - z\| + \alpha_k\|\varphi(z) - z\| \\ &= (1-(1-\alpha)\alpha_k)\|x^k + \theta_k(x^k - x^{k-1}) - z\| + \alpha_k\|\varphi(z) - z\| \\ &\leq (1-(1-\alpha)\alpha_k)\|x^k - z\| + (1-(1-\alpha)\alpha_k)\theta_k\|x^k - x^{k-1}\| + \alpha_k\|\varphi(z) - z\| \\ &= (1-(1-\alpha)\alpha_k)\|x^k - z\| + (1-\alpha)\alpha_k\left(\sigma_k + \frac{\|\varphi(z) - z\|}{1-\alpha}\right),\end{aligned}$$

where

$$\sigma_k = \left(\frac{1-(1-\alpha)\alpha_k}{1-\alpha}\right)\frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\|.$$

By the condition of parameters ξ_k and α_k, we see that

$$\lim_{k\to\infty}\sigma_k = \lim_{k\to\infty}\left(\frac{1-(1-\alpha)\alpha_k}{1-\alpha}\right)\frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| = 0,$$

which implies that the sequence $\{\sigma_k\}$ is bounded. Setting

$$M = \max\left\{\frac{\|\varphi(z) - z\|}{1-\alpha}, \sup_{n\in\mathbb{N}}\sigma_k\right\}$$

and using Lemma 2.5 (1), we thus conclude that the sequence $\{\|x^k - z\|\}$ is bounded, which implies that the sequence $\{x^k\}$ is bounded and so is $\{w^k\}$.

By (15) and $0 \leq \theta_k < 1$ (hence, $\theta_k(1+\theta_k) < 2\theta_k$) we have

$$\|w^k - z\|^2 \leq \|x^k - z\|^2 + \theta_k(\|x^k - z\|^2 - \|x^{k-1} - z\|^2) + 2\theta_k\|x^k - x^{k-1}\|^2.$$

On the other hand, using the inequality (3) with the data $x := z^k - z$, $y := \varphi(w^k) - z$, we have

$$\begin{aligned}\|x^{k+1} - z\|^2 &= \|(1-\alpha_k)(z^k - z) + \alpha_k(\varphi(w^k) - z)\|^2 \\ &\leq (1-\alpha_k)\|z^k - z\|^2 + 2\alpha_k\langle\varphi(w^k) - z, x^{k+1} - z\rangle \\ &\leq (1-\alpha_k)\|w^k - z\|^2 + 2\alpha_k\langle\varphi(w^k) - z, x^{k+1} - z\rangle \\ &\qquad - \left(1 - 2\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\|z^k - y^k\|^2 - \left(1 - 2\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\|x^{k+1} - y^k\|^2 \\ &\leq (1-\alpha_k)\|x^k - z\|^2 + (1-\alpha_k)\theta_k(\|x^k - z\|^2 - \|x^{k-1} - z\|^2) \\ &\qquad + 2(1-\alpha_k)\theta_k\|x^k - x^{k-1}\|^2 + 2\alpha_k\langle\varphi(w^k) - z, x^{k+1} - z\rangle \\ &\qquad - \left(1 - 2\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\|z^k - y^k\|^2 - \left(1 - 2\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\|x^{k+1} - y^k\|^2\end{aligned}$$

Putting $a_k := \|x^k - z\|^2$ for all $n \in \mathbb{N}$ we have

$$\left(1-2\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\|z^k-y^k\|^2+\left(1-2\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\|x^{k+1}-y^k\|^2 \le a_k-a_{k+1}$$
$$+2\alpha_k\langle\varphi(w^k)-z,x^{k+1}-z\rangle+(1-\alpha_k)\theta_k(a_k-a_{k-1})+2(1-\alpha_k)\theta_k\|x^k-x^{k-1}\|^2. \tag{40}$$

Now, we consider two possible cases:

Case 1. Assume that there exists $n_0 \geq 0$ such that for each $k \geq n_0$, $a_{k+1} \leq a_k$. In this case, $\lim_{k\to\infty} a_k$ exists and $\lim_{k\to\infty}(a_k - a_{k+1}) = 0$.

Since $\lim_{k\to\infty}\alpha_k = 0$ and $\lim_{k\to\infty}\theta_k\|x^k - x^{k-1}\|^2 = 0$, it follows from (40) that

$$\lim_{k\to\infty}\left[\left(1-2\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\|z^k-y^k\|^2+\left(1-2\sigma\frac{\lambda_k}{\lambda_{k+1}}\right)\|x^{k+1}-y^k\|^2\right]=0.$$

From the assumption $\inf_{k\to\infty}\beta_k(1-\beta_k-\gamma_k) > 0$, we obtain

$$\lim_{k\to\infty}\|z^k-y^k\|^2=0, \tag{41}$$
$$\lim_{k\to\infty}\|x^{k+1}-y^k\|^2=0.$$

We have

$$\begin{aligned}\|y^k-x^k\| &\le \|y^k-z^k\|+\|z^k-x^k\|\\ &\le \|y^k-z^k\|+\alpha_k\|\varphi(w^k)-w^k\|+(1-\alpha_k)\|w^k-x^k\|\\ &\le \|y^k-z^k\|+\alpha_k\|\varphi(w^k)-w^k\|+(1-\alpha_k)\theta_k\|x^k-x^{k-1}\|.\end{aligned}$$

This means $\lim_{k\to\infty}\|x^k - y^k\| = 0$ Let $\bar{x}$ be an arbitrary weakly cluster point of $\{x^k\}$. Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_l}\}$ of $\{x^k\}$ such that $x^{k_l} \rightharpoonup \bar{x} \in C$ and hence $y^{k_l} \rightharpoonup \bar{x}$. It follows from (41) that $x^{k_l+1} \rightharpoonup \bar{x}$. Following similar arguments as in the proof of Theorem 3.1 we conclude that $\omega_w(x^k) \subset \mathrm{Sol}(f,C)$. Since $z = P_{\mathrm{Sol}(f,C)}(\varphi(z))$, by the characterization of the metric projection (Lemma 2.1 (1)), we get that

$$\limsup_{k\to\infty}\langle\varphi(z)-z,x^{k+1}-z\rangle=\max_{\bar{x}\in\omega_w(x^k)}\langle\varphi(z)-z,\bar{x}-z\rangle\le 0.$$

On the other hand, we see that

$$\begin{aligned}\|x^{k+1}-z\|^2 &= \langle x^{k+1}-z,x^{k+1}-z\rangle\\ &= \alpha_k\langle\varphi(w^k)-\varphi(z),x^{k+1}-z\rangle+\alpha_k\langle\varphi(z)-z,x^{k+1}-z\rangle\\ &\qquad +(1-\alpha_k)\langle z^k-z,x^{k+1}-z\rangle\\ &\le \alpha_k\alpha\|w^k-z\|\|x^{k+1}-z\|+\alpha_k\langle\varphi(z)-z,x^{k+1}-z\rangle\\ &\qquad +(1-\alpha_k)\|z^k-z\|\|x^{k+1}-z\|\\ &\le \alpha_k\alpha\|w^k-z\|\|x^{k+1}-z\|+\alpha_k\langle\varphi(z)-z,x^{k+1}-z\rangle\\ &\qquad +(1-\alpha_k)\|w^k-z\|\|x^{k+1}-z\|\\ &\le (1-\alpha_k(1-\alpha))\|w^k-z\|\|x^{k+1}-z\|+\alpha_k\langle\varphi(z)-z,x^{k+1}-z\rangle\\ &\le (1-\alpha_k(1-\alpha))\left(\frac{\|w^k-z\|^2}{2}+\frac{\|x^{k+1}-z\|^2}{2}\right)+\alpha_k\langle\varphi(z)-z,x^{k+1}-z\rangle,\end{aligned}$$

which gives

$$\begin{aligned}\|x^{k+1}-z\|^2 &\le \frac{1-\alpha_k(1-\alpha)}{1+\alpha_k(1-\alpha)}\|w^k-z\|^2+\frac{2\alpha_k}{1+\alpha_k(1-\alpha)}\langle\varphi(z)-z,x^{k+1}-z\rangle\\ &\le \frac{1-\alpha_k(1-\alpha)}{1+\alpha_k(1-\alpha)}(\|x^k-z\|+\theta_k\|x^k-x^{k-1}\|)^2+\frac{2\alpha_k}{1+\alpha_k(1-\alpha)}\langle\varphi(z)-z,x^{k+1}-z\rangle\\ &=\left(1-\frac{2\alpha_k(1-\alpha)}{1+\alpha_k(1-\alpha)}\right)(\|x^k-z\|^2+2\theta_k\|x^k-x^{k-1}\|\|x^k-z\|+\theta_k^2\|x^k-x^{k-1}\|^2)\\ &\quad+\frac{2\alpha_k}{1+\alpha_k(1-\alpha)}\langle\varphi(z)-z,x^{k+1}-z\rangle. \end{aligned} \tag{42}$$

Put $M_1=\sup_{k\in\mathbb{N}}\|x^k-z\|$ and $\gamma_k=\frac{2\alpha_k(1-\alpha)}{1+\alpha_k(1-\alpha)}$ for all $k\in\mathbb{N}$. It is easily checked that for large enough k, $\gamma_k\in(0,1)$. Without loss of generality, assume $0<\gamma_k<1$ for all $k\ge n_0$ and $\sum_{k=1}^{\infty}\gamma_k=\infty$. From (42), it follows that

$$\begin{aligned}a_{k+1}\le(1-\gamma_k)a_k+2\theta_k\|x^k-x^{k-1}\|M_1+\theta_k^2\|x^k-x^{k-1}\|^2\\ +\frac{2\alpha_k}{1+\alpha_k(1-\alpha)}\langle\varphi(z)-z,x^{k+1}-z\rangle.\end{aligned} \tag{43}$$

We have from the selection of θ_k that $\theta_k\|x^k-x^{k-1}\|\to 0$. Therefore

$$\lim_{k\to\infty}[2\theta_k\|x^k-x^{k-1}\|M_1+\theta_k^2\|x^k-x^{k-1}\|^2]=0. \tag{44}$$

By applying Lemma 2.5 to (43) with the data:

$$\begin{aligned}&\bar{a}_k:=\|x^k-z\|^2,\quad \bar{\delta}_k:=\gamma_k,\quad \bar{c}_k:=0,\\ &\bar{b}_k:=2\theta_k\|x^k-x^{k-1}\|M_1+\theta_k^2\|x^k-x^{k-1}\|^2+\frac{2\alpha_k}{1+\alpha_k(1-\alpha)}\langle\varphi(z)-z,x^{k+1}-z\rangle,\end{aligned}$$

we immediately deduce that the sequence $\{x^k\}$ converges strongly to $z=P_{\mathrm{Sol}(C,f)}(\varphi(z))$.
Case 2. Assume that there exists a subsequence $\{a_{k_l}\}\subset\{a_k\}$ such that $a_{k_l}\le a_{k_l+1}$ for all $l\in\mathbb{N}$. In this case, we can define $\tau:\mathbb{N}\to\mathbb{N}$ by

$$\tau(k)=\max\{i\le k: a_i<a_{i+1}\}.$$

Then we have from Lemma 2.7 that $\tau(k)\to\infty$ as $k\to\infty$ and $a_{\tau(k)}<a_{\tau(k)+1}$. So, we have from (40) that

$$\begin{aligned}&\left(1-2\sigma\frac{\lambda_{\tau(k)}}{\lambda_{\tau(k)+1}}\right)\|z^{\tau(k)}-y^{\tau(k)}\|^2+\left(1-2\sigma\frac{\lambda_{\tau(k)}}{\lambda_{\tau(k)+1}}\right)\|x^{\tau(k)+1}-y^{\tau(k)}\|^2\le a_{\tau(k)}-a_{\tau(k)+1}\\ &\qquad+2\alpha_{\tau(k)}\langle\varphi(w^{\tau(k)})-z,z^{\tau(k)}-z\rangle+(1-\alpha_{\tau(k)})\theta_{\tau(k)}(a_{\tau(k)}-a_{\tau(k)-1})\\ &\qquad+2(1-\alpha_{\tau(k)})\theta_{\tau(k)}\|x^{\tau(k)}-x^{\tau(k)-1}\|^2\\ &\le 2\alpha_{\tau(k)}\langle\varphi(w^{\tau(k)})-z,z^{\tau(k)}-z\rangle+(1-\alpha_{\tau(k)})\theta_{\tau(k)}\|x^{\tau(k)}-x^{\tau(k)-1}\|\left(\sqrt{a_{\tau(k)}}+\sqrt{a_{\tau(k)-1}}\right)\\ &\qquad+2(1-\alpha_{\tau(k)})\theta_{\tau(k)}\|x^{\tau(k)}-x^{\tau(k)-1}\|^2.\end{aligned} \tag{45}$$

Following the same lines as in the proof of Case 1, we get from (45) that

$$\lim_{k\to\infty}\|z^{\tau(k)}-y^{\tau(k)}\|=0,$$
$$\lim_{k\to\infty}\|x^{\tau(k)+1}-y^{\tau(k)}\|=0.$$
$$\lim_{k\to\infty}\|x^{\tau(k)}-y^{\tau(k)}\|=0,$$
$$\limsup_{k\to\infty}\langle\varphi(z)-z,x^{\tau(k)+1}-z\rangle=\max_{\widetilde{z}\in\omega_w(\{x^{\tau(k)}\})}\langle\varphi(z)-z,\widetilde{z}-z\rangle\leq 0 \tag{46}$$

and

$$a_{\tau(k)+1}\leq(1-\gamma_{\tau(k)})a_{\tau(k)}+2\theta_{\tau(k)}\|x^{\tau(k)}-x^{\tau(k)-1}\|M_1+\theta^2_{\tau(k)}\|x^{\tau(k)}-x^{\tau(k)-1}\|^2$$
$$+\frac{2\alpha_{\tau(k)}}{1+\alpha_{\tau(k)}(1-\alpha)}\langle\varphi(z)-z,x^{\tau(k)+1}-z\rangle. \tag{47}$$

Since $a_{\tau(k)}<a_{\tau(k)+1}$, we have from (47) that

$$\gamma_{\tau(k)}a_{\tau(k)}\leq 2\theta_{\tau(k)}\|x^{\tau(k)}-x^{\tau(k)-1}\|M_1+\theta^2_{\tau(k)}\|x^{\tau(k)}-x^{\tau(k)-1}\|^2$$
$$+\frac{2\alpha_{\tau(k)}}{1+\alpha_{\tau(k)}(1-\alpha)}\langle\varphi(z)-z,x^{\tau(k)+1}-z\rangle$$

and hence

$$\|x^{\tau(k)}-z\|^2=a_{\tau(k)}\leq 2\frac{\theta_{\tau(k)}}{\gamma_{\tau(k)}}\|x^{\tau(k)}-x^{\tau(k)-1}\|M_1+\frac{\theta^2_{\tau(k)}}{\gamma_{\tau(k)}}\|x^{\tau(k)}-x^{\tau(k)-1}\|^2$$
$$+\frac{1}{1-\alpha}\langle\varphi(z)-z,x^{\tau(k)+1}-z\rangle \tag{48}$$

Relation (44) and the boundedness of $\{\gamma_k\}$ yield

$$\lim_{k\to\infty}\left[2\frac{\theta_{\tau(k)}}{\gamma_{\tau(k)}}\|x^{\tau(k)}-x^{\tau(k)-1}\|M_1+\frac{\theta^2_{\tau(k)}}{\gamma_{\tau(k)}}\|x^{\tau(k)}-x^{\tau(k)-1}\|^2\right]=0. \tag{49}$$

Combining (46), (48) and (49) gives

$$\limsup_{k\to\infty}\|x^{\tau(k)}-z\|^2\leq 0,$$

and hence

$$\lim_{k\to\infty}\|x^{\tau(k)}-z\|^2=0.$$

From (47), we have

$$\limsup_{k\to\infty}\|x^{\tau(k)+1}-z\|^2\leq\limsup_{k\to\infty}\|x^{\tau(k)}-z\|^2.$$

Thus

$$\lim_{k\to\infty}\|x^{\tau(k)+1}-z\|^2=0.$$

Therefore, by Lemma 2.7, we obtain

$$0\leq\|x^k-z\|\leq\max\{\|x^{\tau(k)}-z\|,\|x^k-z\|\}\leq\|x^{\tau(k)+1}-z\|\to 0.$$

Consequently, $\{x^k\}$ converges strongly to $z=P_{\mathrm{Sol}(f,C)}(\varphi(z))$ and the proof is complete.

5 Application to the variational inequality problem

In this section, we consider the following variational inequality problem (VIP):

$$\text{Find } x^* \in C : \langle Ax^*, y - x^* \rangle \geq 0 \ \ \forall\ y \in C, \tag{50}$$

where $A : \mathcal{H} \to \mathcal{H}$.

Then, for each pair $x, y \in \mathcal{H}$, we define the bifunction f by taking

$$f(x,y) := \begin{cases} \langle Ax, y - x \rangle, & \text{if } x \in C, \\ +\infty, & \text{otherwise.} \end{cases} \tag{51}$$

Suppose that

(C51) A is pseudomonotone on C;

(C52) A is F-hemicontinuous ([25]), i.e., for all $y \in C$, the function $x \mapsto \langle A(x), x - y \rangle$ is weakly lower semicontinuous on C ((or equivalently, $x \mapsto \langle A(x), y - x \rangle$ is weakly upper semicontinuous on C);

(C53) A is Lipschitz continuous on C with a Lipschitz constant L;

(C54) The solution set $\text{Sol}(C,A) = \{x \in C : \langle Ax, y - x \rangle \geq 0 \ \ \forall y \in C\} \neq \emptyset$.

It is easy to see that any weak-to-strong continuous mapping is F-hemicontinuous, but vice-versa not, as the following example shows.

Example 5.1 ([19]) We consider the Hilbert space $l^2 = \{x = (x^i)_{i\in\mathbf{N}} : \sum_{i=1}^{\infty} |x^i|^2 < \infty\}$ and A: $l^2 \to l^2$ be the identity operator. Let $\{x_n\} \subseteq l^2$ be a sequence converging weakly to $\overline{x}$. Since the function $x \longmapsto \|x\|^2$ is continuous and convex, it is weakly lower semicontinuous. Hence,

$$\|\overline{x}\|^2 \leq \liminf_{n\to\infty} \|x_n\|^2,$$

which clearly implies

$$\langle \overline{x}, \overline{x} - y \rangle \leq \liminf_{n\to\infty} \langle x_n, x_n - y \rangle,$$

for all $y \in l^2$, i.e., A is F-hemicontinuous.

On the other hand, we take $x_n = e_n = (0,0,...,0,1,0,...)$ with 1 in the n^{th} position. It is obvious that $e_n \rightharpoonup 0$, but $\{e_n\}$ does not have any strongly convergent subsequence, as $\|e_n - e_m\| = \sqrt{2}$ for $m \neq n$. Therefore, A is not weak-to-strong continuous.

Remark 5.1 To our best knowledge, there is no information on the relationship between weakly continuous operators and F-hemicontinuous operators.

Then it is not hard to see that the variational inequality (50) takes the form of equilibrium problem (51) and the conditions (C1)-(C6) are satisfied for (C51)-(C54). Then Algorithm 3.1 reduces to the following one.

Algorithm 5.1 (Subgradient extragradient method with novel stepsizes for VIPs)

Step 0 (Initialization). Select $\lambda_0 > 0$, $\mu_1 < \mu_0 < \sigma < 1/2$, $\theta \in [0,1)$. Take positive sequences $\{\epsilon_k\}, \{\xi_k\} \subset [0,\infty)$ satisfying

$$\sum_{k=0}^{\infty} \epsilon_k < \infty, \ \sum_{k=0}^{\infty} \xi_k < \infty.$$

Choose initial iterates $x^0, x^1 \in C$ and set $k = 1$.

Step 1. Given the current iterates x^{k-1} and x^k $(k \geq 1)$, choose α_k such that $0 \leq \alpha_k \leq \bar{\alpha}_k$, where

$$\bar{\alpha}_k = \begin{cases} \min\left\{\theta, \dfrac{\xi_k}{\|x^k - x^{k-1}\|}\right\} & \text{if } x^k \neq x^{k-1}, \\ \theta & \text{otherwise.} \end{cases}$$

Compute

$$\begin{cases} w^k = x^k + \alpha_k(x^k - x^{k-1}), \\ y^k = P_C(w^k - \lambda_k A w^k). \end{cases}$$

If $y^k = w^k$ then terminate: x^k is a solution. Otherwise, go to **Step 2**.

Step 2. Construct the half-space

$$T_k = \{x \in \mathcal{H} : \langle w^k - \lambda_k A w^k - y^k, x - y^k \rangle \leq 0\}.$$

Calculate

$$\begin{aligned} x^{k+1} &= P_{T_k}(w^k - Ay^k), \\ \eta_k &= \langle Aw^k - Ay^k, x^{k+1} - y^k \rangle, \ \delta_k = \|w^k - y^k\|^2 + \|y^k - x^{k+1}\|^2 \end{aligned}$$

and update λ_k by

$$\lambda_{k+1} = \begin{cases} \mu_1 \dfrac{\delta_k}{\eta_k}, & \text{if } \eta_k > \frac{\mu_0}{\lambda_k}\delta_k, \\ (1+\epsilon_k)\lambda_k, & \text{otherwise.} \end{cases}$$

Set $k := k+1$, and return to **Step 1**.

The following result is a direct consequence of Theorem 3.1.

Corollary 5.1 *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, $A : \mathcal{H} \to \mathcal{H}$ a mapping satisfying conditions (C51)-(C54). Let $\{x^k\}$, $\{y^k\}$ be the sequences generated by Algorithm 5.1. Then the sequences $\{x^k\}$ and $\{y^k\}$ converge weakly to the same point $x^* \in \mathrm{Sol}(A, C)$.*

6 Numerical results

In this section, we will apply Algorithms 3.1 (NISEM) and 3.2 (SEMNS) to solve an equilibrium problem arising from *Nash-Cournot oligopolistic equilibrium models of electricity markets*. This problem has been investigated in many research papers (see, e.g. [32]). We use the cost function as in [41]. It is a convex but nonsmooth function. Hence the resulting equilibrium problem cannot be transformed into a variational inequality problem. Our numerical experiments are implemented in MATLAB R2022b on a PC Desktop with an Intel(R) Core(TM) i5-1035G1 CPU @ 1.00GHz 1.19 GHz, RAM 4.00 GB.

Example 6.1 (Applications to Nash–Cournot semi-oligopolistic equilibrium models of electricity markets). We consider a Nash–Cournot semi-oligopolistic equilibrium model of electricity markets [41]. Assume that electrical power for a city is provided by N companies and a solar panel system. The total power generation is $\xi = \sum_{k=1}^{N} x_k + a$, where x_k is the power generation level of company n $(k = 1,\dots,N)$ and a is the power generation level of the solar panel system, which is assumed to be a constant. Suppose that the electric price δ is computed by

$$\delta(x) = 378.4 - 2\left(\sum_{k=1}^{N} x_k + a\right),$$

where $x = (x_1,\dots,x_N)^{\mathrm{T}}$ is the power generation of N companies and the production cost of company n is computed by

$$\rho_n(x_k) := \max\{\hat{\rho}_n(x_k), \bar{\rho}_n(x_k)\},$$

where

$$\begin{aligned}\hat{\rho}_n(x_k) :=& \frac{\hat{\mu}_n}{2} x_k^2 + \hat{\alpha}_n x_k + \hat{\beta}_n;\\ \bar{\rho}_n(x_k) :=& \bar{\mu}_n x_k + \frac{\bar{\alpha}_n}{\bar{\alpha}_n + 1} \bar{\beta}_n^{-1/\bar{\alpha}_n} (x_k)^{(\bar{\alpha}_n+1)/\bar{\alpha}_n}.\end{aligned}$$

Let $N = 6$. The constants $\hat{\mu}_n, \hat{\alpha}_n, \hat{\beta}_n, \bar{\mu}_n, \bar{\alpha}_n$ and $\bar{\beta}_n$ are chosen as shown in Table 1. We

j	$\hat{\mu}_n$	$\hat{\alpha}_n$	$\hat{\beta}_n$	$\bar{\mu}_n$	$\bar{\alpha}_n$	$\bar{\beta}_n$
1	0.0400	2.00	0.00	2.0000	1.0000	25.0000
2	0.0350	1.75	0.00	1.7500	1.0000	28.5714
3	0.1250	1.00	0.00	1.0000	1.0000	8.0000
4	0.0116	3.25	0.00	3.2500	1.0000	86.2069
5	0.0500	3.00	0.00	3.0000	1.0000	20.0000
6	0.0500	3.00	0.00	3.0000	1.0000	20.0000

Table 1: Constants of the cost function

define the profit function ν_n of company n as

$$\nu_n(x) := \delta(x)x_k - \rho_n(x_k) = \left(378.4 - 2\left[\sum_{k=1}^{N} x_k + a\right]\right) x_k - \rho_n(x_k).$$

Suppose that $x_k \in [x_n^{\min}, x_n^{\max}]$, where $x_n^{\min}, x_n^{\max}, k = 1,\dots,N$ are given in Table 2.

Let $C := \prod_{k=1}^{N} [x_n^{\min}, x_n^{\max}]$. We find a point $x^* \in C$ such that

$$\begin{aligned}\nu_n(x_1^*,\dots,x_{n-1}^*, y_n, x_{n+1}^*,\dots,x_N^*) \le \nu_n(x_1^*,\dots,x_N^*)\ \forall y\\ = (y_1,\dots,y_N) \in C,\ \forall k = 1,\dots,N.\end{aligned}$$

Following the idea from [32], we can prove that this Nash equilibrium problem can be reformulated as an equilibrium problem:

$$\text{find } x^* \in C \text{ such that } f(x^*, y) \ge 0\ \forall y \in C, \tag{52}$$

n	1	2	3	4	5	6
$x_n^{\min}$	0	0	0	0	0	0
$x_n^{\max}$	90	70	100	60	110	50

Table 2: Values of $x_n^{\min}$, $x_n^{\max}$

where $f(x,y) = \langle (A+\frac{3}{2}B)x + \frac{1}{2}By + s, y-x\rangle + \rho(y) - \rho(x)$, and

$$A := 2\sum_{k=1}^{N} \bar{\sigma}^n(\sigma^n)^{\mathrm{T}},\ B := 2\sum_{k=1}^{N} \sigma^n(\sigma^n)^{\mathrm{T}},$$

$$s := (2a - 378.4)\sum_{k=1}^{N} \sigma^n,\ \rho(x) := \sum_{k=1}^{N} \rho_n(x_k).$$

Here, the vectors $\sigma^n := (\sigma_1^n, \ldots, \sigma_N^n)$ and $\bar{\sigma}^n := (\bar{\sigma}_1^n, \ldots, \bar{\sigma}_N^n)$ are defined by

$$\sigma_m^n = \begin{cases} 1 & \text{iff } n = m, \\ 0 & \text{iff } n \neq m \end{cases}$$

and $\bar{\sigma}_m^n = 1 - \sigma_m^n$ for all $n, m = 1, \ldots, N$. It is easy to see that the function f satisfies all conditions (A1)-(A8). The function ρ is subdifferentiable, and its subdifferential at x is given by $\partial\rho(x) = (\partial\rho_1(x_1), \ldots, \partial\rho_N(x_N))^T$ where, for each $j = 1, \ldots, N$

$$\partial\rho_j(x_j) = \begin{cases} \{\hat{\mu}_j x_j + \hat{\alpha}_j\}, & \text{if } \hat{\rho}_j(x_j) > \bar{\rho}_j(x_j), \\ \left[\hat{\mu}_j x_j + \hat{\alpha}_j, \bar{\mu}_j + \left(\frac{x_j}{\bar{\beta}_j}\right)^{1/\bar{\alpha}_j}\right], & \text{if } \hat{\rho}_j(x_j) = \bar{\rho}_j(x_j), \\ \left\{\bar{\mu}_j + \left(\frac{x_j}{\bar{\beta}_j}\right)^{1/\bar{\alpha}_j}\right\}, & \text{if } \hat{\rho}_j(x_j) < \bar{\rho}_j(x_j). \end{cases}$$

In what follows, we will study the convergence of Algorithm 3.1 and 3.2 and give a comparison with five algorithms, namely, AISEM recommended in [35, Algorithm 2.1], EPSM considered in [34, Algorithm 1], ISEM showed in [40, Algorithm 2.1], NAKM proposed in [39, Algorithm 3.1], SEM presented in [46, Algorithm 3.1].

Test 1. In this experiment, we take $a = 110$ and consider three cases for the starting point

$$x^a = (0,0,0,0,0,0)^T, \qquad x^b = (2,2,2,2,2,2)^T, \qquad x^c = (1,0,1,0,1,0)^T.$$

The parameters are chosen as follows:

In Alg. AISEM, $\mu = 0.5$, $\theta = 0.45(1-\mu)$, $\lambda_1 = 17$, $x_1 = x_0 = x^0$.

In Alg. EPSM, $\theta = 0.05$, $\alpha_k = 0.2\ \ \forall k \geq 0$, $\varrho = 0.55$, $\varkappa_0 = \varkappa_1 = 0.01$, $u_{-1} = u_0 = v_0 = x^0$.

In Alg. ISEM, $\mu = 0.9$, $\alpha = 0.1$, $\tau = 0.88$, $\lambda_1 = 0.01$, $x_0 = x_1 = x^0$.

In Alg. NAKM, $\delta = 2.1$, $\alpha = \frac{\delta - \sqrt{2\delta}}{10\delta}$, $\theta = 0.9$, $\tau = \frac{0.99}{1+\delta}$, $\mu = 0.4$, $\lambda_1 = 0.7$, $x_1 = x_0 = x^0$.

In Alg. SEM, $\mu = 0.99$, $\lambda_0 = 0.7$, $\alpha_k = \frac{1}{10^4(k+5)}\ \ \forall k \geq 1$, $x_0 = x^0$.

In Alg. NISEM, $\theta = 0.9$, $\mu_0 = 0.49$, $\mu_1 = 0.48$, $\alpha = 0.91$, $\lambda_0 = 0.01$, $\epsilon_k = \frac{1}{k^{1.01}}\ \ \forall k \geq 1$, $x^1 = x^0$.

In Alg. SEMNS, $\theta = 0.5$, $\mu_0 = 0.48$, $\mu_1 = 0.47$, $\lambda_0 = 0.2$, $\xi_k = \epsilon_k = \frac{1}{k^{1.01}}$ $\forall k \geq 1$, $x^1 = x^0$.

We use the condition $\|x^k - x^*\| \leq 10^{-5}$ or when the CPU time of the algorithms exceeds $50s$ as the stopping rule for all algorithms, where

$$x^* = (11.3361, 11.4868, 11.3533, 10.8748, 10.7930, 10.7930)^T$$

is the approximate solution of problem (52). The test results are shown in Tables 3, 4, and Figure 1.

Algorithm	Approximate solution x^*						$\|x^k - x^*\|$
AISEM	11.3361	11.4869	11.3533	10.8748	10.7930	10.7930	1.6434e-05
EPSM	11.3362	11.4869	11.3533	10.8749	10.7931	10.7931	1.6500e-04
ISEM	11.3360	11.4866	11.3532	10.8749	10.7932	10.7932	1.1539e-05
NAKM	11.3361	11.4868	11.3533	10.8748	10.7930	10.7930	1.0788e-05
SEM	11.3361	11.4868	11.3533	10.8748	10.7930	10.7930	9.1780e-06
NISEM	11.3361	11.4868	11.3533	10.8748	10.7930	10.7930	9.9631e-06
SENMS	11.3361	11.4868	11.3533	10.8748	10.7930	10.7930	8.6399e-06

Table 3: Results of seven algorithms with $x^0 = x^b$ in Example 6.1

	$x^0 = x^a$			$x^0 = x^b$			$x^0 = x^c$		
	`Times`	`Iter.`	`Error`	`Times`	`Iter.`	`Error`	`Times`	`Iter.`	`Error`
Alg. AISEM	50.1593	434	1.7823e-05	50.0231	641	1.6434e-05	19.2622	148	9.6747e-06
Alg. EPSM	50.0414	423	1.8360e-04	39.5654	439	1.6500e-04	49.1020	365	1.8249e-04
Alg. ISEM	48.6476	565	9.8638e-06	50.0476	557	1.1539e-05	50.2284	367	9.3285e-04
Alg. NAKM	50.0074	432	1.1672e-05	50.0277	558	1.0788e-05	49.4926	388	9.6446e-06
Alg. SEM	17.7749	111	9.1414e-06	25.7378	112	9.1780e-06	19.2080	111	8.7619e-06
Alg. NISEM	10.3225	84	9.8823e-06	9.6250	83	9.9631e-06	12.7461	85	9.6491e-06
Alg. NSEM	10.3658	55	8.5917e-06	9.5109	53	8.6399e-06	10.3028	55	8.7515e-06

Table 4: Comparision of the algorithms in Example 6.1

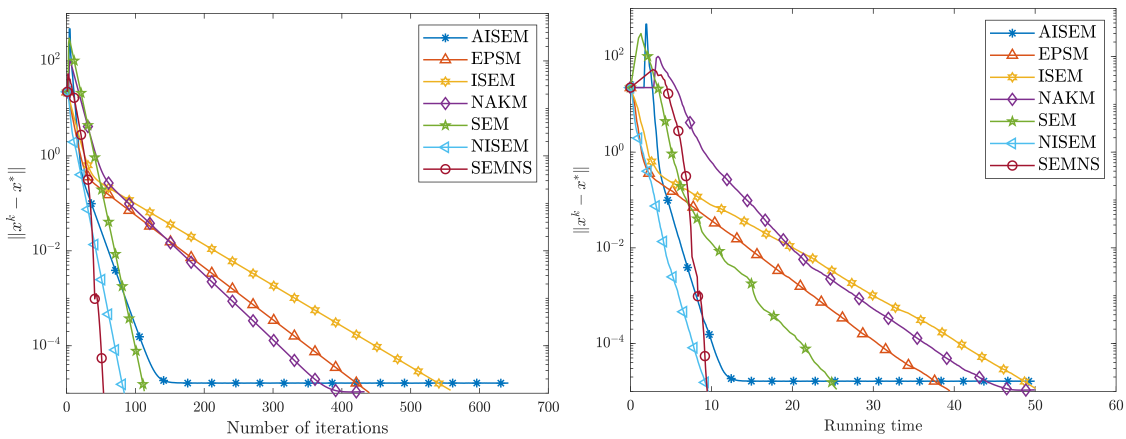

Fig. 1: Comparison of NISEM and SEMNS with some existing algorithms with $x^0 = x^b$

Test 2. In the next numerical experiment, we change the value of the α parameter for Algorithm 3.2 and Algorithm 2.1 in [40]. The results are shown in the Table 5 and Figure

	$x^0 = (1,1,1,1,1,1)^T$			$x^0 = (1,0,1,0,0,0)$		
	Times	Iter.	Error	Times	Iter.	Error
Alg. ISEM: $\alpha = 0.6, \tau = 0.1428$	50.0424	516	0.0168	50.0542	514	0.0238
Alg. ISEM: $\alpha = 0.3, \tau = 0.5568$	50.0751	506	2.0252e-04	50.0035	552	1.3191e-04
Alg. ISEM: $\alpha = 0.2, \tau = 0.7272$	50.0828	594	1.2666e-05	50.0720	570	2.6494e-05
Alg. ISEM: $\alpha = 0.1, \tau = 0.8800$	48.2636	565	9.8638e-06	50.0574	557	1.0657e-05
Alg. NISEM: $\tau = 1, \theta = 0.6, \alpha = 0.1428$	31.2201	235	9.7433e-06	38.8243	237	9.7250e-06
Alg. NISEM: $\tau = 2, \theta = 0.2, \alpha = 0.7800$	10.6962	86	9.7387e-06	12.0562	87	9.6570e-06
Alg. NISEM: $\tau = 3, \theta = 0.05, \alpha = 0.9669$	12.2118	84	9.5251e-06	10.5570	85	9.3631e-06
Alg. NISEM: $\tau = 4, \theta = 0.05, \alpha = 0.9709$	9.6315	84	9.0968e-06	8.4241	84	9.8708e-06

Table 5: Numerical results of NISEM and ISEM algorithms with different values of the parameter α and $x^0 = (1,0,1,0,0,0)^T$

3. It is seen that changing the value of α significantly impacts the computational cost of Algorithm 2.1 in [40].

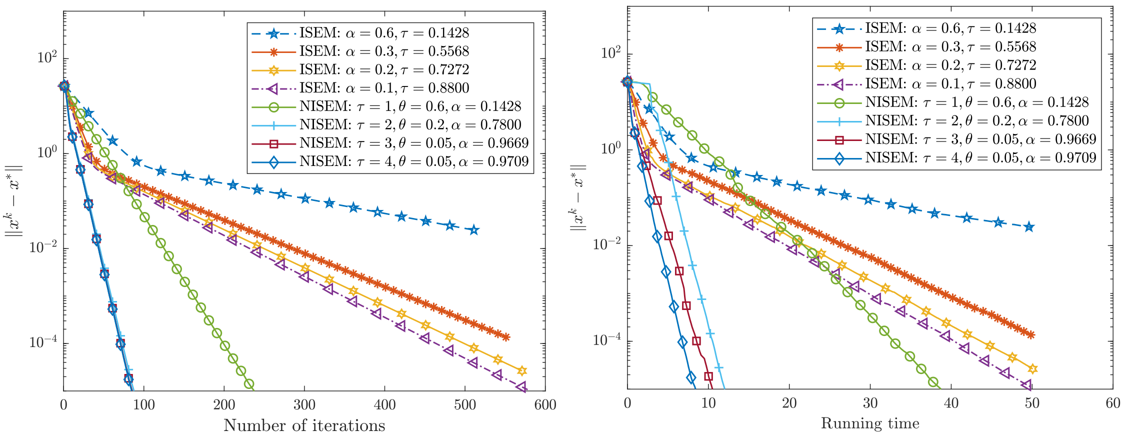

Fig. 2: Numerical results of NISEM and ISEM algorithms with different values of the parameter α

Example 6.2 In this example, we will study the convergence of Algorithm 4.1 (VSEM) and give a comparison with five algorithms, namely EMIEgA considered in [27, Algorithm 3], MIEM showed in [30, Algorithm 1], IEM suggested in [22, Algorithm 3.1], ISEA presented in [38, Algorithm 3.2], and SEM recommended in [46, Algorithm 3.1].

Let $\mathcal{H} := L^2([0,1])$, the Hilbert space of square-integrable real-valued functions on $[0,1]$, equipped with the inner product

$$\langle x, y \rangle := \int_0^1 x(t)y(t)dt \quad x, y \in \mathcal{H},$$

and its induced norm $\|\cdot\|$. Consider problem (1) with $C := \{x \in \mathcal{H} : \|x\| \leq 1\}$ and the bifunctions

$$f : C \times C \to \mathbb{R}, \quad f(x,y) := \left\langle \left(\frac{3}{2} - \|x\|\right)x, y - x \right\rangle + \|y\|^4 - \|x\|^4, \quad \forall x, y \in C.$$

It is easy to see that these bifunctions satisfy all the conditions to ensure the convergence of algorithms. Furthermore, the orthogonal projection onto C has an explicit formula, for example

$$P_C(x(t)) = \begin{cases} x(t), & \text{if } \|x(t)\| \le 1, \\ \frac{x(t)}{\|x(t)\|}, & \text{if } \|x(t)\| > 1. \end{cases}$$

In this experiment, we consider three cases for the starting point

$$x^0 = \frac{1}{100}e^{2t}\sin 3t, \qquad x^0 = \frac{1}{255}t^4e^{-t}, \qquad x^0 = t^2.$$

The parameters are chosen as follows:

In Alg. EMIEgA, $\varkappa = 0.1$, $\zeta_k = \frac{1}{10(k+2)}$, $\epsilon_k = \frac{\log k}{(k+1)^{1.05}}$ $\forall k \ge 1$, $u_{-1} = u_0 = x^0$.

In Alg. MIEM, $\lambda_1 = \tau = \sigma = \eta = 0.6$, $\mu = 0.4$, $\gamma_k = 1 - \frac{1}{k+2}$, $\alpha_k = 0.01 + \frac{1}{k+1}$, $\epsilon_k = \frac{1}{(k+1)^2}$, $\xi_k = 1 + \frac{1}{(k+1)^{1.1}}$, $\rho_k = \frac{1}{(k+1)^{1.1}}$, $\beta_k = \frac{1}{k+1}$ $\forall k \ge 1$, $x_0 = x_1 = x^0$.

In Alg. IEM, $\lambda_1 = 5000$,$\rho = 0.003$, $\mu = 0.9$, $\tau_k = \frac{1}{(k+1)^2}$ $\forall k \ge 1$, $u_0 = u_1 = x^0$.

In Alg. ISEA, $\lambda_1 = 0.1$, $\mu = 0.9$, $\tau_k = \frac{\log k}{(k+1)^{1.05}}$, $\theta_k = 0.01$, $\beta_k = \frac{1}{100(k+5)}$ $\forall k \ge 1$, $x_0 = x_1 = x^0$.

In Alg. SEM, $\lambda_0 = 0.5$, $\mu = 0.9$, $\beta_k = 0.01$, $\alpha_k = \frac{1}{100(k+5)}$ $\forall k \ge 1$, $S = I$, $x_0 = x^0$.

In Alg. VSEM, $\varphi(x) = \frac{1}{8}x$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\mu_1 = 0.15$, $\epsilon_k = \frac{\log k}{(k+1)^{1.05}}$, $\xi_k = \frac{1}{k^{1.05}}$, $\alpha_k = \frac{1}{k^{0.1}}$ $\forall k \ge 1$, $x^1 = x^0$.

We use the estimate $\|x^k - x^*\| \le 10^{-3}$ or when the CPU time exceeds $20s$ as the stopping rule for all algorithms. Numerical results of three algorithms are presented in Table 6 and Figure 3.

	$x^0 = \frac{1}{100}e^{2t}\sin 3t$			$x^0 = \frac{1}{255}t^4e^{-t}$			$x^0 = t^2$		
	Times	Iter.	Error	Times	Iter.	Error	Times	Iter.	Error
EMIEgA	20.0072	164	0.0057	20.0470	163	0.0057	20.0322	162	0.0057
MIEM	20.0255	190	0.0041	20.0565	180	0.0040	20.0448	185	0.0039
IEM	20.0417	131	0.0022	20.1158	127	0.0022	20.0506	129	0.0022
ISEA	20.0860	139	0.1531	20.0712	136	0.0103	20.0297	133	1.6774
SEM	20.0356	141	0.0013	20.0486	144	0.0013	20.0296	133	0.0013
VSEM	2.5160	3	$1.9719e-04$	1.8715	2	$1.1580e-04$	2.1409	5	$4.9019e-04$

Table 6: Comparison of the algorithms of Example 6.2

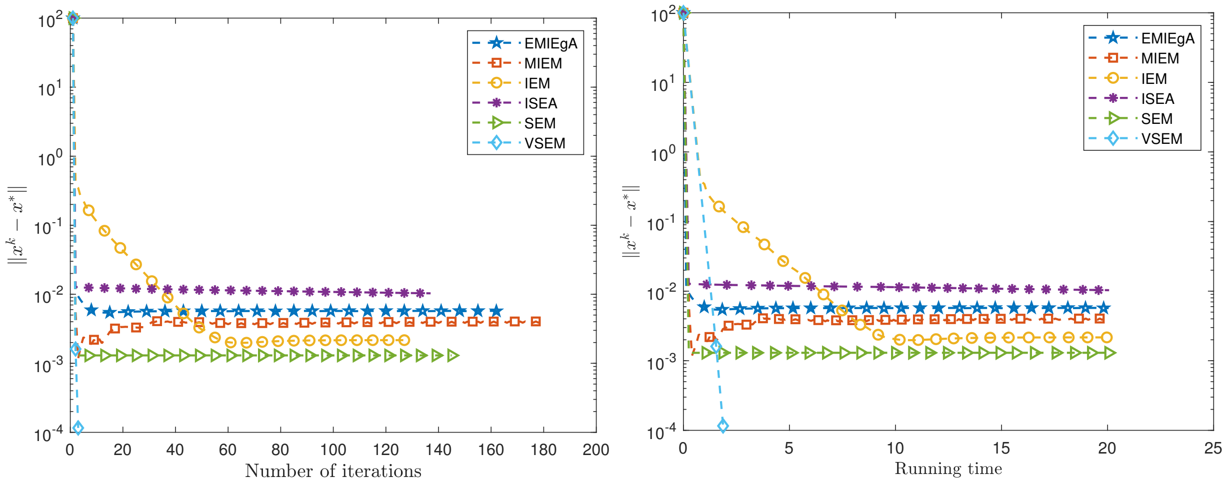

Fig. 3: Comparison of VSEM with some existing algorithms with $x^0 = \frac{1}{255}t^4e^{-t}$

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