NEW RESULTS RELATED TO CUTTERS AND TO AN EXTRAPOLATED BLOCK-ITERATIVE METHOD FOR FINDING A COMMON FIXED POINT OF A COLLECTION OF THEM

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ABSTRACT. Given a Hilbert space and a finite family of operators defined on the space, the common fixed point problem (CFPP) is to find a point in the intersection of the fixed point sets of these operators. Instances of the problem have numerous applications in science and engineering. We consider an extrapolated block-iterative method with dynamic weights for solving the CFPP assuming the operators belong to a wide class of operators called cutters. Global convergence is proved in two different scenarios, one of them is under a seemingly new condition on the weights which is less restrictive than a condition suggested in previous works. In order to establish convergence, we derive various new results of independent interest related to cutters, some of them extend, generalize and clarify previously published results.

1. INTRODUCTION

1.1. **Background:** Given a space X and a finite family of operators $T_i : X \to X$, $i \in I := \{1, 2, \ldots, m\}$, $m \in \mathbb{N}$, the common fixed point problem (CFPP) is the problem of finding a point in the intersection of the fixed point sets of these operators, assuming that the intersection is nonempty. In other words, the CFPP is the following problem:

$$(1.1) \qquad \text{Find } x \in \bigcap_{i \in I} F_i,$$

where $F_i := \text{Fix}(T_i) := \{x \in X \,|\, T_i(x) = x\}$ for every $i \in I$. Various methods have been devised to solve the CFPP, under certain assumptions, such as the ones described in [7, 15, 22, 23, 47, 52] and in some of the references therein; see also the last paragraph of this subsection (for the case $m = 1$, namely when one wants to find, or approximate, a fixed point of a given mapping $T_1 : X \to X$ in various settings, or to consider related aspects such as existence or uniqueness, see, for instance, [10, 33, 37, 48, 50] and the references therein). A particular instance of this problem, when X is a Hilbert space and each operator T_i is the orthogonal projection onto a closed and convex subset C_i, $i \in I$, is the so-called convex feasibility problem (CFP), which has found theoretical and real-world applications in image reconstruction, radiation therapy treatment planning, signal processing, and more: see, for instance, [15, p. 23]

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for a long and not exhaustive list of such applications and [**1**, **4**, **11**, **15**, **21**, **25**, **26**] for a few methods for solving the CFP.

In [**22**] we considered the CFPP under the assumptions that X was a Euclidean space (namely a finite-dimensional real Hilbert space) and each operator T_i, $i \in I$ was a continuous cutter. The class of cutters was introduced (with a different terminology: see the lines after (**2.13**) below) by Bauschke and Combettes in [**6**] and by Combettes in [**29**]. This is a rather wide class which includes, in particular, orthogonal projections, firmly nonexpansive operators, subgradient projections of continuous convex functions having nonempty zero-level-sets, and resolvents of maximally monotone operators [**6**, Proposition 2.3]. See also [**5**, **7**, **9**, **15**, **17**, **22**, **24**, **36**, **42**, **49**, **51**] and Sections **2**–**3** below for more details, examples, results, variants and generalizations related to cutters. The algorithmic scheme that we used in [**22**, Algorithm 1] was inspired by the BIP (Block-Iterative Projections) method of Aharoni and Censor [**1**]. In [**22**] we generalized the BIP method from orthogonal projections to continuous cutters, allowed dynamic weights and certain adaptive perturbations, and still concluded that the whole algorithmic sequence converges globally to a common fixed point of the given cutters (see [**15**, Theorem 5.8.15], [**17**, Theorem 9.27], [**39**, Theorems 4.1 and 4.5], [**44**, Theorem 3.2], and [**47**, Theorems 3.1 and 3.2] for related, but somewhat different, results).

In this work we continue the study done in [**22**] by weakening one of the conditions made there about the relaxation parameters, namely we allow them to belong to a wider interval than the typically used interval $[\tau_1, 2 - \tau_2]$, where $\tau_1, \tau_2 \in (0, 1]$. In other words, we allow extrapolation. The appearance of extrapolation, while intended to accelerate the convergence of the algorithmic sequence by allowing deeper steps, introduces complications which do not appear without extrapolation, and it forces us to make a more restrictive assumption than in [**22**] on either the dynamic weights or on the common fixed point set in order to ensure global convergence. More precisely, we either require the weights to satisfy a certain intermittent condition (Condition **4.5** below) instead of imposing the condition $\sum_{k=0}^{\infty} w_k(i) = \infty$ for each $i \in I$ (Condition **4.4** below) as in [**22**] (following [**1**]), or we require the common fixed point set to have a nonempty interior (and then we allow Condition **4.4**).

We note that several works, apparently starting from [**41**], study extrapolation in the context of the CFPP in various settings (possibly in the particular case of the CFP, possibly in the linear case): see, for instance, [**2**, **3**, **8**, **13**, **14**, **15**, **18**, **19**, **26**, **27**, **29**, **30**, **31**, **32**, **38**, **41**, **43**, **45**, **46**] for a partial list of such works. However, as far as we know, only [**3**, **5**, **8**, **13**, **29**] study extrapolated algorithms in the spirit of our algorithm in the context of the CFPP with dynamic weights and general cutters (see also [**14**, Section 3], [**28**, Section 6.5], and [**30**, Section 3] for closely related by somewhat different extrapolation algorithms with dynamic weights and cutters), and there the condition on the weights is either less general than Condition **4.5** below or is not implied by it and does not imply it, as we explain in Remark **4.6** and Examples **4.7**–**4.8** below. Hence, our convergence theorems are not implied by the convergence results mentioned in [**3**, **5**, **8**, **13**, **29**] (and elsewhere). It should be noted, however, that with the exception of the assumptions on the weights, the settings in

[**3, 5, 8, 13, 29**] are more general than the setting that we consider: for instance, the space there might be infinite-dimensional, the cutters need not be continuous (but rather weakly regular, namely demi-closed at 0), in some of these works appear either strings of cutters (while we consider strings of length 1) or perturbations, and the algorithmic schemes have more general forms than the form of Algorithm **4.1** below.

We wish to end this subsection by commenting about potential computational advantages of our work. Our paper is a theoretical study which does not attempt to show that the method that we discuss is better than other methods (extrapolated or not) from the computational point of view, although it might be the case. In order to prove or disprove this latter possibility, or at least to understand it in a much better manner, one should make exhaustive tests of the many possible specific variants and user-chosen parameters permitted by the general scheme that we discuss, such as the relaxation parameters, the operators and the weights, and, preferably, to include significant real-world applications rather than simple and not-very-convincing demonstrative examples that do not allow to methodologically explore all possibilities. The additional option for extrapolation adds yet another layer of mathematical generality with possible numerical ramifications. All in all, the above-mentioned computational task is beyond our current abilities and beyond the scope of this paper, but we express our hope that the theoretical results that we presented will be tested numerically in the future in one way or another.

1.2. **Contribution:** The contribution of our work is two fold. First, we obtain new results related to cutters such as Corollary **3.9**, Lemma **3.11** and Propositions **3.4** and **3.12** below, some of them extend and generalize previously published results, and we fill a certain gap which appears in [**46**] (see Remark **3.6** below); these results hold in general real Hilbert spaces and without the continuity assumption on the cutters. Second, we use these results in order to obtain finite-dimensional convergence results (Theorems **4.15** and **4.16** below) related to the extrapolated block iterative method (Algorithm **4.1** below) under conditions on the weights which have not been considered yet in the context of extrapolated algorithms for solving the CFPP induced by cutters.

1.3. **Paper layout:** In Section **2** we introduce some preliminary details. In Section **3** we present several results related to cutters. The extrapolated algorithmic scheme is introduced in Section **4**, where we also present conditions on the weights, compare these conditions with previously published conditions, and present the convergence theorems. The paper is concluded in Section **5**.

2. Preliminaries

We use in the sequel the following notations and definitions. Unless otherwise stated, the ambient space is a real Hilbert space X with an inner product $\langle \cdot, \cdot \rangle$ and an induced norm $\| \cdot \|$. Given $\rho > 0$ and $x \in X$, we denote the closed ball with radius ρ centered at x by $B[x, \rho] := \{y \in X \mid \|x - y\| \leq \rho\}$. Given a natural number m, define the index set I as $I := \{1, 2, \ldots, m\}$. A function $w : I \to [0, 1]$ which satisfies $\sum_{i \in I} w(i) = 1$ is called a *weight function*, and if, in addition, $w(i) > 0$ for all $i \in I$,

then w is called a *positive weight function*. We refer to the vector $(w(i))_{i \in I} \in \mathbb{R}^m$ as a *weight vector*, and when w is a positive weight function, then we refer to $(w(i))_{i \in I}$ as a *positive weight vector*. Denote $\widehat{I}_w := \{i \in I \mid w(i) > 0\}$ and let $\widehat{w} : \widehat{I}_w \to (0, 1]$ be the restriction of w to $\widehat{I}_w$, that is, $\widehat{w}(i) := w(i) > 0$ for all $i \in \widehat{I}_w$. Note that $\widehat{I}_w \neq \emptyset$ (since $\sum_{i \in I} w(i) = 1$), that $\widehat{w}$ is a positive weight function defined on $\widehat{I}_w$, and the case $\widehat{w}(i) = 1$ for some $i \in \widehat{I}$ is possible if and only if $\widehat{I}$ is a singleton.

For a given weight function w and given operators $T_i : X \to X$, $i \in I$, we denote for all $x \in X$

$$(2.1) \qquad T_w(x) := \sum_{i \in I} w(i) T_i(x),$$

and define

$$(2.2) \qquad L(x, w) := \begin{cases} \dfrac{\sum_{i \in I} w(i) \|T_i(x) - x\|^2}{\|T_w(x) - x\|^2}, & \text{if } x \neq T_w(x), \\ 1, & \text{otherwise.} \end{cases}$$

Note that $L(x, w) \geq 1$ because of its definition and the convexity of the square of the norm. For each $\lambda \in \mathbb{R}$ (sometimes referred to as the *relaxation/acceleration parameter* whenever it is positive) and each $x \in X$, let

$$(2.3) \qquad T_{w,\lambda}(x) := x + \lambda(T_w(x) - x) = x + \lambda \sum_{i \in I} w(i)(T_i(x) - x).$$

We observe that $T_w = T_{w,1}$ and that $T_{w,\lambda}$ can be written in a block form using $\widehat{I}_w$ and $\widehat{w}$, namely $T_{w,\lambda}(x) = x + \lambda \sum_{i \in \widehat{I}_w} \widehat{w}(i)(T_i(x) - x) = T_{\widehat{w},\lambda}(x)$ for all $x \in X$. In particular, $T_w = T_{\widehat{w}}$.

Given the index set I, suppose that w is a positive weight function. Adopting Pierra's product space formalism [46], we define the product space $\mathbf{X} := X^m$. Its elements are $\mathbf{x} := (x^1, x^2, \ldots, x^m)$, where $x^i \in X$ for all $i \in I$, and the inner product in $\mathbf{X}$ is defined for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ as

$$(2.4) \qquad \langle\langle \mathbf{x}, \ \mathbf{y} \rangle\rangle_w := \sum_{i=1}^{m} w(i) \langle x^i, \ y^i \rangle.$$

The norm, and the distance function in the product space are denoted by $|||\cdot|||_w$ and $d_w((\cdot, \cdot))$, respectively. Observe our notational rule to use bold-face upright letters for quantities in the product space. The *diagonal set* $\mathbf{D}$ in $\mathbf{X}$ is

$$(2.5) \qquad \mathbf{D} := \{\mathbf{x} \in \mathbf{X} \mid x \in X, \ \mathbf{x} = (x, x, \ldots, x)\}.$$

It is easy to see that $\mathbf{D}$ is a closed linear subspace of $\mathbf{X}$. The canonical mapping $\mathbf{J} : X \to \mathbf{D}$, is defined by $\mathbf{J}(x) := (x, x, \ldots, x)$ for every $x \in X$.

Let $T_1, T_2, \ldots, T_m$ be operators from X to itself and let $F_1, F_2, \ldots, F_m$ be their fixed point sets, respectively, that is, $F_i := \text{Fix}(F_i) := \{x \in X \mid T_i(x) = x\}$. We define $\mathbf{T} : \mathbf{X} \to \mathbf{X}$ by

$$(2.6) \qquad \mathbf{T}(\mathbf{x}) := \left(T_1(x^1), T_2(x^2), \ldots, T_m(x^m) \right).$$

It is immediate to verify that the fixed point set $\mathbf{F} := \mathrm{Fix}(\mathbf{T})$ of the operator $\mathbf{T}$ satisfies the following relation:

$$(2.7) \qquad\qquad \mathbf{F} = F_1 \times F_2 \times \cdots \times F_m.$$

In addition, the following equivalence holds for each $x \in X$:

$$(2.8) \qquad\qquad \mathbf{J}(x) \in \mathbf{F} \cap \mathbf{D} \Leftrightarrow x \in F := \bigcap_{i \in I} F_i.$$

This means that $(\mathbf{1.1})$ is equivalent to the following problem:

$$(2.9) \qquad\qquad \text{Find a point } \mathbf{x} \text{ in } \mathbf{F} \cap \mathbf{D} \subseteq \mathbf{X}.$$

For any pair $x, y \in X$, define the set

$$(2.10) \qquad\qquad H(x, y) := \{u \in X \mid \langle x - y, u - y \rangle \leq 0\}$$

which is a half-space unless $x = y$ (and then it is the whole space). Let $T : X \to X$ be an operator. The operator T is called a *cutter* if it satisfies the condition

$$(2.11) \qquad\qquad \emptyset \neq \mathrm{Fix}(T) \subseteq H(x, T(x)), \text{ for all } x \in X,$$

or, equivalently, $\mathrm{Fix}(T) \neq \emptyset$ and

$$(2.12) \qquad \text{for all } q \in \mathrm{Fix}(T) \text{ and all } x \in X \text{ one has } \langle x - T(x), q - T(x) \rangle \leq 0.$$

The set of all fixed points of a cutter is closed and convex because

$$(2.13) \qquad\qquad \mathrm{Fix}(T) = \bigcap_{x \in X} H(x, T(x)),$$

as shown in [**6**, Proposition 2.6(ii)]. As said in Section **1**, the class of cutters was introduced in [**6, 29**], with a different terminology ("class $\mathfrak{T}$"; the name "cutter" was suggested in [**17**]; other names are used in the literature for these operators, such as "firmly quasinonexpansive" [**7**, Definition 4.1(iv) and Proposition 4.2(iv), pp. 69–70], [**30**, Definition 3.5] and "directed operators" [**24**], [**51**]), and there various properties and examples of cutters can be found. If $T_i : X \to X$, $i \in I$ are cutters, then their fixed point sets are nonempty and hence $\mathbf{F} \neq \emptyset$ according to $(\mathbf{2.7})$. This fact, as well as the inequality

$$(2.14) \qquad \langle\langle \mathbf{z} - \mathbf{T}(\mathbf{z}), \mathbf{q} - \mathbf{T}(\mathbf{z}) \rangle\rangle_w = \sum_{i=1}^{m} w(i) \langle z^i - T_i(z^i), q^i - T_i(z^i) \rangle \leq 0,$$

for all $\mathbf{z} \in \mathbf{X}$ and $\mathbf{q} \in \mathbf{F}$, imply that the operator $\mathbf{T}$, defined by $(\mathbf{2.6})$, is a cutter in the product space $\mathbf{X}$.

3. Results related to cutters

In this section we present various results related to cutters. We will use these results in Section **4**. We continue with the notation introduced in Section **2**, and recall that X is a real Hilbert space.

The following lemma is essentially due to Pierra [**46**, Lemma 1.1]. For the sake of completeness, and since both the setting and the formulation of [**46**, Lemma 1.1] are somewhat different from our ones, we present the proof of the lemma.

Lemma 3.1. *Suppose that $\{Q_i\}_{i=1}^m$ are nonempty, closed and convex subsets of X, and let $\mathbf{Q} := Q_1 \times Q_2 \times \cdots \times Q_m$. Then the orthogonal projections onto $\mathbf{Q}$ and onto $\mathbf{D}$, respectively, in the product space $\mathbf{X}$, satisfy, for any $\mathbf{x} := (x^1, x^2, \ldots, x^m) \in \mathbf{X}$, the following two relations:*

$$(3.1) \qquad \mathbf{P_Q}(\mathbf{x}) = \left(P_{Q_1}(x^1), P_{Q_2}(x^2), \ldots, P_{Q_m}(x^m) \right),$$

$$(3.2) \qquad\qquad \mathbf{P_D}(\mathbf{x}) = \mathbf{J}\left(\sum_{i=1}^m w(i) x^i \right).$$

Proof. We start by proving **(3.1)**. Since $f_1(\min\{f_2(\mathbf{z}) \mid \mathbf{z} \in \mathbf{Q}\}) = \min\{f_1(f_2(\mathbf{z})) \mid \mathbf{z} \in \mathbf{Q}\}$ for $f_1(t) := t^2$ and $f_2(\mathbf{z}) := \|\mathbf{x} - \mathbf{z}\|$ for all $t \in [0, \infty)$ and $\mathbf{z} \in \mathbf{X}$, and since $\min\{\sum_{i \in I} a_i \mid a_i \in A_i, i \in I\} = \sum_{i \in I} \min A_i$ for all subsets A_i, $i \in I$ of real numbers such that each one of them has a minimum, we have, using the definition of the orthogonal projection, that

(3.3)

$$\||\mathbf{x} - \mathbf{P_Q}(\mathbf{x})|\|_w^2 = (\min\{\||\mathbf{x} - \mathbf{z}|\|_w \mid \mathbf{z} \in \mathbf{Q}\})^2 = \min\{\||\mathbf{x} - \mathbf{z}|\|_w^2 \mid \mathbf{z} \in \mathbf{Q}\}$$

$$= \min\left\{ \sum_{i \in I} w(i)\|x^i - z^i\|^2 \mid z^i \in Q_i \ \forall i \in I \right\} = \sum_{i \in I} \min\{w(i)\|x^i - z^i\|^2 \mid z^i \in Q_i\}$$

$$= \sum_{i \in I} w(i) \min\{\|x^i - z^i\|^2 \mid z^i \in Q_i\} = \sum_{i \in I} w(i)\|x^i - P_{Q_i}(x^i)\|^2$$

$$= \||(x^i - P_{Q_i}(x^i))_{i \in I}|\|_w^2.$$

Since, as is well known [**7**, Theorem 3.16], the minimum in the definition of the orthogonal projection is attained at a unique point, this point is $\mathbf{z} := (P_{Q_i}(x^i))_{i \in I}$ (and obviously $\mathbf{z} \in \mathbf{Q}$). Hence $\mathbf{P_Q}(\mathbf{x}) = (P_{Q_i}(x^i))_{i \in I}$, namely **(3.1)** holds.

Now we prove **(3.2)**. Any point $\mathbf{z} \in \mathbf{D}$ satisfies $\mathbf{z} = \mathbf{J}(z)$ for some $z \in X$, and so

$$(3.4) \quad \||\mathbf{x} - \mathbf{P_D}(\mathbf{x})|\|_w^2 = \min\{\||\mathbf{x} - \mathbf{z}|\|_w^2 : \mathbf{z} \in \mathbf{D}\} = \min\{\||\mathbf{x} - \mathbf{J}(z)|\|_w^2 : z \in X\}$$

$$= \min\left\{ \sum_{i \in I} w(i)\|x^i - z\|^2 : z \in X \right\}.$$

The minimum in the last expression is attained at the unique point $\widetilde{z}$ where the gradient of the function $g : X \to \mathbb{R}$, defined by $g(z) := \sum_{i \in I} w(i)\|x^i - z\|^2$, $z \in X$, vanishes. Since $\nabla g(z) = 2\sum_{i \in I} w(i)(x^i - z)$ for each $z \in X$ and since $\sum_{i \in I} w(i) = 1$, we conclude that $\widetilde{z} = \sum_{i \in I} w(i) x^i$. Hence **(3.4)** implies that $\mathbf{P_D}(\mathbf{x}) = \mathbf{J}(\widetilde{z}) = \mathbf{J}\left(\sum_{i \in I} w(i) x^i \right)$, as claimed. $\qquad\square$

We continue with the following definition and two propositions.

Definition 3.2. *Let S_1 and S_2 be two subsets of a real Hilbert space Y. A hyperplane Ψ is said to **strictly separate** S_1 **and** S_2 if S_1 is contained in the interior of one half-space induced by Ψ, and S_2 is contained in the interior of the opposite half-space.*

Proposition 3.3. *Let T_1 and T_2 be two cutters defined on a real Hilbert space Y. If there exists some $x \in \mathrm{Fix}(T_1 T_2) \backslash \mathrm{Fix}(T_2)$, then $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) = \emptyset$.*

Proof. Since $x \notin \mathrm{Fix}(T_2)$, both $H(T_2(x), x)$ and $H(x, T_2(x))$ are half-spaces and

$$(3.5) \qquad \Psi := \left\{ u \in Y \mid \left\langle u - \frac{1}{2}(T_2(x) + x), \; T_2(x) - x \right\rangle = 0 \right\}$$

is a hyperplane. We claim that Ψ strictly separates $H(T_2(x), x)$ and $H(x, T_2(x))$. Indeed, if, say, $u \in H(T_2(x), x)$, then $\langle u - x, T_2(x) - x \rangle \leq 0$ (see **(2.10)**), and because $T_2(x) \neq x$, we have

$$\left\langle u - \frac{1}{2}(T_2(x) + x), T_2(x) - x \right\rangle = \frac{1}{2} \langle u - T_2(x), T_2(x) - x \rangle + \frac{1}{2} \langle u - x, T_2(x) - x \rangle$$

$$= \frac{1}{2} \langle u - x, T_2(x) - x \rangle + \frac{1}{2} \langle x - T_2(x), T_2(x) - x \rangle + \frac{1}{2} \langle u - x, T_2(x) - x \rangle$$

$$= \langle u - x, T_2(x) - x \rangle - \frac{1}{2} \|x - T_2(x)\|^2 < 0 + 0 = 0.$$

Hence, u is in the interior of one of the half-spaces whose boundary is Ψ, that is, in the interior of $\{ v \in Y \mid \langle v - 0.5(T_2(x) + x), \; T_2(x) - x \rangle \leq 0 \}$. Similarly, if $u \in H(x, T_2(x))$, then u is in the interior of the other half-space whose boundary is Ψ. Since T_1 and T_2 are cutters and since $T_1 T_2(x) = x$, by denoting $z := T_2(x)$ we obtain the following inclusions:

$$\mathrm{Fix}(T_1) \subseteq H(z, T_1(z)) = H(T_2(x), T_1 T_2(x)) = H(T_2(x), x),$$

$$\mathrm{Fix}(T_2) \subseteq H(x, T_2(x)).$$

Since we already know that the hyperplane Ψ strictly separates $H(x, T_2(x))$ and $H(T_2(x), x)$, we conclude that Ψ strictly separates $\mathrm{Fix}(T_1)$ and $\mathrm{Fix}(T_2)$. Thus, one has $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) = \emptyset$. $\qquad \square$

Proposition 3.4. *Let $\{T_i\}_{i \in I}$ be cutters and $\{F_i\}_{i \in I}$ be their fixed points sets, respectively, and let $w : I \to (0, 1]$ be a positive weight function. If $\cap_{i \in I} F_i \neq \emptyset$ and $x \in X$ satisfies $x \notin \cap_{i \in I} F_i$, then $T_w(x) \neq x$ and $\mathbf{P_D}(\mathbf{T}(\mathbf{x})) \neq \mathbf{x}$, where $\mathbf{x} := \mathbf{J}(x)$.*

Proof. Let $\mathbf{D}$ and $\mathbf{T}$ be as defined above in **(2.5)** and **(2.6)**, respectively, and let $\mathbf{F} := \mathrm{Fix}(\mathbf{T})$. By **(2.8)** and the obvious inclusion $\mathbf{x} = \mathbf{J}(x) \in \mathbf{D}$, we have $\mathbf{x} \notin \mathbf{F}$. Assume, for a contradiction, that $T_w(x) = x$. This equality implies that $\mathbf{J}(T_w(x)) = \mathbf{J}(x) = \mathbf{x}$. From this equality, **(3.2)** and the equality $\mathbf{T}(\mathbf{x}) = (T_i(x))_{i=1}^{m}$, we obtain $\mathbf{P_D}(\mathbf{T}(\mathbf{x})) = \mathbf{J}(T_w(x)) = \mathbf{J}(x) = \mathbf{x}$. The previous lines imply that we can apply Proposition **3.3** with $\mathbf{T}$ as T_2 there and $\mathbf{P_D}$ as T_1 there. This proposition yields $\mathrm{Fix}(\mathbf{P_D}) \cap \mathbf{F} = \emptyset$, namely $\mathbf{D} \cap \mathbf{F} = \emptyset$. This result, together with **(2.8)**, give $\cap_{i \in I} F_i = \emptyset$, a contradiction to the assumption that $\cap_{i \in I} F_i \neq \emptyset$. Therefore, $T_w(x) \neq x$.

It remains to show that $\mathbf{P_D}(\mathbf{T}(\mathbf{x})) \neq \mathbf{x}$. Indeed, since $\mathbf{T}(\mathbf{x}) = (T_i(x))_{i=1}^{m}$, it follows from **(2.6)** and **(3.2)** that $\mathbf{P_D}(\mathbf{T}(\mathbf{x})) = \mathbf{J}(T_w(x))$. As a result, if $\mathbf{P_D}(\mathbf{T}(\mathbf{x})) = \mathbf{x}$, then $\mathbf{J}(T_w(x)) = \mathbf{P_D}(\mathbf{T}(\mathbf{x})) = \mathbf{x} = \mathbf{J}(x)$, and so, from the definition of $\mathbf{J}$, one has $T_w(x) = x$, a contradiction to what we showed in the previous paragraph. Hence, $\mathbf{P_D}(\mathbf{T}(\mathbf{x})) \neq \mathbf{x}$. $\qquad \square$

The following corollary is essentially known (see, for instance, [**29**, Equation (19)]; see [**7**, Proposition 4.47, p. 85] for a more general result). Below we provide a new proof of it, based on Proposition **3.4** above.

Corollary 3.5. *For each $i \in I$ suppose that $T_i : X \to X$ is a cutter having a fixed point set F_i such that $\cap_{i \in I} F_i \neq \emptyset$, and suppose that $w : I \to [0,1]$ is a weight function. Then for all $x \in X$ one has $T_w(x) = x$ if and only if $T_i(x) = x$ for all $i \in \widehat{I}_w$. In other words, $\mathrm{Fix}(T_w) = \cap_{i \in \widehat{I}_w} \mathrm{Fix}(T_i)$.*

Proof. Fix an arbitrary $x \in X$. The triangle inequality, the definitions of $\widehat{I}_w$ and $\widehat{w}$, and the fact that w and $\widehat{w}$ are weight functions, all imply that

$$(3.6) \quad 0 \leq \left\| \sum_{i \in I} w(i) T_i(x) - x \right\| = \left\| \sum_{i \in \widehat{I}_w} \widehat{w}(i)(T_i(x) - x) \right\| \leq \sum_{i \in \widehat{I}_w} \widehat{w}(i) \|T_i(x) - x\|.$$

As a result of **(3.6)**, if $T_i(x) = x$ for all $i \in \widehat{I}_w$, then $\|\sum_{i \in I} w(i) T_i(x) - x\| = 0$ and hence, $T_w(x) = x$. As for the converse direction, suppose that $T_w(x) = x$, and assume, for a contradiction, that $T_i(x) \neq x$ for some $i \in \widehat{I}_w$. Then $x \notin \cap_{i \in \widehat{I}_w} F_i$. Thus, Proposition **3.4** (with $\widehat{w}$ and $\widehat{I}_w$ instead of w and I which appear there, respectively) implies that $x \neq T_{\widehat{w}}(x)$. Since $T_w = T_{\widehat{w}}$, it follows that $x \neq T_w(x)$, a contradiction to our assumption. Thus, $T_i(x) = x$ for all $i \in \widehat{I}_w$, as claimed. $\square$

Now suppose that $\cap_{i \in I} F_i \neq \emptyset$. We define $\mathbf{b}_w : \mathbf{D} \backslash \mathbf{F} \to \mathbf{D}$, by

$$(3.7) \qquad \mathbf{b}_w(\mathbf{x}) := \mathbf{x} + \frac{\||\mathbf{T}(\mathbf{x}) - \mathbf{x}\||_w^2}{\||\mathbf{P_D}(\mathbf{T}(\mathbf{x})) - \mathbf{x}\||_w^2} (\mathbf{P_D}(\mathbf{T}(\mathbf{x})) - \mathbf{x}), \quad \mathbf{x} \in \mathbf{D} \backslash \mathbf{F}$$

where w is the positive weight function which appears in **(2.4)**. By Proposition **3.4** we have $\mathbf{P_D}(\mathbf{T}(\mathbf{x})) \neq \mathbf{x}$ for all $\mathbf{x} \in \mathbf{D} \backslash \mathbf{F}$, a fact which guarantees that $\mathbf{b}_w(\mathbf{x})$ is well-defined.

Remark 3.6. Note that $\mathbf{b}_w$ (from **(3.7)**) generalizes the expression $\mathbf{b}$ which appears in [**46**, Lemma 1.2] with orthogonal projections instead of general cutters as we use here, and with constant weights ($w_k(i) := 1/m$ for all $k \in \mathbb{N} \cup \{0\}$ and $i \in I$) and not general ones as we use here. In addition, and in Proposition **3.4** above, we filled a gap which appears in [**46**, Lemma 1.2] (see also [**8**, Theorem 2.8(i) and Corollary 2.12(i)] and [**29**, Proof of Proposition 2.4] for closely related results): the gap there is the unproven assertion that, with the notation we use here, $\mathbf{P_D}(\mathbf{P_C}(\mathbf{x})) \neq \mathbf{x}$ for all $\mathbf{x} \in \mathbf{D} \backslash \mathbf{C}$, where $\mathbf{C} := C_1 \times \cdots \times C_m$ and C_i is a nonempty, closed and convex subset of X for all $i \in I$; while it is claimed in [**46**, Lemma 1.2] that this assertion was proved in [**46**, Theorem 1.1(i)], as far as we understand, this is not the case. The reason that we indeed filled this gap follows from the observation that if we denote $T_i := P_{C_i}$ for each $i \in I$ and consider the mapping $\mathbf{T}$ from **(2.6)**, then Lemma **3.1** (with $Q_i := C_i$, $i \in I$, and hence $\mathbf{Q} = \mathbf{C}$) guarantees that $\mathbf{T} = \mathbf{P_C}$, and since both $F_i := \mathrm{Fix}(T_i) = C_i$ for every $i \in I$ and $\mathbf{F} := \mathrm{Fix}(\mathbf{T}) = F_1 \times \cdots \times F_m = \mathbf{C}$ (as follows from **(2.7)** and the equality $F_i = C_i$ for every $i \in I$), we conclude from Proposition **3.4** that $\mathbf{P_D}(\mathbf{T}(\mathbf{x})) \neq \mathbf{x}$, namely $\mathbf{P_D}(\mathbf{P_C}(\mathbf{x})) \neq \mathbf{x}$, as required.

The next lemma presents useful observations about $\mathbf{b}_w(\mathbf{x})$. It generalizes and extends [**46**, Lemma 1.2].

Lemma 3.7. *Let $\{T_i\}_{i \in I}$ be cutters and $\{F_i\}_{i \in I}$ their fixed points sets, respectively, and suppose that $w : I \to (0, 1]$ is a positive weight function. Let $\mathbf{D}$ and $\mathbf{T}$ be as defined above in (**2.5**) and (**2.6**), respectively, and let $\mathbf{F} := \mathrm{Fix}(\mathbf{T})$. Assume that $x \in X$, $\mathbf{x} := \mathbf{J}(x) \in \mathbf{D} \backslash \mathbf{F}$ and $\cap_{i \in I} F_i \neq \emptyset$. Then the following assertions hold:*

(i) $\mathbf{b}_w(\mathbf{x})$ is in the intersection of the ray

(3.8) $$\mathbf{L_x} := \{\mathbf{z} \in \mathbf{D} \mid \mathbf{z} = \mathbf{x} + \lambda \left(\mathbf{P_D}\left(\mathbf{T}\left(\mathbf{x}\right)\right) - \mathbf{x}\right), \ \lambda \geq 0\}$$

with the hyperplane

(3.9) $$\mathbf{\Psi} := \{\mathbf{z} \in X^m \mid \langle\langle \mathbf{z} - \mathbf{T}\left(\mathbf{x}\right), \mathbf{T}\left(\mathbf{x}\right) - \mathbf{x}\rangle\rangle_w = 0\}.$$

(ii) If we denote

(3.10) $$\widehat{\lambda}_{x,w} := \frac{|||\mathbf{T}\left(\mathbf{x}\right) - \mathbf{x}|||_w^2}{|||\mathbf{P_D}\left(\mathbf{T}\left(\mathbf{x}\right)\right) - \mathbf{x}|||_w^2},$$

then $\widehat{\lambda}_{x,w} \geq 1$ and

(3.11) $$\widehat{\lambda}_{x,w} = \frac{\sum_{i \in I} w(i) \left\| T_i\left(x\right) - x\right\|^2}{\left\| \sum_{i \in I} w(i)(T_i(x) - x)\right\|^2}.$$

(iii) $\mathbf{D} \cap \mathbf{F} \subseteq \mathbf{D} \cap \mathbf{H}\left(\mathbf{x}, \mathbf{b}_w(\mathbf{x})\right) \subseteq \mathbf{D} \cap \mathbf{H}\left(\mathbf{x}, \mathbf{P_D}\left(\mathbf{T}\left(\mathbf{x}\right)\right)\right).$
(iv) The following inclusions hold:

(3.12) $$\bigcap_{i \in I} F_i \subseteq H\left(x, T_{w, \widehat{\lambda}_{x,w}}(x)\right) \subseteq H\left(x, T_w(x)\right).$$

Proof. We prove the assertions in the order in which they appeared.

Proof of Part (i): Because $\mathbf{P_D}$ is an orthogonal projection onto the linear subspace $\mathbf{D}$, it follows from a basic property of orthogonal projections (see [**7**, Corollary 3.22, p. 55], with $z := 0$) that

(3.13) $$\langle\langle \mathbf{y}, \mathbf{T}\left(\mathbf{x}\right) - \mathbf{P_D}\left(\mathbf{T}\left(\mathbf{x}\right)\right)\rangle\rangle_w = 0, \ \text{for all } \mathbf{y} \in \mathbf{D}.$$

By the definition of $\mathbf{b}_w(\mathbf{x})$ (see (**3.7**)) and simple arithmetic, we have

$$\langle\langle \mathbf{b}_w(\mathbf{x}) - \mathbf{T}\left(\mathbf{x}\right), \mathbf{T}\left(\mathbf{x}\right) - \mathbf{x}\rangle\rangle_w$$
$$= \frac{|||\mathbf{T}\left(\mathbf{x}\right) - \mathbf{x}|||_w^2}{|||\mathbf{P_D}\left(\mathbf{T}\left(\mathbf{x}\right)\right) - \mathbf{x}|||_w^2} \Big(\langle\langle \mathbf{P_D}\left(\mathbf{T}\left(\mathbf{x}\right)\right) - \mathbf{x}, \mathbf{T}\left(\mathbf{x}\right) - \mathbf{P_D}\left(\mathbf{T}\left(\mathbf{x}\right)\right)\rangle\rangle_w$$

(3.14) $$+ \langle\langle \mathbf{P_D}\left(\mathbf{T}\left(\mathbf{x}\right)\right) - \mathbf{x}, \mathbf{P_D}\left(\mathbf{T}\left(\mathbf{x}\right)\right) - \mathbf{x}\rangle\rangle_w \Big) + \langle\langle \mathbf{x} - \mathbf{T}\left(\mathbf{x}\right), \mathbf{T}\left(\mathbf{x}\right) - \mathbf{x}\rangle\rangle_w.$$

From this, along with (**3.13**) and the fact that $\mathbf{P_D}\left(\mathbf{T}\left(\mathbf{x}\right)\right) - \mathbf{x} \in \mathbf{D}$, we have

$$\langle\langle \mathbf{b}_w(\mathbf{x}) - \mathbf{T}\left(\mathbf{x}\right), \mathbf{T}\left(\mathbf{x}\right) - \mathbf{x}\rangle\rangle_w$$

(3.15) $$= \frac{|||\mathbf{T}\left(\mathbf{x}\right) - \mathbf{x}|||_w^2}{|||\mathbf{P_D}\left(\mathbf{T}\left(\mathbf{x}\right)\right) - \mathbf{x}|||_w^2} |||\mathbf{P_D}\left(\mathbf{T}\left(\mathbf{x}\right)\right) - \mathbf{x}|||_w^2 - |||\mathbf{T}\left(\mathbf{x}\right) - \mathbf{x}|||_w^2 = 0.$$

This shows that $\mathbf{b}_w(\mathbf{x}) \in \mathbf{\Psi}$. Since $\mathbf{x} \notin \mathbf{F}$, we have $\mathbf{T}(\mathbf{x}) \neq \mathbf{x}$ and $\widehat{\lambda}_{x,w} > 0$, and since $\mathbf{b}_w(\mathbf{x}) = \mathbf{x} + \widehat{\lambda}_{x,w}(\mathbf{P_D}\,(\mathbf{T}(\mathbf{x})) - \mathbf{x})$, we have $\mathbf{b}_w(\mathbf{x}) \in \mathbf{L_x}$. This completes the proof of Part **(i)**.

Proof of Part (ii): Proposition **3.4** implies that $\mathbf{P_D}\,(\mathbf{T}\,(\mathbf{x})) \neq \mathbf{x}$ for all $\mathbf{x} \in \mathbf{D} \backslash \mathbf{F}$ and so $\widehat{\lambda}_{x,w}$ is well defined. Since $\mathbf{x} \in \mathbf{D}$ and the orthogonal projection is nonexpansive (see, e.g., [**15**, Theorem 2.2.21, p. 76]), we have

$$|||\mathbf{P_D}(\mathbf{T}(\mathbf{x})) - \mathbf{x}|||_w = |||\mathbf{P_D}(\mathbf{T}(\mathbf{x})) - \mathbf{P_D}(\mathbf{x})|||_w \leq |||\mathbf{T}(\mathbf{x}) - \mathbf{x}|||_w.$$

Thus, $\sqrt{\widehat{\lambda}_{x,w}} = |||\mathbf{T}(\mathbf{x}) - \mathbf{x}|||_w / |||\mathbf{P_D}(\mathbf{T}(\mathbf{x}) - \mathbf{x}|||_w \geq 1$, and so $\widehat{\lambda}_{x,w} \geq 1$.

Now we establish **(3.11)**. From the definition of the norm $||| \cdot |||_w$ it follows that

$$(3.16) \qquad |||\mathbf{T}(\mathbf{x}) - \mathbf{x}|||_w^2 = |||(T_i(x) - x)_{i \in I}|||_w^2 = \sum_{i \in I} w(i) \|T_i(x) - x\|^2.$$

In addition, **(3.2)**, the linearity of $\mathbf{P_D}$, the fact that $\mathbf{x} = \mathbf{P_D}(\mathbf{x})$ and the fact that $\sum_{j \in I} w(j) = 1$, all of them imply that

$$|||\mathbf{P_D}(\mathbf{T}(\mathbf{x})) - \mathbf{x}|||_w^2 = |||\mathbf{P_D}(\mathbf{T}(\mathbf{x}) - \mathbf{x})|||_w^2$$

$$= \left|\left|\left| \mathbf{J}\left(\sum_{i \in I} w(i)(T_i(x) - x)\right)\right|\right|\right|_w^2 = \sum_{j \in I} w(j)\left\|\sum_{i \in I} w(i)(T_i(x) - x)\right\|^2$$

$$(3.17) \qquad = \left\|\sum_{i \in I} w(i)(T_i(x) - x)\right\|^2.$$

From **(3.16)** and **(3.17)** we obtain **(3.11)**.

Proof of Part (iii): From the fact that $\mathbf{T}$ is a cutter, from **(3.13)**, **(3.15)** and from the fact that $\mathbf{b}_w(\mathbf{x}) \in \mathbf{D}$, it follows that for all $\mathbf{y} \in \mathbf{F} \cap \mathbf{D}$,

$$0 \leq \langle\langle \mathbf{y} - \mathbf{T}\,(\mathbf{x}), \mathbf{T}\,(\mathbf{x}) - \mathbf{x}\rangle\rangle_w$$

$$= \langle\langle \mathbf{y} - \mathbf{b}_w(\mathbf{x}), \mathbf{T}\,(\mathbf{x}) - \mathbf{x}\rangle\rangle_w + \langle\langle \mathbf{b}_w(\mathbf{x}) - \mathbf{T}\,(\mathbf{x}), \mathbf{T}\,(\mathbf{x}) - \mathbf{x}\rangle\rangle_w$$

$$= \langle\langle \mathbf{y} - \mathbf{b}_w(\mathbf{x}), \mathbf{T}\,(\mathbf{x}) - \mathbf{x}\rangle\rangle_w$$

$$= \langle\langle \mathbf{y} - \mathbf{b}_w(\mathbf{x}), \mathbf{T}\,(\mathbf{x}) - \mathbf{P_D}\,(\mathbf{T}\,(\mathbf{x}))\rangle\rangle_w + \langle\langle \mathbf{y} - \mathbf{b}_w(\mathbf{x}), \mathbf{P_D}\,(\mathbf{T}\,(\mathbf{x})) - \mathbf{x}\rangle\rangle_w$$

$$(3.18) \qquad = \langle\langle \mathbf{y} - \mathbf{b}_w(\mathbf{x}), \mathbf{P_D}\,(\mathbf{T}\,(\mathbf{x})) - \mathbf{x}\rangle\rangle_w.$$

By combining this inequality with **(3.7)** and the assumption that $\mathbf{x} \notin \mathbf{F}$, one has

$$\left\langle\left\langle \mathbf{y} - \mathbf{b}_w(\mathbf{x}), \frac{|||\mathbf{P_D}\,(\mathbf{T}\,(\mathbf{x})) - \mathbf{x}|||_w^2}{|||\mathbf{T}\,(\mathbf{x}) - \mathbf{x}|||_w^2}(\mathbf{b}_w(\mathbf{x}) - \mathbf{x})\right\rangle\right\rangle_w \geq 0, \text{ for all } \mathbf{y} \in \mathbf{F} \cap \mathbf{D},$$

an inequality which implies that $\langle\langle \mathbf{y} - \mathbf{b}_w(\mathbf{x}), \mathbf{b}_w(\mathbf{x}) - \mathbf{x}\rangle\rangle_w \geq 0$, for all $\mathbf{y} \in \mathbf{F} \cap \mathbf{D}$. This, in turn, implies that $\mathbf{F} \cap \mathbf{D} \subseteq \mathbf{D} \cap \mathbf{H}\,(\mathbf{x}, \mathbf{b}_w(\mathbf{x}))$.

Finally, it remains to show that $\mathbf{D} \cap \mathbf{H}\,(\mathbf{x}, \mathbf{b}_w(\mathbf{x})) \subseteq \mathbf{D} \cap \mathbf{H}\,(\mathbf{x}, \mathbf{P_D}\,(\mathbf{T}\,(\mathbf{x})))$. Indeed, let $\mathbf{y} \in \mathbf{D} \cap \mathbf{H}\,(\mathbf{x}, \mathbf{b}_w(\mathbf{x}))$. Then previous lines, as well as **(3.7)** and **(3.10)**, imply that

$$(3.19) \quad 0 \leq \langle\langle \mathbf{y} - \mathbf{b}_w(\mathbf{x}), \mathbf{b}_w(\mathbf{x}) - \mathbf{x}\rangle\rangle_w = \left\langle\!\!\left\langle \mathbf{y} - \mathbf{b}_w(\mathbf{x}), \widehat{\lambda}_{x,w}\left(\mathbf{P_D}(\mathbf{T}(\mathbf{x})) - \mathbf{x}\right)\right\rangle\!\!\right\rangle_w.$$

Now we use this inequality and the fact that $\widehat{\lambda}_{x,w} = 1 + \epsilon$ for some $\epsilon \geq 0$ (as a result of Part **(ii)**), to conclude that

$$\begin{aligned}
0 &\leq \langle\langle \mathbf{y} - \mathbf{b}_w(\mathbf{x}), \mathbf{P_D}(\mathbf{T}(\mathbf{x})) - \mathbf{x}\rangle\rangle_w \\
&= \left\langle\!\!\left\langle \mathbf{y} - (\mathbf{x} + \widehat{\lambda}_{x,w}\left(\mathbf{P_D}(\mathbf{T}(\mathbf{x})) - \mathbf{x}\right)), \mathbf{P_D}(\mathbf{T}(\mathbf{x})) - \mathbf{x}\right\rangle\!\!\right\rangle_w \\
&= \langle\langle \mathbf{y} + \epsilon\mathbf{x} - (1 + \epsilon)\mathbf{P_D}(\mathbf{T}(\mathbf{x})), \mathbf{P_D}(\mathbf{T}(\mathbf{x})) - \mathbf{x}\rangle\rangle_w \\
&= \langle\langle \mathbf{y} - \mathbf{P_D}(\mathbf{T}(\mathbf{x})), \mathbf{P_D}(\mathbf{T}(\mathbf{x})) - \mathbf{x}\rangle\rangle_w + \epsilon\langle\langle \mathbf{x} - \mathbf{P_D}(\mathbf{T}(\mathbf{x})), \mathbf{P_D}(\mathbf{T}(\mathbf{x})) - \mathbf{x}\rangle\rangle_w.
\end{aligned}$$

Therefore,

$$(3.20) \qquad 0 \leq \epsilon |||\mathbf{x} - \mathbf{P_D}(\mathbf{T}(\mathbf{x}))|||_w^2 \leq \langle\langle \mathbf{y} - \mathbf{P_D}(\mathbf{T}(\mathbf{x})), \mathbf{P_D}(\mathbf{T}(\mathbf{x})) - \mathbf{x}\rangle\rangle_w,$$

namely $\mathbf{y} \in \mathbf{D} \cap \mathbf{H}(\mathbf{x}, \mathbf{P_D}(\mathbf{T}(\mathbf{x})))$ (see **(2.10)**), as required.

Proof of Part (iv): The definitions of $\mathbf{b}_w(\mathbf{x})$, $\mathbf{T}$ and $\mathbf{D}$, together with Lemma **3.1** and the linearity of $\mathbf{J}$, imply that

$$(3.21) \qquad\qquad\qquad \mathbf{b}_w(\mathbf{x}) = \mathbf{J}\left(T_{w,\widehat{\lambda}_{x,w}}(x)\right).$$

Given $q \in \cap_{i \in I} F_i$, it follows from **(2.8)** that $\mathbf{J}(q) \in \mathbf{F} \cap \mathbf{D}$. Hence, **(3.21)**, Part **(iii)** and the definition of $\mathbf{J}$ imply that $\mathbf{J}(q) \in \mathbf{D} \cap \mathbf{H}(\mathbf{J}(x), \mathbf{J}(T_{w,\widehat{\lambda}_{x,w}}(x)))$. Thus, the definitions of $\mathbf{J}$ and the inner product $\langle\langle \cdot, \cdot\rangle\rangle_w$ imply that $q \in H(x, T_{w,\widehat{\lambda}_{x,w}}(x))$. Since q was an arbitrary point in $\cap_{i \in I} F_i$, we have $\cap_{i \in I} F_i \subseteq H(x, T_{w,\widehat{\lambda}_{x,w}}(x))$.

It remains to prove the second inclusion in Part **(iv)**. In order to show this, let $u \in H(x, T_{w,\widehat{\lambda}_{x,w}}(x))$ be arbitrary. This fact, as well as **(2.10)** and the fact that $\widehat{\lambda}_{x,w} = 1 + \epsilon$ for some $\epsilon \geq 0$ (see Part **(ii)**), imply that

$$\begin{aligned}
0 &\leq \langle u - T_{w,\widehat{\lambda}_{x,w}}(x), T_{w,\widehat{\lambda}_{x,w}}(x) - x\rangle = \left\langle u - T_{w,\widehat{\lambda}_{x,w}}(x), \widehat{\lambda}_{x,w}\left(T_w(x) - x\right)\right\rangle \\
&= \left\langle u - (x + (1 + \epsilon)\left(T_w(x) - x\right)), \widehat{\lambda}_{x,w}\left(T_w(x) - x\right)\right\rangle \\
&= \left\langle u - T_w(x), \widehat{\lambda}_{x,w}\left(T_w(x) - x\right)\right\rangle - \epsilon\left\langle T_w(x) - x, \widehat{\lambda}_{x,w}\left(T_w(x) - x\right)\right\rangle.
\end{aligned}$$

Consequently,

$$0 \leq \epsilon \|T_w(x) - x\|^2 \leq \langle u - T_w(x), T_w(x) - x\rangle,$$

and hence, $u \in H(x, T_w(x))$, as required. $\qquad\square$

The following proposition essentially appears in [**3**, Corollary 3.4(i)], with a different notation and with a somewhat terse proof. We provide a new proof of it below.

Proposition 3.8. *For each $i \in I$ suppose that $T_i : X \to X$ is a cutter having a fixed point set F_i. Let τ_1 and τ_2 be in $(0, 1]$, let $x \in X$ and let w be a weight function. If $\lambda \in [\tau_1, (2 - \tau_2) L(x, w)]$ and $q \in F := \cap_{i \in I} F_i$, then the following inequality is satisfied:*

$$(3.22) \qquad \|T_{w,\lambda}(x) - q\|^2 \le \|x - q\|^2 - \tau_1 \tau_2 \sum_{i \in I} w(i) \|T_i(x) - x\|^2.$$

Proof. Observe first that since $L(x, w) \ge 1 > 0$ and since $\lambda \in [\tau_1, (2 - \tau_2)L(x, w)]$, we have $2 - \tau_2 \ge \lambda/L(x, w)$ and $\lambda \ge \tau_1$. Thus,

$$(3.23) \qquad\qquad \lambda \left(2 - \frac{\lambda}{L(x, w)} \right) \ge \tau_1 \tau_2.$$

If $x \in F$, then $T_i(x) - x = 0$ for all $i \in I$. Thus $T_{w,\lambda}(x) = x$ and hence **(3.22)** is clear. Suppose now that $x \notin F$. We observe that $T_w(x) = T_{\widehat{w}}(x)$, where $\widehat{w} : \widehat{I}_w \to [0, 1]$ is the restriction of w to $\widehat{I}_w$ (recall that $\widehat{I}_w := \{i \in I \mid w(i) > 0\}$, and, therefore, $\widehat{w}$ is a positive weight function defined on $\widehat{I}_w$). By using Proposition **3.4** with $\widehat{w}$ instead of w (in order to use this proposition we also need to verify that $\cap_{i \in \widehat{I}_w} F_i \ne \emptyset$, which is true because $q \in F \subseteq \cap_{i \in \widehat{I}_w} F_i$), we obtain that $T_{\widehat{w}}(x) \ne x$, and hence $T_w(x) - x \ne 0$. Hence, it follows from **(2.2)** and simple calculations that

$$
\begin{aligned}
\|T_{w,\lambda}(x) - q\|^2 &= \|x + \lambda (T_w(x) - x) - q\|^2 \\
&= \|x - q\|^2 + 2\lambda \langle T_w(x) - x, x - q \rangle + \lambda^2 \|T_w(x) - x\|^2 \\
&= \|x - q\|^2 + 2\lambda \sum_{i \in I} w(i) \langle T_i(x) - x, x - q \rangle \\
(3.24) \qquad &\quad + \frac{\lambda^2}{L(x, w)} \sum_{i \in I} w(i) \|T_i(x) - x\|^2.
\end{aligned}
$$

By adding and subtracting $T_i(x)$ in the inner product of the second summand on the right-hand side of the last equality, we get

$$
\begin{aligned}
\|T_{w,\lambda}(x) - q\|^2 &= \|x - q\|^2 + 2\lambda \sum_{i \in I} w(i) \langle T_i(x) - x, x - T_i(x) \rangle \\
&\quad + 2\lambda \sum_{i \in I} w(i) \langle T_i(x) - x, T_i(x) - q \rangle \\
(3.25) \qquad &\quad + \frac{\lambda^2}{L(x, w)} \sum_{i \in I} w(i) \|T_i(x) - x\|^2.
\end{aligned}
$$

By this result, the assumption that the operators T_i, $i \in I$ are cutters (and hence they obey **(2.12)**) and by **(3.23)**, we get the desired result:

$$\|T_{w,\lambda}(x) - q\|^2 \leq \|x - q\|^2 + 2\lambda \sum_{i \in I} w(i) \langle T_i(x) - x, x - T_i(x) \rangle$$

$$+ \frac{\lambda^2}{L(x,w)} \sum_{i \in I} w(i) \|T_i(x) - x\|^2$$

$$= \|x - q\|^2 - \lambda \left(2 - \frac{\lambda}{L(x,w)} \right) \sum_{i \in I} w(i) \|T_i(x) - x\|^2$$

$$(3.26) \qquad \leq \|x - q\|^2 - \tau_1 \tau_2 \sum_{i \in I} w(i) \|T_i(x) - x\|^2 .$$

$\square$

The next corollary generalizes Corollary **3.5**, and also improves upon [**8**, Corollary 2.12] and [**29**, Proposition 2.4] (assuming the index set I in [**29**, Proposition 2.4] is finite) in the sense that λ can be larger than $L(x,w)$.

Corollary 3.9. *For each $i \in I$ suppose that $T_i : X \to X$ is a cutter having a fixed point set F_i such that $\cap_{i \in I} F_i \neq \emptyset$. Let τ_1 and τ_2 be in $(0,1]$, let $x \in X$ and suppose that $w : I \to [0,1]$ is a weight function. If $\lambda \in [\tau_1, (2 - \tau_2) L(x,w)]$ and if $T_{w,\lambda}(x) = x$, then $T_i(x) = x$ for all $i \in \widehat{I}_w$.*

Proof. Let $q \in F$. Since $T_{w,\lambda}(x) = x$, we conclude from Proposition **3.8** that

$$0 \leq \sum_{i \in \widehat{I}_w} w(i) \|T_i(x) - x\|^2 = \sum_{i \in I} w(i) \|T_i(x) - x\|^2 \leq \frac{\|x - q\|^2 - \|T_{w,\lambda}(x) - q\|^2}{\tau_1 \tau_2} = 0.$$

Since $w(i) > 0$ for all $i \in \widehat{I}_w$, the sum $\sum_{i \in \widehat{I}_w} w(i) \|T_i(x) - x\|^2$ can vanish if and only if $\|T_i(x) - x\|^2 = 0$ for each $i \in \widehat{I}_w$. Thus, $T_i(x) = x$ for all $i \in \widehat{I}_w$, as claimed. $\square$

The next lemmata are used for deriving a certain variant of Proposition **3.8**, namely Proposition **3.12** (these propositions do not seem to imply each other).

Lemma 3.10. *Suppose that $T : X \to X$ is an operator (not necessarily a cutter), $x \in X$ is arbitrary, $\lambda_2 > 0$ and $\lambda_1 \in [0, \lambda_2]$. Then $H(x, T_{\lambda_2}(x)) \subseteq H(x, T_{\lambda_1}(x))$. Moreover, if $x \notin \text{Fix}(T)$ and $0 < \lambda_1 < \lambda_2$, then $H(x, T_{\lambda_2}(x)) \subsetneqq H(x, T_{\lambda_1}(x))$.*

Proof. We start by showing that $H(x, T_{\lambda_2}(x)) \subseteq H(x, T_{\lambda_1}(x))$. Let $z \in H(x, T_{\lambda_2}(x))$ be arbitrary. Then **(2.10)** implies that $\langle z - T_{\lambda_2}(x), x - T_{\lambda_2}(x) \rangle \leq 0$. This inequality,

as well as simple calculations and the facts that $0 \leq \lambda_1 \leq \lambda_2$ and $0 < \lambda_2$, show that

$$(3.27) \quad \langle z - T_{\lambda_1}(x), x - T_{\lambda_1}(x) \rangle = \langle z - (x + \lambda_1(T(x) - x)), x - (x + \lambda_1(T(x) - x)) \rangle$$

$$= \langle z - (x + \lambda_2(T(x) - x) + (\lambda_1 - \lambda_2)(T(x) - x)), -\lambda_1(T(x) - x) \rangle$$

$$= \langle z - (x + \lambda_2(T(x) - x)), -\lambda_1(T(x) - x) \rangle - \langle (\lambda_1 - \lambda_2)(T(x) - x), -\lambda_1(T(x) - x) \rangle$$

$$= (\lambda_1/\lambda_2)\langle z - (x + \lambda_2(T(x) - x)), -\lambda_2(T(x) - x) \rangle + \lambda_1(\lambda_1 - \lambda_2)\|T(x) - x\|^2$$

$$= (\lambda_1/\lambda_2) \langle z - T_{\lambda_2}(x), x - T_{\lambda_2}(x) \rangle + \lambda_1(\lambda_1 - \lambda_2)\|T(x) - x\|^2 \leq 0 + 0 = 0.$$

Hence, $z \in H(x, T_{\lambda_1}(x))$, and since z was an arbitrary point in $H(x, T_{\lambda_2}(x))$, one has $H(x, T_{\lambda_2}(x)) \subseteq H(x, T_{\lambda_1}(x))$, as required.

Finally, if $x \notin \mathrm{Fix}(T)$ and $0 < \lambda_1 < \lambda_2$, then $\lambda_1(\lambda_1 - \lambda_2)\|T(x) - x\|^2 < 0$. Thus, any $z \in H(x, T_{\lambda_2}(x))$ satisfies (3.27) with a strict inequality in the last line of (3.27). This fact, as well as the fact that $z := T_{\lambda_1}(x)$ obviously satisfies $\langle z - T_{\lambda_1}(x), x - T_{\lambda_1}(x) \rangle = 0$, imply that this specific z is in $H(x, T_{\lambda_1}(x))$ and it cannot be in $H(x, T_{\lambda_2}(x))$. Hence, $H(x, T_{\lambda_2}(x)) \subsetneq H(x, T_{\lambda_1}(x))$. $\square$

Lemma 3.11. *Let $\{T_i\}_{i \in I}$ be cutters and $\{F_i\}_{i \in I}$ their fixed points sets, respectively, such that $F := \cap_{i \in I} F_i \neq \emptyset$. Let $w : I \to [0, 1]$ be a weight function. Fix some $x \in X$ and suppose that $\lambda \in [0, \widehat{\lambda}_{x, \widehat{w}}]$, where $\widehat{\lambda}_{x, \widehat{w}}$ satisfies (3.11) (and in Lemma 3.7 we use, instead of I and w, the subset $\widehat{I}_w$ and the positive weight function $\widehat{w} : \widehat{I}_w \to (0, 1]$, respectively). Then $F \subseteq H(x, T_{w, \lambda}(x))$.*

Proof. Since $\widehat{\lambda}_{x, \widehat{w}} > 0$ according to Lemma 3.7(ii), it follows from Lemma 3.10, with T_w instead of the operator T used there, that

$$(3.28) \qquad\qquad H(x, T_{w, \widehat{\lambda}_{x, \widehat{w}}}(x)) \subseteq H(x, T_{w, \lambda}(x)).$$

In addition, from Part **(iv)** of Lemma 3.7 (the inclusion of the first set in the second one in (3.12), where in Lemma 3.7 we use $\widehat{I}_w$ and $\widehat{w}$ instead of I and w, respectively), we have $\cap_{i \in \widehat{I}_w} F_i \subseteq H(x, T_{w, \widehat{\lambda}_{x, \widehat{w}}}(x))$. Since $\widehat{I}_w \subseteq I$, we have $F = \cap_{i \in I} F_i \subseteq \cap_{i \in \widehat{I}_w} F_i$. These inclusions, as well as (3.28), imply that

$$F \subseteq \bigcap_{i \in \widehat{I}_w} F_i \subseteq H(x, T_{w, \widehat{\lambda}_{x, \widehat{w}}}(x)) \subseteq H(x, T_{w, \lambda}(x)).$$

$\square$

Proposition 3.12. *Let $\{T_i\}_{i \in I}$ be cutters and $\{F_i\}_{i \in I}$ their fixed points sets, respectively, such that $F := \cap_{i \in I} F_i \neq \emptyset$. Suppose that τ_1 and τ_2 are in $(0, 1]$. Let $x \in X$ and let $w : I \to [0, 1]$ be a weight function. If $\lambda \in [\tau_1, (2 - \tau_2) L(x, w)]$ and $q \in F$, then the following inequality is satisfied:*

$$(3.29) \qquad\qquad \|T_{w, \lambda}(x) - q\|^2 \leq \|x - q\|^2 - \tau_2 \|T_{w, \lambda}(x) - x\|^2.$$

Proof. From (2.2), (3.11), and the definitions of $\widehat{I}_w$ and $\widehat{w}$, we have

$$(3.30) \qquad\qquad L(x, w) = \widehat{\lambda}_{x, \widehat{w}} \text{ or } L(x, w) = 1.$$

Since $\widehat{\lambda}_{x,\widehat{w}} \geq 1$ according to Part **(ii)** of Lemma **3.7** (in which we use $\widehat{I}_w$ and $\widehat{w}$ instead of I and w, respectively), it follows from **(3.30)** that $L(x,w) \in [1, \widehat{\lambda}_{x,\widehat{w}}]$.

Now we divide the proof into two cases, depending on the value of λ.

Case 1: $\tau_1 \leq \lambda \leq L(x,w)$.

In this case $\lambda \leq L(x,w) \leq \widehat{\lambda}_{x,\widehat{w}}$, and so Lemma **3.11** implies that $F \subseteq H(x, T_{w,\lambda}(x))$. Thus,

$$(3.31) \qquad \langle T_{w,\lambda}(x) - x, T_{w,\lambda}(x) - q \rangle \leq 0.$$

This fact, simple calculations and the fact that $\tau_2 \in (0,1]$, all imply that

$$
\begin{aligned}
(3.32) \quad \|T_{w,\lambda}(x) - q\|^2 &= \|(T_{w,\lambda}(x) - x) + (x - q)\|^2 \\
&= \|x - q\|^2 + 2\langle T_{w,\lambda}(x) - x, x - q \rangle + \|T_{w,\lambda}(x) - x\|^2 \\
&= \|x - q\|^2 + 2\langle T_{w,\lambda}(x) - x, x - T_{w,\lambda}(x) \rangle \\
&\quad + 2\langle T_{w,\lambda}(x) - x, T_{w,\lambda}(x) - q \rangle + \|T_{w,\lambda}(x) - x\|^2 \\
&\leq \|x - q\|^2 - 2\|T_{w,\lambda}(x) - x\|^2 + \|T_{w,\lambda}(x) - x\|^2 \\
&= \|x - q\|^2 - \|T_{w,\lambda}(x) - x\|^2 \leq \|x - q\|^2 - \tau_2\|T_{w,\lambda}(x) - x\|^2.
\end{aligned}
$$

Case 2: $L(x,w) \leq \lambda \leq (2 - \tau_2) L(x,w)$.

In this case, if we let $\alpha := \lambda/L(x,w)$, then simple calculations show that $1 \leq \alpha \leq 2 - \tau_2$ and

$$(3.33) \qquad T_{w,\lambda}(x) = x + \alpha \left(T_{w,L(x,w)}(x) - x \right).$$

Therefore,

$$
\begin{aligned}
\|T_{w,\lambda}(x) - q\|^2 &= \left\| x + \alpha \left(T_{w,L(x,w)}(x) - x \right) - q \right\|^2 \\
&= \|x - q\|^2 + 2\alpha \langle T_{w,L(x,w)}(x) - x, x - q \rangle + \alpha^2 \left\| T_{w,L(x,w)}(x) - x \right\|^2 \\
&= \|x - q\|^2 + 2\alpha \langle T_{w,L(x,w)}(x) - x, x - T_{w,L(x,w)}(x) \rangle \\
(3.34) \quad &\quad + 2\alpha \langle T_{w,L(x,w)}(x) - x, T_{w,L(x,w)}(x) - q \rangle + \alpha^2 \left\| T_{w,L(x,w)}(x) - x \right\|^2.
\end{aligned}
$$

Since $L(x,w) \leq \widehat{\lambda}_{x,\widehat{w}}$, it follows from Lemma **3.11** that $F \subseteq H(x, T_{w,L(x,w)}(x))$, and hence, $\langle T_{w,L(x,w)}(x) - x, T_{w,L(x,w)}(x) - q \rangle \leq 0$. This fact, as well as **(3.33)**, **(3.34)**, and the inequality $\tau_2 \leq 2 - \alpha$, imply that

$$
\begin{aligned}
&\|T_{w,\lambda}(x) - q\|^2 \\
&\leq \|x - q\|^2 + 2\alpha \langle T_{w,L(x,w)}(x) - x, x - T_{w,L(x,w)}(x) \rangle + \alpha^2 \left\| T_{w,L(x,w)}(x) - x \right\|^2 \\
&= \|x - q\|^2 - \alpha (2 - \alpha) \left\| T_{w,L(x,w)}(x) - x \right\|^2 \\
&\leq \|x - q\|^2 - \tau_2 \left\| \alpha \left(T_{w,L(x,w)}(x) - x \right) \right\|^2 \\
&= \|x - q\|^2 - \tau_2 \|T_{w,\lambda}(x) - x\|^2.
\end{aligned}
$$

$\square$

4. The block iterative extrapolated algorithm and the convergence theorems

In this section we present the extrapolated block-iterative algorithm aimed at solving the common fixed point problem (1.1), and show its convergence under certain conditions. In a nutshell, the algorithm generates the iteration x^{k+1} by considering a block $I_k \subseteq I$, calculating the convex combination, for the block I_k, of the differences $T_i(x^k) - x^k$, $i \in I_k$, and implementing an extrapolation in order to reach a *deep step* towards the common fixed point set $\cap_{i \in I} F_i$. In each iterative step the user can choose weights and extrapolation parameters anew, as long as they obey some reasonable conditions.

Algorithm 4.1. (The extrapolated block-iterative algorithm).
Input: *A real Hilbert space X, a positive integer m, an index set $I := \{1, 2, \ldots, m\}$, an arbitrary initialization point $x^0 \in X$, two real numbers τ_1 and τ_2 in the interval $(0, 1]$, a family of cutters $\{T_i\}_{i \in I}$ defined on X with fixed point sets $F_i := \mathrm{Fix}(T_i) = \{x \in X \mid T_i(x) = x\}$ and a nonempty common fixed point set $F := \cap_{i \in I} F_i$, a sequence $\{w_k\}_{k=0}^{\infty}$ of weight functions with respect to I, and a sequence of relaxation parameters $\{\lambda_k\}_{k=0}^{\infty}$ which satisfy $\lambda_k \in \left[\tau_1, (2 - \tau_2) L\left(x^k, w_k\right)\right]$ for each $k \in \mathbb{N} \cup \{0\}$, where L is from (2.2).*

Iterative step: *Given the current iterate x^k, $k \in \mathbb{N} \cup \{0\}$, calculate the next iterate by*

$$(4.1) \qquad x^{k+1} := T_{w_k, \lambda_k}\left(x^k\right) = x^k + \lambda_k \left(\sum_{i \in I} w_k(i) T_i(x^k) - x^k\right),$$

that is, if we denote by $I_k := \widehat{I}_{w_k} = \{i \in I \mid w_k(i) > 0\}$ the k-th block, then

$$(4.2) \qquad x^{k+1} = x^k + \lambda_k \sum_{i \in I_k} w_k(i)(T_i(x^k) - x^k).$$

Remark 4.2. Since $\tau_1/(2 - \tau_2) \leq 1/(2 - 1) \leq L(x^k, w_k)$ for all $k \in \mathbb{N} \cup \{0\}$ according to (2.2), it follows that for all $k \in \mathbb{N} \cup \{0\}$ the interval $[\tau_1, (2 - \tau_2)L(x^k, w_k)]$ is nonempty, and hence one can indeed choose a sequence of relaxation parameters $\{\lambda_k\}_{k=0}^{\infty}$ in this interval. In standard BIP algorithms the parameters $\{\lambda_k\}_{k=0}^{\infty}$ are confined to the interval $[\tau_1, 2 - \tau_2]$ for any user-chosen two real numbers τ_1 and τ_2 in the interval $(0, 1]$, allowing under-relaxation or over-relaxation of the projections. Many papers investigate the role of relaxation parameters, see, e.g., [35]. Here, in Algorithm 4.1, the term $L(x^k, w_k)$ which might be greater than 1, enables "deeper" steps which constitute the extrapolation beyond the usual interval that appears in convergence theorems for projection methods.

Remark 4.3. Algorithm 4.1 becomes fully sequential whenever at each iteration k there is an index $j_k \in I$ such that $w_k(i) = 0$ for all $i \in I \setminus \{j_k\}$ and $w_k(j_k) = 1$, and it becomes fully simultaneous whenever $w_k(i) > 0$ for all $k \in \mathbb{N} \cup \{0\}$ and all $i \in I$. Hence Algorithm 4.1 can be used in both serial and parallel computational settings.

In order to ensure the convergence of Algorithm **4.1**, we will impose, in either Theorem **4.15** or Theorem **4.16** below, one of the following two conditions on the sequence of weights $\{w_k\}_{k=0}^{\infty}$, respectively:

Condition 4.4. *For all $i \in I$ one has $\sum_{k=0}^{\infty} w_k(i) = \infty$.*

Condition 4.5. *There are $s \in \mathbb{N}$ and $\alpha > 0$ such that for all $i \in I$ and all $\ell \in \mathbb{N} \cup \{0\}$ there is some $k \in \{\ell, \ell+1, \ldots, \ell+s-1\}$ such that $w_k(i) \geq \alpha$.*

Remark 4.6. Condition **4.4**, which essentially appeared first in [**1**, p. 171], is strictly more general than Condition **4.5**. Indeed, if $\{w_k\}_{k=0}^{\infty}$ is a sequence of weight functions which satisfies Condition **4.5**, then, in particular, for all $i \in I$ and all $p \in \mathbb{N} \cup \{0\}$ there is some $t_p \in \{ps, ps+1, \ldots, ps+p-1\}$ such that $w_{t_p}(i) \geq \alpha$, and therefore, $\sum_{k=0}^{\infty} w_k(i) \geq \sum_{p=0}^{\infty} w_{t_p}(i) \geq \sum_{p=0}^{\infty} \alpha = \infty$, namely, Condition **4.5** implies Condition **4.4**. On the other hand, the converse is not true since if, for instance, for every $k \in \mathbb{N} \cup \{0\}$ we let $w_k(i) := 1/((k+1)m)$ for each $i \in I \backslash \{m\}$ and $w_k(m) := 1 - (m-1)/((k+1)m)$, then $\{w_k\}_{k=0}^{\infty}$ is a sequence of weight functions which satisfies Condition **4.4** but does not satisfy Condition **4.5**.

However, Condition **4.5**, which seems to be new, is strictly more general than the condition on the weights imposed in either [**3**, Corollary 4.2] (up to a typo in [**3**, Relation (104)], that the weights are constant and not dynamic: in later lines there the weights are assumed to be dynamic) or [**13**, Step 1 (St.1) on page 14232], when in both cases we restrict ourselves to strings of length 1. This condition is the following one:

(4.3)   There are $s \in \mathbb{N}$ and $\alpha > 0$ such that both $\{I_k\}_{k=0}^{\infty}$ is s-intermittent

$$\text{and } w_k(i) \geq \alpha \text{ for all } k \in \mathbb{N} \cup \{0\} \text{ and all } i \in I_k,$$

where by saying that the sequence of blocks $\{I_k\}_{k=0}^{\infty}$ is s-intermittent we mean that $I = I_{\ell} \cup I_{\ell+1} \cup \ldots \cup I_{\ell+s-1}$ for each $\ell \in \mathbb{N} \cup \{0\}$ (it seems that **(4.3)** appeared first in [**16**, Theorem 5.1], although variants of it can be found in previous works such as [**27**, Definition 1.2 Part (c) and Definition 3.1 Relation (3.4)]). Indeed, suppose that **(4.3)** holds, and fix $\ell \in \mathbb{N} \cup \{0\}$ and $i \in I$. Since the sequence of blocks is s-intermittent and $i \in I$, we have $i \in I = I_{\ell} \cup I_{\ell+1} \cup \ldots \cup I_{\ell+s-1}$, and hence there is some $k \in \{\ell, \ell+1, \ldots, \ell+s-1\}$ such that $i \in I_k$. Thus, **(4.3)** implies that $w_k(i) \geq \alpha$, and we conclude that Condition **4.5** holds (with the same s and α as in **(4.3)**). On the other hand, there are cases where Condition **4.5** holds but **(4.3)** does not: see Examples **4.7**–**4.8** below.

Another condition on the weights appears in [**5**, Condition 5.4], [**8**, Algorithm 3.1(4) and Condition 3.2(ii)], and [**29**, Algorithm 6.1(4) and Definition 6.3] (see also [**28**, Definition 3.1 and Algorithm 6.3 (C2)] [**30**, Algorithm 3.9 and Theorem 3.11]). When restricted to the case where I is finite (I in [**5**, Algorithm 5.1], [**8**, Algorithm 3.1] and [**29**, Algorithm 6.1] is allowed to be countable) it essentially says that the sequence of blocks is s-intermittent for some $s \in \mathbb{N}$ and also that there is a positive number $\delta \in (0, 1)$ having the following property: for all $k \in \mathbb{N} \cup \{0\}$ there is an index $j_{\max,k} \in I_k$ for which the maximum $\max\{g_j \mid j \in I_k\}$ is attained (where for each $j \in I_k$ one has

that g_j is a certain nonnegative number depending on j, x^k, and the given cutters) and this $j_{\max,k}$ satisfies $w_k(j_{\max,k}) \geq \delta$. This condition neither implies Condition **4.5** nor is implied by it. Indeed, if we consider the sequence $\{w_k\}_{k=0}^{\infty}$ of weight functions defined in Example **4.8** below, then Condition **4.5** holds, but for the above mentioned condition to hold it must be that $j_{\max,k} = m$ (because for each $j \neq m$ one has $\lim_{k \to \infty, k \neq 0 \,(\mathrm{mod}\ m)} w_k(j) = 0$), which is usually not true. On the other hand, in Condition **4.5** we impose a requirement which should be satisfied by all the indices in I regarding the uniform positive lower bound on the corresponding weights at the indices, while in [**5**, Condition 5.4], [**8**, Algorithm 3.1(4) and Condition 3.2(ii)], and [**29**, Algorithm 6.1(4) and Definition 6.3] this is not the case.

Finally, we note that the concept of intermittent controls was introduced in [**4**, Definition 3.18], following the notion of almost cyclic controls which seems to appear first in [**40**] (even though a more general control appeared before in [**12**, Definition 5]; this latter condition seems to inspire the generalized intermittency conditions which appear in [**5**, Condition 5.4(iii)], [**8**, Condition 3.2(ii)], [**28**, Definition 3.1] and [**29**, Definition 6.3]).

Example 4.7. Suppose that $m > 1$. Let s be an arbitrary even natural number and let $s' := s/2$. For each $t \in \mathbb{N} \cup \{0\}$ choose randomly, say using the uniform distribution on $\{ts', ts' + 1, \ldots, ts' + s' - 1\}$, a number $h_{t,1} \in \{ts', ts' + 1, \ldots, ts' + s' - 1\}$. Now choose (possibly randomly, using the uniform distribution on $[1/(2m), 1/m]$) arbitrary real numbers $w_{h_{t,1}}(i) \in [1/(2m), 1/m]$ for all $i \in I \setminus \{m\}$, and define $w_{h_{t,1}}(m) := 1 - \sum_{j=1}^{m-1} w_{h_{t,1}}(j)$. Now for each $k \in \{ts', ts' + 1, \ldots, ts' + s' - 1\} \setminus \{h_{t,1}\}$ and each $i \in I \setminus \{m\}$ choose an arbitrary (possibly randomly, using the uniform distribution on $[0, 1/m]$) real number $w_k(i) \in [0, 1/m]$, and define $w_k(m) := 1 - \sum_{j=1}^{m-1} w_k(j)$.

By doing this we obtain a sequence $\{w_k\}_{k=0}^{\infty}$ of weight functions which satisfies Condition **4.5** with $\alpha := 1/(2m)$ and s. Indeed, for every $\ell \in \mathbb{N} \cup \{0\}$ let $t := \lceil \ell/s' \rceil$ (where $\lceil \cdot \rceil$ is the strict ceiling function, which assigns to every $r \in \mathbb{R}$ the minimal integer which is strictly greater than r), and let $k := h_{t,1}$. The definition of $h_{t,1}$ implies that $k \in \{ts', ts' + 1, \ldots, ts' + s' - 1\}$. In addition, the definition of t implies that $(t-1)s' \leq \ell < ts'$. Hence,

$$0 < ts' - \ell \leq k - \ell \leq ts' + s' - 1 - (t-1)s' = 2s' - 1 = s - 1,$$

and so $h_{t,1} = k \in \{\ell, \ell + 1, \ldots, \ell + s - 1\}$. The definition of $h_{t,1}$ implies that $w_{h_{t,1}}(i) \in [1/(2m), 1/m]$ if $i \in I \setminus \{m\}$ and

$$w_{h_{t,1}}(m) = 1 - \sum_{j=1}^{m-1} w_{h_{t,1}}(j) \geq 1 - \frac{m-1}{2m} = \frac{1}{2} + \frac{1}{2m} > \frac{1}{2m} = \alpha.$$

It follows that for all $i \in I$ and all $\ell \in \mathbb{N} \cup \{0\}$ there is $k \in \{\ell, \ell + 1, \ldots, \ell + s - 1\}$ (namely, $k := h_{t,1}$ for $t := \lceil \ell/s' \rceil$) such that $w_k(i) \geq \alpha$, that is, Condition **4.5** does hold with the above mentioned s and α, as claimed.

Finally, one observes that since $w_k(i)$ can be an arbitrary number in $[0, 1/m]$ whenever $k \neq h_{t,1}$ and $i \neq m$, there are cases in the choice of the weight functions where they are all positive and therefore all the blocks I_k coincide with I (and hence the

sequence $\{I_k\}_{k=0}^{\infty}$ of blocks is 1-intermittent) and $\inf\{w_k(i) \,|\, k \in \mathbb{N} \cup \{0\}\} = 0$ for all $i \in I\backslash\{m\}$. In these cases (4.3) does not hold.

Example 4.8. Suppose that $m > 1$. Given $k \in \mathbb{N} \cup \{0\}$ and $i \in I\backslash\{m\}$, define $w_k(i) :=$ $1/m$ whenever $k = 0 \,(\text{mod } m)$ and $w_k(i) := 1/(2km)$ otherwise. In addition, define $w_k(m) := 1/m$ if $k = 0 \,(\text{mod } m)$ and $w_k(m) := 1 - (m-1)/(2km)$ otherwise. Then $\{w_k\}_{k=0}^{\infty}$ becomes a sequence of positive weight functions which satisfies Condition **4.5** with $s := m$ and $\alpha := 1/m$, the k-th block is $I_k = I$ and hence the sequence $\{I_k\}_{k=0}^{\infty}$ of blocks is 1-intermittent, but (4.3) does not hold because $\inf\{w_k(i) \,|\, k \in \mathbb{N} \cup \{0\}\} = 0$ for all $i \in I\backslash\{m\}$.

Definition 4.9. *A sequence $\{y^k\}_{k=0}^{\infty}$ in X is said to be Fejér-monotone with respect to some subset $\emptyset \neq S \subseteq X$ if the following condition holds: $\|y^{k+1} - z\| \leq \|y^k - z\|$ for every $k \in \mathbb{N} \cup \{0\}$ and every $z \in S$.*

Proposition 4.10. *Any sequence $\{x^k\}_{k=0}^{\infty}$, generated by Algorithm **4.1**, is Fejér-monotone with respect to the (assumed nonempty) common fixed point set F.*

Proof. By Proposition **3.8**, for any sequence $\{x^k\}_{k=0}^{\infty}$, generated by Algorithm **4.1**, and any $q \in F$, we have

$$(4.4) \qquad \left\|x^{k+1} - q\right\|^2 \leq \left\|x^k - q\right\|^2 - \tau_1\tau_2 \sum_{i \in I} w_k(i) \left\|T_i\left(x^k\right) - x^k\right\|^2.$$

This implies that $\|x^{k+1} - q\| \leq \|x^k - q\|$ for all $k \in \mathbb{N} \cup \{0\}$, and hence $\{x^k\}_{k=0}^{\infty}$ is Fejér-monotone with respect to F. $\qquad\square$

Proposition 4.11. *Suppose that X is finite-dimensional, that Condition **4.4** holds, that all the cutters T_i are continuous, and that $\{x^k\}_{k=0}^{\infty}$ is a sequence generated by Algorithm **4.1**. If this sequence converges to some point $x^* \in X$, then $x^* \in F$.*

Proof. Assume, for a contradiction, that $x^* \notin F = \cap_{i \in I} F_i$, namely that there is some index $i_0 \in I$ such that $x^* \notin F_{i_0}$. This fact and the fact that F_i is closed for each $i \in I$ (and, in particular, for i_0), implies the existence of some $\varepsilon > 0$ such that $B[x^*, \varepsilon] \cap F_{i_0} = \emptyset$. The continuity of T_{i_0} on $B[x^*, \varepsilon]$, as well as the compactness of $B[x^*, \varepsilon]$, the Weierstrass Maximal Value Theorem and the fact that $\|T_{i_0}(x) - x\| > 0$ for all $x \in B[x^*, \varepsilon]$, all imply that there is some $\delta > 0$ such that $\|T_{i_0}(x) - x\| \geq \delta$ for all $x \in B[x^*, \varepsilon]$. Since $x^* = \lim_{k \to \infty} x^k$, there is some $k_0 \in \mathbb{N} \cup \{0\}$ such that $x^k \in B[x^*, \varepsilon]$ for all $k > k_0$. Thus, Proposition **3.8** yields

$$(4.5) \qquad \left\|x^{k+1} - q\right\|^2 \leq \left\|x^k - q\right\|^2 - \tau_1\tau_2\delta^2 w_k(i_0)$$

for all $k > k_0$ and all $q \in F$. Hence, for all $k_0 + 1 \leq \ell \in \mathbb{N}$

$$\sum_{k=k_0+1}^{\ell} w_k(i_0) \leq \frac{1}{\tau_1\tau_2\delta^2} \sum_{k=k_0+1}^{\ell} \left(\|x^k - q\|^2 - \|x^{k+1} - q\|^2\right) \leq \frac{1}{\tau_1\tau_2\delta^2}\|x^{k_0+1} - q\|^2.$$

By letting $\ell \to \infty$ we conclude that $\sum_{k=k_0+1}^{\infty} w_k(i_0)$ is bounded from above, and this is impossible because we assumed that $\sum_{k=0}^{\infty} w_k(i) = \infty$ for every $i \in I$ (Condition **4.4**) and hence $\sum_{k=k_0+1}^{\infty} w_k(i_0) = \infty$. Therefore, the assumption that $x^* \notin F$ is impossible, and hence $x^* \in F$, as required. $\qquad\square$

The following proposition brings together several well-known results, and is used below for proving the convergence of our algorithmic sequence.

Proposition 4.12. *Suppose that $\{y^k\}_{k=0}^\infty$ is a Fejér monotone sequence in X with respect to some nonempty subset C. Then:*

(i) *$\{y^k\}_{k=0}^\infty$ is bounded.*

(ii) *$\{y^k\}_{k=0}^\infty$ has at least one weak sequential cluster point.*

(iii) *For every weak sequential cluster point u of $\{y^k\}_{k=0}^\infty$ and every $q \in C$ one has $\|u - q\| \le \|y^k - q\|$ for all $k \in \mathbb{N} \cup \{0\}$ and $\lim_{k\to\infty} \|y^k - q\| = \|u - q\|$.*

(iv) *If every weak sequential cluster point of $\{y^k\}_{k=0}^\infty$ belongs to C, then $\{y^k\}_{k=0}^\infty$ converges weakly to a point in C.*

(v) *If the interior of C is nonempty, then $\{y^k\}_{k=0}^\infty$ converges strongly to some point in X.*

(vi) *If X is finite-dimensional, then the words "weak" and "weakly" in Parts (ii)–(iv) above can be replaced by the words "strong" and "strongly", respectively.*

Proof. Let $q \in C$ be arbitrary. Since $\{y^k\}_{k=0}^\infty$ is Fejér monotone, one has $\|y^{k+1} - q\| \le \|y^k - q\| \le \ldots \le \|y^0 - q\|$ for all $k \in \mathbb{N} \cup \{0\}$. Thus, $\{y^k\}_{k=0}^\infty$ is in the ball $B[q, \|y^0 - q\|]$. Thus, Part (i) holds. Part (ii) follows immediately from Part (i) since any bounded sequence in a Hilbert space has a weakly convergent subsequence [**7**, Lemma 2.45]. Since $\{\|y^k - q\|\}_{k=0}^\infty$ is monotone decreasing for all $q \in F$ from the definition of Fejér monotonicity, $\lim_{k\to\infty} \|y^k - q\|$ exists and satisfies $\lim_{k\to\infty} \|y^k - q\| \le \|y^t - q\|$ for every $t \in \mathbb{N} \cup \{0\}$, and since the norm is weakly sequentially lower semicontinuous [**34**, II.3.27, p. 68], one has $\|u - q\| \le \liminf_{k\to\infty} \|y^k - q\| = \lim_{k\to\infty} \|y^k - q\|$ whenever u is a weak sequential cluster point of $\{y^k\}_{k=0}^\infty$. Thus, Part (iii) holds. For the proof of Part (iv), see [**7**, Theorem 5.5, p. 92] or [**15**, Corollary 3.3.3, p. 110]. For the proof of Part (v), see [**7**, Proposition 5.10, p. 94]. Part (vi) is immediate since in finite-dimensional spaces a sequence converges weakly if and only if it converges strongly [**7**, Lemma 2.51(ii), p. 39].

Finally, we note that the proofs of Part (iv) and Part (v), respectively, can be found also in [**4**, Theorem 2.16(ii)] and [**4**, Theorem 2.16(iii)], respectively, and while the assertions there are formulated under the assumption that C is nonempty, closed and convex, the proofs actually hold if merely $C \ne \emptyset$.   □

Proposition 4.13. *For any sequence $\{x^k\}_{k=0}^\infty$, which is generated by Algorithm 4.1, we have:*

(i) *$\sum_{k=0}^\infty \|x^{k+1} - x^k\|^2 < \infty$,*

(ii) *$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0$.*

(iii) *$\lim_{t\to\infty} \|x^{h_t} - x^{k_t}\| = 0$ whenever $\{h_t\}_{t=0}^\infty$ and $\{k_t\}_{t=0}^\infty$ are subsequences of natural numbers which have the following properties: $k_t < h_t$ for every $t \in \mathbb{N} \cup \{0\}$ and $\sup\{h_t - k_t \mid t \in \mathbb{N} \cup \{0\}\} < \infty$.*

Proof. We start with Part (i). Take any $q \in F$. Algorithm 4.1 and Proposition 3.12 imply that for all $k \in \mathbb{N} \cup \{0\}$,

$$\left\|x^{k+1} - q\right\|^2 \le \left\|x^k - q\right\|^2 - \tau_2 \left\|x^{k+1} - x^k\right\|^2.$$

Hence, for all $\ell \in \mathbb{N}$,

$$\sum_{k=0}^{\ell} \left\| x^{k+1} - x^k \right\|^2 \leq \frac{1}{\tau_2} \sum_{k=0}^{\ell} \left(\|x^k - q\|^2 - \|x^{k+1} - q\|^2 \right)$$

$$= \frac{1}{\tau_2} \left(\|x^0 - q\|^2 - \|x^{\ell+1} - q\|^2 \right) \leq \frac{1}{\tau_2} \|x^0 - q\|^2.$$

By letting $\ell \to \infty$, we have $\sum_{k=0}^{\infty} \|x^{k+1} - x^k\|^2 \leq \|x^0 - q\|^2/\tau_2 < \infty$, as claimed.

As for Part **(ii)**, it follows from Part **(i)** and well-known results regarding nonnegative series that $\lim_{k\to\infty} \left\| x^{k+1} - x^k \right\|^2 = 0$, and hence $\lim_{k\to\infty} \left\| x^{k+1} - x^k \right\| = 0$.

It remains to prove Part **(iii)**. Let $s := \sup\{h_t - k_t \mid t \in \mathbb{N} \cup \{0\}\}$. By our assumptions on the subsequences $\{h_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=0}^{\infty}$, it follows that s is a natural number. Denote $\beta_{\ell,t} := \|x^{k_t+\ell} - x^{k_t+\ell-1}\|$ for every $\ell \in \{1, 2, \ldots, s\}$ and every $t \in \mathbb{N} \cup \{0\}$. Then each of the s sequences $\{\beta_{\ell,t}\}_{t=0}^{\infty}$, $\ell \in \{1, 2, \ldots, s\}$ is a subsequence of the sequence $\{\|x^{k+1} - x^k\|\}_{k=0}^{\infty}$, and hence, as follows from Part **(ii)**, we have $\lim_{t\to\infty} \beta_{\ell,t} = 0$. Since the triangle inequality and the definition of s imply that $\|x^{h_t} - x^{k_t}\| \leq \sum_{\ell=1}^{h_t-k_t} \|x^{k_t+\ell} - x^{k_t+\ell-1}\| \leq \sum_{\ell=1}^{s} \beta_{\ell,t}$ for all $t \in \mathbb{N} \cup \{0\}$, it follows from previous lines that $\lim_{t\to\infty} \|x^{h_t} - x^{k_t}\| = 0$. $\square$

Proposition 4.14. *Suppose that X is finite-dimensional, that $\{x^k\}_{k=0}^{\infty}$ is generated by Algorithm 4.1, that all the cutters T_i, $i \in I$ are continuous, and that Condition 4.5 holds. If x_∞ is an accumulation point of $\{x^k\}_{k=0}^{\infty}$, then $x_\infty \in F$.*

Proof. Since x_∞ is an accumulation point of $\{x^k\}_{k=0}^{\infty}$, there is a subsequence $\{x^{k_t}\}_{t=0}^{\infty}$ such that $x_\infty = \lim_{t\to\infty} x^{k_t}$. We need to show that $x_\infty \in F_i$ for all $i \in I$. Fix an arbitrary $i \in I$. By Condition 4.5, for all $t \in \mathbb{N} \cup \{0\}$ there is $h_{t,i} \in \{k_t, k_t+1, \ldots, k_t+s-1\}$ such that $w_{h_{t,i}}(i) \geq \alpha$. Proposition 4.13 ensures that $\lim_{t\to\infty} \|x^{h_{t,i}} - x^{k_t}\| = 0$, and hence $\|x^{h_{t,i}} - x_\infty\| \leq \|x^{h_{t,i}} - x^{k_t}\| + \|x^{k_t} - x_\infty\| \xrightarrow[t\to\infty]{} 0 + 0 = 0$. Consider the sequence $\{(w_{h_{t,i}}(1), w_{h_{t,i}}(2), \ldots, w_{h_{t,i}}(m))\}_{t=0}^{\infty}$ of weight vectors in $[0,1]^m$. Since $[0,1]^m$ is a compact subset of $\mathbb{R}^m$, there is an infinite subset N_1 of $\mathbb{N} \cup \{0\}$ and a subsequence $\{(w_{h_{t,i}}(1), w_{h_{t,i}}(2), \ldots, w_{h_{t,i}}(m))\}_{t\in N_1}$ of $\{(w_{h_{t,i}}(1), w_{h_{t,i}}(2), \ldots, w_{h_{t,i}}(m))\}_{t=0}^{\infty}$ which converges to some weight vector $(w_{\infty,i}(1), w_{\infty,i}(2), \ldots, w_{\infty,i}(m))$. This vector satisfies $w_{\infty,i}(i) = \lim_{t\to\infty, t\in N_1} w_{h_{t,i}}(i) \geq \alpha$ because $w_{h_{t,i}}(i) \geq \alpha$ for all $t \in \mathbb{N} \cup \{0\}$ by the choice of $h_{t,i}$.

Now there are two possibilities: either the sequence $\{L(x^{h_{t,i}}, w_{h_{t,i}})\}_{t\in N_1}$ is bounded, or it is unbounded. Consider first the case where this sequence is bounded. Since $\lambda_k \in [\tau_1, (2-\tau_2)L(x^k, w_k)]$ for all $k \in \mathbb{N} \cup \{0\}$, we conclude that $\{\lambda_{h_{t,i}}\}_{t\in N_1}$ is a bounded sequence of positive numbers. Hence, there is a an infinite subset N_2 of N_1 and a subsequence $\{\lambda_{h_{t,i}}\}_{t\in N_2}$ of $\{\lambda_{h_{t,i}}\}_{t\in N_1}$ which converges to some $\lambda_{\infty,i}$ as $t \to \infty, t \in N_2$. Since $\{\lambda_{h_{t,i}}\}_{t\in N_2}$ is also a subsequence of $\{\lambda_k\}_{k\in\mathbb{N}\cup\{0\}}$ and since $\lambda_k \geq \tau_1$ for every $k \in \mathbb{N} \cup \{0\}$, we have $\lambda_{\infty,i} \geq \tau_1 > 0$. Since (4.1) implies that

$$x^{h_{t,i}+1} = T_{w_{h_{t,i}}, \lambda_{h_{t,i}}}(x^{h_{t,i}}) = x^{h_{t,i}} + \lambda_{h_{t,i}} \sum_{j\in I} w_{h_{t,i}}(j)(T_j(x^{h_{t,i}}) - x^{h_{t,i}})$$

and since T_j is continuous for all $j \in I$, we have

$$\lim_{t \to \infty, t \in N_2} x^{h_{t,i}+1} = x_\infty + \lambda_{\infty,i} \sum_{j \in I} w_{\infty,i}(j)(T_j(x_\infty) - x_\infty)$$

$$= x_\infty + \lambda_{\infty,i}(T_{w_{\infty,i}}(x_\infty) - x_\infty).$$

On the other hand, $\lim_{t \to \infty, t \in N_2} \|x^{h_{t,i}+1} - x^{h_{t,i}}\| = 0$ by Proposition **4.13**, and hence one has $\lim_{t \to \infty, t \in N_2} x^{h_{t,i}+1} = \lim_{t \to \infty, t \in N_2}(x^{h_{t,i}+1} - x^{h_{t,i}}) + \lim_{t \to \infty, t \in N_2} x^{h_{t,i}} = x_\infty$. We conclude that $x_\infty = x_\infty + \lambda_{\infty,i}(T_{w_{\infty,i}}(x_\infty) - x_\infty)$, and therefore, using the fact that $\lambda_{\infty,i} > 0$, we have $x_\infty = T_{w_{\infty,i}}(x_\infty)$. But $w_{\infty,i}(i) \geq \alpha > 0$ as explained earlier, namely $i \in \widehat{I}_{w_{\infty,i}}$. Hence, Corollary **3.5** implies that $T_i(x_\infty) = x_\infty$, namely $x_\infty \in F_i$.

It remains to consider the case where $\{L(x^{h_{t,i}}, w_{h_{t,i}})\}_{t \in N_1}$ is unbounded. In this case, since $L(x^{h_{t,i}}, w_{h_{t,i}}) \geq 1 > 0$ for all $t \in N_1$, the unboundedness is from above, and, therefore, there is an infinite set N_2 of N_1 and a subsequence $\{L(x^{h_{t,i}}, w_{h_{t,i}})\}_{t \in N_2}$ of $\{L(x^{h_{t,i}}, w_{h_{t,i}})\}_{t \in N_1}$ for which $\lim_{t \to \infty, t \in N_2} L(x^{h_{t,i}}, w_{h_{t,i}}) = \infty$. Thus, $L(x^{h_{t,i}}, w_{h_{t,i}}) > 1$ for all sufficiently large $t \in N_2$, and hence **(2.2)** implies the equality $L(x^{h_{t,i}}, w_{h_{t,i}}) = \sum_{j \in I} w_{h_{t,i}}(j)\|T_j(x^{h_{t,i}}) - x^{h_{t,i}}\|^2 / \|T_{w_{h_{t,i}}}(x^{h_{t,i}}) - x^{h_{t,i}}\|^2$ for all sufficiently large $t \in N_2$. Since $\{x^{k_t}\}_{t=0}^\infty$ converges, it is bounded (boundedness also follows from the Fejér monotonicity of $\{x^k\}_{k=0}^\infty$), and hence it is contained in some closed ball $B[0, \rho]$ for some $\rho > 0$.

Since the m operators T_j, $j \in I$ are continuous, they are bounded on the compact ball $B[0, \rho]$ by the Weierstrass Extreme Value Theorem. Since $w_k(j) \in [0, 1]$ for all $k \in \mathbb{N} \cup \{0\}$ and all $j \in I$, the previous lines imply that $\sup\{\sum_{j \in I} w_{h_{t,i}}(j)\|T_j(x^{h_{t,i}}) - x^{h_{t,i}}\|^2 \,|\, t \in N_2\} < \infty$. Consequently, the equality $\lim_{t \to \infty, t \in N_2} L(x^{h_{t,i}}, w_{h_{t,i}}) = \infty$ can hold only if $\lim_{t \to \infty, t \in N_2} \|T_{w_{h_{t,i}}}(x^{h_{t,i}}) - x^{h_{t,i}}\| = 0$ (since otherwise there is some $\epsilon > 0$ such that $\|T_{w_{h_{t,i}}}(x^{h_{t,i}}) - x^{h_{t,i}}\| \geq \epsilon$ for every t which belongs to some infinite subset N_3 of N_2, and then $\sup\{L(x^{h_{t,i}}, w_{h_{t,i}}) \,|\, t \in N_3\}$ is bounded by $\sup\{\sum_{j \in I} w_{h_{t,i}}(j)\|T_j(x^{h_{t,i}}) - x^{h_{t,i}}\|^2 \,|\, t \in N_2\}/\epsilon^2 < \infty$, and this contradicts the equality $\lim_{t \to \infty, t \in N_3} L(x^{h_{t,i}}, w_{h_{t,i}}) = \lim_{t \to \infty, t \in N_2} L(x^{h_{t,i}}, w_{h_{t,i}}) = \infty$).

Since $\lim_{t \to \infty, t \in N_2} x^{h_{t,i}} = x_\infty$, one has

$$\lim_{t \to \infty, t \in N_2} T_{w_{h_{t,i}}}(x^{h_{t,i}}) = \lim_{t \to \infty, t \in N_2} \sum_{j \in I} w_{h_{t,i}}(j)T_j(x^{h_{t,i}})$$

$$= \sum_{j \in I} w_{\infty,i}(j)T_j(x_\infty) = T_{w_{\infty,i}}(x_\infty)$$

because of the continuity of the operators T_j, $j \in I$ and the limits $\lim_{t \to \infty, t \in N_2} x^{h_{t,i}} = x_\infty$ and $w_{\infty,i}(j) = \lim_{t \to \infty, t \in N_2} w_{h_{t,i}}(j)$ for all $j \in I$. These equalities, together with the equality $\lim_{t \to \infty, t \in N_2} \|T_{w_{h_{t,i}}}(x^{h_{t,i}}) - x^{h_{t,i}}\| = 0$, imply that $T_{w_{\infty,i}}(x_\infty) = x_\infty$. But $w_{\infty,i}(i) \geq \alpha > 0$ because $w_{h_{t,i}}(i) \geq \alpha$ for all $t \in \mathbb{N} \cup \{0\}$ by the choice of the sequence $\{h_{t,i}\}_{t=0}^\infty$. Hence, $i \in \widehat{I}_{w_{\infty,i}}$ and, therefore, Corollary **3.5** implies that $T_i(x_\infty) = x_\infty$, namely $x_\infty \in F_i$.

Thus, $x_\infty \in F_i$ if either $\{L(x^{h_{t,i}}, w_{h_{t,i}})\}_{t \in N_1}$ is bounded or if it is unbounded. Since i was an arbitrary index in I, we conclude from the previous paragraphs that $x_\infty \in \cap_{i \in I} F_i = F$, as claimed. $\square$

Now we are able to formulate and prove the convergence theorems.

Theorem 4.15. *Suppose that X is a finite-dimensional real Hilbert space and that $\{x^k\}_{k=0}^\infty$ is a sequence generated by Algorithm 4.1, where $\{T_i\}_{i \in I}$ are finitely many continuous cutters defined on X with fixed point sets $\{F_i\}_{i \in I}$ and a common fixed point set $F := \cap_{i \in I} F_i$, where $\{w_k\}_{k=0}^\infty$ is a sequence of weight functions with respect to $I := \{1, 2, \ldots, m\}$ ($m \in \mathbb{N}$), where τ_1 and τ_2 are in the interval $(0, 1]$, and where $\{\lambda_k\}_{k=0}^\infty$ satisfy $\lambda_k \in [\tau_1, (2 - \tau_2) L(x^k, w_k)]$ for all $k \in \mathbb{N} \cup \{0\}$ (and L is defined in (2.2)). If the interior of F is nonempty and Condition 4.4 holds, then $\{x^k\}_{k=0}^\infty$ converges to a point $x^* \in F$.*

Proof. Proposition 4.10 ensures that $\{x^k\}_{k=0}^\infty$ is Fejér monotone with respect to F. Since the interior of F is nonempty, Proposition 4.12(v) ensures that $\{x^k\}_{k=0}^\infty$ converges to a point $x^* \in X$, and since Condition 4.4 holds, Proposition 4.11 ensures that $x^* \in F$, as required. $\square$

Theorem 4.16. *Suppose that X is a finite-dimensional Hilbert space and that $\{x^k\}_{k=0}^\infty$ is a sequence generated by Algorithm 4.1, where $\{T_i\}_{i \in I}$ are finitely many continuous cutters defined on X with fixed point sets $\{F_i\}_{i \in I}$ and a common fixed point set $F := \cap_{i \in I} F_i$, where $\{w_k\}_{k=0}^\infty$ is a sequence of weight functions with respect to $I := \{1, 2, \ldots, m\}$ ($m \in \mathbb{N}$), where τ_1 and τ_2 are in the interval $(0, 1]$, and where $\{\lambda_k\}_{k=0}^\infty$ satisfy $\lambda_k \in [\tau_1, (2 - \tau_2) L(x^k, w_k)]$ for all $k \in \mathbb{N} \cup \{0\}$ (and L is defined in (2.2)). If Condition 4.5 holds, then $\{x^k\}_{k=0}^\infty$ converges to a point $x^* \in F$.*

Proof. Proposition 4.10 ensures that the algorithmic sequence $\{x^k\}_{k=0}^\infty$ is Fejér monotone with respect to F, and hence Parts (ii) and (vi) in Proposition 4.12 ensure that $\{x^k\}_{k=0}^\infty$ has at least one (strong) cluster point. Any such cluster point belongs to F according to Proposition 4.14. Therefore, Parts (iv) and (vi) in Proposition 4.12 imply that $\{x^k\}_{k=0}^\infty$ converges (strongly) to some $x^* \in F$. $\square$

5. Concluding remarks

In this work we presented new results related to a wide class of operators called cutters, and used these results for analyzing an extrapolated block-iterative method aimed at solving the common fixed point problem (CFPP) induced by a finite collection of continuous cutters defined on a finite-dimensional Hilbert space. We showed that the algorithmic sequence produced by the method converges to a solution of the problem under conditions on the dynamic weights which have not been discussed so far in the context of extrapolated algorithms for solving the CFPP induced by cutters. An essential tool in the derivation of our results is the product space formalism of Pierra, a theory which we extended and also clarified a certain issue in it. Since the CFPP has numerous applications in science and engineering, and some instances of it have received a lot of attention over the years, the ideas and results discussed here

can be used in various settings and applications, possibly beyond the ones discussed in this work.

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References

[1] Aharoni; R., and Censor, Y. Block-iterative projection methods for parallel computation of solutions to convex feasibility problems. *Linear Algebra Appl. 120* (1989), 165–175.

[2] Aleyner, A.; and Reich, S. Block-iterative algorithms for solving convex feasibility problems in Hilbert and in Banach spaces. *J. Math. Anal. Appl. 343* (2008), 427–435.

[3] Bargetz, C.; Kolobov, V. I.; Reich, S.; and Zalas, R. Linear convergence rates for extrapolated fixed point algorithms. *Optimization 68* (2019), 163–195.

[4] Bauschke, H. H.; and Borwein, J. M. On projection algorithms for solving convex feasibility problems. *SIAM Review 38* (1996), 367–426.

[5] Bauschke, H. H.; Borwein, J. M.; and Combettes, P. L. Bregman monotone optimization algorithms. *SIAM J. Control Optim. 42* (2003), 596–636.

[6] Bauschke, H. H.; and Combettes, P. L. A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces. *Math. Oper. Res. 26* (2001), 248–264.

[7] Bauschke, H. H., and Combettes, P. L. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, 2 ed. CMS Books in Mathematics. Springer International Publishing, Cham, Switzerland, 2017.

[8] Bauschke, H. H.; Combettes, P. L.; and Kruk, S. G. Extrapolation algorithm for affine-convex feasibility problems. *Numer. Algorithms 41* (2006), 239–274.

[9] Bauschke, H. H.; Wang, C.; Wang, X.; and Xu, J. On the finite convergence of a projected cutter method. *J. Optim. Theory Appl. 165* (2015), 901–916.

[10] Berinde, V. *Iterative Approximation of Fixed Points*, second ed., vol. 1912 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.

[11] Bregman, L. M. The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. *Comput. Math. Math. Phys. 7* (1967), 200–217.

[12] Browder, F. E. Convergence theorems for sequences of nonlinear operators in Banach spaces. *Math. Z. 100* (1967), 201–225.

[13] Buong, N.; and Anh, N. T. Q. Extrapolated simultaneous block-iterative cutter methods and applications. *Math. Methods Appl. Sci. 46* (2023), 14229–14242.

[14] Buong, N.; and Nguyen, N. D. Convergence of extrapolated dynamic string-averaging cutter methods and applications. *East Asian J. Appl. Math. 13* (2023), 257–275.

[15] Cegielski, A. *Iterative Methods for Fixed Point Problems in Hilbert Spaces*, vol. 2057 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012.

[16] Cegielski, A. General method for solving the split common fixed point problem. *J. Optim. Theory Appl. 165* (2015), 385–404.

[17] Cegielski, A.; and Censor, Y. Opial-type theorems and the common fixed point problem. In *Fixed-Point Algorithms for Inverse Problems in Science and Engineering* (New York, NY, 2011), H. H. Bauschke, R. S. Burachik, P. L. Combettes, V. Elser, D. R. Luke, and H. Wolkowicz, Eds., Springer New York, pp. 155–183.

[18] Cegielski, A.; and Censor, Y. Extrapolation and local acceleration of an iterative process for common fixed point problems. *J. Math. Anal. Appl. 394* (2012), 809–818.

[19] Cegielski, A.; and Nimana, N. Extrapolated cyclic subgradient projection methods for the convex feasibility problems and their numerical behaviour. *Optimization 68* (2019), 145–161.

[20] Censor, Y., Chen, W., Combettes, P. L., Davidi, R., and Herman, G. T. On the effectiveness of projection methods for convex feasibility problems with linear inequality constraints. *Comput. Optim. Appl. 51*, 3 (2012), 1065–1088.

[21] Censor, Y.; and Reem, D. Zero-convex functions, perturbation resilience, and subgradient projections for feasibility-seeking methods. *Math. Prog. (Ser. A) 152* (2015), 339–380.

[22] Censor, Y.; Reem, D.; and Zaknoon, M. A generalized block-iterative projection method for the common fixed point problem induced by cutters. *J. Global Optim. 84* (2022), 967–987.

[23] Censor, Y.; and Segal, A. On the string averaging method for sparse common fixed-point problems. *Int. Trans. Oper. Res. 16* (2009), 481–494.

[24] Censor, Y.; and Segal, A. The split common fixed point problem for directed operators. *J. Nonlinear Convex Anal. 16* (2009), 587–600.

[25] Censor, Y.; and Zenios, A. S. *Parallel Optimization: Theory, Algorithms, and Applications.* Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 1997. With a foreword by George B. Dantzig.

[26] Combettes, P. L. The convex feasibility problem in image recovery. vol. 95 of *Advances in Imaging and Electron Physics.* Elsevier, 1996, pp. 155–270.

[27] Combettes, P. L. Hilbertian convex feasibility problem: convergence of projection methods. *Appl. Math. Optim. 35* (1997), 311–330.

[28] Combettes, P. L. Strong convergence of block-iterative outer approximation methods for convex optimization. *SIAM J. Control Optim. 38* (2000), 538–565.

[29] Combettes, P. L. Quasi-Fejérian analysis of some optimization algorithms. In *Inherently Parallel Algorithms in Feasibility and Optimization and their Applications*, D. Butnariu, Y. Censor, and S. Reich, Eds., Elsevier Science Publishers, Amsterdam, The Netherlands, 2001, pp. 115–152.

[30] Combettes, P. L.; and Woodstock, Z. C. Reconstruction of functions from prescribed proximal points. *J. Approx. Theory 268* (2021), Paper No. 105606, 26.

[31] Crombez, G. Finding common fixed points of strict paracontractions by averaging strings of sequential iterations. *J. Nonlinear Convex Anal. 3* (2002), 345–351.

[32] Dos Santos, L. T. A parallel subgradient projections method for the convex feasibility problem. *J. Comput. Appl. Math. 18* (1987), 307–320.

[33] Dugundji, J.; and Granas, A. *Fixed Point Theory.* Springer Monographs in Mathematics. Springer-Verlag, New York, USA, 2003.

[34] Dunford, N.; and Schwartz, J. T. *Linear Operators. I. General Theory.* With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7. Interscience Publishers, Inc., New York; London, 1958.

[35] Elfving, T., Hansen, P. C., and Nikazad, T. Semiconvergence and relaxation parameters for projected SIRT algorithms. *SIAM J. Sci. Comput. 34*, 4 (2012), A2000–A2017.

[36] Kimura, Y.; and Saejung, S. Strong convergence for a common fixed point of two different generalizations of cutter operators. *Linear Nonlinear Anal. 1* (2015), 53–65.

[37] Kirk, W. A.; and Sims, B., Eds. *Handbook of Metric Fixed Point Theory.* Kluwer Academic Publishers, Dordrecht, 2001.

[38] Kiwiel, K. C. Block-iterative surrogate projection methods for convex feasibility problems. *Linear Algebra Appl. 215* (1995), 225–259.

[39] Kolobov, V. I.; Reich, S.; and Zalas, R. Weak, strong, and linear convergence of a double-layer fixed point algorithm. *SIAM J. Optim. 27* (2017), 1431–1458.

[40] Lent, A. Maximum entropy and multiplicative ART. In *Image Analysis and Evaluation, SPSE Conference Proc.* (Toronto, Canada, 1976), R. Shaw, Ed., pp. 249–257.

[41] Merzlyakov, Yu. I. On a relaxation method of solving systems of linear inequalities. *USSR Computational Mathematics and Mathematical Physics 2* (1963), 504–510. Published first in *Ž. Vyčisl. Mat i Mat. Fiz. 2* (1962), 482–487.

[42] Millán, R. D.; Lindstrom, S. B.; and Roshchina, V. Comparing averaged relaxed cutters and projection methods: Theory and examples. In *From Analysis to Visualization* (Cham, 2020), D. H. Bailey, N. S. Borwein, R. P. Brent, R. S. Burachik, J. H. Osborn, B. Sims, and Q. J. Zhu, Eds., Springer International Publishing, pp. 75–98.

[43] Nikazad, T.; and Mirzapour, M. Generalized relaxation of string averaging operators based on strictly relaxed cutter operators. *J. Nonlinear Convex Anal. 18* (2017), 431–450.

[44] Nimana, N.; and Artsawang, N. A strongly convergent simultaneous cutter method for finding the minimal norm solution to common fixed point problem. *Carpathian J. Math. 40* (2024), 155–171.

[45] Ottavy, N. Strong convergence of projection-like methods in Hilbert spaces. *J. Optim. Theory Appl. 56* (1988), 433–461.

[46] Pierra, G. Decomposition through formalization in a product space. *Math. Prog. 28* (1984), 96–115.

[47] Reich, S.; and Zalas, R. A modular string averaging procedure for solving the common fixed point problem for quasi-nonexpansive mappings in Hilbert space. *Numer. Algorithms 72* (2016), 297–323.

[48] Rus, I. A.; Petruşel, A.; and Petruşel, G. *Fixed Point Theory*. Cluj University Press, Cluj-Napoca, 2008.

[49] Wang, F.; and Xu, H.-K. Cyclic algorithms for split feasibility problems in Hilbert spaces. *Nonlinear Anal. 74* (2011), 4105–4111.

[50] Younis, M.; Chen, L.; and Singh, D., Eds. *Recent Developments in Fixed-Point Theory—Theoretical Foundations and Real-World Applications*. Industrial and Applied Mathematics. Springer, Singapore, [2024] ©2024.

[51] Zaknoon, M. *Algorithmic Developments for the Convex Feasibility Problem*. PhD thesis, University of Haifa, Haifa, Israel, 2003.

[52] Zaslavski, A. J. *Approximate Solutions of Common Fixed-Point Problems*, vol. 112 of *Springer Optimization and Its Applications*. Springer, [Cham], 2016.

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