Performance Estimation for Smooth and Strongly Convex Sets

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Abstract

We extend recent computer-assisted design and analysis techniques for first-order optimization over structured functions—known as performance estimation—to apply to structured sets. We prove "interpolation theorems" for smooth and strongly convex sets with Slater points and bounded diameter, showing a wide range of extremal questions amount to structured mathematical programs. Prior function interpolation theorems are recovered as a limit of our set interpolation theory. Our theory provides finite-dimensional formulations of performance estimation problems for algorithms utilizing separating hyperplane oracles, linear optimization oracles, and/or projection oracles of smooth/strongly convex sets. As direct applications of this computer-assisted machinery, we identify the minimax optimal separating hyperplane method and several areas for improvement in the theory of Frank-Wolfe, Alternating Projections, and non-Lipschitz Smooth Optimization. While particular applications and methods are not our primary focus, several simple theorems and numerically supported conjectures are provided.

1 Introduction

Given a proposed algorithm, the study of its worst-case convergence guarantees over some family of problem instances can be framed as a meta-optimization problem. In this work, we will consider examples of first-order optimization methods applied to problems from a structured family of objective functions and constraint sets. Given some measure of performance (e.g., final objective gap), the exact worst-case performance of the method corresponds to computing the maximum of this measure at the algorithm's output over all problem instances. This is known as the Performance Estimation Problem (PEP). In general, optimizing over a family of functions and sets is an infinite-dimensional problem, typically beyond direct approach. Surprisingly, recent advances have shown that PEPs can often be computer-solved and have led to several state-of-the-art results in the design and analysis of optimization algorithms.

Performance estimation was first proposed by Drori and Teboulle in [1], where tractable relaxations of these worst-case PEPs were considered. Shortly afterward, the "interpolation theorems" of Taylor, Hendrickx, and Glineur [2] proved that for many gradient methods and structured function classes, the PEP exactly corresponds to a tractable finite-dimensional problem. Specifically, their function interpolation results showed that the infinite-dimensional optimization problem of finding a worst-case problem instance for N steps of a given gradient method could be equivalently reformulated as a nonconvex Quadratically Constrained Quadratic Program (QCQP), which can often then be reformulated as a Semidefinite Program (SDP). As a result, for a modest fixed number of algorithm steps N, computers can be used to determine an algorithm's worst-case problem instances and, dually, to produce optimal convergence proofs. This computer assistance can provide an invaluable starting point for formally proving improved theory for general N.

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Performance estimation has been applied to many settings and algorithms, including conditional gradient [3], proximal gradient [3], and stochastic gradient descent [4]. Computer PEP solutions directly produce tight numerical convergence guarantees and, in many cases, have led to tight analytical convergence rates. See the growing collection of examples documented in the PEPit toolbox [5][1].

PEPs have a record of directly improving the state of the art in algorithm design by enabling the optimization of parameter selections. In general, the task of finding algorithmic parameters (e.g., stepsizes and momentum sequences) minimizing the worst-case PEP guarantee is a nonconvex problem[2]. We note two recent advances in smooth convex optimization despite this hardness:

- Kim and Fessler [7] proposed an Optimized Gradient Method, improving on Nesterov's accelerated method by a factor of two and attaining the minimax optimal rate by Drori [8].
- Altschuler and Parrilo [9, 10] and Grimmer, Shu, and Wang [11, 12] showed gradient descent with certain fractal stepsize patterns is provably big-O faster than with constant stepsizes.

Both these advances made progress on long-standing, best-known results in large part due to the insights generated from performance estimation and computer assistance.

The primary aim of this work is the development of new interpolation theorems for smooth and strongly convex sets, potentially with Slater points and bounded diameter, paralleling the function-oriented theorems of Taylor et al. [2]. Prior function interpolation theorems in [3] could be used to model simply convex constraint sets via indicator functions but cannot capture the above structural properties. Our theorems enable the formulation of computer-solvable PEPs, not just for optimization over structured families of objectives but also for structured families of constraint sets. Formal definitions of smoothness and strong convexity of closed convex sets and functions (and several equivalent conditions) are given in Section 2. In brief, a closed convex function f is smooth and strongly convex if each subgradient provides certain quadratic upper and lower bounds on f, respectively. Similarly, a closed convex set C is smooth and strongly convex if each normal vector provides certain balls inner and outer approximating C.

1.1 Our Contributions

Our primary contribution is a collection of interpolation theorems for structured sets. Our theorems provide tractable, finite representations for a variety of settings, including smooth and strongly convex sets, sets with bounded diameters, and sets with interior/Slater points. See Theorems 3.1-3.3. These interpolations allow us to express the worst-case performance of many constrained optimization algorithms as numerically solvable performance estimation problems. Theorems 3.7 and 3.8 shows prior function interpolation theory is captured as a limiting case of our set interpolation theory. The remainder of this paper then considers four illustrative application settings, outlined below in brief:

- **Feasibility Problems with Separating Hyperplane Oracle (Section 4).** Given a separating hyperplane oracle for a set C, one can iteratively seek a feasible point. Our interpolation theorems provide a semidefinite programming approach to computing the worst-case termination time for any such procedure as a function of smoothness, strong convexity, and quality of interior point contained in C. Matching our numerically computed guarantees, we prove a simple separating hyperplane algorithm is minimax optimal across any such setting.

- **Frank-Wolfe Convergence Across Numerous Problem Settings (Section 5).** Given a smooth objective and bounded constraint set, the Frank-Wolfe (i.e., conditional gradient)

[1] Available at `https://pepit.readthedocs.io/en/0.3.2/examples.html`.

[2] A branch-and-bound solver for this hard task was recently developed by Das Gupta et al. [6].

method is known to converge at rate $O(1/N)$ [13]. Under the additional assumptions of strong convexity of the objective and constraint set, convergence accelerates to $O(1/N^2)$ [14, 15]. Applying our theory enables us to quantify the suboptimality of existing theory and explore both interim settings and not previously explored settings involving smooth constraint sets. These insights follow from our interpolation theorems, casting the worst-case performance across all of these settings as an SDP with separable nonconvex quadratic equality constraints.

- **Exact Linear Convergence Rates for Alternating Projections (Section 6).** Given two convex sets and a projection oracle for each, one can seek a point in their intersection by the method of alternating projections. This method is well-known to converge linearly whenever the sets possess a nontrivial intersection. We show that this worst-case performance can also be computed as an SDP with separable quadratic equality constraints. From our results, we motivate a conjecture on an exact formula for the method's worst-case linear rate.

- **Gradient Methods for Unconstrained Epismooth Minimization (Section 7).** Finally, our interpolation theorems enable the study of unconstrained minimization of functions with an L-smooth epigraph. Such epismooth functions include, for example, all polynomials with consistent growth, see Section 7.1, whereas the classic model of uniformly Lipschitz gradient only captures quadratic polynomials. We show existing (accelerated) convergence rates for smooth optimization can be lifted to epismooth optimization in an appropriate limiting sense.

Outline. Section 2 defines basic notations and presents the function interpolation theorems of [2]. Section 3 then presents our main result: interpolation theorems for structured sets. Section 4 presents a brief and simple application to separating hyperplane algorithms where computations reduce to SDPs. Section 5 and Section 6 provide insights for Frank-Wolfe and Alternating Projections. Finally, Section 7 applies our theory to the unconstrained optimization of epismooth functions. All numerical results were obtained with `Mosek` [16] or `Gurobi` [17] via `JuMP` [18]. Code for reproducibility is available at `github.com/alanluner/PEPStructuredSets`.

2 Definitions and Preliminaries

Section 2.1 first introduces standard definitions of smooth and strongly convex functions and the function interpolation of Taylor et al. [2]. Section 2.2 then introduces the analogous concepts for sets in preparation for our main results.

2.1 Preliminaries on Smooth and Strongly Convex Functions

Consider a closed convex function $f\colon \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$. We denote $\langle \cdot, \cdot \rangle$ for the Euclidean inner product and use the associated two-norm for all norms throughout. We denote the domain of f by dom f and its subdifferential at some $x \in \text{dom } f$ by $\partial f(x) = \{g \mid f(y) \geq f(x) + \langle g, y - x\rangle \quad \forall y \in \mathbb{R}^d\}$. Each element of the subdifferential is referred to as a subgradient of f at x. Note when f is differentiable, the gradient $\nabla f(x)$ is the unique element of the subdifferential.

Typically, L-smoothness of a function is defined as $\nabla f(x)$ being L-Lipschitz. For the sake of developing other symmetries, here we instead (equivalently) define a closed convex f as being **L-smooth** if for all $x, y \in \mathbb{R}^d$ and $g \in \partial f(x)$, we have

$$f(y) \leq f(x) + \langle g, y - x\rangle + \frac{L}{2}\|y - x\|^2 \ . \tag{2.1}$$

That is, each subgradient provides a quadratic upper bound on f. Mirroring this, we say f is **μ-strongly convex** if for all $x, y \in \mathbb{R}^d$ and $g \in \partial f(x)$, we have

$$f(y) \geq f(x) + \langle g, y - x \rangle + \frac{\mu}{2}\|y - x\|^2 \ . \tag{2.2}$$

That is, each subgradient provides a quadratic lower bound on f. We abuse notation and allow the extreme cases of $L, \mu \in \{0, \infty\}$, taking limits above. Every convex f is vacuously ∞-smooth and 0-strongly convex. The only 0-smooth functions are linear and the only ∞-strongly convex functions are indicators of single points. As a shorthand, we denote the set of all μ-strongly convex and L-smooth functions f by $\mathcal{F}_{\mu,L}$.

Function Interpolation and Performance Estimation. As a motivating example of the performance estimation problem, consider minimizing some $f \in \mathcal{F}_{\mu,L}$ by applying N steps of gradient descent with initial point x_0 and some stepsizes h_k, defined as

$$x_{k+1} = x_k - h_k \nabla f(x_k) \ .$$

Suppose as an initial condition, we are guaranteed the initial distance to a minimizer $\|x_0 - x_\star\|$ is at most one. Then the infinite-dimensional PEP of seeking a problem instance with objective in $\mathcal{F}_{\mu,L}$ achieving the largest final objective gap can be formulated as

$$\begin{cases} \max\limits_{x_i, f} & f(x_N) - f(x_\star) \\ \text{s.t.} & f \in \mathcal{F}_{\mu,L} \\ & \|x_0 - x_\star\| \leq 1 \\ & x_{k+1} = x_k - h_k \nabla f(x_k), \quad \nabla f(x_\star) = 0 \ . \end{cases} \tag{2.3}$$

Note that aside from the constraint $f \in \mathcal{F}_{\mu,L}$, this optimization problem only depends on f via its function value and gradient at finitely points, namely $\{x_0, \dots, x_N, x_\star\}$. Denote these values by $f_i = f(x_i)$ and $g_i = \nabla f(x_i)$ for each $i \in \{0, \dots, N, \star\}$. To reformulate the above problem in these more limited quantities, consider the following definition of function interpolation from [2]:

Definition 2.1. Consider a set of observations $S = \{(x_i, g_i, f_i)\}_{i \in \mathcal{I}}$ where $x_i, g_i \in \mathbb{R}^d$ and $f_i \in \mathbb{R}$ for all $i \in \mathcal{I}$. The set S is **$\mathcal{F}_{\mu,L}$-interpolable** if there exists a function $f \in \mathcal{F}_{\mu,L}$ such that $g_i \in \partial f(x_i)$ and $f(x_i) = f_i$ for all $i \in \mathcal{I}$.

Function interpolability describes whether there exists an appropriate function that matches the given finite set of function values and subgradients and interpolates between them (See Figure 2). In terms of the working example (2.3), this can then be expressed finitely as

$$\begin{cases} \max\limits_{x_i, g_i, f_i} & f_N - f_\star \\ \text{s.t.} & \{(x_i, g_i, f_i)\}_{i \in \{0, \dots, N, \star\}} \text{ is } \mathcal{F}_{\mu,L}\text{-interpolable} \\ & \|x_0 - x_\star\| \leq 1 \\ & x_{k+1} = x_k - h_k g_k, \quad g_\star = 0 \ . \end{cases} \tag{2.4}$$

Although finite, this formulation is still not particularly tractable as the interpolability constraint is, at first glance, not approachable. The critical insight of Taylor et al. [2, Theorem 4] was establishing explicit necessary and sufficient conditions for function interpolability.

Theorem 2.2. ([2, Theorem 4]) *A set $S = \{(x_i, g_i, f_i)\}_{i\in\mathcal{I}}$ is $\mathcal{F}_{\mu,L}$-interpolable if and only if the following quadratic condition holds for all $i, j \in \mathcal{I}$:*

$$f_i - f_j + \langle g_i, x_j - x_i\rangle + \frac{1}{2(1-\mu/L)}\left(\frac{1}{L}\|g_i - g_j\|^2 + \mu\|x_i - x_j\|^2 - 2\frac{\mu}{L}\langle g_i - g_j, x_i - x_j\rangle\right) \le 0 \ . \quad (2.5)$$

Plugging this into (2.4) yields an explicit QCQP exactly describing gradient descent's worst-case behavior over the considered family of structured functions. Semidefinite programming lifting techniques (and duality) can be further used to gain insights from this new formulation.

It is worth noting that there are many collections of necessary constraints on the set of observation data S. Any standard inequality for μ-strongly convex, L-smooth functions must necessarily hold, for example, the definitions (2.1) and (2.2). However, as Taylor et al. [2] highlight, these other inequalities prove not to be sufficient for interpolation. Hence, identification of the right inequalities, e.g., (2.5), is key for the "if and only if" nature of interpolation theory and, consequently, its power in enabling computer assistance. Similarly, in Section 3, we find that while there are many necessary constraints for set interpolability, a careful selection is needed for sufficiency.

2.2 Preliminaries on Smooth and Strongly Convex Sets

We now switch our focus to smoothness and strong convexity of sets. These notions mirror their function counterparts; see [19–21] as classic references. These properties are, however, quite under-explored in the contemporary first-order method literature compared to their function counterparts, which are the backbone of much modern convergence theory. In part, this work aims to provide tools to eventually alleviate this discrepancy.

Consider a nonempty closed convex set $C \subseteq \mathbb{R}^d$. We denote the interior of a set C by int C and its boundary by bdry C. We denote the normal cone of C at some $z \in \text{bdry } C$ by $N_C(z) = \{v \mid \langle v, x - z\rangle \le 0 \quad \forall x \in C\}$ and refer to individual elements as normal vectors. Let $B(x, r)$ be the closed ball of radius r centered at x. We define $\text{diam}(C) = \sup\{\|x - y\| \mid x, y \in C\}$ and the δ-interior of C by $\text{int}_\delta C = \{x \mid B(x, \delta) \subseteq C\}$. Note $\text{int}_0 C = C$. Lastly, we define the Minkowski sum of two sets C_1 and C_2 as

$$C_1 + C_2 = \{x + y \mid x \in C_1, y \in C_2\} \ .$$

We now define smoothness and strong convexity with respect to sets. For a fuller discussion of the equivalent definitions of these properties and their analogs with smooth and strongly convex functions, we refer readers to [22, Section 2]. Overall, one can recover the analogous definitions for functions by replacing unit normal vectors with gradients, bounding balls with bounding quadratics, and sets with epigraphs. A set C is **β-smooth** if for any $z \in \text{bdry } C$ and unit vector $n \in N_C(z)$,

$$B(z - \frac{1}{\beta}n, \frac{1}{\beta}) \subseteq C \ .$$

A set C is **α-strongly convex** if for any $z \in \text{bdry } C$ and unit vector $n \in N_C(z)$,

$$B(z - \frac{1}{\alpha}n, \frac{1}{\alpha}) \supseteq C \ .$$

Again, we allow the limiting cases of $\alpha, \beta \in \{0, \infty\}$. Every convex set is 0-strongly convex and ∞-smooth. The only ∞-strongly convex sets are singletons and the only 0-smooth sets are halfspaces. As a shorthand, we let $\mathcal{C}_{\alpha,\beta,D}$ denote the set of all closed convex sets C that are α-strongly convex, β-smooth, and have $\text{diam}(C) \le D$.

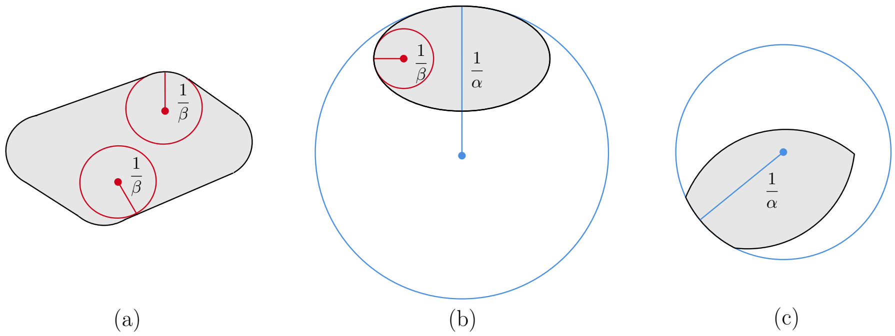

Figure 1: Example closed convex sets with red inner approximating balls from smoothness and blue outer approximating balls from strong convexity. Namely, (a) is β-smooth but not strongly convex, (b) is β-smooth and α-strongly convex, and (c) is α-strongly convex but not smooth.

Figure 1 showcases several examples of smooth and/or strongly convex sets. As additional classic examples with these properties, any vector p-norm ball and Schatten matrix p-norm ball is smooth if $p \in [2, \infty)$ and strongly convex if $p \in (1, 2]$ [14, Lemma 3]. Moreover, level sets of any smooth and/or strongly convex function typically inherit this structure [22, Lemma 6].

Useful Characterizations and Lemmas on Structured Sets. Below are several equivalent characterizations of set smoothness and strong convexity that will be useful in our analysis. These characterizations parallel classic results for functions.

Proposition 2.3. ([22, Proposition 3]) *For any closed convex set C, the following are equivalent:*

1. *C is β-smooth.*
2. *For any $z_1, z_2 \in \operatorname{bdry} C$, with unit normal vectors $n_1 \in N_C(z_1)$ and $n_2 \in N_C(z_2)$,*

$$\langle z_1 - z_2, n_1 - n_2 \rangle \geq \frac{1}{\beta} \|n_1 - n_2\|^2 \ .$$

3. *There exists a closed convex set C_0 such that $C_0 + B(0, \frac{1}{\beta}) = C$.*

Proposition 2.4. ([21, Theorem 2.1]) *For any closed convex set C, the following are equivalent:*

1. *C is α-strongly convex.*
2. *For any $z_1, z_2 \in \operatorname{bdry} C$, with unit normal vectors $n_1 \in N_C(z_1)$ and $n_2 \in N_C(z_2)$,*

$$\langle z_1 - z_2, n_1 - n_2 \rangle \leq \frac{1}{\alpha} \|n_1 - n_2\|^2 \ .$$

3. *There exists a closed convex set C_0 such that $C_0 + C = B(0, \frac{1}{\alpha})$.*

Many more equivalent characterizations of strongly convex sets are given in [21]. Building on the above equivalences, the following pair of lemmas will also help with our upcoming theory.

Lemma 2.5. *A set C is β-smooth with $\operatorname{diam}(C) \leq D$ if and only if $C = C_0 + B(0, \frac{1}{\beta})$ for some convex set C_0 with $\operatorname{diam}(C_0) \leq D - \frac{2}{\beta}$.*

Proof. We know from Proposition 2.3 that C is β-smooth if and only if there exists a convex set C_0 such that $C = C_0 + B(0, \frac{1}{\beta})$. So the result follows as

$$\begin{aligned}\operatorname{diam}(C) &= \sup\{\|x + \frac{1}{\beta}n_x - (y + \frac{1}{\beta}n_y)\| \mid x, y \in C_0, \|n_x\| \le 1, \|n_y\| \le 1\} \\ &= \sup\{\|x - y\| \mid x, y \in C_0\} + \frac{2}{\beta} = \operatorname{diam}(C_0) + \frac{2}{\beta} .\end{aligned}$$

□

Lemma 2.6. *A set C is α-strongly convex and β-smooth if and only if $C = C_0 + B(0, \frac{1}{\beta})$ for some γ-strongly convex set C_0, with $\gamma := (\frac{1}{\alpha} - \frac{1}{\beta})^{-1} = \frac{\alpha\beta}{\beta-\alpha}$.*

Proof. This follows from property 3 in both Propositions 2.3 and 2.4. Namely, C is α-strongly convex and β-smooth if and only if there exist convex sets $C_0^{\mathtt{sm}}$ and $C_0^{\mathtt{sc}}$ such that $C + C_0^{\mathtt{sc}} = B(0, \frac{1}{\alpha})$ and $C = C_0^{\mathtt{sm}} + B(0, \frac{1}{\beta})$. Together, these properties imply $C_0^{\mathtt{sm}} + B(0, \frac{1}{\beta}) + C_0^{\mathtt{sc}} = B(0, \frac{1}{\alpha})$, and so $C_0^{\mathtt{sm}} + C_0^{\mathtt{sc}} = B(0, \frac{1}{\gamma})$, i.e., $C_0^{\mathtt{sm}}$ is γ-strongly convex. Conversely, if $C = C_0 + B(0, \frac{1}{\beta})$ for some γ-strongly convex C_0, it immediately follows that C is β-smooth. Further some convex C_0' must exist with $C_0 + C_0' = B(0, \frac{1}{\gamma})$. Hence $C + C_0' = B(0, \frac{1}{\alpha})$, i.e., C is α-strongly convex. □

Set Interpolation. Finally, we define set interpolability, paralleling the function interpolability of Definition 2.1. In addition to smoothness and strong convexity, we model two additional structures: diameter bounds and various interior/Slater point conditions, which occur widely in constrained optimization theory.

We consider a set of observations S defined by two types of data: points $z_i \in \mathbb{R}^d$ on the boundary of the constraint set with nonzero normal vectors $v_i \in \mathbb{R}^d$ and points $x_k \in \mathbb{R}^d$ in the δ_k-interior of the set. Note the index sets $i \in \mathcal{I}$ and $k \in \mathcal{K}$ need not be of the same size or even both be nonempty. Given target strong convexity α, smoothness β, and diameter D, we can then formalize whether given observation data can be interpolated by some such set as follows.

Definition 2.7. Consider a set of observations $S = (\{(z_i, v_i)\}_{i\in\mathcal{I}}, \{(x_k, \delta_k)\}_{k\in\mathcal{K}})$, with $z_i, v_i, x_k \in \mathbb{R}^d$ and $\delta_k \in \mathbb{R}_{\ge 0}$ for all $i \in \mathcal{I}$ and all $k \in \mathcal{K}$. The set S is **$\mathcal{C}_{\alpha,\beta,D}$-interpolable** if there exists $C \in \mathcal{C}_{\alpha,\beta,D}$ such that $z_i \in C$ and $v_i \in N_C(z_i)$ for all $i \in \mathcal{I}$ and $x_k \in \operatorname{int}_{\delta_k} C$ for all $k \in \mathcal{K}$.

Note that this definition allows $\delta_k = 0$ (denoting the corresponding x_k simply must be a member of C) and all limiting values $\alpha, \beta, D \in \{0, \infty\}$.

This work's primary contribution is identifying computationally tractable, verifiable equivalent conditions for set interpolability. We present these results in Section 3. Before presenting our general theory, we first highlight an illustrative corollary of our Theorem 3.3 for the simplified setting where $\alpha = 0$ and $\beta = \infty$. In this case, interpolability corresponds to determining if a bounded diameter convex set exists with the required normals $v_i \in N_C(z_i)$ and δ_k-interior points x_k. Our theory provides the following characterization, slightly generalizing the indicator function result of [3, Theorem 3.6].

Corollary 2.8. *A set $S = (\{(z_i, v_i)\}_{i\in\mathcal{I}}, \{(x_k, \delta_k)\}_{k\in\mathcal{K}})$ is $\mathcal{C}_{0,\infty,D}$-interpolable if and only if for all $i, j \in \mathcal{I}$ and $k, l \in \mathcal{K}$,*

$$\begin{aligned}&\langle n_i, z_j - z_i\rangle \le 0\\ &\langle n_i, x_k + \delta_k n_i - z_i\rangle \le 0\\ &\|x_k - x_l\| \le D - \delta_k - \delta_l\\ &\|z_i - x_k\| \le D - \delta_k\end{aligned}$$

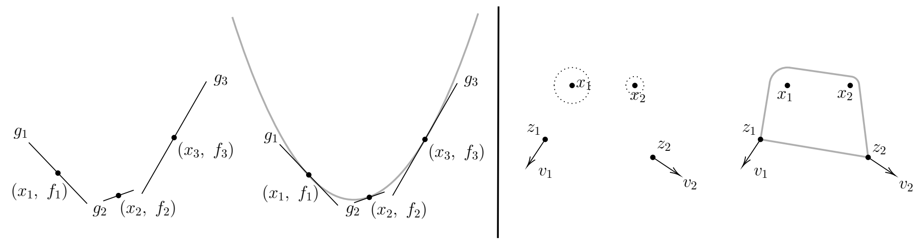

Figure 2: Left: Function interpolation of a collection of function values f_i and subgradients g_i evaluated at x_i. Right: Set interpolation of a collection of boundary points z_i with normal vectors v_i and δ_k-interior points x_k.

$$\|z_i - z_j\| \leq D$$

where $n_i = \frac{v_i}{\|v_i\|}$ *for all* $i \in \mathcal{I}$.

When these conditions hold, a simple, explicit construction of the interpolating convex, bounded set exists: $C = \text{conv}(\{z_i\}_{i\in\mathcal{I}}, \{B(x_k, \delta_k)\}_{k\in\mathcal{K}})$. Figure 2 shows an example of this simplified case's construction. Designing and analyzing such a construction is the primary step in proving the sufficiency of given interpolation conditions. Our constructions grow substantially in complexity from this easy case, but remain explicit. As a result, one can always recover a (smooth, strongly convex, bounded diameter) set that exactly interpolates any observed normal vectors and interior points satisfying our theory's conditions.

3 Main Result: Set Interpolation Theory

This section presents and proves our main results on set interpolability.

We first define a set of conditions that will be necessary throughout our theory. Consider a set of observations $S = (\{(z_i, v_i)\}_{i\in\mathcal{I}}, \{(x_k, \delta_k)\}_{k\in\mathcal{K}})$ and denote the associated unit normal vectors by $n_i = \frac{v_i}{\|v_i\|}$ for all $i \in \mathcal{I}$. Due to the added complexity of our set interpolation model, our conditions involve the introduction of auxiliary parameters $\{w_k\}_{k\in\mathcal{K}} \subset \mathbb{R}^d$ along with a constant factor $\lambda \in (0, 1]$, typically equal to one, enabling strengthened versions of these conditions. We find that for any $\mathcal{C}_{\alpha,\beta,D}$-interpolable observation data $S = (\{(z_i, v_i)\}_{i\in\mathcal{I}}, \{(x_k, \delta_k)\}_{k\in\mathcal{K}})$, there must exist auxiliary parameters $\{w_k\}_{k\in\mathcal{K}}$ such that for all $i, j \in \mathcal{I}$ and $k, l \in \mathcal{K}$,

$$\|z_i - \frac{1}{\alpha}n_i - (z_j - \frac{1}{\beta}n_j)\| \leq \frac{1}{\gamma} \tag{Interp1}$$

$$\|z_i - \frac{1}{\alpha}n_i - w_k\| \leq \frac{1}{\gamma} - s_k \tag{Interp2}$$

$$\|x_k - w_k\| \leq \frac{1}{\beta} - \delta_k + s_k \tag{Interp3}$$

$$\|z_i - \frac{1}{\beta}n_i - (z_j - \frac{1}{\beta}n_j)\| \leq \lambda(D - \frac{2}{\beta}) \tag{Interp4}$$

$$\|z_i - \frac{1}{\beta}n_i - w_k\| \leq \lambda(D - \frac{2}{\beta}) - s_k \tag{Interp5}$$

$$\|w_k - w_l\| \leq \lambda(D - \frac{2}{\beta}) - s_k - s_l \tag{Interp6}$$

where $\gamma = (\frac{1}{\alpha} - \frac{1}{\beta})^{-1}$ and $s_k = \max\{0, \delta_k - \frac{1}{\beta}\}$ for all $k \in \mathcal{K}$. In the case $\alpha = \beta$, we will use the

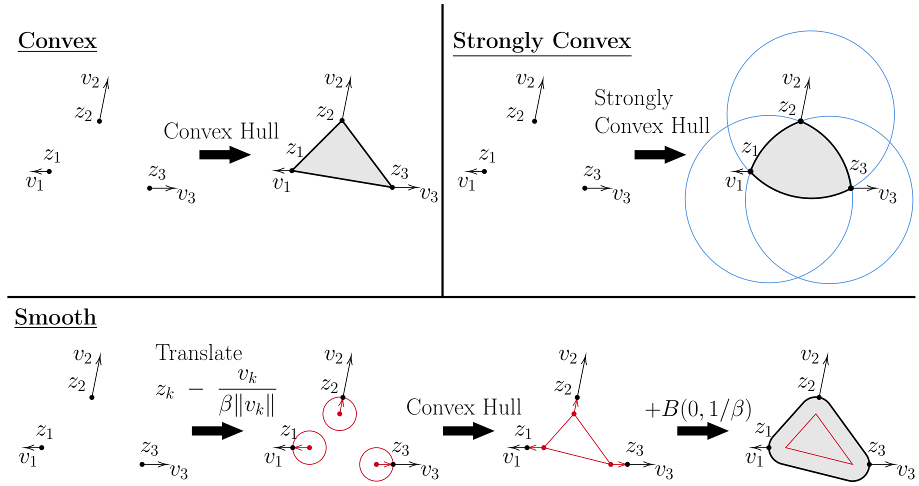

Figure 3: Examples of construction of interpolating sets in cases with no Slater points ($\mathcal{K} = \emptyset$).

convention that $\frac{1}{\gamma} = 0$. As a shorthand, we write $S \in \text{Interp}(\alpha, \beta, D; \lambda)$ if $\{w_k\}_{k\in\mathcal{K}} \subset \mathbb{R}^d$ exist satisfying all six conditions.

Note that if one only enforces smoothness and strong convexity (i.e., $\mathcal{K} = \emptyset, D = \infty$), only a single condition (Interp1) is needed, mirroring the single interpolation condition (2.5) of Taylor et al. [2]. In particular, (Interp2), (Interp3), (Interp5), and (Interp6) only apply when given interior points; (Interp4)-(Interp6) only apply when given a bound on diameter.

Our three main theorems below show that these conditions are always necessary for interpolation (Theorem 3.1); they become sufficient if the diameter bound is tightened by a factor of at most $\sqrt{2}$ (Theorem 3.2); they are necessary and sufficient if either the strong convexity or diameter bound is omitted (Theorem 3.3). Immediately afterward, we discuss the necessity of this $\sqrt{2}$ gap between our first two theorems and provide several convenient corollaries. Figure 3 presents sample constructions for convex, smooth, and strongly convex interpolating sets for the case where $\mathcal{K}$ is empty. These examples are special cases of the constructions used in our sufficiency proofs.

The following theorems, proven in the following subsections, consider a set of observations $S = (\{(z_i, v_i)\}_{i\in\mathcal{I}}, \{(x_k, \delta_k)\}_{k\in\mathcal{K}})$. Recall d denotes the dimension of space being considered, i.e., $z_i, v_i, x_k \in \mathbb{R}^d$.

Theorem 3.1. *If S is $\mathcal{C}_{\alpha,\beta,D}$-interpolable, then $S \in \text{Interp}(\alpha, \beta, D; 1)$.*

Theorem 3.2. *If $S \in \text{Interp}(\alpha, \beta, D; \sqrt{\frac{d+1}{2d}})$, then S is $\mathcal{C}_{\alpha,\beta,D}$-interpolable. In particular, if $S \in \text{Interp}(\alpha, \beta, D; \frac{1}{\sqrt{2}})$, then S is $\mathcal{C}_{\alpha,\beta,D}$-interpolable.*

Theorem 3.3. *If $\alpha = 0$ or $D \geq \frac{2}{\alpha}$, then S is $\mathcal{C}_{\alpha,\beta,D}$-interpolable if and only if $S \in \text{Interp}(\alpha, \beta, D; 1)$.*

Remark 1 (On the Computational Cost of Auxiliary Variables)**.** The function interpolation result of Theorem 2.2 provides a direct means to algebraically check if a given collection of observation data $\{(x_i, f_i, g_i)\}_{i\in\mathcal{I}}$ is interpolable. One must check $O(|\mathcal{I}|^2)$ quadratic inequalities. Our set interpolation theory, in its most general form, does not present such an easy algebraic check. Instead, one must determine whether $\{w_k\}_{k\in\mathcal{K}}$ exist satisfying the needed conditions. This corresponds to solving a second-order cone program with $O(|\mathcal{K}|)$ variables and $O(|\mathcal{I}|^2 + |\mathcal{K}|^2)$ constraints. In cases with no or very few Slater points ($|\mathcal{K}| = 0$ or small), this cost reduces back to checking $O(|\mathcal{I}|^2)$ inequalities.

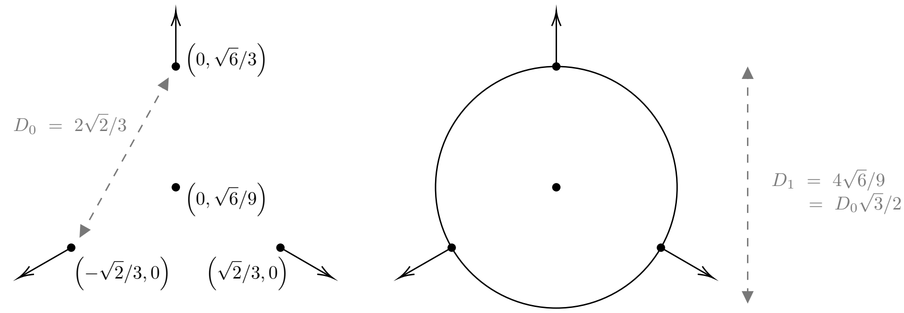

Figure 4: Example of slackness in diameter constraint for the 2-simplex (after projecting into $\mathbb{R}^2$ and translating).

Remark 2 (On the Tightness of the Gap Between Theorems 3.1 and 3.2)**.** The gap between $\lambda = 1$ and $\lambda = \sqrt{\frac{d+1}{2d}} \geq 1/\sqrt{2}$ in Theorems 3.1 and 3.2 is fundamental: Consider the regular d-simplex in $\mathbb{R}^{d+1}$, with $d > 1$. Placing the centroid at 0, our vertices become $(\frac{d}{d+1}, \frac{-1}{d+1}, \dots, \frac{-1}{d+1})$, $(\frac{-1}{d+1}, \frac{d}{d+1}, \frac{-1}{d+1} \dots, \frac{-1}{d+1})$, etc. Taking $\mathcal{I} = [0:d]$, we choose $\{z_i\}_{i\in\mathcal{I}}$ to be these $d+1$ vertices and $v_i = \frac{z_i}{\|z_i\|}$ for all $i \in \mathcal{I}$. Further, set $\mathcal{K} = \emptyset$, $S = (\{(v_i, z_i)\}_{i\in\mathcal{I}}, \emptyset)$, and $\alpha = \beta = \sqrt{\frac{d+1}{d}}$. Observing that

$$\|z_i - \frac{1}{\alpha}n_i - (z_j - \frac{1}{\beta}n_j)\| = \|z_i - \frac{1}{\beta}n_i - (z_j - \frac{1}{\beta}n_j)\| = 0 \leq \sqrt{2} - 2\sqrt{\frac{d}{d+1}} = \sqrt{2} - \frac{2}{\beta}$$

for all $i \neq j$, (Interp1) and (Interp4) are satisfied, so $S \in \text{Interp}(\alpha, \beta, \sqrt{2}; 1)$. However, we claim that S is not $\mathcal{C}_{\alpha,\beta,\sqrt{2}}$-interpolable. Since $\alpha = \beta$, any interpolating set must be a ball of radius $r := \frac{1}{\alpha} = \frac{1}{\beta} = \sqrt{\frac{d}{d+1}}$. Observe that all vertices z_i satisfy $\|z_i\| = r$. Therefore $B(0, r)$ is the unique α-strongly convex, β-smooth set that interpolates S. So S is not $\mathcal{C}_{\alpha,\beta,\sqrt{2}}$-interpolable as any interpolating set must have diameter at least $2r = 2\sqrt{\frac{d}{d+1}} > \sqrt{2}$, exactly matching the gap in our theorems. See Figure 4 demonstrating this for $d = 2$.

Remark 3 (On the Special Case of Interpolating with Strongly Convex Sets)**.** If we take β to ∞, that is, entirely relaxing the requirement of smoothness, then our interpolation conditions reduce to

$$\|z_i - \frac{1}{\alpha}n_i - z_j\| \leq \frac{1}{\alpha} \tag{3.1}$$

$$\|z_i - \frac{1}{\alpha}n_i - x_k\| \leq \frac{1}{\alpha} - \delta_k \tag{3.2}$$

$$\|z_i - z_j\| \leq \lambda D \tag{3.3}$$

$$\|z_i - x_k\| \leq \lambda D - \delta_k \tag{3.4}$$

$$\|x_k - x_l\| \leq \lambda D - \delta_k - \delta_l \tag{3.5}$$

with no more dependence on auxiliary parameters $\{w_k\}_{k\in\mathcal{K}}$. As an immediate corollary, we have the following result for strongly convex sets.

Corollary 3.4. *Let $S = (\{(z_i, v_i)\}_{i\in\mathcal{I}}, \{(x_k, \delta_k)\}_{k\in\mathcal{K}})$. If S is $\mathcal{C}_{\alpha,\infty,D}$-interpolable, then for all $i, j \in \mathcal{I}$ and $k, l \in \mathcal{K}$, (3.1-3.5) are satisfied for $\lambda = 1$. Conversely, if for all $i, j \in \mathcal{I}$ and $k, l \in \mathcal{K}$, (3.1-3.5) are satisfied for $\lambda = 1/\sqrt{2}$, then S is $\mathcal{C}_{\alpha,\infty,D}$-interpolable .*

Remark 4 (On the Special Case of Interpolating with Smooth Convex Sets)**.** We can similarly relax the strong convexity condition, taking α to zero and deriving simpler conditions for smooth convex interpolation from Theorem 3.3. Suppose that $\|z_i - \frac{1}{\alpha}n_i - (z_j - \frac{1}{\beta}n_j)\| \leq \frac{1}{\gamma}$. Then

$$\begin{aligned}
&\langle z_i - \frac{1}{\alpha}n_i - z_j + \frac{1}{\beta}n_j, z_i - \frac{1}{\alpha}n_i - z_j + \frac{1}{\beta}n_j\rangle \leq \frac{1}{\gamma^2} \\
\Leftrightarrow\quad &\|z_i - \frac{1}{\beta}n_i - z_j + \frac{1}{\beta}n_j\|^2 - 2\langle\frac{1}{\gamma}n_i, z_i - \frac{1}{\beta}n_i - z_j + \frac{1}{\beta}n_j\rangle + \frac{1}{\gamma^2} \leq \frac{1}{\gamma^2} \\
\Leftrightarrow\quad &\langle n_i, z_j - \frac{1}{\beta}n_j - z_i + \frac{1}{\beta}n_i\rangle \leq -\frac{\gamma}{2}\|z_i - \frac{1}{\beta}n_i - z_j + \frac{1}{\beta}n_j\|^2 \ .
\end{aligned}$$

Taking the limit as $\alpha \to 0$ and correspondingly $\gamma \to 0$, this inequality becomes

$$\langle n_i, z_j - \frac{1}{\beta}n_j - z_i + \frac{1}{\beta}n_i\rangle \leq 0 \tag{3.6}$$

in place of (Interp1). Similarly, for w_k, we obtain

$$\langle n_i, w_k + s_k n_i - z_i + \frac{1}{\beta}n_i\rangle \leq 0 \tag{3.7}$$

in place of (Interp2). We can then state a corollary for smooth sets.

Corollary 3.5. *Let $S = (\{(z_i, v_i)\}_{i\in\mathcal{I}}, \{(x_k, \delta_k)\}_{k\in\mathcal{K}})$. S is $\mathcal{C}_{\beta,0,D}$-interpolable if and only if there exist $\{w_k\}_{k\in\mathcal{K}} \subset \mathbb{R}^d$ such that for all $i, j \in \mathcal{I}$ and $k, l \in \mathcal{K}$,* (3.6) *and* (3.7) *along with* (Interp3)-(Interp6) *are satisfied for $\lambda = 1$.*

Taking the limit as β tends to ∞ yields our previously stated Corollary 2.8 for nonsmooth, non-strongly convex interpolation.

3.1 Proof of Theorem 3.1

Suppose that S is $\mathcal{C}_{\alpha,\beta,D}$-interpolable. Then there exists some $C \in \mathcal{C}_{\alpha,\beta,D}$ such that $v_i \in N_C(z_i)$, $z_i \in C$, and $x_k \in \mathrm{int}_{\delta_k}C$ for all $i \in \mathcal{I}, k \in \mathcal{K}$. From Lemma 2.6, we know that $C = C_0 + B(0, \frac{1}{\beta})$ for some γ-strongly convex set C_0, where $\frac{1}{\gamma} + \frac{1}{\beta} = \frac{1}{\alpha}$. In particular, as shown in [22], we can write C_0 explicitly with the Minkowski difference $C_0 = C - B(0, \frac{1}{\beta}) := \{x \mid x + B(0, \frac{1}{\beta}) \subseteq C\}$.

As a first intermediate result, we claim for each $i \in \mathcal{I}$,

$$v_i \in N_{C_0}(z_i - \frac{v_i}{\beta\|v_i\|}) \ . \tag{3.8}$$

This follows since having $v_i \in N_C(z_i)$ ensures for all $y \in C$, $\langle v_i, y - z_i\rangle \leq 0$. So noting that every $p \in C_0$ has $p + \frac{v_i}{\beta\|v_i\|} \in C$, it follows that $\langle v_i, p - (z_i - \frac{v_i}{\beta\|v_i\|})\rangle \leq 0$ for all $p \in C_0$. From the definition of C_0, $z_i - \frac{v_i}{\beta\|v_i\|} \in C_0$. Together these yield (3.8).

Since C_0 is γ-strongly convex, (3.8) implies that $C_0 \subseteq B(z_i - \frac{v_i}{\beta\|v_i\|} - \frac{v_i}{\gamma\|v_i\|}, \frac{1}{\gamma})$. Since $z_j - \frac{v_j}{\beta\|v_j\|} \in C_0$ for all $j \in \mathcal{I}$, we have $z_j - \frac{v_j}{\beta\|v_j\|} \in B(z_i - \frac{v_i}{\beta\|v_i\|} - \frac{v_i}{\gamma\|v_i\|}, \frac{1}{\gamma})$. So (Interp1) holds as

$$\|z_i - \frac{v_i}{\alpha\|v_i\|} - (z_j - \frac{v_j}{\beta\|v_j\|})\| = \|z_i - \frac{v_i}{\beta\|v_i\|} - \frac{v_i}{\gamma\|v_i\|} - (z_j - \frac{v_j}{\beta\|v_j\|})\| \leq \frac{1}{\gamma} \ .$$

Now, we verify (Interp2) and (Interp3) for each $k \in \mathcal{K}$ by considering two cases:

Case 1: $x_k \in C_0$**.** In this case, we let $w_k = x_k$. Suppose that $\delta_k > \frac{1}{\beta}$, so $s_k := \max\{0, \delta_k - \frac{1}{\beta}\} = \delta_k - \frac{1}{\beta}$. We claim that $x_k \in \text{int}_{s_k} C_0$. Suppose that for some ζ, $x_k + (\delta_k - \frac{1}{\beta})\frac{\zeta}{\|\zeta\|} \notin C_0$. Then $x_k + (\delta_k - \frac{1}{\beta})\frac{\zeta}{\|\zeta\|} + \frac{1}{\beta}\frac{\zeta}{\|\zeta\|} = x_k + \delta_k \frac{\zeta}{\|\zeta\|} \notin C$. But since $x_k \in \text{int}_{\delta_k} C$, this is a contradiction, so we must have $x_k \in \text{int}_{s_k} C_0$. Now suppose $\delta_k \leq \frac{1}{\beta}$. Then $s_k = 0$, so $x_k \in \text{int}_{s_k} C_0$ is true by assumption. Since $x_k = w_k$, we have shown that $w_k + B(0, s_k) \subseteq C_0$. When combined with the previous fact that $C_0 \subseteq B(z_i - \frac{v_i}{\beta\|v_i\|} - \frac{v_i}{\gamma\|v_i\|}, \frac{1}{\gamma})$, it follows that

$$\|z_i - \frac{v_i}{\alpha\|v_i\|} - w_k\| \leq \frac{1}{\gamma} - s_k \ .$$

This is equivalent to (Interp2). Finally, noting $\|w_k - x_k\| = 0 \leq \max\{0, \frac{1}{\beta} - \delta_k\} = \frac{1}{\beta} - \delta_k + s_k$, we conclude (Interp3) is satisfied.

Case 2: $x_k \notin C_0$**.** Since $C = C_0 + B(0, \frac{1}{\beta})$ and $x_k \notin C_0$, we must have $\delta_k < \frac{1}{\beta}$. We therefore have $s_k = 0$, and we set w_k as the orthogonal projection of x_k onto C_0 and $\zeta = \frac{x_k - w_k}{\|x_k - w_k\|}$ as the corresponding normal vector to C_0. Since $w_k \in C_0$, the same reasoning as above leveraging (3.8) implies (Interp2). We know $x_k \in \text{int}_{\delta_k} C$, so it follows that $w_k + \zeta(\|w_k - x_k\| + \delta_k) \in C$. However, since $\zeta \in N_{C_0}(w_k)$, it follows that $\|w_k - x_k\| + \delta_k \leq \frac{1}{\beta}$. Recalling $s_k = 0$, that is exactly (Interp3).

Finally, we verify the diameter conditions (Interp4), (Interp5), and (Interp6). For all $i \in \mathcal{I}, k \in \mathcal{K}$, we have shown $z_i - \frac{v_i}{\beta\|v_i\|} \in C_0$ and in either case above, we have that $w_k \in \text{int}_{s_k} C_0$. Hence the diameter bound of $\text{diam}(C_0) \leq D - \frac{2}{\beta}$ from Lemma 2.5 yields the remaining three conditions for $\lambda = 1$.

3.2 Proof of Theorem 3.2

Suppose our six interpolation conditions hold for $\lambda = \sqrt{\frac{d+1}{2d}} \geq 1/\sqrt{2}$. We begin by constructing our interpolating set C. First, we construct C_0 as the γ-strongly convex hull of $\{B(w_k, s_k)\}_{k\in\mathcal{K}}$ and $\{z_i - \frac{v_i}{\beta\|v_i\|}\}_{i\in\mathcal{I}}$. More formally, by [20, Proposition 2.5], we set $C_0 = \bigcap_{y\in Y} B(y, \frac{1}{\gamma})$, where

$$Y = \{y \mid \|z_i - \frac{v_i}{\beta\|v_i\|} - y\| \leq \frac{1}{\gamma} \quad \forall i \in \mathcal{I}, \quad \|w_k - y\| \leq \frac{1}{\gamma} - s_k \quad \forall k \in \mathcal{K}\} \ .$$

Observe that each ball $B(y, \frac{1}{\gamma})$ is γ-strongly convex, and this property is preserved under intersections [19, Proposition 2], so C_0 is γ-strongly convex. We then define $C = C_0 + B(0, \frac{1}{\beta})$. By Lemma 2.6, we see that C is α-strongly convex and β-smooth. All that remains is to verify this set C correctly interpolates the given observation data S and has the desired diameter bound D.

Verification that $z_i \in C$ **and** $v_i \in N_C(z_i)$ **for each** $i \in \mathcal{I}$**.** From the definition of Y, $z_i - \frac{v_i}{\beta\|v_i\|} \in B(y, \frac{1}{\gamma})$ for all $y \in Y$. Hence $z_i - \frac{v_i}{\beta\|v_i\|} \in C_0$. Since $C = C_0 + B(0, \frac{1}{\beta})$, it follows that $z_i \in C$. Further, by (Interp1), $\|z_i - \frac{v_i}{\alpha\|v_i\|} - (z_j - \frac{v_j}{\beta\|v_j\|})\| \leq \frac{1}{\gamma}$ for all $j \in \mathcal{I}$, and by (Interp2), $\|z_i - \frac{v_i}{\alpha\|v_i\|} - w_k\| \leq \frac{1}{\gamma} - s_k$ for all $k \in \mathcal{K}$. Hence $z_i - \frac{v_i}{\alpha\|v_i\|} \in Y$. Letting $B_i = B(z_i - \frac{v_i}{\alpha\|v_i\|}, \frac{1}{\gamma})$, observe that $v_i \in N_{B_i}(z_i - \frac{v_i}{\alpha\|v_i\|} + \frac{v_i}{\gamma\|v_i\|}) = N_{B_i}(z_i - \frac{v_i}{\beta\|v_i\|})$. Combining this with the fact that $C_0 \subseteq B_i$ since $z_i - \frac{v_i}{\alpha\|v_i\|} \in Y$, one has that $v_i \in N_{C_0}(z_i - \frac{v_i}{\beta\|v_i\|})$. Finally, since $C = C_0 + B(0, \frac{1}{\beta})$, it follows that $v_i \in N_C(z_i)$ as well.

Verification that $x_k \in \text{int}_{\delta_k} C$ **for each** $k \in \mathcal{K}$**.** Next, we consider w_k and x_k. By definition of Y, we see that for all $y \in Y$, $w_k \in \text{int}_{s_k} B(y, \frac{1}{\gamma})$. Consequently, by construction, $w_k \in \text{int}_{s_k} C_0$. Then using the fact that $\|x_k - w_k\| \leq \frac{1}{\beta} + s_k - \delta_k$ by (Interp3), for any ζ, it follows that

$$\|x_k + \delta_k \frac{\zeta}{\|\zeta\|} - (w_k + s_k \frac{\zeta}{\|\zeta\|})\| \leq \|x_k - w_k\| + |\delta_k - s_k| = \|x_k - w_k\| + \delta_k - s_k$$

$$\leq \frac{1}{\beta} + s_k - \delta_k + (\delta_k - s_k) = \frac{1}{\beta} \ .$$

Since $w_k \in \mathrm{int}_{s_k} C_0$, for any nonzero ζ, one has $\hat{w}_k := w_k + s_k \frac{\zeta}{\|\zeta\|} \in C_0$. Hence $\|x_k + \delta_k \frac{\zeta}{\|\zeta\|} - \hat{w}_k\| \leq \frac{1}{\beta}$ from which one can conclude for any ζ, $x_k + \delta_k \frac{\zeta}{\|\zeta\|} \in C$ and consequently $x_k \in \mathrm{int}_{\delta_k} C$.

Verification of diameter bound $\mathrm{diam}(C) \leq D$**.** By Lemma 2.5, it is sufficient to show that $\mathrm{diam}(C_0) \leq D - \frac{2}{\beta}$. Since C_0 is γ-strongly convex, we already know $\mathrm{diam}(C_0) \leq \frac{2}{\gamma}$. So, all that remains is to prove the needed bound when $D - \frac{2}{\beta} < \frac{2}{\gamma}$. Jung's theorem [23] states that given a compact set $X \subseteq \mathbb{R}^d$, there exists a closed ball B with radius $R = \mathrm{diam}(X)\sqrt{\frac{d}{2(d+1)}}$ containing X. Applied to the set $X = \{z_i - \frac{v_i}{\beta\|v_i\|}\}_{i\in\mathcal{I}} \cup \{B(w_k, s_k)\}_{k\in\mathcal{K}}$, which has diameter at most $\sqrt{\frac{d+1}{2d}}(D - \frac{2}{\beta})$ by (Interp4), (Interp5), and (Interp6), there must exist $q \in \mathbb{R}^d$ such that $X \subseteq B(q, R)$ where $R = \left(\sqrt{\frac{d+1}{2d}}(D - \frac{2}{\beta})\right)\sqrt{\frac{d}{2(d+1)}} = \frac{1}{2}(D - \frac{2}{\beta})$. Hence the distance from q to any point in X is at most $\frac{1}{2}(D - \frac{2}{\beta}) < 1/\gamma$. As a result, a neighborhood of q lies in Y. Namely, $B(q, \frac{1}{\gamma} - \frac{1}{2}(D - \frac{2}{\beta})) \subseteq Y$. So,

$$C_0 = \bigcap_{y\in Y} B\left(y, \frac{1}{\gamma}\right) \subseteq \bigcap_{y\in B(q, \frac{1}{\gamma} - \frac{1}{2}(D-\frac{2}{\beta}))} B\left(y, \frac{1}{\gamma}\right) = B\left(q, \frac{1}{2}(D - \frac{2}{\beta})\right) \ ,$$

proving the needed bound $\mathrm{diam}(C_0) \leq D - \frac{2}{\beta}$.

3.3 Proof of Theorem 3.3

The forward direction is already proven by Theorem 3.1, so we only need to prove the reverse.

First consider the case of $D \geq \frac{2}{\alpha}$. By our argument in Theorem 3.2, if our interpolation conditions hold for $\lambda = 1$, then there exists an α-strongly convex, β-smooth set C (of unspecified diameter) that interpolates S. However, since any α-strongly convex set C must satisfy $C \subseteq B(z - \frac{1}{\alpha}n, \frac{1}{\alpha})$ for $z \in \mathrm{bdry}\ C$ with unit normal vector n, we see that $\mathrm{diam}(C) \leq \mathrm{diam}(B(z - \frac{1}{\alpha}n, \frac{1}{\alpha})) = \frac{2}{\alpha} \leq D$. Therefore, S is $\mathcal{C}_{\alpha,\beta,D}$-interpolable.

Now consider the case of $\alpha = 0$. Suppose our six interpolation conditions hold for $\lambda = 1$. As shown in Remark 4, our conditions (Interp1) and (Interp2) reduce to (3.6) and (3.7). Define $C_0 = \mathrm{conv}(\{z_i - \frac{v_i}{\beta\|v_i\|}\}_{i\in\mathcal{I}}, \{B(w_k, s_k)\}_{k\in\mathcal{K}})$. Clearly C_0 is convex with $w_k \in \mathrm{int}_{s_k} C_0$ and $z_i - \frac{v_i}{\beta\|v_i\|} \in C_0$. We then construct $C = C_0 + B(0, \frac{1}{\beta})$. From Proposition 2.3, C is β-smooth.

Next, we show that $z_i \in C$ and $v_i \in N_C(z_i)$. Since $z_i - \frac{v_i}{\beta\|v_i\|} \in C_0$, we immediately have that $z_i \in C$. Consider any $y \in C$ and let $p \in C_0$ be such that $y = p + \frac{1}{\beta}\zeta$ for some $\|\zeta\| \leq 1$. Since C_0 is a convex hull, we write p as the convex combination

$$p = \sum_{j\in\mathcal{I}} \sigma_j(z_j - \frac{v_j}{\beta\|v_j\|}) + \sum_{k\in\mathcal{K}} \phi_k(w_k + s_k\xi_k)$$

where $\sigma_j, \phi_k \geq 0$ and $\sum_j \sigma_j + \sum_k \phi_k = 1$, and $\|\xi_k\| \leq 1$. We then have

$$\begin{aligned}
\langle v_i, y - z_i\rangle &= \langle v_i, p + \frac{1}{\beta}\zeta - z_i\rangle \leq \langle v_i, p - (z_i - \frac{v_i}{\beta\|v_i\|})\rangle \\
&= \sum_{j\in\mathcal{I}} \sigma_j \langle v_i, z_j - \frac{v_j}{\beta\|v_j\|} - (z_i - \frac{v_i}{\beta\|v_i\|})\rangle + \sum_{k\in\mathcal{K}} \phi_k \langle v_i, w_k + s_k\xi_k - (z_i - \frac{v_i}{\beta\|v_i\|})\rangle \\
&\leq \sum_{j\in\mathcal{I}} \sigma_j \langle v_i, z_j - \frac{v_j}{\beta\|v_j\|} - (z_i - \frac{v_i}{\beta\|v_i\|})\rangle + \sum_{k\in\mathcal{K}} \phi_k \langle v_i, w_k + s_k\frac{v_i}{\|v_i\|} - (z_i - \frac{v_i}{\beta\|v_i\|})\rangle
\end{aligned}$$

≤ 0

where the last inequality follows from (3.6) and (3.7). This holds for any $y \in C$, so $v_i \in N_C(z_i)$.

By the same argument as in Theorem 3.2, we have that $x_k \in \text{int}_{\delta_k} C$. Lastly, since C_0 is a convex hull, we can see by (Interp4), (Interp5), and (Interp6) that

$$\text{diam}(C_0) = \text{diam}\left(\{z_i - \frac{v_i}{\beta\|v_i\|}\}_{i\in\mathcal{I}} \,\cup\, \{B(w_k, s_k)\}_{k\in\mathcal{K}}\right) \leq D - \frac{2}{\beta} .$$

Then, by Lemma 2.5, we conclude that $\text{diam}(C) \leq D$.

3.4 Function Interpolation as a Limit of Set Interpolation

Here, we demonstrate that we can recover the function interpolation of [2] as a limit of our set interpolation theory. Given $\{(x_i, g_i, f_i)\}_{i\in\mathcal{I}}$, we define

$$\begin{aligned}\hat{Q}^{i,j}_{\mu,L}(x, g, f) &= f_j - f_i - \langle g_i, x_j - x_i\rangle \\ &\quad - \frac{1}{2(1-\mu/L)}\left(\frac{1}{L}\|g_i - g_j\|^2 + \mu\|x_i - x_j\|^2 - 2\frac{\mu}{L}\langle g_i - g_j, x_i - x_j\rangle\right) .\end{aligned}$$

Recall from Theorem 2.2 that $\hat{Q}^{i,j}_{\mu,L}(x, g, f) \geq 0$ for all $i, j \in \mathcal{I}$ if and only if $\{(x_i, g_i, f_i)\}_{i\in\mathcal{I}}$ is $\mathcal{F}_{\mu,L}$-interpolable. Next, let $z_i = (x_i, f_i)$ and $v_i = (g_i, -1)$, and let $n_i = \frac{v_i}{\|v_i\|}$. We further define

$$\begin{aligned}Q^{i,j}_{\mu,L}(x, g, f) &= \frac{\mu}{2(1-\mu/L)}\left((\frac{1}{\mu} - \frac{1}{L})^2 - \|z_j - \frac{1}{L}n_j - z_i + \frac{1}{\mu}n_i\|^2\right) \\ &= \frac{L-\mu}{2L\mu} - \frac{\mu}{2(1-\mu/L)}\left(\|x_j - \frac{g_i}{L\|v_j\|^2} - x_i + \frac{g_i}{\mu\|v_i\|^2}\|^2\right. \\ &\qquad\left. + (f_j + \frac{1}{L\|v_j\|} - f_i - \frac{1}{\mu\|v_i\|})^2\right) .\end{aligned}$$

We can similarly define our limiting cases $Q^{i,j}_{\mu,\infty}$ and $Q^{i,j}_{0,L}$ using Corollary 3.4 and Corollary 3.5. From Theorem 3.2, we know that $Q^{i,j}_{\mu,L}(x, g, f) \geq 0$ for all $i, j \in \mathcal{I}$ if and only if $(\{((x_i, f_i), (g_i, -1))\}_{i\in\mathcal{I}}, \emptyset)$ is $\mathcal{C}_{\mu,L,\infty}$-interpolable. We can relate these conditions through the following lemma, proven in Appendix A.1 by a direct Taylor series expansion, and resulting theorem.

Lemma 3.6. *For any $\eta \geq 0$,*

$$Q^{i,j}_{\eta^2\mu,\eta^2 L}(\frac{x}{\eta}, \eta g, f) = \hat{Q}^{i,j}_{\mu,L}(x, g, f) + \eta^2 c(x, g, f, \mu, L, \eta)$$

where $c(x, g, f, \mu, L, \eta)$ is bounded as $\eta \to 0$.

Theorem 3.7. $\lim_{\eta\to 0} Q^{i,j}_{\eta^2\mu,\eta^2 L}(\frac{x}{\eta}, \eta g, f) \geq 0$ *if and only if $\{(x_i, g_i, f_i)\}_{i\in\mathcal{I}}$ is $\mathcal{F}_{\mu,L}$-interpolable.*

Proof. This follows directly from Lemma 3.6 by taking the limit with η. □

Note for any differentiable convex function f, $(\nabla f(x), -1) \in N_{\text{epi}f}(x, f(x))$. Then, by definition, epi$f$ being an L-smooth set is equivalent to the following upper bound holding for all x, y:

$$f(y) \leq \mathfrak{b}_x(y; L) := f(x) + \frac{1}{L\sqrt{\|\nabla f(x)\|^2 + 1}} - \sqrt{\frac{1}{L^2} - \|y - x + \frac{\nabla f(x)}{L\sqrt{\|\nabla f(x)\|^2 + 1}}\|^2} \qquad (3.9)$$

where we say $\mathfrak{b}_x(y;L) = \infty$ outside of its domain. This ball upper bound $\mathfrak{b}_x(y;L)$ dominates the analogous quadratic bound for smooth functions. That is, for all x, y, we have

$$\mathfrak{b}_x(y;L) \geq \mathfrak{q}_x(y;L) := f(x) + \langle \nabla f(x), y - x\rangle + \frac{L}{2}\|x-y\|^2 \ .$$

Combining this observation with our lemma above, we can strengthen our claim when restricting to smooth (not necessarily strongly convex) functions.

Theorem 3.8. *The set of observations $\{(x_i, g_i, f_i)\}_{i\in\mathcal{I}}$ is $\mathcal{F}_{0,L}$-interpolable if and only if $(\{((\frac{x_i}{\eta}, f_i), (\eta g_i, -1))\}_{i\in\mathcal{I}}, \emptyset)$ is $\mathcal{C}_{0,\eta^2 L,\infty}$-interpolable for all $\eta > 0$.*

Proof. ($\Rightarrow$) Suppose that $\{(x_i, g_i, f_i)\}_{i\in\mathcal{I}}$ is $\mathcal{F}_{0,L}$-interpolable. So there exists an L-smooth convex function f such that $f(x_i) = f_i$ and $\nabla f(x_i) = g_i$. Observe that for any $\eta > 0$, $\hat{f}$ defined by $\hat{f}(x) = f(\eta x)$ is an $\eta^2 L$-smooth convex function with $\hat{f}(\frac{x_i}{\eta}) = f(x_i) = f_i$ and $\nabla \hat{f}(\frac{x_i}{\eta}) = \eta\nabla f(\eta\frac{x_i}{\eta}) = \eta g_i$. Then for all x, y, $\hat{f}(y) \leq \mathfrak{q}_x(y;\eta^2 L) \leq \mathfrak{b}_x(y;\eta^2 L)$. As a result, epi$\hat{f}$ is $\eta^2 L$-smooth. Finally, since $(\nabla \hat{f}(x), -1) \in N_{\text{epi}\hat{f}}(x, \hat{f}(x))$, epi$\hat{f}$ interpolates $(\{((\frac{x_i}{\eta}, f_i), (\eta g_i, -1))\}_{i\in\mathcal{I}}, \emptyset)$.

($\Leftarrow$) Suppose that $(\{((\frac{x_i}{\eta}, f_i), (\eta g_i, -1))\}_{i\in\mathcal{I}}, \emptyset)$ is $\mathcal{C}_{0,\eta^2 L,\infty}$-interpolable for all $\eta > 0$. By our interpolability theorems, we know that $Q^{i,j}_{0,L}(\frac{x}{\eta}, \eta g, f) \geq 0$ for all i, j. Applying, Lemma 3.6 yields

$$\hat{Q}^{i,j}_{0,L}(x,g,f) = Q^{i,j}_{0,\eta^2 L}(\frac{x}{\eta}, \eta g, f) + \eta^2 c(x,g,f,\mu,L,\eta) \geq \eta^2 c(x,g,f,\mu,L,\eta) \ .$$

Considering this as η tends to 0 implies $\hat{Q}^{i,j}_{0,L}(x,g,f) \geq 0$ from which Theorem 2.2 ensures $\{(x_i, g_i, f_i)\}_{i\in\mathcal{I}}$ is $\mathcal{F}_{0,L}$-interpolable. □

3.5 Example Application: Optimal Interpolations as an SOCP

As a simple direct application of our theory, consider computing optimal interpolations of a set of observation data. Given points $\{\hat{z}_i\}_{i\in\mathcal{I}}$ with normal vectors $\{\hat{v}_i\}_{i\in\mathcal{I}}$ and points $\{\hat{x}_k\}_{k\in\mathcal{K}}$ required to be at least δ_k interior, one can consider several naturally related computational questions:

- *Given a diameter bound $D \in \mathbb{R}_+$, what is the smoothest set interpolating the data?*
- *Given a smoothness bound $\beta \in \mathbb{R}_+$, what is the most strongly convex set interpolating the data?*
- *Alternatively, given a strong convexity bound $\alpha \in \mathbb{R}_+$, how feasible can a given point x_1 be to an interpolating set (i.e., what is the maximum δ_1 still facilitating an interpolation)?*

These questions and their various permutations all correspond to optimizing some respective parameters such that $(\{(\hat{z}_i, \hat{v}_i)\}_{i\in\mathcal{I}}, \{(\hat{x}_k, \delta_k)\}_{k\in\mathcal{K}})$ is $\mathcal{C}_{\alpha,\beta,D}$-interpolable. Our interpolation theorems allow these to be formulated as second-order cone optimization problems.

For example, consider just the first question above, seeking the smoothest set interpolating the given data. To simplify notation, let $\hat{n}_i = \frac{\hat{v}_i}{\|\hat{v}_i\|}$. Applying Theorem 3.3, this amounts to

$$\begin{cases} \min\limits_{\beta, w_k} & \beta \\ \text{s.t.} & \langle \hat{n}_i, \hat{z}_j - \frac{1}{\beta}\hat{n}_j - (\hat{z}_i - \frac{1}{\beta}\hat{n}_i)\rangle \leq 0 & \forall i,j\in\mathcal{I} \\ & \langle \hat{v}_i, w_k - (\hat{z}_i - \frac{1}{\beta}\hat{n}_i)\rangle \leq 0 & \forall i\in\mathcal{I}, k\in\mathcal{K} \\ & \|\hat{x}_k - w_k\| \leq \frac{1}{\beta} & \forall k\in\mathcal{K} \\ & \|\hat{z}_i - \frac{1}{\beta}\hat{n}_i - (\hat{z}_j - \frac{1}{\beta}\hat{n}_j)\| \leq D - \frac{2}{\beta} & \forall i,j\in\mathcal{I} \\ & \|\hat{z}_i - \frac{1}{\beta}\hat{n}_i - w_k\| \leq D - \frac{2}{\beta} & \forall i\in\mathcal{I}, k\in\mathcal{K} \\ & \|w_k - w_l\| \leq D - \frac{2}{\beta} & \forall k,l\in\mathcal{K}\ . \end{cases}$$

Since $\hat{v}_i$, $\hat{z}_i$, and $\hat{x}_k$ are all fixed, we can see this is simply a second-order cone program (SOCP) with $d|\mathcal{K}|+1$ variables (namely, $1/\beta$ and $\{w_k\}_{k\in\mathcal{K}}$) and $2\binom{|\mathcal{I}|}{2}+\binom{|K|}{2}+|\mathcal{K}|\,(2|\mathcal{I}|+1)$ constraints. Therefore one can calculate a numerical global solution using standard methods, i.e., in polynomial time using an interior point method. Once optimal $\beta, \{w_k\}_{k\in\mathcal{K}}$ are known, the actual set construction follows our set construction in Theorem 3.3. Specifically, given the solution β and $\{w_k\}_{k\in\mathcal{K}}$, the optimally smoothed set is then given by $C = \mathrm{conv}(\{\hat{z}_i - \frac{1}{\beta}\hat{n}_i\}_{i\in\mathcal{I}}, \{w_k\}_{k\in\mathcal{K}}) + B(0, \frac{1}{\beta})$.

4 Application: Separating Hyperplane Algorithm as an SDP

In the remainder of this paper, we provide four applications of our interpolation theorems. In this section, we consider a family of simple algorithms computing an element of a convex set by iteratively using a separating hyperplane oracle. Our interpolation theorems allow us to quantify the worst-case stopping time of such a method over any smooth and/or strongly convex set with a Slater point via semidefinite programming. Motivated from and confirming resulting numerics, we identify a simple minimax optimal separating hyperplane algorithm.

General Separating Hyperplane Algorithms and Problem Instances. For any closed convex set $C \subseteq \mathbb{R}^d$, we denote the set of (unit) separating hyperplanes of C at some $x \in \mathbb{R}^d$ by

$$SH_C(x) = \{n \in \mathbb{R}^d \mid \|n\| = 1,\ \langle n, y - x\rangle \leq 0 \quad \forall y \in C\} \ .$$

Note this set is nonempty exactly when $x \notin \mathrm{int}\ C$. Here, our primary interest is in designing algorithms constructing a member of int C using a sequence of separating hyperplane oracle queries. As a general form of method with N queries, consider any iteration producing points x_i via

$$x_{i+1} = x_0 - \sum_{j=0}^{i} H_{i,j} n_j, \qquad n_i \in SH_C(x_i) \ , \qquad \forall i = 0, \dots, N-1 \tag{4.1}$$

parameterized by the lower triangular matrix of stepsizes H. Note this method must halt once $x_i \in \mathrm{int}\ C$ as $SH_C(x_i)$ is empty. As a general form of problem instances, for fixed α, β, δ, R, consider any strongly convex, smooth sets $C \in \mathcal{C}_{\alpha,\beta,\infty}$ containing some $q \in \mathrm{int}_\delta C$ and any initialization x_0 with $\|x_0 - q\| \leq R$. Since no diameter bound is enforced on C, our interpolation conditions are necessary and sufficient (i.e., Theorem 3.3 applies).

Guarantees from Performance Estimation. The question of whether a problem instance exists where a proposed algorithm (defined by a lower triangular matrix H) can fail to construct some $x_i \in \mathrm{int}\ C$ for $i = 0, \dots, N$ within its N steps corresponds to the following proposition:

$$\exists x_i, n_i, q, C \qquad \text{s.t.} \qquad \begin{cases} C \in \mathcal{C}_{\alpha,\beta,\infty} \\ B(q,\delta) \subseteq C \\ n_i \in SH_C(x_i) \\ \|x_0 - q\| \leq R \\ x_{i+1} = x_0 - \sum_{j=0}^{i} H_{i,j} n_j \ . \end{cases}$$

A successful algorithm design would be a selection of H such that no solution to the above system exists, as a failure to have $n_k \in SH_C(x_k)$ exist for some k implies $x_k \in \mathrm{int}\ C$. Noting that every separating hyperplane n_i must be normal to the set C at some z_i, this system can be rewritten in terms of interpolation with $\mathcal{I} = \{0, \dots, N, \star\}$ as

$$\exists x_i, z_i, n_i, q \qquad \text{s.t.} \qquad \begin{cases} (\{(n_i, z_i)\}_{i\in\mathcal{I}}, \{(q, \delta)\}) \text{ is } \mathcal{C}_{\alpha,\beta,\infty}\text{-interpolable} \\ n_i \in SH_C(x_i) \\ \|x_0 - q\| \le R \\ x_{i+1} = x_0 - \sum_{j=0}^{i} H_{i,j} n_j \ . \end{cases}$$

Theorem 3.3 enables this decision problem to be described as a (rather cumbersome) system of quadratic inequalities using our first three interpolation conditions (Interp1)-(Interp3):

$$\exists x_i, z_i, n_i, q, w \qquad \text{s.t.} \qquad \begin{cases} \|z_i - \frac{1}{\alpha} n_i - (z_j - \frac{1}{\beta} n_j)\|^2 \le \frac{1}{\gamma^2} \\ \|z_i - \frac{1}{\alpha} n_i - w\|^2 \le (\frac{1}{\gamma} - s)^2 \\ \|q - w\|^2 \le (\frac{1}{\beta} - \delta + s)^2 \\ \langle n_i, z_i - x_i \rangle \le 0 \\ \|n_i\|^2 = 1 \\ \|x_0 - q\|^2 \le R^2 \\ x_{i+1} = x_0 - \sum_{j=0}^{i} H_{i,j} n_j \end{cases} \tag{4.2}$$

where $\gamma = (\frac{1}{\alpha} - \frac{1}{\beta})^{-1}$ and $s = \max\{0, \delta - \frac{1}{\beta}\}$. Without loss of generality, we fix $q = 0$. Assuming $d \ge 2N + 4$, a Gram matrix reformulation of this quadratic system yields an equivalent semidefinite programming feasibility problem with variables G capturing every quadratic term in the original variables x_0, z_i, n_i, w, i.e.,

$$\Lambda = [x_0|z_0|z_1|\dots|z_N|n_0|n_1|\dots|n_N|w] \in \mathbb{R}^{d\times(2N+4)} \ ,$$
$$G = \Lambda^T\Lambda \in \mathbb{S}_+^{2N+4} \ .$$

Appendix B.1 presents this reformulation in full for the sake of completeness. Such reformulations are widespread in the existing PEP literature. Hence, we can efficiently certify whether a proposed separating hyperplane algorithm is guaranteed to construct a feasible point.

A Simple Constant Step Algorithm is Minimax Optimal. Consider the simple separating hyperplane method fixing H to be diagonal with constant value h, that is

$$x_{i+1} = x_i - h n_i \ , \qquad n_i \in SH_C(x_i) \ , \qquad \forall i = 0, \dots, N-1 \ . \tag{4.3}$$

For a given family of problem instances fixing α, β, δ, R, we can compute the number of iterations needed to guarantee a strictly feasible point by growing N until the corresponding SDP becomes infeasible. Denote this maximal number of steps needed to ensure an interior point is found by $N_{\max}$. As a stepsize rule, consider the constant stepsize $h = \max\{\delta, \frac{1}{\beta}\}$. Figure 5 shows $N_{\max}$ as δ and R vary under this stepsize. From these numerical results, one can readily identify a formula $N_{\max} = \lfloor \frac{(R+h-\delta)^2}{h^2} \rfloor$.

This simple method and numerically observed rate turn out to be the minimax optimal method and guarantee for any separating hyperplane method of the general form (4.1) seeking a strictly feasible point in a smooth and/or strongly convex set. The following theorem formalizes this and generalizes to allow bounds on set diameter $D \ge \max\{\delta, 2/\beta\}$. Its proof is deferred to Appendix B.2.

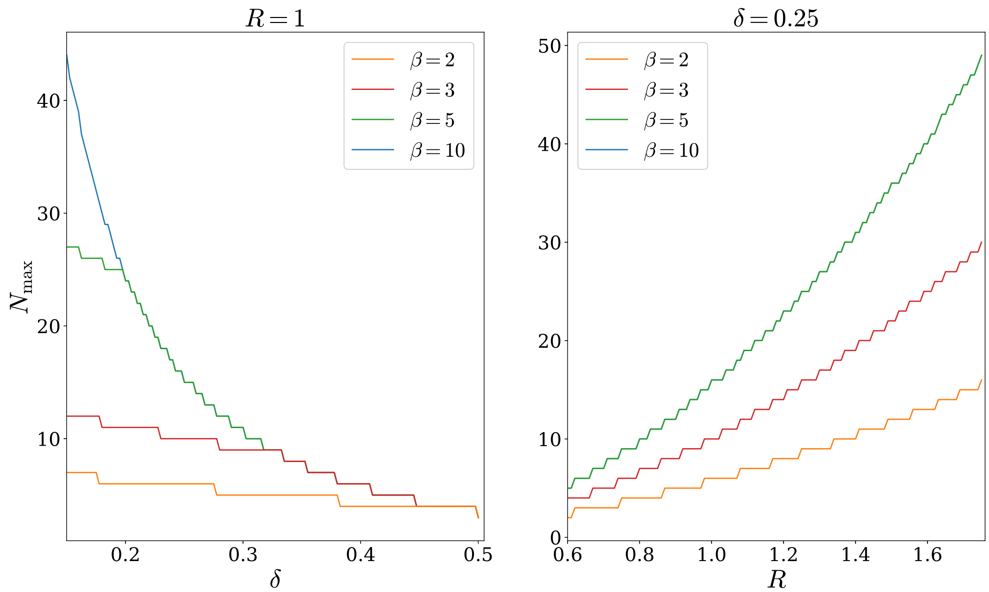

Figure 5: Numerical PEP result for $N_{\max}$ given a constant stepsize $h = \max\{\delta, \frac{1}{\beta}\}$ as δ varies on the left and R on the right. In the second plot, the line for $\beta = 10$ is covered by the line for $\beta = 5$.

Theorem 4.1. *For any $C \in \mathcal{C}_{\alpha,\beta,D}$ with $q \in \text{int}_\delta C$ and $\|x_0 - q\| \le R$, the iteration (4.3) with constant stepsize $h = \max\{\delta, \frac{1}{\beta}\}$ must halt with $x_i \in \text{int } C$ by iteration $i \le N_{\max} := \lfloor \frac{(R+h-\delta)^2}{h^2} \rfloor$. Moreover, this constant stepsize method is minimax optimal among all separating hyperplane algorithms. That is, there exists C, q, x_0 as above such that for any method (4.1), $x_0, \dots, x_{N_{\max}-1} \notin \text{int } C$.*

In this case, the proof of this minimax optimality theorem is sufficiently simple that one could have reached these conclusions without computer-assistance. The numerical insights generated in the following three sections escalate in complexity beyond what is reasonable to do "by-hand".

5 Application: Frank-Wolfe as an SDP with Nonconvex Constraints

In this section, we consider performance estimation of the Frank-Wolfe (conditional gradient) method on smooth and strongly convex sets. This method applies to constrained optimization problems

$$\min_{x \in C} f(x) \tag{5.1}$$

where f is at least convex and L-smooth and C is at least a convex set with diameter at most D. At various points, we will consider problem classes with these properties and additional, optional structures like μ-strong convexity of f and α-strong convexity and β-smoothness of C. Hence, a family of problems is parameterized by $(\mu, L, \alpha, \beta, D)$. A particular problem instance is then defined by (f, C, x_0). We will let $x_\star \in C$ denote a minimizer of (5.1) which always exists by compactness.

(Generalized) Frank-Wolfe Algorithm Definition. Given a problem instance, we consider two types of algorithms, both using a gradient oracle for f and a linear minimization oracle for C.

The Frank-Wolfe method generates iterates $x_1, \dots, x_N \in C$ via

$$z_k \in \underset{y\in C}{\operatorname{argmin}}\langle \nabla f(x_k), y\rangle \qquad \text{(FW)}$$
$$x_{k+1} = (1-h_k)x_k + h_k z_k$$

for some stepsize schedule $h = (h_0, \dots, h_{N-1})$. Such stepsizes are sometimes called "open-loop". Typically, stepsizes follow $h_k = \frac{2}{k+2}$ or more generally $h_k = \frac{\ell}{k+\ell}$ for some $\ell > 0$. In all cases, we will assume that $0 \le h_k \le 1$ for all k, ensuring x_{k+1} remains a convex combination of x_0 and all z_i.

More generally, we consider "Generalized Frank-Wolfe" methods allowed to select each iterate as any convex combination of x_0 and the extreme points z_i seen. Given a lower triangular matrix of weights H, such a method iterates

$$z_k \in \underset{y\in C}{\operatorname{argmin}}\langle \nabla f(x_k), y\rangle \qquad \text{(GFW)}$$
$$x_{k+1} = (1-\sum_{i=0}^{k} H_{k,i})x_k + \sum_{i=0}^{k} H_{k,i} z_i \ .$$

Clearly, this model includes the vanilla Frank-Wolfe method above. Further, this allows for additional freedom to incorporate and explore a wide variety of momentum-type schemes[3].

Frank-Wolfe PEP Definition and Computation. For a fixed algorithm, determined by fixing h or H above, its worst-case performance after N steps can be formulated as a performance estimation problem below. For example, the Frank-Wolfe method's worst-case objective gap seen is given by

$$p_{\mathrm{FW}}(N, h; \mu, L, \alpha, \beta, D) = \begin{cases} \max\limits_{f, C, x_0} & \min\limits_{i=0,\dots,N} f(x_i) - f(x_\star) \\ \text{s.t.} & x_{k+1} = (1-h_k)x_k + h_k z_k \\ & z_k \in \operatorname{argmin}_{y\in C}\langle \nabla f(x_k), y\rangle \\ & x_0 \in C \\ & C \in \mathcal{C}_{\alpha,\beta,D} \\ & f \in \mathcal{F}_{\mu,L} \\ & x_\star \in \operatorname{argmin}_{y\in C} f(y) \ . \end{cases} \qquad (5.2)$$

A similar definition follows for the generalized method with stepsize matrix H. We leverage two computational approaches, formalized in Section 5.4 with full details in Appendix C:

(i) Our interpolation theorems allow (5.2)—and a nearly identical PEP for Generalized Frank-Wolfe methods—to be formulated as a semidefinite program with additional separable nonconvex equality constraints. See (5.7). The complexity of these nonconvex constraints limits the scale of problems able to be solved. In our numerics below, we solved these instances via a global method for up to $N \approx 5$ and by a local method for up to $N \approx 15$. Numerically, our local solutions always agreed with our global solutions when both were tractable.

(ii) We also consider a semidefinite programming relaxation of this PEP, see (5.13), allowing the computations of upper bounds on the PEP for larger values of $N \approx 50$. This relaxation, denoted $p_{\mathrm{FW,relaxed}}$, and in particular the dual of the relaxed SDP, facilitates the local optimization of the stepsizes h and H to improve performance by solving the minimax problems

$$\min_h \ p_{\mathrm{FW,relaxed}}(N, h; \mu, L, \alpha, \beta, D) \quad \text{or} \quad \min_H \ p_{\mathrm{FW,relaxed}}(N, H; \mu, L, \alpha, \beta, D) \ . \qquad (5.3)$$

[3]A similar H-matrix generalization in the context of gradient descent enables one to describe a wide range of momentum schemes including Nesterov's fast method, facilitating big-O improvements in convergence rates.

These become purely nonlinear minimization problems when $p_{\text{FW,relaxed}}$ is replaced by its dual minimization problem.

Organization of Results. The following three subsections present positive (and negative) numerical guidance on where existing theory can be improved and new theory can be developed. As mentioned above, details on the reformulation of the above performance estimation problems and solving them are deferred to Section 5.4. In brief,

—Section 5.1 considers problems with non-strongly convex constraint sets ($\alpha = 0$), highlighting opportunities to improve current convergence theory via stepsize optimization and smoothness.

—Section 5.2 considers problems with strongly convex functions ($\mu > 0$) and either strongly convex constraint sets ($\alpha > 0$) or interior optimal points ($x_\star \in \text{int}_\delta C$, $\delta > 0$), highlighting several gaps and opportunities in current accelerated $O(1/N^2)$ convergence rates theory [14, 15].

—Section 5.3 surveys 24 different settings given by enumerating every combination of assumptions on f, C, and $x_\star$ that our theory can support. We numerically find one so-far unstudied setting where convergence appears faster than $O(1/N)$ but no new settings clearly supporting a $O(1/N^2)$ rate.

5.1 Improved Convergence from Set Smoothness and Stepsize Design ($\alpha = 0$)

We first consider the performance of Frank-Wolfe over β-smooth sets and with various improved stepsize strategies. To focus on these two effects, we fix $\alpha = 0$, not requiring any strong convexity of our sets and enabling the necessary and sufficient interpolation Theorem 3.3 to apply. These numerical results highlight areas for future theoretical development.

5.1.1 Improvements from Stepsize Design Given Nonsmooth Constraints ($\beta = \infty$) As a first (warm-up) case, which is already well-studied, we consider the convergence of Frank-Wolfe on general nonsmooth constraint sets ($\beta = \infty$). The best known upper bound on convergence rates in this setting is due to Jaggi [13], establishing that Frank-Wolfe with the "standard" stepsize sequence $h_k = \frac{2}{k+2}$ has $p_{\text{FW}}(N, h; 0, L, 0, \infty, D) \leq \frac{2LD^2}{N+2}$. That is, for any L-smooth convex f and convex C with diameter at most D, the objective gap converges at rate at least $\frac{2LD^2}{N+2}$. Note that while the rate proven by Jaggi describes the objective gap at the terminal iterate, it holds as an equally valid bound for our objective $\min_i f(x_i) - f(x_\star)$. The best known lower bound on convergence rates in this setting is due to Lan [24], establishing that a method using a gradient oracle for f and a linear optimization oracle for C cannot guarantee convergence better than $\frac{LD^2}{4N}$. Hence, there is a factor of eight gap between our best-known method guarantee and lower bounding hard instance.

The minimax optimal performance of a Frank-Wolfe or Generalized Frank-Wolfe method lies somewhere between these two bounds being given by solving either

$$\min_h \ p_{\text{FW}}(N, h; 0, L, 0, \infty, D) \qquad \text{or} \qquad \min_H \ p_{\text{FW}}(N, H; 0, L, 0, \infty, D)\ .$$

Figure 6's first plot shows gaps between existing upper and lower bounds, and performance estimation solves for the standard stepsizes $h_k = \frac{2}{k+2}$ and for numerically, locally, minimax optimal stepsizes $\tilde{h}$ for (FW) and stepsize matrices $\tilde{H}$ for (GFW). Note this case, having $\alpha = 0, \beta = \infty$, does not require our set interpolation theory. Using only function interpolation theory, the gap between the standard stepsizes and their known upper bound has been previously plotted in [3, Section 4.5]. Minimax optimal stepsizes can be produced using the branch-and-bound software of [25]. Even in this well-studied limiting setting of nonsmooth sets, slackness exists in modern theory and improvements follow from stepsize design and further gains from stepsize matrix design.

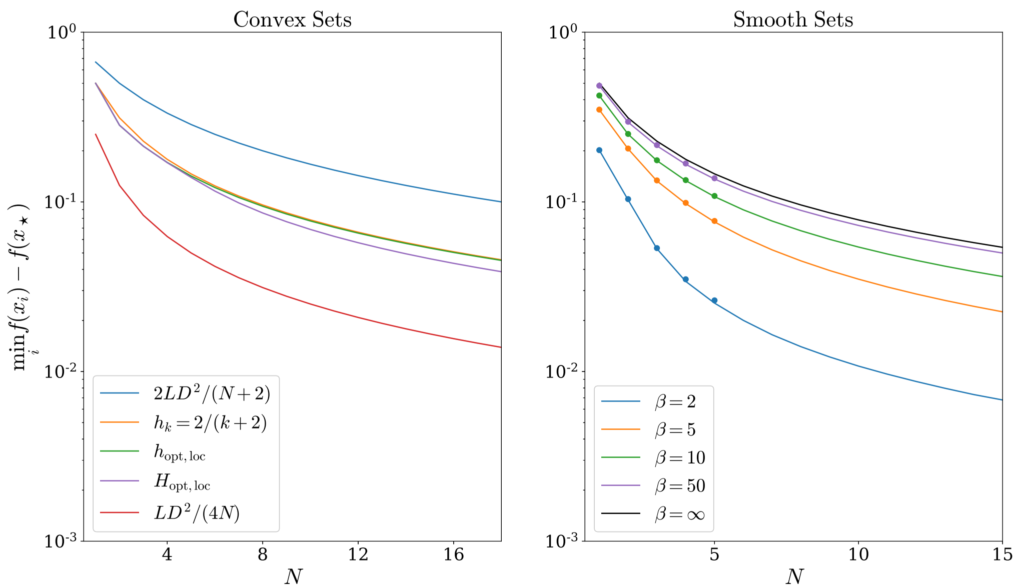

Figure 6: Fixed $L = D = 1$. Left: For nonsmooth convex constraint sets, plots of existing upper and lower bounds on the convergence of Frank-Wolfe and locally optimized PEP results using both standard and numerically optimized stepsizes. Right: For β-smooth convex constraints (where no current theory exists) with varied β, locally optimal PEP results for the standard stepsizes $h_k = 2/(k+2)$ are shown up to $N = 15$. Results for smooth sets are obtained via local optimization giving a lower bound but verified by computing a global upper bound for small N, as indicated by individual points on the plot.

5.1.2 Improvements from Smoothness of Constraint Sets ($\beta < \infty$) To date, no theory exists answering whether Frank-Wolfe's convergence benefits from smoothness in the constraint set C. Our interpolation theorems provide an immediate method to answer this question by estimating the performance gains (if any) from this additional structure. To do so requires formulating $p_{\text{FW}}(N, h; 0, L, 0, \beta, D)$ as a tractable mathematical program. Our numerics use our first approach: (i) both global and local solves of an exact nonconvex PEP formulation.

Figure 6's second plot provides an immediate positive answer to the question of whether smoothness of constraints improves Frank-Wolfe's performance, even fixing the use of the standard stepsizes $h_k = 2/(k+2)$. Local solves for our exact, nonconvex PEP formulation show up to an order of magnitude improvement in performance by $N = 15$ over the convex setting $\beta = \infty$.

5.1.3 Improvements from Stepsize Design Given Smooth Constraints ($\beta < \infty$) The previous two experiments showed improvements from optimizing stepsizes and from smoothness. Further benefits follow from optimizing stepsizes for the smooth constrained setting. Here, we use our second approach: (ii) global solves of a relaxed SDP upper bound. For this section we restrict to the case that $\nabla f(x_\star) \neq 0$ (i.e., the set does not contain an unconstrained minimum).

The additional structural assumption of smoothness of the constraint sets adds separable nonconvex equality constraints to the semidefinite program describing p_{FW}. As a result, it no longer has an exactly matching dual problem. To maintain a pure semidefinite programming form, we focus here on the relaxation $p_{\text{FW,relaxed}}$ (again see (5.13) for a formal definition). By virtue of

Table 1: Comparison of improved stepsize sequences with the standard stepsize $h_k = \frac{2}{k+2}$ at $N = 10$ for various levels of smoothness β. These stepsizes and corresponding PEP results apply to the case where $\nabla f(x_\star) \neq 0$. Improved stepsizes $\tilde{h}$ and $\tilde{H}$ were produced by local optimization, minimizing the relaxed SDP upper bound, see Section 5.4 for formal definitions. PEP results (solutions to $p_{\rm FW}$ using $\tilde{h}$ and $\tilde{H}$) were then obtained via local optimization of the exact PEP formulation.

	Standard h_k	Improved $\tilde{h}$		Improved $\tilde{H}$	
β	PEP Result	PEP Result	% Improvement	PEP Result	% Improvement
2	0.00330	0.00318	3.6	0.00171	48.2
5	0.03429	0.03435	-0.2	0.02848	16.9
10	0.05392	0.05374	0.3	0.04636	14.0
50	0.07297	0.07210	1.2	0.06381	12.6
∞	0.07829	0.07715	1.5	0.06877	12.2

$p_{\rm FW,relaxed}$ being an SDP, evaluation of the worst-case performance upper bound can be cast as a dual minimization problem. Consequently, the minimax problem seeking the improved stepsizes h minimizing $p_{\rm FW,relaxed}$ can be formulated as a single nonlinear minimization problem.

For a given smoothness level β, we denote by $\tilde{h}$ the improved stepsizes minimizing the Frank-Wolfe method's performance upper bound $p_{\rm FW,relaxed}$ and denote by $\tilde{H}$ the improved lower triangular stepsize matrix minimizing the Generalized Frank-Wolfe method's upper bound. Numerically, we compute estimates of these by locally solving the associated nonlinear minimization problem. Table 1 shows the level of improvement in worst-case performance attained by these improved stepsizes for $N = 10$ when compared to the standard selection $h_k = 2/(k+2)$. We find that optimizing stepsizes offers limited gains, at most a 3.6 percent improvement in the worst case, whereas optimizing over stepsize matrices offered larger gains, in the best case almost a 50 percent speed up. This hints that the design of Generalized Frank-Wolfe methods may be a fruitful direction in seeking to computationally benefit from any smoothness present in constraint sets.

5.2 Gaps in Accelerated Theory on Strongly Convex Functions ($\mu > 0$)

Next, we address settings where the objective function is strongly convex ($\mu > 0$). In [14], Garber and Hazan proved that for strongly convex functions, Frank-Wolfe attains an accelerated convergence rate of $O(1/N^2)$ if the constraint set C is strongly convex ($\alpha > 0$), or if $x_\star \in \mathrm{int}_\delta C$ ($\delta > 0$). Their result relied on stepsize selection by linesearch, selecting h_k via

$$h_k = \operatorname*{argmin}_{h\in[0,1]} \; h\langle z_k - x_k, \nabla f(x_k)\rangle + h^2 \frac{L}{2}\|z_k - x_k\|^2 \ .$$

Wirth et al. later proved in [15] that similar $O(1/N^2)$ convergence rates were attained in these settings using the open-loop sequence $h_k = \frac{4}{k+4}$. Specifically, in [15, Theorem E.1], for α-strongly convex sets they show that if $f \in \mathcal{F}_{\mu,L}$ then the iterates of Frank-Wolfe satisfy

$$f(x_N) - f(x_\star) \leq \frac{\frac{128L^2}{\alpha^2\mu} + 8LD^2}{(N+2)^2} \ . \tag{5.4}$$

Additionally, in [15, Theorem 3.6], they show that if $f \in \mathcal{F}_{\mu,L}$ and $x_\star \in \mathrm{int}_\delta C$, then letting $M = \lceil 64LD^2/(\mu\delta^2)\rceil$, for any $N \geq M$, the iterates of Frank-Wolfe satisfy

$$f(x_N) - f(x_\star) \leq \max\left\{\frac{(M+3)^2}{(N+2)^2}(f(x_M) - f(x_\star)), \quad \frac{\frac{128L^2D^6}{\mu\delta^4} + 8LD^2}{(N+2)^2}\right\} \ . \tag{5.5}$$

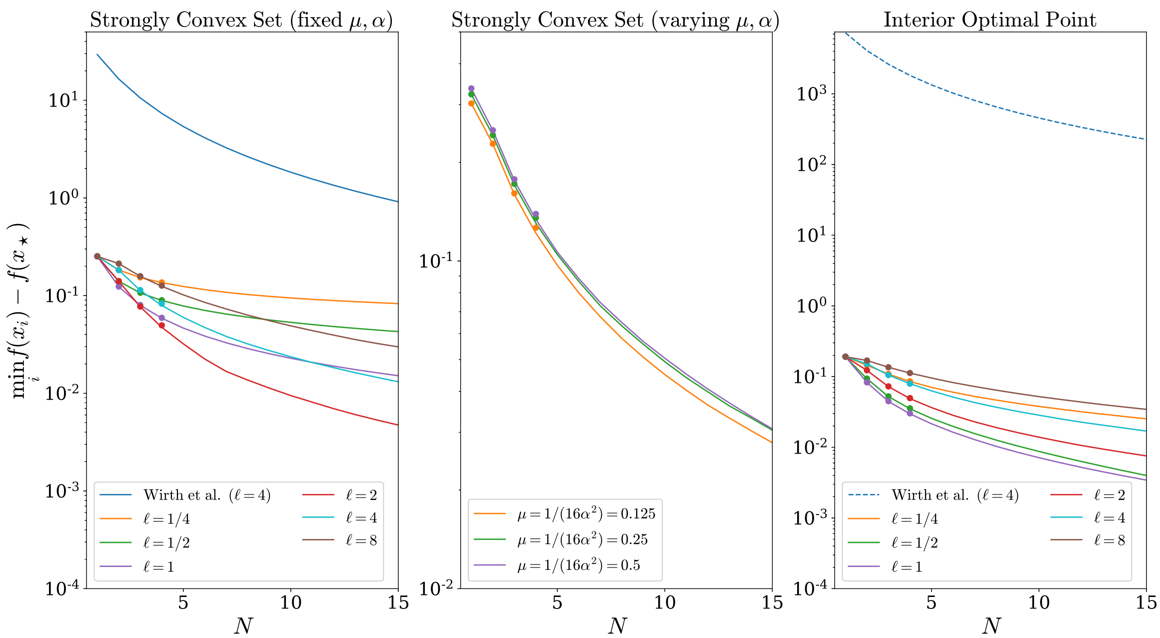

Figure 7: Frank-Wolfe PEP results for μ-strongly convex, L-smooth functions with $L = D = 1$. Left: Comparing PEP solves over α-strongly convex sets for different choices of $h_k = \frac{\ell}{k+\ell}$ and the bound (5.4), fixing $\mu = 0.5$ and $\alpha = 1.0$. Middle: Comparing PEP solves over α-strongly convex sets with the value $\mu\alpha^2$ fixed and $\ell = 4$. Right: Comparing PEP solves with $x_\star \in \text{int}_\delta C$ for different choices of $h_k = \frac{\ell}{k+\ell}$ with $\mu = 0.5$ and $\delta = 0.25$. Although the bound (5.5) does not apply until $N \geq 256$, it is included, dotted. All results above are obtained via local optimization but verified globally for small N, as indicated by individual points on the plot.

Wirth et al. emphasize that the acceleration in (5.5) is only guaranteed after an initial "burn-in" phase of M iterations. For further background, we direct the reader to the recent survey [26].

The following pair of subsections compare the two convergence rate bounds above due to [15] with our numerical estimates of the actual worst-case performance. We note that Wirth et al. [15] also showed that if, instead of assuming μ-strong convexity, one assumes that f has uniformly bounded gradient norm ($\|\nabla f(x)\| \geq \lambda > 0$ for all $x \in C$), this attains an $O(1/N^{\ell/2})$ rate for steps $h_k = \frac{\ell}{k+\ell}$ for integers $\ell \geq 4$. This was subsequently improved to $O(1/N^{\ell})$ for integers $\ell \geq 2$ in [27]. However, since the function interpolation theory of Theorem 2.2 does not support ensuring gradients are uniformly bounded, such settings are excluded from our consideration. In addition, [24] showed that with a strongly convex function, one can attain a linear convergence rate using an "enhanced" linear optimization oracle. In this method, at each iteration one instead solves the modified subproblem $z_k = \text{argmin}_{y \in C}\{\langle \nabla f(x_k), y\rangle \mid \|y - x_k\| \leq r_k\}$ for some $r_k > 0$. Given the more complex optimality condition of this subproblem, our PEP results are not well-suited to modeling such algorithms and hence they are beyond our scope.

5.2.1 Improvements for the Accelerated Strongly Convex Set Rate of (5.4)

In this setting of strongly convex sets with bounded diameter, we can no longer apply our necessary and sufficient interpolation of Theorem 3.3. As a consequence, by applying our interpolation conditions of Interp$(\alpha, 0, D; 1)$, we only obtain upper bounds on the method's worst-case performance.

In Figure 7, we compare the PEP result for $h_k = \frac{4}{k+4}$ ($\ell = 4$) with the convergence rate (5.4). These approximate PEP results outperform the guarantee of [15] by about two orders of magnitude, indicating significant room for improvement in the constants of (5.4). Additionally, we include in Figure 7 the performance for other values of ℓ (with $h_k = \frac{\ell}{k+\ell}$). While our results only extend to $N = 15$, in this region, the upper bound of (5.4) appears to hold for many other values of ℓ. The method with $\ell = 2$ performed best among the tested values and, in particular, outperforms $\ell = 4$. This suggests that similar acceleration theory likely holds for these other stepsize sequences.

Lastly, we examine the form of (5.4). This convergence rate depends on the value $\mu\alpha^2$, but not μ or α individually. In Figure 7, we vary μ and α while keeping the $\mu\alpha^2$ fixed; the results show that the performance is not constant as μ varies. This behavior indicates that the convergence rate of (5.4) does not match the form of the optimal rate. While outside the scope of this paper, further testing of the PEP results with each of the parameters μ, L, α, β, D could elucidate the general form of the tight convergence rates. Moreover, truly tight theory, as emphasized by [27], ought to only depend on affine invariant quantities, which the above parameters are not.

5.2.2 Improvements for the Accelerated Interior Optimal Point Rate of (5.5) We repeat this analysis for the setting of a strongly convex function with an interior optimal point instead of strongly convex constraint set. Note then our necessary and sufficient Theorem 3.3. Recall that the rate (5.5) is only active after iterate $M = \lceil 64LD^2/(\mu\delta^2) \rceil \geq 256$. We therefore cannot compare directly with (5.5) due to our computational limits around $N = 15$. Nevertheless, our results in Figure 7 indicate that tighter analysis may be possible. An $O(1/N^2)$ rate appears to hold for various ℓ, with $\ell = 1$ outperforming all other tested values of ℓ, including the choice $\ell = 4$ of [15].

5.3 Survey of Smooth/Strongly Convex Settings for Acceleration

As discussed in Section 5.2, Frank-Wolfe's convergence is known to accelerate in a few settings: given a strongly convex f and C or given a strongly convex f and $x_\star \in \mathrm{int}_\delta C$. However, it remains undetermined if acceleration can be achieved in any intermediate settings, including any case with smooth constraint sets. The existing $O(1/N)$ lower bound of Lan [24] uses the simplex as a constraint set, so either smoothness or strong convexity has the potential to break this bound.

Using our expanded performance estimation toolbox, we explore 24 permutations of problem instances: C being smooth or not; C being strongly convex or not; f being strongly convex or not; the minimizer having $g_\star \neq 0$ (i.e., being on the exterior of the constraint set), having $g_\star = 0$ with $x_\star$ simply being in the set, or having $g_\star = 0$ with $x_\star$ strictly interior to C. This enables us to assess if any other settings aside from those in Section 5.2 achieved $O(1/N^2)$ convergence. In Figure 8, we show our results for each of these settings for $N = 2 \dots 15$ along with an estimated rate $O(1/N^m)$ with m determined by regression on the values from $N = 8 \dots 15$.

Our results show no clear evidence of any intermediate problem settings that achieve acceleration, typically having $m \approx 1$. However, in the settings with smooth functions, strongly convex sets, and an interior optimal point, our results seem to noticeably outperform the standard $O(1/N)$ rate, instead being closer to $O(1/N^{1.2})$. This presents an intriguing area for future analysis. Given the acceleration result (5.5) of [15] required a burn-in phase, it remains possible that acceleration in some interim settings surveyed is hidden by a lengthy burn-in phase and missed by our survey.

5.4 PEP Formulations and Computational Approaches

In the remainder of this section, we formalize the computational details underlying all of the previously presented numerical insights. First, we use both set and function interpolation theorems

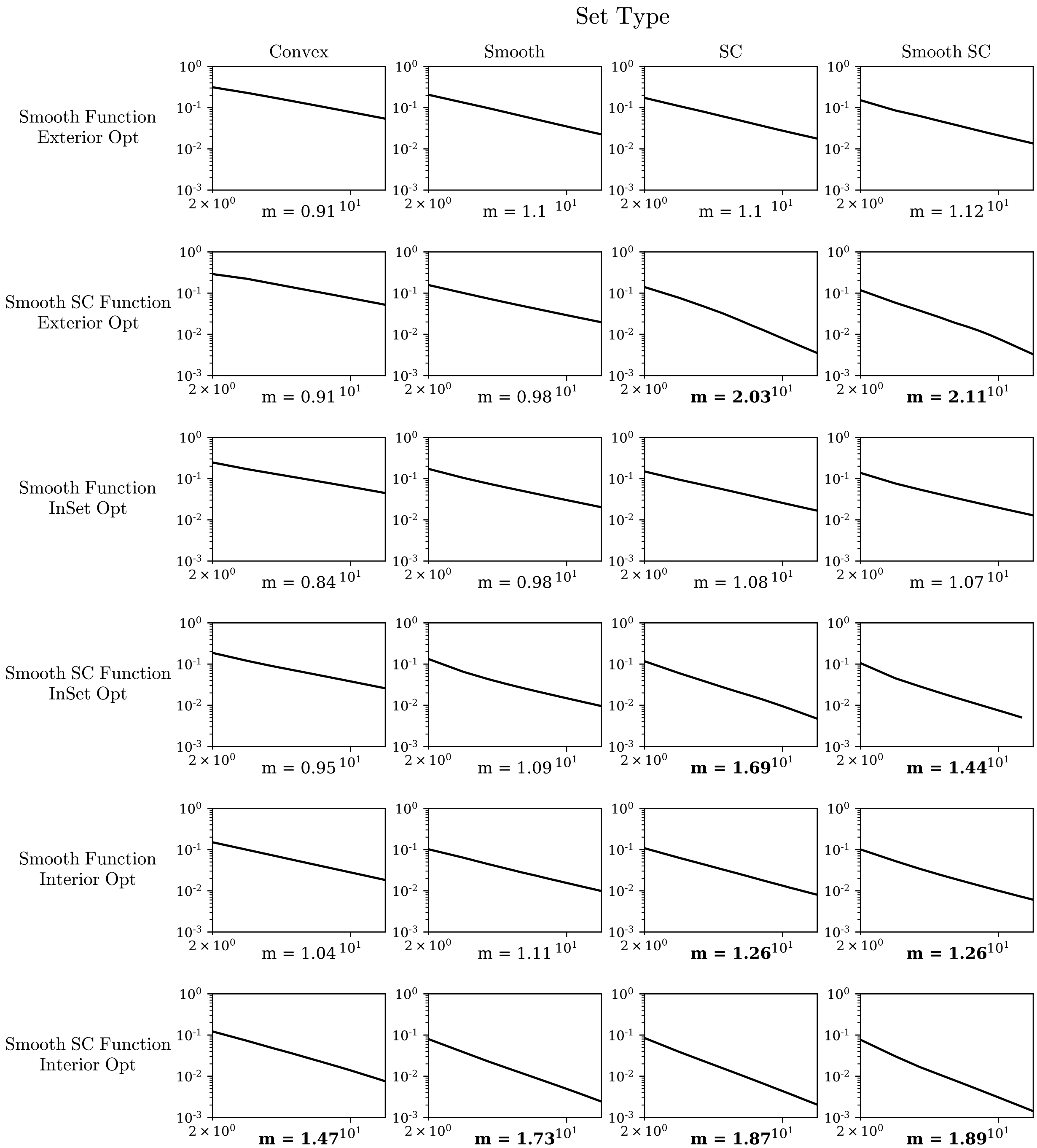

Figure 8: Survey of Frank-Wolfe's performance with standard stepsizes on 24 different problem classes. "InSet" describes the case $x_\star \in C$ (equivalently, $\delta = 0$). We highlight in bold the problem settings in which the estimated rate $O(1/N^m)$ noticeably exceeds the standard $O(1/N)$ rate.

to formulate the PEP (5.2) as an SDP with separable nonconvex equality constraints. Then we present the optimization methods used to generate all of our numerics: (i) local and global approaches solving this exact formulation and (ii) global optimization of a relaxed pure SDP formulation.

5.4.1 Deriving Finite-Dimensional PEP Formulations. The process of reformulating PEP (5.2) using interpolation theory closely follows our derivation in Section 4. The critical difference is that in this setting, we end up with quadratic equality constraints in our SDP.

Denote $x_\star$ as a minimizer of (5.1) and $f_\star = f(x_\star)$, and let $\hat{x}$ denote a global minimum of f. Note that the behavior of the Frank-Wolfe algorithm depends significantly on the location of $\hat{x}$: $\hat{x} \in C$ or $\hat{x} \notin C$. This dependence is highlighted in the classic results of [28, 29] and continues to be widely discussed in the literature. In particular, as discussed in Section 5.2, for strongly convex objective f, when $\hat{x} \in \operatorname{int} C$, the convergence rate improves from $O(1/N)$ to $O(1/N^2)$ [14, 15]. We note that the location of $\hat{x}$ determines our optimality conditions for $x_\star$ and, consequently, our formulation of the PEP. Hence we must consider two separate settings, when $-g_\star \in N_C(x_\star)$ with $g_\star \neq 0$, making $\hat{x}$ lie outside C, or when $g_\star = \nabla f(x_\star) = 0$, making $x_\star$ an unconstrained minimizer of f.

In the results below, let $\mathcal{I} = [0 : N-1]$ and $\mathcal{K} = [0 : N]$, along with $\mathcal{I}_\star = \mathcal{I} \cup \{\star\}$ and $\mathcal{K}_\star = \mathcal{K} \cup \{\star\}$.

Proposition 5.1. *Consider any points z_i, $x_\star$, and $x_{i+1} = (1-h_i)x_i + h_i z_i$ and define*

$$S_{g_\star \neq 0} = (\{(z_i, -g_i)\}_{i\in\mathcal{I}} \cup \{(x_\star, -g_\star)\},\ \{(x_k, 0)\}_{k\in\mathcal{K}})$$

where $g_\star \neq 0$. Then there exist $f \in \mathcal{F}_{\mu,L}$ and $C \in \mathcal{C}_{\alpha,\beta,D}$ satisfying $z_i \in \operatorname{argmin}_{y\in C}\langle \nabla f(x_i), y\rangle$ for all $i \in \mathcal{I}$, $x_\star \in \operatorname{argmin}_{y\in C} f(y)$, and $x_0 \in C$ if and only if $S_{g_\star\neq 0}$ is $\mathcal{C}_{\alpha,\beta,D}$-interpolable and $\{(x_k, g_k, f_k)\}_{k\in\mathcal{K}_\star}$ is $\mathcal{F}_{\mu,L}$-interpolable, where $f(x_k) = f_k$ and $\nabla f(x_k) = g_k$ for all $k \in \mathcal{K}_\star$.

Proof. ($\Rightarrow$) Suppose that there exist $f \in \mathcal{F}_{\mu,L}$ and $C \in \mathcal{C}_{\alpha,\beta,D}$ such that $x_\star \in \operatorname{argmin}_{y\in C} f(y)$ with $\nabla f(x_\star) \neq 0$ and applying Frank-Wolfe with $x_0 \in C$ yields $z_i \in \operatorname{argmin}_{y\in C}\langle \nabla f(x_i), y\rangle$ for all $i \in I$. By construction, setting $g_k = \nabla f(x_k)$ and $f(x_k) = f_k$ for all $k \in \mathcal{K}_\star$ ensures $\{(x_k, g_k, f_k)\}_{k\in\mathcal{K}_\star}$ is $\mathcal{F}_{\mu,L}$-interpolable. Since C is convex, the optimality condition defining $z_i \in \operatorname{argmin}_{y\in C}\langle g_i, y\rangle$ implies that $z_i \in C$ and $-g_i \in N_C(z_i)$. Similarly, the optimality condition for $x_\star$ ensures that $-g_\star \in N_C(x_\star)$. Additionally, given that $0 \leq h_k \leq 1$ for all k, we see that $x_k \in \operatorname{conv}(\{x_0, z_0, \dots, z_k\})$. Since $x_0 \in C$ and $z_k \in C$ for all k, this yields $x_k \in C$ for all $k \in \mathcal{K}$. Hence $S_{g_\star\neq 0}$ is $\mathcal{C}_{\alpha,\beta,D}$-interpolable.

($\Leftarrow$) Suppose that $S_{g_\star\neq 0}$ is $\mathcal{C}_{\alpha,\beta,D}$-interpolable and $\{(x_k, g_k, f_k)\}_{k\in\mathcal{K}_\star}$ is $\mathcal{F}_{\mu,L}$-interpolable. That is, there exists a μ-strongly convex, L-smooth function f such that that $f(x_k) = f_k$ and $\nabla f(x_k) = g_k$ for all $k \in \mathcal{K}_\star$ and an α-strongly convex, β-smooth set C such that $z_i \in C$, $x_i \in C$, and $-g_i \in N_C(z_i)$ for all $i \in \mathcal{I}_\star$. Since C is convex, we know that $-\nabla f(x_i) \in N_C(z_i)$ implies that $z_i \in \operatorname{argmin}_{y\in C}\langle \nabla f(x_i), y\rangle$. Similarly, $-\nabla f(x_\star) \in N_C(x_\star)$ implies that $z_i \in \operatorname{argmin}_{y\in C} f(y)$. Hence the desired f and C exist. □

Proposition 5.2. *Consider any points z_i, $x_\star$, and $x_{i+1} = (1-h_i)x_i + h_i z_i$ and define*

$$S^\delta_{g_\star=0} = (\{(z_i, -g_i)\}_{i\in\mathcal{I}},\ \{(x_k, 0)\}_{k\in\mathcal{K}} \cup \{(x_\star, \delta)\})$$

and $g_\star = 0$. Then there exist $f \in \mathcal{F}_{\mu,L}$ and $C \in \mathcal{C}_{\alpha,\beta,D}$ satisfying $z_i \in \operatorname{argmin}_{y\in C}\langle \nabla f(x_i), y\rangle$ for all $i \in \mathcal{I}$ with $x_0 \in C$ and $x_\star \in \operatorname{argmin} f(y)$ with $x_\star \in \operatorname{int}_\delta C$ if and only if $S^\delta_{g_\star=0}$ is $\mathcal{C}_{\alpha,\beta,D}$-interpolable and $\{(x_k, g_k, f_k)\}_{k\in\mathcal{K}_\star}$ is $\mathcal{F}_{\mu,L}$-interpolable, where $f(x_k) = f_k$ and $\nabla f(x_k) = g_k$ for all $k \in \mathcal{K}_\star$.

Proof. We apply the same approach as above, but now with the assumption that $g_\star = 0$. The modified result follows from the fact that $g_\star = 0$ if and only if $x_\star \in \operatorname{argmin} f(y)$ (that is, $x_\star$ is the unconstrained minimizer). □

These straightforward results (along with Theorems 2.2 and 3.3) allow us to reformulate the PEP for Frank-Wolfe for the two separate cases $\nabla f(x_\star) = 0$ and $\nabla f(x_\star) \neq 0$. To solve (5.2) without any assumptions on $x_\star$, one must solve the PEP under each assumption separately and then take the maximum of the two results. Here, we present the resulting formulation assuming $\nabla f(x_\star) \neq 0$ (and therefore $g_\star \neq 0$). An essentially identical derivation applies when $g_\star = 0$ from Proposition 5.2. In our results, we found that the case where $\nabla f(x_\star) \neq 0$ (i.e., the set does not contain an unconstrained minimum), typically—but not always—resulted in a worse performance guarantee.

By Proposition 5.1, (5.2) equals

$$p_{\mathrm{FW}}(N, h; \mu, L, \alpha, \beta, D) = \begin{cases} \max\limits_{x_k, g_k, f_k, z_i} & \min\limits_k f_k - f_\star \\ \text{s.t.} & x_{i+1} = (1 - h_i)x_i + h_i z_i \quad \forall i \in \mathcal{I} \\ & \{(x_k, g_k, f_k)\}_{k \in \mathcal{K}_\star} \text{ is } \mathcal{F}_{\mu,L}\text{-interpolable} \\ & S_{g_\star \neq 0} \text{ is } \mathcal{C}_{\alpha,\beta,D}\text{-interpolable}\ . \end{cases} \tag{5.6}$$

Applying interpolation theorems for the function and set constraints above yields an explicit finite-dimensional formulation. For ease of exposition, in the derivation to follow, we consider the specific case of optimizing L-smooth functions over β-smooth sets with several details deferred to Appendix C. Using the appropriate function and set interpolation theorems, analogous derivations follow for any combination of smoothness and strong convexity of functions and sets.

Letting $\theta = (h, \mu, L, \alpha, \beta, D)$, we obtain

$$p_{\mathrm{FW}}(N; \theta) = \begin{cases} \max\limits_{\substack{x_k, g_k, f_k \\ w_k, z_i, n_i}} & f_{\min} - f_\star & \\ \text{s.t.} & f_{\min} - f_k \leq 0 & \forall k \in \mathcal{K} \\ & f_k - f_l + \langle g_k, x_l - x_k \rangle + \frac{1}{2L}\|g_k - g_l\|^2 \leq 0 & \forall k, l \in \mathcal{K}_\star \\ & x_{i+1} = (1 - h_i)x_i + h_i z_i & \forall i \in \mathcal{I} \\ & \langle -g_i, z_j - \frac{1}{\beta}n_j - (z_i - \frac{1}{\beta}n_i) \rangle \leq 0 & \forall i, j \in \mathcal{I}_\star \\ & \langle -g_i, w_k - (z_i - \frac{1}{\beta}n_i) \rangle \leq 0 & \forall i \in \mathcal{I}_\star, k \in \mathcal{K}_\star \\ & \|x_k - w_k\|^2 \leq \frac{1}{\beta^2} & \forall k \in \mathcal{K}_\star \\ & \|z_i - \frac{1}{\beta}n_i - (z_j - \frac{1}{\beta}n_j)\|^2 \leq (D - \frac{2}{\beta})^2 & \forall i, j \in \mathcal{I}_\star \\ & \|z_i - \frac{1}{\beta}n_i - w_k\|^2 \leq (D - \frac{2}{\beta})^2 & \forall i \in \mathcal{I}_\star, k \in \mathcal{K}_\star \\ & \|w_k - w_l\|^2 \leq (D - \frac{2}{\beta})^2 & \forall k, l \in \mathcal{K}_\star \\ & \langle g_i, n_i \rangle \leq 0 & \forall i \in \mathcal{I}_\star \\ & \|n_i\|^2 = 1 & \forall i \in \mathcal{I}_\star \\ & \langle g_i, n_i \rangle^2 = \|g_i\|^2 & \forall i \in \mathcal{I}_\star\ . \end{cases} \tag{5.7}$$

Note that the last three constraints above ($\langle g_i, n_i \rangle \leq 0$, $\|n_i\|^2 = 1$, $\langle g_i, n_i \rangle^2 = \|g_i\|^2$) are equivalent to the condition $n_i = \frac{-g_i}{\|g_i\|}$. We will assume without loss of generality that $x_\star = 0$ and $f_\star = 0$. Then, one can repeat the "standard" Gram matrix reformulation approach discussed in Section 4. Assuming $d \geq 4N + 6$, one can derive an equivalent problem with variables G and F as

$$\begin{aligned} F &= [f_0 | f_1 | \dots | f_N | f_{\min}] \in \mathbb{R}^{1 \times (N+2)} \\ \Lambda &= [x_0 | g_\star | g_0 | g_1 | \dots | g_N | z_0 | \dots | z_{N-1} | n_\star | n_0 | \dots | n_{N-1} | w_\star | w_0 | \dots | w_N] \in \mathbb{R}^{d \times (4N+6)} \\ G &= \Lambda^T \Lambda \in \mathbb{S}_+^{4N+6}\ . \end{aligned}$$

This Grammian change-of-variables reformulates all of the constraints of (5.7) as linear constraints in our new variables, with the exception of the equality constraint $\langle g_i, n_i \rangle^2 = \|g_i\|^2$ which is quadratic in G. This constraint is nonconvex, and therefore cannot be expressed through a convex SDP.

5.4.2 Local Optimization of (5.7) Yielding Lower Bounds Note the sole source of nonconvexities in (5.7) in terms of F, G is the constraints $\langle g_i, n_i\rangle^2 = \|g_i\|^2$. Let $(F, G) \in \mathfrak{S}(\theta)$ denote the semidefinite programming feasible region given by all other constraints and the additional linear constraints $\langle -g_i, n_j\rangle \leq \langle -g_i, n_i\rangle$ for all $i, j \in \mathcal{I}_\star$ (note these added constraints are implied by the nonconvex equalities $\langle g_i, n_i\rangle^2 = \|g_i\|^2$). Appendix C.1 provides a formal definition of this relaxed feasible region. By introducing new variables ψ_i, we can decompose these quadratic constraints into two linear constraints in G. Given $\psi = (\psi_\star, \psi_0, \ldots, \psi_{N-1}) \in \mathbb{R}^{N+1}_{\geq 0}$, consider

$$\hat{p}_{\mathrm{FW}}(\psi;\theta) = \begin{cases} \max\limits_{F,G} & Fe_{N+2} \\ \text{s.t.} & (F,G) \in \mathfrak{S}(\theta) \\ & \|g_i\|^2 = \psi_i \qquad \forall i \in \mathcal{I}_\star \\ & \langle -g_i, n_i\rangle = \sqrt{\psi_i} \qquad \forall i \in \mathcal{I}_\star \end{cases} \tag{5.8}$$

where e_i is i-th standard unit basis vector in $\mathbb{R}^{N+2}$. Note given ψ, this problem is now an SDP (and hence globally solvable) and its solutions are always feasible solutions to the original problem (5.7), providing lower bounds on $p_{\mathrm{FW}}(N;\theta)$. Moreover,

$$\max_{\psi} \hat{p}_{\mathrm{FW}}(\psi;\theta) = p_{\mathrm{FW}}(N;\theta) \ . \tag{5.9}$$

Solving (5.9) remains nonconvex but is now unconstrained and of lower dimension than the original formulation. Numerically, we perform zeroth-order local maximization of $\hat{p}_{\mathrm{FW}}(\psi,\theta)$ with respect to ψ, yielding a lower bound for (5.7). This approach was computationally feasible to run up to $N \leq 15$.

5.4.3 Global Optimization of (5.7) Yielding Upper Bounds Complementing the above local computation of lower bounds, we consider globally optimizing (5.7) via a simple branch-and-bound-type procedure. Again, the primary difficulty is resolving the nonconvex constraints $\langle g_i, n_i\rangle^2 = \|g_i\|^2$. Our approach is based on approximating the nonconvex region defined by this constraint by the union of several convex regions containing the original constraint, see Figure 9. The maximum value over all of these regions provides an upper bound on $p_{\mathrm{FW}}(N;\theta)$. We then globally optimize by iterative refinement of this union of convex feasible regions for each i. Given this procedure scales exponentially in N, we found it is often only practical for $N \leq 5$.

Observe that by Cauchy-Schwarz, we have $\langle g_i, n_i\rangle^2 \leq \|n_i\|^2\|g_i\|^2 = \|g_i\|^2$. Next, given $M > 0$, for each i, we define a partition as a set $T_i = \{t_i^{(0)}, t_i^{(1)}, \ldots, M, \infty\}$ where $0 = t_i^{(0)} \leq t_i^{(1)} \leq \cdots \leq M$. We define a full partition T as the tuple $T = (T_\star, T_0, \ldots, T_{N-1})$. Last, we define a slice $\mathcal{S}$ with respect to our full partition T as a set of intervals

$$\mathcal{S} = \left\{ \left[t_\star^{(r_\star)}, t_\star^{(r_\star+1)}\right], \left[t_0^{(r_0)}, t_0^{(r_0+1)}\right], \ldots, \left[t_{N-1}^{(r_{N-1})}, t_{N-1}^{(r_{N-1}+1)}\right] \right\} \tag{5.10}$$

where $r_i \in \mathbb{N}$ and $t_i^{(r_i)}, t_i^{(r_i+1)} \in T_i$ for all i. Effectively, for each partition T_i, a slice $\mathcal{S}$ selects one interval to consider for each i. For a specific slice $\mathcal{S}$, we consider the subproblem

$$\tilde{p}_{\mathrm{FW}}(\mathcal{S};\theta) = \begin{cases} \max\limits_{F,G} & Fe_{N+2} \\ \text{s.t.} & (F,G) \in \mathfrak{S}(\theta) \\ & \langle -g_i, n_i\rangle \geq \frac{1}{\sqrt{t_i^{(r_i)}}+\sqrt{t_i^{(r_i+1)}}}\|g_i\|^2 + \frac{\sqrt{t_i^{(r_i)}t_i^{(r_i+1)}}}{\sqrt{t_i^{(r_i)}}+\sqrt{t_i^{(r_i+1)}}} \qquad \forall i \in \mathcal{I}_\star \end{cases} \tag{5.11}$$

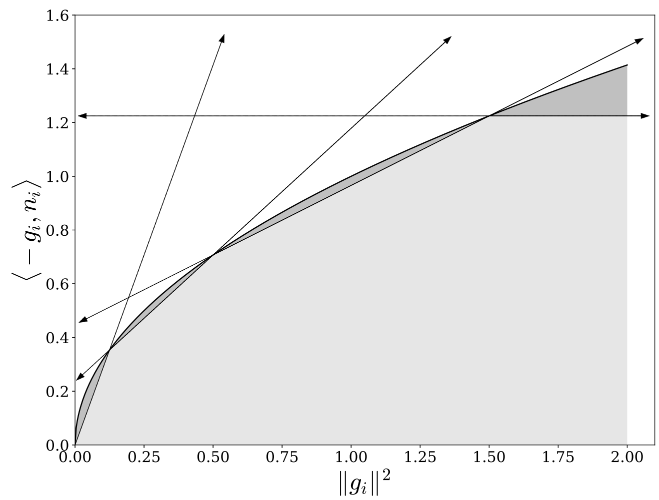

Figure 9: Subregions defined by each slice $\mathcal{S}$ from a partition T.

whose feasible region contains the original nonconvex constraint set for values $t_i^{(r_i)} \le \|g_i\|^2 \le t_i^{(r_i+1)}$. We will use the convention that if $t_i^{(r_i+1)} = \infty$, then our constraint becomes $\langle -g_i, n_i\rangle \ge \sqrt{t_i^{(r_i)}}$.

As illustrated in Figure 9, each slice defines a convex problem approximating part of the original nonconvex problem. Denoting the set of all slices with respect to T by $\mathcal{T}$, we can see that for any partition T, the feasible region of (5.7) is fully contained in the union of the feasible regions of all slices $\mathcal{S} \in \mathcal{T}$. Therefore, we can upper bound the solution to (5.7) by

$$p_{\mathrm{FW}}(\theta) \le \max_{\mathcal{S}\in\mathcal{T}} \tilde{p}_{\mathrm{FW}}(\mathcal{S};\theta)\ . \tag{5.12}$$

In Appendix C.2, we present a detailed algorithm for iteratively branching on the partition $\mathcal{T}$ to yield increasingly accurate upper bounds on the underlying nonconvex PEP. When tractable, we found these numerical upper bounds (denoted by dots in our figures) always closely aligned with the previous method's lower bounds, together certifying tight bounds on the true optimal value.

5.4.4 Relaxed SDP Optimization Yielding Weakened Upper Bounds

Rather than using the above branch-and-bound scheme to closely approximate the feasible region given by intersecting $(F, G) \in \mathfrak{G}(\theta)$ with nonconvex equality constraints, one can simply omit the nonconvex constraints. Doing so gives a convex SDP which is an upper bound on (5.7). Namely,

$$p_{\mathrm{FW}}(\theta) \le p_{\mathrm{FW,relaxed}}(N;\theta) = \begin{cases} \max\limits_{F,G} & Fe_{N+2} \\ \text{s.t.} & (F,G) \in \mathfrak{G}(\theta)\ . \end{cases} \tag{5.13}$$

See (C.1) for an expanded statement of this SDP. While excluded from our numerics above, this SDP relaxation enables computations of upper bounds on convergence rates up to larger values of N (~ 50). Moreover, as shown in Section 5.1.3, considering its equivalent dual minimization problem enables local optimization via blackbox minimization methods of the minimax problems (5.3).

6 Application: Alternating Projections as an SDP with Nonconvex Constraints

As a next application, we consider performance estimation for the method of alternating projections. Our performance estimation for this method takes the same form as our Frank-Wolfe problems, corresponding to an SDP with quadratic equality constraints arising from our interpolation theory. Below, we derive a lower bound on the method's linear convergence and then numerically support a conjecture that this bound is exactly the worst-case performance.

Problem and Alternating Projection Definitions. We consider the problem of seeking a point x minimizing the distance to the intersection of two sets C_1 and C_2. We will assume the sets C_1 and C_2 are closed, convex, and β-smooth with no assumptions of strong convexity or bounded diameter. As a result, our necessary and sufficient interpolation theory from Theorem 3.3 applies. Further, we assume their intersection is nontrivial, meaning a point q exists in the δ-interior of both sets (with $\delta > 0$), and that the initialization has $\|x_0 - q\| \leq R$.

We define the projection of x onto a convex set C by

$$\mathrm{proj}_C(x) = \operatorname*{argmin}_{y \in C} \|x - y\|^2 \ .$$

Note the useful property that $x - \mathrm{proj}_C(x) \in N_C(\mathrm{proj}_C(x))$. Given a problem instance (C_1, C_2, x_0), alternating projections proceeds by iterating

$$x_{k+1} = \mathrm{proj}_{C_2}(\mathrm{proj}_{C_1}(x_k)) \ .$$

Convergence Theory. This is an old and well-studied algorithm. Gubin et al. [25] established convergence guarantees for convex sets and conditions ensuring the distance between x_k and the intersection set $C_1 \cap C_2$ decreases at a linear rate. Further works [30,31] have shown many variations of this linear rate by assuming transversality of the intersecting sets, or other related properties. In [32], the authors proved various tight convergence rates for semialgebraic sets (sets defined by polynomial inequalities) and lines. If one omits the nontrivial intersection condition we impose, only sublinear convergence in terms of $\mathrm{dist}(x_N, C_1)$ can be guaranteed and has been studied using performance estimation by Taylor [33] and Zamani and Glineur [34]. The latter work gave an exactly tight analysis of the convergence of this weaker measure. However, to the best of our knowledge, under our performance measure $\mathrm{dist}(x_N, C_1 \cap C_2)$, no tight convergence rate theory exists for alternating projections between two general convex sets with a nontrivial intersection. Here, rather than improving on existing theory, we use performance estimation to describe the exact linear worst-case convergence rate in our considered, general setting.

We conjecture that two halfspaces in $\mathbb{R}^2$ as shown in Figure 10 constitute the worst-case instance in our defined problem class. This is a standard problem instance for slow convergence of alternating projections (see [25]). Optimizing the angle of intersection between these halfspaces gives the following lower bound on the worst-case performance of alternating projections.

Proposition 6.1. *For any δ, β, R, there exists a problem instance (C_1, C_2, x_0) such that*

$$\mathrm{dist}(x_N, C_1 \cap C_2) \geq \max_{c \in [0,1]} c^{2N-1} \left(\sqrt{R^2 - \delta^2} - \frac{\delta(c+1)}{\sqrt{1-c^2}} \right) \ . \tag{6.1}$$

Considering $c = 1 - \frac{4\delta^2}{R^2}$, this lower bound is at least a linear lower bound of $\Omega((1 - \frac{4\delta^2}{R^2})^{2N-1})$.

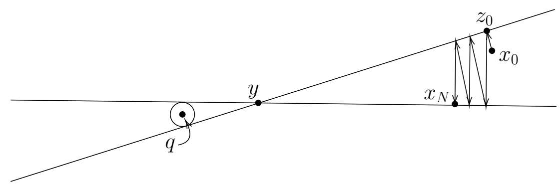

Figure 10: Alternating projections for two intersecting halfspaces.

Proof. Denote the halfspaces $C_1 = \{(y,z) \mid z \le 0\}$ and $C_2 = \{(y,z) \mid \langle n,(y,z)\rangle \le \rho\}$ for some $n \in \mathbb{R}^2$ with $\|n\| = 1$ and $\rho \in \mathbb{R}$. Suppose C_2 forms an angle $\phi > 0$ with C_1 (i.e. $\langle (0,-1), n\rangle = \cos\phi$). Note if $\phi > \frac{\pi}{2}$, then after one iteration, $x_1 \in C_1 \cap C_2$. We therefore assume $\phi \le \frac{\pi}{2}$. Appendix D.2 calculates the performance for this specific problem instance, parameterized by $c = \cos\phi$, finding that

$$\|x_N - y\| = c^{2N-1}\left(\sqrt{R^2-\delta^2} - \frac{\delta(c+1)}{\sqrt{1-c^2}}\right) . \tag{6.2}$$

Then the claimed performance lower bound follows by maximizing over all $c \in [0,1]$. □

We note that in the proof above, ϕ is equivalent to the Friedrich's angle between C_1 and C_2. Moreover, our calculation (6.2) is simply an application of the classic result of [35, 36] that $\|x_N - \mathrm{proj}_C(x_0)\| \le \|x_0\| \cos^{2N-1}(\phi)$. Note when $R \le \sqrt{2}\delta$, the bound in (6.1) evaluates to zero. Beyond this edge case, the optimal value of (6.1) lacks an analytical solution. Table 2 compares numerical evaluations of this maximum with performance estimation solves over all C_1, C_2. Numerically, halfspaces appear to attain the worst-case PEP, motivating the following conjecture.

Conjecture 6.2. *For any closed convex β-smooth sets C_1, C_2, with $q \in \mathrm{int}_\delta(C_1 \cap C_2)$ and $\|x_0 - q\| \le R$, the iterates x_k of alternating projections satisfy*

$$\mathrm{dist}(x_N, C_1 \cap C_2) \le \max_{c\in[0,1]} c^{2N-1}\left(\sqrt{R^2-\delta^2} - \frac{\delta(c+1)}{\sqrt{1-c^2}}\right) .$$

Numerical Performance Estimation Validation of Conjecture 6.2. To simplify notation, we define $z_k = \mathrm{proj}_{C_1}(x_k)$ (and therefore $x_{k+1} = \mathrm{proj}_{C_2}(z_k)$). We also denote $u_i = x_i - z_i$ and $v_k = z_{k-1} - x_k$. We then have that for all i, k

$$u_i \in N_{C_1}(z_i), \qquad v_k \in N_{C_2}(x_k)$$

and our iteration becomes

$$z_i = x_0 - \sum_{j=0}^{i} u_j - \sum_{j=1}^{i} v_j, \qquad x_k = x_0 - \sum_{j=0}^{k-1} u_j - \sum_{j=1}^{k} v_j .$$

Define $C = C_1 \cap C_2$. We will assume that there exists $q \in \mathrm{int}_\delta C$ and that $\|x_0 - q\| \le R$ for some chosen $R > 0$. Since $\mathrm{int}\, C$ is nonempty, this means that for any $x \in C$

$$N_C(x) = N_{C_1}(x) + N_{C_2}(x) \tag{6.3}$$

with the above expression again denoting the Minkowski sum.

Table 2: PEP results from global method with precision $\Delta = 10^{-4}$ for convex alternating projections compared with lower bound (6.1) ($R = 1, \delta = 0.01$).

N	1	2	3	4	5	6
PEP Result ($\Delta = 10^{-4}$)	0.79715	0.71849	0.67394	0.64162	0.61593	0.59449
SDP Matrix Size	10	14	18	22	26	30
Computation Time (min)	0.2	0.1	0.2	1.1	6.4	47.9
Lower Bound (6.1)	0.79711	0.71834	0.67368	0.64157	0.61579	0.59444
Difference	4e-5	1.5e-4	2.6e-4	5.0e-5	1.4e-4	5e-5

Let $y = \mathrm{proj}_C(x_N)$ and further define $x_\star$ and $z_\star$ such that $x_\star = z_\star = y$. By (6.3), we therefore have that $x_N - y \in N_C(y) = N_{C_1}(y) + N_{C_2}(y)$. So we can write $x_N = y + u_\star + v_\star$ for some $u_\star \in N_{C_1}(y)$ and $v_\star \in N_{C_2}(y)$. Next, observe that $B(q, \delta) \subseteq C_1, C_2$ if and only if $B(q, \delta) \subseteq C$, so $q \in \mathrm{int}_\delta C$ if and only if $q \in \mathrm{int}_\delta C_1$ and $q \in \mathrm{int}_\delta C_2$. Lastly, we define $\mathcal{I}_\star = [0 : N - 1] \cup \{\star\}$ and $\mathcal{K}_\star = [1 : N] \cup \{\star\}$.

We consider the worst-case performance of alternating projections in this setting, as measured by the distance from our final iterate x_N to C. We formalize this as

$$p_{\mathrm{AP}}(N; \delta, R, \beta) = \begin{cases} \max\limits_{C_1, C_2, x_0, q} & \|x_N - \mathrm{proj}_C(x_N)\|^2 \\ \text{s.t.} & C_1, C_2 \in \mathcal{C}_{0,\beta,\infty} \\ & B(q, \delta) \subseteq C \\ & x_{k+1} = \mathrm{proj}_{C_2}(\mathrm{proj}_{C_1}(x_k)) \\ & \|x_0 - q\| \leq R\ . \end{cases} \tag{6.4}$$

We apply our definition of interpolability to rewrite as

$$p_{\mathrm{AP}}(N; \delta, R, \beta) = \begin{cases} \max\limits_{\substack{x_k, z_i, v_k \\ u_i, y, q}} & \|x_N - y\|^2 \\ \text{s.t.} & (\{(u_i, z_i)\}_{i \in \mathcal{I}_\star}, \{(q, \delta)\}) \text{ is } \mathcal{C}_{0,\beta,\infty}\text{-interpolable} \\ & (\{(v_k, z_k)\}_{k \in \mathcal{K}_\star}, \{(q, \delta)\}) \text{ is } \mathcal{C}_{0,\beta,\infty}\text{-interpolable} \\ & y = \mathrm{proj}_C(x_N) \\ & \|x_0 - q\| \leq R\ . \end{cases} \tag{6.5}$$

Following the same procedure as done for Frank-Wolfe, one can derive an equivalent SDP with nonconvex equality constraints. Then the same local and global optimization methods discussed in Section 5.4.3 can be applied. For completeness, we include these calculations in Appendix D.1. In Table 2, we show the results of (6.5) using our global method with $\beta = \infty$ for various N. These results support our conjecture since our solves with $\beta = \infty$ upper bound the performance for all $\beta \in \mathbb{R}_+$ and agree with our lower bound numerically.

7 Application: Gradient Methods for Epismooth Functions as an SDP with a Rank-1 Constraint

As a final application, we show our set interpolation theory can provide novel insights even in the context of unconstrained minimization. In particular, consider a gradient method applied to

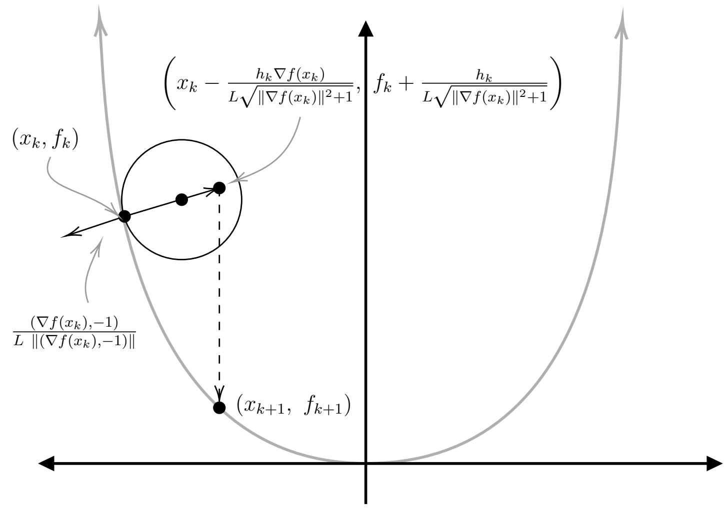

Figure 11: Modified gradient method for epismooth functions (GM).

minimize some differentiable, convex function $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$, defined as

$$x_{i+1} = x_i - \sum_{j=0}^{i} H_{i,j} \nabla f(x_j) \tag{GM}$$

with stepsizes $H_{i,j} \geq 0$. Note this model captures simple methods like gradient descent via a diagonal H and more complex momentum methods like Nesterov's accelerated method [37] and the optimal gradient method of [7]. Much of the modern theory for gradient methods assumes f is L-smooth. This is a major restriction from the family of all differentiable functions, ruling out, for instance, all polynomials with degree greater than two.

Here, we consider applying first-order methods to a larger class of differentiable functions, namely functions with a smooth epigraph. We say f is **L-epigraphically smooth** (**L-epismooth**) if its epigraph epi $f = \{(x,t) \mid f(x) \leq t\}$ is an L-smooth set. Theorem 3.8 with $\eta = 1$ establishes that all L-smooth functions are L-epismooth, so this is a strictly broader class of functions. Note epismoothness of f does not guarantee that f has a full domain of $\mathbb{R}^d$ (see the ball-pen function in Figure 12). Consequently, one cannot guarantee the iteration (GM) is well-defined in general, so additional care is needed in algorithm design. To handle this, we consider modified gradient methods of the form

$$x_{i+1} = x_0 - \frac{1}{L} \sum_{j=0}^{i} H_{i,j} \frac{\nabla f(x_j)}{\sqrt{\|\nabla f(x_j)\|^2 + 1}}\ . \tag{Epi-GM}$$

Normalizing our stepsizes above allows methods to keep $x_{i+1} - x_i$ bounded even as $\nabla f(x_k)$ becomes arbitrarily large (as shown in Figure 11). This property can then be used to ensure the iterates do not leave the domain of f: For example, fixing H to be the identity matrix, the above iteration becomes the gradient descent iteration $x_{i+1} = x_i - \frac{\nabla f(x_i)}{L\sqrt{\|\nabla f(x_i)\|^2+1}}$, which repeatedly moves to the minimizer of the majorizing ball upper bound of f, ensuring progress every iteration.

7.1 Example Epismooth Functions

To illustrate the breadth of this class of functions, the following proposition provides a useful means of checking if a function is **locally** epismooth.

Proposition 7.1. *For any $f \in \mathcal{C}^2$ and x in the domain of f, let*

$$M(x) = \frac{(I + \nabla f(x)\nabla f(x)^T)^{-1/2}(\nabla^2 f(x))(I + \nabla f(x)\nabla f(x)^T)^{-1/2}}{\sqrt{\|\nabla f(x)\|^2 + 1}} \ . \tag{7.1}$$

Then f is locally L-epismooth at x if and only if $L \geq \lambda_{\max}(M)$.

Proof. From the definition of smoothness, f is L-epismooth if and only if for all x, y, $f(y) \leq \mathfrak{b}_x(y; L)$ as defined in (3.9). Observe that $\mathfrak{b}_x(x; L) = f(x)$ and $\nabla \mathfrak{b}_x(x; L) = \nabla f(x)$. Therefore, the upper bound $\mathfrak{b}_x(y; L)$ holds locally if and only if $\nabla^2 f(x) \preceq \nabla^2 \mathfrak{b}_x(x; L)$. We can compute $\nabla^2 \mathfrak{b}_x(x; L) = L\sqrt{\|\nabla f(x)\|^2 + 1}(I + \nabla f(x)\nabla f(x)^T)$. Then rearranging and taking the maximum eigenvalue, we obtain our result. □

Using this result, we can identify simple functions that are not smooth in the function sense of uniformly Lipschitz gradient but satisfy local epismoothness on their domain. For example, consider

$$f(x) = \begin{cases} -\log x & \text{if } x > 0 \\ \infty & \text{if } x \leq 0 \ . \end{cases}$$

Using (7.1), we can calculate $\lambda_{\max}(M(x)) = \frac{\frac{1}{x^2}}{(1+\frac{1}{x^2})^{3/2}} \leq \frac{2}{3\sqrt{3}}$ for all $x > 0$. So f is locally $\frac{2}{3\sqrt{3}}$-epismooth everywhere on its domain. As a more general class of examples, epismoothness also holds for any convex polynomial with "consistent growth", defined for a polynomial $q(x)$ of degree m by the existence of $c_0, c_1 > 0$ such that for all x, $q(x) \geq c_0 + c_1\|x\|^{m-1}$. For example, $q(x) = \|Ax - b\|_4^4$. By convexity, such a polynomial must satisfy

$$\|\nabla q(x)\| \geq \frac{q(x) - q(0)}{\|x\|} \geq \frac{c_0 + c_1\|x\|^{m-1} - q(0)}{\|x\|} \geq c_1\|x\|^{m-2} + o(1)$$

as $\|x\| \to \infty$. Noting $\lambda_{\max}(\nabla^2 q(x)) \leq d_0 + d_1\|x\|^{m-2}$ for some $d_0, d_1 > 0$, $\lambda_{\max}(M(x)) \leq \frac{d_1}{c_1} + o(1)$. Then, by continuity, there must exist some L, such that q is locally L-epismooth everywhere.

7.2 Worst-Case Performance Characterizations of Epismooth Gradient Methods

The worst-case performance of a gradient method defined by a predetermined stepsize matrix H on a smooth function and on an epismooth function are closely related. First, we formulate these as performance estimation problems. In both cases, we assume the initial point x_0 satisfies $\|x_0 - x_\star\| \leq R$ for some minimizer $x_\star$ of f. Lastly, to ensure that x_0 is in the domain of f (i.e., $f(x_0) < \infty$), we further require that $R \leq \frac{1}{L}$. Then the worst-case performance of (Epi-GM) for a given matrix H on an L-epismooth function is

$$p_{\text{ES}}(L, R) = \begin{cases} \max\limits_{x_i, f} & f(x_N) - f(x_\star) \\ \text{s.t.} & \text{epi} f \in \mathcal{C}_{0,L,\infty} \\ & x_{i+1} = x_0 - \frac{1}{L}\sum_{j=0}^{i} H_{i,j}\frac{\nabla f(x_j)}{\sqrt{\|\nabla f(x_j)\|^2+1}} \qquad \forall i \in [0 : N-1] \\ & \|x_0 - x_\star\| \leq R \\ & \nabla f(x_\star) = 0, \quad f(x_\star) = 0, \quad x_\star = 0 \ . \end{cases} \tag{7.2}$$

For simplicity, we have suppressed the parameters N and H from our notation for p_{ES}. Similarly, denote the PEP for performance of (GM) given H on an L-smooth function by

$$p_{\mathrm{S}}(L,R) = \begin{cases} \max\limits_{x_i,f} & f(x_N) - f(x_\star) \\ \text{s.t.} & f \in \mathcal{F}_{0,L} \\ & x_{i+1} = x_0 - \frac{1}{L}\sum_{j=0}^{i} H_{i,j}\nabla f(x_j) \qquad \forall i \in [0:N-1] \\ & \|x_0 - x_\star\| \le R, \quad \nabla f(x_\star) = 0 \ . \end{cases}$$

Both of these settings possess useful rescaling properties, proof deferred to Appendix E.1.

Lemma 7.2. *For all $\eta > 0$ and any choice of N, L, R, the following rescaling properties hold:*

$$p_{\mathrm{ES}}(L,\eta R) = \eta p_{\mathrm{ES}}(\eta L, R) \ ,$$
$$p_{\mathrm{S}}(L,\eta R) = \eta^2 p_{\mathrm{S}}(L,R) \ ,$$
$$p_{\mathrm{S}}(\eta L, R) = \eta p_{\mathrm{S}}(L,R) \ .$$

Observe that together the rescaling properties for p_{S} imply $p_{\mathrm{S}}(L,\eta R) = \eta p_{\mathrm{S}}(\eta L, R)$, matching that of the epismooth setting. However, notably, the individual rescaling properties for p_{S} do not hold for p_{ES}. These individual rescaling properties of p_S establish that it suffices to characterize $p_{\mathrm{S}}(1,1)$ to fully understand p_{S}. Since this does not hold for p_{ES}, any future works characterizing the behavior of epismooth minimization methods must meaningfully depend on the ratio of L and R.

Under an appropriate regularity condition, we find that the convergence of any epismooth minimization method converges to its classic smooth convergence rate as the ratio between L and R grows. That is, if initialized sufficiently close to $x_\star$, perhaps from some initial "burn-in" procedure, guarantees proven for L-smooth minimization can be lifted to L-epismooth minimization.

Formally, this result relies on the following regularity condition, ensuring that if initialized sufficiently close to $x_\star$, the method's N iterates remain bounded.

Definition 7.3. A gradient method defined by H is **eventually-epismooth-stable** if for any L, there exist constants $\bar{R}, C > 0$ such that applying (Epi-GM) to any L-epismooth function with $\|x_0 - x_\star\| \le R < \bar{R}$ must have $\|x_i - x_\star\| \le CR$, $\|g_i\| \le CLR$, and $|f_i| \le CLR^2$ for all $i = 0, \dots, N$.

Given this regularity condition, the asymptotic performance of any gradient method for epismooth minimization is characterized as follows, proof deferred to Appendix E.2.

Theorem 7.4. *For any gradient method defined by H that is eventually-epismooth-stable,*

$$\lim_{\eta\to 0} \frac{p_{\mathrm{ES}}(L,\eta R)}{\eta^2} = \lim_{\eta\to 0} \frac{p_{\mathrm{S}}(L,\eta R)}{\eta^2} = p_{\mathrm{S}}(L,R) \ .$$

Informally, *eventually, epismooth functions behave like smooth functions.* That is, after running some initial burn-in method to produce x_0 with $\|x_0 - x_\star\|$ sufficiently small, optimization strategies for smooth functions such as momentum [7,37] or recent long-step gradient descent techniques [10,12] should be applicable to epismooth functions. One simple example of an eventually-epismooth-stable algorithm is setting $H = \mathrm{diag}(h, \dots, h)$ for some $h \in (0,2)$. One can directly show such an iteration guarantees descent and a nonincreasing distance to $x_\star$, as illustrated in Figure 11, ensuring the needed bounds. In the following section, we provide numerical evidence that acceleration schemes like momentum and long steps [10] can also be lifted to epismooth optimization. A direct analysis quantifying how long of a burn-in phase would be needed for these accelerated methods to apply stably is beyond our scope and left as a future direction.

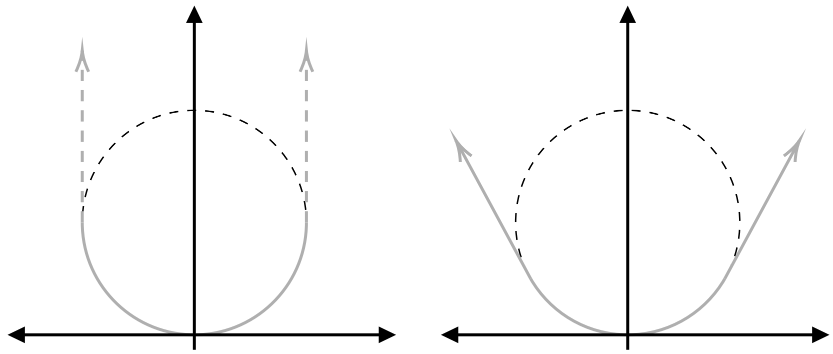

Figure 12: Left: Ball-pen function. Right: Ball-Huber function.

7.3 Numerical Validation of Theorem 7.4 for (Accelerated) Gradient Methods

We now apply our PEP formulation to the epismooth setting with the goal of observing both its short-term deviation and its long-term convergence to the behavior of smooth functions.

Without loss of generality, we assume that our minimizer $x_\star = 0$. Denote $b_i = \frac{\nabla f(x_i)}{\sqrt{\|\nabla f(x_i)\|^2+1}} \in \mathbb{R}^d$ and $t_i = \frac{-1}{\sqrt{\|\nabla f(x_i)\|^2+1}} \in \mathbb{R}$. Then the iteration (Epi-GM) becomes $x_{i+1} = x_0 - \frac{1}{L}\sum_{j=0}^{i} H_{i,j} b_k$. From our optimality conditions, we immediately get $(b_\star, t_\star) = (\vec{0}, -1)$. Moreover, observe that for all i, (b_i, t_i) is a unit normal vector to epi$f$ at $(x_i, f(x_i))$. Applying Theorem 3.3 to (7.2) gives a finite-dimensional problem with $\mathcal{I} = [0:N-1]$ and $\mathcal{K} = [0:N]$ (and $\mathcal{I}_\star = \mathcal{I} \cup \{\star\}$ and $\mathcal{K}_\star = \mathcal{K} \cup \{\star\}$) of

$$p_{\mathrm{ES}}(L,R) = \begin{cases} \max\limits_{x_k, f_k, b_k, t_k} & f_N - f_\star \\ \text{s.t.} & x_{i+1} = x_0 - \frac{1}{L}\sum_{j=0}^{i} H_{i,j} b_j \qquad \forall i \in \mathcal{I} \\ & \|x_0 - x_\star\|^2 \le R^2 \\ & (b_\star, t_\star) = (\vec{0}, -1), \qquad (x_\star, f_\star) = (\vec{0}, 0) \\ & \|b_k\|^2 + \|t_k\|^2 = 1 \qquad \forall k \in \mathcal{K} \\ & \langle b_k, x_l - \frac{1}{L}b_l - x_k + \frac{1}{L}b_k\rangle \\ & \quad + \langle t_k, f_l - \frac{1}{L}t_l - f_k + \frac{1}{L}t_k\rangle \le 0 \qquad \forall k, l \in \mathcal{K}_\star \\ & t_k \le 0 \qquad \forall k \in \mathcal{K}\ . \end{cases} \tag{7.3}$$

The second-to-last constraint above is simply $-Q^{i,j}_{0,L}(x, g, f) \le 0$, and the final constraint comes as epigraphs are unbounded along the direction $(\vec{0}, 1)$, so every normal vector $(b, t) \in \mathbb{R}^{d+1}$ must have $t \le 0$.

Following the same Grammian reformulations presented in previous sections, Appendix E.3 provides a reformulation as the following SDP with an additional rank-1 constraint of

$$(p_{\mathrm{ES}}(L,R))^2 = \begin{cases} \max\limits_{F,G,v} & \mathrm{Tr}\left(F\left(e_{N+1}e_{N+1}^T\right)\right) \\ \text{s.t.} & (F, G) \in \mathfrak{S}(L,R) \\ & F = vv^T \end{cases} \tag{7.4}$$

where e_i denotes the i-th standard unit vector in $\mathbb{R}^{N+1}$. Unfortunately, the rank-1 constraint ($F = vv^T$) makes this problem nonconvex. Instead, we can only approach it as a QCQP using black-box optimization software [17, 18]. Below we use this to validate Theorem 7.4 for gradient descent and two accelerated schemes.

Gradient Descent's Epismooth Performance. Consider the epismooth gradient descent

$$x_{i+1} = x_i - \frac{h_i \nabla f(x_i)}{L\sqrt{\|\nabla f(x_i)\|^2 + 1}} \tag{Epi-GD}$$

corresponding to $H = \mathrm{diag}(h_0, \dots, h_{N-1})$. In standard smooth gradient descent iterating $x_{i+1} = x_i - h_i/L\ \nabla f(x_i)$, as discussed in [2,38], the worst-case problem instance is often either the quadratic $f(x) = \frac{L}{2}x^2$ or the Huber function, defined by

$$f(x) = \begin{cases} \frac{L}{2}x^2 & \text{if } |x| \le \tau \\ L\tau|x| - \frac{L\tau^2}{2} & \text{if } |x| > \tau \end{cases} \tag{7.5}$$

where $\tau = \frac{1}{2N\sum_{i=0}^{N-1} h_i}$, depending on the stepsize sequence. In fact, it was conjectured by [1][4] that the optimal constant stepsize $h_i = h$ for gradient descent is given by the unique positive solution to

$$\frac{1}{2Nh+1} = (1-h)^{2N} , \tag{7.6}$$

which precisely balances the performance on the quadratic and the Huber function. We find no such simple balancing of extremes applies to give an optimal epismooth gradient descent stepsize.

Analogous to quadratic and Huber functions, we define the ball-pen function by

$$f(x) = \begin{cases} \frac{1}{L} - \sqrt{\frac{1}{L^2} - x^2} & \text{if } |x| \le \frac{1}{L} \\ \infty & \text{otherwise} \end{cases} \tag{7.7}$$

and the ball-Huber function as

$$f(x) = \begin{cases} \frac{1}{L} - \sqrt{\frac{1}{L^2} - x^2} & |x| \le \tau \\ \frac{\tau}{\sqrt{\frac{1}{L^2}-\tau^2}}|x| + \left(\frac{1}{L} - \frac{\frac{1}{L^2}}{\sqrt{\frac{1}{L^2}-\tau^2}}\right) & |x| > \tau \end{cases} \tag{7.8}$$

for some $\tau > 0$, illustrated in Figure 12. In Appendix E.4 and Appendix E.5, we calculate the worst-case performance for both of these problem instances, providing the following lower bound.

Proposition 7.5. *Consider N iterations of* (Epi-GD) *over an L-epismooth set, with stepsize sequence $h = (h_0, \dots, h_{N-1})$ and initial point x_0 with $\|x_0 - x_\star\| \le R$. For any $L, R > 0$, with $R \le \frac{1}{L}$,*

$$p_{\mathrm{ES}}(L,R) \ge \max\left\{ \frac{1}{L} - \sqrt{\frac{1}{L^2} - R^2\left(\prod_{i=0}^{N-1}(1-h_i)^2\right)}, \frac{1}{L} - \frac{\frac{1}{L^2} - \tau(R - \tau\bar{h})}{\sqrt{\frac{1}{L^2} - \tau^2}} \right\}$$

where $\bar{h} = \sum_{i=0}^{N-1} h_i$ and τ is the unique solution in $(0,R)$ to $-L^2\tau^3\bar{h} + (2\bar{h}+1)\tau - R = 0$.

Figure 13's first plot shows the worst-case behavior of one step of (Epi-GD) on epismooth functions for various stepsizes with $L = R = 1$. We see an intermediate regime for h where the worst-case problem instance is neither the ball-Huber nor the ball-pen. In fact, for these values of h, we observed that the worst-case problem instances had dimension greater than one, indicating more complex worst-case epismooth functions are necessary. This presents a notable contrast with the simpler behavior of smooth gradient descent with constant stepsizes, highlighting a discrepancy between smooth and epismooth worst cases when $\eta = 1$.

[4] Recent mild progress on this conjecture was made by [39].

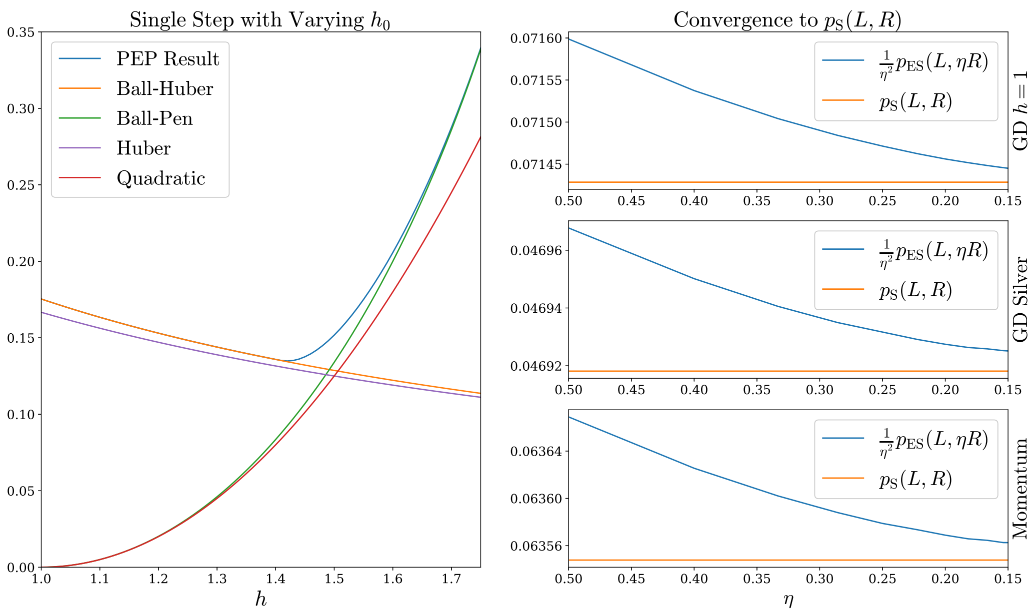

Figure 13: Left: PEP result for one step of (Epi-GD) for various constant stepsizes h and comparison with lower bounds. There is an intermediate region where the worst-case is neither ball-pen nor ball-Huber. Right: Convergence of $\frac{1}{\eta^2}p_{ES}(L,\eta R)$ to the standard smooth result $p_S(L,R)$ as η decreases, fixing $N=3$. Data is shown for three different algorithms (Epi-GM): gradient descent with $h=1$, gradient descent with silver stepsizes [10], and Nesterov fast gradient method [37].

Momentum and Long Step Methods' Epismooth Performance. In Figure 13's second set of plots, the worst-case performance of several classic smooth optimization methods is shown as η approaches 0. We consider the stepsize matrices H corresponding to classic gradient descent (with $h_i = 1$), gradient descent with Silver stepsizes, and Nesterov's fast gradient method. Note the corresponding epismooth method differs slightly from its smooth counterpart as the update (Epi-GM) is used instead of (GM). These results show the similarity of their long-term behavior, agreeing with Theorem 3.8. This provides evidence that these classic momentum methods are eventually-epismooth-stable and, hence, can provide accelerations for the optimization of a wider family of problems than they were originally designed for.

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A Deferred Interpolation Proofs

A.1 Proof of Lemma 3.6

We will make use of the following equations from Taylor series expansion:

$$\frac{1}{\|v_i\|} = \frac{1}{\sqrt{\eta^2\|g_i\|^2+1}} = 1 - \frac{1}{2}\eta^2\|g_i\|^2\xi^{-3/2} = 1 - \frac{1}{2}\eta^2\|g_i\|^2 + \frac{3}{8}\eta^4\|g_i\|^4\hat{\xi}^{-5/2}$$

where $\xi, \hat{\xi} \in [1, 1+\eta^2\|g_i\|^2]$. We use a similar expansion for $\frac{1}{\|v_j\|}$ using ζ and $\hat{\zeta}$. Letting $\rho = \frac{\mu}{1-\mu/L}$,

$$
\begin{aligned}
Q^{i,j}_{\eta^2\mu,\eta^2 L}(\frac{x}{\eta}, \eta g, f) &= \frac{\eta^2\rho}{2}\left((\frac{1}{\eta^2\mu} - \frac{1}{\eta^2 L})^2 - \|z_i - \frac{1}{\eta^2\mu}n_i - z_j + \frac{1}{\eta^2 L}n_j\|^2 \right) \\
&= \frac{\eta^2\rho}{2}\left(\frac{1}{\eta^4\rho^2} - \|\frac{x_i}{\eta} - \frac{\eta g_i}{\eta^2\mu\|v_i\|} - \frac{x_j}{\eta} + \frac{\eta g_j}{\eta^2 L\|v_j\|}\|^2 - \|f_i + \frac{1}{\eta^2\mu\|v_i\|} - f_j - \frac{1}{\eta^2 L\|v_j\|}\|^2 \right) \\
&= \frac{1}{2\eta^2\rho} - \frac{\rho}{2}\|x_i - x_j\|^2 - \frac{\rho\|g_i\|^2}{2\mu^2\|v_i\|^2} - \frac{\rho\|g_j\|^2}{2L^2\|v_j\|^2} + \frac{\rho}{L\mu\|v_i\|\|v_j\|}\langle g_i, g_j\rangle - \frac{\rho}{\mu\|v_i\|}\langle g_i, x_j - x_i\rangle \\
&\quad + \frac{\rho}{L\|v_j\|}\langle g_j, x_j - x_i\rangle - \frac{\eta^2\rho}{2}(f_i - f_j)^2 - \frac{\rho}{2\eta^2 L^2\|v_j\|^2} - \frac{\rho}{2\eta^2\mu^2\|v_i\|^2} + \frac{\rho}{\eta^2 L\mu\|v_i\|\|v_j\|} \\
&\quad + \frac{\rho}{\mu\|v_i\|}(f_j - f_i) - \frac{\rho}{L\|v_j\|}(f_j - f_i) \\
&= -\frac{\rho}{2}\|x_i - x_j\|^2 + \frac{\rho}{L\mu\|v_i\|v_j\|}\langle g_i, g_j\rangle - \frac{\rho}{\mu\|v_i\|}\langle g_i, x_j - x_i\rangle + \frac{\rho}{L\|v_j\|}\langle g_j, x_j - x_i\rangle \\
&\quad - \frac{\eta^2\rho}{2}(f_i - f_j)^2 + \frac{\rho}{\eta^2 L\mu\|v_i\|\|v_j\|} + \frac{\rho}{\mu\|v_i\|}(f_j - f_i) - \frac{\rho}{L\|v_j\|}(f_j - f_i) - \frac{\rho}{\eta^2 L\mu} \\
&= -\frac{\rho}{2}\|x_j - x_i\|^2 - \frac{\rho}{2L\mu}\|g_i - g_j\|^2 - \frac{\rho}{\mu}\langle g_i, x_j - x_i\rangle + \frac{\rho}{L}\langle g_j, x_j - x_i\rangle + f_j - f_i \\
&\quad + \eta^2\left[\frac{\rho}{L\mu}\langle g_i, g_j\rangle \left(-\frac{1}{2}\|g_i\|^2\xi^{-3/2} - \frac{1}{2}\|g_j\|^2\zeta^{-3/2} + \frac{1}{4}\eta^2\|g_i\|^2\|g_j\|^2\xi^{-3/2}\zeta^{-3/2} \right) \right. \\
&\quad + \frac{\rho}{\mu}\langle g_i, x_j - x_i\rangle \left(\frac{1}{2}\|g_i\|^2\xi^{-3/2} \right) + \frac{\rho}{L}\langle g_j, x_j - x_i\rangle \left(-\frac{1}{2}\|g_j\|^2\zeta^{-3/2} \right) \\
&\quad + \frac{\rho}{L\mu}\left(\frac{1}{4}\|g_i\|^2\|g_j\|^2 + \frac{3}{8}\|g_i\|^4\hat{\xi}^{-5/2} + \frac{3}{8}\|g_j\|^4\hat{\zeta}^{-5/2} - \frac{3}{16}\eta^2\|g_i\|^4\|g_j\|^2\hat{\xi}^{-5/2} \right. \\
&\qquad \left. - \frac{3}{16}\eta^2\|g_i\|^2\|g_j\|^4\hat{\zeta}^{-5/2} + \frac{9}{64}\eta^4\|g_i\|^4\|g_j\|^4\hat{\xi}^{-5/2}\hat{\zeta}^{-5/2} \right) \\
&\quad \left. + \frac{\rho}{\mu}(f_j - f_i)\left(-\frac{1}{2}\|g_i\|^2\xi^{-3/2} \right) + \frac{\rho}{L}(f_j - f_i)\left(\frac{1}{2}\|g_j\|^2\zeta^{-3/2} \right) - \frac{\rho}{2}(f_i - f_j)^2 \right]
\end{aligned}
$$

where for the last equality we apply our Taylor expansion and rearrange terms. We are left with $\hat{Q}^{i,j}_{\mu,L}(x, g, f)$ in the first line, along with error terms all scaled by η^2. Grouping all error terms into $c(x, g, f, \mu, L, \eta)$, $Q^{i,j}_{\eta^2\mu,\eta^2 L}(\frac{x}{\eta}, \eta g, f) = \hat{Q}^{i,j}_{\mu,L}(x, g, f) + \eta^2 c(x, g, f, \mu, L, \eta)$. Observe that while c has some dependence on η (via ξ, $\hat{\xi}$, etc.), it is bounded as $\eta \to 0$.

B Deferred Calculations on Separating Hyperplane Algorithms

B.1 A Semidefinite Programming Reformulation of Stopping Decision Problem

We follow the approach in many previous works [2,3,40] by converting our finite-dimensional problem (4.2) into an SDP. We follow similar notation to that of [6] and introduce it below:

$$
\begin{aligned}
&\Lambda = [x_0|z_0|z_1|\dots|z_N|n_0|n_1|\dots|n_N|w] \in \mathbb{R}^{d\times(2N+4)} \\
&G = \Lambda^T\Lambda \in \mathbb{S}^{2N+4}_+
\end{aligned}
$$

Using the standard unit basis vectors e_i, we define special selection vectors for extracting particular elements of our matrix G:

$$\begin{aligned}
&\mathbf{x_0} = e_1 \in \mathbb{R}^{2N+4} && \\
&\mathbf{z_i} = e_{i+2} \in \mathbb{R}^{2N+4} && i \in [0:N] \\
&\mathbf{n_i} = e_{i+(N+3)} \in \mathbb{R}^{2N+4} && i \in [0:N] \\
&\mathbf{w} = e_{2N+4} \in \mathbb{R}^{2N+4} && \\
&\mathbf{x_{i+1}} = \mathbf{x_0} - \textstyle\sum_{j=0}^{i} H_{i,j}\mathbf{n_j} \in \mathbb{R}^{2N+4} && i \in [0:N-1]\ .
\end{aligned}$$

This construction yields the useful identities $\Lambda\mathbf{x_i} = x_i$, $\Lambda\mathbf{z_i} = z_i$, $\Lambda\mathbf{n_i} = n_i$, and $\Lambda\mathbf{w} = w$ (lastly, recall our assumption that $q = 0$). Denote the symmetric outer product by $x \odot y$ where

$$x \odot y = \frac{1}{2}xy^T + \frac{1}{2}yx^T\ .$$

Then we can directly express the standard dot product of our vectors as $\mathrm{Tr}\left(G\left(\mathbf{x_i} \odot \mathbf{z_j}\right)\right) = \langle x_i, z_j\rangle$ for any $i, j \in [0:N]$ (and this holds analogously for other vector combinations). Lastly, we invoke the assumption that the problem dimension d satisfies $d \geq 2N + 4$ to guarantee that any identified G can be factorized into some $\Lambda \in \mathbb{R}^{d\times(2N+4)}$, which is common practice throughout PEP literature [1].

We can then write our feasibility condition (4.2) as

$$\exists G \in \mathbb{S}_+^{2N+4} \text{ s.t. } \begin{cases} G \succeq 0 \\ \mathrm{Tr}\left(G\left((\mathbf{z_i} - \frac{1}{\alpha}\mathbf{n_i} - \mathbf{z_j} + \frac{1}{\beta}\mathbf{n_j}) \odot (\mathbf{z_i} - \frac{1}{\alpha}\mathbf{n_i} - \mathbf{z_j} + \frac{1}{\beta}\mathbf{n_j})\right)\right) \leq \frac{1}{\gamma^2} \\ \mathrm{Tr}\left(G\left((\mathbf{z_i} - \frac{1}{\alpha}\mathbf{n_i} - \mathbf{w}) \odot (\mathbf{z_i} - \frac{1}{\alpha}\mathbf{n_i} - \mathbf{w})\right)\right) \leq (\frac{1}{\gamma} - s)^2 \\ \mathrm{Tr}\left(G\left(\mathbf{w} \odot \mathbf{w}\right)\right) \leq (\frac{1}{\beta} - \delta + s)^2 \\ \mathrm{Tr}\left(G\left(\mathbf{n_i} \odot (\mathbf{z_i} - \mathbf{x_i})\right)\right) \leq 0 \\ \mathrm{Tr}\left(G\left(\mathbf{n_i} \odot \mathbf{n_i}\right)\right) = 1 \\ \mathrm{Tr}\left(G\left(\mathbf{x_0} \odot \mathbf{x_0}\right)\right) \leq R^2 \end{cases} \tag{B.1}$$

with $\gamma = (\frac{1}{\alpha} - \frac{1}{\beta})^{-1}$ and $s = \max\{0, \delta - \frac{1}{\beta}\}$. This is now an easily solvable SDP feasibility problem.

B.2 Proof of Theorem 4.1

We first claim that there exists $\hat{q}$ such that $B(\hat{q}, h) \subseteq C$ and $\|q - \hat{q}\| \leq h - \delta$. If $\frac{1}{\beta} \leq \delta$, then $h = \delta$ and we simply let $\hat{q} = q$ and the claim is trivial. Otherwise, if $\frac{1}{\beta} > \delta$, let $z \in \mathrm{argmin}_{y\in\mathrm{bdry}\ C}\|y - q\|$. The optimality of z ensures $z - q \in N_C(z)$ and hence, by smoothness of C, $B(z - \frac{1}{\beta}\frac{z-q}{\|z-q\|}, \frac{1}{\beta}) \subseteq C$. If $\|z - q\| > \frac{1}{\beta}$, then $B(q, \frac{1}{\beta}) \subseteq C$, so we again let $\hat{q} = q$, with $h = \frac{1}{\beta}$. Lastly, suppose $\|z - q\| \leq \frac{1}{\beta}$. Then setting $\hat{q} = z - \frac{1}{\beta}\frac{z-q}{\|z-q\|}$ yields

$$\|q - \hat{q}\| = \|z - \frac{1}{\beta}\frac{z - q}{\|z - q\|} - q\| = \frac{1}{\beta} - \|z - q\| \leq \frac{1}{\beta} - \delta = h - \delta$$

where the inequality follows from $z \in \mathrm{bdry}\ C$, so $\|z - q\| \geq \delta$. Since $B(\hat{q}, \frac{1}{\beta}) \subseteq C$, this proves our first claim.

Proof of Claimed Stopping Time. Suppose that by iteration k, the method has not yet terminated. Since $x_k \notin \mathrm{int}\ C$, $x_k \notin \mathrm{int}\ B(\hat{q}, h)$. By definition of n_k as a separating hyperplane, we know $\langle n_k, y - x_k\rangle \leq 0$ for all $y \in B(\hat{q}, h)$. Considering $y = \hat{q} + hn_k$, this yields $\langle n_k, x_k - \hat{q}\rangle \geq h$. Using this fact, every iteration k satisfies

$$\|x_{k+1} - \hat{q}\|^2 = \|x_k - \hat{q}\|^2 + h^2 - 2h\langle n_k, x_k - \hat{q}\rangle \leq \|x_k - \hat{q}\|^2 - h^2 \ .$$

Inductively applying this ensures that if all $x_0, \dots, x_N \notin \text{int } C$, then

$$\|x_N - \hat{q}\|^2 \leq \|x_0 - \hat{q}\|^2 - Nh^2 \leq (R + h - \delta)^2 - Nh^2 \tag{B.2}$$

where we use the fact that if $\delta > \frac{1}{\beta}$, then $R + h - \delta = R \geq \|x_0 - q\| = \|x_0 - \hat{q}\|$, and if $\delta \leq \frac{1}{\beta}$, then $\|x_0 - \hat{q}\| \leq \|x_0 - q\| + \|q - \hat{q}\| \leq R + (\frac{1}{\beta} - \delta) = R + h - \delta$. Noting $x_N \notin \text{int } C$ implies that $x_N \notin \text{int } B(\hat{q}, h)$, i.e. $\|x_N - \hat{q}\| \geq h$, it follows from $(R + h - \delta)^2 - Nh^2 \geq \|x_N - \hat{q}\|^2 \geq h^2$ that $N \leq \frac{(R+h-\delta)^2}{h^2} - 1$. Hence by after $N = \lfloor \frac{(R+h-\delta)^2}{h^2} \rfloor$ iterations, some iterate x_k must have lied in the interior of C, halting the algorithm.

Proof of Matching Lower Bound. Finally, we establish an exactly matching lower bound on any separating hyperplane method of the form (4.1). For simplicity, we will assume that $N = \frac{(R+h-\delta)^2}{h^2}$ is an integer, however, one can adjust our construction to handle non-integer cases. Below we construct a hard problem instance such that for any separating hyperplane method, $x_k \notin \text{int } C$ for all $k \leq N - 1$ and moreover, ensuring x_k will only have nonzero entries in its first k entries.

Let $x_0 = 0$, $q = (\delta, \dots, \delta) \in \mathbb{R}^N$, and $\hat{q} = (h, \dots, h)$. Further define $C = B(\hat{q}, h) \subseteq \mathbb{R}^N$ and observe that $q \in \text{int}_\delta C$. Next, we calculate $\|x_0 - q\| = \delta\sqrt{N} = \frac{\delta(R+h-\delta)}{h} \leq R$ so our initial condition is satisfied. For each k, we can select $n_k = -e_{k+1}$, the negative unit basis vector in $\mathbb{R}^N$. Note this is a valid separating hyperplane due to x_k's support being its first k entries as $\langle n_k, x_k\rangle = 0 \geq \langle n_k, y\rangle$ for all $y \in C$. Moreover, this choice of n_k ensures the support of x_{k+1} will only increase by one. Noting all $k \leq N - 1$ have final entry equal to zero, $\|x_k - \hat{q}\| \geq h$ and so $x_k \notin \text{int } C$. Therefore no separating hyperplane method method can identify an interior point faster than the above simple constant stepsize method.

C Deferred Calculations on Frank-Wolfe Methods

C.1 Derivation of SDP for Frank-Wolfe

As shown in Section 4, and following a similar procedure to that of Appendix B.1, we apply the standard assumption that $d \geq 4N + 6$, and define our Grammian matrix variables G and F as

$$\begin{aligned}
&F = [f_0|f_1|\dots|f_N|f_{\min}] \in \mathbb{R}^{1\times(N+2)} \\
&\Lambda = [x_0|g_\star|g_0|g_1|\dots|g_N|z_0|\dots|z_{N-1}|n_\star|n_0|\dots|n_{N-1}|w_\star|w_0|\dots|w_N] \in \mathbb{R}^{d\times(4N+6)} \\
&G = \Lambda^T\Lambda \in \mathbb{S}_+^{4N+6} \ .
\end{aligned}$$

We define new selection vectors based on our new matrices F and Λ:

$$
\begin{aligned}
&\mathbf{x}_\star = 0 \in \mathbb{R}^{4N+6} && \\
&\mathbf{x_0} = e_1 \in \mathbb{R}^{4N+6} && \\
&\mathbf{g}_\star = e_2 \in \mathbb{R}^{4N+6} && \\
&\mathbf{g_k} = e_{k+3} \in \mathbb{R}^{4N+6} && k \in \mathcal{K} \\
&\mathbf{z_i} = e_{i+(N+4)} \in \mathbb{R}^{4N+6} && i \in \mathcal{I} \\
&\mathbf{n}_\star = e_{2N+4} \in \mathbb{R}^{4N+6} && \\
&\mathbf{n_i} = e_{i+(2N+5)} \in \mathbb{R}^{4N+6} && i \in \mathcal{I} \\
&\mathbf{w}_\star = e_{3N+5} \in \mathbb{R}^{4N+6} && \\
&\mathbf{w_k} = e_{k+(3N+6)} \in \mathbb{R}^{4N+6} && k \in \mathcal{K} \\
&\mathbf{x_{i+1}} = (1-h_i)\mathbf{x_i} + h_i \mathbf{z_i} && i \in \mathcal{I} \\
&\mathbf{f}_\star = 0 \in \mathbb{R}^{N+2} && \\
&\mathbf{f_k} = e_{k+1} \in \mathbb{R}^{N+2} && k \in \mathcal{K} \\
&\mathbf{f}_{\min} = e_{N+2} \in \mathbb{R}^{N+2}\ .
\end{aligned}
$$

Once again, we have the useful identities $\Lambda\mathbf{x_i} = x_i$, $\Lambda\mathbf{g_i} = g_i$, $\Lambda\mathbf{z_i} = z_i$, $\Lambda\mathbf{n_i} = n_i$, and $\Lambda\mathbf{w_i} = w_i$. In addition, following the notation of [6], we define

$$
\begin{aligned}
A_{i,j} &= \mathbf{g_j} \odot (\mathbf{x_i} - \mathbf{x_j}) \in \mathbb{S}^{4N+6} \\
B_{i,j} &= (\mathbf{x_i} - \mathbf{x_j}) \odot (\mathbf{x_i} - \mathbf{x_j}) \in \mathbb{S}^{4N+6} \\
C_{i,j} &= (\mathbf{g_i} - \mathbf{g_j}) \odot (\mathbf{g_i} - \mathbf{g_j}) \in \mathbb{S}^{4N+6} \\
a_{i,j} &= \mathbf{f_j} - \mathbf{f_i} \in \mathbb{R}^{N+2}\ .
\end{aligned}
$$

To handle the nonconvex constraint $\langle g_i, n_i \rangle^2 = \|g_i\|^2$, recall our relaxation included the linear constraints $\langle -g_i, n_j \rangle \leq \langle -g_i, n_i \rangle \quad \forall i, j \in \mathcal{I}$. We substitute this into (5.7), then using our new variables and special matrices we define our relaxation, denoted $p_{\text{FW,relaxed}}(N;\theta)$, as

$$
\left\{
\begin{array}{lll}
\max\limits_{F,G} & F a_{\star,\min} & \\
\text{s.t.} & F a_{k,\min} \leq 0 & \forall k \in \mathcal{K} \\
& F a_{l,k} + \operatorname{Tr} G A_{l,k} + \frac{1}{2L} \operatorname{Tr} G C_{l,k} \leq 0 & \forall k, l \in \mathcal{K}_\star \\
& G \succeq 0 & \\
& \operatorname{Tr}\left(G\left(-\mathbf{g_i} \odot (\mathbf{z_j} - \frac{1}{\beta}\mathbf{n_j} - \mathbf{z_i} + \frac{1}{\beta}\mathbf{n_i})\right)\right) \leq 0 & \forall i, j \in \mathcal{I}_\star \\
& \operatorname{Tr}\left(G\left(-\mathbf{g_i} \odot (\mathbf{w_k} - \mathbf{z_i} + \frac{1}{\beta}\mathbf{n_i})\right)\right) \leq 0 & \forall i \in \mathcal{I}_\star, k \in \mathcal{K}_\star \\
& \operatorname{Tr}\left(G\left((\mathbf{x_k} - \mathbf{w_k}) \odot (\mathbf{x_k} - \mathbf{w_k})\right)\right) \leq \frac{1}{\beta^2} & \forall k \in \mathcal{K}_\star \\
& \operatorname{Tr}\left(G\left((\mathbf{z_i} - \frac{1}{\beta}\mathbf{n_i} - \mathbf{w_k}) \odot (\mathbf{z_i} - \frac{1}{\beta}\mathbf{n_i} - \mathbf{w_k})\right)\right) \leq (D - \frac{2}{\beta})^2 & \forall i \in \mathcal{I}_\star, k \in \mathcal{K}_\star \\
& \operatorname{Tr}\left(G\left((\mathbf{w_k} - \mathbf{w_l}) \odot (\mathbf{w_k} - \mathbf{w_l})\right)\right) \leq (D - \frac{2}{\beta})^2 & \forall k, l \in \mathcal{K}_\star \\
& \operatorname{Tr}\left(G\left((\mathbf{z_i} - \frac{1}{\beta}\mathbf{n_i} - \mathbf{z_j} + \frac{1}{\beta}\mathbf{n_j}) \odot \right.\right. & \\
& \qquad \left.\left.(\mathbf{z_i} - \frac{1}{\beta}\mathbf{n_i} - \mathbf{z_j} + \frac{1}{\beta}\mathbf{n_j})\right)\right) \leq (D - \frac{2}{\beta})^2 & \forall i, j \in \mathcal{I}_\star \\
& \operatorname{Tr}\left(G\left(\mathbf{n_i} \odot \mathbf{n_i}\right)\right) = 1 & \forall i \in \mathcal{I}_\star \\
& \operatorname{Tr}\left(G\left(\mathbf{g_i} \odot \mathbf{n_i}\right)\right) \leq 0 & \forall i \in \mathcal{I}_\star \\
& \operatorname{Tr}\left(G\left(-\mathbf{g_i} \odot (\mathbf{n_j} - \mathbf{n_i})\right)\right) \leq 0 & \forall i, j \in \mathcal{I}_\star
\end{array}
\right. \tag{C.1}
$$

C.2 Global Optimization Algorithm for Frank-Wolfe PEP

We now combine our upper and lower bounding subproblems, (5.13) and (5.8), respectively, into Algorithm 1 to solve (5.7) to a given precision Δ. To simplify notation, we denote

$$P(\mathcal{T};\theta) = \max_{\mathcal{S}\in\mathcal{T}} \tilde{p}_{\text{FW}}(\mathcal{S};\theta)$$

and let $P_G(\mathcal{T};\theta)$ and $P_F(\mathcal{T};\theta)$ denote the optimal solutions G and F that attain $P(\mathcal{T};\theta)$.

We assume that some reasonable heuristic is used for `choose_split_index` when determining j, the index of the partition to be refined. In practice, we chose j to be the index with the maximum distance from the target curve. That is, given the current solution G, set

$$j = \operatorname*{argmax}_i \sqrt{\operatorname{Tr}\left(G(\mathbf{g_i}\odot\mathbf{g_i})\right)} - \operatorname{Tr}\left(G(-\mathbf{g_i}\odot\mathbf{n_i})\right) \ .$$

Then the partition T_j is split. This process, formalized in Algorithm 1, terminates whenever the upper bound from the resulting $\mathcal{T}_k$ is within Δ of the given lower bound c_{lower}. Note this procedure is not guaranteed to terminate (c_{lower} from the above local solve may be too slack). In practice up to $N \leq 5$, our local solves were always sufficiently accurate to terminate with $\Delta \leq 10^{-4}$. Note this procedure's runtime can grow exponentially, limiting further verification of local solve accuracies.

D Deferred Calculations on Alternating Projections

D.1 Derivation of SDP for Alternating Projections

We assume without loss of generality that $y = 0$. Then applying our interpolation results in Theorem 3.3, we can reformulate (6.5), denoted $p_{\text{AP}}(N;\delta,R,\beta)$, as

$$\left\{\begin{array}{lllll}
\max\limits_{\substack{x_0,u_i,m_i,v_k\\ n_k,q,w_1,w_2}} & \|x_N\|^2 & & \\
\text{s.t.} & z_i = x_0 - \sum_{j=0}^{i} u_j - \sum_{j=1}^{i} v_j & & \forall i\in\mathcal{I} \\
& x_k = x_0 - \sum_{j=0}^{k-1} u_j - \sum_{j=1}^{k} v_j & & \forall k\in\mathcal{K} \\
& \langle u_i, z_j - \frac{1}{\beta}m_j - z_i + \frac{1}{\beta}m_i\rangle \leq 0 & & \forall i,j\in\mathcal{I}_\star \\
& \langle v_k, x_l - \frac{1}{\beta}n_l - x_k + \frac{1}{\beta}n_k\rangle \leq 0 & & \forall k,l\in\mathcal{K}_\star \\
& \langle u_i, w_1 + s m_i - z_i + \frac{1}{\beta}m_i\rangle \leq 0 & & \forall i\in\mathcal{I}_\star \\
& \langle v_k, w_2 + s n_k - x_k + \frac{1}{\beta}n_k\rangle \leq 0 & & \forall k\in\mathcal{K}_\star \\
& \|q - w_1\|^2 \leq (\frac{1}{\beta} - \delta + s)^2 & & \\
& \|q - w_2\|^2 \leq (\frac{1}{\beta} - \delta + s)^2 & & \\
& x_N = u_\star + v_\star & & \\
& \|x_0 - q\|^2 \leq R^2 & & \\
& x_\star = z_\star = 0 & & \\
& \|m_i\|^2 = 1, & \|n_k\|^2 = 1 & \forall i\in\mathcal{I}_\star, k\in\mathcal{K}_\star \\
& \langle u_i, m_i\rangle \geq 0, & \langle v_k, n_k\rangle \geq 0 & \forall i\in\mathcal{I}_\star, k\in\mathcal{K}_\star \\
& \langle u_i, m_i\rangle^2 = \|u_i\|^2, & \langle v_k, n_k\rangle^2 = \|v_k\|^2 & \forall i\in\mathcal{I}_\star, k\in\mathcal{K}_\star
\end{array}\right. \tag{D.1}$$

where $s = \max\{0, \frac{1}{\beta} - \delta\}$.

Following our the same approach outlined in Appendix B.1 and Appendix C.1, we define

$$\Lambda = [x_0|u_\star|u_0|\dots|u_{N-1}|v_\star|v_1|\dots|v_N|m_\star|m_0|\dots|m_{N-1}|n_\star|n_1|\dots|n_N|q|w_1|w_2] \in \mathbb{R}^{d\times(4N+8)}$$

Algorithm 1 Branch and Cut

Input: M, N, Δ, c_{lower}, $\theta = (\mu, L, \alpha, \beta, D)$
Initialize $T_j = \begin{pmatrix} 0 & M & \infty \end{pmatrix}$ for all $j = \star, 0, \ldots, N-1$
Set $\mathcal{T}_0 = \texttt{get\_all\_slices}(T)$ where $T = (T_\star, \ldots, T_{N-1})$
for $k = 1, 2, \ldots$ **do**
    $T_j = \texttt{split}(T_j)$ for $j = \texttt{choose\_split\_index}(P_G(\mathcal{T}_k; \theta))$
    $\mathcal{T}_k = \texttt{get\_all\_slices}(T)$ where $T = (T_\star, \ldots, T_{N-1})$
    if $P(\mathcal{T}_k; \theta) - c_{\text{lower}} \leq \Delta$ **return:** $P(\mathcal{T}_k; \theta)$ **end if**
end for

and $G = \Lambda^T \Lambda \in \mathbb{S}_+^{4N+8}$. We define our selection vectors $\mathbf{x_i}$, $\mathbf{u_i}$, $\mathbf{m_i}$, $\mathbf{v_k}$, $\mathbf{z_k}$, $\mathbf{n_k}$, $\mathbf{q}$, and $\mathbf{w_j}$ analogously to Appendix C.1. Once again, we are faced with the issue that the constraints $\langle u_i, m_i \rangle^2 = \|u_i\|^2$ and $\langle v_k, n_k \rangle^2 = \|v_k\|^2$ cannot be expressed in a convex SDP. We take the same approach as in Appendix C.1 defining our various subproblems and relaxations, similar to (5.13), (5.11), and (5.8). For reference, we include the relaxation problem $p_{\text{AP,relaxed}}(N; \delta, \beta)$ below. The analogues to (5.8) and (5.11) are excluded for brevity but are easily constructed from (D.2). We can then apply our Algorithm 1 to globally estimate alternating projection's performance.

$$\begin{cases} \max\limits_{G} & \operatorname{Tr}\left(G\left(\mathbf{x_N} \odot \mathbf{x_N}\right)\right) & \\ \text{s.t.} & G \succeq 0 & \\ & \operatorname{Tr}\left(G\left(\mathbf{u_i} \odot (\mathbf{z_j} - \frac{1}{\beta}\mathbf{m_j} - \mathbf{z_i} + \frac{1}{\beta}\mathbf{m_i})\right)\right) \leq 0 & \forall i, j \in \mathcal{I}_\star \\ & \operatorname{Tr}\left(G\left(\mathbf{v_k} \odot (\mathbf{x_l} - \frac{1}{\beta}\mathbf{n_l} - \mathbf{x_k} + \frac{1}{\beta}\mathbf{n_k})\right)\right) \leq 0 & \forall k, l \in \mathcal{K}_\star \\ & \operatorname{Tr}\left(G\left(\mathbf{u_i} \odot (\mathbf{w_1} + s\mathbf{m_i} - \mathbf{z_i} + \frac{1}{\beta}\mathbf{m_i})\right)\right) \leq 0 & \forall i \in \mathcal{I}_\star \\ & \operatorname{Tr}\left(G\left(\mathbf{v_k} \odot (\mathbf{w_2} + s\mathbf{n_k} - \mathbf{x_k} + \frac{1}{\beta}\mathbf{n_k})\right)\right) \leq 0 & \forall k \in \mathcal{K}_\star \\ & \operatorname{Tr}\left(G\left((\mathbf{q} - \mathbf{w_1}) \odot (\mathbf{q} - \mathbf{w_1})\right)\right) \leq (\frac{1}{\beta} - \delta + s)^2 & \\ & \operatorname{Tr}\left(G\left((\mathbf{q} - \mathbf{w_2}) \odot (\mathbf{q} - \mathbf{w_2})\right)\right) \leq (\frac{1}{\beta} - \delta + s)^2 & \\ & \operatorname{Tr}\left(G\left((\mathbf{x_N} - \mathbf{u_\star} - \mathbf{v_\star}) \odot (\mathbf{x_N} - \mathbf{u_\star} - \mathbf{v_\star})\right)\right) = 0 & \\ & \operatorname{Tr}\left(G\left((\mathbf{x_0} - \mathbf{q}) \odot (\mathbf{x_0} - \mathbf{q})\right)\right) \leq R^2 & \\ & \operatorname{Tr}\left(G\left(\mathbf{m_i} \odot \mathbf{m_i}\right)\right) = 1 & \forall i \in \mathcal{I}_\star \\ & \operatorname{Tr}\left(G\left(\mathbf{n_k} \odot \mathbf{n_k}\right)\right) = 1 & \forall k \in \mathcal{K}_\star \\ & \operatorname{Tr}\left(G\left(-\mathbf{u_i} \odot \mathbf{m_i}\right)\right) \leq 0 & \forall i \in \mathcal{I}_\star \\ & \operatorname{Tr}\left(G\left(-\mathbf{v_k} \odot \mathbf{n_k}\right)\right) \leq 0 & \forall k \in \mathcal{K}_\star \\ & \operatorname{Tr}\left(G\left(\mathbf{u_i} \odot (\mathbf{m_j} - \mathbf{m_i})\right)\right) \leq 0 & \forall i, j \in \mathcal{I}_\star \\ & \operatorname{Tr}\left(G\left(\mathbf{v_k} \odot (\mathbf{n_l} - \mathbf{n_k})\right)\right) \leq 0 & \forall k, l \in \mathcal{K}_\star \end{cases} \tag{D.2}$$

D.2 Calculation of Worst-Case Performance of Intersecting Halfspaces

Let $x_0 = z_0 = (r\cos^2\phi, r\cos\phi\sin\phi)$. After one iteration, one has $x_1 = (r\cos^2\phi, 0)$. Ultimately, alternating projections produces $x_N = (r\cos^{2N}\phi, 0)$. Hence $\|x_N - y\| = r\cos^{2N}\phi$. We choose q such that $B(q, \delta)$ is inscribed by the two boundary lines of our halfspaces as $q = (\frac{-\delta(\cos\phi+1)}{\sin\phi}, -\delta)$. Calculating an expression for $R^2 = \|x_0 - q\|^2$, in terms of r and ϕ yields

$$R^2 = \|x_0 - q\|^2 = \left(r\cos^2\phi + \frac{\delta(\cos\phi + 1)}{\sin\phi}\right)^2 + (r\cos\phi\sin\phi + \delta)^2 \ .$$

Equivalently, $r = \frac{\sqrt{R^2-\delta^2}-\delta(\cot\phi+\cos\phi\cot\phi+\sin\phi)}{\cos\phi}$. Letting $c = \cos\phi$, $r = \frac{\sqrt{R^2-\delta^2}}{c} - \frac{\delta}{\sqrt{1-c^2}} - \frac{\delta}{c\sqrt{1-c^2}}$. Then using the fact that $\|x_N - y\| = r\cos^{2N}\phi$ gives the claimed $\|x_N - y\| = c^{2N-1}\left(\sqrt{R^2-\delta^2} - \frac{\delta(c+1)}{\sqrt{1-c^2}}\right)$.

E Deferred Calculations on Epismooth Gradient Descent

E.1 Proof of Lemma 7.2

Denote $b_i = \frac{\nabla f(x_i)}{\sqrt{\|\nabla f(x_i)\|^2+1}} \in \mathbb{R}^d$ and $t_i = \frac{-1}{\sqrt{\|\nabla f(x_i)\|^2+1}} \in \mathbb{R}$. Then this follows from a simple change of variables, $\tilde{x}_i = \frac{x_i}{\eta}$ and $\tilde{f}_i = \frac{f_i}{\eta}$, as

$$
\begin{aligned}
p_{\mathrm{ES}}(L, \eta R) &= \begin{cases} \max\limits_{x_i, b_i, f_i} & f_N \\ \text{s.t.} & x_{i+1} = x_0 - \frac{1}{L}\sum_{j=0}^{i} H_{i,j} b_j & \forall i \in \mathcal{I} \\ & \|x_0 - x_\star\|^2 \le \eta^2 R^2 \\ & b_\star = 0, \quad f_\star = 0, \quad x_\star = 0 \\ & t_k = -\sqrt{1 - \|b_k\|^2} & \forall k \in \mathcal{K} \\ & \langle b_k, x_l - \frac{1}{L} b_l - x_k + \frac{1}{L} b_k \rangle \\ & \quad + \langle t_k, f_l - \frac{1}{L} t_l - f_k + \frac{1}{L} t_k \rangle \le 0 & \forall k, l \in \mathcal{K}_\star \end{cases} \\
&= \begin{cases} \max\limits_{\tilde{x}_i, b_i, \tilde{f}_i} & \eta \tilde{f}_N \\ \text{s.t.} & \tilde{x}_{i+1} = \tilde{x}_0 - \frac{1}{\eta L}\sum_{j=0}^{i} H_{i,j} b_j & \forall i \in \mathcal{I} \\ & \|\tilde{x}_0 - \tilde{x}_\star\|^2 \le R^2 \\ & g_\star = 0, \quad \tilde{f}_\star = 0, \quad \tilde{x}_\star = 0 \\ & t_k = -\sqrt{1 - \|b_k\|^2} & \forall k \in \mathcal{K} \\ & \langle b_k, \tilde{x}_l - \frac{1}{\eta L} b_l - \tilde{x}_k + \frac{1}{\eta L} b_k \rangle \\ & \quad + \langle t_k, \tilde{f}_l - \frac{1}{\eta L} t_l - \tilde{f}_k + \frac{1}{\eta L} t_k \rangle \le 0 & \forall k, l \in \mathcal{K}_\star \end{cases} \\
&= \eta\, p_{\mathrm{ES}}(\eta L, R) \ .
\end{aligned}
$$

Similarly, for p_{S}, the first result follows using the change of variables $\tilde{x}_i = \frac{x_i}{\eta}$, $\tilde{g}_i = \frac{g_i}{\eta}$ and $\tilde{f}_i = \frac{f_i}{\eta^2}$, and the second from $\tilde{g}_i = \frac{g_i}{\eta}$ and $\tilde{f}_i = \frac{f_i}{\eta}$.

E.2 Proof of Theorem 7.4

From Lemma 7.2, we know that $\frac{1}{\eta^2} p_{\mathrm{S}}(L, \eta R) = p_{\mathrm{S}}(L, R)$ for all η. So the second equality in our statement is immediate. We focus on showing that $\lim_{\eta\to 0} \frac{p_{\mathrm{ES}}(L,\eta R)}{\eta^2} = p_{\mathrm{S}}(L, R)$.

Let $\rho_\eta := (x^{(\eta)}, g^{(\eta)}, f^{(\eta)})$ denote the sequence of minimizers of $p_{\mathrm{ES}}(\eta^2 L, \frac{R}{\eta})$ as $\eta \to 0$. We consider the rescaled sequence $s_\eta = (\eta x^{(\eta)}, \frac{g^{(\eta)}}{\eta}, f^{(\eta)})$. By our method being eventually-epismooth-stable and the fact that ρ_η is a solution to $p_{\mathrm{ES}}(\eta^2 L, \frac{R}{\eta})$, we must have $\|x_i^{(\eta)}\| \le C\frac{R}{\eta}$, $\|g_i^{(\eta)}\| \le C\eta^2 L \frac{R}{\eta}$, and $|f_i^{(\eta)}| \le C\eta^2 L \frac{R^2}{\eta^2}$ for all i. Consequently, we have $\|\eta x_i^{(\eta)}\| \le CR$, $\|\frac{g_i^{(\eta)}}{\eta}\| \le CLR$, and $|f_i^{(\eta)}| \le CLR^2$ for all i. This shows that s_η belongs to a compact set for all η. We consider $\limsup_{\eta\to 0} p_{\mathrm{ES}}(\eta L, \frac{R}{\eta})$ and define η_k as the subsequence attaining the lim sup. By compactness, consider a further subsequence $\eta_{k'}$ such that $s_{\eta_{k'}}$ converges to some limit point s^*.

Next, we claim that s^* is a feasible point for $p_{\text{S}}(L,R)$. We know that each $\rho_\eta = (x^{(\eta)}, g^{(\eta)}, f^{(\eta)})$ satisfies the constraints

$$\begin{cases} x_{i+1}^{(\eta)} = x_0^{(\eta)} - \frac{1}{\eta^2 L}\sum_{j=0}^{i} H_{i,j}\frac{g_j^{(\eta)}}{\sqrt{\|g_j^{(\eta)}\|^2+1}} \\ \|x_0^{(\eta)} - x_\star^{(\eta)}\| \le \frac{R^2}{\eta^2} \\ Q_{0,\eta^2 L}^{i,j}(x^{(\eta)}, g^{(\eta)}, f^{(\eta)}) \ge 0 \ . \end{cases}$$

Then rescaling with $\tilde{x}^{(\eta)} = \eta x^{(\eta)}$, $\tilde{g}^{(\eta)} = \frac{g^{(\eta)}}{\eta}$, and $\tilde{f}^{(\eta)} = f^{(\eta)}$ yields

$$\begin{cases} \tilde{x}_{i+1}^{(\eta)} = \tilde{x}_0^{(\eta)} - \frac{1}{L}\sum_{j=0}^{i} H_{i,j}\frac{\tilde{g}_j^{(\eta)}}{\sqrt{\eta^2\|\tilde{g}_j^{(\eta)}\|^2+1}} \\ \|\tilde{x}_0^{(\eta)} - \tilde{x}_\star^{(\eta)}\| \le R^2 \\ Q_{0,\eta^2 L}^{i,j}(\frac{\tilde{x}^{(\eta)}}{\eta}, \eta\tilde{g}^{(\eta)}, \tilde{f}^{(\eta)}) \ge 0 \ . \end{cases}$$

Applying Theorem 3.7, the continuity of our constraints, $s^* = (\tilde{x}^*, \tilde{g}^*, \tilde{f}^*)$ must satisfy

$$\begin{cases} \tilde{x}_{i+1}^* = \tilde{x}_0^* - \frac{1}{L}\sum_{j=0}^{i} H_{i,j}\tilde{g}_j^* \\ \|\tilde{x}_0^* - \tilde{x}_\star^*\| \le R^2 \\ \hat{Q}_{0,L}^{i,j}(\tilde{x}^*, \tilde{g}^*, \tilde{f}^*) \ge 0 \ . \end{cases}$$

Therefore s^* is a feasible solution to $p_{\text{S}}(L,R)$ and consequently $p_{\text{S}}(L,R) \ge \tilde{f}_N^* - \tilde{f}_\star^*$.

We know from the proof of Theorem 3.7 that for all η, $p_{\text{ES}}(\eta^2 L, \frac{R}{\eta}) \ge p_{\text{S}}(\eta^2 L, \frac{R}{\eta})$. Combining this with our rescaling result in Lemma 7.2, we have

$$\liminf_{\eta\to 0} p_{\text{ES}}(\eta^2 L, \frac{R}{\eta}) \ge \liminf_{\eta\to 0} p_{\text{S}}(\eta^2 L, \frac{R}{\eta}) = p_{\text{S}}(L,R) \ .$$

Combining our results above, we can squeeze the limit as follows

$$\liminf_{\eta\to 0} p_{\text{ES}}(\eta^2 L, \frac{R}{\eta}) \ge p_{\text{S}}(L,R) \ge \tilde{f}_N^* - \tilde{f}_\star^* = \lim_{k\to\infty} f_N^{(\eta_k')} - f_\star^{(\eta_k')} = \limsup_{\eta\to 0} p_{\text{ES}}(\eta^2 L, \frac{R}{\eta}) \ .$$

Therefore, $\lim_{\eta\to 0} p_{\text{ES}}(\eta^2 L, \frac{R}{\eta}) = p_{\text{S}}(L,R)$. Applying Lemma 7.2 twice gives the final claim.

E.3 Derivation of SDP for Epismooth Gradient Methods

We define

$$\Lambda = [x_0|b_0|\dots|b_N] \in \mathbb{R}^{d\times(N+2)}$$
$$v = (f_0,\dots,f_N,t_\star,t_0,\dots,t_N) \in \mathbb{R}^{1\times(2N+3)}$$

with $G = \Lambda^T\Lambda \in \mathbb{S}_+^{N+2}$ and $F = v^T v \in \mathbb{S}_+^{2N+3}$. We define $\mathbf{x_i}$ and $\mathbf{b_i}$ as selection vectors relative to G, similar to Appendix B, and we define $\mathbf{f_i}$ and $\mathbf{t_i}$ as selection vectors relative to F. We can then encode our finite optimization problem (7.3) as an SDP with an additional rank-1 constraint:

$$(p_{\mathrm{ES}}(L,R))^2 = \begin{cases} \max\limits_{F,G,v} & \mathrm{Tr}\left(F\left(\mathbf{f_N}\odot\mathbf{f_N}\right)\right) \\ \text{s.t.} & G\succeq 0 \\ & F\succeq 0 \\ & \mathrm{Tr}\left(G\left((\mathbf{x_0}-\mathbf{x_\star})\odot(\mathbf{x_0}-\mathbf{x_\star})\right)\right)\le R^2 \\ & \mathrm{Tr}\left(G\left(\mathbf{b_k}\odot\mathbf{b_k}\right)+F\left(\mathbf{t_k}\odot\mathbf{t_k}\right)\right)=1 & \forall k\in\mathcal{K}_\star \\ & \mathrm{Tr}\Big(G\left(\mathbf{b_k}\odot(\mathbf{x_l}-\tfrac{1}{L}\mathbf{b_l}-\mathbf{x_k}+\tfrac{1}{L}\mathbf{b_k})\right) \\ & \quad +F\left(\mathbf{t_k}\odot(\mathbf{f_l}-\tfrac{1}{L}\mathbf{t_l}-\mathbf{f_k}+\tfrac{1}{L}\mathbf{t_k})\right)\Big)\le 0 & \forall k,l\in\mathcal{K}_\star \\ & \mathrm{Tr}\left(F(\mathbf{t_\star}\odot\mathbf{t_\star})\right)=1 \\ & \mathrm{Tr}\left(F(\mathbf{t_\star}\odot\mathbf{f_0})\right)\le 0 \\ & \mathrm{Tr}\left(F(\mathbf{t_k}\odot\mathbf{f_0})\right)\le 0 & \forall k\in\mathcal{K} \\ & F=v^Tv\ . \end{cases} \tag{E.1}$$

In order to enforce $t_\star=-1$ in (E.1), we use the two constraints $t_\star^2=1$ and $t_\star f_0\le 0$. The first ensures that $t_\star=\pm1$. By convexity of f, we know that $f_0\ge f_\star=0$. If $f_0=0$, then $x_0=0$, and consequently $f(x_N)=0$, so this is irrelevant for the worst-case instance. So we know effectively that $f_0>0$ and conclude that $t_\star=-1$. Similarly, to enforce that $t_k\le 0$, we require $t_kf_0\le 0$.

E.4 Worst-Case Performance of Ball-Pen Function

We define f as in (7.7). This yields $\nabla f(x)=\frac{x}{\sqrt{\frac{1}{L^2}-x^2}}$ for $|x|<\frac{1}{L}$. Then for any x_k, we can calculate

$$b_k=\frac{\nabla f(x_k)}{\sqrt{\|\nabla f(x_k)\|^2+1}}=\frac{x_k}{\sqrt{\frac{1}{L^2}-x_k^2}}\,\frac{\sqrt{\frac{1}{L^2}-x_k^2}}{\frac{1}{L}}=Lx_k\ .$$

Iterating $x_{k+1}=x_k-\frac{h_k}{L}b_k$ we get $x_{k+1}=(1-h_k)x_k$. Hence with, $x_0=R$, $x_k=R\left(\prod_{i=0}^{k-1}(1-h_i)\right)$, giving a final objective gap as claimed of $f(x_N)-f(x_\star)=f(x_N)=\frac{1}{L}-\sqrt{\frac{1}{L^2}-R^2\left(\prod_{i=0}^{k-1}(1-h_i)^2\right)}$.

E.5 Worst-Case Performance of Ball-Huber Function

We consider the Ball-Huber from (7.8) for undetermined τ and we assume that $x_0>\tau$. For $|x|>\tau$, we have $\nabla f(x)=\frac{\tau}{\sqrt{\frac{1}{L^2}-\tau^2}}$, so for any x_k with $|x_k|>\tau$ we can again calculate

$$b_k=\frac{\nabla f(x_k)}{\sqrt{\|\nabla f(x_k)\|^2+1}}=\frac{\frac{\tau}{\sqrt{\frac{1}{L^2}-\tau^2}}}{\sqrt{\frac{\tau^2}{\frac{1}{L^2}-\tau^2}+1}}=L\tau\ .$$

Our iteration becomes $x_k=x_0-\tau\sum_{i=0}^{k-1}h_i=R-\tau\sum_{i=0}^{k-1}h_i$. Defining $\bar h=\sum_{i=0}^{k-1}h_i$, we get

$$f(x_N)=\frac{1}{L}-\frac{\frac{1}{L^2}-\tau(R-\tau\bar h)}{\sqrt{\frac{1}{L^2}-\tau^2}}\ .$$

Hence this provides a lower bound for all τ. The choice specified in Proposition 7.5 corresponds to the optimality condition for maximizing this quantity with respect to τ.