FULLY FIRST-ORDER METHODS FOR DECENTRALIZED BILEVEL OPTIMIZATION

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ABSTRACT

This paper focuses on decentralized stochastic bilevel optimization (DSBO) where agents only communicate with their neighbors. We propose Decentralized Stochastic Gradient Descent and Ascent with Gradient Tracking (DSGDA-GT), a novel algorithm that only requires first-order oracles that are much cheaper than second-order oracles widely adopted in existing works. We further provide a finite-time convergence analysis showing that for n agents collaboratively solving the DSBO problem, the sample complexity of finding an ϵ-stationary point in our algorithm is $\mathcal{O}(n^{-1}\epsilon^{-7})$, which matches the currently best-known results of the single-agent counterpart with linear speedup. The numerical experiments demonstrate both the communication and training efficiency of our algorithm.

1 Introduction

Bilevel optimization (BO) has recently gained growing attention in the machine learning community due to its effectiveness in various applications such as hyperparameter optimization [17, 44, 19, 41], meta-learning [2, 20, 51], reinforcement learning [64, 27], and many others [52]. Mathematically, the bilevel optimization problem can be formulated as follows

$$\min_{x\in\mathbb{R}^p} \quad \Phi(x) = f(x, y^*(x)), \quad \text{s.t. } y^*(x) = \arg\min_{y\in\mathbb{R}^q} g(x,y) \tag{1}$$

where g is the lower-level (LL) function and is usually assumed to be strongly convex with respect to y for all x, and f is the upper-level (UL) function which is possibly non-convex. A natural strategy to solve problem (1) is to estimate $\nabla\Phi(x)$ (which we call hypergradient), and then perform hypergradient descent on x. Under certain smoothness assumptions, the hypergradient exists and has the following closed-form expression by implicit function theorem [22]:

$$\nabla\Phi(x) = \nabla_x f(x, y^*(x)) + \nabla y^*(x)^\top \nabla_y g(x, y^*(x)) \tag{2}$$

where we have

$$\nabla y^*(x)^\top = -\nabla^2_{xy} g(x, y^*(x))(\nabla^2_y g(x, y^*(x)))^{-1}. \tag{3}$$

Two major challenges are obvious from the hypergradient expression in (2) – one may not have direct access to $y^*(x)$ and it is usually expensive to directly invert a Hessian matrix $\nabla^2_y g(x, y^*(x))$, which may further require some approximation of the Hessian inverse. This suggests that one should carefully handle these two sources of large bias in estimating (2). State-of-the-art techniques to estimate (2) include AID-based methods [16, 48, 23, 22, 24, 31], ITD-based methods [16, 44, 20, 24, 31], Neumann-series-based methods [22, 6, 27, 31], and SGD-based methods [3, 10, 8, 26]. Although the sample complexity of BO has been proven to match the lower bound under mild assumptions [8, 26], it is worth noting all these works require Jacobian-vector product oracles, which largely restrict the applicability of

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Table 1: We compare our Algorithm 1 with existing DSBO algorithms including DSBO-JHIP [9], GBDSBO [61], MA-DSBO [7], and D-SOBA [35]. "Cost / Iter" represents the per-iteration computational and communication cost. "Complexity" represents the oracle complexity as well as the communication rounds required to find an ϵ-stationary point. "Oracles" represents the oracles needed in the algorithms. We use "Jacobian", "JVP", and "Grad" to denote oracles of Jacobian matrices, Jacobian-vector products, and gradients respectively. "Heterogeneity" corresponds to data heterogeneity, and "Bounded" indicates the requirement of an additional assumption that the data heterogeneity is bounded across agents, i.e., $\left\|\nabla f_i - \frac{1}{n}\sum_{i=1}^n \nabla f_i\right\|$ is bounded uniformly for all i. In deep learning architectures, the computation of a Jacobian-vector product can take four times the time taken by computing a gradient and may require three times more memory than computing a gradient [11].

Algorithm	Cost / Iter	Complexity	Oracles	Heterogeneity
DSBO-JHIP	$\mathcal{O}(d^2)$	$\tilde{\mathcal{O}}(\epsilon^{-6})$	JVP, Grad	Bounded
GBDSBO	$\mathcal{O}(d^2)$	$\tilde{\mathcal{O}}(n^{-1}\epsilon^{-4})$	Jacobian, Grad	Bounded
MA-DSBO	$\mathcal{O}(d)$	$\tilde{\mathcal{O}}(\epsilon^{-4})$	JVP, Grad	Bounded
D-SOBA	$\mathcal{O}(d)$	$\mathcal{O}(n^{-1}\epsilon^{-4})$	JVP, Grad	Bounded
DSGDA-GT	$\mathcal{O}(d)$	$\mathcal{O}(n^{-1}\epsilon^{-7})$	Grad	Unbounded

such algorithms. To mitigate this issue, another line of research has been dedicated to tackling Problem (1) by using first-order information only [36, 5].

To accelerate the optimization process of BO algorithms, there is a flurry of work extending the single-agent training setting to the multi-agent ones such as decentralized training [43, 9, 61, 21, 18, 35] and federated learning [58, 28, 63]. Designing provably convergent and efficient algorithms for these types of problems is even harder, as we need to handle the heterogeneity from various sources of data and achieve consensus among different agents. Existing decentralized stochastic bilevel optimization (DSBO) algorithms mainly utilize second-order information to approximate the hypergradient, and then apply updates in a decentralized manner on top of it. This paper aims to propose and evaluate the fully first-order methods for DSBO problems. Our contributions can be summarized as follows.

1.1 Our contributions

- We propose Decentralized Stochastic Gradient Descent and Ascent with Gradient Tracking (DSGDA-GT), a fully first-order algorithm for solving the DSBO problem with a constant batch size and unbounded data heterogeneity. Our algorithm greatly improves the per-iteration time and space complexity compared to existing works, which heavily depend on second-order information of the objectives.
- We provide a finite-time analysis, which indicates that our algorithm is capable of finding an ϵ-stationary point within $\mathcal{O}(n^{-1}\epsilon^{-7})$ first-order oracle complexity, which matches the current best-known result in the single-agent counterpart and achieves a linear speedup effect in the decentralized setting. In addition, our analysis of the double-loop and two-timescale decentralized optimization is of independent interest.
- We conduct experiments on both synthetic and real-world datasets, comparing the performance of our algorithm against existing state-of-the-art baselines. The empirical results demonstrate that our methods exhibit superior generalization performance and greater efficiency compared to the others.

1.2 Related work

Bilevel optimization. The study of bilevel optimization can be traced back to [55]. Recently, there is a flurry of work proposing novel BO algorithms with provable convergence rates [22, 24, 27, 6, 10] and implementing BO in large-scale problems in the machine learning community [48, 41]. It is gaining popularity due to its capability to handle different types of problems with a hierarchical structure. One line of theoretical work aims at settling the sample complexity of finding a stationary point in BO [22, 27, 30, 6, 3, 10, 8, 26] when second-order oracles like Jacobian-vector products are accessible. Despite the fact that the complexity of computing a matrix-vector product oracle is roughly the same as that of a gradient [47], such oracles are still time-consuming and difficult to implement, especially when it comes to neural network models, which require additional efforts in developing machine learning libraries to efficiently compute the hypergradient [25, 13, 14, 4]. Motivated by this, some recent works propose novel algorithms to avoid accessing second-order information of the problem, such as fully first-order method [36, 5], which reformulates the bilevel problem as a single-level one treating the lower-level problem as a penalty term, zeroth-order method [54, 62, 1], which estimates the hypergradient via finite-difference approximation, etc.

Decentralized optimization. Decentralized optimization has been studied extensively in control community [60, 15]. When it comes to large-scale machine learning problems, the decentralized training was revealed to have its own advantages in terms of privacy protection, robustness, scalability, and linear speedup effect [39, 57]. Theoretical investigations include analyzing the sample complexity [39, 57], effects of network topology [46], compression techniques [56, 33], etc.

Decentralized stochastic bilevel optimization (DSBO) arises naturally when the data of a bilevel problem is distributed among different agents connected by a communication network. Extending BO from single-agent training to distributed training is non-trivial, as the hypergradient estimation involves Hessian inverse estimation, which requires the information of each local function pair (f_i, g_i). Some efforts are trying to overcome this obstacle in the distributed setting, for example, decentralized setting [43, 9, 61, 21, 18, 35] and federated learning setting [58, 28, 63]. However, all these works require access to matrix-vector products, i.e., second-order information, that are sometimes unavailable.

2 Preliminaries

Problem setup. In decentralized stochastic bilevel optimization (DSBO), we aim to solve the BO problem via multiple agents or devices in a distributed manner. Specifically, there are n different agents communicating over a decentralized network, which can be represented by a graph whose vertices denote local agents and each edge indicates the neighboring relationship between end points of it. The formal description of the DSBO problem is

$$\min_{x\in\mathbb{R}^p} \quad \Phi(x) = \frac{1}{n}\sum_{i=1}^{n} f_i(x, y^*(x)) \quad \text{s.t. } y^*(x) = \arg\min_{y} \frac{1}{n}\sum_{i=1}^{n} g_i(x,y) \tag{4}$$

where the lower and upper functions $f_i(x,y) = \mathbb{E}_{\xi\sim\Xi_i}[F(x,y;\xi)]$ and $g_i(x,y) = \mathbb{E}_{\psi\sim\Psi_i}[G(x,y;\psi)]$ are only accessible to the agent i. We assume that each agent only has access to stochastic gradient oracles of local functions (f_i, g_i), and they can only communicate with their neighbors to exchange information so that they can collaboratively solve the problem. It is worth noting that according to the hypergradient expression in (2) and (3), we can obtain

$$\nabla\Phi(x) = \left(\frac{1}{n}\sum_{i=1}^{n}\nabla_x f_i(x,y^*(x))\right) + \nabla y^*(x)^\top \left(\frac{1}{n}\sum_{i=1}^{n}\nabla_y f_i(x,y^*(x))\right)$$

$$\nabla y^*(x)^\top = -\left(\frac{1}{n}\sum_{i=1}^{n}\nabla^2_{xy} g_i(x,y^*(x))\right)\left(\frac{1}{n}\sum_{i=1}^{n}\nabla^2_{y} g_i(x,y^*(x))\right)^{-1}.$$

We can clearly see that the main challenge of solving DSBO problems lies in estimating $\nabla y^*(x)^\top$, and there have been some efforts along this line [9, 7, 61, 35]. They all require access to Jacobian-vector products, which are not available in our setting.

Notation. For convenience, we first introduce our notation conventions. $\mathbf{1}_n$ denotes the all-one vector in $\mathbb{R}^n$. $\|\cdot\|$ represents ℓ^2-norm for vectors and Frobenius norm for matrices. $\|\cdot\|_2$ denotes the spectral norm for matrices. We use bar notation over a variable to represent the average of the variables of all agents. We use $\mathcal{O}$ and Θ to denote big-O and big-Theta notation, i.e.,

$$f(x) = \mathcal{O}(g(x)), \text{when } |f(x)| \le C|g(x)| \text{ for some constant } C \text{ independent of } f, g,$$
$$f(x) = \Theta(g(x)), \text{when } C_1|g(x)| \le |f(x)| \le C_2|g(x)| \text{ for some constants } C_1, C_2 \text{ independent of } f, g.$$

The notion of stationarity in this paper is defined as follows.

Definition 1. Suppose we are given the output sequence $\{\bar{x}_1, \bar{x}_2, ..., \bar{x}_S\}$ of an algorithm for Problem (4). We say it finds an ϵ-stationary point, when

$$\min_{1\le s\le S} \mathbb{E}\left[\|\nabla\Phi(\bar{x}_s)\|\right] \le \epsilon.$$

2.1 Fully first-order hypergradient estimation

To effectively approximate $(\nabla^2_y g)^{-1}\nabla_y f$ in the expression of the hypergradient in (2), classical stochastic algorithms either require Neumann series methods [22, 27, 6], or approximating the solution of a linear system via minimizing a quadratic function [3, 10, 8, 26]. All of them require Hessian-vector products. To avoid the computation of second-order information, we consider the following min-max formulation shown in [36, 5] to design a fully first-order method for DBSO.

Min-max reformulation. Note that in (4) the lower-level can be viewed as a constraint of the upper-level problem, and thus it is tempting to reformulate the DSBO problem as:

$$\min_{x\in\mathbb{R}^p, y\in\mathbb{R}^q} \frac{1}{n}\sum_{i=1}^n f_i(x,y), \quad \text{s.t. } \frac{1}{n}\sum_{i=1}^n g_i(x,y) - \min_z \frac{1}{n}\sum_{i=1}^n g_i(x,z) = 0. \tag{5}$$

In this formulation, we introduce an auxiliary variable z to transform the lower problem $y^*(x) = \arg\min_y \frac{1}{n}\sum_{i=1}^n g_i(x,y)$ into the constraint $\frac{1}{n}\sum_{i=1}^n g_i(x,y) - \min_z \frac{1}{n}\sum_{i=1}^n g_i(x,z) = 0$, where y serves as a proxy of $y^*(x)$. By adding the constraint in (5) as a penalty term with a factor α to the upper-level function, the DSBO problem can be reformulated as follows:

$$\min_{x\in\mathbb{R}^p, y\in\mathbb{R}^q} \max_z \mathcal{L}^\alpha(x,y,z) \tag{6}$$

where

$$\mathcal{L}^\alpha(x,y,z) := \frac{1}{n}\sum_{i=1}^n (f_i(x,y) + \alpha(g_i(x,y) - g_i(x,z))) \tag{7}$$

and $z \in \mathbb{R}^q$ is the lower variable whose optimum value is still $y^*(x)$, while $y \in \mathbb{R}^q$, $\alpha > 0$ is the multiplier. In this way, the approximation of both lower constraint and upper optimum can be obtained during the same optimization process, and α controls the priority.

Equivalence between Problems (4) and (6). We overload the notation in (6) and define

$$\Omega^\alpha(x,y) = \max_z \mathcal{L}^\alpha(x,y,z), \quad z_*(x) := \arg\max_z \mathcal{L}^\alpha(x,y,z) = \arg\min_z \frac{1}{n}\sum_{i=1}^n g_i(x,z),$$
$$\Gamma^\alpha(x) = \min_y \Omega^\alpha(x,y), \quad y_*^\alpha(x) := \arg\min_y \Omega^\alpha(x,y).$$

Note that solving for z does not require α to be present in the problem. The max part is essentially $\min_z g(x,z)$. The optimality metric of Problem (6) is defined as

$$\|\nabla\Gamma^\alpha(x)\| \le \epsilon, \tag{8}$$

which is commonly used in non-convex strongly-concave (NCSC) min-max optimization [40]. Moreover, we have the following Lemma 2.1 characterizing the relationship between the optimality of the min-max problem defined above and the first-order stationarity of problem (4). We omit the proof and the details can be found in lemma 4.1 of [5]).

Lemma 2.1. *Under Assumption 1, if $\alpha \ge 2\ell_{f,1}/\mu_g$, then*

$$(a.)\ \|\nabla\Phi(x) - \nabla\Gamma^\alpha(x)\| \le \mathcal{O}\left(\frac{\kappa^3}{\alpha}\right); \quad (b.)\ \left\|\nabla^2\Gamma^\alpha(x)\right\| \le \mathcal{O}(\kappa^3) \tag{9}$$

where $\kappa, \mu_g, \ell_{f,1}$ are defined in Section 4.

Lemma 2.1 (a.) implies that when $\alpha \sim 1/\epsilon$, the stationary point of Problem (6) is also a stationary point of Problem (4). Note that Lemma 2.1 (b.) clarifies that the gradient Lipschitz constant of $\Gamma^\alpha(x)$ does not depend on the multiplier α when α is larger than a certain threshold.

3 Algorithm

In this section, we introduce the main ingredients of our algorithmic framework.

3.1 Decentralized optimization with gradient tracking

In decentralized optimization, the gradient tracking (GT) technique was proposed to improve the convergence rates of decentralized optimization algorithms [60, 15, 45, 50]. It was later shown, under mild assumptions, to have unique advantages in handling unbounded gradient similarity caused by data heterogeneity [66, 42, 49, 34]. Thus, we will incorporate this technique into our algorithms to mitigate the data heterogeneity effect. It is worth noting that the implementation of Algorithm 2 has one communication round in each iteration, and one can also adopt multi-consensus techniques such as FastMix [65] and Chebyshev-type communication [53] to enhance consensus among agents.

3.2 Proposed algorithm

To solve the equivalent decentralized min-max problem (6), we are ready to present our main Algorithm 1 named decentralized stochastic gradient descent ascent with gradient tracking (DSGDA-GT). It adopts a double-loop structure widely used in bilevel optimization literature [22, 30, 6].

We first perform the T-step inner-loop decentralized training with gradient tracking (in Algorithm 2) to update lower variables y, z. As shown in line 6 of Algorithm 2, we use $u_{t+1}^{(i)}$ to track the stochastic gradients of the local agent i, which provably achieves linear speedup without assuming data similarity assumption [49, 34]. Since the inner variables y, z are independent of each other, the two T-step inner-loop updates can be performed synchronously. In the inner-loop subroutines: when setting $T = 1$, Algorithm 1 immediately becomes a single-loop algorithm, while choosing large T could potentially bring better convergence rates [32, 5, 37]. Thus, this seemingly complex framework offers more flexibility than the single-loop counterpart.

In each outer iteration (indexed by s), we run stochastic gradient descent with gradient tracking specifically for the upper variable x. The gradient track update for agent i is obtained in line 8 of Algorithm 1 utilizing additional variable set $v_{s+1}^{(i)}$. Note that Algorithm 1 may involve unequal stepsizes for x, y, and z to accommodate their distinct objectives, as dictated by their theoretical properties.

Algorithm 1 Decentralized stochastic gradient descent ascent with gradient tracking (DSGDA-GT)

1: **Input:** $x_0, y_0, z_0, \alpha, \eta_x, \eta_y, \eta_z, S, T$.
2: **Initialization:** $x_0^{(i)} = x_0, y_0^{(i)} = y_0, z_0^{(i)} = z_0, v_0^{(i)} = \delta_0^{(i)} = 0$ on node i.
3: **for** $s = 0 : S - 1$ **do**
4:   **for** $i = 1 : n$ **do**
5:     $y_{s+1}^{(i)}, u_{s+1,y}^{(i)}, h_{s+1,y}^{(i)} = \texttt{Inner Loop}(y_s^{(i)}, \eta_y, f_i(x_s^{(i)}, \cdot) + \alpha g_i(x_s^{(i)}, \cdot), u_{s,y}^{(i)}, h_{s,y}^{(i)}, T)$
6:     $z_{s+1}^{(i)}, u_{s+1,z}^{(i)}, h_{s+1,z}^{(i)} = \texttt{Inner Loop}(z_s^{(i)}, \eta_z, g_i(x_s^{(i)}, \cdot), u_{s,z}^{(i)}, h_{s,z}^{(i)}, T)$
7:     $\delta_{s+1}^{(i)} = \nabla_x f_i(x_s^{(i)}, y_s^{(i)}; \xi_s^{(i)}) + \alpha \left(\nabla_x g_i(x_s^{(i)}, y_s^{(i)}; \psi_s^{(i)}) - \nabla_x g_i(x_s^{(i)}, z_s^{(i)}; \psi_s^{(i)}) \right)$
8:     $v_{s+1}^{(i)} = \sum_{j=1}^n w_{ij} v_s^{(j)} + \delta_{s+1}^{(i)} - \delta_s^{(i)}$
9:     $x_{s+1}^{(i)} = \sum_{j=1}^n w_{ij} x_s^{(i)} - \eta_x v_{s+1}^{(i)}$
10:   **end for**
11: **end for**
12: **Output:** $x_S^{(i)}, y_S^{(i)}, z_S^{(i)}$ on each node.

Algorithm 2 `Inner Loop` $(\theta_0, \gamma, \phi_i(x, \theta), u_0, h_0, T)$

1: **Input:** $\theta_0, \gamma, \phi_i(x, \theta), u_0, h_0, T$.
2: **Initialization:** $u_0^{(i)}, h_0^{(i)}$ **on node** i **satisfying** $\bar{u}_0 = \bar{h}_0$.
3: **for** $t = 0 : T - 1$ **do**
4:   **for** $i = 1 : n$ **do**
5:     $h_{t+1}^{(i)} = \nabla \phi_i(x^{(i)}, \theta_t^{(i)}; \zeta_t^{(i)})$
6:     $u_{t+1}^{(i)} = \sum_{j=1}^n w_{ij} u_t^{(i)} + h_{t+1}^{(i)} - h_t^{(i)}$
7:     $\theta_{t+1}^{(i)} = \sum_{j=1}^n w_{ij} \theta_t^{(i)} - \gamma u_{t+1}^{(i)}$
8:   **end for**
9: **end for**
10: **Output:** $\theta_T^{(i)}, u_T^{(i)}, h_{T+1}^{(i)}$ on each node.

4 Theoretical results

In this section, we provide a convergence analysis of our algorithms. We first introduce the following assumptions, which are standard in both bilevel and distributed optimization literature, as follows.

Assumption 1. (Smoothness) The objectives f_i and g_i for each agent i satisfy:

(1) The UL objective $f_i(x, y)$ is $\ell_{f,0}$-Lipschitz continuous in y; $\ell_{f,1}$-gradient Lipschitz, and $\ell_{f,2}$-Hessian Lipschitz.

(2) The LL objective $g_i(x, y)$ is $\ell_{g,1}$-gradient Lipschitz, $\ell_{g,2}$-Hessian Lipschitz, and μ_g-strongly convex in y.

In this paper, we consider the well-conditioned bilevel problem which is sufficient under Assumption 1(2) [22]. Here we define the condition number $\kappa = \max\{\ell_{f,0}, \ell_{f,1}, \ell_{g,1}, \ell_{g,2}\}/\mu_g$ which aligns with Definition 3.1 in [5].

Under Assumption 1, $f_i + \alpha g_i$ is $\mu_g\alpha/2$-strongly convex in y if $\alpha \geq 2\ell_{f,1}/\mu_g$. The technical lemmas for functions $\mathcal{L}^\alpha(x, y, z)$ and $\Gamma^\alpha(x)$ and their optimal functions $z_*(x)$ and $y_*^\alpha(x)$ in the nonconvex-(strongly-convex)-(strongly-concave) min-max setting can be found in Appendix A.1.

Assumption 2. (Bounded variance) Denote by $\mathcal{F}_s$ the σ-algebra generated by all iterates with subscripts up to s. All stochastic oracles are unbiased with bounded variance. The stochastic oracles of iterates with subscript s are independent under $\mathcal{F}_s$.

Remark 1. The assumptions for objectives f_i, g_i are similar to those of Theorem 4.1 in [36], except for the boundedness requirement on ∇g_i as stated in [36]. In comparison to the assumptions made in [5], the Hessian Lipschitz condition of f_i is required to ensure the smoothness of $y_*^\alpha(x)$ (see Lemma A.11 in Appendix), which is necessary for the consensus analysis of Y when the inner-loop step $T = 1$. It is worth noting that this higher-order smoothness assumption in f_i can be further relaxed by incorporating the moving-average technique used in [8, 35].

Assumption 3. (Network topology) $\mathbf{W} = (w_{ij}) \in \mathbb{R}^{n\times n}$ is symmetric and doubly stochastic, and its eigenvalues $\lambda_n \leq ... \leq \lambda_1 = 1$ satisfy $\rho := \max\{|\lambda_2|, |\lambda_n|\} < 1$.

Assumption 4. There exists a constant c_δ such that in Algorithm 1 we have

$$\mathbb{E}\left[\left\|\bar{\delta}_{s+1}\right\|^2 | \mathcal{F}_s\right] \leq c_\delta \alpha^2.$$

Note that Assumption 4 holds provided that Assumptions 1 and 2 hold and $\|\nabla_x g_i(x, y)\|$ is bounded since

$$\mathbb{E}\left[\left\|\bar{\delta}_{s+1}\right\|^2 | \mathcal{F}_s\right] = \left\|\mathbb{E}\left[\bar{\delta}_{s+1}|\mathcal{F}_s\right]\right\|^2 + \mathbb{E}\left[\left\|\bar{\delta}_{s+1} - \mathbb{E}\left[\bar{\delta}_{s+1}|\mathcal{F}_s\right]\right\|^2 | \mathcal{F}_s\right]$$

which is of order $\mathcal{O}(\alpha^2)$. A similar assumption is also used in bilevel optimization literature (see Assumption 3.7 in [10]).

Now we are ready to present the convergence results of our algorithms.

Theorem 4.1. *Suppose Assumptions 1, 2, 3, and 4 hold, and parameters α and step sizes are chosen such that*

$$\alpha = \Theta\left((nS)^{1/7}\right), \eta_x = \Theta\left(\frac{n^{2/7}}{S^{5/7}}\right), \eta_y = \Theta\left(\frac{n^{2/7}}{S^{5/7}}\right), \eta_z = \Theta\left(\frac{n^{3/7}}{S^{4/7}}\right)$$

and further assume a warm-start for variables y, z such that

$$\max\left(\left\|\bar{y}_0 - y_{*,0}^\alpha\right\|^2, \left\|\bar{z}_0 - z_{*,0}\right\|^2\right) = \mathcal{O}\left(1/\alpha\right) \tag{10}$$

Consider Algorithm 1 with $T = 1$ and $S \geq n^{4/3}$, we have

$$\min_{0\leq s\leq S-1} \mathbb{E}\left[\|\nabla\Phi(\bar{x}_s)\|\right] \leq \mathcal{O}\left(\frac{1}{(nS)^{1/7}}\right), \min_{0\leq s\leq S-1} \frac{\mathbb{E}\left[\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n\|\right]}{n} \leq \mathcal{O}\left(\frac{1}{n^{1/14}S^{4/7}}\right).$$

As a byproduct of Theorem 4.1, we have the following Corollary that gives the sample complexity of finding an ϵ-stationary point.

Corollary 4.2. *Under the same conditions of Theorem 4.1, the stochastic first-order oracles needed in Algorithm 1 for finding an ϵ-stationary point is $\mathcal{O}(n^{-1}\epsilon^{-7})$.*

We highlight that the warm-start condition (10) can be satisfied via running Algorithm 2 as another subroutine. Note that the sample complexity (per node) of achieving (10) is $\mathcal{O}(n^{-1}\alpha) = \mathcal{O}(n^{-6/7}S^{1/7})$ according to Lemma B.2, and for $S = \mathcal{O}(n^{-1}\epsilon^{-7})$ we know this requires $\mathcal{O}(n^{-1}\epsilon^{-1})$ additional stochastic oracles, which do not affect the final sample complexity. Note that we also obtain the linear speedup effect in the sample complexity bound, i.e., the samples required on each node is $\mathcal{O}(n^{-1}\epsilon^{-7})$.

Remark 2. When considering $\mathcal{O}(1)$ batch size setting, if we set $n = 1$, which represents the single-agent training scenario, then the sample complexity of finding an ϵ-stationary point of Algorithm 1 matches that of [36]. It is worth noting that the large-batch and inner-loop $T \gg 1$ settings can also be covered by our analysis, however, it does not yield the desired improvement by a simple extension of [5] and [37] due to the consensus error in the upper variable x. With stronger assumptions such as mean-squared smoothness [37, 62] and large batch sizes [5] imposed, we anticipate the sample complexity can be further improved, and we leave this as an interesting future work.

4.1 Proof sketch

In this section, we highlight the main steps of analyzing the proposed algorithms and the novelty of our analysis as compared to the existing ones.

By the smoothness of $\Gamma^\alpha(x)$ in Lemma 2.1, we first get the descent inequality over the variable x:

$$\begin{aligned}
&\mathbb{E}\left[\Gamma^\alpha(\bar{x}_{s+1})|\mathcal{F}_s\right]-\Gamma^\alpha(\bar{x}_s)\\
\le&-\frac{\eta_x}{2}\left\|\nabla\Gamma^\alpha(\bar{x}_s)\right\|^2-\left(\frac{\eta_x}{2}-\frac{\eta_x^2\ell_\Gamma}{2}\right)\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2+\frac{\ell_\Gamma\eta_x^2\sigma_x^2}{2n}\\
&+\underbrace{\frac{3\eta_x\ell_{x,1}^2}{2n}\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2}_{\text{outer-loop error}}+\underbrace{\frac{3\eta_x\ell_{y,1}^2}{2n}\left\|\mathbf{Y}_s-y_*^\alpha(\bar{x}_s)\mathbf{1}_n^\top\right\|^2+\frac{3\eta_x\alpha^2\ell_{z,1}^2}{2n}\left\|\mathbf{Z}_s-z_*(\bar{x}_s)\mathbf{1}_n^\top\right\|^2}_{\text{inner-loop error}}.
\end{aligned}$$

This, together with (2.1), indicates that to theoretically bound $\|\nabla\Phi(\bar{x}_s)\|$, we need to carefully estimate the error induced by the inner-loop variables y, z and the outer-loop variable x.

Inner-loop error. Take y for example, motivated by the decomposition

$$\left\|\mathbf{Y}_s-y_*^\alpha(\bar{x}_s)\mathbf{1}_n^\top\right\|^2\le\underbrace{\left\|\mathbf{Y}_s-\bar{y}_s\mathbf{1}_n^\top\right\|^2}_{\text{Consensus error}}+\underbrace{n\left\|\bar{y}_s-y_*^\alpha(\bar{x}_s)\right\|^2}_{\text{Convergence error}},$$

we separately analyze the consensus and convergence of inner variables y, z in Section B.

Outer-loop error. Note that due to the double-loop and two-timescale nature of our algorithm, the analysis of the inner-loop error, which gives a recursive relation between $\left\|\mathbf{Y}_{s+1}-y_*^\alpha(\bar{x}_{s+1})\mathbf{1}_n^\top\right\|$ and $\left\|\mathbf{Y}_s-y_*^\alpha(\bar{x}_s)\mathbf{1}_n^\top\right\|$ (see Lemma B.2, same for z), cannot be directly incorporated into the outer-loop analysis. We provide a novel analysis to balance these two sources of error in Section C.

We highlight that different from classical analysis of decentralized stochastic gradient tracking techniques for optimizing strongly convex functions [49] which only requires all stepsizes to have the same order of magnitude in terms of S (i.e., single-timescale), our convergence analysis requires careful design of stepsize choices for η_x, η_y, η_z to handle the consensus error and convergence error induced by both the inner and outer loops. Different from the existing analysis of double-loop DSBO algorithm [7], we provide a fine-grained analysis in Section C that is of independent interest.

5 Experiments

In this section, we investigate the empirical performance of Algorithm 1. Following the basic experimental setup in existing works [48, 24, 30, 7, 35], we consider the following hyperparameter optimization problem under the decentralized setting.

$$\min_{\lambda\in\mathbb{R}^p}\frac{1}{n}\sum_{i=1}^n\mathcal{L}_{\text{val}}^{(i)}(\lambda,\omega^*(\lambda)),\quad \text{s.t. }\ \omega^*(\lambda)=\arg\min_{w\in\mathbb{R}^q}\frac{1}{n}\sum_{i=1}^n\mathcal{L}_{\text{train}}^{(i)}(\lambda,\omega). \tag{11}$$

Here, agent i has access to validation dataset $\mathcal{D}_{\text{val}}^{(i)}$ and the training dataset $\mathcal{D}_{\text{train}}^{(i)}$, that are used to evaluate $\mathcal{L}_{\text{val}}$ and $\mathcal{L}_{\text{train}}^{(i)}$ respectively. We aim at learning the best hyperparameters λ, under the constraint that the model parameters ω are optimal. All experiments are conducted on a computer with Intel Core i7-11370H Processor. We use 8 cores to simulate 8 agents ($n = 8$), and the communication steps are conducted with mpi4py [12] module. We compare our Algorithm 1 with MA-DSBO [7] and D-SOBA [35], two DSBO algorithms that only require first-order oracles and matrix-vector product oracles. We note that both DSBO-JHIP [9] and Gossip-DSBO [61] require computing and communicating Jacobian matrices, and are inefficient [7] as reported by [7]. Hence we do not include them as baseline algorithms.

We would like to highlight that for hyperparameter optimization problems, the validation datasets that produce the upper-level functions f_i are relatively much smaller than the training datasets for the lower-level functions g_i. It is thus more reasonable to update the hyperparameters less frequently than the model parameters, which indicates that our double-loop DSBO Algorithm 1 offers more flexibility in this type of problem than single-loop ones.

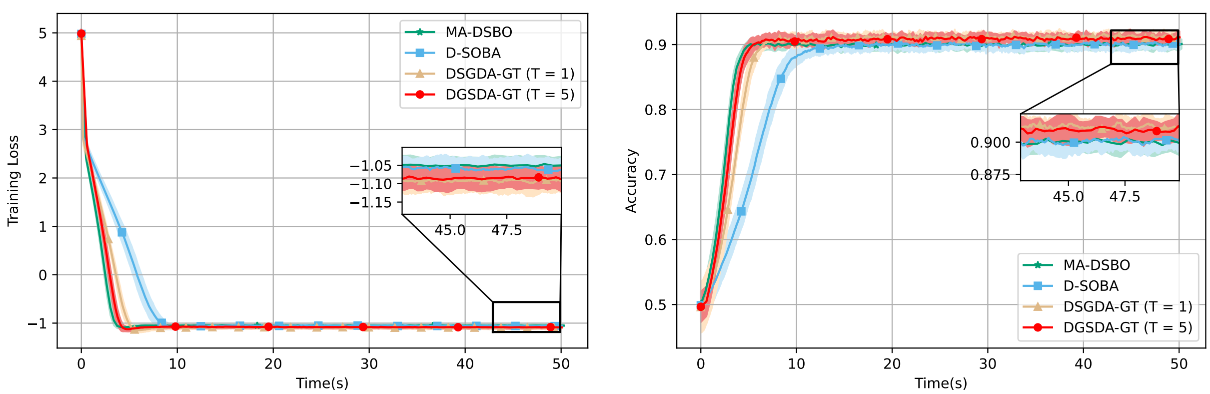
5.1 Synthetic data

To validate the efficiency of Algorithm 1, we first consider a simple binary classification problem with synthetic data. Specifically, we consider problem (11), with functions $(\mathcal{L}_{\text{val}}^{(i)}, \mathcal{L}_{\text{train}}^{(i)})$ as follows.

$$\mathcal{L}_{\text{val}}^{(i)}(\lambda, \omega) = \frac{1}{|\mathcal{D}_{\text{val}}^{(i)}|} \sum_{(x_e, y_e) \in \mathcal{D}_{\text{val}}^{(i)}} \psi(y_e x_e^\top \omega),$$

$$\mathcal{L}_{\text{train}}^{(i)}(\lambda, \omega) = \frac{1}{|\mathcal{D}_{\text{train}}^{(i)}|} \sum_{(x_e, y_e) \in \mathcal{D}_{\text{train}}^{(i)}} \psi(y_e x_e^\top \omega) + \frac{1}{2} \sum_{i=1}^{d} e^{\lambda_i} \omega_i^2,$$

where $\psi(x) = \log(1 + e^{-x})$. We have $x_e \sim \mathcal{N}(0, i^2 I_d)$ and $y_e = \text{sgn}(x_e^\top \omega + 0.1 \cdot z)$, where $\text{sgn}(\cdot)$ is the sign function that outputs 1 for a positive input and 0 otherwise. z is the noise vector generated from standard normal distribution. This gives a regularized logistic regression problem, which is widely used in bilevel optimization literature [48, 24]. We plot the training loss and test accuracy over wall-clock time in Figures 1(a) and 1(b), from which we can observe that our methods achieve the lowest training loss and best accuracy in a relatively short amount of time. Interestingly, when all curves stabilize, the test accuracy of our Algorithm is better than the ones that require second-order information. This may indicate fully first-order methods have better generalization performance than second-order ones.



(a) (b)

Figure 1: Training loss and test accuracy of ℓ^2-regularized logistic regression on synthetic data. The vertical axis of Figure 1(a) is in log scale.

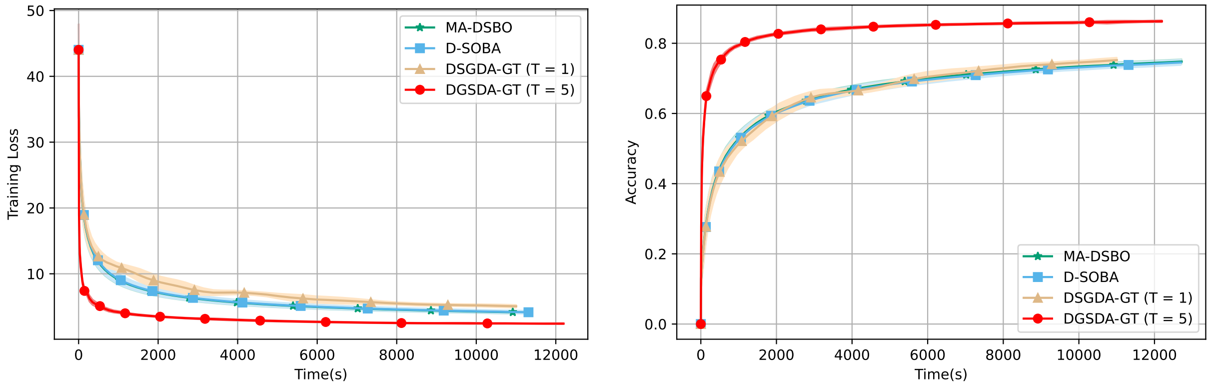
5.2 Real-world data

We then test the performance of our algorithm on real-world data – MNIST [38], with functions $(\mathcal{L}_{\text{val}}^{(i)}, \mathcal{L}_{\text{train}}^{(i)})$ defined as

$$\mathcal{L}_{\text{val}}^{(i)}(\lambda, \omega) = \frac{1}{|\mathcal{D}_{\text{val}}^{(i)}|} \sum_{(x_e, y_e) \in \mathcal{D}_{\text{val}}^{(i)}} L(x_e^\top \omega, y_e),$$

$$\mathcal{L}_{\text{train}}^{(i)}(\lambda, \omega) = \frac{1}{|\mathcal{D}_{\text{train}}^{(i)}|} \sum_{(x_e, y_e) \in \mathcal{D}_{\text{train}}^{(i)}} L(x_e^\top \omega, y_e) + \frac{1}{cd} \sum_{i=1}^{c} \sum_{j=1}^{d} e^{\lambda_j} \omega_{ij}^2,$$

where we denote by L the cross-entropy loss, and $(c, d) = (10, 784)$ represent the number of classes and number of features. We plot the training loss and test accuracy with respect to training time in Figure 2. Our Algorithm 1 with different settings is consistently better than existing ones in terms of training loss and accuracy. Moreover, we can observe better generalization performance of the fully first-order algorithm over the second-order algorithms under the same training time. Our Algorithm also provides more flexibility, in the sense that we can set the number of inner-loop iterations T to be greater than 1, which gives a double-loop algorithm, which has been proven beneficial over the fully single-loop ones both theoretically [6, 32] and also empirically in our Figures 2(a) and 2(b).



(a) (b)

Figure 2: ℓ^2-regularized logistic regression on MNIST.

6 Conclusion

In this paper, we propose a novel algorithm called Decentralized Stochastic Gradient Descent Ascent with Gradient Tracking (DSGDA-GT) for solving decentralized stochastic bilevel optimization problems. The proposed algorithm only requires the first-order gradient oracle, making it more efficient compared to the existing methods that involve second-order oracles. We provide the first-order oracle complexity $\mathcal{O}(n^{-1}\epsilon^{-7})$ to find an ϵ stationary point, which matches the well-known result in the single agent method [36]. In the future, it will be interesting to improve the convergence rate of the fully first-order methods under stronger assumptions and large-batch settings. Moreover, investigating the fundamental limits and analyzing the lower bound of such problems is an area of independent interest.

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Contents

A Appendix / Auxiliary lemmas for theoretical results

In this section, we analyze the convergence of Algorithm 1. For convenience, we first introduce our notational conventions. $\mathbf{1}_n$ denotes the all-one vector in $\mathbb{R}^n$. $\|\cdot\|$ represents ℓ^2-norm for vectors and Frobenius norm for matrices. $\|\cdot\|_2$ denotes the spectral norm for matrices.

$$
\begin{aligned}
&\mathbf{X}_s = \left(x_s^{(1)}, ..., x_s^{(n)}\right), \mathbf{Y}_s = \left(y_s^{(1)}, ..., y_s^{(n)}\right), \mathbf{Z}_s = \left(z_s^{(1)}, ..., z_s^{(n)}\right),\\
&\mathbf{V}_s = \left(v_s^{(1)}, ..., v_s^{(n)}\right), \mathbf{\Delta}_s = \left(\delta_s^{(1)}, ..., \delta_s^{(n)}\right).\\
&\bar{x}_s = \frac{1}{n}\mathbf{X}_s\mathbf{1}_n = \frac{1}{n}\sum_{i=1}^n x_s^{(i)},\ \bar{y}_s = \frac{1}{n}\mathbf{Y}_s\mathbf{1}_n = \frac{1}{n}\sum_{i=1}^n y_s^{(i)},\ \bar{z}_s = \frac{1}{n}\mathbf{Z}_s\mathbf{1}_n = \frac{1}{n}\sum_{i=1}^n z_s^{(i)},\\
&\bar{v}_s = \frac{1}{n}\mathbf{V}_s\mathbf{1}_n = \frac{1}{n}\sum_{i=1}^n v_s^{(i)},\ \bar{\delta}_s = \frac{1}{n}\mathbf{\Delta}_s\mathbf{1}_n = \frac{1}{n}\sum_{i=1}^n \delta_s^{(i)}.\\
&y_*^\alpha(x) := \arg\min_y\ \Omega^\alpha(x,y),\ z_*(x) := \arg\min_z\ g(x,z),\\
&y_{*,s}^\alpha := \arg\min_y\ \Omega^\alpha(\bar{x}_s,y),\ z_{*,s} := \arg\min_z\ g(\bar{x}_s,z).\\
&\mathcal{F}_s = \sigma\left(\bigcup_{i=1}^n \left\{x_0^{(i)}, y_0^{(i)}, z_0^{(i)}, v_0^{(i)}, ..., x_s^{(i)}, y_s^{(i)}, z_s^{(i)}, v_s^{(i)}\right\}\right).
\end{aligned}
$$

In the following analysis, the symbol $\lesssim$ indicates that there exists an absolute constant C such that LHS $\leq C$ RHS, and for simplicity, omitting C does not affect the order of RHS.

Note that suppose that f_i and g_i for each agent i satisfy the variance bounded condition in Assumption 2, we might let

$$
\mathbb{E}\left[\|\nabla F_i(x,y;\xi) - \nabla f_i(x,y)\|^2\right] \leq \sigma_f^2; \quad \mathbb{E}\left[\|\nabla G_i(x,y;\psi) - \nabla g_i(x,y)\|^2\right] \leq \sigma_g^2. \tag{12}
$$

The following technical lemmas are standard.

Lemma A.1. *For any $m, n \in \mathbb{N}_+$ and matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$ and $c > 0$, we have:*

$$
\|\mathbf{A}+\mathbf{B}\|^2 \leq (1+c)\|\mathbf{A}\|^2 + (1+c^{-1})\|\mathbf{B}\|^2.
$$

Lemma A.2. *For any $p, q, r \in \mathbb{N}_+$ and matrices $\mathbf{A} \in \mathbb{R}^{p\times q}, \mathbf{B} \in \mathbb{R}^{q\times r}$, we have:*

$$
\|\mathbf{AB}\| \leq \min\left(\|\mathbf{A}\|_2 \cdot \|\mathbf{B}\|, \|\mathbf{A}\| \cdot \|\mathbf{B}^\top\|_2\right).
$$

Lemma A.3. *For three sequences $\{a_n\}_{n=0}^\infty, \{b_n\}_{n=0}^\infty,\ \{\tau_n\}_{n=-1}^\infty$, and a constant r satisfying*

$$
a_{k+1} \leq ra_k + b_k,\ a_k \geq 0,\ b_k \geq 0,\ 0 = \tau_{-1} \leq \tau_{k+1} \leq \tau_k \leq 1,\ 0 < r < 1, \tag{13}
$$

for all $k \geq 0$. Then for any $K > 0$, we have

$$
a_k \leq r^k a_0 + \sum_{i=0}^{k-1} r^{k-1-i} b_i, \tag{14}
$$

$$
\sum_{k=0}^K \tau_k a_k \leq \frac{1}{1-r}\left(\tau_0 a_0 + \sum_{k=0}^K \tau_k b_k\right). \tag{15}
$$

Proof. (of Lemma A.3) To prove (14), notice that we have $\frac{a_i}{r^i} \leq \frac{a_{i-1}}{r^{i-1}} + \frac{b_{i-1}}{r^i}$, and thus taking summation for $1 \leq i \leq k$ on both sides completes the proof. To prove (15), note that we have

$$
(1-r)\sum_{k=0}^K \tau_k a_k \leq \sum_{k=0}^K \tau_k(a_k - a_{k+1} + b_k) = \sum_{k=0}^K (\tau_k - \tau_{k-1})a_k - \tau_K a_{K+1} + \sum_{k=0}^K \tau_k b_k \leq \tau_0 a_0 + \sum_{k=0}^K \tau_k b_k,
$$

where the inequalities use (13), and the equality uses summation by parts. ∎

Lemma A.4. *For the sequence $\{x_n\}_{n=1}^N$ and constant $r \in (0,1)$, then*

$$\sum_{s=0}^{N}\sum_{n=0}^{s} r^{s-n}x_n = \sum_{n=0}^{N}\sum_{s=n}^{N} r^{s-n}x_n \le \frac{1}{1-r}\sum_{n=0}^{N} x_n.$$

Lemma A.5. *Suppose Assumption 3 holds. For any $m \in \mathbb{N}+$, we have*

$$\left\|\mathbf{W}^m - \frac{\mathbf{1}_n\mathbf{1}_n^\top}{n}\right\|_2 \le \rho^m.$$

Lemma A.6. *Suppose $f(x)$ is μ-strongly convex and ℓ-smooth. For any x and $\gamma < \frac{2}{\mu+\ell}$, define $x^+ = x - \gamma\nabla f(x)$, $x^* = \arg\min_x f(x)$. Then we have*

$$\|x^+ - x^*\| \le (1-\gamma\mu)\|x - x^*\|.$$

Proof. See, e.g., Lemma 10 in [50]. □

Lemma A.7. *Suppose Assumption 3 holds. We have for all $0 \le s \le S-1$ that*

$$\bar{v}_s = \bar{\delta}_s.$$

Proof. (of Lemma A.7) We first note that each $v_{s+1}^{(i)}$ is introduced in the gradient tracking step of Algorithm 1, i.e.,

$$v_{s+1}^{(i)} = \sum_{j=1}^{n} w_{ij}v_s^{(j)} + \delta_{s+1}^{(i)} - \delta_s^{(i)}$$

Computing the average on both sides and using the fact that $\mathbf{W}$ is doubly stochastic, we have

$$\bar{v}_{s+1} = \bar{v}_s + \bar{\delta}_{s+1} - \bar{\delta}_s.$$

Hence, $\bar{v}_s = \bar{\delta}_s$ given the initialization $\bar{v}_0 = \bar{\delta}_0$. □

A.1 Properties of min-max functions and its optimal functions

Suppose Assumption 1 hold, the functions $\mathcal{L}^\alpha(x,y,z)$ and $\Gamma^\alpha(x)$ satisfy the following properties.

Lemma A.8. *Under Assumption 1, the followings hold:*

(i) $\mathcal{L}^\alpha(x,y,z)$ is $\mu_g\alpha$-strongly concave w.r.t. z;

(ii) $\mathcal{L}^\alpha(x,y,z)$ is $\mu_g\alpha/2$-strongly convex w.r.t. y if $\alpha > 2\ell_{f,1}/\mu_g$.

The results of Lemma A.8 can be found in [36] and Lemma B.1 of [5]. From Lemma B.7 in [5], the following result holds for $\Gamma^\alpha(x)$:

Lemma A.9. *Under Assumption 1, if $\alpha > 2\ell_{f,1}/\mu_g$, then $\Gamma^\alpha(x)$ is ℓ_Γ-smooth, where $\ell_\Gamma = \mathcal{O}(\kappa^3)$ is a constant that is independent on α.*

Moreover, the functions $y_*^\alpha(x)$ and $z_*(x)$ satisfy the following properties.

Lemma A.10. *Under Assumption 1, we have*

$$\|y_*^\alpha(x) - y^*(x)\| \le \frac{C_0}{\alpha}$$

where $C_0 = \ell_{f,0}/\mu_g$.

The result in Lemma A.10 follows from Lemma B.2 of [5].

Lemma A.11. *Under Assumption 1, if $\alpha > 2\ell_{f,1}/\mu_g$, then we have*

(i) $z_*(x)$ is κ-Lipschitz continuous;

(ii) $y_*^\alpha(x)$ is $\ell_{y_*,0}$-Lipschitz continuous where $\ell_{y_*,0} = 3\kappa$.

Claim (i) in Lemma A.11 can be found in Lemma 2.2 of [22] and Claim (ii) implies from Lemma 3.2 (setting $\lambda_1 = \lambda_2$) of [36].

Lemma A.12. *Under Assumption 1, if $\alpha > 2\ell_{f,1}/\mu_g$,*

(i) $y_*^\alpha(x)$ *is* $\ell_{\nabla y_*}$*-smooth where* $\ell_{\nabla y_*} = \mathcal{O}\left(\frac{\kappa^2}{\mu_g}\left(\frac{\ell_{f,2}}{\alpha} + \ell_{g,2}\right)\right)$

(ii) $z_*(x)$ *is* $\ell_{\nabla z_*}$*-smooth where* $\ell_{\nabla z_*} = \mathcal{O}\left(\frac{\kappa^2}{\mu_g}(\ell_{g,1}+1)\right)$

Following Lemma A.3 of [36] and recalling the Lipschitz continuous property of $y_*^\alpha(x)$, we have the first claim (i) is correct. Note that to ensure the smoothness of $y_*^\alpha(x)$, we need to assume the Hessian-Lipschitz of f. Similarly, recalling the Lipschitz continuity of $z_*(x)$ from Lemma A.11, the function $z_*(x)$ is gradient Lipschitz, that is Claim (ii) holds.

B Appendix / Analysis of Algorithm 2

In Algorithm 1, the updates for $y_s^{(i)}$ and $z_s^{(i)}$ are essentially T-step decentralized stochastic gradient descent with gradient tracking (see Algorithm 2). Hence, their convergence and the consensus can be analyzed through the following technical lemma.

Lemma B.1. *Suppose $\phi_i(x,\theta)$ in Algorithm 2 is ℓ-smooth and μ-strongly convex. The stochastic oracle $h_{t+1}^{(i)} = \nabla_\theta \phi_i(x^{(i)}, \theta_t^{(i)}; \zeta_t^{(i)})$ is unbiased with variance bounded by σ^2, and is independent of $h_{t+1}^{(j)}$ conditioning on all iterates with subscripts up to t. Define*

$$\mathbf{\Theta}_t = \left(\theta_t^{(1)}, ..., \theta_t^{(n)}\right), \ \mathbf{H}_t = \left(h_t^{(1)}, ..., h_t^{(n)}\right), \ \bar{x} = \frac{1}{n}\sum_{i=1}^n x^{(i)}, \ \phi(x,\theta) = \frac{1}{n}\sum_{i=1}^n \phi_i(x,\theta)$$

$$\theta_* = \arg\min_\theta \frac{1}{n}\sum_{i=1}^n \phi_i(\bar{x},\theta), \ \mathcal{G}_t = \sigma\left(\bigcup_{i=1}^n \{\theta_0^{(i)}, h_0^{(i)}, ..., \theta_t^{(i)}, h_t^{(i)}, x^{(i)}\}\right).$$

If $\gamma < \frac{1}{\ell} \le \frac{2}{\mu+\ell}$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left\|\bar{\theta}_{t+1} - \theta_*\right\|^2 \mid \mathcal{G}_t\right] \\
\le&(1-\gamma\mu)\left\|\bar{\theta}_t - \theta_*\right\|^2 + \frac{2\gamma\ell^2}{\mu n}\left(\left\|\mathbf{X} - \bar{x}\mathbf{1}_n^\top\right\|^2 + \left\|\mathbf{\Theta}_t - \bar{\theta}_t\mathbf{1}_n^\top\right\|^2\right) + \frac{\gamma^2\sigma^2}{n},
\end{aligned} \tag{16a}$$

$$\left\|\mathbf{\Theta}_{t+1} - \bar{\theta}_{t+1}\mathbf{1}_n^\top\right\|^2 \le \frac{1+\rho^2}{2}\left\|\mathbf{\Theta}_t - \bar{\theta}_t\mathbf{1}_n^\top\right\|^2 + \frac{(1+\rho^2)\gamma^2}{1-\rho^2}\left\|\mathbf{U}_{t+1} - \bar{u}_{t+1}\mathbf{1}_n^\top\right\|^2, \tag{16b}$$

$$\begin{aligned}
&\mathbb{E}\left[\left\|\mathbf{U}_{t+1} - \bar{u}_{t+1}\mathbf{1}_n^\top\right\|^2\right] \\
\le&\left(\frac{1+\rho^2}{2} + \frac{6\ell^2\gamma^2(1+\rho^2)}{1-\rho^2}\right)\mathbb{E}\left[\left\|\mathbf{U}_t - \bar{u}_t\mathbf{1}_n^\top\right\|^2\right] + \frac{36(1+\rho^2)\ell^2}{1-\rho^2}\mathbb{E}\left[\left\|\mathbf{\Theta}_{t-1} - \bar{\theta}_{t-1}\mathbf{1}^\top\right\|^2\right] \\
&+\frac{12(1+\rho^2)\ell^4\gamma^2}{1-\rho^2}\mathbb{E}\left[\left\|\mathbf{X} - \bar{x}\mathbf{1}_n^\top\right\|^2\right] + \frac{12n(1+\rho^2)\ell^4\gamma^2}{1-\rho^2}\mathbb{E}\left[\left\|\bar{\theta}_{t-1} - \theta_*\right\|^2\right] + \frac{12n(1+\rho^2)\sigma^2}{1-\rho^2}.
\end{aligned} \tag{16c}$$

Proof. (of Lemma B.1) At each step, we have

$$\mathbf{U}_{t+1} = \mathbf{U}_t\mathbf{W} + \mathbf{H}_{t+1} - \mathbf{H}_t, \mathbf{\Theta}_{t+1} = \mathbf{\Theta}_t\mathbf{W} - \gamma\mathbf{U}_{t+1}, \ \bar{\theta}_{t+1} = \bar{\theta}_t - \gamma\bar{u}_{t+1} = \bar{\theta}_t - \gamma\bar{h}_{t+1}. \tag{17}$$

To prove the first inequality (16a), we have

$$\begin{aligned}
&\bar{\theta}_{t+1} - \theta_* = \bar{\theta}_t - \gamma\bar{h}_{t+1} - \theta_* \\
&= \bar{\theta}_t - \theta_* - \gamma\nabla_\theta\phi(\bar{x},\bar{\theta}_t) - \gamma\left(\mathbb{E}[\bar{h}_{t+1} \mid \mathcal{G}_t] - \nabla_\theta\phi(\bar{x},\bar{\theta}_t)\right) - \gamma\left(\bar{h}_{t+1} - \mathbb{E}[\bar{h}_{t+1} \mid \mathcal{G}_t]\right).
\end{aligned}$$

This implies

$$\begin{aligned}
&\mathbb{E}\left[\left\|\bar{\theta}_{t+1} - \theta_*\right\|^2 \mid \mathcal{G}_t\right] \\
=&\left\|\bar{\theta}_t - \theta_* - \gamma\nabla_\theta\phi(\bar{x},\bar{\theta}_t) - \gamma\left(\mathbb{E}[\bar{h}_{t+1} \mid \mathcal{G}_t] - \nabla_\theta\phi(\bar{x},\bar{\theta}_t)\right)\right\|^2 + \gamma^2\mathbb{E}\left[\left\|\bar{h}_{t+1} - \mathbb{E}[\bar{h}_{t+1} \mid \mathcal{G}_t\right\|^2 \mid \mathcal{G}_t\right]
\end{aligned}$$

$$\leq (1+\gamma\mu)\left\|\bar{\theta}_t - \theta_* - \gamma\nabla_\theta\phi(\bar{x},\bar{\theta}_t)\right\|^2 + \left(1+\frac{1}{\gamma\mu}\right)\gamma^2\left\|\mathbb{E}[\bar{h}_{t+1}\mid\mathcal{G}_t] - \nabla_\theta\phi(\bar{x},\bar{\theta}_t)\right\|^2$$
$$+\gamma^2\mathbb{E}\left[\left\|\bar{h}_{t+1} - \mathbb{E}[\bar{h}_{t+1}\mid\mathcal{G}_t]\right\|^2 \mid \mathcal{G}_t\right], \tag{18}$$

where the first equality holds by the unbiasedness of $h_{t+1}^{(i)}$. We use Lemma A.6 to estimate the first term of (18):

$$\left\|\bar{\theta}_t - \theta_* - \gamma\nabla_\theta\phi(\bar{x},\bar{\theta}_t)\right\|^2 \leq (1-\gamma\mu)^2\left\|\bar{\theta}_t - \theta_*\right\|^2.$$

Then we focus on the second term of (18):

$$\begin{aligned}\left\|\mathbb{E}[\bar{h}_{t+1}\mid\mathcal{G}_t] - \nabla_\theta\phi(\bar{x},\bar{\theta}_t)\right\|^2 &= \left\|\frac{1}{n}\sum_{i=1}^n \nabla_\theta\phi_i(x^{(i)},\theta_t^{(i)}) - \nabla_\theta\phi_i(\bar{x},\bar{\theta}_t)\right\|^2\\ &\leq \frac{1}{n}\sum_{i=1}^n\left\|\nabla_\theta\phi_i(x^{(i)},\theta_t^{(i)}) - \nabla_\theta\phi_i(\bar{x},\bar{\theta}_t)\right\|^2\\ &\leq \frac{\ell^2}{n}\sum_{i=1}^n\left(\left\|x^{(i)}-\bar{x}\right\|^2 + \left\|\theta_t^{(i)} - \bar{\theta}_t\right\|^2\right)\end{aligned}$$

where the last inequality follows from the Lipschitz smoothness of each ϕ_i. Next, we estimate the third term of (18):

$$\begin{aligned}&\mathbb{E}\left[\left\|\bar{h}_{t+1} - \mathbb{E}[\bar{h}_{t+1}\mid\mathcal{G}_t]\right\|^2 \mid \mathcal{G}_t\right] = \mathbb{E}\left[\left\|\frac{1}{n}\sum_{i=1}^n\left(h_{t+1}^{(i)} - \mathbb{E}\left[h_{t+1}^{(i)}\mid\mathcal{G}_t\right]\right)\right\|^2\middle|\mathcal{G}_t\right]\\ =&\frac{1}{n^2}\sum_{i=1}^n\mathbb{E}\left[\left\|h_{t+1}^{(i)} - \mathbb{E}\left[h_{t+1}^{(i)}\mid\mathcal{G}_t\right]\right\|^2\middle|\mathcal{G}_t\right] + \frac{1}{n^2}\sum_{j\neq i}\mathbb{E}\left[\left\langle h_{t+1}^{(i)} - \mathbb{E}\left[h_{t+1}^{(i)}\mid\mathcal{G}_t\right], h_{t+1}^{(j)} - \mathbb{E}\left[h_{t+1}^{(j)}\mid\mathcal{G}_t\right]\right\rangle\middle|\mathcal{G}_t\right]\\ \leq&\frac{\sigma^2}{n}\end{aligned} \tag{19}$$

where the inequality uses the bounded variance, unbiasedness, and the independence of different stochastic oracles. Substituting the above results into (18), we have

$$\mathbb{E}\left[\left\|\bar{\theta}_{t+1} - \theta_*\right\|^2 \mid \mathcal{G}_t\right] \leq (1-\gamma\mu)\left\|\bar{\theta}_t - \theta_*\right\|^2 + \left(1+\frac{1}{\gamma\mu}\right)\frac{\gamma^2\ell^2}{n}\left(\left\|\mathbf{X} - \bar{x}\mathbf{1}_n^\top\right\|^2 + \left\|\mathbf{\Theta}_t - \bar{\theta}_t\mathbf{1}_n^\top\right\|^2\right) + \frac{\sigma^2\gamma^2}{n}.$$

The first inequality (16a) holds due to the step-size $\gamma < \frac{1}{\ell} \leq \frac{1}{\mu}$. Now for the second inequality (16b), by (17) we have

$$\begin{aligned}\mathbf{\Theta}_{t+1} - \bar{\theta}_{t+1}\mathbf{1}_n^\top &= \mathbf{\Theta}_t\mathbf{W} - \gamma\mathbf{U}_{t+1} - (\bar{\theta}_t - \gamma\bar{u}_{t+1})\mathbf{1}_n^\top\\ &= \left(\mathbf{\Theta}_t - \bar{\theta}_t\mathbf{1}_n^\top\right)\left(\mathbf{W} - \frac{\mathbf{1}_n\mathbf{1}_n^\top}{n}\right) - \gamma\left(\mathbf{U}_{t+1} - \bar{u}_{t+1}\mathbf{1}_n^\top\right).\end{aligned} \tag{20}$$

By Lemmas A.1 and A.5 we know for any $c > 0$,

$$\left\|\mathbf{\Theta}_{t+1} - \bar{\theta}_{t+1}\mathbf{1}_n^\top\right\|^2 \leq (1+c)\rho^2\left\|\mathbf{\Theta}_t - \bar{\theta}_t\mathbf{1}_n^\top\right\|^2 + (1+c^{-1})\gamma^2\left\|\mathbf{U}_{t+1} - \bar{u}_{t+1}\mathbf{1}_n^\top\right\|^2.$$

We set $c = \frac{1-\rho^2}{2\rho^2}$ and obtain the second inequality (16b). Finally, for the third inequality (16c), we have from (17) that

$$\begin{aligned}\mathbf{U}_{t+1} - \bar{u}_{t+1}\mathbf{1}_n^\top &= \mathbf{U}_t\mathbf{W} + \mathbf{H}_{t+1} - \mathbf{H}_t - (\bar{u}_t + \bar{h}_{t+1} - \bar{h}_t)\mathbf{1}_n^\top\\ &= \left(\mathbf{U}_t - \bar{u}_t\mathbf{1}_n^\top\right)\left(\mathbf{W} - \frac{\mathbf{1}_n\mathbf{1}_n^\top}{n}\right) + (\mathbf{H}_{t+1} - \mathbf{H}_t)\left(\mathbf{I}_n - \frac{\mathbf{1}_n\mathbf{1}_n^\top}{n}\right).\end{aligned} \tag{21}$$

which, together with Lemmas A.1, A.2 and A.5, and $\left\|\mathbf{I}_n - \frac{\mathbf{1}_n\mathbf{1}_n^\top}{n}\right\|_2 \leq 1$, implies

$$\left\|\mathbf{U}_{t+1} - \bar{u}_{t+1}\mathbf{1}_n^\top\right\|^2 \leq \frac{1+\rho^2}{2}\left\|\mathbf{U}_t - \bar{u}_t\mathbf{1}_n^\top\right\|^2 + \frac{1+\rho^2}{1-\rho^2}\left\|\mathbf{H}_{t+1} - \mathbf{H}_t\right\|^2.$$

To bound $\|\mathbf{H}_{t+1} - \mathbf{H}_t\|$, we have

$$\mathbf{H}_{t+1} - \mathbf{H}_t = \mathbf{H}_{t+1} - \mathbb{E}\left[\mathbf{H}_{t+1}\mid\mathcal{G}_t\right] - \left(\mathbf{H}_t - \mathbb{E}\left[\mathbf{H}_t\mid\mathcal{G}_{t-1}\right]\right) + \mathbb{E}\left[\mathbf{H}_{t+1}\mid\mathcal{G}_t\right] - \mathbb{E}\left[\mathbf{H}_t\mid\mathcal{G}_{t-1}\right] \tag{22}$$

and thus

$$\begin{aligned}\mathbb{E}\left[\|\mathbf{H}_{t+1}-\mathbf{H}_t\|^2\right] &\leq 3\mathbb{E}\left[\|\mathbf{H}_{t+1}-\mathbb{E}\left[\mathbf{H}_{t+1}\mid\mathcal{G}_t\right]\|^2+\|\mathbf{H}_t-\mathbb{E}\left[\mathbf{H}_t\mid\mathcal{G}_{t-1}\right]\|^2+\|\mathbb{E}\left[\mathbf{H}_{t+1}\mid\mathcal{G}_t\right]-\mathbb{E}\left[\mathbf{H}_t\mid\mathcal{G}_{t-1}\right]\|^2\right]\\ &\leq 6n\sigma^2+3\mathbb{E}\left[\|\mathbb{E}\left[\mathbf{H}_{t+1}\mid\mathcal{G}_t\right]-\mathbb{E}\left[\mathbf{H}_t\mid\mathcal{G}_{t-1}\right]\|^2\right] \end{aligned} \tag{23}$$

in which we bound $\|\mathbb{E}\left[\mathbf{H}_{t+1}\mid\mathcal{G}_t\right]-\mathbb{E}\left[\mathbf{H}_t\mid\mathcal{G}_{t-1}\right]\|$ via the following inequalities:

$$\|\mathbb{E}\left[\mathbf{H}_{t+1}\mid\mathcal{G}_t\right]-\mathbb{E}\left[\mathbf{H}_t\mid\mathcal{G}_{t-1}\right]\|^2=\sum_{i=1}^n\left\|\nabla_\theta\phi_i(x^{(i)},\theta_t^{(i)})-\nabla_\theta\phi_i(x^{(i)},\theta_{t-1}^{(i)})\right\|^2\leq\ell^2\left\|\mathbf{\Theta}_t-\mathbf{\Theta}_{t-1}\right\|^2.$$

$$\begin{aligned}\|\mathbf{\Theta}_{t+1}-\mathbf{\Theta}_t\|^2 &=\left\|\left(\mathbf{\Theta}_t-\bar{\theta}_t\mathbf{1}_n^\top\right)(\mathbf{W}-\mathbf{I})-\gamma\mathbf{U}_{t+1}\right\|^2\leq 2\left\|\left(\mathbf{\Theta}_t-\bar{\theta}_t\mathbf{1}_n^\top\right)(\mathbf{W}-\mathbf{I})\right\|^2+2\gamma^2\left\|\mathbf{U}_{t+1}\right\|^2\\ &\leq 8\left\|\mathbf{\Theta}_t-\bar{\theta}_t\mathbf{1}^\top\right\|^2+2\gamma^2\left\|\mathbf{U}_{t+1}-\bar{u}_{t+1}\mathbf{1}_n^\top\right\|^2+2\gamma^2\left\|\bar{u}_{t+1}\mathbf{1}_n^\top\right\|^2.\end{aligned}$$

$$\mathbb{E}\left[\|\bar{u}_{t+1}\|^2\mid\mathcal{G}_t\right]=\mathbb{E}\left[\left\|\bar{h}_{t+1}-\mathbb{E}\left[\bar{h}_{t+1}\mid\mathcal{G}_t\right]\right\|^2\mid\mathcal{G}_t\right]+\left\|\mathbb{E}\left[\bar{h}_{t+1}\mid\mathcal{G}_t\right]\right\|^2\leq\frac{\sigma^2}{n}+\left\|\mathbb{E}\left[\bar{h}_{t+1}\mid\mathcal{G}_t\right]\right\|^2.$$

$$\begin{aligned}\left\|\mathbb{E}\left[\bar{h}_{t+1}\mid\mathcal{G}_t\right]\right\|^2 &=\left\|\mathbb{E}\left[\bar{h}_{t+1}\mid\mathcal{G}_t\right]-\nabla_\theta\phi(\bar{x},\bar{\theta}_t)+\nabla_\theta\phi(\bar{x},\bar{\theta}_t)-\nabla_\theta\phi(\bar{x},\theta_*)\right\|^2\\ &\leq\frac{2\ell^2}{n}\left(\left\|\mathbf{X}-\bar{x}\mathbf{1}_n^\top\right\|^2+\left\|\mathbf{\Theta}_t-\bar{\theta}_t\mathbf{1}_n^\top\right\|^2\right)+2\ell^2\left\|\bar{\theta}_t-\theta_*\right\|^2.\end{aligned}$$

Combining all the inequalities above, we obtain

$$\begin{aligned}&\mathbb{E}\left[\|\mathbb{E}\left[\mathbf{H}_{t+1}\mid\mathcal{G}_t\right]-\mathbb{E}\left[\mathbf{H}_t\mid\mathcal{G}_{t-1}\right]\|^2\right]\\ \leq&\ell^2\mathbb{E}\left[8\left\|\mathbf{\Theta}_{t-1}-\bar{\theta}_{t-1}\mathbf{1}^\top\right\|^2+2\gamma^2\left\|\mathbf{U}_t-\bar{u}_t\mathbf{1}_n^\top\right\|^2+2\gamma^2\left\|\bar{u}_t\mathbf{1}_n^\top\right\|^2\right]\\ \leq&8\ell^2\mathbb{E}\left[\left\|\mathbf{\Theta}_{t-1}-\bar{\theta}_{t-1}\mathbf{1}^\top\right\|^2\right]+2\ell^2\gamma^2\mathbb{E}\left[\left\|\mathbf{U}_t-\bar{u}_t\mathbf{1}_n^\top\right\|^2\right]+2\ell^2\gamma^2\left(\sigma^2+n\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{h}_t\mid\mathcal{G}_t\right]\right\|^2\right]\right)\\ \leq&(8\ell^2+4\ell^4\gamma^2)\mathbb{E}\left[\left\|\mathbf{\Theta}_{t-1}-\bar{\theta}_{t-1}\mathbf{1}^\top\right\|^2\right]+2\ell^2\gamma^2\mathbb{E}\left[\left\|\mathbf{U}_t-\bar{u}_t\mathbf{1}_n^\top\right\|^2\right]+4\ell^4\gamma^2\mathbb{E}\left[\left\|\mathbf{X}-\bar{x}\mathbf{1}_n^\top\right\|^2\right]\\ &+4\ell^4\gamma^2n\mathbb{E}\left[\left\|\bar{\theta}_{t-1}-\theta_*\right\|^2\right]+2\ell^2\gamma^2\sigma^2.\end{aligned}$$

and thus

$$\begin{aligned}&\mathbb{E}\left[\left\|\mathbf{U}_{t+1}-\bar{u}_{t+1}\mathbf{1}_n^\top\right\|^2\right]\\ \leq&\frac{1+\rho^2}{2}\mathbb{E}\left[\left\|\mathbf{U}_t-\bar{u}_t\mathbf{1}_n^\top\right\|^2\right]+\frac{1+\rho^2}{1-\rho^2}\mathbb{E}\left[\|\mathbf{H}_{t+1}-\mathbf{H}_t\|^2\right]\\ \leq&\frac{1+\rho^2}{2}\mathbb{E}\left[\left\|\mathbf{U}_t-\bar{u}_t\mathbf{1}_n^\top\right\|^2\right]+\frac{1+\rho^2}{1-\rho^2}\left(6n\sigma^2+3\left\{(8\ell^2+4\ell^4\gamma^2)\mathbb{E}\left[\left\|\mathbf{\Theta}_{t-1}-\bar{\theta}_{t-1}\mathbf{1}^\top\right\|^2\right]\right.\right.\\ &\left.\left.+2\ell^2\gamma^2\mathbb{E}\left[\left\|\mathbf{U}_t-\bar{u}_t\mathbf{1}_n^\top\right\|^2\right]+4\ell^4\gamma^2\mathbb{E}\left[\left\|\mathbf{X}-\bar{x}\mathbf{1}_n^\top\right\|^2\right]+4\ell^4\gamma^2n\mathbb{E}\left[\left\|\bar{\theta}_{t-1}-\theta_*\right\|^2\right]+2\ell^2\gamma^2\sigma^2\right\}\right)\\ =&\left(\frac{1+\rho^2}{2}+\frac{6\ell^2\gamma^2(1+\rho^2)}{1-\rho^2}\right)\mathbb{E}\left[\left\|\mathbf{U}_t-\bar{u}_t\mathbf{1}_n^\top\right\|^2\right]+\frac{3(1+\rho^2)(8\ell^2+4\ell^4\gamma^2)}{1-\rho^2}\mathbb{E}\left[\left\|\mathbf{\Theta}_{t-1}-\bar{\theta}_{t-1}\mathbf{1}^\top\right\|^2\right]\\ &+\frac{1+\rho^2}{1-\rho^2}\left(12\ell^4\gamma^2\mathbb{E}\left[\left\|\mathbf{X}-\bar{x}\mathbf{1}_n^\top\right\|^2\right]+12n\ell^4\gamma^2\mathbb{E}\left[\left\|\bar{\theta}_{t-1}-\theta_*\right\|^2\right]+6(n+\ell^2\gamma^2)\sigma^2\right).\end{aligned}$$

The third inequality (16c) holds by noticing that $\gamma<\frac{1}{\ell}$. We have completed the proof. □

Based on Lemma B.1, after T-steps, Algorithm 2 achieves the following result.

Lemma B.2. *Under the same conditions as Lemma B.1. Suppose the stepsize γ satisfies*

$$\gamma\leq\mathcal{O}\left(\min\left\{\frac{1-\rho^2}{\ell},\frac{(1-\rho^2)\sqrt{\mu/\ell}}{\ell},\frac{(1-\rho^2)^2}{\ell}\right\}\right). \tag{24}$$

Define the constants

$$e_\theta = 1-\gamma\mu,\ e_{\rho,1} = \frac{\rho^2+1}{2},\ e_{\rho,2} = \frac{\rho^2+3}{4}$$

$$C_{x,1} = \left(\frac{\ell^2}{\mu} + \frac{\gamma^4\ell^6}{\mu(1-\rho^2)^4}\right), C_{x,2} = \frac{\ell^4}{(1-\rho^2)^4}\left(\frac{\ell^2}{\mu^2}+1\right), C_{x,3} = \frac{\left(\frac{\ell^2}{\mu^2}+1\right)\ell^4}{(1-\rho^2)^2},$$

$$C_{\sigma,1} = \left(\frac{\gamma\ell^2 n}{\mu\left(1-\rho^2\right)^4}+1\right), C_{\sigma,2} = \frac{\gamma^3\ell^4}{\mu(1-\rho^2)^4} + \frac{n}{(1-\rho^2)^4}, C_{\sigma,3} = \frac{1}{(1-\rho^2)^2}\left(\frac{\gamma^3\ell^4}{n\mu}+1\right)$$

then consider Algorithm 2, for any $T \geq 1$, we have

$$\mathbb{E}\left[\left\|\bar{\theta}_T - \theta_*\right\|^2\right] \leq e_\theta^T\left(1+\frac{\gamma^4\ell^6}{\mu^2(1-\rho^2)^4}\right)\mathbb{E}\left[\left\|\bar{\theta}_0-\theta_*\right\|^2\right] + \frac{e_\theta^{T-1}\gamma\ell^2}{\mu(1-\rho^2)}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{\Theta}_0 - \bar{\theta}_0\mathbf{1}_n^\top\right\|^2\right]$$
$$+ \frac{e_\theta^{T-1}\gamma^3\ell^2}{\mu(1-\rho^2)^3 n}\mathbb{E}\left[\left\|\mathbf{U}_1 - \bar{u}_1\mathbf{1}_n^\top\right\|^2\right] + \min\left(T, \frac{1}{\mu\gamma}\right)\left(\frac{C_{x,1}\gamma}{n}\mathbb{E}\left[\left\|\mathbf{X} - \bar{x}\mathbf{1}_n^\top\right\|^2\right] + \frac{C_{\sigma,1}}{n}\gamma^2\sigma^2\right);$$

$$\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{\Theta}_T - \bar{\theta}_T\mathbf{1}_n^\top\right\|^2\right] \leq e_{\rho,1}^T\left(1+\frac{\gamma^2\ell^2}{(1-\rho^2)^4}\right)\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{\Theta}_0 - \bar{\theta}_0\mathbf{1}_n^\top\right\|^2\right] + \frac{e_{\rho,1}^{T-1}\gamma^2}{(1-\rho^2)^2}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_1 - \bar{u}_1\mathbf{1}_n^\top\right\|^2\right]$$
$$+ \frac{e_{\rho,1}^{T-1}\gamma^3\ell^4}{\mu(1-\rho^2)^3}\mathbb{E}\left[\left\|\bar{\theta}_0-\theta_*\right\|^2\right] + \frac{C_{x,2}\gamma^4}{n}\mathbb{E}\left[\left\|X - \bar{x}\mathbf{1}_n^\top\right\|^2\right] + \frac{C_{\sigma,2}\gamma^2\sigma^2}{n};$$

and

$$\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_{T+1} - \bar{u}_{T+1}\mathbf{1}_n^\top\right\|^2\right] \leq e_{\rho,2}^T\left(1+\frac{\ell^2\gamma^2}{(1-\rho^2)^4}\right)\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_1 - \bar{u}_1\mathbf{1}_n^\top\right\|^2\right] + \frac{\ell^2 e_{\rho,2}^{T-1}}{(1-\rho^2)^3}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{\Theta}_0 - \bar{\theta}_0\mathbf{1}_n^\top\right\|^2\right]$$
$$+ \frac{e_{\rho,2}^{T-1}\ell^4\gamma}{\mu(1-\rho^2)}\mathbb{E}\left[\left\|\bar{\theta}_0-\theta_*\right\|^2\right] + \frac{C_{x,3}\gamma^2}{n}\mathbb{E}\left[\left\|\mathbf{X} - \bar{x}\mathbf{1}_n^\top\right\|^2\right] + C_{\sigma,3}\sigma^2.$$

Proof. (of Lemma B.2) We define the vector function Ω_t

$$\Omega_t = \left(\mathbb{E}\left[\left\|\bar{\theta}_t - \theta_*\right\|^2\right], \frac{1}{n}\mathbb{E}\left[\left\|\mathbf{\Theta}_t - \bar{\theta}_t\mathbf{1}_n^\top\right\|^2\right], \frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_{t+1} - \bar{u}_{t+1}\mathbf{1}_n^\top\right\|^2\right]\right)^\top \tag{25}$$

and an 3×3 matrix M

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \tag{26}$$

where

$$M_{11} = 1-\gamma\mu; \quad M_{12} = \frac{\gamma\ell^2}{\mu}; \quad M_{13} = 0$$
$$M_{21} = 0; \quad M_{22} = \frac{1+\rho^2}{2}; \quad M_{23} = \frac{\gamma^2}{1-\rho^2}$$
$$M_{31} = \frac{\ell^4\gamma^2}{(1-\rho^2)}; \quad M_{32} = \frac{\ell^2}{(1-\rho^2)}; \quad M_{33} = \frac{1+\rho^2}{2} + \frac{6\ell^2\gamma^2(1+\rho^2)}{1-\rho^2}. \tag{27}$$

By the results of Lemma B.1, for any t, we have

$$\Omega_{t+1} \leq M\Omega_t + \tilde{C} \tag{28}$$

where

$$\tilde{C} = \left(\frac{\gamma\ell^2}{\mu n}\mathbb{E}\left[\left\|X - \bar{x}\mathbf{1}_n^\top\right\|^2\right] + \frac{\gamma^2\sigma^2}{n}, 0, \frac{\ell^4\gamma^2}{(1-\rho^2)n}\mathbb{E}\left[\left\|\mathbf{X} - \bar{x}\mathbf{1}_n^\top\right\|^2\right] + \frac{\sigma^2}{(1-\rho^2)}\right)^T. \tag{29}$$

Note that we omit the constant factor to simplify the definitions of matrix M and $\tilde{C}$. For sufficient small stepsize $\gamma \le \mathcal{O}\left(\frac{1-\rho^2}{\ell}\right)$, we have $M_{33} \le \frac{3+\rho^2}{4}$. For simplicity, we overload the notation and set $\Omega_t = (a_t, b_t, c_t)^\top$ and $\tilde{C} = (d_1, d_2, d_3)^\top$. Note that we have

$$
\begin{aligned}
a_{t+1} &\le M_{11}a_t + M_{12}b_t + d_1 \\
b_{t+1} &\le M_{22}b_t + M_{23}c_t \\
c_{t+1} &\le M_{31}a_t + M_{32}b_t + M_{33}c_t + d_3
\end{aligned}
$$

and thus we apply Lemma A.3 ((14) to a_t and (15) to a_t, b_t, c_t) to get

$$
a_{t+1} \le M_{11}^{t+1}a_0 + M_{12}M_{11}^t \sum_{i=0}^t \frac{b_i}{M_{11}^i} + M_{11}^t \sum_{i=0}^t \frac{d_1}{M_{11}^i} \tag{a*}
$$

$$
\sum_{i=0}^t \frac{a_i}{M_{11}^i} \le \frac{1}{1-M_{11}}\left(a_0 + M_{12}\left(\sum_{i=0}^t \frac{b_i}{M_{11}^i}\right) + \left(\sum_{i=0}^t \frac{d_1}{M_{11}^i}\right)\right) \tag{a}
$$

$$
\sum_{i=0}^t \frac{b_i}{M_{11}^i} \le \frac{1}{1-M_{22}}\left(b_0 + M_{23}\left(\sum_{i=0}^t \frac{c_i}{M_{11}^i}\right)\right) \tag{b}
$$

$$
\sum_{i=0}^t \frac{c_i}{M_{11}^i} \le \frac{1}{1-M_{33}}\left(c_0 + M_{31}\left(\sum_{i=0}^t \frac{a_i}{M_{11}^i}\right) + M_{32}\left(\sum_{i=0}^t \frac{b_i}{M_{11}^i}\right) + \left(\sum_{i=0}^t \frac{d_3}{M_{11}^i}\right)\right) \tag{c}
$$

$$
\begin{aligned}
\sum_{i=0}^t \frac{c_i}{M_{11}^i} \le{}& \frac{c_0}{1-M_{33}} + \frac{M_{31}}{1-M_{33}}\frac{a_0}{1-M_{11}} + \frac{M_{31}}{1-M_{33}}\frac{M_{12}}{1-M_{11}}\left(\sum_{i=0}^t \frac{b_i}{M_{11}^i}\right) \\
&+ \frac{M_{32}}{1-M_{33}}\left(\sum_{i=0}^t \frac{b_i}{M_{11}^i}\right) + \frac{M_{31}}{1-M_{33}}\frac{1}{1-M_{11}}\left(\sum_{i=0}^t \frac{d_1}{M_{11}^i}\right) + \frac{1}{1-M_{33}}\left(\sum_{i=0}^t \frac{d_3}{M_{11}^i}\right)
\end{aligned} \tag{$\tilde{c}$}
$$

Incorporating (a) into (c) gives ($\tilde{c}$), the coefficient of $\sum_{i=0}^t \frac{b_i}{M_{11}^i}$ in ($\tilde{c}$) is denoted by R_0

$$
R_0 = \frac{M_{31}}{1-M_{33}}\frac{M_{12}}{1-M_{11}} + \frac{M_{32}}{1-M_{33}} \sim \Theta\left(\frac{\ell^6\gamma^2}{(1-\rho^2)^2\mu^2} + \frac{\gamma^2}{(1-\rho^2)}\right) \sim \Theta\left(\frac{\ell^6\gamma^2}{(1-\rho^2)^2\mu^2}\right), \tag{30}
$$

and then doing the operations on the two inequalities $\frac{(1-M_{22})}{M_{23}} \times$ (b) + ($\tilde{c}$) gives

$$
\begin{aligned}
\left(\frac{1-M_{22}}{M_{23}} - R_0\right)\sum_{i=0}^t \frac{b_i}{M_{11}^i} \le{}& \frac{b_0}{M_{23}} + \frac{c_0}{1-M_{33}} + \frac{M_{31}}{1-M_{33}}\frac{a_0}{1-M_{11}} \\
&+ \frac{M_{31}}{1-M_{33}}\frac{1}{1-M_{11}}\left(\sum_{i=0}^t \frac{d_1}{M_{11}^i}\right) + \frac{1}{1-M_{33}}\left(\sum_{i=0}^t \frac{d_3}{M_{11}^i}\right).
\end{aligned} \tag{31}
$$

Let

$$
R_1 = \frac{1-M_{22}}{M_{23}} - R_0 = \Theta\left(\frac{(1-\rho^2)^2}{\gamma^2} - \frac{\ell^6\gamma^2}{(1-\rho^2)^2\mu^2}\right). \tag{32}
$$

For sufficient small stepsize $\gamma \le (1-\rho^2)\sqrt{\mu/\ell}/\ell$, we have $R_1 \ge \frac{(1-\rho^2)^2}{2\gamma^2}$. Then

$$
\begin{aligned}
\sum_{i=0}^t \frac{b_i}{M_{11}^i} \le{}& \frac{1}{R_1}\left(\frac{b_0}{M_{23}} + \frac{c_0}{1-M_{33}} + \frac{M_{31}}{1-M_{33}}\frac{a_0}{1-M_{11}}\right) \\
&+ \frac{1}{R_1}\left(\frac{M_{31}}{1-M_{33}}\frac{1}{1-M_{11}}\left(\sum_{i=0}^t \frac{d_1}{M_{11}^i}\right) + \frac{1}{1-M_{33}}\left(\sum_{i=0}^t \frac{d_3}{M_{11}^i}\right)\right).
\end{aligned} \tag{33}
$$

Then incorporating (33) into (a*), then

$$
a_{t+1} \le M_{11}^{t+1}a_0 + M_{11}^t\frac{M_{12}}{R_1}\left(\frac{M_{31}}{1-M_{33}}\frac{1}{1-M_{11}}\left(\sum_{i=0}^t \frac{d_1}{M_{11}^i}\right) + \frac{1}{1-M_{33}}\left(\sum_{i=0}^t \frac{d_3}{M_{11}^i}\right)\right)
$$

$$+ M_{11}^t \frac{M_{12}}{R_1} \left(\frac{b_0}{M_{23}} + \frac{c_0}{1 - M_{33}} + \frac{M_{31}}{1 - M_{33}} \frac{a_0}{1 - M_{11}} \right) + M_{11}^t \sum_{i=0}^{t} \frac{d_1}{M_{11}^i}. \tag{34}$$

Incorporating the definitions of a_t, M and d_1, d_3, we have

$$\begin{aligned}
&\mathbb{E}\left[\left\|\bar{\theta}_T - \theta_*\right\|^2\right] \\
&\le (1-\mu\gamma)^T \mathbb{E}\left[\left\|\bar{\theta}_0 - \theta_*\right\|^2\right] + \min\left(T, \frac{1}{\mu\gamma}\right)\left(\frac{\gamma\ell^2}{\mu n}\mathbb{E}\left[\left\|X - \bar{x}\mathbf{1}_n^\top\right\|^2\right] + \frac{\gamma^2\sigma^2}{n}\right) \\
&+ \min\left(T, \frac{1}{\mu\gamma}\right) \frac{\gamma^3\ell^2}{\mu\left(1-\rho^2\right)^3}\left(\frac{\ell^4\gamma^2}{(1-\rho^2)n}\mathbb{E}\left[\left\|\mathbf{X} - \bar{x}\mathbf{1}_n^\top\right\|^2\right] + \frac{\sigma^2}{(1-\rho^2)}\right) \\
&+ (1-\mu\gamma)^{T-1}\left(\frac{\gamma\ell^2}{\mu(1-\rho^2)n}\mathbb{E}\left[\left\|\mathbf{\Theta}_0 - \bar{\theta}_0\mathbf{1}_n^\top\right\|^2\right] + \frac{\gamma^3\ell^2}{\mu(1-\rho^2)^3 n}\mathbb{E}\left[\left\|\mathbf{U}_1 - \bar{u}_1\mathbf{1}_n^\top\right\|^2\right]\right) \\
&+ (1-\mu\gamma)^{T-1} \frac{\gamma^4\ell^6}{\mu^2(1-\rho^2)^4}\mathbb{E}\left[\left\|\bar{\theta}_0 - \theta_*\right\|^2\right].
\end{aligned} \tag{35}$$

Following the same process for sequence a_t, we may achieve the estimation for b_t. We apply Lemma A.3 ((14) to b_t and (15) to a_t, c_t) to get

$$b_{t+1} \le M_{22}^{t+1} b_0 + M_{23} M_{22}^t \sum_{i=0}^{t} \frac{c_i}{M_{22}^i} \tag{b*}$$

$$\sum_{i=0}^{t} \frac{b_i}{M_{22}^i} \le \frac{1}{1-M_{22}}\left(b_0 + M_{23}\sum_{i=0}^{t}\frac{c_i}{M_{22}^i}\right) \tag{b'}$$

$$\sum_{i=0}^{t} \frac{a_i}{M_{22}^i} \le \frac{1}{1-M_{11}}\left(a_0 + M_{12}\sum_{i=0}^{t}\frac{b_i}{M_{22}^i} + \sum_{i=0}^{t}\frac{d_1}{M_{22}^i}\right) \tag{a'}$$

$$\sum_{i=0}^{t} \frac{c_i}{M_{22}^i} \le \frac{1}{1-M_{33}}\left(c_0 + M_{31}\left(\sum_{i=0}^{t}\frac{a_i}{M_{22}^i}\right) + M_{32}\left(\sum_{i=0}^{t}\frac{b_i}{M_{22}^i}\right) + \left(\sum_{i=0}^{t}\frac{d_3}{M_{22}^i}\right)\right). \tag{c'}$$

Firstly, we incorporate (a') into (c') and get that

$$\begin{aligned}
\sum_{i=0}^{t} \frac{c_i}{M_{22}^i} \le &\left(\frac{M_{31}}{1-M_{33}}\frac{a_0}{1-M_{11}} + \frac{M_{31}}{1-M_{33}}\frac{M_{12}}{1-M_{11}}\sum_{i=0}^{t}\frac{b_i}{M_{22}^i} + \frac{M_{31}}{1-M_{33}}\frac{1}{1-M_{11}}\sum_{i=0}^{t}\frac{d_1}{M_{22}^i}\right) \\
&+ \frac{c_0}{1-M_{33}} + \frac{M_{32}}{1-M_{33}}\left(\sum_{i=0}^{t}\frac{b_i}{M_{22}^i}\right) + \frac{1}{1-M_{33}}\left(\sum_{i=0}^{t}\frac{d_3}{M_{22}^i}\right).
\end{aligned} \tag{36}$$

Let

$$R_3 = \frac{M_{31}}{1-M_{33}}\frac{M_{12}}{1-M_{11}} + \frac{M_{32}}{1-M_{33}} \sim \Theta\left(\frac{\ell^6\gamma^2}{\mu^2(1-\rho^2)^2} + \frac{\ell^2}{(1-\rho^2)^2}\right) \sim \Theta\left(\frac{\ell^2}{(1-\rho^2)^2}\right), \tag{37}$$

then we do the operations $R_3 \times$ (b') + (36), we have

$$\begin{aligned}
\left(1 - R_3\frac{M_{23}}{1-M_{22}}\right)\sum_{i=0}^{t}\frac{c_i}{M_{22}^i} \le &\left(\frac{M_{31}}{1-M_{33}}\frac{a_0}{1-M_{11}} + \frac{M_{31}}{1-M_{33}}\frac{1}{1-M_{11}}\sum_{i=0}^{t}\frac{d_1}{M_{22}^i}\right) \\
&+ \frac{R_3 b_0}{1-M_{22}} + \frac{c_0}{1-M_{33}} + \frac{1}{1-M_{33}}\left(\sum_{i=0}^{t}\frac{d_3}{M_{22}^i}\right).
\end{aligned} \tag{38}$$

We can select sufficient small step-size $\gamma \le (1-\rho^2)^2/\ell$ such that the coefficient $1 - R_3\frac{M_{23}}{1-M_{22}} \ge 1/2$. Incorporating the above inequality to (b*) get that

$$b_{t+1} \le M_{22}^{t+1} b_0 + 2M_{22}^t M_{23}\left(\frac{R_3 b_0}{1-M_{22}} + \frac{c_0}{1-M_{33}} + \frac{M_{31}}{1-M_{33}}\frac{1}{1-M_{11}}\left(a_0 + \sum_{i=0}^{t}\frac{d_1}{M_{22}^i}\right)\right)$$

$$+ \frac{2M_{22}^t M_{23}}{1-M_{33}} \left(\sum_{i=0}^{t} \frac{d_3}{M_{22}^i} \right), \tag{39}$$

then we thus substitute the definitions b_t, M, d_1, d_2 into the above inequality:

$$\begin{aligned}
&\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{\Theta}_T - \bar{\theta}_T \mathbf{1}_n^\top\right\|^2\right] \\
\leq &\left(\frac{\rho^2+1}{2}\right)^T \frac{1}{n}\mathbb{E}\left[\left\|\mathbf{\Theta}_0 - \bar{\theta}_0 \mathbf{1}_n^\top\right\|^2\right] \\
&+ \min\left(T, \frac{2}{1-\rho^2}\right) \frac{\gamma^3\ell^4}{\mu(1-\rho^2)^3}\left(\frac{\gamma\ell^2}{\mu n}\mathbb{E}\left[\left\|\mathbf{X} - \bar{x}\mathbf{1}_n^\top\right\|^2\right] + \frac{\gamma^2\sigma^2}{n}\right) \\
&+ \min\left(T, \frac{4}{1-\rho^2}\right) \frac{\gamma^2}{(1-\rho^2)^2}\left(\frac{\ell^4\gamma^2}{(1-\rho^2)n}\mathbb{E}\left[\left\|\mathbf{X} - \bar{x}\mathbf{1}_n^\top\right\|^2\right] + \frac{\sigma^2}{(1-\rho^2)}\right) \\
&+ \left(\frac{\rho^2+1}{2}\right)^T \left(\frac{\gamma^2\ell^2}{(1-\rho^2)^4 n}\mathbb{E}\left[\left\|\mathbf{\Theta}_0 - \bar{\theta}_0 \mathbf{1}_n^\top\right\|^2\right] + \frac{\gamma^2}{(1-\rho^2)^2 n}\mathbb{E}\left[\left\|\mathbf{U}_1 - \bar{u}_1 \mathbf{1}_n^\top\right\|^2\right]\right) \\
&+ \left(\frac{\rho^2+1}{2}\right)^T \frac{\gamma^3\ell^4}{\mu(1-\rho^2)^3}\mathbb{E}\left[\left\|\bar{\theta}_0 - \theta_*\right\|^2\right].
\end{aligned}$$

Since for the sequence c_t, we have the recursive formulation $c_{t+1} \leq M_{33}c_t + M_{31}a_t + M_{32}b_t + d_3$. Applying Lemma A.3 ((14) to c_t and (15) to a_t, b_t, c_t), we have

$$\sum_{i=0}^{t} \frac{a_i}{M_{33}^i} \leq \frac{1}{1-M_{11}}\left(a_0 + M_{12}\sum_{i=0}^{t}\frac{b_i}{M_{33}^i} + \sum_{i=0}^{t}\frac{d_1}{M_{33}^i}\right) \tag{a”}$$

$$\sum_{i=0}^{t} \frac{b_i}{M_{33}^i} \leq \frac{1}{1-M_{22}}\left(b_0 + M_{23}\sum_{i=0}^{t}\frac{c_i}{M_{33}^i}\right) \tag{b”}$$

$$\sum_{i=0}^{t} \frac{c_i}{M_{33}^i} \leq \frac{1}{1-M_{33}}\left(c_0 + M_{31}\sum_{i=0}^{t}\frac{a_i}{M_{33}^i} + M_{32}\sum_{i=0}^{t}\frac{b_i}{M_{33}^i} + \sum_{i=0}^{t}\frac{d_3}{M_{33}^i}\right) \tag{c”}$$

$$c_{t+1} \leq M_{33}^{t+1}c_0 + M_{31}M_{33}^t\sum_{i=0}^{t}\frac{a_i}{M_{33}^i} + M_{32}M_{33}^t\sum_{i=0}^{t}\frac{b_i}{M_{33}^i} + M_{33}^t\sum_{i=0}^{t}\frac{d_3}{M_{33}^i}. \tag{$c*$}$$

Incorporating (a”) into ($c*$) gives

$$\begin{aligned}
c_{t+1} \leq M_{33}^{t+1}c_0 + \frac{M_{31}M_{33}^t}{1-M_{11}}\sum_{i=0}^{t}\left(a_0 + \sum_{i=0}^{t}\frac{d_1}{M_{33}^i}\right) + \left(M_{32} + \frac{M_{31}M_{12}}{1-M_{11}}\right)M_{33}^t\sum_{i=0}^{t}\frac{b_i}{M_{33}^i} \\
+ M_{33}^t\sum_{i=0}^{t}\frac{d_3}{M_{33}^i}.
\end{aligned} \tag{40}$$

Then incorporating (a”) into (c”), we have

$$\begin{aligned}
\sum_{i=0}^{t}\frac{c_i}{M_{33}^i} \leq \frac{1}{1-M_{33}}\left(c_0 + \frac{M_{31}a_0}{1-M_{11}} + \left(\frac{M_{31}M_{12}}{1-M_{11}} + M_{32}\right)\sum_{i=0}^{t}\frac{b_i}{M_{33}^i} + \frac{M_{31}}{1-M_{11}}\sum_{i=0}^{t}\frac{d_1}{M_{33}^i}\right) \\
+ \frac{1}{1-M_{33}}\sum_{i=0}^{t}\frac{d_3}{M_{33}^i}.
\end{aligned} \tag{41}$$

Combining (41) and (b”) and doing the operations $\frac{1-M_{22}}{M_{23}} \times$ (b”) + (41) gives:

$$\begin{aligned}
&\left(\frac{1-M_{22}}{M_{23}} - \frac{1}{1-M_{33}}\left(\frac{M_{31}M_{12}}{1-M_{11}} + M_{32}\right)\right)\sum_{i=0}^{t}\frac{b_i}{M_{33}^i} \\
\leq &\frac{b_0}{M_{23}} + \frac{1}{1-M_{33}}\left(c_0 + \frac{M_{31}a_0}{1-M_{11}} + \frac{M_{31}}{1-M_{11}}\sum_{i=0}^{t}\frac{d_1}{M_{33}^i} + \sum_{i=0}^{t}\frac{d_3}{M_{33}^i}\right).
\end{aligned} \tag{42}$$

Define R_5, R_6 and for sufficient small stepsize $\gamma \leq \min\left((1-\rho^2)/\ell, (1-\rho^2)(\mu/\ell)^{1/3}/\ell\right)$

$$R_5 = \frac{M_{31}M_{12}}{1-M_{11}} + M_{32} \sim \Theta\left(\frac{\ell^2}{1-\rho^2}\right),$$
$$R_6 = \frac{1-M_{22}}{M_{23}} - \frac{1}{1-M_{33}}\left(\frac{M_{31}M_{12}}{1-M_{11}} + M_{32}\right) \sim \Theta\left(\frac{(1-\rho^2)^2}{\gamma^2}\right).$$

Thus

$$\sum_{i=0}^{t} \frac{b_i}{M_{33}^i} \leq \frac{1}{R_6}\frac{b_0}{M_{23}} + \frac{1}{R_6}\frac{1}{1-M_{33}}\left(c_0 + \frac{M_{31}a_0}{1-M_{11}} + \frac{M_{31}}{1-M_{11}}\sum_{i=0}^{t}\frac{d_1}{M_{33}^i} + \sum_{i=0}^{t}\frac{d_3}{M_{33}^i}\right).$$

Applying the above inequality into (40) gives

$$\begin{aligned}
& c_{t+1} \\
\leq & M_{33}^{t+1}c_0 + \frac{M_{31}M_{33}^t}{1-M_{11}}\sum_{i=0}^{t}\left(a_0 + \sum_{i=0}^{t}\frac{d_1}{M_{33}^i}\right) + M_{33}^t\sum_{i=0}^{t}\frac{d_3}{M_{33}^i} \\
& + \left(M_{32} + \frac{M_{31}M_{12}}{1-M_{11}}\right)\frac{M_{33}^t}{R_6}\left(\frac{b_0}{M_{23}} + \frac{1}{1-M_{33}}\left(c_0 + \frac{M_{31}a_0}{1-M_{11}} + \frac{M_{31}}{1-M_{11}}\sum_{i=0}^{t}\frac{d_1}{M_{33}^i} + \sum_{i=0}^{t}\frac{d_3}{M_{33}^i}\right)\right) \\
\leq & M_{33}^{t+1}c_0 + \frac{M_{31}M_{33}^t a_0}{1-M_{11}} + R_5\frac{M_{33}^t}{R_6}\left(\frac{b_0}{M_{23}} + \frac{1}{1-M_{33}}\left(c_0 + \frac{M_{31}a_0}{1-M_{11}}\right)\right) \\
& + M_{33}^t\left(\frac{M_{31}}{1-M_{11}} + \frac{R_5}{R_6}\frac{M_{31}}{1-M_{11}}\right)\sum_{i=0}^{t}\frac{d_1}{M_{33}^i} + \left(1 + \frac{R_5}{R_6}\frac{1}{1-M_{33}}\right)M_{33}^t\sum_{i=0}^{t}\frac{d_3}{M_{33}^i}.
\end{aligned} \tag{43}$$

We thus substitute the value of c_t, M, d_1, d_2 and get the simplified result

$$\begin{aligned}
& \frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_{T+1} - \bar{u}_{T+1}\mathbf{1}_n^\top\right\|^2\right] \\
\leq\ & e_{\rho,2}^T\left(1 + \frac{\ell^2\gamma^2}{(1-\rho^2)^4}\right)\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_1 - \bar{u}_1\mathbf{1}_n^\top\right\|^2\right] + \frac{\ell^2 e_{\rho,2}^{T-1}}{(1-\rho^2)^3}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{\Theta}_0 - \bar{\theta}_0\mathbf{1}_n^\top\right\|^2\right] \\
& + \frac{e_{\rho,2}^{T-1}\ell^4\gamma}{\mu(1-\rho^2)}\mathbb{E}\left[\left\|\bar{\theta}_0 - \theta_*\right\|^2\right] + \left(\frac{\ell^2}{\mu^2} + 1\right)\frac{\gamma^2\ell^4}{(1-\rho^2)^2 n}\mathbb{E}\left[\left\|\mathbf{X} - \bar{x}\mathbf{1}_n^\top\right\|^2\right] \\
& + \left(\frac{\gamma^3\ell^4}{n\mu} + 1\right)\frac{\sigma^2}{(1-\rho^2)^2}.
\end{aligned}$$

The proof is complete. □

To get the recursive result of the T-step inner-loop, we need to carefully estimate $\mathbb{E}\left[\left\|\mathbf{U}_1 - \bar{u}_1\mathbf{1}_n^\top\right\|^2\right]$.

Remark. Note that by (21) we have

$$\mathbb{E}\left[\left\|\mathbf{U}_1 - \bar{u}_1\mathbf{1}_n^\top\right\|^2\right] \leq \mathbb{E}\left[\left\|\mathbf{U}_0 - \bar{u}_0\mathbf{1}_n^\top\right\|^2\right] + \frac{1}{1-\rho^2}\mathbb{E}\left[\left\|\mathbf{H}_1 - \mathbf{H}_0\right\|^2\right].$$

Following (22) and (23), for

$$z_{s+1}^{(i)}, u_{s+1,z}^{(i)}, h_{s+1,z}^{(i)} = \texttt{Inner Loop}(z_s^{(i)}, \eta_z, g_i(x_s^{(i)}, \cdot), u_{s,z}^{(i)}, h_{s,z}^{(i)}, T)$$

we know that $\mathbf{H}_0$ is initialized by $\mathbf{H}_{T+1}$, the output in the previous T-steps inner-loop update (see Algorithm 2). Hence, we know

$$\begin{aligned}
\mathbb{E}\left[\left\|\mathbf{H}_1 - \mathbf{H}_0\right\|^2\right] \leq & 6n\sigma_z^2 + 3\sum_{i=1}^{n}\mathbb{E}\left[\left\|\nabla_y g_i(x_s^{(i)}, z_s^{(i)}) - \nabla_y g_i(x_{s-1}^{(i)}, z_s^{(i)})\right\|^2\right] \\
\leq & 6n\sigma_z^2 + 9\ell_{g,1}^2\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2 + \left\|\mathbf{X}_{s-1} - \bar{x}_{s-1}\mathbf{1}_n^\top\right\|^2 + n\left\|\bar{x}_s - \bar{x}_{s-1}\right\|^2\right].
\end{aligned}$$

Combining the above conclusions we know for $s \geq 1$

$$\mathbb{E}\left[\left\|\mathbf{U}_1 - \bar{u}_1\mathbf{1}_n^\top\right\|^2\right] \leq \mathbb{E}\left[\left\|\mathbf{U}_0 - \bar{u}_0\mathbf{1}_n^\top\right\|^2\right] + \frac{6n\sigma_z^2}{1-\rho^2} + \frac{9\ell_{g,1}^2}{1-\rho^2}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2 + \left\|\mathbf{X}_{s-1} - \bar{x}_{s-1}\mathbf{1}_n^\top\right\|^2 + n\left\|\bar{x}_s - \bar{x}_{s-1}\right\|^2\right]. \tag{44}$$

For $s = 0$, we provide a more careful estimation. Following (22) and (23), for

$$z_1^{(i)}, u_{1,z}^{(i)}, h_{1,z}^{(i)} = \texttt{Inner Loop}(z_0^{(i)}, \eta_z, g_i(x_0^{(i)}, \cdot), u_{0,z}^{(i)}, h_{0,z}^{(i)}, T)$$

We know $H_0 = 0$, thus

$$\begin{aligned}\mathbb{E}\left[\|\mathbf{H}_1 - \mathbf{H}_0\|^2\right] &= \mathbb{E}\left[\|\mathbf{H}_1\|^2\right] = \sum_{i=1}^n \mathbb{E}\left[\left\|\nabla_y g_i(x_0^{(i)}, z_0^{(i)}; \xi_j)\right\|^2\right] \\ &\leq 2n\sigma_z^2 + 2\sum_{i=1}^n \mathbb{E}\left[\left\|\nabla_y g_i(x_0^{(i)}, z_0^{(i)})\right\|^2\right] := 2n(\sigma_z^2 + \ell_{f,0}^2).\end{aligned} \tag{45}$$

C Appendix / Consensus and convergence analysis for Y, Z, X

As a direct result of Lemma B.2, we first get the estimations for the consensus of y and z.

Lemma C.1. *Suppose Assumptions 1, 2, 3, and 4 hold, we have the following estimations for the consensus of y and z:*

$$\begin{aligned}\frac{1}{2n}\sum_{s=0}^S \mathbb{E}\left[\left\|\mathbf{Z}_s - z_{*,s}\mathbf{1}_n^\top\right\|^2\right] \leq\ & C_{z_*,0}\Delta_{z_*,0} + C_{Z,0}\Delta_{Z,0} + C_{U_z,0}\Delta_{U_z,0} + C_{z,v}\sum_{s=0}^S \mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right] \\ & + C_{z,vs}\sum_{s=0}^S \mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right] + C_{z,x}\frac{1}{n}\sum_{s=0}^S \mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2\right] + S\cdot C_{z,\sigma}\sigma_z^2 \\ \frac{1}{2n}\sum_{s=0}^S \mathbb{E}\left[\left\|\mathbf{Y}_s - y_{*,s}^{\alpha}\mathbf{1}_n^\top\right\|^2\right] \leq\ & C_{y_*,0}\Delta_{y_*,0} + C_{Y,0}\Delta_{Y,0} + C_{U_y,0}\Delta_{U_y,0} + C_{y,v}\sum_{s=0}^S \mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right] \\ & + C_{y,vs}\sum_{s=0}^S \mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right] + C_{y,x}\sum_{s=0}^S \frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2\right] + SC_{y,\sigma}\sigma_y^2\end{aligned}$$

where the constants $C_{z_*,0}, C_{Z,0}, C_{U_z,0}, \Delta_{z_*,0}, \Delta_{Z,0}, \Delta_{U_z,0}, C_{z,v}, C_{z,vs}, C_{z,x}, C_{z,\sigma}$ are defined in (66) and $C_{y_*,0}, C_{Y,0}, C_{U_y,0}, \Delta_{y_*,0}, \Delta_{Y,0}, \Delta_{U_y,0}, C_{y,v}, C_{y,vs}, C_{y,x}, C_{y,\sigma}$ are defined in (87).

Proof. (of Lemma C.1) We train the variables y, z with T-steps b-batch gradient descent for $b \geq 1$. Note that from Algorithm 1 and by Lemma A.7 we know the updates of $\bar{y}_s, \bar{z}_s$ take the form

$$\bar{y}_{s+1} = \bar{y}_s - \eta_y\bar{v}_{s+1,y} = \bar{y}_s - \eta_{s,y}\bar{\delta}_{s+1,y} \tag{46}$$

$$\bar{z}_{s+1} = \bar{z}_s - \eta_z\bar{v}_{s+1,z} = \bar{z}_s - \eta_z\bar{\delta}_{s+1,z}. \tag{47}$$

The variable z is to optimize the objective g_i which is μ_g-strongly convex and $\ell_{g,1}$-smooth. The stochastic gradient $h_{t+1,z}$ of updating z is supposed to be variance-bounded by $\sigma_z^2 = \sigma_g^2$. By Lemma B.2 and Inequality (44), we know if the step-size η_z satisfies that

$$\eta_z \leq \mathcal{O}\left(\min\left\{\frac{1-\rho^2}{\ell_{g,1}}, \frac{(1-\rho^2)\sqrt{\mu_g}}{\ell_{g,1}\sqrt{\ell_{g,1}}}, \frac{(1-\rho^2)^2}{\ell_{g,1}}\right\}\right) \tag{48}$$

then for $s \geq 1$

$$\begin{aligned}&\mathbb{E}\left[\left\|\bar{z}_{s+1} - z_{*,s+1}\right\|^2\right] \\ &\leq (1-\mu_g\eta_z)^T\left(1 + \frac{\eta_z^4\ell_{g,1}^6}{\mu_g^2(1-\rho^2)^4}\right)\mathbb{E}\left[\left\|\bar{z}_s - z_{*,s+1}\right\|^2\right] + (1-\mu_g\eta_z)^{T-1}\frac{\eta_z\ell_{g,1}^2}{\mu_g(1-\rho^2)}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{Z}_s - \bar{z}_s\mathbf{1}_n^\top\right\|^2\right]\end{aligned}$$

$$
\begin{aligned}
&+\min\left(T,\frac{1}{\mu_g\eta_z}\right)\left(\frac{C_{x,1}\eta_z}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]+\frac{C_{\sigma,1}}{n}\eta_z^2\sigma_z^2\right)\\
&+\frac{(1-\mu_g\eta_z)^{T-1}\eta_z^3\ell_{g,1}^2}{\mu_g(1-\rho^2)^3 n}\mathbb{E}\left[\left\|\mathbf{U}_{s,z}-\bar{u}_{s,z}\mathbf{1}_n^\top\right\|^2\right]+(1-\mu_g\eta_z)^{T-1}\frac{\eta_z^3\ell_{g,1}^2}{\mu_g(1-\rho^2)^4}6\sigma_z^2\\
&+\frac{(1-\mu_g\eta_z)^{T-1}\eta_z^3\ell_{g,1}^2}{\mu_g(1-\rho^2)^4}\left(9\ell_{g,1}^2\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2+\left\|\mathbf{X}_{s-1}-\bar{x}_{s-1}\mathbf{1}_n^\top\right\|^2+n\left\|\bar{x}_s-\bar{x}_{s-1}\right\|^2\right]\right)\\
&\lesssim e_z^T\mathbb{E}\left[\left\|\bar{z}_s-z_{*,s+1}\right\|^2\right]+\frac{e_z^{T-1}\eta_z}{1-\rho^2}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{Z}_s-\bar{z}_s\mathbf{1}_n^\top\right\|^2\right]+\frac{e_z^{T-1}\eta_z^3}{(1-\rho^2)^3}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_{s,z}-\bar{u}_{s,z}\mathbf{1}_n^\top\right\|^2\right]\\
&+\min\left(T,\frac{1}{\mu_g\eta_z}\right)\left(\frac{C_{x,1}\eta_z}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]+\frac{C_{\sigma,1}}{n}\eta_z^2\sigma_z^2\right)\\
&+\frac{e_z^{T-1}\eta_z^3}{(1-\rho^2)^4}\left(\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_{s-1}-\bar{x}_{s-1}\mathbf{1}_n^\top\right\|^2+n\left\|\bar{x}_s-\bar{x}_{s-1}\right\|^2\right]\right)
\end{aligned}\tag{49}
$$

where $e_z=1-2\mu_g\eta_z/3$ and for simplicity we choose sufficient small η_z and any $T\geq 1$ such that

$$
\begin{aligned}
(1-\mu_g\eta_z)^T\left(1+\frac{\eta_z^4\ell_{g,1}^6}{\mu_g^2(1-\rho^2)^4}\right)&\leq\left(1-\frac{2\mu_g\eta_z}{3}\right)^T\\
(1-\mu_g\eta_z)^{T-1}\frac{\eta_z^3\ell_{g,1}^4}{\mu_g(1-\rho^2)^4}&\leq\frac{\eta_z^3\ell_{g,1}^4}{\mu_g(1-\rho^2)^4}\leq\min\left(T,\frac{1}{\mu_z\eta_z}\right)C_{x,1}\eta_z\\
(1-\mu_g\eta_z)^{T-1}\frac{\eta_z\ell_{g,1}^2 n}{\mu_g(1-\rho^2)^4}&\leq\frac{\eta_z\ell_{g,1}^2 n}{\mu_g(1-\rho^2)^4}\leq C_{\sigma,1}
\end{aligned}\tag{50}
$$

with constant $C_{x,1}=\mathcal{O}(1)$ and $C_{\sigma,1}=\mathcal{O}(\eta_z n+1)$ for variable z. We also have

$$
\begin{aligned}
\left\|\bar{z}_s-z_{*,s+1}\right\|^2&=\left\|\bar{z}_s-z_{*,s}\right\|^2+\left\|z_{*,s}-z_{*,s+1}\right\|^2+2\left\langle z_{*,s}-\bar{z}_s,z_{*,s+1}-z_{*,s}\right\rangle\\
&\leq\left\|\bar{z}_s-z_{*,s}\right\|^2+\eta_x^2\ell_{z_*}^2\left\|\bar{v}_{s+1}\right\|^2+2\left\langle z_{*,s}-\bar{z}_s,z_{*,s+1}-z_{*,s}\right\rangle
\end{aligned}
$$

When we consider the convergence of the consensus convergence $\mathbf{Z}$ and $\mathbf{U}$, we only need the following inequality

$$
\left\|\bar{z}_s-z_{*,s+1}\right\|^2\leq 2\left\|\bar{z}_s-z_{*,s}\right\|^2+2\left\|z_{*,s}-z_{*,s+1}\right\|^2=2\left\|\bar{z}_s-z_{*,s}\right\|^2+2\eta_x^2\ell_{z_*}^2\left\|\bar{v}_{s+1}\right\|^2.\tag{51}
$$

To ensure convergence of variable $\bar{z}_s$, it is necessary to carefully estimate the cross-term. For any $a_1,a_2>0$, we have

$$
\begin{aligned}
&\mathbb{E}\left[2\left\langle z_{*,s}-\bar{z}_s,z_{*,s+1}-z_{*,s}\right\rangle|\mathcal{F}_s\right]\\
&=\mathbb{E}\left[2\left\langle z_{*,s}-\bar{z}_s,\left\langle\nabla z_*(\bar{x}_s),\bar{x}_{s+1}-\bar{x}_s\right\rangle\right\rangle|\mathcal{F}_s\right]+\mathbb{E}\left[2\left\langle z_{*,s}-\bar{z}_s,z_{*,s+1}-z_{*,s}-\left\langle\nabla z_*(\bar{x}_s),\bar{x}_{s+1}-\bar{x}_s\right\rangle\right\rangle|\mathcal{F}_s\right]\\
&\leq 2\eta_x\ell_{z_*}\left\|\bar{z}_s-z_{*,s}\right\|\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|+\mathbb{E}\left[2\left\|\bar{z}_s-z_{*,s}\right\|\left\|z_{*,s+1}-z_{*,s}-\left\langle\nabla z_*(\bar{x}_s),\bar{x}_{s+1}-\bar{x}_s\right\rangle\right\||\mathcal{F}_s\right]\\
&\leq\eta_x\ell_{z_*}\left(a_1\left\|\bar{z}_s-z_{*,s}\right\|^2+\frac{1}{a_1}\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right)+\ell_{\nabla z_*}\eta_x^2\mathbb{E}\left[\left\|\bar{z}_s-z_{*,s}\right\|\left\|\bar{v}_{s+1}\right\|^2\middle|\mathcal{F}_s\right]\\
&\leq\eta_x\ell_{z_*}\left(a_1\left\|\bar{z}_s-z_{*,s}\right\|^2+\frac{1}{a_1}\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right)+\ell_{\nabla z_*}\eta_x^2\left(\frac{a_2}{2}\mathbb{E}\left[\left\|\bar{z}_s-z_{*,s}\right\|^2\left\|\bar{v}_{s+1}\right\|^2|\mathcal{F}_s\right]+\frac{1}{2a_2}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\middle|\mathcal{F}_s\right]\right)\\
&\leq\eta_x\ell_{z_*}\left(a_1\left\|\bar{z}_s-z_{*,s}\right\|^2+\frac{1}{a_1}\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right)+\ell_{\nabla z_*}\eta_x^2\left(\frac{a_2c_\delta\alpha^2}{2}\left\|\bar{z}_s-z_{*,s}\right\|^2+\frac{1}{2a_2}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\middle|\mathcal{F}_s\right]\right)
\end{aligned}
$$

where the second inequality uses smoothness of $z^*(\cdot)$. Note that here we carefully analyze the cross term by using the method introduced in [6]. Note that this type of analysis utilizes Taylor expansion that leads to better error bound, and can be avoided by using the Moving-Average trick in [8]. Combining the above inequalities, we have

$$
\begin{aligned}
\mathbb{E}\left[\left\|\bar{z}_s-z_{*,s+1}\right\|^2\right]\leq&\left(1+a_1\eta_x\ell_{z_*}+\frac{a_2\ell_{\nabla z_*}c_\delta\eta_x^2\alpha^2}{2}\right)\mathbb{E}\left[\left\|\bar{z}_s-z_{*,s}\right\|^2\right]\\
&+\frac{\eta_x\ell_{z^*}}{a_1}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right]+\eta_x^2\left(\ell_{z_*}^2+\frac{\ell_{\nabla z_*}}{2a_2}\right)\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right]
\end{aligned}\tag{52}
$$

Choosing η_x,η_z and T such that

$$
r_z^T=\left(1+a_1\eta_x\ell_{z_*}+\frac{a_2\ell_{\nabla z_*}c_\delta\eta_x^2\alpha^2}{2}\right)\left(1-\frac{2\mu_g\eta_z}{3}\right)^T\leq\left(1-\frac{\mu_g\eta_z}{3}\right)^T.\tag{53}
$$

Combining (49), (52) and (53) gives

$$
\begin{aligned}
&\mathbb{E}\left[\|\bar{z}_{s+1}-z_{*,s+1}\|^2\right]\\
\lesssim & r_z^T\mathbb{E}\left[\|\bar{z}_s-z_{*,s}\|^2\right]+\frac{e_z^{T-1}\eta_z}{1-\rho^2}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{Z}_s-\bar{z}_s\mathbf{1}_n^\top\right\|^2\right]+\frac{e_z^{T-1}\eta_z^3}{(1-\rho^2)^3}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_{s,z}-\bar{u}_{s,z}\mathbf{1}_n^\top\right\|^2\right]\\
&+e_z^T\frac{\eta_x\ell_{z^*}}{a_1}\mathbb{E}\left[\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\|^2\right]+e_z^T\eta_x^2\left(\ell_{z_*}^2+\frac{\ell_{\nabla z_*}}{2a_2}\right)\mathbb{E}\left[\|\bar{v}_{s+1}\|^2\right]+\frac{e_z^T\eta_z^3\eta_x^2}{(1-\rho^2)^4}\mathbb{E}\left[\|\bar{v}_s\|^2\right]\\
&+\min\left(T,\frac{1}{\mu_g\eta_z}\right)\left(\frac{C_{x,1}\eta_z}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]+\frac{C_{\sigma,1}}{n}\eta_z^2\sigma_z^2\right)+\frac{e_z^T\eta_z^3}{(1-\rho^2)^4}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_{s-1}-\bar{x}_{s-1}\mathbf{1}_n^\top\right\|^2\right].
\end{aligned}\tag{54}
$$

where $e_z=1-2\mu_g\eta_z/3$ and $r_z\le 1-\mu_g\eta_z/3$. Recalling the result of Lemma B.2 for the consensus of z and incorporating Inequalities (44) and (51), we have

$$
\begin{aligned}
&\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{Z}_{s+1}-\bar{z}_{s+1}\mathbf{1}_n^\top\right\|^2\right]\\
\lesssim&\left(1+\frac{\eta_z^2\ell_{g,1}^2}{(1-\rho^2)^4}\right)\frac{e_{\rho,1}^T}{n}\mathbb{E}\left[\left\|\mathbf{Z}_s-\bar{z}_s\mathbf{1}_n^\top\right\|^2\right]+\frac{e_{\rho,1}^T\eta_z^3}{(1-\rho^2)^3}\mathbb{E}\left[\|\bar{z}_s-z_{*,s}\|^2\right]+\frac{e_{\rho,1}^T\eta_z^2}{(1-\rho^2)^2n}\mathbb{E}\left[\left\|\mathbf{U}_{s,z}-\bar{u}_{s,z}\mathbf{1}_n^\top\right\|^2\right]\\
&+\frac{\eta_z^2}{(1-\rho^2)n}\left(C_{x,2}\eta_z^2+\frac{e_{\rho,1}^T}{(1-\rho^2)^2}\right)\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]+\frac{e_{\rho,1}^T\eta_z^2}{(1-\rho^2)^3}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_{s-1}-\bar{x}_{s-1}\mathbf{1}_n^\top\right\|^2\right]\\
&+\eta_z^2\left(\frac{C_{\sigma,2}}{n}+\frac{e_{\rho,1}^T}{(1-\rho^2)^2}\right)\sigma_z^2+\frac{e_{\rho,1}^T\eta_z^2}{(1-\rho^2)^3}\eta_x^2\mathbb{E}\left[\|\bar{v}_s\|^2\right]+\frac{e_{\rho,1}^T\eta_z^3\eta_x^2}{(1-\rho^2)^3}\mathbb{E}\left[\|\bar{v}_{s+1}\|^2\right]
\end{aligned}\tag{55}
$$

for sufficient small $\eta_z\le(1-\rho^2)^3/\ell_{g,1}$ such that $\left(1+\frac{\eta_z^2\ell_{g,1}^2}{(1-\rho^2)^4}\right)e_{\rho,1}\le\frac{3+\rho^2}{4}$, then $\left(1+\frac{\eta_z^2\ell_{g,1}^2}{(1-\rho^2)^4}\right)e_{\rho,1}^T\le e_{\rho,2}^T$ for any $T\ge 1$. Similarly, we recall the result of Lemma B.2 for the consensus convergence $\mathbf{U}$ of z and incorporate Inequalities (44) and (51)

$$
\begin{aligned}
&\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_{s+1,z}-\bar{u}_{s+1,z}\mathbf{1}_n^\top\right\|^2\right]\\
\lesssim& e_{\rho,2}^T\left(1+\frac{\ell_{g,1}^2\eta_z^2}{(1-\rho^2)^4}\right)\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_{s,z}-\bar{u}_{s,z}\mathbf{1}_n^\top\right\|^2\right]+\frac{e_{\rho,2}^{T-1}\eta_z}{(1-\rho^2)}\mathbb{E}\left[\|\bar{z}_s-z_{*,s}\|^2\right]+\frac{e_{\rho,2}^{T-1}}{(1-\rho^2)^3}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{Z}_s-\bar{z}_s\mathbf{1}_n^\top\right\|^2\right]\\
&+\left(C_{x,3}\eta_z^2+\frac{e_{\rho,2}^T}{1-\rho^2}\right)\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]+\frac{e_{\rho,2}^T}{(1-\rho^2)n}\mathbb{E}\left[\left\|\mathbf{X}_{s-1}-\bar{x}_{s-1}\mathbf{1}_n^\top\right\|^2\right]\\
&+\left(C_{\sigma,3}+\frac{e_{\rho,2}^T}{1-\rho^2}\right)\sigma_z^2+\frac{e_{\rho,2}^T\eta_x^2}{1-\rho^2}\mathbb{E}\left[\|\bar{v}_s\|^2\right]+\frac{e_{\rho,2}^{T-1}\eta_z\eta_x^2}{(1-\rho^2)}\mathbb{E}\left[\|\bar{v}_{s+1}\|^2\right]
\end{aligned}\tag{56}
$$

for sufficient small $\eta_z\le(1-\rho^2)^3/\ell_{g,1}$ such that $r_{U,z}=e_{\rho,2}\left(1+\frac{\ell_{g,1}^2\eta_z^2}{(1-\rho^2)^4}\right)\le\frac{4+\rho^2}{5}<1$ for any $T\ge 1$. Here we use the same idea in Lemma B.2 and define the vector function $\Omega_{Z,s}$:

$$
\Omega_{Z,s}=\left(\mathbb{E}\left[\|\bar{z}_s-z_{*,s}\|^2\right],\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{Z}_s-\bar{z}_s\mathbf{1}_n^\top\right\|^2\right],\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_{s,z}-\bar{u}_{s,z}\mathbf{1}_n^\top\right\|^2\right]\right)\tag{57}
$$

and an 3×3 matrix M_Z

$$
M_Z=\begin{pmatrix}M_{11}&M_{12}&M_{13}\\M_{21}&M_{22}&M_{23}\\M_{31}&M_{32}&M_{33}\end{pmatrix}\tag{58}
$$

where

$$
\begin{aligned}
&M_{11}=r_z^T;\quad M_{12}=\frac{e_z^{T-1}\eta_z}{1-\rho^2};\quad M_{13}=\frac{e_z^{T-1}\eta_z^3}{(1-\rho^2)^3}\\
&M_{21}=\frac{e_{\rho,1}^T\eta_z^3}{(1-\rho^2)^3};\quad M_{22}=e_{\rho,2}^T;\quad M_{23}=\frac{e_{\rho,1}^T\eta_z^2}{(1-\rho^2)^2}
\end{aligned}
$$

$$M_{31} = \frac{e_{\rho,2}^{T-1}\eta_z}{(1-\rho^2)}; \quad M_{32} = \frac{e_{\rho,2}^{T-1}}{(1-\rho^2)^3}; \quad M_{33} = r_{U,z}^T. \tag{59}$$

By the above inequalities (54), (55) and (56), we have

$$\Omega_{Z,s+1} \leq M_Z \Omega_{Z,s} + \tilde{C}_{z,s} \tag{60}$$

where $\tilde{C}_{z,s} \in \mathbb{R}^3$ is defined by

$$\begin{aligned}
\tilde{C}_{z,s}[1] = &\min\left(T, \frac{1}{\mu_g \eta_z}\right)\left(\frac{C_{x,1}\eta_z}{n}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2\right] + \frac{C_{\sigma,1}}{n}\eta_z^2\sigma_z^2\right) \\
&+ \frac{e_z^T\eta_z^3}{(1-\rho^2)^4}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_{s-1} - \bar{x}_{s-1}\mathbf{1}_n^\top\right\|^2\right] + e_z^T\frac{\eta_x \ell_{z^*}}{a_1}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right] \\
&+ e_z^T\eta_x^2\left(\ell_{z_*}^2 + \frac{\ell_{\nabla z_*}}{2a_2}\right)\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right] + \frac{e_z^T\eta_z^3\eta_x^2}{(1-\rho^2)^4}\mathbb{E}\left[\left\|\bar{v}_s\right\|^2\right] \\
\tilde{C}_{z,s}[2] = &\frac{\eta_z^2}{(1-\rho^2)n}\left(C_{x,2}\eta_z^2 + \frac{e_{\rho,1}^T}{(1-\rho^2)^2}\right)\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2\right] \\
&+ \frac{e_{\rho,1}^T\eta_z^2}{(1-\rho^2)^3}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_{s-1} - \bar{x}_{s-1}\mathbf{1}_n^\top\right\|^2\right] \\
&+ \eta_z^2\left(\frac{C_{\sigma,2}}{n} + \frac{e_{\rho,1}^T}{(1-\rho^2)^2}\right)\sigma_z^2 + \frac{e_{\rho,1}^T\eta_z^2}{(1-\rho^2)^3}\eta_x^2\mathbb{E}\left[\left\|\bar{v}_s\right\|^2\right] + \frac{e_{\rho,1}^T\eta_z^3\eta_x^2}{(1-\rho^2)^3}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right] \\
\tilde{C}_{z,s}[3] = &\left(C_{x,3}\eta_z^2 + \frac{e_{\rho,2}^T}{1-\rho^2}\right)\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2\right] + \frac{e_{\rho,2}^T}{(1-\rho^2)n}\mathbb{E}\left[\left\|\mathbf{X}_{s-1} - \bar{x}_{s-1}\mathbf{1}_n^\top\right\|^2\right] \\
&+ \left(C_{\sigma,3} + \frac{e_{\rho,2}^T}{1-\rho^2}\right)\sigma_z^2 + \frac{e_{\rho,2}^T\eta_x^2}{1-\rho^2}\mathbb{E}\left[\left\|\bar{v}_s\right\|^2\right] + \frac{e_{\rho,2}^{T-1}\eta_z\eta_x^2}{(1-\rho^2)}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right].
\end{aligned}$$

For simplicity, we also overload the notation and set $\Omega_{Z,s} = (a_s, b_s, c_s)^\top$ and $\tilde{C}_{z,s} = (d_{1,s}, d_{2,s}, d_{3,s})^\top$. Note that we have

$$\begin{aligned}
a_{s+1} &\leq M_{11}a_s + M_{12}b_s + M_{13}c_s + d_{1,s} \\
b_{s+1} &\leq M_{21}a_s + M_{22}b_s + M_{23}c_s + d_{2,s} \\
c_{s+1} &\leq M_{31}a_s + M_{32}b_s + M_{33}c_s + d_{3,2},
\end{aligned}$$

thus we apply Lemma A.3 ((15) to a_s, b_s, c_s and let $\tau_k = 1$)

$$\sum_{i=0}^s a_i \leq \frac{1}{1-M_{11}}\left(a_0 + M_{12}\sum_{i=0}^s b_i + M_{13}\sum_{i=0}^s c_i + \sum_{i=0}^s d_{1,i}\right) \tag{$z:a$}$$

$$\sum_{i=0}^s b_i \leq \frac{1}{1-M_{22}}\left(b_0 + M_{21}\sum_{i=0}^s a_i + M_{23}\sum_{i=0}^s c_i + \sum_{i=0}^s d_{2,i}\right) \tag{$z:b$}$$

$$\sum_{i=0}^s c_i \leq \frac{1}{1-M_{33}}\left(c_0 + M_{31}\sum_{i=0}^s a_i + M_{32}\sum_{i=0}^s b_i + \sum_{i=0}^s d_{3,i}\right). \tag{$z:c$}$$

Incorporating ($z:c$) into ($z:a$) and ($z:b$), let $Q_1 = \frac{M_{13}}{1-M_{11}}\frac{1}{1-M_{33}}$, we have

$$\begin{aligned}
(1 - Q_1M_{31})\sum_{i=0}^s a_i \leq &\frac{a_0}{1-M_{11}} + c_0Q_1 + \left(Q_1M_{32} + \frac{M_{12}}{1-M_{11}}\right)\sum_{i=0}^s b_i \\
&+ Q_1\sum_{i=0}^s d_{3,i} + \frac{1}{1-M_{11}}\sum_{i=0}^s d_{1,i}.
\end{aligned} \tag{$z:a'$}$$

Let $Q_2 = \frac{M_{23}}{1-M_{22}}\frac{1}{1-M_{33}}$, we have

$$(1 - Q_2M_{32})\sum_{i=0}^s b_i \leq \frac{b_0}{1-M_{22}} + Q_2c_0 + \left(Q_2M_{31} + \frac{M_{21}}{1-M_{22}}\right)\sum_{i=0}^s a_i$$

$$+ Q_2 \sum_{i=0}^{s} d_{3,i} + \frac{1}{1-M_{22}} \sum_{i=0}^{s} d_{2,i}. \tag{$z:b'$}$$

Then we make the operations on the sum of a_s and b_s: that $\left(Q_2 M_{31} + \frac{M_{21}}{1-M_{22}}\right) \times (z:a') + (1-Q_1M_{31}) \times (z:b')$, then

$$\begin{aligned}
&\left((1-Q_1M_{31})(1-Q_2M_{32}) - \left(Q_2M_{31} + \frac{M_{21}}{1-M_{22}}\right)\left(Q_1M_{32} + \frac{M_{12}}{1-M_{11}}\right)\right)\sum_{i=0}^{s} b_i \\
&\le \left(Q_2M_{31} + \frac{M_{21}}{1-M_{22}}\right)\left(\frac{a_0}{1-M_{11}} + c_0Q_1\right) + (1-Q_1M_{31})\left(\frac{b_0}{1-M_{22}} + Q_2c_0\right) \\
&+ \left(Q_1Q_2M_{31} + \frac{M_{21}Q_1}{1-M_{22}} + Q_2(1-Q_1M_{31})\right)\sum_{i=0}^{s} d_{3,i} + \frac{(1-Q_1M_{31})}{1-M_{22}}\sum_{i=0}^{s} d_{2,i} \\
&+ \frac{\left(Q_2M_{31} + \frac{M_{21}}{1-M_{22}}\right)}{1-M_{11}}\sum_{i=0}^{s} d_{1,i}.
\end{aligned} \tag{61}$$

Note that to simplify the calculations and also cover the two cases: $T=1$ and $T \gg 1$, we set

$$\begin{aligned}
&\frac{1}{1-M_{11}} = \frac{1}{1-r_z^T} \sim \max\left(\frac{1}{\mu\gamma}, 2\right); \\
&\frac{1}{1-M_{22}} \sim \frac{1}{1-M_{33}} \sim \frac{1}{1-\left(\frac{3+\rho^2}{4}\right)^T} \sim \max\left(\frac{1}{1-\rho^2}, 2\right).
\end{aligned}$$

Then

$$\begin{aligned}
Q_1 &= \frac{M_{13}}{1-M_{11}}\frac{1}{1-M_{33}} \sim \Theta\left(\frac{e_z^T\eta_z^3}{(1-\rho^2)^3}\max\left(\frac{1}{\mu_g\eta_z(1-\rho^2)}, 4\right)\right), \\
Q_2 &= \frac{M_{23}}{1-M_{22}}\frac{1}{1-M_{33}} \sim \Theta\left(\frac{e_{\rho,1}^T\eta_z^2}{(1-\rho^2)^2}\max\left(\frac{1}{(1-\rho^2)^2}, 4\right)\right).
\end{aligned}$$

For any $T \ge 1$ we choose sufficient small stepsize $\eta_z \le \min\left((1-\rho^2)^{3.5}, (1-\rho^2)\mu_g\right)$ such that

$$\begin{aligned}
1 - Q_1M_{31} \sim 1 - \Theta\left(\max\left(\frac{\eta_z^3}{\mu_g(1-\rho^2)^5}, \frac{\eta_z^4}{(1-\rho^2)^4}\right)\right) &\ge \frac{2}{3}, \\
1 - Q_2M_{23} \sim 1 - \Theta\left(\max\left(\frac{e_{\rho,1}^{2T}\eta_z^2}{(1-\rho^2)^7}, \frac{e_{\rho,1}^{2T}\eta_z^2}{(1-\rho^2)^5}\right)\right) &\ge \frac{2}{3}, \\
\left((1-Q_1M_{31})(1-Q_2M_{32}) - \left(Q_2M_{31} + \frac{M_{21}}{1-M_{22}}\right)\left(Q_1M_{32} + \frac{M_{12}}{1-M_{11}}\right)\right) &\ge \frac{1}{3},
\end{aligned}$$

we have

$$\begin{aligned}
\sum_{i=0}^{s} b_i &\le 3\left(Q_2M_{31} + \frac{M_{21}}{1-M_{22}}\right)\left(\frac{a_0}{1-M_{11}} + c_0Q_1\right) + 3(1-Q_1M_{31})\left(\frac{b_0}{1-M_{22}} + Q_2c_0\right) \\
&+ 3\left(Q_1Q_2M_{31} + \frac{M_{21}Q_1}{1-M_{22}} + Q_2(1-Q_1M_{31})\right)\sum_{i=0}^{s} d_{3,i} + 3\frac{1-Q_1M_{31}}{1-M_{22}}\sum_{i=0}^{s} d_{2,i} \\
&+ 3\frac{\left(Q_2M_{31} + \frac{M_{21}}{1-M_{22}}\right)}{1-M_{11}}\sum_{i=0}^{s} d_{1,i} \\
&\lesssim \frac{e_{\rho,1}^T\eta_z^2}{\mu_g(1-\rho^2)^4}a_0 + \max\left(\frac{1}{1-\rho^2}, 2\right)b_0 + \frac{e_{\rho,1}^T\eta_z^2}{(1-\rho^2)^4}c_0 + \frac{e_{\rho,1}^T\eta_z^2}{(1-\rho^2)^4}\sum_{i=0}^{s} d_{3,i} \\
&+ \max\left(\frac{1}{1-\rho^2}, 2\right)\sum_{i=0}^{s} d_{2,i} + \left(\frac{1}{\mu_g} + \frac{e_{\rho,2}^T}{(1-\rho^2)}\right)\frac{e_{\rho,1}^T\eta_z^2}{(1-\rho^2)^4}\sum_{i=0}^{s} d_{1,i}.
\end{aligned} \tag{62}$$

Then we substitute the above result w.r.t. b_i to Inequality ($z : a'$), then

$$
\begin{aligned}
\sum_{i=0}^{s} a_i &\leq \frac{3}{2}\frac{a_0}{1-M_{11}} + \frac{3}{2}c_0 Q_1 + \frac{3}{2}\left(Q_1 M_{32} + \frac{M_{12}}{1-M_{11}}\right)\sum_{i=0}^{s} b_i \\
&+ \frac{3}{2}Q_1 \sum_{i=0}^{s} d_{3,i} + \frac{3}{2}\frac{1}{1-M_{11}}\sum_{i=0}^{s} d_{1,i} \\
&\lesssim \frac{a_0}{1-M_{11}} + c_0 Q_1 + \left(Q_1 M_{32} + \frac{M_{12}}{1-M_{11}}\right)\left(Q_2 M_{31} + \frac{M_{21}}{1-M_{22}}\right)\left(\frac{a_0}{1-M_{11}} + c_0 Q_1\right) \\
&+ \left(Q_1 M_{32} + \frac{M_{12}}{1-M_{11}}\right)(1 - Q_1 M_{31})\left(\frac{b_0}{1-M_{22}} + Q_2 c_0\right) \\
&+ \left[\left(Q_1 M_{32} + \frac{M_{12}}{1-M_{11}}\right)\left(Q_1 Q_2 M_{31} + \frac{M_{21}Q_1}{1-M_{22}} + Q_2\left(1 - Q_1 M_{31}\right)\right) + Q_1\right]\sum_{i=0}^{s} d_{3,i} \\
&+ \left(Q_1 M_{32} + \frac{M_{12}}{1-M_{11}}\right)\frac{1 - Q_1 M_{31}}{1-M_{22}}\sum_{i=0}^{s} d_{2,i} \\
&+ \left[\left(Q_1 M_{32} + \frac{M_{12}}{1-M_{11}}\right)\frac{\left(Q_2 M_{31} + \frac{M_{21}}{1-M_{22}}\right)}{1-M_{11}} + \frac{1}{1-M_{11}}\right]\sum_{i=0}^{s} d_{1,i} \\
&\lesssim \max\left(\frac{1}{\mu_g \eta_z}, 2\right) a_0 + \frac{e_z^T}{\mu_g(1-\rho^2)} b_0 + \frac{e_z^T \eta_z^2}{\mu_g(1-\rho^2)^5} c_0 + \frac{e_z^T \eta_z^2}{\mu_g(1-\rho^2)^5}\sum_{i=0}^{s} d_{3,i} \\
&+ \frac{e_z^T}{\mu_g(1-\rho^2)^2}\sum_{i=0}^{s} d_{2,i} + \max\left(\frac{1}{\mu_g \eta_z}, 2\right)\sum_{i=0}^{s} d_{1,i}.
\end{aligned} \tag{63}
$$

We then substitute the definitions of a_t, b_t and the matrix M_Z and $d_{1,s}, d_{2,s}, d_{3,s}$. Note that

$$
\frac{1}{2n}\mathbb{E}\left[\left\|\mathbf{Z}_s - z_{*,s}\mathbf{1}_n^\top\right\|^2\right] \leq \mathbb{E}\left[\left\|\bar{z}_s - z_{*,s}\right\|^2 + \frac{1}{n}\left\|\mathbf{Z}_s - \bar{z}_s \mathbf{1}_n^\top\right\|^2\right] := a_s + b_s \tag{64}
$$

Now we can give the estimation for the sum of z:

$$
\begin{aligned}
\frac{1}{2n}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbf{Z}_s - z_{*,s}\mathbf{1}_n^\top\right\|^2\right] &:= \sum_{s=0}^{S}(a_s + b_s) \\
&\lesssim \frac{e_{\rho,1}^T \eta_z^2}{\mu_g(1-\rho^2)^4} a_0 + \max\left(\frac{1}{1-\rho^2}, 2\right) b_0 + \frac{e_{\rho,1}^T \eta_z^2}{(1-\rho^2)^4} c_0 + \frac{e_{\rho,1}^T \eta_z^2}{(1-\rho^2)^4}\sum_{i=0}^{s} d_{3,i} \\
&+ \max\left(\frac{1}{1-\rho^2}, 2\right)\sum_{i=0}^{s} d_{2,i} + \left(\frac{1}{\mu_g} + \frac{e_{\rho,2}^T}{(1-\rho^2)}\right)\frac{e_{\rho,1}^T \eta_z^2}{(1-\rho^2)^4}\sum_{i=0}^{s} d_{1,i} \\
&+ \max\left(\frac{1}{\mu_g \eta_z}, 2\right) a_0 + \frac{e_z^T}{\mu_g(1-\rho^2)} b_0 + \frac{e_z^T \eta_z^2}{\mu_g(1-\rho^2)^5} c_0 + \frac{e_z^T \eta_z^2}{\mu_g(1-\rho^2)^5}\sum_{i=0}^{s} d_{3,i} \\
&+ \frac{e_z^T}{\mu_g(1-\rho^2)^2}\sum_{i=0}^{s} d_{2,i} + \max\left(\frac{1}{\mu_g \eta_z}, 2\right)\sum_{i=0}^{s} d_{1,i} \\
&\lesssim \max\left(\frac{e_{\rho,1}^T \eta_z^2}{\mu_g(1-\rho^2)^4}, \frac{1}{\mu_g \eta_z}\right) a_0 + \max\left(\frac{1}{1-\rho^2}, \frac{e_z^T}{\mu_g(1-\rho^2)}\right) b_0 \\
&+ \max\left(e_{\rho,1}^T, \frac{e_z^T}{\mu_g(1-\rho^2)}\right)\frac{\eta_z^2 c_0}{(1-\rho^2)^4} + \max\left(e_{\rho,1}^T, \frac{e_z^T}{\mu_g(1-\rho^2)}\right)\frac{\eta_z^2}{(1-\rho^2)^4}\sum_{s=0}^{S} d_{3,s} \\
&+ \max\left(1, \frac{e_z^T}{\mu_g}\right)\frac{\sum_{s=0}^{S} d_{2,i}}{(1-\rho^2)^2} + \max\left(\frac{1}{\mu_g \eta_z}, 2\right)\sum_{s=0}^{S} d_{1,s}.
\end{aligned}
$$

We thus substitute the definition of $a_0, b_0, c_0, d_{1,s}, d_{2,s}, d_{3,s}$ then

$$
\begin{aligned}
&\frac{1}{2n}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbf{Z}_s - z_{*,s}\mathbf{1}_n^\top\right\|^2\right] \\
&\lesssim \max\left(\frac{e_{\rho,1}^T\eta_z^2}{\mu_g(1-\rho^2)^4}, \frac{1}{\mu_g\eta_z}\right)\mathbb{E}\left[\left\|\bar{z}_0 - z_{*,0}\right\|^2\right] + \max\left(\frac{1}{1-\rho^2}, \frac{e_z^T}{\mu_g(1-\rho^2)}\right)\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{Z}_0 - \bar{z}_0\mathbf{1}_n^\top\right\|^2\right] \\
&+ \max\left(e_{\rho,1}^T, \frac{e_z^T}{\mu_g(1-\rho^2)}\right)\frac{\eta_z^2}{(1-\rho^2)^4}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_{0,z} - \bar{u}_{0,z}\mathbf{1}_n^\top\right\|^2\right] \\
&+ \max\left(e_{\rho,1}^T, \frac{e_z^T}{\mu_g(1-\rho^2)}\right)\frac{\eta_z^2}{(1-\rho^2)^4}\left\{\left(C_{x,3}\eta_z^2 + \frac{e_{\rho,2}^T}{1-\rho^2}\right)\sum_{s=0}^{S}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2\right]\right. \\
&+ \frac{e_{\rho,2}^T}{(1-\rho^2)n}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2\right] + S\left(C_{\sigma,3} + \frac{e_{\rho,2}^T}{1-\rho^2}\right)\sigma_z^2 + \frac{e_{\rho,2}^T\eta_x^2}{1-\rho^2}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right] \\
&\left.+ \frac{e_{\rho,2}^{T-1}\eta_z\eta_x^2}{(1-\rho^2)}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right]\right\} + \frac{e_z^T}{\mu_g(1-\rho^2)^2}\left\{\frac{\eta_z^2\left(C_{x,2}\eta_z^2 + \frac{e_{\rho,1}^T}{(1-\rho^2)^2}\right)}{(1-\rho^2)}\sum_{s=0}^{S}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2\right]\right. \\
&+ \frac{e_{\rho,1}^T\eta_z^2}{(1-\rho^2)^3}\frac{1}{n}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2\right] + S\eta_z^2\left(\frac{C_{\sigma,2}}{n} + \frac{e_{\rho,1}^T}{(1-\rho^2)^2}\right)\sigma_z^2 \\
&\left.+ \frac{e_{\rho,1}^T\eta_z^2}{(1-\rho^2)^3}\eta_x^2\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right] + \frac{e_{\rho,1}^T\eta_z^3\eta_x^2}{(1-\rho^2)^3}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right]\right\} \\
&+ \max\left(\frac{1}{\mu_g\eta_z}, 2\right)\left\{\min\left(T, \frac{1}{\mu_g\eta_z}\right)\left(\frac{C_{x,1}\eta_z}{n}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2\right] + S\frac{C_{\sigma,1}}{n}\eta_z^2\sigma_z^2\right)\right. \\
&+ \frac{e_z^T\eta_z^3}{(1-\rho^2)^4}\sum_{s=0}^{S-1}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}^\top\right\|^2\right] + e_z^T\frac{\eta_x\ell_{z^*}}{a_1}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right] \\
&\left.+ e_z^T\eta_x^2\left(\ell_{z_*}^2 + \frac{\ell_{\nabla z_*}}{2a_2}\right)\sum_{s=0}^{S}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right] + \frac{e_z^T\eta_z^3\eta_x^2}{(1-\rho^2)^4}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right]\right\} \\
&\lesssim C_{z_*,0}\Delta_{z_*,0} + C_{Z,0}\Delta_{Z,0} + C_{U_z,0}\Delta_{U_z,0} + C_{z,v}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right] \\
&+ C_{z,vs}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right] + C_{z,x}\frac{1}{n}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}^\top\right\|^2\right] + S\cdot C_{z,\sigma}\sigma_z^2
\end{aligned} \tag{65}
$$

where the constants are given by

$$
\begin{aligned}
&e_z = 1 - \frac{2\mu_g\eta_z}{3}, r_z \le 1 - \frac{\mu_g\eta_z}{3}, e_{\rho,1} = \frac{\rho^2+1}{2}, a_1 > 0, a_2 > 0, \\
&\Delta_{z_*,0} = \mathbb{E}\left[\left\|\bar{z}_0 - z_{*,0}\right\|^2\right], \Delta_{Z,0} = \frac{1}{n}\mathbb{E}\left[\left\|\mathbf{Z}_0 - \bar{z}_0\mathbf{1}_n^\top\right\|^2\right], \\
&\Delta_{U_z,0} = \frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_{0,z} - \bar{u}_{0,z}\mathbf{1}_n^\top\right\|^2\right] = \mathcal{O}(1), \\
&C_{z_*,0} = \max\left(\frac{e_{\rho,1}^T\eta_z^2}{\mu_g(1-\rho^2)^4}, \frac{1}{\mu_g\eta_z}\right) = \mathcal{O}\left(\frac{1}{\eta_z}\right), C_{Z,0} = \max\left(\frac{1}{1-\rho^2}, \frac{e_z^T}{\mu_g(1-\rho^2)}\right) = \mathcal{O}(1), \\
&C_{U_z,0} = \max\left(e_{\rho,1}^T, \frac{e_z^T}{\mu_g(1-\rho^2)}\right)\frac{\eta_z^2}{(1-\rho^2)^4} = \mathcal{O}\left(\frac{\eta_z^2}{(1-\rho^2)^4}\right), \\
&C_{z,v} = \max\left(\frac{1}{\mu_g\eta_z}, 2\right)e_z^T\frac{\eta_x\ell_{z^*}}{a_1} = \mathcal{O}\left(\frac{e_z^T\eta_x}{a_1\eta_z}\right),
\end{aligned}
$$

$$
\begin{aligned}
C_{z,vs} &= \mathcal{O}\left(\frac{e_z^T\eta_x^2}{\eta_z}\left(1+\frac{1}{a_2}\right)+\frac{e_z^T\eta_z^2\eta_x^2}{(1-\rho^2)^4}\right),\\
C_{z,x} &= \mathcal{O}\left(\frac{e_z^T\eta_z^4}{(1-\rho^2)^7}+\frac{e_{\rho,2}^T e_z^T\eta_z^2}{(1-\rho^2)^5}+\min\left(T,\frac{1}{\eta_z}\right)\right),\\
C_{z,\sigma} &= \mathcal{O}\left(\left(\frac{\eta_z}{n}+\frac{\eta_z^2}{(1-\rho^2)^4}\right)\min\left(T,\frac{1}{\eta_z}\right)+\frac{e_z^T\eta_z^2}{(1-\rho^2)^2}\right).
\end{aligned}
\tag{66}
$$

Following the same reasoning in (49), (52), (55), (53), (56), and (65) we may obtain a similar conclusion for y. The variable y is to optimize the objective, $f+\alpha g$ with respect to y which is $\frac{\alpha\mu_g}{2}$-strongly convex and $\frac{3\alpha\ell_{g,1}}{2}$-smooth. The stochastic gradient $h_{t+1,y}$ of updating y is variance-bounded by $\sigma_y^2=\sigma_f^2+\alpha^2\sigma_g^2$. Let $\mu=\frac{\alpha\mu_g}{2}, \ell=\frac{3\alpha\ell_{g,1}}{2}$ in Lemma B.2. If the step-size η_y satisfies that

$$
\eta_y \le \mathcal{O}\left(\frac{1}{\alpha}\min\left\{\frac{1-\rho^2}{\ell_{g,1}},\frac{(1-\rho^2)\sqrt{\mu_g}}{\ell_{g,1}\sqrt{\ell_{g,1}}},\frac{(1-\rho^2)^2}{\ell_{g,1}}\right\}\right),
\tag{67}
$$

we have

$$
\begin{aligned}
&\mathbb{E}\left[\left\|\bar{y}_{s+1}-y^\alpha_{*,s+1}\right\|^2\right]\\
&\le \left(1-\frac{\alpha\mu_g\eta_y}{2}\right)^T\left(1+\frac{\alpha^4\eta_y^4\ell_{g,1}^6}{\mu_g^2(1-\rho^2)^4}\right)\mathbb{E}\left[\left\|\bar{y}_s-y^\alpha_{*,s+1}\right\|^2\right]\\
&+\frac{\left(1-\frac{\alpha\mu_g\eta_y}{2}\right)^T\alpha\eta_y\ell_{g,1}^2}{\mu_g(1-\rho^2)}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{Y}_s-\bar{y}_s\mathbf{1}_n^\top\right\|^2\right]\\
&+\frac{\left(1-\frac{\alpha\mu_g\eta_y}{2}\right)^T\alpha\eta_y^3\ell_{g,1}^2}{\mu_g(1-\rho^2)^3 n}\mathbb{E}\left[\left\|\mathbf{U}_{s,y}-\bar{u}_{s,y}\mathbf{1}^\top\right\|^2\right]+\min\left(T,\frac{1}{\alpha\mu_g\eta_y}\right)\frac{C_{x,1}\eta_y}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]\\
&+\min\left(T,\frac{1}{\alpha\mu_g\eta_y}\right)\frac{C_{\sigma,1}}{n}\eta_y^2\sigma_y^2+\frac{\left(1-\frac{\alpha\mu_g\eta_y}{2}\right)^T\alpha\eta_y^3\ell_{g,1}^2}{\mu_g(1-\rho^2)^3}\cdot 6\sigma_y^2\\
&+\frac{\left(1-\frac{\alpha\mu_g\eta_y}{2}\right)^T\alpha\eta_y^3\ell_{g,1}^2}{\mu_g(1-\rho^2)^3}\left(\alpha^2\ell_{g,1}^2\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}^\top\right\|^2+\left\|\mathbf{X}_{s-1}-\bar{x}_{s-1}\mathbf{1}^\top\right\|^2+n\left\|\bar{x}_s-\bar{x}_{s-1}\right\|^2\right]\right)\\
&\lesssim e_y^T\mathbb{E}\left[\left\|\bar{y}_s-y^\alpha_{*,s+1}\right\|^2\right]+\frac{e_y^T\alpha\eta_y}{1-\rho^2}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{Y}_s-\bar{y}_s\mathbf{1}_n^\top\right\|^2\right]+\frac{e_y^T\alpha\eta_y^3}{(1-\rho^2)^3}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_{s,y}-\bar{u}_{s,y}\mathbf{1}^\top\right\|^2\right]\\
&+\min\left(T,\frac{1}{\alpha\mu_g\eta_y}\right)\left(\frac{C_{x,1}\eta_y}{n}\mathbb{E}\left[\left\|\mathbf{X}-\bar{x}\mathbf{1}_n^\top\right\|^2\right]+\frac{C_{\sigma,1}}{n}\eta_y^2\sigma_y^2\right)\\
&+\frac{e_y^T\alpha^3\eta_y^3}{(1-\rho^2)^3}\left(\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_{s-1}-\bar{x}_{s-1}\mathbf{1}_n^\top\right\|^2\right]+\eta_x^2\mathbb{E}\left[\left\|\bar{v}_s\right\|^2\right]\right)
\end{aligned}
\tag{68}
$$

where $e_y=1-\frac{\alpha\mu_g\eta_y}{3}$ and for simplicity we choose sufficient small $\eta_y\le\mathcal{O}\left((1-\rho^2)\mu_g/(\alpha\ell_{g,1}^2)\right)$ and for any $T\ge 1$ such that

$$
\begin{aligned}
&\left(1-\frac{\alpha\mu_g\eta_y}{2}\right)^T\left(1+\frac{\alpha^4\eta_y^4\ell_{g,1}^6}{\mu_g^2(1-\rho^2)^4}\right)\le\left(1-\frac{\alpha\mu_g\eta_y}{3}\right)^T\\
&\frac{\left(1-\frac{\alpha\mu_g\eta_y}{2}\right)^T\alpha^3\eta_y^3\ell_{g,1}^4}{\mu_g(1-\rho^2)^3}\le\min\left(T,\frac{1}{\alpha\mu_g\eta_y}\right)C_{x,1}\eta_y, \text{where } C_{x,1}\sim\mathcal{O}\left(\frac{\alpha\ell_{g,1}^2}{\mu_g}+\frac{\alpha^5\eta_y^4\ell_{g,1}^6}{\mu_g(1-\rho^2)^4}\right)\\
&\frac{\left(1-\frac{\alpha\mu_g\eta_y}{2}\right)^T\alpha\eta_y\ell_{g,1}^2}{\mu_g(1-\rho^2)^3}\le\frac{\alpha\eta_y\ell_{g,1}^2}{\mu_g(1-\rho^2)^3}\le\frac{C_{\sigma,1}}{n}, \text{where } C_{\sigma,1}\sim\mathcal{O}\left(\frac{\alpha\eta_y\ell_{g,1}^2 n}{\mu_g(1-\rho^2)^4}+1\right).
\end{aligned}
\tag{69}
$$

Recalling the inequality (51) for z, we also have a similar result for y. When we consider the convergence of the consensus convergence $\mathbf{Y}_s$ and $\mathbf{U}_{s,y}$, we only need the following inequality

$$
\left\|\bar{y}_s-y^\alpha_{*,s+1}\right\|^2\le 2\left\|\bar{y}_s-y^\alpha_{*,s}\right\|^2+2\left\|y^\alpha_{*,s}-y^\alpha_{*,s+1}\right\|^2=2\left\|\bar{y}_s-y^\alpha_{*,s}\right\|^2+2\eta_x^2\ell_{y_*}^2\left\|\bar{v}_{s+1}\right\|^2.
\tag{70}
$$

For the convergence of variable $\bar{y}_s$, we need a careful estimate about $\left\|\bar{y}_s - y^{\alpha}_{*,s+1}\right\|^2$ just as $\bar{z}$,

$$\mathbb{E}\left[\left\|\bar{y}_s - y^{\alpha}_{*,s+1}\right\|^2\right] \leq \left(1 + a_1\eta_x\ell_{y_*} + \frac{a_2\ell_{\nabla y_*}c_\delta\eta_x^2\alpha^2}{2}\right)\mathbb{E}\left[\left\|\bar{y}_s - y^{\alpha}_{*,s}\right\|^2\right] + \frac{\eta_x\ell_{y^*}}{a_1}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right] + \eta_x^2\left(\ell_{y_*}^2 + \frac{\ell_{\nabla y_*}}{2a_2}\right)\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right] \tag{71}$$

and we properly choose η_x, η_y and T such that

$$r_y = \left(1 + a_1\eta_x\ell_{y_*} + \frac{a_2\ell_{\nabla y_*}c_\delta\eta_x^2\alpha^2}{2}\right)\left(1 - \frac{\mu_g\eta_y\alpha}{3}\right)^T \leq 1 - \frac{\mu_g\eta_y\alpha}{6}. \tag{72}$$

Combining the above inequalities, we have

$$\begin{aligned}
&\mathbb{E}\left[\left\|\bar{y}_{s+1} - y^{\alpha}_{*,s+1}\right\|^2\right] \\
\lesssim\, & r_y^T\mathbb{E}\left[\left\|\bar{y}_s - y^{\alpha}_{*,s}\right\|^2\right] + \frac{e_y^T\alpha\eta_y}{1-\rho^2}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{Y}_s - \bar{y}_s\mathbf{1}_n^\top\right\|^2\right] + \frac{e_y^T\alpha\eta_y^3}{(1-\rho^2)^3}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_{s,y} - \bar{u}_{s,y}\mathbf{1}^\top\right\|^2\right] \\
&+ \min\left(T, \frac{1}{\alpha\mu_g\eta_y}\right)\left(\frac{C_{x,1}\eta_y}{n}\mathbb{E}\left[\left\|\mathbf{X} - \bar{x}\mathbf{1}_n^\top\right\|^2\right] + \frac{C_{\sigma,1}}{n}\eta_y^2\sigma_y^2\right) + e_y^T\frac{\eta_x\ell_{y^*}}{a_1}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right] \\
&+ e_y^T\eta_x^2\left(\ell_{y_*}^2 + \frac{\ell_{\nabla y_*}}{2a_2}\right)\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right] + \frac{e_y^T\alpha^3\eta_y^3}{(1-\rho^2)^3}\left(\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_{s-1} - \bar{x}_{s-1}\mathbf{1}_n^\top\right\|^2\right] + \eta_x^2\mathbb{E}\left[\left\|\bar{v}_s\right\|^2\right]\right)
\end{aligned} \tag{73}$$

where $e_y = 1 - \frac{\alpha\mu_g\eta_y}{3}$ and $r_y \leq 1 - \frac{\alpha\mu_g\eta_y}{6}$. Recalling the result of Lemma B.2 for the consensus of y and incorporating Inequalities (44) and (70), we have

$$\begin{aligned}
&\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{Y}_{s+1} - \bar{y}_{s+1}\mathbf{1}_n^\top\right\|^2\right] \\
\lesssim & \left(1 + \frac{\alpha^2\eta_y^2\ell_{g,1}^2}{(1-\rho^2)^4}\right)\frac{e_{\rho,1}^T}{n}\mathbb{E}\left[\left\|\mathbf{Y}_s - \bar{y}_s\mathbf{1}_n^\top\right\|^2\right] + \frac{e_{\rho,1}^T\alpha^3\eta_y^3}{(1-\rho^2)^3}\mathbb{E}\left[\left\|\bar{y}_s - y^{\alpha}_{*,s}\right\|^2\right] + \frac{e_{\rho,1}^T\eta_y^2}{(1-\rho^2)^2 n}\mathbb{E}\left[\left\|\mathbf{U}_{s,y} - \bar{u}_{s,y}\mathbf{1}_n^\top\right\|^2\right] \\
&+ \frac{\eta_y^2}{(1-\rho^2)n}\left(C_{x,2}\eta_y^2 + \frac{e_{\rho,1}^T\alpha^2}{(1-\rho^2)^2}\right)\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2\right] + \frac{e_{\rho,1}^T\alpha^2\eta_y^2}{(1-\rho^2)^3}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_{s-1} - \bar{x}_{s-1}\mathbf{1}^\top\right\|^2\right] \\
&+ \left(\frac{C_{\sigma,2}}{n} + \frac{e_{\rho,1}^T}{(1-\rho^2)^2}\right)\eta_y^2\sigma_y^2 + \frac{e_{\rho,1}^T\alpha^2\eta_y^2}{(1-\rho^2)^3}\eta_x^2\mathbb{E}\left[\left\|\bar{v}_s\right\|^2\right] + \frac{e_{\rho,1}^T\alpha^3\eta_y^3\eta_x^2}{(1-\rho^2)^3}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right].
\end{aligned} \tag{74}$$

for sufficient small η_y such that $\left(1 + \frac{\alpha^2\eta_y^2\ell_{g,1}^2}{(1-\rho^2)^4}\right)e_{\rho,1} \leq e_{\rho,2} = \frac{3+\rho^2}{4}$, then $\left(1 + \frac{\alpha^2\eta_y^2\ell_{g,1}^2}{(1-\rho^2)^4}\right)e_{\rho,1}^T \leq e_{\rho,2}^T$ for any $T \geq 1$. Similarly, we apply the result of Lemma B.2 to the consensus $U_{s,y}$ for y and incorporate Inequalities (44) and (70)

$$\begin{aligned}
&\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_{s+1,y} - \bar{u}_{s+1,y}\mathbf{1}_n^\top\right\|^2\right] \\
\lesssim\, & e_{\rho,2}^T\left(1 + \frac{\alpha^2\ell_{g,1}^2\eta_y^2}{(1-\rho^2)^4}\right)\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_{s,y} - \bar{u}_{s,y}\mathbf{1}_n^\top\right\|^2\right] + + \frac{e_{\rho,2}^{T-1}\alpha^3\eta_y}{(1-\rho^2)}\mathbb{E}\left[\left\|\bar{y}_s - y^{\alpha}_{*,s}\right\|^2\right] \\
&+ \frac{\alpha^2 e_{\rho,2}^{T-1}}{(1-\rho^2)^3}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{Y}_s - \bar{y}_s\mathbf{1}_n^\top\right\|^2\right] + \left(C_{x,3}\eta_y^2 + \frac{e_{\rho,2}^T\alpha^2}{1-\rho^2}\right)\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2\right] \\
&+ \frac{e_{\rho,2}^T\alpha^2}{(1-\rho^2)n}\mathbb{E}\left[\left\|\mathbf{X}_{s-1} - \bar{x}_{s-1}\mathbf{1}^\top\right\|^2\right] + \left(C_{\sigma,3} + \frac{e_{\rho,2}^T}{1-\rho^2}\right)\sigma_y^2 \\
&+ \frac{e_{\rho,2}^T}{1-\rho^2}\eta_x^2\alpha^2\mathbb{E}\left[\left\|\bar{v}_s\right\|^2\right] + \frac{e_{\rho,2}^T}{(1-\rho^2)}\alpha^3\eta_y\eta_x^2\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right].
\end{aligned} \tag{75}$$

for sufficiently small $\eta_y \leq (1-\rho^2)^2/(\alpha\ell_{g,1})$, we have $r_{U,y} := \left(1 + \frac{\alpha^2\ell_{g,1}^2\eta_y^2}{(1-\rho^2)^4}\right)e_{\rho,2} := \frac{4+\rho^2}{5}$.

Combining the above results for $\mathbb{E}\left[\left\|\bar{y}_{s+1}-y^{\alpha}_{*,s+1}\right\|^2\right]$,$\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{Y}_{s+1}-\bar{y}_{s+1}\mathbf{1}_n^{\top}\right\|^2\right]$, and $\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_{s+1,y}-\bar{u}_{s+1,y}\mathbf{1}_n^{\top}\right\|^2\right]$, we follow the same procedure for variable z and define the vector function $\Omega_{Y,s}$:

$$\Omega_{Y,s}=\left(\mathbb{E}\left[\left\|\bar{y}_s-y^{\alpha}_{*,s}\right\|^2\right],\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{Y}_s-\bar{y}_s\mathbf{1}_n^{\top}\right\|^2\right],\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_{s,y}-\bar{u}_{s,y}\mathbf{1}_n^{\top}\right\|^2\right]\right) \tag{76}$$

and an 3×3 matrix M_Y

$$M_Y=\begin{pmatrix}M_{11} & M_{12} & M_{13}\\ M_{21} & M_{22} & M_{23}\\ M_{31} & M_{32} & M_{33}\end{pmatrix} \tag{77}$$

where

$$\begin{aligned}
&M_{11}=r_y^T;\quad M_{12}=\frac{e_y^T\alpha\eta_y}{1-\rho^2};\quad M_{13}=\frac{e_y^T\alpha\eta_y^3}{(1-\rho^2)^3}\\
&M_{21}=\frac{e_{\rho,1}^T\alpha^3\eta_y^3}{(1-\rho^2)^3};\quad M_{22}=e_{\rho,2}^T;\quad M_{23}=\frac{e_{\rho,1}^T\eta_y^2}{(1-\rho^2)^2}\\
&M_{31}=\frac{e_{\rho,2}^T\alpha^3\eta_y}{(1-\rho^2)};\quad M_{32}=\frac{\alpha^2e_{\rho,2}^T}{(1-\rho^2)^3};\quad M_{33}=r_{U,y}^T.
\end{aligned} \tag{78}$$

By the inequalities (73), (74) and (75), we have

$$\Omega_{Y,s+1}\le M_Y\Omega_{Y,s}+\tilde{C}_{y,s} \tag{79}$$

where $\tilde{C}_{y,s}\in\mathbb{R}^3$ are defined as below:

$$\begin{aligned}
\tilde{C}_{y,s}[1]=&\min\left(T,\frac{1}{\alpha\mu_g\eta_y}\right)\left(\frac{C_{x,1}\eta_y}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}\mathbf{1}_n^{\top}\right\|^2\right]+\frac{C_{\sigma,1}}{n}\eta_y^2\sigma_y^2\right)\\
&+\frac{e_y^T\alpha^3\eta_y^3}{(1-\rho^2)^3}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_{s-1}-\bar{x}_{s-1}\mathbf{1}_n^{\top}\right\|^2\right]+e_y^T\frac{\eta_x\ell_{y^*}}{a_1}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right]\\
&+e_y^T\eta_x^2\left(\ell_{y^*}^2+\frac{\ell_{\nabla y_*}}{2a_2}\right)\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right]+\frac{e_y^T\alpha^3\eta_y^3}{(1-\rho^2)^3}\eta_x^2\mathbb{E}\left[\left\|\bar{v}_s\right\|^2\right]\\
\tilde{C}_{y,s}[2]=&\frac{\eta_y^2}{(1-\rho^2)n}\left(C_{x,2}\eta_y^2+\frac{e_{\rho,1}^T\alpha^2}{(1-\rho^2)^2}\right)\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^{\top}\right\|^2\right]\\
&+\frac{e_{\rho,1}^T\alpha^2\eta_y^2}{(1-\rho^2)^3}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_{s-1}-\bar{x}_{s-1}\mathbf{1}^{\top}\right\|^2\right]+\left(\frac{C_{\sigma,2}}{n}+\frac{e_{\rho,1}^T}{(1-\rho^2)^2}\right)\eta_y^2\sigma_y^2\\
&+\frac{e_{\rho,1}^T\alpha^2\eta_y^2}{(1-\rho^2)^3}\eta_x^2\mathbb{E}\left[\left\|\bar{v}_s\right\|^2\right]+\frac{e_{\rho,1}^T\alpha^3\eta_y^3\eta_x^2}{(1-\rho^2)^3}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right]\\
\tilde{C}_{y,s}[3]=&\left(C_{x,3}\eta_y^2+\frac{e_{\rho,2}^T\alpha^2}{1-\rho^2}\right)\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^{\top}\right\|^2\right]+\frac{e_{\rho,2}^T\alpha^2}{(1-\rho^2)n}\mathbb{E}\left[\left\|\mathbf{X}_{s-1}-\bar{x}_{s-1}\mathbf{1}^{\top}\right\|^2\right]\\
&+\left(C_{\sigma,3}+\frac{e_{\rho,2}^T}{1-\rho^2}\right)\sigma_y^2+\frac{e_{\rho,2}^T}{1-\rho^2}\eta_x^2\alpha^2\mathbb{E}\left[\left\|\bar{v}_s\right\|^2\right]+\frac{e_{\rho,2}^T}{(1-\rho^2)}\alpha^3\eta_y\eta_x^2\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right].
\end{aligned} \tag{80}$$

For simplicity we overload the same notation and set $\Omega_{Y,s}=(a_s,b_s,c_s)^{\top}$ and $\tilde{C}_{y,s}=(d_{1,s},d_{2,s},d_{3,s})^{\top}$. We thus obtain a similar conclusion for y.

$$Q_1\sim\frac{e_y^T\alpha\eta_y^3}{(1-\rho^2)^4}\max\left(\frac{1}{\alpha\mu_g\eta_y},2\right);\quad Q_2\sim\frac{e_{\rho,1}^T\eta_y^2}{(1-\rho^2)^4}. \tag{81}$$

For sufficient small $\eta_y\le(1-\rho^2)^{3.5}/(\alpha)$ such that

$$1-Q_1M_{31}\ge\frac{2}{3},\qquad 1-Q_2M_{32}\ge\frac{2}{3},$$

$$\left((1-Q_1M_{31})(1-Q_2M_{32})-\left(Q_2M_{31}+\frac{M_{21}}{1-M_{22}}\right)\left(Q_1M_{32}+\frac{M_{12}}{1-M_{11}}\right)\right)\geq\frac{1}{3}. \tag{82}$$

Then

$$\begin{aligned}\sum_{i=0}^{s}b_i &\lesssim \frac{e_{\rho,1}^T\alpha^2\eta_y^2}{\mu_g(1-\rho^2)^5}a_0+\max\left(\frac{1}{1-\rho^2},2\right)b_0+\frac{e_{\rho,1}^T\eta_y^2}{(1-\rho^2)^4}c_0+\frac{e_{\rho,1}^T\eta_y^2}{(1-\rho^2)^4}\sum_{i=0}^{s}d_{3,i}\\ &\quad+\max\left(\frac{1}{1-\rho^2},2\right)\sum_{i=0}^{s}d_{2,i}+\frac{e_{\rho,1}^T\alpha^2\eta_y^2}{(1-\rho^2)^5}\sum_{i=0}^{s}d_{1,i},\end{aligned} \tag{83}$$

$$\begin{aligned}\sum_{i=0}^{s}a_i &\lesssim \max\left(\frac{1}{\alpha\mu_g\eta_y},2\right)a_0+\frac{e_y^T}{\mu_g(1-\rho^2)^2}b_0+\frac{e_y^T\eta_y^2}{\mu_g(1-\rho^2)^4}c_0+\frac{e_y^T\eta_y^2}{(1-\rho^2)^5}\sum_{i=0}^{s}d_{3,i}\\ &\quad+\frac{e_y^T}{\mu_g(1-\rho^2)^2}\sum_{i=0}^{s}d_{2,i}+\max\left(\frac{1}{\alpha\mu_g\eta_y},2\right)\sum_{i=0}^{s}d_{1,i}.\end{aligned} \tag{84}$$

Combining the above inequalities we have

$$\begin{aligned}&\frac{1}{2n}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbf{Y}_s-y_{*,s}^{\alpha}\mathbf{1}^\top\right\|^2\right]\leq\sum_{s=0}^{S}\mathbb{E}\left[\left\|\bar{y}_s-y_{*,s}^{\alpha}\right\|^2\right]+\sum_{s=0}^{S}\left[\frac{1}{n}\left\|\mathbf{Y}_s-\bar{y}_{s+1}\mathbf{1}_n^\top\right\|^2\right]:=\sum_{s=0}^{S}(a_s+b_s)\\
&\lesssim\frac{e_{\rho,1}^T\alpha^2\eta_y^2}{\mu_g(1-\rho^2)^5}a_0+\max\left(\frac{1}{1-\rho^2},2\right)b_0+\frac{e_{\rho,1}^T\eta_y^2}{(1-\rho^2)^4}c_0+\frac{e_{\rho,1}^T\eta_y^2}{(1-\rho^2)^4}\sum_{i=0}^{s}d_{3,i}\\
&+\max\left(\frac{1}{1-\rho^2},2\right)\sum_{i=0}^{s}d_{2,i}+\frac{e_{\rho,1}^T\alpha^2\eta_y^2}{(1-\rho^2)^5}\sum_{i=0}^{s}d_{1,i}\\
&+\max\left(\frac{1}{\alpha\mu_g\eta_y},2\right)a_0+\frac{e_y^T}{\mu_g(1-\rho^2)^2}b_0+\frac{e_y^T\eta_y^2}{\mu_g(1-\rho^2)^4}c_0+\frac{e_y^T\eta_y^2}{(1-\rho^2)^5}\sum_{i=0}^{s}d_{3,i}\\
&+\frac{e_y^T}{\mu_g(1-\rho^2)^2}\sum_{i=0}^{s}d_{2,i}+\max\left(\frac{1}{\alpha\mu_g\eta_y},2\right)\sum_{i=0}^{s}d_{1,i}\\
&\lesssim\max\left(\frac{1}{\alpha\mu_g\eta_y},2\right)a_0+\max\left(1,\frac{e_y^T}{\mu_g(1-\rho^2)}\right)\frac{b_0}{1-\rho^2}+\max\left(e_{\rho,1}^T,\frac{e_y^T}{\mu_g}\right)\frac{\eta_y^2}{(1-\rho^2)^4}c_0\\
&+\max\left(e_{\rho,1}^T,\frac{e_y^T}{(1-\rho^2)}\right)\frac{\eta_y^2}{(1-\rho^2)^4}\sum_{i=0}^{s}d_{3,i}+\max\left(\frac{1}{1-\rho^2},\frac{e_y^T}{\mu_g(1-\rho^2)^2}\right)\sum_{i=0}^{s}d_{2,i}+\max\left(\frac{1}{\alpha\mu_g\eta_y},2\right)\sum_{i=0}^{s}d_{1,i}\\
&\lesssim\max\left(\frac{1}{\alpha\mu_g\eta_y},2\right)\Delta_{y_*,0}+\max\left(1,\frac{e_y^T}{\mu_g(1-\rho^2)}\right)\frac{\Delta_{Y,0}}{1-\rho^2}++\max\left(e_{\rho,1}^T,\frac{e_y^T}{\mu_g}\right)\frac{\eta_y^2}{(1-\rho^2)^4}\Delta_{U_y,0}\\
&+\max\left(e_{\rho,1}^T,\frac{e_y^T}{(1-\rho^2)}\right)\frac{\eta_y^2}{(1-\rho^2)^4}\left\{\left(C_{x,3}\eta_y^2+\frac{e_{\rho,2}^T\alpha^2}{1-\rho^2}\right)\sum_{s=0}^{S}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]\right.\\
&+\frac{e_{\rho,2}^T\alpha^2}{(1-\rho^2)}\sum_{s=0}^{S-1}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_{s-1}-\bar{x}_{s-1}\mathbf{1}_n^\top\right\|^2\right]+S\left(C_{\sigma,3}+\frac{e_{\rho,2}^T}{1-\rho^2}\right)\sigma_y^2\\
&\left.+\frac{e_{\rho,2}^T}{1-\rho^2}\eta_x^2\alpha^2\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right]+\frac{e_{\rho,2}^T}{(1-\rho^2)}\alpha^3\eta_y\eta_x^2\sum_{s=0}^{S}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right]\right\}\\
&+\max\left(\frac{1}{1-\rho^2},\frac{e_y^T}{\mu_g(1-\rho^2)^2}\right)\left\{\frac{\eta_y^2}{(1-\rho^2)}\left(C_{x,2}\eta_y^2+\frac{e_{\rho,1}^T\alpha^2}{(1-\rho^2)^2}\right)\sum_{s=0}^{S}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]\right.\\
&+\frac{e_{\rho,1}^T\alpha^2\eta_y^2}{(1-\rho^2)^3}\sum_{s=0}^{S-1}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]+S\left(\frac{C_{\sigma,2}}{n}+\frac{e_{\rho,1}^T}{(1-\rho^2)^2}\right)\eta_y^2\sigma_y^2\end{aligned}$$

$$
\begin{aligned}
&+ \frac{e_{\rho,1}^T\alpha^2\eta_y^2}{(1-\rho^2)^3}\eta_x^2 \sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right] + \frac{e_{\rho,1}^T\alpha^3\eta_y^3\eta_x^2}{(1-\rho^2)^3}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right]\Bigg\} \\
&+ \max\left(\frac{1}{\alpha\mu_g\eta_y}, 2\right)\Bigg\{\min\left(T, \frac{1}{\alpha\mu_g\eta_y}\right)\left(\frac{C_{x,1}\eta_y}{n}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}\mathbf{1}_n^\top\right\|^2\right] + S\frac{C_{\sigma,1}}{n}\eta_y^2\sigma_y^2\right) \\
&+ \frac{e_y^T\alpha^3\eta_y^3}{(1-\rho^2)^3}\sum_{s=0}^{S-1}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2\right] + e_y^T\frac{\eta_x\ell_{y^*}}{a_1}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right] \\
&+ e_y^T\eta_x^2\left(\ell_{y_*}^2 + \frac{\ell_{\nabla y_*}}{2a_2}\right)\sum_{s=0}^{S}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right] + \frac{e_y^T\alpha^3\eta_y^3}{(1-\rho^2)^3}\eta_x^2\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right]\Bigg\}
\end{aligned} \tag{85}
$$

where we use these notations to simplify the inequality

$$
\Delta_{y_*,0} = \mathbb{E}\left[\left\|\bar{y}_0 - y_{*,0}^\alpha\right\|^2\right], \Delta_{Y,0} = \frac{1}{n}\mathbb{E}\left[\left\|\mathbf{Y}_0 - \bar{y}_0\mathbf{1}_n^\top\right\|^2\right], \Delta_{U_y,0} = \frac{1}{n}\mathbb{E}\left[\left\|\mathbf{U}_{0,y} - \bar{u}_{0,y}\mathbf{1}_n^\top\right\|^2\right].
$$

We further re-arrange the above inequality and get that

$$
\begin{aligned}
&\sum_{s=0}^{S}\frac{1}{2n}\mathbb{E}\left[\left\|\mathbf{Y}_s - y_{*,s}^\alpha\mathbf{1}^\top\right\|^2\right] \\
&\lesssim C_{y_*,0}\Delta_{y_*,0} + C_{Y,0}\Delta_{Y,0} + C_{U_y,0}\Delta_{U_y,0} + C_{y,v}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right] + C_{y,vs}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right] \\
&+ C_{y,x}\sum_{s=0}^{S}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2\right] + SC_{y,\sigma}\sigma_y^2.
\end{aligned} \tag{86}
$$

where the constants are given by

$$
\begin{aligned}
C_{y_*,0} &= \max\left(\frac{1}{\alpha\mu_g\eta_y}, 2\right), C_{Y,0} = \max\left(1, \frac{e_y^T}{\mu_g(1-\rho^2)}\right)\frac{1}{1-\rho^2}, \\
C_{U_y,0} &= \max\left(e_{\rho,1}^T, \frac{e_y^T}{\mu_g}\right)\frac{\eta_y^2}{(1-\rho^2)^4} \\
C_{y,v} &= \max\left(\frac{1}{\alpha\mu_g\eta_y}, 2\right)e_y^T\frac{\eta_x\ell_{y^*}}{a_1} \sim \mathcal{O}\left(\frac{e_y^T\eta_x}{a_1\alpha\eta_y}\right) \\
C_{y,vs} &= \frac{e_y^T\eta_y^2}{(1-\rho^2)^5}\left(\frac{e_{\rho,2}^2\eta_x^2\alpha^2}{1-\rho^2} + \frac{e_{\rho,2}^2\eta_x^2\alpha^3\eta_y}{1-\rho^2}\right) + \frac{e_y^T}{(1-\rho^2)^2}\left(\frac{e_{\rho,1}^T\alpha^2\eta_y^2\eta_x^2}{(1-\rho^2)^3} + \frac{e_{\rho,1}^T\alpha^3\eta_y^3\eta_x^2}{(1-\rho^2)^3}\right) \\
&+ \max\left(\frac{1}{\alpha\mu_g\eta_y}, 2\right)\left(e_y^T\eta_x^2\left(\ell_{y_*}^2 + \frac{\ell_{\nabla y_*}}{2a_2}\right) + \frac{e_y^T\alpha^3\eta_y^3}{(1-\rho^2)^3}\eta_x^2\right) \\
&\sim \mathcal{O}\left(\frac{e_y^Te_{\rho,2}^T}{(1-\rho^2)^6}\left(\eta_x^2\alpha^2 + \eta_x^2\alpha^3\eta_y\right) + \frac{e_y^T\eta_x^2}{\alpha\eta_y}\left(1 + \frac{1}{a_2} + \frac{\alpha^3\eta_y^3}{(1-\rho^2)^3}\right)\right) \\
C_{y,x} &= \frac{e_y^T\eta_y^2}{(1-\rho^2)^5}\left(\left(C_{x,3}\eta_y^2 + \frac{e_{\rho,2}^T\alpha^2}{1-\rho^2}\right) + \frac{e_{\rho,2}^T\alpha^2}{(1-\rho^2)}\right) \\
&+ \frac{e_y^T}{(1-\rho^2)^2}\left(\frac{\eta_y^2}{(1-\rho^2)}\left(C_{x,2}\eta_y^2 + \frac{e_{\rho,1}^T\alpha^2}{(1-\rho^2)^2}\right) + \frac{e_{\rho,1}^T\alpha^2\eta_y^2}{(1-\rho^2)^3}\right) + \\
&+ \max\left(\frac{1}{\alpha\mu_g\eta_y}, 2\right)\left(\min\left(T, \frac{1}{\alpha\mu_g\eta_y}\right)C_{x,1}\eta_y + \frac{e_y^T\alpha^3\eta_y^3}{(1-\rho^2)^3}\right) \\
&\sim \mathcal{O}\left(\min\left(T, \frac{1}{\alpha\eta_y}\right) + \frac{e_y^T\alpha^4\eta_y^4}{(1-\rho^2)^7} + \frac{e_{\rho,1}^T\alpha^2\eta_y^2}{(1-\rho^2)^6} + \min\left(T, \frac{1}{\alpha\eta_y}\right)\frac{\alpha^4\eta_y^4}{(1-\rho^2)^4}\right)
\end{aligned}
$$

$$\begin{aligned}
C_{y,\sigma} &= \frac{e_y^T \eta_y^2}{(1-\rho^2)^5}\left(C_{\sigma,3} + \frac{e_{\rho,2}^T}{1-\rho^2}\right) + \frac{e_y^T}{(1-\rho^2)^2}\left(\frac{C_{\sigma,2}}{n} + \frac{e_{\rho,1}^T}{(1-\rho^2)^2}\right)\eta_y^2 \\
&\quad + \frac{1}{\alpha\eta_y}\min\left(T, \frac{1}{\alpha\eta_y}\right)\frac{C_{\sigma,1}}{n}\eta_y^2 \\
&\sim \mathcal{O}\left(\min\left(T, \frac{1}{\alpha\eta_y}\right)\frac{\eta_y}{\alpha}\left(\frac{\alpha\eta_y}{(1-\rho^2)^4} + \frac{1}{n}\right) + \frac{e_y^T\eta_y^2}{(1-\rho^2)^7}\right). \qquad (87)
\end{aligned}$$

The proof is complete. □

Next, we derive the consensus analysis for the upper-level variable x.

Lemma C.2. *Suppose Assumptions 1, 2, 3, and 4 hold, consider Algorithm 1, by properly choosing η_x such that*

$$\eta_x \le \mathcal{O}\left(\min\left\{\frac{(1-\rho^2)}{\alpha\ell_{f,1}}, \frac{(1-\rho^2)^2}{\alpha\ell_{g,1}}\right\}\right),$$

we have

$$\begin{aligned}
&\frac{1}{2n}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbf{X}_{s+1} - \bar{x}_{s+1}\mathbf{1}_n^\top\right\|^2\right] \\
&\lesssim \frac{\eta_x^2}{(1-\rho^2)^3}\left(C_{vz}C_{z,v} + C_{vy}C_{y,v}\right)\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{i+1}|\mathcal{F}_i\right]\right\|^2\right] \\
&+ \frac{\eta_x^2}{(1-\rho^2)^3}\left(4C_{vz}C_{z,vs} + 4C_{vy}C_{y,vs} + \frac{C_{vv}}{n}\right)\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{i+1}\right\|^2\right] \\
&+ \frac{\eta_x^2 C_{vz}}{(1-\rho^2)^3}\left(C_{z_*,0}\Delta_{z_*,0} + C_{Z,0}\Delta_{Z,0} + C_{U_z,0}\Delta_{U_z,0}\right) \\
&+ \frac{C_{vy}\eta_x^2}{(1-\rho^2)^3}\left(C_{y_*,0}\Delta_{y_*,0} + C_{Y,0}\Delta_{Y,0} + C_{U_y,0}\Delta_{U_y,0}\right) \\
&+ \frac{\eta_x^2 S}{(1-\rho^2)^3}\left(C_{vz}C_{z,\sigma}\sigma_z^2 + C_{vy}C_{y,\sigma}\sigma_y^2\right) + \frac{S\eta_x^2\sigma_x^2}{(1-\rho^2)^3} + \frac{\eta_x^2\ell_{f,0}^2(1+\rho^2)}{(1-\rho^2)^3}
\end{aligned}$$

where $\sigma_x^2 = \sigma_f^2 + 2\alpha^2\sigma_g^2$, $\sigma_y^2 = \sigma_f^2 + \alpha^2\sigma_g^2$, $\sigma_z^2 = \sigma_g^2$, and other constants are defined in (66) and (87) of Lemma C.1.

Proof. (of Lemma C.2) Note that in Algorithm 1 we have

$$\mathbf{X}_{s+1} = \mathbf{X}_s\mathbf{W} - \eta_x\mathbf{V}_{s+1},\ \mathbf{V}_{s+1} = \mathbf{V}_s\mathbf{W} + \Delta_{s+1} - \Delta_s,\ \bar{x}_{s+1} = \bar{x}_s - \eta_x\bar{v}_{s+1},\ \bar{v}_s = \bar{\delta}_s.$$

Thus we know by Lemmas A.1 (with $c = \frac{1-\rho^2}{2\rho^2}$) and A.5,

$$\left\|\mathbf{X}_{s+1} - \bar{x}_{s+1}\mathbf{1}_n^\top\right\|^2 \le \frac{1+\rho^2}{2}\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2 + \frac{(1+\rho^2)\eta_x^2}{1-\rho^2}\left\|\mathbf{V}_{s+1} - \bar{v}_{s+1}\mathbf{1}_n^\top\right\|^2.$$

We also have

$$\begin{aligned}
\mathbf{V}_{s+1} - \bar{v}_{s+1}\mathbf{1}_n^\top &= \mathbf{V}_s\mathbf{W} + \Delta_{s+1} - \Delta_s - (\bar{v}_s + \bar{\delta}_{s+1} - \bar{\delta}_s)\mathbf{1}_n^\top \\
&= \left(\mathbf{V}_s - \bar{v}_s\mathbf{1}_n^\top\right)\left(\mathbf{W} - \frac{\mathbf{1}_n\mathbf{1}_n^\top}{n}\right) + (\Delta_{s+1} - \Delta_s)\left(\mathbf{I}_n - \frac{\mathbf{1}_n\mathbf{1}_n^\top}{n}\right)
\end{aligned}$$

which, together with Lemmas A.1, A.2 and A.5, and $\left\|\mathbf{I}_n - \frac{\mathbf{1}_n\mathbf{1}_n^\top}{n}\right\|_2 \le 1$, implies

$$\left\|\mathbf{V}_{s+1} - \bar{v}_{s+1}\mathbf{1}_n^\top\right\|^2 \le \frac{1+\rho^2}{2}\left\|\mathbf{V}_s - \bar{v}_s\mathbf{1}_n^\top\right\|^2 + \frac{1+\rho^2}{1-\rho^2}\left\|\Delta_{s+1} - \Delta_s\right\|^2 \qquad (88)$$

To bound $\|\Delta_{s+1} - \Delta_s\|$, we have

$$\Delta_{s+1} - \Delta_s = \Delta_{s+1} - \mathbb{E}\left[\Delta_{s+1} \mid \mathcal{F}_s\right] - \left(\Delta_s - \mathbb{E}\left[\Delta_s \mid \mathcal{F}_{s-1}\right]\right) + \mathbb{E}\left[\Delta_{s+1} \mid \mathcal{F}_s\right] - \mathbb{E}\left[\Delta_s \mid \mathcal{F}_{s-1}\right],$$

and thus

$$
\begin{aligned}
&\mathbb{E}\left[\|\Delta_{s+1}-\Delta_s\|^2\right] \\
\leq &3\mathbb{E}\left[\|\Delta_{s+1}-\mathbb{E}\left[\Delta_{s+1}\mid\mathcal{F}_s\right]\|^2+\|\Delta_s-\mathbb{E}\left[\Delta_s\mid\mathcal{F}_{s-1}\right]\|^2+\|\mathbb{E}\left[\Delta_{s+1}\mid\mathcal{F}_s\right]-\mathbb{E}\left[\Delta_s\mid\mathcal{F}_{s-1}\right]\|^2\right] \\
\leq &6n\sigma_x^2+3\mathbb{E}\left[\|\mathbb{E}\left[\Delta_{s+1}\mid\mathcal{F}_s\right]-\mathbb{E}\left[\Delta_s\mid\mathcal{F}_{s-1}\right]\|^2\right],
\end{aligned}
\tag{89}
$$

where the stochastic gradient Δ_{s+1} of updating variable x is variance-bounded by $\sigma_x^2=\sigma_f^2+2\alpha^2\sigma_g^2$. We then bound $\|\mathbb{E}\left[\Delta_{s+1}\mid\mathcal{F}_s\right]-\mathbb{E}\left[\Delta_s\mid\mathcal{F}_{s-1}\right]\|$ via the following inequalities:

$$
\begin{aligned}
&\|\mathbb{E}\left[\Delta_{s+1}\mid\mathcal{F}_s\right]-\mathbb{E}\left[\Delta_s\mid\mathcal{F}_{s-1}\right]\|^2 \\
\leq &\sum_{i=1}^n 3\left\|\nabla_x f_i(x_s^{(i)},y_s^{(i)})-\nabla_x f_i(x_{s-1}^{(i)},y_{s-1}^{(i)})\right\|^2+\sum_{i=1}^n 3\alpha^2\left\|\nabla_x g_i(x_s^{(i)},y_s^{(i)})-\nabla_x g_i(x_{s-1}^{(i)},y_{s-1}^{(i)})\right\|^2 \\
&+\sum_{i=1}^n 3\alpha^2\left\|\nabla_x g_i(x_s^{(i)},z_s^{(i)})-\nabla_x g_i(x_{s-1}^{(i)},z_{s-1}^{(i)})\right\|^2 \\
\leq &(3+6\alpha^2)\ell_{f,1}^2\|\mathbf{X}_s-\mathbf{X}_{s-1}\|^2+(3+3\alpha^2)\ell_{g,1}^2\|\mathbf{Y}_s-\mathbf{Y}_{s-1}\|^2+3\alpha^2\ell_{g,1}^2\|\mathbf{Z}_s-\mathbf{Z}_{s-1}\|^2.
\end{aligned}
\tag{90}
$$

Note that we also have for $\|\mathbf{X}_s-\mathbf{X}_{s-1}\|$,

$$
\begin{aligned}
\|\mathbf{X}_{s+1}-\mathbf{X}_s\|^2=&\left\|\left(\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right)(\mathbf{W}-\mathbf{I})-\eta_x\mathbf{V}_{s+1}\right\|^2 \\
\leq &2\left\|\left(\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right)(\mathbf{W}-\mathbf{I})\right\|^2+2\eta_x^2\|\mathbf{V}_{s+1}\|^2 \\
\leq &8\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}^\top\right\|^2+2\eta_x^2\left\|\mathbf{V}_{s+1}-\bar{v}_{s+1}\mathbf{1}_n^\top\right\|^2+2n\eta_x^2\|\bar{v}_{s+1}\|^2
\end{aligned}
\tag{91}
$$

for $\|\mathbf{Y}_s-\mathbf{Y}_{s-1}\|$,

$$
\begin{aligned}
\|\mathbf{Y}_s-\mathbf{Y}_{s-1}\|^2=&\left\|\mathbf{Y}_s-y_*^\alpha(\bar{x}_s)\mathbf{1}_n^\top-\mathbf{Y}_{s-1}+y_*^\alpha(\bar{x}_{s-1})\mathbf{1}_n^\top+y_*^\alpha(\bar{x}_s)\mathbf{1}_n^\top-y_*^\alpha(\bar{x}_{s-1})\mathbf{1}_n^\top\right\|^2 \\
\leq &3\left\|\mathbf{Y}_s-y_*^\alpha(\bar{x}_s)\mathbf{1}_n^\top\right\|^2+3\left\|\mathbf{Y}_{s-1}-y_*^\alpha(\bar{x}_{s-1})\mathbf{1}_n^\top\right\|^2+3n\ell_{y_*}^2\eta_x^2\|\bar{v}_s\|^2
\end{aligned}
\tag{92}
$$

and for $\|\mathbf{Z}_s-\mathbf{Z}_{s-1}\|$,

$$
\begin{aligned}
\|\mathbf{Z}_s-\mathbf{Z}_{s-1}\|^2=&\left\|\mathbf{Z}_s-z_*(\bar{x}_s)\mathbf{1}_n^\top-\mathbf{Z}_{s-1}+z_*(\bar{x}_{s-1})\mathbf{1}_n^\top+z_*(\bar{x}_s)\mathbf{1}_n^\top-z_*(\bar{x}_{s-1})\mathbf{1}_n^\top\right\|^2 \\
\leq &3\left\|\mathbf{Z}_s-z_*(\bar{x}_s)\mathbf{1}_n^\top\right\|^2+3\left\|\mathbf{Z}_{s-1}-z_*(\bar{x}_{s-1})\mathbf{1}_n^\top\right\|^2+3n\ell_{z_*}^2\eta_x^2\|\bar{v}_s\|^2.
\end{aligned}
\tag{93}
$$

Combining all the inequalities above and setting $\alpha\geq 1$, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\|\mathbb{E}\left[\Delta_{s+1}\mid\mathcal{F}_s\right]-\mathbb{E}\left[\Delta_s\mid\mathcal{F}_{s-1}\right]\|^2\right] \\
\leq &9\alpha^2\ell_{f,1}^2\left(8\left\|\mathbf{X}_{s-1}-\bar{x}_{s-1}\mathbf{1}^\top\right\|^2+2\eta_x^2\left\|\mathbf{V}_s-\bar{v}_s\mathbf{1}_n^\top\right\|^2+2n\eta_x^2\|\bar{v}_s\|^2\right) \\
&+6\alpha^2\ell_{g,1}^2\left(3\left\|\mathbf{Y}_s-y_*^\alpha(\bar{x}_s)\mathbf{1}_n^\top\right\|^2+3\left\|\mathbf{Y}_{s-1}-y_*^\alpha(\bar{x}_{s-1})\mathbf{1}_n^\top\right\|^2+3n\ell_{y_*}^2\eta_x^2\|\bar{v}_s\|^2\right) \\
&+3\alpha^2\ell_{g,1}^2\left(3\left\|\mathbf{Z}_s-z_*(\bar{x}_s)\mathbf{1}_n^\top\right\|^2+3\left\|\mathbf{Z}_{s-1}-z_*(\bar{x}_{s-1})\mathbf{1}_n^\top\right\|^2+3n\ell_{z_*}^2\eta_x^2\|\bar{v}_s\|^2\right)
\end{aligned}
$$

and thus for $s\geq 1$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\left\|\mathbf{V}_{s+1}-\bar{v}_{s+1}\mathbf{1}_n^\top\right\|^2\right] \\
\leq &\frac{1+\rho^2}{2}\mathbb{E}\left[\left\|\mathbf{V}_s-\bar{v}_s\mathbf{1}_n^\top\right\|^2\right]+\frac{1+\rho^2}{1-\rho^2}\mathbb{E}\left[\|\Delta_{s+1}-\Delta_s\|^2\right] \\
\leq &\frac{1+\rho^2}{2}\mathbb{E}\left[\left\|\mathbf{V}_s-\bar{v}_s\mathbf{1}_n^\top\right\|^2\right]+\frac{6n(1+\rho^2)\sigma_x^2}{1-\rho^2} \\
&+\frac{3(1+\rho^2)}{1-\rho^2}\Bigg\{3\alpha^2\ell_{f,1}^2\left(8\left\|\mathbf{X}_{s-1}-\bar{x}_{s-1}\mathbf{1}^\top\right\|^2+2\eta_x^2\left\|\mathbf{V}_s-\bar{v}_s\mathbf{1}_n^\top\right\|^2+2n\eta_x^2\|\bar{v}_s\|^2\right) \\
&+6\alpha^2\ell_{g,1}^2\left(\left\|\mathbf{Y}_s-y_*^\alpha(\bar{x}_s)\mathbf{1}_n^\top\right\|^2+\left\|\mathbf{Y}_{s-1}-y_*^\alpha(\bar{x}_{s-1})\mathbf{1}_n^\top\right\|^2+n\ell_{y_*}^2\eta_x^2\|\bar{v}_s\|^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ 3\alpha^2\ell_{g,1}^2\left(\left\|\mathbf{Z}_s - z_*(\bar{x}_s)\mathbf{1}_n^\top\right\|^2 + \left\|\mathbf{Z}_{s-1} - z_*(\bar{x}_{s-1})\mathbf{1}_n^\top\right\|^2 + n\ell_{z_*}^2\eta_x^2\left\|\bar{v}_s\right\|^2\right)\Bigg\} \\
&= C_v\mathbb{E}\left[\left\|\mathbf{V}_s - \bar{v}_s\mathbf{1}_n^\top\right\|^2\right] + C_{vx}\mathbb{E}\left[\left\|\mathbf{X}_{s-1} - \bar{x}_{s-1}\mathbf{1}^\top\right\|^2\right] + C_{vy}\mathbb{E}\left[\left\|\mathbf{Y}_s - y_*^\alpha(\bar{x}_s)\mathbf{1}_n^\top\right\|^2 + \left\|\mathbf{Y}_{s-1} - y_*^\alpha(\bar{x}_{s-1})\mathbf{1}_n^\top\right\|^2\right] \\
&+ C_{vz}\mathbb{E}\left[\left\|\mathbf{Z}_s - z_*(\bar{x}_s)\mathbf{1}_n^\top\right\|^2 + \left\|\mathbf{Z}_{s-1} - z_*(\bar{x}_{s-1})\mathbf{1}_n^\top\right\|^2\right] + C_{vv}\mathbb{E}\left[\left\|\bar{v}_s\right\|^2\right] + \frac{6n(1+\rho^2)\sigma_x^2}{1-\rho^2}. \qquad (94)
\end{aligned}
$$

where the constants are given by

$$
\begin{aligned}
C_v &= \left(\frac{1+\rho^2}{2} + \frac{18\alpha^2\ell_{f,1}^2\eta_x^2(1+\rho^2)}{1-\rho^2}\right), \\
C_{vx} &= \frac{72\alpha^2\ell_{f,1}^2(1+\rho^2)}{1-\rho^2} = \mathcal{O}\left(\frac{\alpha^2}{1-\rho^2}\right),\ C_{vy} = \frac{18\alpha^2\ell_{g,1}^2(1+\rho^2)}{1-\rho^2} = \mathcal{O}\left(\frac{\alpha^2}{1-\rho^2}\right), \\
C_{vz} &= \frac{9\alpha^2\ell_{g,1}^2(1+\rho^2)}{1-\rho^2} = \mathcal{O}\left(\frac{\alpha^2}{1-\rho^2}\right), \\
C_{vv} &= \frac{3n\alpha^2\eta_x^2(1+\rho^2)(6\ell_{f,1}^2 + 6\ell_{g,1}^2\ell_{y_*}^2 + 3\ell_{g,1}^2\ell_{z_*}^2)}{1-\rho^2} = \mathcal{O}\left(\frac{n\alpha^2\eta_x^2}{1-\rho^2}\right).
\end{aligned}
$$

Especially, for $s = 0$, by (88) and $\Delta_0 = 0$ and $V_0 = 0$, if we initialize $y_0^{(i)} = z_0^{(i)}$ at each agent, we have

$$
\begin{aligned}
\mathbb{E}\left[\left\|\mathbf{V}_1 - \bar{v}_1\mathbf{1}_n^\top\right\|^2\right] &\le \frac{1+\rho^2}{2}\left\|\mathbf{V}_0 - \bar{v}_0\mathbf{1}_n^\top\right\|^2 + \frac{1+\rho^2}{1-\rho^2}\left\|\Delta_1 - \Delta_0\right\|^2 \\
&\le \frac{1+\rho^2}{2}\left\|\mathbf{V}_0 - \bar{v}_0\mathbf{1}_n^\top\right\|^2 + \frac{1+\rho^2}{1-\rho^2}\sum_{i=1}^n\left(\delta_1^{(i)}\right)^2 \\
&\le \frac{1+\rho^2}{2}\left\|\mathbf{V}_0 - \bar{v}_0\mathbf{1}_n^\top\right\|^2 + \frac{n\ell_{f,0}^2(1+\rho^2)}{1-\rho^2} \le \frac{n\ell_{f,0}^2(1+\rho^2)}{1-\rho^2}. \qquad (95)
\end{aligned}
$$

Combining the result at the initial iteration $s = 0$ and telescoping the inequality (94) for $s = 1, 2, \cdots, S$, we have

$$
\begin{aligned}
&\sum_{i=0}^S\mathbb{E}\left[\left\|\mathbf{V}_{s+1} - \bar{v}_{s+1}\mathbf{1}_n^\top\right\|^2\right] \\
\le& C_v\sum_{i=0}^S\mathbb{E}\left[\left\|\mathbf{V}_s - \bar{v}_s\mathbf{1}_n^\top\right\|^2\right] + C_{vx}\sum_{i=0}^{S-1}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}^\top\right\|^2\right] + C_{vy}\sum_{i=0}^S\mathbb{E}\left[\left\|\mathbf{Y}_s - y_*^\alpha(\bar{x}_s)\mathbf{1}_n^\top\right\|^2\right] \\
&+ C_{vy}\sum_{i=0}^{S-1}\mathbb{E}\left[\left\|\mathbf{Y}_s - y_*^\alpha(\bar{x}_s)\mathbf{1}_n^\top\right\|^2\right] + \frac{n\ell_{f,0}^2(1+\rho^2)}{1-\rho^2} + \frac{6nS(1+\rho^2)\sigma_x^2}{1-\rho^2} \\
&+ C_{vz}\sum_{i=0}^S\mathbb{E}\left[\left\|\mathbf{Z}_s - z_*(\bar{x}_s)\mathbf{1}_n^\top\right\|^2\right] + C_{vz}\sum_{i=0}^{S-1}\mathbb{E}\left[\left\|\mathbf{Z}_s - z_*(\bar{x}_s)\mathbf{1}_n^\top\right\|^2\right] + C_{vv}\sum_{i=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right] \\
\le& C_v\sum_{i=0}^S\mathbb{E}\left[\left\|\mathbf{V}_s - \bar{v}_s\mathbf{1}_n^\top\right\|^2\right] + C_{vx}\sum_{i=0}^{S-1}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}^\top\right\|^2\right] + 2C_{vy}\sum_{i=0}^S\mathbb{E}\left[\left\|\mathbf{Y}_s - y_*^\alpha(\bar{x}_s)\mathbf{1}_n^\top\right\|^2\right] \\
&+ 2C_{vz}\sum_{i=0}^S\mathbb{E}\left[\left\|\mathbf{Z}_s - z_*(\bar{x}_s)\mathbf{1}_n^\top\right\|^2\right] + C_{vv}\sum_{i=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right] + \frac{n\ell_{f,0}^2(1+\rho^2)}{1-\rho^2} + \frac{6nS(1+\rho^2)\sigma_x^2}{1-\rho^2}. \qquad (96)
\end{aligned}
$$

Applying the consensus convergence estimations of y, z in Lemma C.1 (see (65) and (86)) into (96), we have

$$
\begin{aligned}
&\sum_{i=0}^S\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{V}_{s+1} - \bar{v}_{s+1}\mathbf{1}_n^\top\right\|^2\right] \\
\lesssim& C_v\sum_{i=0}^S\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{V}_s - \bar{v}_s\mathbf{1}_n^\top\right\|^2\right] + C_{vx}\sum_{i=0}^{S-1}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2\right] + C_{vv}\sum_{i=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&+4C_{vz}\Big\{C_{z_*,0}\Delta_{z_*,0}+C_{Z,0}\Delta_{Z,0}+C_{U_z,0}\Delta_{U_z,0}+C_{z,v}\sum_{s=0}^{S}\mathbb{E}\left[\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\|^2\right]\\
&\quad+C_{z,vs}\sum_{s=0}^{S-1}\mathbb{E}\left[\|\bar{v}_{s+1}\|^2\right]+C_{z,x}\frac{1}{n}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]+S\cdot C_{z,\sigma}\sigma_z^2\Big\}+\frac{6S(1+\rho^2)\sigma_x^2}{1-\rho^2}\\
&+4C_{vy}\Big\{C_{y_*,0}\Delta_{y_*,0}+C_{Y,0}\Delta_{Y,0}+C_{U_y,0}\Delta_{U_y,0}+C_{y,vs}\sum_{s=0}^{S}\mathbb{E}\left[\|\bar{v}_{s+1}\|^2\right]\\
&\quad+C_{y,v}\sum_{s=0}^{S}\mathbb{E}\left[\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\|^2\right]+C_{y,x}\sum_{s=0}^{S}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]+SC_{y,\sigma}\sigma_y^2\Big\}+\frac{\ell_{f,0}^2(1+\rho^2)}{1-\rho^2}\\
&\lesssim C_v\sum_{i=0}^{S}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{V}_s-\bar{v}_s\mathbf{1}_n^\top\right\|^2\right]+\left(4C_{vz}C_{z,vs}+4C_{vy}C_{y,vs}+\frac{C_{vv}}{n}\right)\sum_{s=0}^{S}\mathbb{E}\left[\|\bar{v}_{s+1}\|^2\right]\\
&+\left(2C_{vz}C_{z,v}+2C_{vy}C_{y,v}\right)\sum_{s=0}^{S}\mathbb{E}\left[\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\|^2\right]+\left(C_{vx}+4C_{vz}C_{z,x}+4C_{vy}C_{y,x}\right)\sum_{i=0}^{S}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]\\
&+4C_{vz}\left(C_{z_*,0}\Delta_{z_*,0}+C_{Z,0}\Delta_{Z,0}+C_{U_z,0}\Delta_{U_z,0}\right)+4C_{vy}\left(C_{y_*,0}\Delta_{y_*,0}+C_{Y,0}\Delta_{Y,0}+C_{U_y,0}\Delta_{U_y,0}\right)\\
&+4S\left(C_{vz}C_{z,\sigma}\sigma_z^2+C_{vy}C_{y,\sigma}\sigma_y^2\right)+\frac{\ell_{f,0}^2(1+\rho^2)}{1-\rho^2}+\frac{6S(1+\rho^2)\sigma_x^2}{1-\rho^2}
\end{aligned}\tag{97}
$$

For sufficient small η_x such that

$$
C_v=\left(\frac{1+\rho^2}{2}+\frac{18\alpha^2\ell_{f,1}^2\eta_x^2(1+\rho^2)}{1-\rho^2}\right)\le\frac{3+\rho^2}{4},\tag{98}
$$

since $v_0^{(i)}=0$ for each agent i, dividing the both side of (99) by $1-C_v$ then

$$
\begin{aligned}
&\frac{1}{n}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbf{V}_{s+1}-\bar{v}_{s+1}\mathbf{1}_n^\top\right\|^2\right]\\
&\lesssim\frac{4}{1-\rho^2}\left(4C_{vz}C_{z,vs}+4C_{vy}C_{y,vs}+\frac{C_{vv}}{n}\right)\sum_{s=0}^{S}\mathbb{E}\left[\|\bar{v}_{s+1}\|^2\right]\\
&+\frac{4}{1-\rho^2}\left(4C_{vz}C_{z,v}+4C_{vy}C_{y,v}\right)\sum_{s=0}^{S}\mathbb{E}\left[\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\|^2\right]\\
&+\frac{16}{1-\rho^2}\left(C_{vx}+4C_{vz}C_{z,x}+4C_{vy}C_{y,x}\right)\sum_{i=0}^{S}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]\\
&+\frac{4C_{vz}}{1-\rho^2}\left(C_{z_*,0}\Delta_{z_*,0}+C_{Z,0}\Delta_{Z,0}+C_{U_z,0}\Delta_{U_z,0}\right)\\
&+\frac{16C_{vy}}{1-\rho^2}\left(C_{y_*,0}\Delta_{y_*,0}+C_{Y,0}\Delta_{Y,0}+C_{U_y,0}\Delta_{U_y,0}\right)+\frac{16S}{1-\rho^2}\left(C_{vz}C_{z,\sigma}\sigma_z^2+C_{vy}C_{y,\sigma}\sigma_y^2\right)\\
&+\frac{\ell_{f,0}^2(1+\rho^2)}{(1-\rho^2)^2}+\frac{S\sigma_x^2}{(1-\rho^2)^2}.
\end{aligned}\tag{99}
$$

Recalling that

$$
\left\|\mathbf{X}_{s+1}-\bar{x}_{s+1}\mathbf{1}_n^\top\right\|^2\le\frac{1+\rho^2}{2}\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2+\frac{(1+\rho^2)\eta_x^2}{1-\rho^2}\left\|\mathbf{V}_{s+1}-\bar{v}_{s+1}\mathbf{1}_n^\top\right\|^2,\tag{100}
$$

we have

$$
\frac{1}{n}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbf{X}_{s+1}-\bar{x}_{s+1}\mathbf{1}_n^\top\right\|^2\right]\le\frac{1+\rho^2}{2}\sum_{s=0}^{S}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]+\frac{(1+\rho^2)\eta_x^2}{1-\rho^2}\sum_{s=0}^{S}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{V}_{s+1}-\bar{v}_{s+1}\mathbf{1}_n^\top\right\|^2\right]
$$

$$\lesssim\left(\frac{1+\rho^2}{2}+\frac{16\eta_x^2}{(1-\rho^2)^2}\left(C_{vx}+4C_{vz}C_{z,x}+4C_{vy}C_{y,x}\right)\right)\sum_{s=0}^{S}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]$$
$$+\frac{(1+\rho^2)\eta_x^2}{(1-\rho^2)^2}\left(C_{vz}C_{z,v}+C_{vy}C_{y,v}\right)\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{i+1}|\mathcal{F}_i\right]\right\|^2\right]$$
$$+\frac{\eta_x^2}{(1-\rho^2)^2}\left(4C_{vz}C_{z,vs}+4C_{vy}C_{y,vs}+\frac{C_{vv}}{n}\right)\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{i+1}\right\|^2\right]$$
$$+\frac{\eta_x^2C_{vz}}{(1-\rho^2)^2}\left(C_{z_*,0}\Delta_{z_*,0}+C_{Z,0}\Delta_{Z,0}+C_{U_z,0}\Delta_{U_z,0}\right)$$
$$+\frac{C_{vy}\eta_x^2}{(1-\rho^2)^2}\left(C_{y_*,0}\Delta_{y_*,0}+C_{Y,0}\Delta_{Y,0}+C_{U_y,0}\Delta_{U_y,0}\right)$$
$$+\frac{\eta_x^2S}{1-\rho^2}\left(C_{vz}C_{z,\sigma}\sigma_z^2+C_{vy}C_{y,\sigma}\sigma_y^2\right)+\frac{\eta_x^2\ell_{f,0}^2(1+\rho^2)}{(1-\rho^2)^2}+\frac{S\eta_x^2\sigma_x^2}{(1-\rho^2)^2}. \tag{101}$$

Properly choosing $\eta_x\le\mathcal{O}\left((1-\rho^2)^2/(\alpha\ell_{g,1})\right)$ such that

$$\frac{1+\rho^2}{2}+\frac{16\eta_x^2}{(1-\rho^2)^2}\left(C_{vx}+4C_{vz}C_{z,x}+4C_{vy}C_{y,x}\right)\le\frac{3+\rho^2}{4} \tag{102}$$

and due to that $x_0^{(i)}=x_0$ for each agent, we have

$$\frac{1}{n}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbf{X}_{s+1}-\bar{x}_{s+1}\mathbf{1}_n^\top\right\|^2\right]\lesssim\frac{\eta_x^2}{(1-\rho^2)^3}\left(C_{vz}C_{z,v}+C_{vy}C_{y,v}\right)\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{i+1}|\mathcal{F}_i\right]\right\|^2\right]$$
$$+\frac{\eta_x^2}{(1-\rho^2)^3}\left(4C_{vz}C_{z,vs}+4C_{vy}C_{y,vs}+\frac{C_{vv}}{n}\right)\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{i+1}\right\|^2\right]$$
$$+\frac{\eta_x^2C_{vz}}{(1-\rho^2)^3}\left(C_{z_*,0}\Delta_{z_*,0}+C_{Z,0}\Delta_{Z,0}+C_{U_z,0}\Delta_{U_z,0}\right)$$
$$+\frac{C_{vy}\eta_x^2}{(1-\rho^2)^3}\left(C_{y_*,0}\Delta_{y_*,0}+C_{Y,0}\Delta_{Y,0}+C_{U_y,0}\Delta_{U_y,0}\right)$$
$$+\frac{\eta_x^2S}{(1-\rho^2)^3}\left(C_{vz}C_{z,\sigma}\sigma_z^2+C_{vy}C_{y,\sigma}\sigma_y^2\right)+\frac{S\eta_x^2\sigma_x^2}{(1-\rho^2)^3}+\frac{\eta_x^2\ell_{f,0}^2(1+\rho^2)}{(1-\rho^2)^3}. \tag{103}$$

where $\sigma_x^2=\sigma_f^2+2\alpha^2\sigma_g^2$, $\sigma_y^2=\sigma_f^2+\alpha^2\sigma_g^2$, $\sigma_z^2=\sigma_g^2$. Note that the symbol $\lesssim$ indicates that there exists an absolute constant C such that LHS $\le C$ RHS. Now we have completed the proof. ☐

D Appendix / Convergence complexity in Theorem 4.1

To derive the convergence rate in Theorem 4.1, we first have the following sufficient decrease lemma.

Lemma D.1. *Suppose Assumptions 1 and 2 hold. The stepsize η_x satisfies $\eta_x\le 1/\ell_\Gamma$ for all s. We have*

$$\frac{\eta_x}{2}\left\|\nabla\Gamma^\alpha(\bar{x}_s)\right\|^2+\left(\frac{\eta_x}{2}-\frac{\eta_x^2\ell_\Gamma}{2}\right)\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2$$
$$\le\Gamma^\alpha(\bar{x}_s)-\mathbb{E}\left[\Gamma^\alpha(\bar{x}_{s+1})|\mathcal{F}_s\right]+\frac{\eta_x}{2}\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]-\nabla\Gamma^\alpha(\bar{x}_s)\right\|^2+\frac{\ell_\Gamma\eta_x^2\sigma^2}{2n}.$$

Proof. (of Lemma D.1) Note that in Algorithm 1 we have

$$\bar{x}_{s+1}=\bar{x}_s-\eta_x\bar{v}_{s+1}. \tag{104}$$

By smoothness of $\Gamma^\alpha(x)$, we have

$$\Gamma^\alpha(\bar{x}_{s+1})-\Gamma^\alpha(\bar{x}_s)-\langle\nabla\Gamma^\alpha(\bar{x}_s),\bar{x}_{s+1}-\bar{x}_s\rangle\le\frac{\ell_\Gamma}{2}\left\|\bar{x}_{s+1}-\bar{x}_s\right\|^2 \tag{105}$$

and thus

$$\mathbb{E}\left[\Gamma^\alpha(\bar{x}_{s+1})|\mathcal{F}_s\right] - \Gamma^\alpha(\bar{x}_s) + \eta_x \left\langle \nabla\Gamma^\alpha(\bar{x}_s), \mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\rangle \leq \frac{\ell_\Gamma \eta_x^2}{2}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\middle|\mathcal{F}_s\right] \tag{106}$$

which implies

$$\begin{aligned}&\frac{\eta_x}{2}\left(\|\nabla\Gamma^\alpha(\bar{x}_s)\|^2 + \|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\|^2 - \|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right] - \nabla\Gamma^\alpha(\bar{x}_s)\|^2\right)\\ \leq&\Gamma^\alpha(\bar{x}_s) - \mathbb{E}\left[\Gamma^\alpha(\bar{x}_{s+1})|\mathcal{F}_s\right] + \frac{\ell_\Gamma\eta_x^2\sigma^2}{2n} + \frac{\ell_\Gamma\eta_x^2}{2}\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\|^2 .\end{aligned} \tag{107}$$

Rearranging terms on both sides completes the proof. □

Note that following the analysis in [6], $\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\|^2$ should not be thrown away since it will be used later in the analysis. Another commonly used approach is to incorporate the moving-average updates in the algorithm, which has been used in distributed optimization [59, 29].

Now we analyze $\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right] - \nabla\Gamma^\alpha(\bar{x}_s)\|$.

Lemma D.2. *Suppose Assumptions 1, 2, and 3 hold. We have*

$$\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right] - \nabla\Gamma^\alpha(\bar{x}_s)\|^2 \leq \frac{3\ell_{x,1}^2}{n}\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2 + \frac{3\ell_{y,1}^2}{n}\left\|\mathbf{Y}_s - y_*^\alpha(\bar{x}_s)\mathbf{1}_n^\top\right\|^2 + \frac{3\alpha^2\ell_{z,1}^2}{n}\left\|\mathbf{Z}_s - z_*(\bar{x}_s)\mathbf{1}_n^\top\right\|^2$$

where $\ell_{x,1}^2 = \ell_{f,1}^2 + 2\alpha^2\ell_{g,1}^2$, $\ell_{y,1}^2 = \ell_{f,1}^2 + \alpha^2\ell_{g,1}^2$, *and* $\ell_{z,1}^2 = \ell_{g,1}^2$.

Proof. (of Lemma D.2) By Lemma A.7 we know it suffices to analyze $\left\|\mathbb{E}\left[\bar{\delta}_{s+1}\middle|\mathcal{F}_s\right] - \nabla\Gamma^\alpha(\bar{x}_s)\right\|$.

$$\begin{aligned}&\left\|\mathbb{E}\left[\bar{\delta}_{s+1}\middle|\mathcal{F}_s\right] - \nabla_x\mathcal{L}^\alpha(\bar{x}_s, y_*^\alpha(\bar{x}_s), z_*(\bar{x}_s))\right\|^2\\ =&\left\|\frac{1}{n}\sum_{i=1}^n\mathbb{E}\left[\delta_{s+1}^{(i)}\middle|\mathcal{F}_s\right] - \nabla_x\mathcal{L}^\alpha(\bar{x}_s, y_*^\alpha(\bar{x}_s), z_*(\bar{x}_s))\right\|^2\\ =&\frac{3}{n^2}\left\|\sum_{i=1}^n\nabla_x f_i(x_s^{(i)}, y_s^{(i)}) - \nabla_x f_i(\bar{x}_s, y_*^\alpha(\bar{x}_s))\right\|^2 + \frac{3\alpha^2}{n^2}\left\|\sum_{i=1}^n\nabla_x g_i(x_s^{(i)}, y_s^{(i)}) - \nabla_x g_i(\bar{x}_s, y_*^\alpha(\bar{x}_s))\right\|^2\\ &+\frac{3\alpha^2}{n^2}\left\|\sum_{i=1}^n\nabla_x g_i(x_s^{(i)}, z_s^{(i)}) - \nabla_x g_i(\bar{x}_s, z_*(\bar{x}_s))\right\|^2\\ \leq&\frac{3\ell_{f,1}^2}{n}\sum_{i=1}^n\left(\left\|x_s^{(i)} - \bar{x}_s\right\|^2 + \left\|y_s^{(i)} - y_*^\alpha(\bar{x}_s)\right\|^2\right) + \frac{3\alpha^2\ell_{g,1}^2}{n}\sum_{i=1}^n\left(\left\|x_s^{(i)} - \bar{x}_s\right\|^2 + \left\|y_s^{(i)} - y_*^\alpha(\bar{x}_s)\right\|^2\right)\\ &+\frac{3\alpha^2\ell_{g,1}^2}{n}\sum_{i=1}^n\left(\left\|x_s^{(i)} - \bar{x}_s\right\|^2 + \left\|z_s^{(i)} - z_*(\bar{x}_s)\right\|^2\right)\\ =&\frac{3\left(\ell_{f,1}^2 + 2\alpha^2\ell_{g,1}^2\right)}{n}\left\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n^\top\right\|^2 + \frac{3\left(\ell_{f,1}^2 + \alpha^2\ell_{g,1}^2\right)}{n}\left\|\mathbf{Y}_s - y_*^\alpha(\bar{x}_s)\mathbf{1}_n^\top\right\|^2 + \frac{3\alpha^2\ell_{g,1}^2}{n}\left\|\mathbf{Z}_s - z_*(\bar{x}_s)\mathbf{1}_n^\top\right\|^2 .\end{aligned}$$

□

Theorem 4.1. *Suppose Assumptions 1, 2, 3, and 4 hold, and parameters α and step sizes are chosen such that*

$$\alpha = \Theta\left((nS)^{1/7}\right), \eta_x = \Theta\left(\frac{n^{2/7}}{S^{5/7}}\right), \eta_y = \Theta\left(\frac{n^{2/7}}{S^{5/7}}\right), \eta_z = \Theta\left(\frac{n^{3/7}}{S^{4/7}}\right)$$

and further assume a warm-start for variables y, z such that

$$\max\left(\left\|\bar{y}_0 - y_{*,0}^\alpha\right\|^2, \|\bar{z}_0 - z_{*,0}\|^2\right) = \mathcal{O}\left(1/\alpha\right) \tag{10}$$

Consider Algorithm 1 with $T = 1$ and $S \geq n^{4/3}$, we have

$$\min_{0\leq s\leq S-1}\mathbb{E}\left[\|\nabla\Phi(\bar{x}_s)\|\right] \leq \mathcal{O}\left(\frac{1}{(nS)^{1/7}}\right), \min_{0\leq s\leq S-1}\frac{\mathbb{E}\left[\|\mathbf{X}_s - \bar{x}_s\mathbf{1}_n\|\right]}{n} \leq \mathcal{O}\left(\frac{1}{n^{1/14}S^{4/7}}\right).$$

Proof. (of Theorem 4.1) Incorporating the result of Lemma D.2 into the sufficient decrease condition in Lemma D.1 and telescoping the inequality from $s=0$ to $S-1$, we have

$$\begin{aligned}
&\sum_{s=0}^{S-1}\frac{\eta_x}{2}\left\|\nabla\Gamma^\alpha(\bar{x}_s)\right\|^2+\left(\frac{\eta_x}{2}-\frac{\eta_x^2\ell_\Gamma}{2}\right)\sum_{s=0}^{S-1}\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\\
\le&\sum_{s=0}^{S-1}\Gamma^\alpha(\bar{x}_s)-\mathbb{E}\left[\Gamma^\alpha(\bar{x}_{s+1})|\mathcal{F}_s\right]+\frac{\eta_x}{2}\sum_{s=0}^{S-1}\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]-\nabla\Gamma^\alpha(\bar{x}_s)\right\|^2+\sum_{s=0}^{S-1}\frac{\ell_\Gamma\eta_x^2\sigma_x^2}{2n}\\
\le&\Gamma^\alpha(\bar{x}_0)-\mathbb{E}\left[\Gamma^\alpha(\bar{x}_S)|\mathcal{F}_S\right]+\frac{\eta_x}{2}\sum_{s=0}^{S-1}\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]-\nabla\Gamma^\alpha(\bar{x}_s)\right\|^2+S\frac{\ell_\Gamma\eta_x^2\sigma_x^2}{2n}\\
\le&\Delta_{x,0}+\frac{\eta_x}{2}\sum_{s=0}^{S-1}\left(\frac{3\ell_{x,1}^2}{n}\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2+\frac{3\ell_{y,1}^2}{n}\left\|\mathbf{Y}_s-y_*^\alpha(\bar{x}_s)\mathbf{1}_n^\top\right\|^2+\frac{3\alpha^2\ell_{z,1}^2}{n}\left\|\mathbf{Z}_s-z_*(\bar{x}_s)\mathbf{1}_n^\top\right\|^2\right)\\
&+S\frac{\ell_\Gamma\eta_x^2\sigma_x^2}{2n}
\end{aligned}\tag{108}$$

where $\Delta_{x,0}=\Gamma^\alpha(\bar{x}_0)-\Gamma^\alpha(x^*)$. Incorporating the consensus results of Y and Z in Lemma C.1, we have

$$\begin{aligned}
&\sum_{s=0}^{S-1}\frac{\eta_x}{2}\left\|\nabla\Gamma^\alpha(\bar{x}_s)\right\|^2+\left(\frac{\eta_x}{2}-\frac{\eta_x^2\ell_\Gamma}{2}\right)\sum_{s=0}^{S-1}\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\\
\lesssim&\Delta_{x,0}+S\frac{\ell_\Gamma\eta_x^2\sigma_x^2}{2n}+\frac{3\eta_x\ell_{y,1}^2}{2}\Bigg\{C_{y_*,0}\Delta_{y_*,0}+C_{Y,0}\Delta_{Y,0}+C_{U_y,0}\Delta_{U_y,0}+C_{y,vs}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right]\\
&+C_{y,v}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right]+C_{y,x}\sum_{s=0}^{S-1}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]+SC_{y,\sigma}\sigma_y^2\Bigg\}\\
&+\frac{3\eta_x\alpha^2\ell_{z,1}^2}{2}\Bigg\{C_{z_*,0}\Delta_{z_*,0}+C_{Z,0}\Delta_{Z,0}+C_{U_z,0}\Delta_{U_z,0}+C_{z,v}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right]\\
&+C_{z,vs}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right]+C_{z,x}\frac{1}{n}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}^\top\right\|^2\right]+S\cdot C_{z,\sigma}\sigma_z^2\Bigg\}\\
&+\frac{3\eta_x\ell_{x,1}^2}{2}\sum_{s=0}^{S-1}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]\\
\lesssim&\Delta_{x,0}+\frac{3\eta_x\alpha^2}{2}\left(C_{y_*,0}\Delta_{y_*,0}+C_{Y,0}\Delta_{Y,0}+C_{U_y,0}\Delta_{U_y,0}+C_{z_*,0}\Delta_{z_*,0}+C_{Z,0}\Delta_{Z,0}+C_{U_z,0}\Delta_{U_z,0}\right)\\
&+\frac{3\eta_x\alpha^2}{2}\left(C_{y,vs}+C_{z,vs}\right)\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right]+\frac{3\eta_x\alpha^2}{2}\left(C_{y,v}+C_{z,v}\right)\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right]\\
&+\frac{3\eta_x\alpha^2}{2}\left(C_{y,x}+C_{z,x}+1\right)\sum_{s=0}^{S-1}\frac{1}{n}\mathbb{E}\left[\left\|\mathbf{X}_s-\bar{x}_s\mathbf{1}_n^\top\right\|^2\right]+\frac{S\eta_x^2\sigma_x^2}{2n}+\frac{S\eta_x\alpha^2}{2}\left(C_{y,\sigma}\sigma_y^2+C_{z,\sigma}\sigma_z^2\right)
\end{aligned}\tag{109}$$

where $\ell_{x,1}^2=\mathcal{O}(\alpha^2)$, $\ell_{y,1}^2=\mathcal{O}(\alpha^2)$ and $\ell_{z,1}^2=\mathcal{O}(1)$. For simplicity, we let $C_X=C_{y,x}+C_{z,x}+1$. Then incorporating the sum w.r.t $\mathbf{X}_s$ in Lemma C.2 and dividing $\eta_x S$ on both side, we achieve that

$$\begin{aligned}
&\frac{1}{2S}\sum_{s=0}^{S-1}\left\|\nabla\Gamma^\alpha(\bar{x}_s)\right\|^2+\left(\frac{1}{2S}-\frac{\eta_x\ell_\Gamma}{2S}\right)\sum_{s=0}^{S-1}\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\\
\lesssim&\frac{\Delta_{x,0}}{\eta_x S}+\frac{\alpha^2}{S}\left(C_{y_*,0}\Delta_{y_*,0}+C_{Y,0}\Delta_{Y,0}+C_{U_y,0}\Delta_{U_y,0}+C_{z_*,0}\Delta_{z_*,0}+C_{Z,0}\Delta_{Z,0}+C_{U_z,0}\Delta_{U_z,0}\right)\\
&+\frac{\alpha^2}{S}\left(C_{y,vs}+C_{z,vs}\right)\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{s+1}\right\|^2\right]+\frac{\alpha^2}{S}\left(C_{y,v}+C_{z,v}\right)\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right]
\end{aligned}$$

$$+\frac{\eta_x\sigma_x^2}{2n}+\frac{\alpha^2}{2}\left(C_{y,\sigma}\sigma_y^2+C_{z,\sigma}\sigma_z^2\right)+\frac{\alpha^2 C_X}{S}\Bigg\{\frac{\eta_x^2\left(C_{vz}C_{z,v}+C_{vy}C_{y,v}\right)}{(1-\rho^2)^3}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{i+1}|\mathcal{F}_i\right]\right\|^2\right]$$
$$+\frac{\eta_x^2}{(1-\rho^2)^3}\left(4C_{vz}C_{z,vs}+4C_{vy}C_{y,vs}+\frac{C_{vv}}{n}\right)\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{i+1}\right\|^2\right]$$
$$+\frac{\eta_x^2C_{vz}}{(1-\rho^2)^3}\left(C_{z_*,0}\Delta_{z_*,0}+C_{Z,0}\Delta_{Z,0}+C_{U_z,0}\Delta_{U_z,0}\right)$$
$$+\frac{C_{vy}\eta_x^2}{(1-\rho^2)^3}\left(C_{y_*,0}\Delta_{y_*,0}+C_{Y,0}\Delta_{Y,0}+C_{U_y,0}\Delta_{U_y,0}\right)$$
$$+\frac{\eta_x^2S}{(1-\rho^2)^3}\left(C_{vz}C_{z,\sigma}\sigma_z^2+C_{vy}C_{y,\sigma}\sigma_y^2\right)+\frac{S\eta_x^2\sigma_x^2}{(1-\rho^2)^3}+\frac{\eta_x^2\ell_{f,0}^2(1+\rho^2)}{(1-\rho^2)^3}\Bigg\}$$
$$\lesssim\frac{\Delta_{x,0}}{\eta_xS}+\frac{\alpha^2}{S}\left((C_{y,v}+C_{z,v})+C_X\frac{\eta_x^2}{(1-\rho^2)^3}\left(C_{vz}C_{z,v}+C_{vy}C_{y,v}\right)\right)\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{i+1}|\mathcal{F}_i\right]\right\|^2\right]$$
$$+\frac{\alpha^2}{S}\left((C_{y,vs}+C_{z,vs})+C_X\frac{\eta_x^2}{(1-\rho^2)^3}\left(4C_{vz}C_{z,vs}+4C_{vy}C_{y,vs}+\frac{C_{vv}}{n}\right)\right)\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\bar{v}_{i+1}\right\|^2\right]$$
$$+\frac{\eta_x\sigma_x^2}{2n}+\frac{\alpha^2}{2}\left(C_{y,\sigma}\sigma_y^2+C_{z,\sigma}\sigma_z^2\right)+\alpha^2C_X\left(\frac{\eta_x^2}{(1-\rho^2)^3}\left(C_{vz}C_{z,\sigma}\sigma_z^2+C_{vy}C_{y,\sigma}\sigma_y^2\right)+\frac{\eta_x^2\sigma_x^2}{(1-\rho^2)^3}\right)$$
$$+\frac{\alpha^2}{S}C_X\frac{\eta_x^2\ell_{f,0}^2(1+\rho^2)}{(1-\rho^2)^3}+\frac{\alpha^2}{S}\left(1+C_X\frac{C_{vy}\eta_x^3}{(1-\rho^2)^3}\right)\left(C_{y_*,0}\Delta_{y_*,0}+C_{Y,0}\Delta_{Y,0}+C_{U_y,0}\Delta_{U_y,0}\right)$$
$$+\frac{\alpha^2}{S}\left(1+C_X\frac{\eta_x^2C_{vz}}{(1-\rho^2)^3}\right)\left(C_{z_*,0}\Delta_{z_*,0}+C_{Z,0}\Delta_{Z,0}+C_{U_z,0}\Delta_{U_z,0}\right). \tag{110}$$

Incorporating the inequality that

$$\mathbb{E}\left[\left\|\bar{v}_i\right\|^2\right]\le\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_i|\mathcal{F}_i\right]\right\|^2\right]+\mathcal{O}\left(\frac{\sigma_x^2}{n}\right), \tag{111}$$

we have

$$\frac{1}{2S}\sum_{s=0}^{S-1}\left\|\nabla\Gamma^\alpha(\bar{x}_s)\right\|^2+\left(\frac{1}{2S}-\frac{\eta_x\ell_\Gamma}{2S}\right)\sum_{s=0}^{S-1}\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2$$
$$\lesssim\frac{\Delta_{x,0}}{\eta_xS}+\frac{\alpha^2}{S}\left((C_{y,v}+C_{z,v})+C_X\frac{\eta_x^2}{(1-\rho^2)^3}\left(C_{vz}C_{z,v}+C_{vy}C_{y,v}\right)\right)\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{i+1}|\mathcal{F}_i\right]\right\|^2\right]$$
$$+\frac{\alpha^2}{S}\left((C_{y,vs}+C_{z,vs})+\frac{C_X\eta_x^2}{(1-\rho^2)^3}\left(4C_{vz}C_{z,vs}+4C_{vy}C_{y,vs}+\frac{C_{vv}}{n}\right)\right)\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{i+1}|\mathcal{F}_i\right]\right\|^2\right]$$
$$+\frac{\alpha^2}{S}\left((C_{y,vs}+C_{z,vs})+C_X\frac{\eta_x^2}{(1-\rho^2)^3}\left(4C_{vz}C_{z,vs}+4C_{vy}C_{y,vs}+\frac{C_{vv}}{n}\right)\right)\frac{S\sigma_x^2}{n}$$
$$+\frac{\eta_x\sigma_x^2}{2n}+\frac{\alpha^2}{2}\left(C_{y,\sigma}\sigma_y^2+C_{z,\sigma}\sigma_z^2\right)+\frac{\alpha^2C_X\eta_x^2}{(1-\rho^2)^3}\left(C_{vz}C_{z,\sigma}\sigma_z^2+C_{vy}C_{y,\sigma}\sigma_y^2+\sigma_x^2\right)$$
$$+\frac{\alpha^2}{S}C_X\frac{\eta_x^2\ell_{f,0}^2(1+\rho^2)}{(1-\rho^2)^3}+\frac{\alpha^2}{S}\left(1+C_X\frac{C_{vy}\eta_x^3}{(1-\rho^2)^3}\right)\left(C_{y_*,0}\Delta_{y_*,0}+C_{Y,0}\Delta_{Y,0}+C_{U_y,0}\Delta_{U_y,0}\right)$$
$$+\frac{\alpha^2}{S}\left(1+C_X\frac{\eta_x^2C_{vz}}{(1-\rho^2)^3}\right)\left(C_{z_*,0}\Delta_{z_*,0}+C_{Z,0}\Delta_{Z,0}+C_{U_z,0}\Delta_{U_z,0}\right). \tag{112}$$

We consider Algorithm 1 with $T=1$ and variance $\sigma_x^2\sim\Theta\left(\alpha^2\sigma^2\right),\sigma_y^2\sim\Theta\left(\alpha^2\sigma^2\right),\sigma_z^2=\Theta\left(\sigma^2\right)$ (defined in Lemma C.2), to ensure that

$$r_z=\left(1+a_1\eta_x\ell_{z_*}+\frac{a_2\ell_{\nabla z_*}c_\delta\eta_x^2\alpha^2}{2}\right)\left(1-\frac{2\mu_g\eta_z}{3}\right)^T\le1-\frac{\mu_g\eta_z}{3}, \tag{113}$$

we set

$$a_1 \lesssim \frac{\eta_z}{\eta_x}; \quad a_2 \lesssim \frac{\eta_z}{\eta_x^2\alpha^2}. \tag{114}$$

Similarly, to guarantee that

$$r_y = \left(1 + a_1\eta_x\ell_{y_*} + \frac{a_2\ell_{\nabla y_*}c_\delta\eta_x^2\alpha^2}{2}\right)\left(1 - \frac{\mu_g\eta_y\alpha}{3}\right)^T \le 1 - \frac{\mu_g\eta_y\alpha}{6}. \tag{115}$$

we might choose

$$a_1 \lesssim \frac{\alpha\eta_y}{\eta_x}; \quad a_2 \lesssim \frac{\alpha\eta_y}{\eta_x^2\alpha^2}. \tag{116}$$

Thus to make sure the two conditions (113) and (115) both hold, we choose $\alpha\eta_y$ and η_z are in the same scale, then a_1 and a_2 are well-defined. Setting

$$\alpha\eta_y \sim \eta_z \gg \frac{1}{S}, \tag{117}$$

and C_X can be simplified as

$$\begin{aligned} C_X &= C_{y,x} + C_{z,x} + 1 \\ &\sim \mathcal{O}\left(\min\left(T, \frac{1}{\alpha\eta_y}\right) + \frac{e_y^T\alpha^4\eta_y^4}{(1-\rho^2)^7} + \frac{e_{\rho,1}^T\alpha^2\eta_y^2}{(1-\rho^2)^6} + \min\left(T, \frac{1}{\alpha\eta_y}\right)\frac{\alpha^4\eta_y^4}{(1-\rho^2)^4}\right) \\ &+ \mathcal{O}\left(\frac{e_z^T\eta_z^4}{(1-\rho^2)^7} + \frac{e_{\rho,2}^T e_z^T\eta_z^2}{(1-\rho^2)^5} + \min\left(T, \frac{1}{\eta_z}\right)\right) + 1 \\ &\sim \mathcal{O}(1). \end{aligned} \tag{118}$$

Combining this and replacing $e_y^T, e_{\rho,1}^T, e_{\rho,2}^T$ and e_z^T with 1 and let $a_1 \sim \frac{\alpha\eta_y}{\eta_x} \sim \frac{\eta_z}{\eta_x}$ and $a_2 \sim \frac{\eta_z}{\eta_x^2\alpha^2} \sim \frac{\alpha\eta_y}{\eta_x^2\alpha^2}$, the inequality (112) can be simplified

$$\begin{aligned} &\frac{1}{2S}\sum_{s=0}^{S-1}\|\nabla\Gamma^\alpha(\bar{x}_s)\|^2 + \left(\frac{1}{2S} - \frac{\eta_x\ell_\Gamma}{2S}\right)\sum_{s=0}^{S-1}\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\|^2 \\ \lesssim &\frac{\Delta_{x,0}}{\eta_x S} + \frac{\alpha^2}{S}\left(\left(\frac{\eta_x^2}{\alpha^2\eta_y^2} + \frac{\eta_x^2}{\eta_z^2}\right) + \frac{\eta_x^2}{(1-\rho^2)^4}\left(\frac{\alpha^2\eta_x^2}{\eta_z^2} + \frac{\eta_x^2}{\eta_y^2}\right)\right)\sum_{s=0}^{S-1}\mathbb{E}\left[\|\mathbb{E}\left[\bar{v}_{i+1}|\mathcal{F}_i\right]\|^2\right] \\ &+\frac{\alpha^2}{S}\left(\eta_x^2\alpha^2 + \frac{\eta_x^2}{\alpha\eta_y} + \frac{\eta_x^4}{\eta_y^2} + \frac{\alpha^2\eta_x^4}{\eta_z^2}\right)\sum_{s=0}^{S-1}\mathbb{E}\left[\|\mathbb{E}\left[\bar{v}_{i+1}|\mathcal{F}_i\right]\|^2\right] \\ &+\frac{\alpha^2}{S}\frac{\eta_x^4\alpha^2}{(1-\rho^2)^3}\left(\alpha^2 + \frac{1}{\alpha\eta_y} + \frac{\eta_x^2}{\eta_y^2} + \frac{\alpha^2\eta_x^2}{\eta_z^2}\right)\sum_{s=0}^{S-1}\mathbb{E}\left[\|\mathbb{E}\left[\bar{v}_{i+1}|\mathcal{F}_i\right]\|^2\right] \\ &+\left(\left(\eta_x^2\alpha^2 + \frac{\eta_x^2}{\alpha\eta_y} + \frac{\eta_x^4}{\eta_y^2} + \frac{\alpha^2\eta_x^4}{\eta_z^2}\right) + \frac{\eta_x^4\alpha^2}{(1-\rho^2)^3}\left(\alpha^2 + \frac{1}{\alpha\eta_y} + \frac{\eta_x^2}{\eta_y^2} + \frac{\alpha^2\eta_x^2}{\eta_z^2}\right)\right)\frac{\alpha^4\sigma^2}{n} \\ &+\frac{\eta_x\alpha^2\sigma^2}{n} + \frac{\alpha^2\sigma^2}{2}\left(\frac{\alpha\eta_y}{n} + \frac{\eta_z}{n}\right) + \alpha^2\left(\frac{\eta_x^2\alpha^2\sigma^2}{(1-\rho^2)^4}\left(\frac{\alpha\eta_y}{n} + \frac{\eta_z}{n}\right) + \frac{\eta_x^2\alpha^2\sigma^2}{(1-\rho^2)^3}\right) \\ &+\frac{\alpha^2}{S}\frac{\eta_x^2}{(1-\rho^2)^3} + \frac{\alpha^2}{S}\left(1 + \frac{\alpha^2\eta_x^3}{(1-\rho^2)^4}\right)\left(\frac{1}{\alpha\eta_y}\Delta_{y_*,0} + \Delta_{Y,0} + \eta_y^2\Delta_{U_y,0}\right) \\ &+\frac{\alpha^2}{S}\left(1 + \frac{\eta_x^2\alpha^2}{(1-\rho^2)^3}\right)\left(\frac{1}{\eta_z}\Delta_{z_*,0} + \Delta_{Z,0} + \eta_z^2\Delta_{U_z,0}\right) \end{aligned} \tag{119}$$

where

$$C_{y,vs} \sim \eta_x^2\alpha^2 + \frac{\eta_x^2}{\alpha\eta_y} + \frac{\eta_x^4}{\eta_y^2}; \quad C_{z,vs} \sim \frac{\eta_x^2}{\eta_z} + \frac{\alpha^2\eta_x^4}{\eta_z^2}$$

$$C_{vz} \sim \frac{\alpha^2}{1-\rho^2}; \quad C_{vy} \sim \frac{\alpha^2}{1-\rho^2}; \quad C_{y,\sigma} \sim \frac{\eta_y}{n\alpha} + \eta_y^2; \quad C_{z,\sigma} \sim \frac{\eta_z}{n} + \eta_z^2$$

Now we choose the parameters α and stepsizes as follows.

$$\alpha = \Theta\left((nS)^{1/7}\right), \eta_x = \eta_y = \Theta\left(\frac{n^{2/7}}{S^{5/7}}\right), \alpha\eta_y = \eta_z = \Theta\left(\frac{n^{3/7}}{S^{4/7}}\right), \tag{120}$$

and set a warm-start for y, z such that

$$\Delta_{y_*,0}, \Delta_{z_*,0} \leq \mathcal{O}\left(1/\alpha\right). \tag{121}$$

By properly choosing the parameters as in (120), (121) and $S \geq n^{4/3}$, the coefficient of $\frac{1}{S}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2\right]$ of RHS in (119) is smaller than that of LHS. We thus conclude

$$\begin{aligned}
&\sum_{s=0}^{S-1}\frac{1}{2S}\mathbb{E}\left[\left\|\nabla\Gamma^\alpha(\bar{x}_s)\right\|^2\right] + \frac{1}{2S}\sum_{s=0}^{S-1}\left\|\mathbb{E}\left[\bar{v}_{s+1}|\mathcal{F}_s\right]\right\|^2 \\
\lesssim &\frac{\Delta_{x,0}}{(nS)^{2/7}} + \frac{1}{n^{9/7}S^{6/7}} + \frac{\sigma^2}{(nS)^{2/7}} + \frac{1}{(nS)^{2/7}} = \mathcal{O}\left(\frac{1}{(nS)^{2/7}}\right).
\end{aligned} \tag{122}$$

Note that the difference $\Delta_{x,0} = \Gamma^\alpha(x_0) - \Gamma^\alpha(x^*)$ can be controlled by a constant which is independent with α:

$$\begin{aligned}
\Delta_{x,0} =&\Gamma^\alpha(x_0) - \Gamma^\alpha(x^*) = \mathcal{L}^\alpha(x_0, y_*^\alpha(x_0), z_*(x_0)) - \mathcal{L}^\alpha(x^*, y_*^\alpha(x^*), z_*(x^*)) \\
=&f(x_0, y_*^\alpha(x_0)) - f(x^*, y_*^\alpha(x^*)) + \alpha\left(g(x_0, y_*^\alpha(x_0)) - g(x_0, z_*(x_0))\right) \\
&+ \alpha\left(g(x^*, y_*^\alpha(x^*)) - g(x^*, z_*(x^*))\right) \\
=&f(x_0, y_*(x_0)) - f(x^*, y_*(x^*)) + f(x_0, y_*^\alpha(x_0)) - f(x_0, y_*(x_0)) \\
&+ f(x^*, y_*(x^*)) - f(x^*, y_*^\alpha(x^*)) + \alpha\left(g(x_0, y_*^\alpha(x_0)) - g(x_0, z_*(x_0))\right) \\
&+ \alpha\left(g(x^*, y_*^\alpha(x^*)) - g(x^*, z_*(x^*))\right) \\
\leq&\Phi(x_0) - \Phi(x^*) + \ell_{f,0}\left\|y_*^\alpha(x_0) - y_*(x_0)\right\| + \ell_{f,0}\left\|y_*^\alpha(x^*) - y_*(x^*)\right\| \\
&+ \alpha\frac{\ell_{g,1}}{2}\left\|y_*^\alpha(x_0) - z_*(x_0)\right\|^2 + \alpha\frac{\ell_{g,1}}{2}\left\|y_*^\alpha(x^*) - z_*(x^*)\right\|^2 \\
\leq&\Phi(x_0) - \Phi(x^*) + \frac{2\ell_{f,0}C_0}{\alpha} + 2\alpha\frac{\ell_{g,1}}{2}\frac{C_0^2}{\alpha^2} \\
\leq&\Phi(x_0) - \Phi(x^*) + \frac{2\ell_{f,0}C_0\mu_g}{\ell_{f,1}} + \frac{\ell_{g,1}C_0^2\mu_g}{\ell_{f,1}} = \Phi(x_0) - \Phi(x^*) + \mathcal{O}\left(\kappa^2\ell_{g,1}\right),
\end{aligned} \tag{123}$$

where the first inequality follows from the gradient-Lipschitz of g and the Lipschitz continuity of f in y, and the second inequality uses Lemma A.10.

We then recall the relationship of $\left\|\nabla\Phi(\bar{x}_s)\right\|^2$ and $\left\|\nabla\Gamma^\alpha(\bar{x}_s)\right\|^2$ in Lemma 2.1, we have

$$\sum_{s=0}^{S-1}\frac{1}{2S}\mathbb{E}\left[\left\|\nabla\Phi(\bar{x}_s)\right\|^2\right] \leq 2\sum_{s=0}^{S-1}\frac{1}{2S}\mathbb{E}\left[\left\|\nabla\Gamma^\alpha(\bar{x}_s)\right\|^2\right] + \frac{2}{\alpha^2} \leq \mathcal{O}\left(\frac{1}{(nS)^{2/7}}\right).$$

We then notice that

$$\left(\frac{1}{S}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\nabla\Phi(\bar{x}_s)\right\|\right]\right)^2 \leq \frac{1}{S}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\nabla\Phi(\bar{x}_s)\right\|\right]^2 \leq \frac{1}{S}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\nabla\Phi(\bar{x}_s)\right\|^2\right]$$

where the first inequality uses Cauchy-Schwarz inequality and the second one uses Jensen's inequality. Hence we know

$$\frac{1}{S}\sum_{s=0}^{S-1}\mathbb{E}\left[\left\|\nabla\Phi(\bar{x}_s)\right\|\right] \leq \mathcal{O}\left(\frac{1}{(nS)^{1/7}}\right).$$

Moreover, we notice that from Lemma C.2, (120), and (121) we know

$$\begin{aligned}
&\frac{1}{n}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbf{X}_{s+1} - \bar{x}_{s+1}\mathbf{1}_n^\top\right\|^2\right] \\
\lesssim &\frac{\eta_x^2}{(1-\rho^2)^3}\left(C_{vz}C_{z,v} + C_{vy}C_{y,v}\right)\sum_{i=0}^{S}\mathbb{E}\left[\left\|\mathbb{E}\left[\bar{v}_{i+1}|\mathcal{F}_i\right]\right\|^2\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\eta_x^2}{(1-\rho^2)^3}\left(4C_{vz}C_{z,vs}+4C_{vy}C_{y,vs}+\frac{C_{vv}}{n}\right)\sum_{i=0}^{S-1}\mathbb{E}\left[\|\bar{v}_{i+1}\|^2\right]\\
&+\frac{\eta_x^2 C_{vz}}{(1-\rho^2)^3}\left(C_{z_*,0}\Delta_{z_*,0}+C_{Z,0}\Delta_{Z,0}+C_{U_z,0}\Delta_{U_z,0}\right)\\
&+\frac{C_{vy}\eta_x^2}{(1-\rho^2)^3}\left(C_{y_*,0}\Delta_{y_*,0}+C_{Y,0}\Delta_{Y,0}+C_{U_y,0}\Delta_{U_y,0}\right)\\
&+\frac{\eta_x^2 S}{(1-\rho^2)^3}\left(C_{vz}C_{z,\sigma}\sigma_z^2+C_{vy}C_{y,\sigma}\sigma_y^2\right)+\frac{S\eta_x^2\sigma_x^2}{(1-\rho^2)^3}+\frac{\eta_x^2\ell_{f,0}^2(1+\rho^2)}{(1-\rho^2)^3}\\
&\lesssim\frac{\eta_x^2}{(1-\rho^2)^4}\left(\frac{\alpha^2\eta_x^2}{\eta_z^2}+\frac{\eta_x^2}{\eta_y^2}\right)\sum_{i=0}^{S}\mathbb{E}\left[\|\mathbb{E}\left[\bar{v}_{i+1}|\mathcal{F}_i\right]\|^2\right]\\
&+\frac{\eta_x^4\alpha^2}{(1-\rho^2)^3}\left(\alpha^2+\frac{1}{\alpha\eta_y}+\frac{\eta_x^2}{\eta_y^2}+\frac{\alpha^2\eta_x^2}{\eta_z^2}\right)\sum_{s=0}^{S-1}\mathbb{E}\left[\|\bar{v}_{i+1}\|^2\right]\\
&+\frac{\alpha^2\eta_x^3}{(1-\rho^2)^4}\left(\frac{1}{\alpha\eta_y}\Delta_{y_*,0}+\Delta_{Y,0}+\eta_y^2\Delta_{U_y,0}\right)\\
&+\frac{\eta_x^2\alpha^2}{(1-\rho^2)^3}\left(\frac{1}{\eta_z}\Delta_{z_*,0}+\Delta_{Z,0}+\eta_z^2\Delta_{U_z,0}\right)\\
&+\frac{\alpha^2\eta_x^2 S}{(1-\rho^2)^4}\left(\left(\frac{\eta_z}{n}+\eta_z^2\right)+\left(\frac{\eta_y}{n\alpha}+\eta_y^2\right)\alpha^2\right)\sigma^2+\frac{S\alpha^2\eta_x^2\sigma^2}{(1-\rho^2)^3}+\frac{\eta_x^2\ell_{f,0}^2(1+\rho^2)}{(1-\rho^2)^3}\\
&\lesssim\frac{\eta_x^2}{(1-\rho^2)^4}\sum_{i=0}^{S}\mathbb{E}\left[\|\mathbb{E}\left[\bar{v}_{i+1}|\mathcal{F}_i\right]\|^2\right]+\frac{\eta_x^4\alpha^2}{(1-\rho^2)^3}\left(\alpha^2+\frac{1}{\alpha\eta_x}\right)\sum_{s=0}^{S-1}\mathbb{E}\left[\|\bar{v}_{i+1}\|^2\right]\\
&+\frac{\eta_x^2}{(1-\rho^2)^4}+\frac{\eta_x}{(1-\rho^2)^3}+\frac{\alpha^3\eta_x^3 S\sigma^2}{n(1-\rho^2)^4}+\frac{S\alpha^2\eta_x^2\sigma^2}{(1-\rho^2)^3}+\frac{\eta_x^2\ell_{f,0}^2(1+\rho^2)}{(1-\rho^2)^3}.
\end{aligned}
$$

Using (111) and (122) in the above inequality, we know

$$
\begin{aligned}
&\frac{1}{nS}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbf{X}_{s+1}-\bar{x}_{s+1}\mathbf{1}_n^\top\right\|^2\right]\\
=&\mathcal{O}\left(\frac{n^{2/7}}{S^{12/7}}\right)+\mathcal{O}\left(\frac{n^{12/7}}{S^{16/7}}\left(\frac{n^{2/7}}{S^{12/7}}+\frac{S^{2/7}}{n^{5/7}}\right)\right)+\mathcal{O}\left(\frac{n^{2/7}}{S^{12/7}}+\frac{n^{6/7}}{S^{8/7}}+\frac{n^{4/7}}{S^{17/7}}\right)=\mathcal{O}\left(\frac{n^{6/7}}{S^{8/7}}\right).
\end{aligned}
$$

This indicates that

$$
\begin{aligned}
&\frac{1}{n}\left(\min_{0\le s\le S-1}\mathbb{E}\left[\|X_s-\bar{x}_s\mathbf{1}_n\|\right]\right)^2\le\frac{1}{n}\left(\frac{1}{S}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbf{X}_{s+1}-\bar{x}_{s+1}\mathbf{1}_n^\top\right\|\right]\right)^2\\
\le&\frac{1}{nS}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbf{X}_{s+1}-\bar{x}_{s+1}\mathbf{1}_n^\top\right\|\right]^2\le\frac{1}{nS}\sum_{s=0}^{S}\mathbb{E}\left[\left\|\mathbf{X}_{s+1}-\bar{x}_{s+1}\mathbf{1}_n^\top\right\|^2\right]=\mathcal{O}\left(\frac{n^{6/7}}{S^{8/7}}\right)
\end{aligned}
$$

which gives

$$
\min_{0\le s\le S-1}\frac{1}{n}\mathbb{E}\left[\|X_s-\bar{x}_s\mathbf{1}_n\|\right]=\mathcal{O}\left(\frac{1}{n^{1/14}S^{4/7}}\right).
$$

□