On Subproblem Tradeoffs in Decomposition for Multiobjective Optimization

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Abstract

Multiobjective optimization is widely used in applications for modeling and solving complex decision-making problems. To help resolve computational and cognitive difficulties associated with problems which have more than 3 or 4 objectives, we propose a decomposition and coordination methodology to support decision making for large multiobjective optimization problems (MOPs) with global, quasi-global, and local variables. Since the MOPs are decomposable into subproblems, the methodology allows the decision maker (DM) to quantify tradeoffs between the subproblems rather than only between specific objectives associated with them. To coordinate the subproblems, we extend the theory of achievement scalarizing functions which allows for the subproblems to be autonomously coordinated without the DM's participation. However, we do not totally exclude DMs, by proposing a hybrid coordination method where autonomous coordination is used to aid them in an interactive procedure to explore the subproblem tradeoffs. Finally, we demonstrate the effectiveness of our work on a humanitarian aid case study.

Keywords: Multiobjective optimization; Multicriteria decision making; Autonomous decision making; Achievement Scalarizing Functions; Complex Systems; Decomposition

1 Introduction

Multiobjective programs (MOPs) model decision problems governed by multiple and conflicting criteria or objectives that arise in many areas of human activity such as management or engineering. In the presence of conflict, a unique optimal decision is not available. Rather, the decision maker (DM) is presented with a set of non-improvable decisions known as efficient solutions and with the outcomes of these decisions known as Pareto (nondominated) criterion vectors. The final goal for the DM is to apply personal preferences, that are not contained in the MOP model, and select a preferred efficient solution as the final decision to be implemented. Solving MOPs therefore involves an optimization stage to compute the efficient and/or the Pareto set or their representations, and a decision stage to conduct a search for a preferred efficient solution and/or Pareto outcome ([5, 35, 20, 53]).

The difficulty to perform the optimization stage results from the size of the MOP and the type of variables and objective and constraint functions in the mathematical model. Since this stage relies on the capabilities of algorithms, certain types of variables and functions make MOPs harder to solve than the MOP's size does. For example, algorithms to compute the efficient set for convex MOPs with continuous variables have been well established regardless of the number of objective and constraint functions or variables ([46]). This is not the case for other types of MOPs such as convex or nonconvex problems with mixed-integer variables. For example, state-of-the-art algorithms for MOPs with mixed-integer variables and linear functions are available only for problems with two or three criteria ([22]).

The decision stage faces other difficulties. The search for a preferred efficient solution is likely to be manageable for bi- or triobjective programs but becomes challenging for MOPs with more objectives regardless

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of their type. In lower-dimensional objective spaces, a DM is likely to have the knowledge of the decision problem beyond the MOP model to identify a preferred efficient solution and the Pareto set can be easily represented graphically to further assist the DM. However, in higher dimensions, the DM may experience a cognitive burden resulting from too much information at once, or too many simultaneous tasks, resulting in not being able to process the information.

MOPs with more than three criteria are named "many-objective problems" to recognize the difficulties they cause ([52]). To resolve the challenges of "many-objective problems," one of the main research directions has been to decompose the original MOP into subproblems (sub-MOPs), each with a smaller number of criteria. The sub-MOPs are then coordinated to guarantee that by only computing their efficient sets, the efficient set of the MOP can be retrieved. A recent review of decomposition and coordination (D&C) approaches to MOPs is given in [47]. Below we review the studies that have given motivation for the current work.

The D&C methods for MOPs with global variables proposed in [15] and [14] rely on approximate efficiency and are supported with tradeoffs between two objective functions. In [15], the Lagrange multipliers associated with a single-objective problem related to two sub-MOPs provide the tradeoff value at a feasible solution with respect to two objective functions, each belonging to another sub-MOP. In [14], *a priori* tradeoffs are provided by the DM. Complex MOPs with local and global variables and constraints are defined by means of graphs in [9]. The concepts of (approximate) superior solutions and dominance between subsystems are introduced to complement the classical concepts of (approximate) efficient solutions and dominance between criterion vectors. The subsystems are coordinated by computing a compromise solution that may not be necessarily superior for every subsystem but which is as close as possible to the superior sets of all subsystems with respect to a distance measure such as a norm.

Furthermore, recent applied studies give evidence of the significance and relevance of D&C methods customized to specific real-life applications such as automotive design in [24, 54, 8] and food bank network redesign in [34].

The overall goal of this paper is to develop a D&C methodology to support decision making for complex MOPs composed of sub-MOPs, and allow the DM to quantify tradeoffs between the sub-MOPs rather than only between specific objectives associated with them. To accomplish this, we first extend the theory of achievement scalarizing functions (ASFs) that have successfully been used to support decision making under multiple criteria ([48, 51, 50, 38]). This extension leads us to define and measure a new type of subproblem tradeoffs.

Second, we propose a model of complex MOPs that are decomposable into interacting sub-MOPs, and show how efficient solutions for the complex MOP may be obtained from efficient solutions for the sub-MOPs.

Third, we design a coordination methodology that offers three ways a complex system can be coordinated during the decision stage to arrive at a preferred efficient solution. In particular, we propose a coordination procedure which provides a feasible solution that is as close as possible to the efficient sets of all sub-MOPs, as similarly proposed in [15] and [9]. The closeness, however, is measured by the introduction of subsystem tradeoffs that are provided by a bilevel MOP solved over the efficient sets of the sub-MOPs. We recognize that bilevel MOPs are hard to solve ([13, 12, 3, 29, 43]), but believe this difficulty is worthwhile since the proposed bilevel MOP enriches the decision stage with valuable information that other D&C methods cannot provide. However, we propose an auxiliary MOP which circumvents the difficulties while still conceptually employing the bilevel MOP.

The paper is structured as follows. In Section 2, the new theory on achievement scalarizing functions is developed. An MOP with many criteria and a decomposable structure is presented in Section 3. In Section 4, the new theory on ASFs is applied to the decomposable MOP leading to further theoretical developments on the newly defined subproblem tradeoffs. In Section 5, the new concepts are employed in an interactive decision procedure, which combines the hierarchical coordination and a newly proposed autonomous coordination, to assist DMs in exploring their alternatives. We apply our new concepts and methods to a case study of disaster relief in Section 6, while Section 7 concludes the paper.

1.1 Preliminaries

We begin with a generic MOP and the optimality concept based on the efficiency of solutions and the nondominance of their images. Let $f_i(x) : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, p$ and let $X \subseteq \mathbb{R}^n$ be a nonempty set. A generic multiobjective optimization problem is of the form

$$\begin{aligned} \min_x \quad & f(x) = [f_1(x),\ \dots, f_p(x)] \\ \text{s.t.} \quad & x \in X \end{aligned} \tag{MOP}$$

We denote the outcome set by $Y = f(X) = \{f(x) \in \mathbb{R}^p \mid x \in X\}$ and use the following standard definition for vector ordering. Let $u = (u_1, \dots, u_p),\ v = (v_1, \dots, v_p) \in \mathbb{R}^p$. We say that $u < v$ if $u_i < v_i$ for each $i = 1, \dots, p$; $u \leq v$ if $u_i \leq v_i$ for each $i = 1, \dots, p$ and $u \neq v$; and $u \leqq v$ if $u_i \leq v_i$ for each $i = 1, \dots, p$. We also define the cones $\mathbb{R}^p_{\geqq} = \{y \in \mathbb{R}^p \mid y \geqq 0\}$ and $\mathbb{R}^p_{>} = \{y \in \mathbb{R}^p \mid y > 0\}$. We denote the boundary of a set $S \subseteq \mathbb{R}^p$ by $\partial(S)$. Finally, we use efficiency for determining the optimality of a feasible solution for (MOP).

Definition 1. Let $x \in X \subseteq \mathbb{R}^n$ be feasible for (MOP). We say that x is a **(weakly) efficient solution** for (MOP) if there is no $x' \in X$ such that $f(x')(<) \leq f(x)$. We say that x is a **strictly efficient solution** for (MOP) if there is no $x' \in X,\ x' \neq x$, such that $f(x') \leqq f(x)$. If x is a(n) (weakly/strictly) efficient solution, we say that $f(x)$ is a **(weak/strict) Pareto point**.

We denote the set of all (weakly/strictly) efficient solutions of (MOP) by $E_{(w/s/\cdot)}(X)$ and denote the set of all (weak/strict) Pareto points by $P_{(w/s/\cdot)}(Y) = f(E_{(w/s/\cdot)}(X))$. It will also be useful to have a notion of relaxed efficiency.

Definition 2. Let $x \in X$ be feasible for (MOP) and $\varepsilon \in \mathbb{R}^p_{\geqq}$. We say that x is a **(weakly) $\boldsymbol{\varepsilon}$-efficient solution** for MOP if there is no feasible $x' \in X$ such that $f(x')(<) \leq f(x) - \varepsilon$. If x is (weakly) ε-efficient, we say that $f(x)$ is a **(weak) $\boldsymbol{\varepsilon}$-Pareto point**.

We denote the set of (weakly) ε-efficient solutions by $E_{(w)}(X, \varepsilon) \subseteq X$ and the set of (weak) ε-Pareto solutions by $P_{(w)/\cdot}(Y, \varepsilon)$.

Finally, the following are helpful definitions in multiobjective optimization.

Definition 3 ([26])**.** Let $S_1, S_2 \subseteq \mathbb{R}^p$ be nonempty sets. We say that S_1 **dominates** S_2 and write $S_1 \leqq S_2$ if and only if the **upper-type set relation** holds: for each $s_1 \in S_1$ there exists $s_2 \in S_2$ such that $s_1 \leqq s_2$.

Definition 4 ([41])**.** Let Y be a nonempty set in $\mathbb{R}^p$. We say that $P(Y)$ is **externally stable** if for every $y \in Y \setminus P(Y)$, there exists $\hat{y} \in P(Y)$ such that $y \in \hat{y} + \mathbb{R}^p_{\geqq}$.

Theorem 1 ([41])**.** *If Y a nonempty compact set in $\mathbb{R}^p$ then $P(Y)$ is externally stable.*

Definition 5. The **ideal point** of (MOP) is a point $y^I \in \mathbb{R}^p$ such that for each component $i \in \{1, \dots, p\}$, $y^I_i = \min\{f_i(x) \mid x \in X\}$.

The following well known result describes the relationship between $P(Y)$ and y^I.

Lemma 1. *Let y^I be the ideal point of (MOP). $P(Y) = \{y^I\}$ if and only if $y^I \in Y$.*

A classical method of computing efficient solutions of (MOP) is by minimizing a weighted-sum of the objective functions.

Theorem 2 ([19, 11])**.** *Let $\omega = (\omega_1, \dots, \omega_p) \in \mathbb{R}^p_{\geq}$. If $\hat{x} \in X$ is an optimal solution to the **weighted-sum scalarization** of (MOP), $\min\{\sum_{i=1}^p \omega_i f_i(x) \mid x \in X\}$, then $\hat{x}$ is a weakly efficient solution for (MOP). Furthermore, if $\omega \in \mathbb{R}^p_{>}$, then $\hat{x}$ is an efficient solution for (MOP).*

With these preliminaries, in the next section we extend the theory of achievement scalarizing functions to allow variable reference points.

2 Bivariate Achievement Scalarizing Functions

Achievement scalarizing functions (ASFs), introduced in [48, 49], seek to find efficient solutions for (MOP) by scalarizing the objective functions of (MOP). As such, they work in a similar way to other scalarizations including the weighted-sum, ϵ-constraint, Chebyshev, and many other methods ([46, 23]). However, ASFs differ from these methods since they are not norms. Instead, they measure the "distance" between the outcome set and a single, fixed reference point by using specific properties. In contrast to the scalarization methods based on norms, they allow the reference point to be located anywhere in the objective space and have the property of preserving vector ordering, while norms do not preserve this ordering.

Due to their flexibility and proven high utility in decision making ([37, 36, 30]), ASFs have been developed to gain new features. They are modified to become additive to have even more flexibility ([40]), or to carry two weight vectors rather than one to be able to work better with achievable or not achievable reference points ([33]). They are parametrized to allow changing the reference point or weighting coefficients ([38]). They also belong to a class of biaffine functions used in decision making with stochastic preferences ([39]). Interestingly, [32] shows how to construct a set of equivalent reference points, all of which select the same Pareto point during scalarization.

Independently of ASFs, the use of multiple reference points is introduced in [44] to model DMs' preferences and the decision stage is reduced to a biobjective problem seeking a compromise between the distances to the sets of desirable and avoidable reference points and using a utility function as a distance measure.

In this work, we continue the idea of a set of reference points to provide further extensions to the theory of ASFs. We make these functions bivariate and allow the reference point to be itself a variable over a given reference set in the objective space. To ensure efficiency is achieved, our bivariate ASFs (BASFs) maintain the properties of order preservation and order representation similar to ASFs ([49]).

Definition 6. Let $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be a continuous function and $y, y', r \in \mathbb{R}^p$.

1. We say that σ is
   i. **order preserving** if $y \leqq y'$ implies $\sigma(y, r) \leq \sigma(y', r)$.
   ii. **strictly order preserving** if $y < y'$ implies $\sigma(y, r) < \sigma(y', r)$;
   iii. **strongly order preserving** if $y \leq y'$ implies $\sigma(y, r) < \sigma(y', r)$.
2. We say that σ is **order representing** if for each $r \in \mathbb{R}^p$,
$$S(r) = \{y \in \mathbb{R}^p \mid \sigma(y, r) < 0\} = r - \mathbb{R}^p_>$$
3. If $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ is (strongly/strictly) order preserving and order representing then we call σ a **(strong/strict) bivariate achievement scalarizing function (BASF)**.

Proposition 1. *If $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ is an order representing function then for all $r \in \mathbb{R}^p$ and for all $y \in \partial(r - \mathbb{R}^p_{\geqq})$, $\sigma(y, r) = 0$.*

Proof. Let $r \in \mathbb{R}^p$ and $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be an order representing function. Let $y \in \partial(r - \mathbb{R}^p_{\geqq})$. Without loss of generality, let $y = r - \lambda e$, where $\lambda \geq 0$ and $e = (1, 0, \ldots, 0)$. Let $\{y_m\} \subseteq S(r) = r - \mathbb{R}^p_>$ be a sequence which converges to y. Observe that since σ is continuous and y_m converges to y, it must be that for any $\varepsilon > 0$ there exists M such that for all $m > M$, $-\varepsilon + \sigma(y, r) < \sigma(y_m, r) < \varepsilon + \sigma(y, r)$. Furthermore, since $y_m \in S(r)$, $\sigma(y_m, r) < 0$ for all m.

Since $y \notin r - \mathbb{R}^p_>$, it must that $\sigma(y, r) \geq 0$. If $\sigma(y, r) > 0$ then letting $\varepsilon = \frac{\sigma(y,r)}{2}$, there exists M' such that for all $m > M'$, $\frac{\sigma(y,r)}{2} < \sigma(y_m, r) < \frac{3\sigma(y,r)}{2}$, which contradicts the fact that $\sigma(y_m, r) < 0$. Thus, it must be that $\sigma(y, r) = 0$. □

Proposition 2. *Let $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be a strict BASF. Let $y, r \in \mathbb{R}^p$.*

1. *$\sigma(y, r) < 0$ if and only if $y \in r - \mathbb{R}^p_>$*
2. *$\sigma(y, r) = 0$ if and only if $y \in \partial(r - \mathbb{R}^p_{\geqq})$*
3. *$\sigma(y, r) > 0$ if and only if $y \in (r - \mathbb{R}^p_{\geqq})^C$*

Proof. Let $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be a strict BASF and let $y, r \in \mathbb{R}^p$.

1. Observe that $\sigma(y, r) < 0$ if and only if $y \in r - \mathbb{R}^p_>$ since σ is a BASF.

2. First, we show that $\sigma(y, r) = 0$ implies $y \in \partial(r - \mathbb{R}^p_{\geqq})$ by the contrapositive. To that end, suppose $y \notin \partial(r - \mathbb{R}^p_{\geqq})$. Then $y \in r - \mathbb{R}^p_>$ or $y \in (r - \mathbb{R}^p_{\geqq})^C$. In the former, since σ is a BASF then $\sigma(y, r) < 0$. In the latter, let $y' \in (y - \mathbb{R}^p_>) \cap \partial(r - \mathbb{R}^p_{\geqq})$. This means that $y' < y$, and since σ is strictly order preserving, then $\sigma(y', r) < \sigma(y, r)$. Furthermore, by Proposition 1, $\sigma(y', r) = 0$. Thus, $0 < \sigma(y, r)$. Conversely, by Proposition 1 the result holds.

3. This follows directly from parts 1 and 2 above.

□

Remark 1. The definition of order preserving is the same as the definition of monotone or increasing functions. We use the term "order preserving" to follow the terminology used in [49] and to emphasize that multiobjective optimization relies on a vector ordering. Since we make ASFs bivariate, we adopt the notation of writing σ as a function of both y and r.

In Example 1, we compare an ASF and a BASF by depicting their level sets.

Example 1. Consider the well-known strictly order preserving ASF given by $\sigma(y, r) = \max\limits_{1 \leq i \leq p} \{\lambda_i(y_i - r_i)\}$ ([38, 33, 11]). The level curves of σ for $p = 2, \lambda = (1, 1)$ and $r = (1, 1)$ are depicted in Figure 1. The level sets of a BASF defined as $\sigma(y, r) = \max_{i=1,2}\{y_1 - r_1, y_2 - r_2\}$ for $-10 \leq y_1, y_2 \leq 10,\ -10 \leq r_1 \leq 10$, and $r_2 = 0$ are depicted in Figure 2.

To scalarize (MOP) using a BASF, let $R \subseteq \mathbb{R}^p$ be a nonempty set, let (MOP) have the nonempty feasible set X, and let $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be a BASF. We scalarize (MOP) in the following way.

$$\begin{aligned} \min_{x,r} \quad & \sigma(f(x), r) \\ \text{s.t.} \quad & (x, r) \in X \times R \end{aligned} \tag{σ-MOP}$$

We examine the role the two properties of order representation and preservation play in (σ-MOP). In the two subsequent propositions, we prove a necessary and a sufficient conditions for an optimal solution to (σ-MOP) to provide an efficient solution to (MOP). Each condition only requires that one of these properties hold.

Lemma 2. *Let (MOP) and (σ-MOP) be given. If $R \cap Y$ is nonempty and R dominates $P(Y)$, then $R \cap Y$ is a subset of $P(Y)$.*

Proof. Let $R \cap Y \neq \emptyset$ and $R \leqq P(Y)$. Suppose $r \in R \cap Y$. Since $R \leqq P(Y)$, there exists $f(x) \in P(Y)$ such that $r \leqq f(x)$. By definition of Pareto nondominance, it must be that $r = f(x)$. Therefore, $r \in P(Y)$. □

Proposition 3 (Necessary condition)**.** *Let $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be order representing and (MOP) and (σ-MOP) be given and let the conditions of Lemma 2 hold. If $x \in X$ is an efficient solution for (MOP) such that $f(x) \in R \cap Y$ then $(x, f(x))$ is an optimal solution for (σ-MOP) with optimal value 0.*

Proof. Let $R \cap Y \neq \emptyset$ and $R \leqq P(Y)$. Let $x \in E(X)$ such that $f(x) \in R \cap Y$ and let $(\bar{x}, \bar{r})$ be an optimal solution for (σ-MOP). Then $\sigma(f(\bar{x}), \bar{r}) \leq \sigma(f(x), f(x))$ and by Proposition 1, $\sigma(f(x), f(x)) = 0$. Thus, $\sigma(f(\bar{x}), \bar{r}) \leq 0$.

We show that $\sigma(f(\bar{x}), \bar{r}) \not< 0$. Towards a contradiction, suppose $\sigma(f(\bar{x}), \bar{r}) < 0$. Since σ is order representing, $f(\bar{x}) \in \bar{r} - \mathbb{R}^p_>$. This implies $f(\bar{x}) < \bar{r}$. Since $R \leqq P(Y)$, there exists $\bar{y} \in P(Y)$ such that $f(\bar{x}) < \bar{r} \leqq \bar{y}$, implying $f(\bar{x}) < \bar{y}$, contradicting $\bar{y} \in P(Y)$. Thus, it must be that $\sigma(f(\bar{x}), \bar{r}) = 0 = \sigma(f(x), f(x))$. Since $(\bar{x}, \bar{r})$ is an optimal solution for (σ-MOP) with optimal value 0, it must be that $(x, f(x))$ is also an optimal solution with optimal value 0. □

The following corollary shows a special case of (σ-MOP).

Corollary 1. *Let (MOP) be given and let $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be order representing. Define (σ-MOP) with $R = P(Y)$. Then for all efficient solutions x for (MOP), $(x, f(x))$ is an optimal solution for (σ-MOP) with optimal value 0.*

Proof. By Proposition 3, for all $x \in E(X)$ such that $f(x) \in R \cap Y = P(Y) \cap Y = P(Y)$, $(x, f(x))$ is an optimal solution for (σ-MOP) with optimal value 0. □

Proposition 4 (Sufficient condition)**.** *Let $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be a function.*

1. *If $(\bar{x}, \bar{r})$ is a unique optimal solution for (σ-MOP) and σ is order preserving and then $\bar{x}$ is a strictly efficient solution for (MOP).*
2. *If $(\bar{x}, \bar{r})$ is an optimal solution for (σ-MOP) and σ is strictly order preserving then $\bar{x}$ is a weakly efficient solution for (MOP).*
3. *If $(\bar{x}, \bar{r})$ is an optimal solution for (σ-MOP) and σ is strongly order preserving then $\bar{x}$ is an efficient solutionf for (MOP).*

Proof. Let $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be a function and let $(\bar{x}, \bar{r})$ be an optimal solution for (σ-MOP).

We prove case 1 and note that cases 2 and 3 follow analogously. Let σ be order preserving and $(\bar{x}, \bar{r})$ the unique optimal solution of (σ-MOP). Towards a contradiction, suppose $\bar{x} \notin E_s(X)$. Then there exists $x \in X$ with $x \neq \bar{x}$ such that $f(x) \leqq f(\bar{x})$. Since σ is order preserving, $\sigma(f(x), \bar{r}) \leq \sigma(f(\bar{x}), \bar{r})$. Since $(\bar{x}, \bar{r})$ is optimal for (σ-MOP), it must be that $\sigma(f(x), \bar{r}) = \sigma(f(\bar{x}), \bar{r})$, which implies that $(x, \bar{r})$ is also an optimal solution for (σ-MOP), contradicting the fact that $(\bar{x}, \bar{r})$ is the unique optimal solution. □

Remark 2. Observe that letting $R = \{r\}$ for a fixed $r \in \mathbb{R}^p$ is precisely equivalent to fixing a reference point and Propositions 3 and 4 reduce to the case discussed in [49].

Propositions 3 and 4 show that the order representing and preservation are complementary in the sense that they separately "secure" the efficiency of the (MOP) solutions computed with a BASF and the non-negativity of the BASF values. These two properties turn out to be fundamental for the developments that follow.

The results of this section are used in the defining of subproblem tradeoffs in Section 4 and a coordination method in Section 5. In the next section we turn to decomposable complex MOPs.

3 Complex System Modeling and Decomposition

In this work, we assume a specific structure for (MOP) that represents the complex system or all-in-one problem, denoted by (AiO). The (AiO) consists of multiobjective subproblems with decision variables specific to one, more, or all suproblems.

3.1 Structure of the All-in-One Problem

The following notation describes this structure and is illustrated on an (AiO) example with four subproblems.

Definition 7. Let $N \in \mathbb{N}$ be the number of subproblems in (MOP). Define the set $[N] = \{1, 2, \ldots, N\}$ and let $i, k \in [N]$. For $k \in [N]$, let $\binom{N}{k}$ denote the binomial coefficient, which gives the number of subsets of $[N]$ of cardinality k. Furthermore, for every $S \subseteq [N]$, let $m_S \in \mathbb{Z}_{>0}$.

1. We define a vector variable $x^i_S \in \mathbb{R}^{m_S}$ for each $i \in [N]$ such that $S \subseteq [N]$ and $i \in S$. Note that S is an index for the decision variable.
2. For $i \in [N]$, define the vector of vectors $x^i_{\binom{N}{k}} = (x^i_S)_{\substack{S \subseteq [N] \\ |S|=k \\ i \in S}}$. We have
$$x^i_{\binom{N}{k}} \in \prod_{\substack{S \subseteq [N] \\ |S|=k \\ i \in S}} \mathbb{R}^{m_S}.$$

3. In general, for each $i \in [N]$, let

$$x^i = (x^i_{\binom{N}{N}}, x^i_{\binom{N}{N-1}}, \ldots, x^i_{\binom{N}{1}}) = (x^i_{\binom{N}{k}})_{k=N,\ldots,1}$$

4. To reference a specific variable we write x_S, where $S \subseteq [N]$. If more specificity is needed, i.e., if the subproblem of the variable needs to be denoted, we write $x^i_{\binom{N}{k},S}$.

5. We call variables $x^i_{\binom{N}{N}}$ **global variables** and $x^i_{\binom{N}{1}}$ **local variables**. For $2 \le k \le N-1$, we call $x^i_{\binom{N}{k}}$ **quasi-global variables**.

6. We have:

   (a) $x^i_{\binom{N}{k}} \in \prod_{\substack{S\subseteq[N]\\ |S|=k\\ i\in S}} \mathbb{R}^{m_S}$;

   (b) $x^i = (x^i_{\binom{N}{N}}, x^i_{\binom{N}{N-1}}, \ldots, x^i_{\binom{N}{1}}) = (x^i_{\binom{N}{k}})_{k=N,\ldots,1} \in \prod_{k=N}^{1} \prod_{\substack{S\subseteq[N]\\ |S|=k\\ i\in S}} \mathbb{R}^{m_S}$;

   (c) $x = (x^1, \ldots, x^N) = (x^i)_{i=1,\ldots,N} \in \prod_{i=1}^{N} \prod_{\substack{S\subseteq[N]\\ i\in S}} \mathbb{R}^{m_S}$;

   (d) $x = (x_S)_{S\subseteq[N]}$

7. In any notation, repetitions are dropped. In other words, any variables sharing the same index $S \subseteq [N]$, are written only once.

Example 2. Suppose (AiO) has four subproblems, so $N = 4$. For the third subproblem, $i = 3$,

$$x^3_{\binom{4}{1}} = x_3 \qquad x^3_{\binom{4}{2}} = (x_{13}, x_{23}, x_{34})$$
$$x^3_{\binom{4}{3}} = (x_{123}, x_{134}, x_{234}) \qquad x^3_{\binom{4}{4}} = x_{1234}.$$

Note that $x^3 = (x_{1234}, x_{123}, x_{134}, x_{234}, x_{13}, x_{23}, x_{34}, x_3)$. On the other hand, consider the second subproblem, $i = 2$,

$$x^2_{\binom{4}{1}} = x_2 \qquad x^2_{\binom{4}{2}} = (x_{12}, x_{23}, x_{24})$$
$$x^2_{\binom{4}{3}} = (x_{123}, x_{124}, x_{234}) \qquad x^2_{\binom{4}{4}} = x_{1234}.$$

Similarly, $x^2 = (x_{1234}, x_{123}, x_{124}, x_{234}, x_{12}, x_{23}, x_{24}, x_2)$. Observe that x^2 and x^3 have overlapping variables. For example, $x^3_{\binom{4}{3},123} = x^2_{\binom{4}{3},123} = x_{123}$. Finally, observe that

$$x = (x_{1234}, x_{123}, x_{124}, x_{134}, x_{234}, x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}, x_1, x_2, x_3, x_4).$$

This notation is compact yet informative because it captures all possible locations of the decision variables in the subproblems and therefore describes the relationship between subproblems of a complex system MOP. Given the constant N denoting the number of subproblems, the index $S \subseteq [N]$ denotes which subproblems the variable x_S appears in.

We are now in a position to define the structure of the MOP under consideration. For $i \in [N]$, let $f^i : \prod_{\substack{S\subseteq[N]\\ i\in S}} \mathbb{R}^{m_S} \to \mathbb{R}^{p_i}$ and let $\emptyset \ne X^i \subseteq \prod_{\substack{S\subseteq[N]\\ i\in S}} \mathbb{R}^{m_S}$. Let $\emptyset \ne X \subseteq \prod_{i=1}^{N} X^i$. Then the **All-in-One** complex multiobjective problem is the following.

$$\min_{x} \quad f(x) = [f^1(x^1), \ldots,\ f^i(x^i), \ldots,\ f^n(x^N)] \tag{AiO}$$

$$\text{s. t.} \quad x = (x^1, \ldots, x^i, \ldots, x^N) \in X$$

We let $Y^i = f^i(X^i)$ for each $i \in [N]$ and $Y = f(X)$.

3.2 Decomposition

We decompose (AiO) by duplicating the global and quasi-global variables in each subproblem. For $i \in [N]$, the i^{th} subproblem is given by the following.

$$\begin{aligned} \min_{z^i_N, \ldots, z^i_2, x^i_{\binom{N}{1}}} \quad & f^i(z^i_N, \ldots, z^i_2, x^i_{\binom{N}{1}}) \\ \text{s. t.} \quad & (z^i_N, \ldots, z^i_2, x^i_{\binom{N}{1}}) \in X^i \end{aligned} \tag{SP$_i$}$$

where $z^i_k = x^i_{\binom{N}{k}}$ for $k \in \{2, \ldots, N\}$. We essentially take all of the variables, constraints, and objective functions which have i in its index and include them in (SP$_i$). The following results relate (weakly) efficient solutions for (SP$_i$) to (weakly) efficient solutions for (AiO).

Proposition 5. *Let $i \in [N]$ and let $(\hat{z}^i_N, \ldots, \hat{z}^i_2, \hat{x}^i_{\binom{N}{1}})$ be a weakly efficient solution for (SP$_i$). If there exists $\hat{x}^j \in X^j$ for every $j \in [N] \setminus \{i\}$ such that $\hat{x} = (\hat{z}^i_N, \ldots, \hat{z}^i_2, \hat{x}^i_{\binom{N}{1}}, \hat{x}^j)_{j \in [N]\setminus\{i\}}$ is feasible for (AiO), then $\hat{x}$ is a weakly efficient solution for (AiO).*

Proof. Let $(\hat{z}^i_N, \ldots, \hat{z}^i_2, \hat{x}^i_{\binom{N}{1}}) \in E_w(X^i)$ and suppose there exists $\hat{x}^j \in X^j$ for all $j \in [N] \setminus \{i\}$ such that $\hat{x} = (\hat{z}^i_N, \ldots, \hat{z}^i_2, \hat{x}^i_{\binom{N}{1}}, \hat{x}^j)_{j \in [N]\setminus\{i\}}$ is feasible for (AiO). Towards a contradiction, suppose $\hat{x} \notin E_w(X)$. Then there exists $x \in X$ such that $[f^1(x^1), \ldots, f^i(x^i), \ldots, f^N(x^N)] < [f^1(\hat{x}^1), \ldots, f^i(\hat{z}^i_N, \ldots, \hat{z}^i_2, \hat{x}^i_{\binom{N}{1}}), \ldots, f^N(\hat{x}^N)]$ which implies that $f^i(x^i) < f^i(\hat{z}^i_N, \ldots, \hat{z}^i_2, \hat{x}^i_{\binom{N}{1}})$, contradicting the weak efficiency of $(\hat{z}^i_N, \ldots, \hat{z}^i_2, \hat{x}^i_{\binom{N}{1}})$ for (SP$_i$). □

Proposition 6. *For each $i \in [N]$, let $(\hat{z}^i_N, \ldots, \hat{z}^i_2, \hat{x}^i_{\binom{N}{1}})$ be an ε^i-efficient solution for (SP$_i$) with $\varepsilon^i \in \mathbb{R}^{p_i}_{\geqq}$. If for every $i, j \in [N]$, $2 \leq k \leq N$ and $S \subseteq [N]$ such that $|S| = \binom{N}{k}$ and $i, j \in S$, it holds that $\hat{z}^i_{k,S} = \hat{z}^j_{k,S} = \hat{x}_S$, then $\hat{x} = (\hat{x}_S)_{S \subseteq [N]} = (\hat{x}^1, \ldots, \hat{x}^N)$ is an $\varepsilon = (\varepsilon^1, \ldots, \varepsilon^N)$-efficient solution for (AiO).*

Proof. For each $i \in [N]$ let $(\hat{z}^i_N, \ldots, \hat{z}^i_2, \hat{x}^i_{\binom{N}{1}}) \in E(X^i, \varepsilon^i)$ be such that for every $i, j \in [N], 2 \leq k \leq N$ and for every $S \subseteq [N]$ with $|S| = \binom{N}{k}$, we have that $\hat{z}^i_{k,S} = \hat{z}^j_{k,S} = \hat{x}_S$. Towards a contradiction, suppose $\hat{x} = (\hat{x}^1, \ldots, \hat{x}^N) \notin E(X, \varepsilon)$, with $\varepsilon = (\varepsilon^1, \ldots, \varepsilon^N)$. Then there exists $x = (x^1, \ldots, x^N) \in X$ such that $[f^1(x^1), \ldots, f^i(x^i), \ldots,\ f^N(x^N)] \leq [f^1(\hat{x}^1) - \varepsilon^1, \ldots, f^i(\hat{x}^i) - \varepsilon^i, \ldots, f^N(\hat{x}^N) - \varepsilon^N]$. This implies that there exists $i \in [N]$ such that $f^i(x^i) \leq f^i(\hat{x}^i) - \varepsilon^i = f^i(\hat{z}^i_N, \ldots, \hat{z}^i_2, \hat{x}^i_{\binom{N}{1}}) - \varepsilon^i$, contradicting the fact that $(\hat{z}^i_N, \ldots, \hat{z}^i_2, \hat{x}^i_{\binom{N}{1}}) \in E(X^i, \varepsilon^i)$. □

Corollary 2. *For each $i \in [N]$, let $(\hat{z}^i_N, \ldots, \hat{z}^i_2, \hat{x}^i_{\binom{N}{1}})$ be an efficient solution for (SP$_i$). Suppose that for every $i, j \in [N]$, $2 \leq k \leq N$, and $S \subseteq [N]$ with $|S| = \binom{N}{k}$ and $i, j \in S$, we have that $\hat{z}^i_{k,S} = \hat{z}^j_{k,S} = \hat{x}_S$. Then $\hat{x} = (\hat{x}_S)_{S \subseteq [N]} = (\hat{x}^1, \ldots, \hat{x}^N)$ is an efficient solution for (AiO).*

Proof. Note that being efficient is equivalent to being ε-efficient for $\varepsilon = 0$. Thus, let $\varepsilon^i = 0$ for all $i \in [N]$ and apply the previous proposition. □

Proposition 7. *Let $\hat{x} = (\hat{x}^1, \ldots, \hat{x}^N) \in X$. If $\hat{x}^i$ is an efficient solution for* (SP$_i$) *for every $i \in [N]$, then $\hat{x}$ is an efficient solution for* (AiO)*.*

Proof. Let $\hat{x}$ be defined as above and suppose $\hat{x}^i \in E(X^i)$ for every $i \in [N]$. Towards a contradiction, suppose $\hat{x} \notin E(X)$. Then there exists $x \in X$ such that $[f^1(x^1), \ldots, f^i(x^i), \ldots, f^N(x^N)] \leq [f^1(\hat{x}^1), \ldots, f^i(\hat{x}^i), \ldots, f^N(\hat{x}^N)]$. Thus, there exists $i \in [N]$ such that $f^i(x) \leq f^i(\hat{x}^i)$, contradicting the fact that for all $i \in [N]$, $\hat{x}^i \in E(X^i)$. □

Remark 3. Notice that in Proposition 5, we assume that we have a weakly efficient solution for a single subproblem (SP$_i$) such that we can find values for the rest of variables which make the concatenation of this weakly efficient solution with these values feasible for (AiO). If this is the case, then it must be that this concatenated vector must also be weakly efficient for (AiO). On the other hand, in Corollary 2, we only assume that we have efficient solutions for each respective subproblem. If it happens that the duplicated variables are all equal, this forces feasibility for (AiO), and therefore the concatenated solution must be efficient for (AiO). Finally, in Proposition 7, we assume the feasibility of a solution for (AiO) first. If it happens that the projection of this solution into the decision space of each subproblem lands it in the respective efficient sets, then it must be that the solution is also efficient for (AiO).

4 Subproblem Tradeoffs

We apply the theory of Section 2 to construct an auxiliary multiobjective problem which measures the tradeoffs between *subproblems*, as opposed to tradeoffs between objectives, of (AiO). The measurement of tradeoffs between subproblems is accomplished by measuring the "distance" between the projection, into the objective space of each subproblem, of the image of a feasible point for (AiO) and the Pareto set of each subproblem. The goal is to select a feasible solution for (AiO) whose image is closest to the Pareto set of each respective subproblem. As such, it would be natural to use a norm to measure this distance. However, since norms are not order preserving, we turn to BASFs for measuring the distance. For each $i \in [N]$, let $\sigma_i : \mathbb{R}^{p_i} \times \mathbb{R}^{p_i} \to \mathbb{R}$ be a (strict) BASF. The **subproblem tradeoff problem** is formulated as

$$\begin{aligned} \min_{x, s^1, \ldots, s^N} \quad & [\sigma_1(f^1(x^1), s^1), \ldots, \sigma_N(f^N(x^N), s^N)] \\ \text{s.t.} \quad & x = (x^1, \ldots, x^N) \in X \\ & s = (s^1, \ldots, s^N) \in \prod_{i=1}^N P(Y^i). \end{aligned} \tag{SPTP}$$

(SPTP) is a bilevel multiobjective optimization problem in which optimization with respect to N objective functions is performed over the Pareto sets of the N subproblems of (AiO). We denote the feasible set of (SPTP) by Ξ and the efficient set by $E(\Xi)$. Similarly, the outcome set of (SPTP) is denoted by Σ and the Pareto set by $P(\Sigma)$. Furthermore, we call the i^{th} component of the image of a feasible point $(x, s) \in \Xi$ the σ_i-value of (x, s) for $i \in [N]$. Note that the new variables $(s^1, \ldots, s^N) \in \prod_{i=1}^N P(Y^i)$ are reference points for each BASF. In particular, for $i \in [N]$, s^i is a Pareto point for subproblem i. Furthermore, observe that there are N different sets of reference points.

The following propositions present important properties of (SPTP), which we use to define a notion of subproblem tradeoffs.

Proposition 8. *The ideal point of* (SPTP) *is nonnegative.*

Proof. Observe that for each $i \in [N]$ the reference points for σ_i is a Pareto outcome for subproblem i. Furthermore, $\text{Proj}_{X^i}(X) \subseteq X^i$. Thus, for each $i \in [N]$, $\min\{\sigma_i(f^i(x^i), s^i) | (x^1, \ldots, x^i, \ldots x^N) \in X,\ s^i \in P(Y^i)\} = \min\{\sigma_i(f^i(x^i), s^i) | x^i \in \text{Proj}_{X^i}(X),\ s^i \in P(Y^i)\} \geq \min\{\sigma_i(f^i(x^i), s^i) | x^i \in X^i,\ s^i \in P(Y^i)\}$, which by Corollary 1, $\min\{\sigma_i(f^i(x^i), s^i) | x^i \in X^i,\ s^i \in P(Y^i)\} = 0$. Therefore, the ideal point of (SPTP) is nonnegative. □

Remark 4. Observe that Proposition 8 ensures that Σ is entirely nonnegative. Put another way, the set $\{0\}$ dominates Σ, $\{0\} \leqq \Sigma$.

Lemma 3. *For all $i \in [N]$, let σ_i be a strict BASF and let $(\hat{x}, \hat{s})$ be feasible for (SPTP). For any $i \in [N]$, $\sigma_i(f^i(\hat{x}^i), \hat{s}^i) = 0$ if and only if $f^i(\hat{x}^i) = \hat{s}^i$.*

Proof. Since $\hat{s}^i \in P(Y^i)$, it must be that $f^i(\hat{x}^i) = \hat{s}^i$. By Proposition 2, $\sigma_i(f^i(\hat{x}^i), \hat{s}^i) = 0$ if and only if $f^i(\hat{x}^i) \in \partial(\hat{s}^i - \mathbb{R}^{p_i}_{\geqq})$. □

Proposition 9. *The following statements are equivalent.*

1. *$P(\Sigma) = \{0\}$.*
2. *If $(\hat{x}, \hat{s})$ is efficient for (SPTP), then $\hat{x}^i$ is efficient for (SP$_i$) for all $i \in [N]$.*
3. *For any $\omega \in \mathbb{R}^N_{>}$, the optimal value of the weighted-sum scalarization of (SPTP) is 0.*

Proof. Proof. (1) $\iff$ (2) : First, suppose $P(\Sigma) = \{0\}$. Thus for all $(\hat{x}, \hat{s}) \in E(\Xi)$ and for every $i \in [N]$, $\sigma_i(f^i(\hat{x}^i), \hat{s}^i) = 0$. Since σ_i is a BASF, it must be that $f^i(\hat{x}^i) = \hat{s}^i$ and since $\hat{s}^i \in P(Y^i)$ then $\hat{x}^i \in E(X^i)$. Conversely, let $z \in P(\Sigma)$. Then there exists $(\hat{x}, \hat{s}) \in E(\Xi)$ such that $z = [\sigma_1(f^1(\hat{x}^1), \hat{s}^1), \ldots, \sigma_N(f^N(\hat{x}^N), \hat{s}^N)]$. By assumption, $\hat{x}^i \in E(X^i)$ for each $i \in [N]$. Thus, the point $(\hat{x}, f(\hat{x}))$ is feasible for (SPTP). Note that by Proposition 2, $0 = [\sigma_1(f^1(\hat{x}^1), f^1(\hat{x}^1)), \ldots, \sigma_N(f^N(\hat{x}^N), f^N(\hat{x}^N))] \leqq [\sigma_1(f^1(\hat{x}^1), \hat{s}^1), \ldots, \sigma_N(f^N(\hat{x}^N), \hat{s}^N)] = z$. Since $(\hat{x}, \hat{s}) \in E(\Xi)$, equality must hold. So $z = 0$.

(1) $\iff$ (3) : Suppose $P(\Sigma) = \{0\}$. Let $(\hat{x}, \hat{s})$ be an optimal solution for the weighted-sum scalarization of (SPTP) with weight vector $\omega \in \mathbb{R}^N_{>}$. By [19], $(\hat{x}, \hat{s})$ is an efficient solution of (SPTP). Thus $[\sigma_1(f^1(\hat{x}^1, \hat{s}^1), \ldots, f^N(f^N(\hat{x}^N), \hat{s}^N)] = 0$ since $P(\Sigma) = \{0\}$. This implies that the optimal value of the weighted-sum scalarization is $\omega_1\sigma_1(f^1(\hat{x}^1), \hat{s}^1) + \cdots + \omega_N\sigma_N(f^N(\hat{x}^N), \hat{s}^N) = 0$. Conversely, suppose that the optimal value of the weighted-sum scalarization of (SPTP) with weight vector $\omega \in \mathbb{R}^N_{>}$ is 0. Let $(\hat{x}, \hat{s})$ be an optimal solution for the weighted-sum scalarization. Then we have that $\omega_1\sigma_1(f^1(\hat{x}^1), \hat{s}^1) + \cdots + \omega_n\sigma_N(f^N(\hat{x}^N), \hat{s}^N) = 0$. Since for all $(x, s) \in \Xi$, $(\sigma_1(f^1(x^1), s^1), \ldots, \sigma_N(f^N(x^N), s^N)) \geqq 0$, it must be that $\sigma_i(f^i(\hat{x}^i), \hat{s}^i) = 0$ for every $i \in [N]$. Furthermore, since this is holds for every $\omega \in \mathbb{R}^N_{>}$, it must be that $P(\Sigma) = \{0\}$. □

Recalling that the purpose of decomposition is to aid decision makers in finding suitable efficient solutions for their original problem modeled by (AiO), the next proposition guarantees that efficient solutions for (SPTP) are in fact efficient for (AiO).

Proposition 10. *For each $i \in [N]$, let $\sigma_i : \mathbb{R}^{p_i} \times \mathbb{R}^{p_i} \to \mathbb{R}$ and $(\hat{x}, \hat{s})$ be feasible for (SPTP). If for every $i \in [N]$:*

1. *σ_i is order preserving and $(\hat{x}, \hat{s})$ is a strictly efficient solution for (SPTP) then $\hat{x}$ is a strictly efficient solution for (AiO).*
2. *σ_i is strictly order preserving and $(\hat{x}, \hat{s}^1, \ldots, \hat{s}^N)$ is a weakly efficient solution for (SPTP), then $\hat{x}$ is a weakly efficient solution for (AiO).*
3. *σ_i is strongly order preserving and $(\hat{x}, \hat{s})$ is a weakly efficient solution for (SPTP) then $\hat{x}$ is an efficient solution for (AiO).*

Proof. We prove part 1 and note that parts 2 and 3 follow analogously. Let $\sigma_i : \mathbb{R}^{p_i} \times \mathbb{R}^{p_i} \to \mathbb{R}$ be order preserving for every $i \in [N]$ and let $(\hat{x}, \hat{s}) \in E_s(\Xi)$. Towards a contradiction, suppose $\hat{x} \notin E_s(X)$. Then there exists $x = (x^1, \ldots, x^N) \in X$ with $x \neq \hat{x}$ such that for every $i \in N, f^i(x^i) \leqq f^i(\hat{x}^i)$. Since σ_i is order preserving for every $i \in [N]$, it must be that $\sigma_i(f^i(x^i), \hat{s}^i) \leqq \sigma_i(f^i(\hat{x}^i), \hat{s}^i)$. Note that $(x, \hat{s}) \neq (\hat{x}, \hat{s})$. But this contradicts the fact that $(\hat{x}, \hat{s})$ is a strictly efficient solution for (SPTP). □

Remark 5. Proposition 10 implies that coordination can, in principle, be performed without a (strict) BASF, since only (strictly/strongly) order preserving is needed to ensure efficiency for (AiO). However, in this case Proposition 3 and Proposition 8 do not hold and (SPTP) loses the unique property that for each $i \in [N]$, $\sigma_i(f^i(x), s^i) \geq 0$ for all $x \in X$ and for all $s^i \in P(Y^i)$. It is the nonnegativity of Σ guaranteed by the use of BASFs which allows for subproblem tradeoff analysis, which we develop in what follows.

The primary utility of (SPTP) is seen in the objective functions. For every subproblem $i \in N$, the value of $\sigma_i(f^i(x^i), s^i)$ serves as a measurement of the performance of an (AiO) feasible solution, x^i, for subproblem i with respect to the Pareto set, $P(Y^i)$, of this subproblem. An efficient solution $(\hat{x}, \hat{s})$ for (SPTP) provides a feasible solution $\hat{x}$ to (AiO) and a Pareto point $\hat{s}^i$ for (SP_i) that is closest to $f^i(\hat{x}^i)$ with respect to the BASF σ_i. The best performance is achieved when $\hat{x}^i$ is efficient for subproblem i, that is, $f^i(\hat{x}^i) = \hat{s}^i$. As indicated in Proposition 9, the case when all subproblems perform at their best can be discovered by solving the weighted-sum scalarization of (SPTP) for any positive weight vector. Additionally, (SPTP) allows the use of different BASFs for the subproblems, which facilitates the application of different preferences to every subproblem or the participation of multiple DMs in the decision process.

Given the properties of (SPTP), we are ready to define subproblem tradeoffs.

Definition 8. Let $i, j \in [N]$ with $i \neq j$ and $(\hat{x}, \hat{s}) \in \Xi$ be a(n) (weakly) efficient solution for (SPTP). Then if $\sigma_j(f^j(\hat{x}^j), f^j(\hat{s}^j)) \neq 0$,

$$\mathcal{ST}_{ij}(\hat{x}, \hat{s}) = \frac{\sigma_i(f^i(\hat{x}^i), \hat{s}^i)}{\sigma_j(f^j(\hat{x}^j), \hat{s}^j)}$$

is the **subproblem tradeoff** at $(\hat{x}, \hat{s})$ with respect to subproblems i and j.

Remark 6. There are four cases for the value of $\mathcal{ST}_{ij}(\hat{x}, \hat{s})$.

Case 1: $\mathcal{ST}_{ij}(\hat{x}, \hat{s}) = 0$: $\hat{x}$ is efficient for subproblem i but not subproblem j;

Case 2: $0 < \mathcal{ST}_{ij}(\hat{x}, \hat{s}) < 1$: $\hat{x}$ performs "better" in subproblem i than in subproblem j;

Case 3: $\mathcal{ST}_{ij}(\hat{x}, \hat{s}) = 1$: there is no tradeoff between subproblem i and subproblem j, i.e., $\hat{x}$ performs "equally well" in subproblem i as in subproblem j;

Case 4: $\mathcal{ST}_{ij}(\hat{x}, \hat{s}) > 1$: $\hat{x}$ performs "worse" in subproblem i than in subproblem j.

The subproblem tradeoffs compare the performance of an (AiO)-feasible solution between two subproblems in their entirety because the BASF associated with each subproblem measures the performance of that solution with respect to each subproblem's Pareto set. Furthermore, the subproblem tradeoffs may serve as additional information supporting the decision stage of multiobjective optimization. Utilizing these tradeoffs, DMs may approach the decision problem more holistically than when considering only standard tradeoffs and individual objective functions. Thus, even if decomposition is not needed for computational reasons, it can still be beneficial for a DM because decomposition gives her access to measuring the subproblem tradeoffs which are otherwise unavailable.

4.1 A Mixed-Binary Formulation of (SPTP)

As recognized in Section 1, bilevel multiobjective optimization problems are challenging to solve. However, given the benefits of (SPTP), we suggest a pragmatic reformulation of (SPTP) which finds an ε-efficient solution of (SPTP). This reformulation is motivated by the availability of a variety methods for efficiently finding a finite representations of the Pareto sets $P(Y^i)$ ([16, 25, 27]) and the ubiquity of powerful computational tools that are available to DMs. The reformulation introduces auxiliary binary variables which select a reference point from a representation of the Pareto set of each subproblem.

Let $i \in [N]$ and $\mathcal{P}^i = \{p^{i,1}, \ldots, p^{i,j}, \ldots, p^{i,|\mathcal{P}^i|}\} \subseteq P(Y^i)$ be a finite subset of $P(Y^i)$. Let $\gamma^i \in \{0,1\}^{|\mathcal{P}^i|}$ be such that $\gamma^i_j = 1$ if the j^{th} element of $\mathcal{P}^i$ is selected as a reference point, and $\gamma^i_j = 0$ otherwise, for $1 \leq j \leq |\mathcal{P}|^i$. Then the reformulation of (SPTP) is the following.

$$\begin{aligned} \min_{\substack{x^1,\ldots,x^N\\ s^1,\ldots,s^N\\ \gamma^1,\ldots,\gamma^N}} \quad & [\sigma_1(f^1(x^1),s^1),\ldots,\sigma_N(f^N(x^N),s^N)] && (\text{MB-SPTP}(\textstyle\prod_{i=1}^N \mathcal{P}^i))\\ \text{s.t.} \quad & x=(x^1,\ldots,x^N)\in X && (1)\\ & s^i=\sum_{j=1}^{|\mathcal{P}^i|}\gamma_j^i p^{i,j} && (2)\\ & \sum_{j=1}^{|\mathcal{P}^i|}\gamma_j^i=1 && (3)\\ & \gamma^i\in\{0,1\}^{|\mathcal{P}^i|} && (4)\\ & i\in[N] && (5)\end{aligned}$$

We denote the feasible set and outcome set of (MB-SPTP($\prod_{i=1}^N \mathcal{P}^i$)) by $\Xi^{\mathcal{P}}$ and $\Sigma^{\mathcal{P}}$, respectively. In (1), we ensure that x is feasible for (AiO), while constraints (2) and (3) select the reference point. In particular, (2) forces the reference point s^i to be equal to a Pareto point from the finite set of Pareto points in $\mathcal{P}^i$, while (3) ensures that only one Pareto point is selected. The following propositions show that an efficient solution to (MB-SPTP($\prod_{i=1}^N \mathcal{P}^i$)) is ε-efficient for (SPTP) for a specified value of $\varepsilon\in\mathbb{R}^N_{\geqq}$.

Proposition 11. *If $(\hat{x},\hat{s},\hat{\gamma})$ is an efficient solution for (MB-SPTP($\prod_{i=1}^N \mathcal{P}^i$)) then $(\hat{x},\hat{s})$ is an ε-efficient solution for (SPTP), where $\varepsilon\in\mathbb{R}^N_{\geqq}$ with $\varepsilon_i=\max\{\sigma_1(f^1(\hat{x}^1),f^1(\hat{s}^1)),\ldots,\sigma_N(f^N(\hat{x}^N),f^N(\hat{s}^N))\}$ for all $i\in[N]$.*

Proof. Let $(\hat{x},\hat{s},\hat{\gamma})$ be an efficient solution for (MB-SPTP($\prod_{i=1}^N \mathcal{P}^i$)) and let ε be defined as above. Towards a contradiction, suppose $(\hat{x},\hat{s})$ is not an ε-efficient solution for (SPTP). Then there exists $(x,s)\in\Xi$ such that $[\sigma_1(f^1(x^1),f^1(s^1)),\ldots,\sigma_N(f^N(x^N),f^N(s^N))]\leq[\sigma_1(f^1(\hat{x}^1),f^1(\hat{s}^1))-\varepsilon_1,\ldots,\sigma_N(f^N(\hat{x}^N),f^N(\hat{s}^N))-\varepsilon_N]$. Note that for each $i\in N$, $\sigma_i(f^i(\hat{x}^i),f^i(\hat{s}^i))\leq\varepsilon_i$. Thus, $[\sigma_1(f^1(x^1),f^1(s^1)),\ldots,\sigma_N(f^N(x^N),f^N(s^N))]\leq[\sigma_1(f^1(\hat{x}^1),f^1(\hat{s}^1))-\varepsilon_1,\ldots,\sigma_N(f^N(\hat{x}^N),f^N(\hat{s}^N))-\varepsilon_N]\leqq 0$. But since (x,s) is feasible for (SPTP), this contradicts the fact that $0\leqq P(\Sigma)$. Thus, $(\hat{x},\hat{s})$ must be ε-efficient. □

Proposition 12. *Every efficient solution of (MB-SPTP($\prod_{i=1}^N \mathcal{P}^i$)) is an ε-efficient solution of (SPTP), where $\varepsilon\in\mathbb{R}^N_{\geqq}$ with $\varepsilon_i=\max\{\sigma_i(f^i(x^i),s^i)\mid(x,s)\in E(\Xi^{\mathcal{P}})\}$, for $i\in[N]$.*

Proof. Let $(\hat{x},\hat{s},\hat{\gamma})$ be an efficient solution for (MB-SPTP($\prod_{i=1}^N \mathcal{P}^i$)) and define $\varepsilon\in\mathbb{R}^N$ by

$$\varepsilon_i=\max\{\sigma_i(f^i(x^i),s^i)\mid(x,s)\in E(\Xi^{\mathcal{P}})\}$$

for each $i\in[N]$. Towards a contradiction, suppose $(\hat{x},\hat{s})$ is not ε-efficient for (SPTP). Then there exists $(x,s)\in\Xi$ such that $[\sigma_1(f^1(x^1),s^1),\ldots,\sigma_N(f^N(x^N),s^N)]\leq[\sigma_1(f^1(\hat{x}^1),\hat{s}^1)-\varepsilon_1,\ldots,\sigma_N(f^N(\hat{x}^N),\hat{s}^N)-\varepsilon_N]$. By definition of ε, for each $i\in[N]$, $\sigma_i(f^i(\hat{x}^i),\hat{s}^i)\leq\varepsilon_i$. Thus, $[\sigma_1(f^1(x^1),s^1),\ldots,\sigma_N(f^N(x^N),s^N)]\leq[\sigma_1(f^1(\hat{x}^1),\hat{s}^1)-\varepsilon_1,\ldots,\sigma_N(f^N(\hat{x}^N),\hat{s}^N)-\varepsilon_N]\leqq 0$, which contradicts that $0\leqq\Sigma$. Therefore, $(\hat{x},\hat{s})$ is ε-efficient. □

We proceed to show that as $\mathcal{P}^i$ grows, thus providing better representations of $P(Y^i)$, the ε-efficient solutions found by (MB-SPTP($\prod_{i=1}^N \mathcal{P}^i$)) has that ε goes to zero. In the remaining portions of this section, we use the following notation. For every $i\in[N]$, let $\mathcal{P}^i,\mathcal{Q}^i\subseteq P(Y^i)$ be nonempty finite sets such that $\mathcal{P}^i\subset\mathcal{Q}^i$. Without loss of generality, assume that $\mathcal{P}^i=\{p^{i,1},\ldots,p^{i,|\mathcal{P}^i|}\}$ and $\mathcal{Q}^i=\{p^{i,1},\ldots,p^{i,|\mathcal{P}^i|},q^{i,|\mathcal{P}^i|+1},\ldots,q^{i,|\mathcal{Q}^i|}\}$. Let $\Xi^{\mathcal{P}}$ be the feasible set and $\Sigma^{\mathcal{P}}$ be the outcome set for (MB-SPTP($\prod_{i=1}^N \mathcal{P}^i$)). Similarly, let $\Xi^{\mathcal{Q}}$ be the feasible set and let $\Sigma^{\mathcal{Q}}$ be the outcome set for (MB-SPTP($\prod_{i=1}^N \mathcal{Q}^i$).

Lemma 4. *For all (x,s,γ) feasible for (MB-SPTP($\prod_{i=1}^N \mathcal{P}^i$)), there exists $\tilde{\gamma}\in\{0,1\}^{|\mathcal{Q}^i|}$ such that $(x,s,\tilde{\gamma})$ is feasible for (MB-SPTP($\prod_{i=1}^N \mathcal{Q}^i$)).*

Proof. Let $(x, s, \gamma) \in \Xi^{\mathcal{P}}$. Define $\tilde{\gamma}$ by the following: for each $i \in [N]$,

$$\tilde{\gamma}^i = \begin{cases} \gamma^i_j, & \text{if } 1 \leq j \leq |\mathcal{P}^i| \\ 0, & |\mathcal{P}^i| + 1 \leq j \leq |\mathcal{Q}^i|. \end{cases}$$

Indeed, for each $i \in [N]$,

$$s^i = \sum_{j=1}^{|\mathcal{P}^i|} \gamma^i_j \cdot p^{i,j} + \sum_{j=|\mathcal{P}^i|+1}^{|\mathcal{Q}^i|} 0 \cdot q^{i,j}.$$

Therefore, $(x, s, \tilde{\gamma}) \in \Xi^{\mathcal{Q}}$. $\square$

Remark 7. Observe that $\Sigma^{\mathcal{P}}$ is a finite set, therefore it is compact. This means that $\Sigma^{\mathcal{P}}$ is externally stable. Thus, for any $y \in \Sigma^{\mathcal{P}}$, there exists $y' \in P(\Sigma^{\mathcal{P}})$ such that $y' \leqq y$.

Lemma 5. *The following two statements hold.*

1. $\Sigma^{\mathcal{P}} \subseteq \Sigma^{\mathcal{Q}}$
2. $P(\Sigma^{\mathcal{P}}) + \mathbb{R}^N_{\geqq} \subseteq P(\Sigma^{\mathcal{Q}}) + \mathbb{R}^N_{\geqq}$

Proof.

1. Let $y \in \Sigma^{\mathcal{P}}$. Then there exists $(x, s, \gamma) \in \Xi^{\mathcal{P}}$ such that $y = [\sigma_1(f^1(x^1), s^1), \ldots, \sigma_N(f^N(x^N), s^N)]$. By Lemma 4, there exists γ' such that $(x, s, \gamma') \in \Xi^{\mathcal{Q}}$. Therefore, $y = [\sigma_1(f^1(x^1), s^1), \ldots, \sigma_N(f^N(x^N), s^N)] \in \Sigma^{\mathcal{Q}}$.

2. Let $y \in P(\Sigma^{\mathcal{P}}) + \mathbb{R}^N_{\geqq}$. Then there exists $y^{\mathcal{P}} \in P(\Sigma^{\mathcal{P}})$ and $d^{\mathcal{P}} \in \mathbb{R}^N_{\geqq}$ such that $y = y^{\mathcal{P}} + d^{\mathcal{P}}$. Since $P(\Sigma^{\mathcal{P}}) \subseteq \Sigma^{\mathcal{P}} \subseteq \Sigma^{\mathcal{Q}}$, then $y^{\mathcal{P}} \in \Sigma^{\mathcal{Q}}$. Therefore, either $y^{\mathcal{P}} \in P(\Sigma^{\mathcal{Q}})$ or $y^{\mathcal{P}} \in \Sigma^{\mathcal{Q}} \setminus P(\Sigma^{\mathcal{Q}})$. If $y^{\mathcal{P}} \in P(\Sigma^{\mathcal{Q}})$ then $y = y^{\mathcal{P}} + d^{\mathcal{P}} \in P(\Sigma^{\mathcal{Q}}) + \mathbb{R}^N_{\geqq}$. On the other hand, let $y^{\mathcal{P}} \in \Sigma^{\mathcal{Q}} \setminus P(\Sigma^{\mathcal{Q}})$. Since $\Sigma^{\mathcal{Q}}$ is finite and therefore compact, $P(\Sigma^{\mathcal{Q}})$ is externally stable. Thus, there exists $y^{\mathcal{Q}} \in P(\Sigma^{\mathcal{Q}})$ and $d^{\mathcal{Q}} \in \mathbb{R}^N_{\geqq}$ such that $y^{\mathcal{P}} = y^{\mathcal{Q}} + d^{\mathcal{Q}}$. Therefore, $y = y^{\mathcal{P}} + d^{\mathcal{P}} = y^{\mathcal{Q}} + (d^{\mathcal{Q}} + d^{\mathcal{P}})$, which means $y \in P(\Sigma^{\mathcal{Q}}) + \mathbb{R}^N_{\geqq}$.

$\square$

Proposition 13. *For every efficient solution (x, s, γ) for (MB-SPTP($\prod_{i=1}^N \mathcal{P}^i$)), there exists an efficient solution $(\hat{x}, \hat{s}, \hat{\gamma})$ for (MB-SPTP($\prod_{i=1}^N \mathcal{Q}^i$)) such that*

$$[\sigma_1(f^1(\hat{x}^1), \hat{s}^1), \ldots, \sigma_N(f^N(\hat{x}^N), \hat{s}^N] \leqq [\sigma_1(f^1(x^1), s^1), \ldots, \sigma_N(f^N(x^N), s^N)].$$

Proof. Let $(x, s, \gamma) \in E(\Xi^{\mathcal{P}})$ and let $p = [\sigma_1(f^1(x^1), s^1), \ldots, \sigma_N(f^N(x^N), s^N)]$. By Lemma 5, $P(\Sigma^{\mathcal{P}}) + \mathbb{R}^N_{\geqq} \subseteq P(\Sigma^{\mathcal{Q}}) + \mathbb{R}^N_{\geqq}$, and so it must be that $p \in P(\Sigma^{\mathcal{Q}}) + \mathbb{R}^N_{\geqq}$. So there exists $q \in P(\Sigma^{\mathcal{Q}})$ such that $q \leqq p$. Since $q \in P(\Sigma^{\mathcal{Q}})$, there exists $(\hat{x}, \hat{s}, \hat{\gamma}) \in E(\Xi^{\mathcal{Q}})$ such that $q = [\sigma_1(f^1(\hat{x}^1), \hat{s}^1), \ldots, \sigma_N(f^N(\hat{x}^N), \hat{s}^N)]$. Therefore, it must be that $[\sigma_1(f^1(\hat{x}^1), \hat{s}^1), \ldots, \sigma_N(f^N(\hat{x}^N), \hat{s}^N)] \leqq \sigma_1(f^1(x^1), s^1), \ldots, \sigma_N(f^N(x^N), s^N)]$ $\square$

Remark 8. For $i \in [N]$, Proposition 13 shows that as $|\mathcal{P}^i|$ grows, ε decreases. In particular, the better the attained representation of $P(Y^i)$, the better (MB-SPTP($\prod_{i=1}^N \mathcal{P}^i$)) will approximate the efficient set of (SPTP). To find a representation of $P(Y^i)$, there are a variety of methods available in the literature, including exact methods and genetic algorithms ([7, 2]). Although (MB-SPTP($\prod_{i=1}^N \mathcal{P}^i$)) is a multiobjective mixed-binary optimization problem, there are several methods in the literature for solving such problems and we refer the reader to a survey of such methods (specifically for the mixed-binary linear case) in [22].

4.2 (SPTP) in the Linear Case

When (AiO) is a linear problem, we may express (SPTP) as a single-level optimization problem. This is possible since the Pareto set of a linear multiobjective problem is the union of maximal nondominated faces of the outcome set. This observation makes it possible for each reference point s^i, $i \in [N]$, to be written as a convex combination of the extreme points of a maximal nondominated face, where the specific face is selected by a binary variable.

We formulate the linear (SPTP) as follows. Since we assume that the (AiO) is linear, for each $i \in [N]$ we may write $f^i(x^i) = C^i x^i$, for some matrix C^i of appropriate dimension. Without loss of generality, let the feasible set be $X = \{x = (x^1, \ldots, x^N) \mid Ax = b,\ x \geqq 0\}$, for real-valued matrix A and vector b, both also of appropriate dimensions. For each $i \in [N]$, let N_i be the number of maximal faces defining the Pareto set of subproblem i and let $t_{i,j}$ be the number of extreme points defining the j^{th} face of subproblem i, for $j \in [N_i]$. Define $E^{i,j} = \{e^{i,j,1}, \ldots, e^{i,j,t_{i,j}}\}$ to be the set of extreme points of the j^{th} maximal nondominated face of subproblem i. Note that $\bigcup_{j=1}^{N_i} \operatorname{conv}\left(E^{i,j}\right) = P(Y^i)$. Define binary variables

$$\gamma^{i,j} = \begin{cases} 1, & \text{if face } j \text{ is selected,} \\ 0, & \text{otherwise} \end{cases}.$$

for $1 \leq j \leq N_i$. In particular, let $\gamma^i = (\gamma^{i,1}, \ldots, \gamma^{i,N_i})$. Finally, define variables $0 \leq \lambda^{i,j,k} \leq 1$ to be the weight for the k^{th} extreme point in the convex combination defining the j^{th} maximal face of the Pareto set of subproblem i. Let λ^i be the vector of all weights of all extreme points of all maximal nondominated faces in subproblem i. Then (SPTP) in the linear case is

$$\min_{\substack{x^1,\ldots,x^N \\ s^1,\ldots,s^N \\ \lambda^1,\ldots,\lambda^N \\ \gamma^1,\ldots,\gamma^N}} \quad \left[\sigma_1(C^1x^1, s^1), \ldots, \sigma_N(C^Nx^N, s^N)\right] \tag{L-SPTP}$$

$$\text{s.t.} \quad Ax = b \tag{6}$$

$$s^i = \sum_{j=1}^{N_i} \gamma^{i,j} \left(\sum_{k=1}^{t_{i,j}} \lambda^{i,j,k} e^{i,j,k} \right), \tag{7}$$

$$\sum_{k=1}^{t_{i,j}} \lambda^{i,j,k} = 1 \tag{8}$$

$$0 \leq \lambda^{i,j,k} \leq 1 \tag{9}$$

$$\sum_{j=1}^{N_i} \gamma^{i,j} = 1 \tag{10}$$

$$\gamma^{i,j} \in \{0,1\} \tag{11}$$

$$i \in [N], j \in [N_i], k \in [t_{i,j}] \tag{12}$$

where $e^{i,j,k} \in E^{i,j}$ for all $i \in [N], j \in N_i, k \in [t_{i,j}]$. In (L-SPTP), we define the objective functions, which are the BASFs scalarizing each subproblem. Constraint (6) ensures that x is feasible for (AiO). The constraint in (7) selects the reference point in subproblem i, while constraints (8)-(9) ensure that the reference point s^i is in fact a convex combination of extreme points and constraints (10)-(11) ensure that only one Pareto point is selected for the reference point. Observe that in the case of biobjective subproblems, maximal faces may be found using the parametric simplex method ([11]). Thus, in the linear case, it is beneficial to decompose (AiO) into biobjective subproblems.

We acknowledge that (L-SPTP), although not bilevel, includes N bilinear constraints. However, with a suitable choice of $\sigma_1, \ldots, \sigma_N$ and scalarization of (L-SPTP), no additional nonlinear terms will be introduced to the resulting single-objective problem. Furthermore, modern computational solvers, such as `Gurobi`, are powerful enough to handle such constraints. There are also extensive studies on bilinear optimization in the single-objective case ([1, 21, 18, 6]).

Having completed the presentation of the theory, in the subsequent sections we employ this theory in a methodology to support decision making.

5 Coordination

Coordination ensures that the efficient solutions found in decomposition and (SPTP), respectively, satisfy the preferences of the DM. In this section, we consider three methods of coordination. First, we use (SPTP) to autonomously coordinate all subproblems. This autonomous coordination is independent of DM participation. We then present hierarchical coordination, an extension of the method in [15] and [47], which uses a preferred outcome as an "anchor" point and uses relaxations, which are proposed by the DM, on its performance to find improved solutions in the other subproblems, all the while remaining ε-efficient for the anchor subproblem. Observe that hierarchical coordination does require the active participation of the DM. Finally, we propose a hybrid decision-making procedure which uses autonomous coordination to suggest anchor points and relaxations to the DM to begin the interactive hierarchical coordination process.

5.1 Autonomous Coordination

In addition to measuring the tradeoffs between subsystems, we may use (SPTP) to coordinate the subproblems of (AiO) to obtain its efficient solutions without directly solving (AiO). In Algorithm 1, we present autonomous coordination. First, (AiO) is given and suitable (strict) BASFs $\sigma_1, \dots, \sigma_N$ are provided. In step 2, (AiO) is decomposed into N subproblems. (SPTP) is formulated and solved in Step 3. In Step 4, a DM may use any MCDM tool to explore the Pareto set $P(\Sigma)$ to select an outcome which is coordinated between all of the subproblems. Finally, Step 5 outputs the efficient solution $(\hat{x}, \hat{s})$, where $\hat{x}$ is guaranteed to be (weakly/strictly) efficient for (AiO) by Proposition 10, and the components of $\hat{s}$ are Pareto points for each respective subproblem.

Algorithm 1 Autonomous Coordination.

1: **input:** All-in-One multiobjective optimization problem and BASFs $\sigma_1, \dots, \sigma_N$.
2: Decompose (AiO) into subproblems (SP$_1$), ..., (SP$_N$).
3: Solve (SPTP) and let $P(\Sigma)$ be the Pareto set of (SPTP) (or a representation of the Pareto set).

$$\begin{aligned} \min_{x, s^1, \dots, s^N} \quad & [\sigma_1(f^1(x^1), s^1), \dots, \sigma_N(f^N(x^N), s^N)] \\ \text{s.t.} \quad & x = (x^1, \dots, x^N) \in X \\ & s = (s^1, \dots, s^N) \in \prod_{i=1}^{N} P(Y^i). \end{aligned} \tag{SPTP}$$

4: Select a point of interest in $P(\Sigma)$, with preimage $(\hat{x}, \hat{s})$.
5: **output:** $(\hat{x}, \hat{s})$. By Proposition 10, $\hat{x}$ is a(n) (weakly/strictly) efficient solution for (AiO). Furthermore, for $\hat{s} = (\hat{s}^1, \dots, \hat{s}^N)$, $\hat{s}^i$ is an efficient solution for (SP$_i$), $i \in [N]$.

5.2 Hierarchical Coordination

We extend the hierarchical coordination as proposed in [15] and [47] for the new case presented here of global, quasi-global, and local variables with subproblem feasible sets X^i, $i \in [N]$. Without loss of generality, we assume that Subproblem 1 is preferred to Subproblem 2, Subproblem 2 is preferred to Subproblem 3, and so on. The hierarchical procedure is as follows. The procedure coordinates subproblems (SP$_1$), ..., (SP$_k$), $1 \leq k \leq N$, using DM selected anchor points and relaxations in each of the k subproblems. In general, the k^{th} coordination problem, for $k \in \{2, \dots, N\}$, is given by the following.

$$\begin{aligned} \min_{x^1,\ldots,x^k} \ & f^k(x^k) && \text{(HCOP}_{1\cdots k}\text{)} \\ \text{s. t.} \ & (x^1,\ldots,x^k) \in X^1 \times \cdots \times X^k \\ & f^j(x^j) \leqq f^j(x^{j*}) + \varepsilon^{j*} \\ & j \in \{1,\ldots,k-1\} \end{aligned}$$

Note that x^{j*} is a component of an efficient solution for (HCOP$_{1\cdots j}$) for each $j \in \{2,\ldots,k-1\}$, while x^{1*} is an efficient solution for (SP$_1$). Furthermore, $\varepsilon^{j*} \in \mathbb{R}^{p_j}_{\geqq}$ for each $j \in \{1,\ldots,k-1\}$. The following proposition shows how weakly efficient solutions for (HCOP$_{1\cdots k}$) can be used to construct weakly efficient solutions for (AiO).

Proposition 14. *Let $i \in \{2,\ldots,N\}$ and let $(\hat{x}^1,\ldots,\hat{x}^i)$ be a weakly efficient solution for (HCOP$_{1\cdots k}$) such that there exists $\hat{x}^{i+1},\ldots,\hat{x}^N \in \mathbb{R}^{n_{i+1}} \times \cdots \times \mathbb{R}^{n_N}$ such that $\hat{x} = (\hat{x}^1,\ldots,\hat{x}^i,\hat{x}^{i+1},\ldots,\hat{x}^N)$ is feasible for (AiO). Then $\hat{x}$ is a weakly efficient solution for (AiO).*

Proof. Towards a contradiction, suppose $(\hat{x}^1,\ldots,\hat{x}^i,\hat{x}^{i+1},\ldots,\hat{x}^N) \notin E_w(X)$. Then there exists $x = (x^1,\ldots,x^N) \in X$ such that $[f^1(x^1),\ldots,f^i(x^i),\ldots,f^N(x^N)] < [f^1(\hat{x}^1),\ldots,f^i(\hat{x}^i),\ldots,f^N(\hat{x}^N)]$ Observe that $(x^1,\ldots,x^{k-1},x^k)$ is feasible for (HCOP$_{1\cdots k}$) since for each $j \in \{1,\ldots,k-1\}$, $f^j(x^j) < f^j(\hat{x}^j) \leqq f^j(x^{j*}) + \varepsilon^{j*}$, and $(x^1,\ldots,x^k) \in X^1 \times \cdots \times X^k$. But this contradicts the weak efficiency of $(\hat{x}^1,\ldots,\hat{x}^k)$ for (HCOP$_{1\cdots k}$). Thus, it must be that $(\hat{x}^1,\ldots,\hat{x}^i,\ldots,\hat{x}^N)$ is a weakly efficient solution for (AiO). □

Table 1 presents a comparison of autonomous and hierarchical coordination contrasting computational and *a priori* requirements for both procedures.

5.3 Hybrid Coordination

Autonomous and hierarchcical coordination can be combined into an interactive decision-making procedure by using autonomous coordination to suggest anchor points and relaxations to the DM for hierarchical coordination. A proposed decision making procedure is listed in Algorithm 2.

In step 1, the DM inputs (AiO) and a collection of BASFs. Step 2 decomposes the problem, while Step 3 finds at least a representation of $P(\Sigma)$, which will be used to formulate the subproblem rankings and selection of the first anchor point and relaxation in Steps 4 and Step 6. Notably, Steps 6-7 select anchor points, relaxations, and find the (weak) Pareto set of the first hierarchical coordination problem until the DM is satisfied with the performance in subproblems 1 and 2. Next is the main loop, Steps 9-16. During this loop, the DM progressively solves the autonomous coordination problem with respect to the remaining objective functions (Step 11), and selects anchor points, relaxations, and solves (HCOP$_{1\cdots i+1}$) until the DM is satisfied (Steps 13-14). At the end, the output is a preferred (weakly) efficient solution for (HCOP$_{1\cdots N}$), which is guaranteed to be (weakly) efficient for (AiO) by Proposition 14.

6 Application

The generic nature of the D&C methodology presented here provides a DM with flexibility in application. For instance, notice that the theory for BASFs, decomposition, and coordination is agnostic concerning the convexity of the underlying multiobjective problem. Thus, in principle, given a multiobjective problem which is decomposable as described in Section 3, the DM may readily apply the D&C methodology presented here so long as she has access to sufficiently powerful solvers and computational tools.

To see this D&C methodology at work, we apply our theory of decomposition and coordination to a case of humanitarian aid by extending the Continuous Multiobjective Multidimensional Knapsack Problem ([28, 17, 4]). First, we describe the mathematical structure of this extension of the knapsack problem and then apply it to a manufactured case study of humanitarian aid management.

Algorithm 2 Hybrid Coordination.

input: All-in-One multiobjective optimization problem and BASFs $\sigma_1, \dots, \sigma_N$.
Decompose (AiO) into subproblems (SP_1), ..., (SP_N).
Solve (SPTP). (Or find representation of $P(\Sigma)$.)
Select a point of interest $\hat{\sigma} = (\sigma_1(f^1(\hat{x}), \hat{s}^1), \dots, \sigma_N(f^N(\hat{x}), \hat{s}^N)) \in P(\Sigma)$. Rank subproblems according to increasing value of the components of $\hat{\sigma}$. Without loss of generality, assume (SP_1) $\succ \cdots \succ$ (SP_N).
repeat
    Select an efficient solution $(\hat{x}, \hat{s})$ for (SPTP). Define $\hat{s}^1$ as the anchor point and $\varepsilon = (|f^{11}(\hat{x}) - \hat{s}^{11}|, \dots, |f^{1p_1}(\hat{x}) - \hat{s}^{1p_1}|)$ as the relaxation for (HCOP_{12}).
    Solve (HCOP_{12}).
until Decision maker is satisfied with (HCOP_{12})
for $i = 2, \dots, N-1$ **do**
    Let $P(Y(\text{HCOP}_{1\cdots i}))$ be the Pareto set of ($\text{HCOP}_{1\cdots k}$).
    Solve

$$\begin{aligned} \min_{x, s^i, \dots, s^N} \quad & \begin{bmatrix} \sigma_i(f^i(x), s^i) \\ \vdots \\ \sigma_N(f^N(x^N), s^N) \end{bmatrix} && && (\text{SPTP}_{i\cdots N}) \\ \text{s.t.} \quad & x = (x^1, \dots, x^N) \in X \\ & f^k(x^k) \leqq \hat{s}^k + \varepsilon^k && k \in \{1, \dots, i-1\} \\ & s^i \in P(Y(\text{HCOP}_{1\cdots i})) \\ & s^\ell \in P(Y^\ell) && \ell \in \{i+1, \dots, N\} \end{aligned}$$

    repeat
        Select a(n) (weakly) efficient solution $(\hat{x}, \hat{s}^i, \dots, \hat{s}^N)$ for ($\text{SPTP}_{i\cdots N}$). Define $\hat{s}^i$ as the anchor point and $\varepsilon = (|f^{i1}(\hat{x}) - \hat{s}^{i1}|, \dots, |f^{ip_i}(\hat{x}) - \hat{s}^{ip_i}|)$ as the relaxation.
        Solve ($\text{HCOP}_{1\cdots i+1}$).
    until Decision maker is satisfied with ($\text{HCOP}_{1\cdots i+1}$)
end for
output: $\hat{x}$, a preferred weakly efficient solution of ($\text{HCOP}_{1\dots N}$), and $\hat{s}$, a vector of anchor point in the Pareto set of each subproblem. Any weakly efficient solution for ($\text{HCOP}_{1\cdots N}$) is weakly efficient for (AiO) by Proposition 14.

6.1 Structure of Continuous Multiobjective Multidimensional Knapsack Problem with Complicating Constraints

The Continuous Multiobjective Multidimensional Knapsack Problem (CMOMDKP) modifies the classical knapsack problem in three key ways. First, it makes all variables continuous rather than discrete. Next, the scalar knapsack capacity constraint is extended to a vector constraint, which models the capacities of each dimension of the knapsack. It also adds more objective functions which model, for example, maximizing the value of items packed and minimizing costs to pack the chosen items. In this work, we consider another extension to CMOMDKP: the addition of complicating, but useful, constraints, which can, for example, model requirements on the minimum number of items that are packed in a dimension of the knapsack. The generic form of CMOMDKP with complicating constraints may be written in the following form.

$$\begin{aligned} \min \quad & Fx && \text{(13a)} \\ \text{s.t.} \quad & Wx \leqq b_{\text{cap}} && \text{(13b)} \\ & Ax \geqq b_{\text{req}} && \text{(13c)} \\ & 0 \leqq x \leqq u && \text{(13d)} \end{aligned}$$

In Equation (13a), the matrix F describes the objective functions while in Equation (13b), the matrix W and vector b_{cap} yield the standard CMOMDKP weight and capacity constraints for each dimension of the knapsack. However, Equation (13c) introduces the matrix A and the vector b_{req} to describe the complicating constraints. Finally, Equation (13d) describes inventory bounds on the items at hand.

Depending on the locations of the variables in CMOMDKP, the DM may be able to re-write the problem in the decomposable formulation presented in Section 3. If such a transformation is possible, then all of the theory of BASFs, decomposition, and coordination shown here may be applied to CMOMDKP. In the next section, we consider an application of CMOMDKP to a humanitarian aid management problem. This application has the underlying mathematical structure for the proposed decomposition, and we use the theory of BASFs and the coordination procedure developed here to work through a possible decision-making scenario.

6.2 Application to Humanitarian Aid

In an emergency crisis, delivering humanitarian aid to an affected area is of critical importance. We show that our decision making procedure is a helpful tool for making such consequential decisions.

We consider a scenario where a humanitarian aid agency must provide a rapid response to a natural disaster by sending an immediate delivery of goods to the affected area. Once the area is secured, more aid will follow. The agency has three modes of transportation: sea ($i = 1$), land ($i = 2$), and air ($i = 3$). They also have various goods that can be delivered in different combinations on each mode of transportation. Figure 4 shows how the goods may be shipped on each transportation type. Each mode of transportation has its own value of, and incurs its own cost on, the goods delivered.

Our D&C methodology is germane to this problem since the DM is uninterested in how the goods are to be packed; rather, she is only interested that the goods be *delivered*, albeit in such a way that the value of the goods for the victims is maximized and the cost of delivery is minimized. Thus, the DM does not want to compare tradeoffs between individual cost and value functions, but rather wants to compare tradeoffs between *pairs* of cost and value functions. Since each mode of transportation has its own pair of cost and value functions, this amounts to the DM performing an analysis between transportation systems. This implies the need for a higher-level view for the DM to analyze, which is precisely what subproblem tradeoffs provide. Each subproblem corresponds to a different mode of transportation, and the DM will select the mode of transportation, or subproblem, that she finds to be the most beneficial for the situation at hand.

6.2.1 Mathematical Structure of Humanitarian Aid Problem

Many humanitarian problems have been modeled by classic operations research models, including variations of the knapsack problem ([10, 42, 45]). Here, we model this problem with an instance of CMOMDKP. Each dimension of the knapsack models a specific mode of transportation, while each objective measures either the cost or value of the goods delivered with respect to the mode of transportation. The constraints of the knapsack problem ensure that no one mode of transportation is over-packed, but we note that two modes of transportation have some complicating constraints, which require that at least some number of goods be packed.

Observe that this problem has the structure necessary for decomposition and coordination to be applied. We define the AiO as the knapsack problem presented in Section 6.1. The values for $F, W, A, b_{\text{cap}}, b_{\text{req}},$ and u and the strict BASFs used for autonomous and hybrid coordination are in Table 2. We adopt the notation in Section 3 to make the decomposable structure apparent and the assignment of goods to variables is shown in Table 3. Table 4 shows the decomposition of AiO into subproblems. Note that in the decomposition, the functions denoted by c^i, for $i = 1, 2, 3$, are cost functions, which model the cost to pack items in knapsack i, while each v^i models the value of items packed in knapsack i.

In what follows, we implement CMOMDKP and perform all numerical optimization using `JuMP v1.22.2` in `Julia` ([31]) and `Gurobi 11.0.2` as our optimization solver.

6.2.2 Analysis with Autonomous Coordination

To perform autonomous coordination, the DM solves (L-SPTP) as described in Section 4.2. In Figure 5, the blue circles are the Pareto extreme points and the blue lines connecting them are the maximal nondominated faces in the outcome space of each subproblem. Of the Pareto points of (L-SPTP), the DM has selected 3 points of interest. These three points are denoted in the legend of Figure 5. Table 5 lists the preimages and outcome values, while Table 6 lists the subproblem tradeoff values and σ_i-values for each of these points.

Figure 5 provides helpful information for the DM since it shows the image of Point 1, $(\hat{x}, \hat{s})$, Point 2, $(\tilde{x}, \tilde{s})$, and Point 3, $(\breve{x}, \breve{s})$ in each subproblem. For example, consider Point 1. For each Subproblem $i,\ i \in [3]$, Point 1 is represented graphically by two points: $\hat{s}^i$ and $f^i(\hat{x}^i)$. The former is plotted on the Pareto set of the subproblem, while the latter is connected by a black line (for visual purposes only). However, in Subproblem 2, $f^2(\hat{x}^2)$ is on the Pareto set of Subproblem 2 since $f^2(\hat{x}^2) = \hat{s}^2$. Similar observations may be made of the other points in each subproblem. The values of Points 1, 2, and 3 are listed in Table 5. For $i \in [3]$, by observing the location of $(f^i(\hat{x}^i), \hat{s}^i), (f^i(\tilde{x}^i), \tilde{s}^i), (f^i(\breve{x}^i), \breve{s}^i)$, the DM has the ability to visualize the "distance" between $f^i(\hat{x}^i), f^i(\tilde{x}^i), f^i(\breve{x}^i)$ and the Pareto set of Subproblem i.

Observe that for Point 1, $\sigma_2(f^2(\hat{x}^2), \hat{s}^2) = \sigma_3(f^3(\hat{x}^3, \hat{s}^3) = 0$, which means that $\hat{x}^2$ and $\hat{x}^3$ are efficient solutions for Subproblems 2 and 3, respectively. On the other hand, $\sigma_1(f^1(\hat{x}^1), \hat{s}^1) = 1.75$. These subsystem tradeoffs imply that sending 2 units of food and 2 units of fuel over land is a better choice than sending the same aid package over sea. Similarly, the aid package of 2 units of food sent over air is also a better choice than sending aid over sea.

In the case of Point 2, observe that $\sigma_2(f^2(\tilde{x}^2), \tilde{s}^2) = 0$. This means that sending 1.11 units of food and 2 units of fuel over land is an efficient solution with respect to Subproblem 2. However, since $\sigma_1(f^1(\tilde{x}^1), \tilde{s}^1) = 0.975 < 1.84 = \sigma_3(f^3(\tilde{x}^3), \tilde{s}^3)$, sending 1.11 units of food and 2 units of fuel over sea is a better option than sending 1.11 units of food and 0.88571 units of plasma over air. In fact, using the subproblem tradeoff value $\mathcal{ST}_{13} \approx 0.53$, this aid package sent over sea is 53% better than sending it over air. Although the same package (1.11 units of food and 2 units of fuel) may be sent either by sea or air, since $\sigma_2(f^2(\tilde{x}^2), \tilde{s}^2) = 0$, the DM ought to choose the land option for delivery.

Finally, for Point 3, note that $\sigma_2(f^2(\breve{x}^2), \breve{s}^2) = 0.53115 < 0.875 = \sigma_1(f^1(\breve{x}^1, \breve{s}^1)$, which shows that sending 1 unit of food, 3.13 units of fuel, 2.4 units of medicine, and 0.425 units of building materials over land is a better decision than sending 1 unit of food and 3.13 units of fuel by sea. A similar observation may be made when comparing land to air, since $\sigma_2(f^2(\breve{x}^2), \breve{s}^2) = 0.53115 < 1.4 = \sigma_3(f^3(\breve{x}^3), \breve{s}^3)$. However, observe that $\sigma_1(f^1(\breve{x}^1), \breve{s}^1) = 0.875 < 1.4 = \sigma_3(f^3(\breve{x}^3), \breve{s}^3)$, which shows that delivery by sea is better than air. It is better to send the aid package of 1 unit of food, 3.13 units of fuel by sea than the corresponding aid package by air. Since air is not a good option in any case, the DM needs to decide whether to deliver aid by land or sea. The subproblem tradeoff value $\mathcal{ST}_{12} \approx 1.65$ shows that delivering aid by land is 61% better than delivering aid by sea.

6.2.3 Analysis with Hybrid Coordination

The DM now coordinates the subproblems using hybrid cooridination, which is listed in Algorithm 2.

Step 1: Ranking subproblems. First, the DM determines a preference ranking of the subproblems. This may be done using the points of interest, Points 1, 2, and 3, and the subproblem tradeoff values. Notice that Point 1 is efficient for Subproblems 2 and 3 since $\sigma_2(f^2(\hat{x}^2), \hat{s}^2) = \sigma_3(f^3(\hat{x}^3), \hat{s}^3) = 0$. Thus, Point 1 does not provide much direction in ranking the subproblems since there is no determination between whether to prefer Subproblem 2 or Subproblem 3. On the other hand, consider Point 3. Observe that $\sigma_2(f^2(\breve{x}^2), \breve{s}^2) = 0.53115 < \sigma_1(f^1(\breve{x}^1), \breve{s}^1) = 0.875 < \sigma_3(f^3(\breve{x}^3), \breve{s}^3) = 1.4$. With a slight relaxation on Point 3 in Subproblem 2, there is an opportunity to improve performance in Subproblem 1 and then Subproblem 3. This suggests the ranking $\mathrm{SP}_2 \succ \mathrm{SP}_1 \succ \mathrm{SP}_3$. Similarly, Point 2 has that $\sigma_2(f^2(\tilde{x}^2), \tilde{s}^2) = 0 < \sigma_1(f^1(\tilde{x}^1), \tilde{s}^1) = 0.975 < \sigma_3(f^3(\tilde{x}^3), \tilde{s}^3) = 1.84$, also implying the ranking of $\mathrm{SP}_2 \succ \mathrm{SP}_1 \succ \mathrm{SP}_3$. However, since $\sigma_2(f^2(\tilde{x}^2), \tilde{s}^2) = 0$, Point 2 is efficient for Subproblem 2, making Point 2 a suitable anchor.

Step 2: Select anchor point and relaxation for ($\mathbf{HCOP_{21}}$). As mentioned in the previous step, Point 2 is selected as the anchor point for the first coordination problem. The DM must now select a suitable relaxation. We may observe that across the 3 points, the largest σ_2-value is found with Point 3, where $\sigma_2(f^2(\breve{x}^2), \breve{s}^2) = 0.53115$. This largest σ_2-value suggests the relaxation we may place on the anchor point. We use $\breve{s}^2$ and $f^2(\breve{x}^2)$ to define the relaxation. Let $\varepsilon_{21} = f^2(\breve{x}^2) - \breve{s}^2 = (0.53, 0.53)$. Observe that since Point 2 is efficient for Subproblem 2, it is also ε_{21}-efficient.

Step 3: Find representation of ($\mathbf{HCOP_{21}}$). The DM formulates (HCOP_{21}) and solves it for its Pareto set. Figure 6 depicts the Pareto set of (HCOP_{21}) projected into the outcome space of Subproblems 1 and 2.

Step 4: Select anchor point and relaxation for ($\mathbf{HCOP_{231}}$). To find a new anchor point and relaxation in the outcome space of Subproblem 1, the DM formulates and solves (SPTP_{13}). Figure 7 shows the representation of the efficient set of (SPTP_{13}) in the outcome space of each subproblem. The DM selects the point $\dot{s}^1 = (5.57143, -9.28572)$ as the anchor point in the outcome space of Subproblem 1 and the corresponding relaxation of $f^1(\dot{x}^1) - \dot{s}^1 = (2, 2)$.

Step 5: Find representation of ($\mathbf{HCOP_{231}}$). The DM formulates and solves the problem (HCOP_{213}). Figure 8 shows the Pareto set of (HCOP_{213}) projected in the outcome spaces of Subproblems 1, 2, and 3, respectively. Observe that the DM has the opportunity to select an (AiO) feasible design which is within her allowed relaxations for Subproblems 2 and 1, and is also efficient for Subproblem 3, since part of the Pareto set of (HCOP_{213}) intersects with the Pareto set of Subproblem 3. For example, the middle point of the set of (HCOP_{213}) corresponds to an aid package of 2.29 units of food, 1 unit of fuel, 0.05 units of medicine, 0 units of dock materials, 1 unit of building materials, and 0 units of plasma. The DM may choose to deliver this aid package by land, since Subproblem 2 was used to begin the hybrid coordination. What is unique about hybrid coordination, however, is that whatever aid package is finally selected by the DM, its performance is well within her preferences across all three modes of transportation. Such information may be invaluable since, for example, if the land delivery fails, the DM may attempt a re-delivery by sea or air using the same aid package with no need to go through the decision making process again.

Step 6: Repeat until satisfaction. The DM may go back and forth between any of these steps until she is satisfied with the performances of her selected (AiO) decision within each subproblem. Once she is satisfied, she may select an output and the corresponding aid package is returned. The DM now has actionable information and may begin the process of delivering aid.

Hybrid coordination may be readily put into a "black-box", so that the DM need never personally formulate (HCOP) or (SPTP) at any point. All that is required to provide to the DM are a good representation of the Pareto sets of each subproblem, outputs on potential anchor points and relaxations, and the ability to move back and forth between every step of the hybrid decision making procedure.

7 Conclusion

Although multiobjective optimization is difficult mathematically and computationally, MOPs continue to be relevant due to their ability to address complex, real-life decision problems.

First, we consider a multiobjective optimization problem, called the All-in-One (AiO) problem, with global, quasi-global, and local variables. We show how to effectively decompose (AiO) into a set of subproblems with fewer objective functions in each subproblem. This decomposition makes optimization significantly easier. Furthermore, we show how efficient solutions for these subproblems may be used to construct efficient solutions for (AiO). This decomposition and coordination (D&C) methodology only assumes structure on the *locations* of the variables, not properties of the variables, constraints, or objective functions themselves. Thus, a DM may apply whatever optimization algorithms which are best suited given the properties of the variables, constraints, or objective functions.

Decomposition also assists a DM in selecting an efficient solution. Since each subproblem has fewer objective functions, visualization of performances within each subproblem is easier, as well as removing much of the cognitive load of needing to consider many objectives at the same time.

Next, we continue to make contributions to the decision-making stage by theoretical extensions to a powerful scalarization technique called achievement scalarizing functions (ASFs). We extend ASFs to become bivariate achievement scalarizing functions (BASFs). In so doing, we allow the reference point to itself be a variable for optimization, which gives a DM the ability to measure subproblem tradeoffs, rather than only considering each subproblem individually or measuring tradeoffs between individual objective functions. This higher level of analysis allows a DM to think of the (AiO) more holistically.

We use subproblem tradeoffs to construct an auxiliary multiobjective problem which autonomously coordinates all the subproblems in order to suggest decisions which are efficient for (AiO) but which also perform well in each individual subproblem. The autonomous nature of this coordination removes all of the cognitive burden on a DM; whatever decision she selects, she may be confident that it is mathematically "the best" choice she could have made. However, if she desires to engage directly with the selection of a decision, we propose an interactive decision making procedure which uses subproblem tradeoffs to select anchor points and suggest relaxations to improve performances in other subproblems. This interactive procedure is all performed in the outcome space, so that a DM is always concerned with performance. She may go back and forth between any step of this interactive procedure, all the while given helpful guidance by our subproblem tradeoffs and BASFs in selecting anchor points and relaxations.

In order to demonstrate the effectiveness of our D&C framework, we consider the application of delivering aid in the event of a disaster. We decompose the problem, apply autonomous coordination, and work through the hybrid coordination procedure. We believe that our decomposition methodology makes this critical decision problem tractable, mathematically and cognitively, in a way which was previously not possible.

For our future work, we desire to pursue two directions that this research has pointed out. First, investigations on the application of BASFs are needed to understand how the choice of BASF affects autonomous and hybrid coordination, and the final solution selected by a DM. Second, since autonomous coordination requires multiobjective bilevel optimization in general, and multiobjective bilinear optimization in the linear case, continued work is needed to improve methods for these types of optimization problems. We believe that our contribution here, along with these future investigations, will continue to assist DMs in making informed decisions.

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Appendix

A Tables

	Autonomous	**Hierarchical**
Representation of Pareto sets necessary	No	Yes
Anchor point necessary	No	Yes
Ranking of subproblems necessary	No	Yes
Interactive procedure	No	Yes
Minimum # of optimization problems solved	1	$2(N-1)$

Table 1: A comparison of hierarchical and autonomous coordination.

$$F = \begin{bmatrix} 2 & 3 & 0 & 15 & 0 & 0 \\ -1 & -5 & 0 & -7 & 0 & 0 \\ 1 & 4 & 8 & 0 & 3 & 0 \\ -1 & -3 & -5 & 0 & -2 & 0 \\ 1 & 0 & 6 & 0 & 0 & 12 \\ -1 & 0 & -4 & 0 & 0 & -6 \end{bmatrix} \qquad W = \begin{bmatrix} 1/2 & 2 & 0 & 3 & 0 & 0 \\ 1/2 & 2 & 1 & 0 & 2 & 0 \\ 1/2 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$b_{\text{cap}} = \begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix} \qquad b_{\text{req}} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad u = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 1 \\ 1 \end{bmatrix}$$

$$\sigma_i(y, r) = \max_{j=1,\ldots,p_i} \{y_i - r_i\}, \; i \in [3]$$

Table 2: Data for the humanitarian aid example.

Transportation	**Subproblem**
Sea	1
Land	2
Air	3
Goods	**Variable Name**
Food	x_{123}
Fuel	x_{12}
Medicine	x_{23}
Floating docks	x_1
Building materials	x_2
Plasma	x_3

Table 3: Subproblems and variables in humanitarian aid example.

	Objective functions	Constraints
Subproblem 1	$\min \quad f^1(x^1) = \begin{bmatrix} c^1(x^1) = 2x_{123} + 3x_{12} + 15x_1 \\ -v^1(x^1) = -1x_{123} - 5x_{12} - 7x_1 \end{bmatrix}$	$1/2x_{123} + 2x_{12} + 3x_1 \leq 7$ $0 \leq x_{123}, x_{12}, x_1 \leq 5$
Subproblem 2	$\min \quad f^2(x^2) = \begin{bmatrix} c^2(x^2) = x_{123} + 4x_{12} + 8x_{23} + 3x_2 \\ -v^2(x^2) = -x_{123} - 3x_{12} - 5x_{23} - 2x_2 \end{bmatrix}$	$1/2x_{123} + 2x_{12} + x_{23} + 2x_2 \leq 10$ $x_{12} + x_2 \geq 2$ $0 \leq x_{123}, x_{12}, x_{23} \leq 5$ $0 \leq x_2 \leq 1$
Subproblem 3	$\min \quad f^3(x^3) = \begin{bmatrix} c^3(x^3) = x_{123} + 6x_{23} + 12x_3 \\ -v^3(x^3) = -x_{123} - 4x_{23} - 6x_3 \end{bmatrix}$	$1/2x_{123} + x_{23} + 2x_3 \leq 5$ $x_{123} + x_3 \geq 2$ $0 \leq x_{123}, x_{23} \leq 5$ $0 \leq x_3 \leq 1$

Table 4: Decomposition for humanitarian aid example.

	Point 1:$(\hat{\mathbf{x}}, \hat{\mathbf{s}})$	**Point 2:**$(\tilde{\mathbf{x}}, \tilde{\mathbf{s}})$	**Point 3:**$(\check{\mathbf{x}}, \check{\mathbf{s}})$
Food: $\mathbf{x_{123}}$	2	1.11	1
Fuel: $\mathbf{x_{12}}$	2	2	3.13
Medicine: $\mathbf{x_{23}}$	0	0	2.4
Dock Materials: $\mathbf{x_1}$	0	0	0
Building Materials: $\mathbf{x_2}$	0	0	0.425
Plasma: $\mathbf{x_3}$	0	0.88571	1
$\mathbf{s^1}$	(8.25, -13.75)	(7.25, -12.09)	(10.5, -17.5)
$\mathbf{s^2}$	(10, -8)	(9.11, -7.11)	(33.44, -23.76)
$\mathbf{s^3}$	(2, -2)	(9.90, -8.27)	(26, -18)
$(\mathbf{f^{11}}(\mathbf{x^1}), \mathbf{f^{12}}(\mathbf{x^1})$	(10, -12)	(8.23, -11.11)	(11.38, -16.63)
$(\mathbf{f^{21}}(\mathbf{x^2}), \mathbf{f^{22}}(\mathbf{x^2}))$	(10, -8)	(9.11, -7.11)	(33.98, -23.23)
$(\mathbf{f^{31}}(\mathbf{x^3}), \mathbf{f^{32}}(\mathbf{x^3}))$	(2, -2)	(11.74, -6.43)	(27.4, -16.6)

Table 5: Values of Points 1, 2, and 3.

	Point 1	**Point 2**	**Point 3**
$\mathcal{ST}_{12}$	-	-	1.6474
$\mathcal{ST}_{13}$	-	0.5299	0.6250
$\mathcal{ST}_{23}$	-	0	0.3794
$\sigma_1(f^1(x^1), s^1)$	1.75	0.975	0.875
$\sigma_2(f^2(x^2), s^2)$	0	0	0.53115
$\sigma_3(f^3(x^3), s^3)$	0	1.84	1.4

Table 6: Subproblem Tradeoffs and σ_i-values for Points 1, 2, and 3.

B Figures

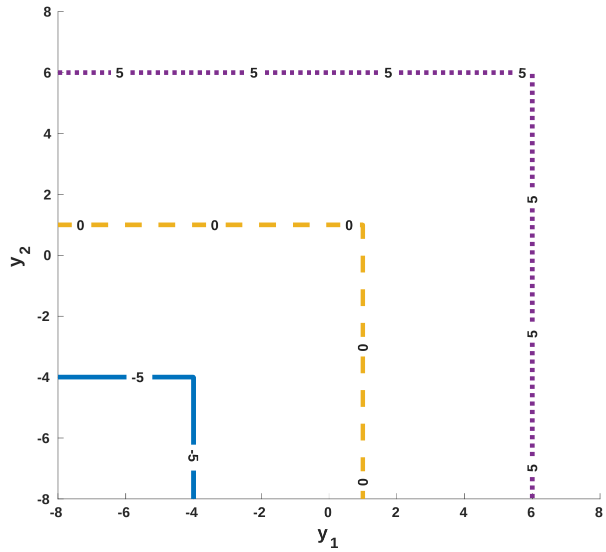

Figure 1: Level curves of $\sigma(y, r) = \max_{i=1,2}\{y_1 - r_1, y_2 - r_2\}$ for fixed $r = (1, 1)$.

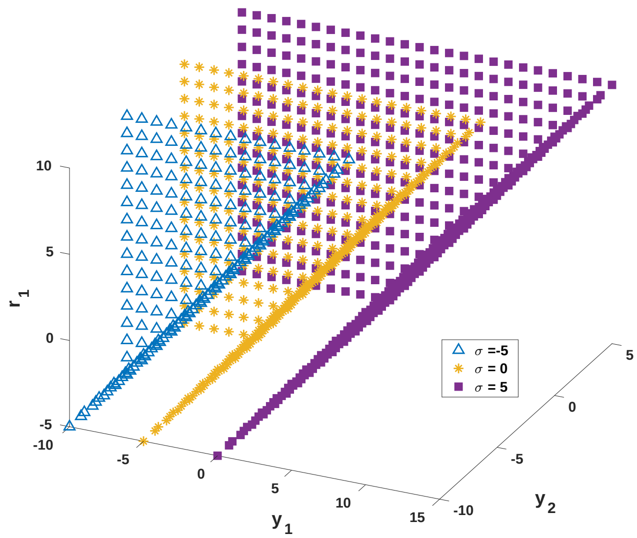

Figure 2: Level surfaces of $\sigma(y, r) = \max_{i=1,2}\{y_1 - r_1, y_2 - r_2\}$ for $-10 \leq y_1, y_2 \leq 10$, $-10 \leq r_1 \leq 10$, and $r_2 = 0$.

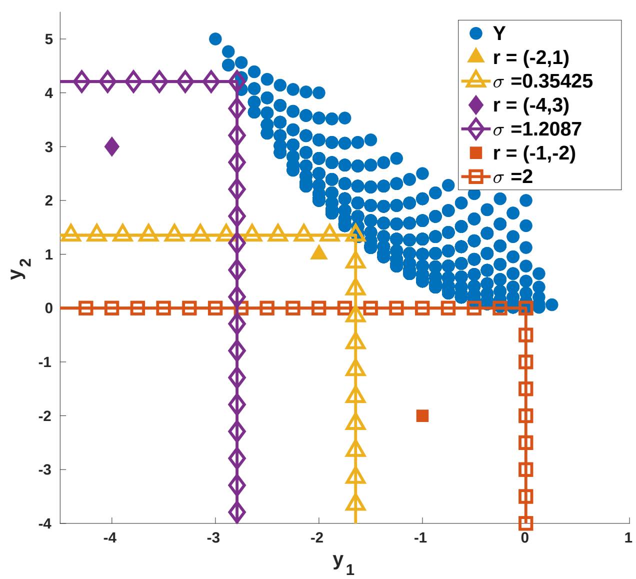

Figure 3: (σ-MOP) selects the reference point which has the smallest value level curve. In this case, (σ-MOP) selects $r = (-2, 1)$ with $\sigma = 0.35425$.

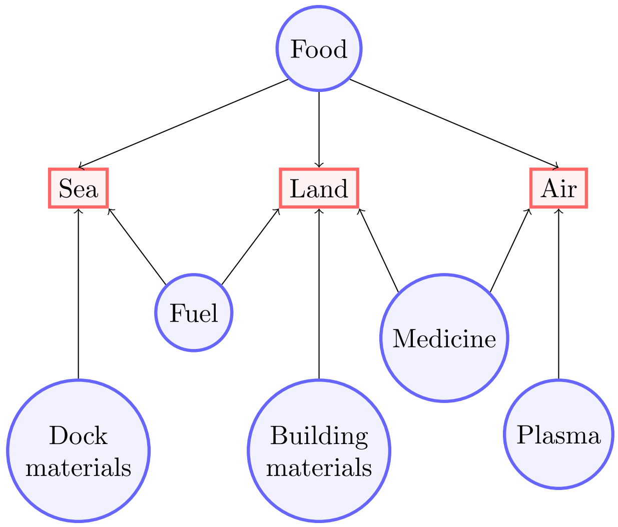

Figure 4: Modes of transportation (squares) and goods to be delievered (circles) in the humanitarian aid example.

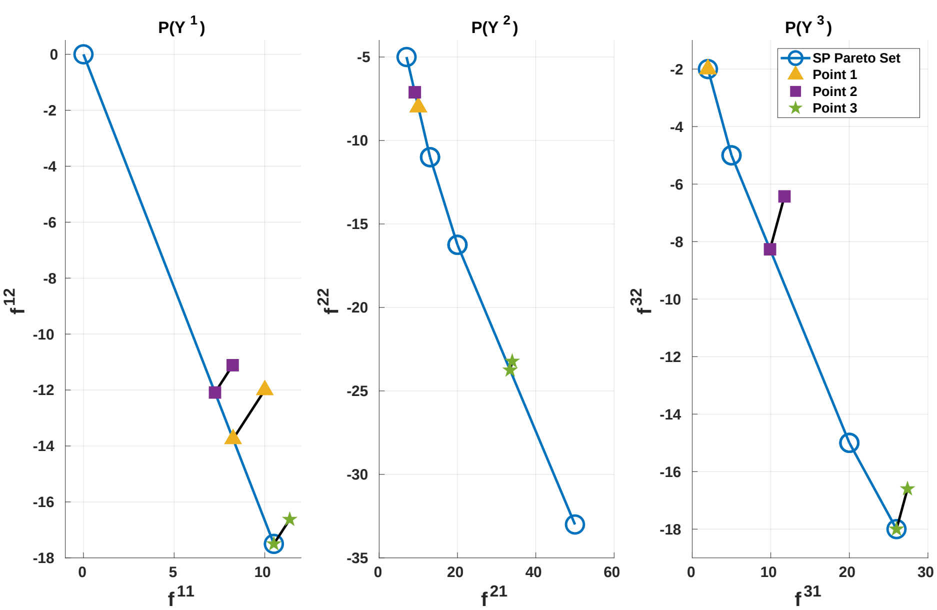

Figure 5: Autonomous Coordination: Results of (SPTP).

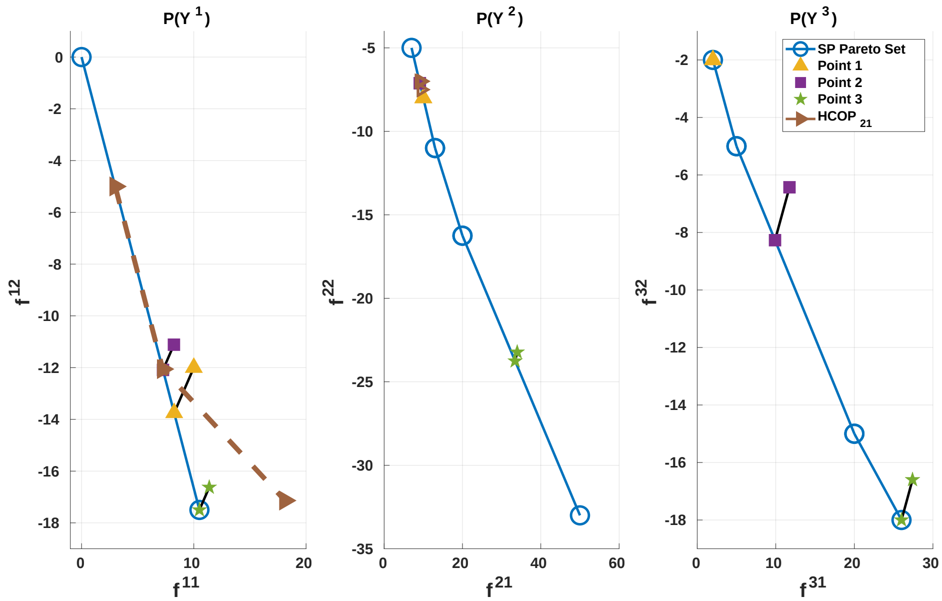

Figure 6: Hybrid Coordination: Results of ($\mathbf{HCOP_{21}}$).

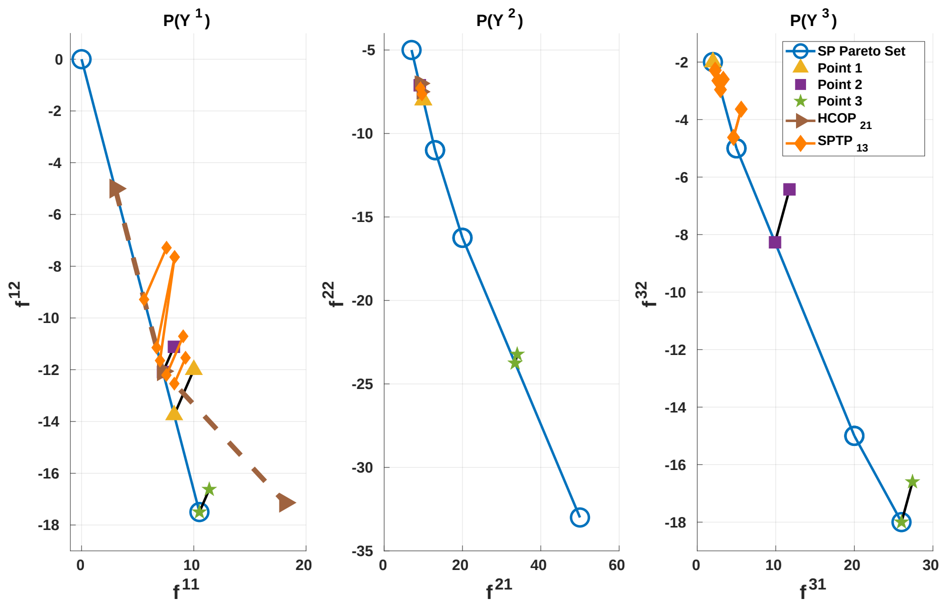

Figure 7: Hybrid Coordination: Results of ($\mathbf{HCOP_{21}}$) and ($\mathbf{SPTP_{31}}$).

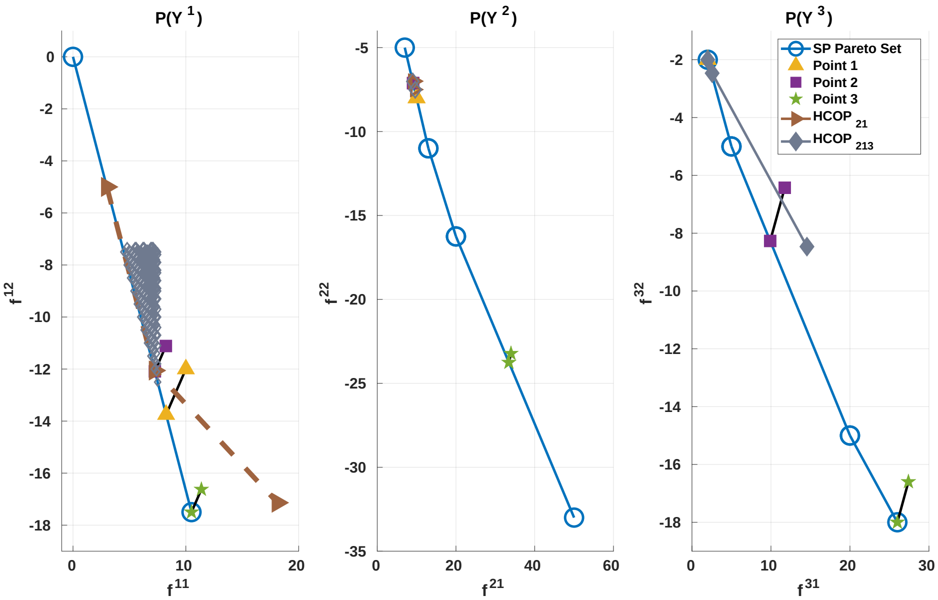

Figure 8: Hybrid Coordination: Results of ($\mathrm{HCOP}_{\mathbf{21}}$), ($\mathrm{SPTP}_{\mathbf{31}}$), and ($\mathrm{HCOP}_{\mathbf{213}}$).

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