Closest Assignment Constraints for Hub Disruption Problems

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Abstract

Supply chains and logistics can be well represented with hub networks. Operations of these hubs can be disrupted due to unanticipated occurrences or attacks. This study includes *Closest assignment Constraints* related to hub disruption problems, which can be used in single-level reformulation of the bilevel model. In this study, We propose new sets of constraints for closer assignments of non-hub nodes to hub nodes. We analysis those alternate sets of constraints and present the dominance relations among them. Experiment results show that the effectiveness of constraints can enhance tightness of models. Finally, We present a case study based on bicycle network.

Keywords: Closest assignment constraints; Disruption;Hub Network; Bilevel optimization

1 Introduction

Hub disruption problems study the problem of identifying critical hubs in a hub network, which when failed results in worst case damage to the network operator. These problems are typically modeled as bilevel static Stackelberg games between two decision makers namely, an attacker(natural occurrences like calamities) and a defender. The attacker is the Stackelberg *leader* in this problem and it creates disruption in set of hubs (deactivate) such that the defender's(stackelberg *follower*) post-disruption operational objective is worsened(Liberatore, Scaparra, & Daskin 2012). The objectives studied in hub interdiction problems can be: maximizing the minimum demand weighted cost of transportation (Ghaffarinasab & Motallebzadeh 2018, Lei 2013, Ramamoorthy, Jayaswal, Sinha, & Vidyarthi 2018, Ullmert, Ruzika, & Schöbel 2020), or minimizing the maximum demand covered, or maximizing the objective of minimizing worst case transportation cost between a source node and destination node. These problems are called as: r-hub median interdiction problem, r-hub maximal covering interdiction problem or as r-hub center interdiction problem.

These problems are typically solved through reformulating the problem as a single level problem using either i) a duality based approach (Ramamoorthy et al. 2018), or ii) using closest assignment constraints (CAC) (Ghaffarinasab & Motallebzadeh 2018, Lei 2013), which captures the lower level objective. In this paper, we propose several alternate closest assignment constraints for the r-hub Median disruption problem(r-HDP). We also study their theoretical properties and perform computational experiments to validate the theoretical results. In the following section, we studied the background literature, related to our study. In section 3, we describe the r-hub median disruption problem followed by its single level reformulation with various alternate set of closest assignment constraints. In Section 4, we study the dominance relationships between the various closest assignment constraints. In section 5, we present computational results followed by case study at section 6 and conclusion at section 7.

2 Literature Review

Disruption problems can be categorized based on several attributes: according to the components face failure, it is broadly classified into failure in Network, failure in facility and hubs.

In most network related problems, the interdictor tries to maximize the cost to the defender by attacking the arc capacity or disrupting the arcs. McMasters and Mustin (1970) studied network interdiction problem to generate defense plan for military combat force. They proposed a solution approach capable of generating an optimal interdiction plan for the opposing force, which targets to minimize the network's capacity within a predefined budget. InWood (1993),Interdictor attacks network arcs to reduce the maximum flow of drugs and chemicals. Cormican, Morton, and Wood (1998) extended this problem by adding uncertainty in arc capacity. In this model, the interdictor wants to minimize the expected maximum flow of the network. Another variant of network interdiction problems is shortest path interdiction, which was studied in Israeli and Wood (2002). They modelled the problem as to maximize the shortest path by destroying arcs.Morton, Pan, and Saeger (2007) appeared with an interdiction strategy to install sensors in transportation paths in nuclear smuggling networks. They included stochastic issues in the model. Altner, Ergun, and Uhan (2010), Lozano, Smith, and Kurz (2017) are some literature dealing with different network interdiction issues.Interested readers are recommended to study the survey literature of Smith and Song (2020) for further knowledge.

In interdiction, interdictor has complete information about the set of decision maker's facilities or hubs, out of which attacker interdicts a set of facilities or hubs to create maximum disruption to the decision maker. Interdiction of facilities (set of critical nodes) is studied in Church, Scaparra, and Middleton (2004) as r-interdiction median problems (r-IMP)and r- Interdiction covering problem. These binary integer programming (BIP) problems are one of the initial works, considering the underlying problem of p-hub median problem and maximal covering problem, respectively. Church and Scaparra (2007) extended this model as a mixed integer programming model and included fortification issues. The objective is to fortify some facilities out of a set of facilities to minimize the interdiction effects. They also commented that the fortification of those facilities mainly improves the system, which is considered as critical during interdiction. InScaparra and Church (2008a), the fortification problem is studied as bilevel mixed integer programming problem. They used implicit enumeration to solve the model. It is seen in Scaparra and Church (2008b) that, r-Interdiction Median problem with Fortification (r-IMF) can be formulated as a maximal covering problem. Aksen, Piyade, and Aras (2010) proposed budget constrained r-interdiction-fortification problem. They modeled this problem as Bilevel Binary Integer Programming problem. Closest Assignment Constraints (CACs) are used in the model.Liberatore et al. (2012) includes ripple effects in the fortification problem. In this literature, it is observed that the disruption effects of a facility are not consolidated at a point; rather, these spread over the whole network.Scaparra and Church (2012) modeled Capacitated r-Interdiction Median Problem (CRIMP) as tri-level model. Facility interdiction issues in fixed charge location problem is considered in Aksen and Aras (2012). Aksen, Aras, and Piyade (2013) appeared with a bilevel p-median protection problem with capacity expansion option. Furthermore, Losada, Scaparra, Church, and Daskin (2012) took stochastic approach for interdiction model with different disruption intensity levels. Facility protection problem considering the time horizon is seen in Parvasi, Tavakkoli-Moghaddam, Bashirzadeh, Taleizadeh, and Baboli (2019). This literature proposed hybrid metaheuristics for different levels to solve the problem. Lei (2013) proposed the hub median interdiction problem (HIMP) and studied the difference with the Flow interdiction problem (FIM). In FIM problem, the O-D flows are completely disrupted by interdiction of nodes in the network, which is not the same in HMIP problem. Hub Median Interdiction Problem (HMIP) is seen in Parvaresh, Husseini, Golpayegany, and Karimi (2014). They modelled this as a bilevel binary integer programming problem and solved it using the tabu search heuristic method. Ghaffarinasab and Motallebzadeh (2018) interdiction in p-hub median, p-hub maximal covering and p-hub centre problems. Ghaffarinasab and Atayi (2018) also appeared with hub interdiction median problem with fortification problem. In Ramamoorthy et al. (2018), hub fortification problem followed by hub interdiction problem was solved by decomposition methods. The implicit enumeration method is also used in this literature. Lei (2019), Quadros, Roboredo, and Pessoa (2018), Ullmert et al. (2020) are some literature that deal with hub interdiction problems considering different issues and solution methods.

Under the available literature, it is required to have a study on CACs related to hub disruption problems and prepare new sets of these constraints to enhance the model performance and tightness.

3 Problem Description and Model Formulation

In a network $G = (N, A)$, set of nodes N represents the set of origins (i) and destinations(j) (called an O-D pair), whose flows are routed through one or two transshipment nodes between them, known as hubs. Let H denote the set of p hubs present in the system. Every O-D pair (i, j) is routed through some hub pair (k, m) such that the transportation distance is $D_{ijkm} = \alpha d_{ij} + \omega d_{km} + \gamma d_{mj}$, where d_{ab} denotes the distance between nodes a and b, and α, ω, γ are the discount factors of collection, transshipment, and distribution respectively. k is index for one hub within a route, $k \in H$ and m is index for another hub within the same route, $m \in H$. W_{ij} denotes demand to be routed from source i to destination j. Here, the assumption is that distance is linearly proportional to cost.Hence, cost between node pair (a, b) is d_{ab}. Disruption in first level, occurs at a set of r out of the located p hubs such that the defender's post disruption objective of minimizing the cost of transportation is maximized. In decision variables, we considers y_k= 1 if hub k is surviving after disruption, 0 otherwise; X_{ijkm} is fraction of flows between OD pair (i, j) through hubs (k, m). In the following subsection, we present the bilevel formulation of the r-HMDP.

3.1 Mathematical Formulation:

With the above notation, the Bi-level formulation for Hub interdiction problem is given below:

$$[HMDP_{BL}] : \max_{y} Z \tag{1}$$

$$\text{s.t.} \sum_{k \in H} y_k = p - r; \qquad \forall k \in H \tag{2}$$

$$y_k \in \{0, 1\} \qquad \forall k \in H \tag{3}$$

$$Z = \min_{X} \sum_{i \in N} \sum_{j \in N} \sum_{k \in H} \sum_{m \in H} W_{ij} D_{ijkm} X_{ijkm} \tag{4}$$

$$\text{s.t.} \sum_{k \in H} \sum_{m \in H} X_{ijkm} = 1 \qquad \forall i \in N \; j \in N \tag{5}$$

$$\sum_{m \in H} X_{ijkm} + \sum_{m \in H m \neq k} X_{ijmk} \leqslant y_k \qquad \forall i, j \in N \; k \in H \tag{6}$$

$$X_{ijkm} \geqslant 0 \qquad \forall i, j \in N \; k, m \in H \tag{7}$$

In the $HMDP$ formulation, (1)-(3) represent the attacker's problem while, (4)-(7) is the defender's problem. The attacker's objective is to maximize the system cost by creating disruption at r hubs. Upper level constraint (2) represents the remaining $(p - r)$ surviving hubs in post disruption situation. Objective function (4) shows the operator, who is trying to minimize the overall cost by routing through remaining hubs($y_k = 1$).Lower level constraint (5) states that the complete traffic flow between each OD pair (i, j) should be through combination of hubs (k, m). Constraint (6) shows that origin node i can only route traffic to the destination node j through k

if the hub k is active during disruption.(3) and (7) states the binary and continuous nature of variables respectively.

The allocation of node pair (i,j) to hub (k,m) is significantly influenced by the presence of Closest Assignment Constraints (CACs).CACs play a important role as a solution to hub location problem and maintains the traffic routing between two non-hub nodes.Single level reformulation of the above problem with CACs is given below:

$$[HMDP_{SL}] : \max_{y,X} \sum_{i\in N}\sum_{j\in N}\sum_{k\in H}\sum_{m\in H} W_{ij} D_{ijkm} X_{ijkm} \tag{8}$$

$$\text{s.t.} \sum_{k\in H} y_k = p - r; \qquad \forall k \in H \tag{9}$$

$$\sum_{k\in H}\sum_{m\in H} X_{ijkm} = 1 \qquad \forall i \in N\ j \in N \tag{10}$$

$$\sum_{m\in H} X_{ijkm} + \sum_{m\in H m\neq k} X_{ijmk} \leqslant y_k \qquad \forall i,j \in N\ k \in H \tag{11}$$

$$set\ of\ CACs \tag{12}$$

$$X_{ijkm} \geqslant 0 \qquad \forall i,j \in N\ k,m \in H \tag{13}$$

$$y_k \in \{0,1\} \qquad \forall k \in H \tag{14}$$

3.2 Closest Assignment Constraints:

CACs replace the lower level objective of r-HMDP by enforcing the flows between any O-D pair (i,j) go through the cheapest path. In literature, different types of CACs are studied to model single level version of r-HMIP. Specifically, Ramamoorthy et al. (2018) studied several version of CACs. For comparison with the CACs presented in this paper, we take the proposed CACs presented as most effective in Ramamoorthy et al. (2018).

$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} \leqslant 2 - y_k - y_m \qquad \forall\ i,j \in N; (k,m) \in H'_{ij}. \tag{CAC1}$$

where $E_{ijkm} = \{(q,s)|\ d_{ijqs} > d_{ijkm}\ \}$, $H'_{ij} = \{H''_{ijkm}|(k,m) \in H, k \leqslant m\}$ and

$$H''_{ijkm} = \begin{cases} (k,m) \text{ if } d_{ijkm} \leqslant d_{ijmk} \\ (m,k) \text{ if } d_{ijkm} > d_{ijmk}. \end{cases}$$

The constraint states that when the hubs k and m are located, the flow between any O-D pair (i,j) should not happen through paths that are costlier than d_{ijkm}. For any pair of hubs k and m, out of the four possible paths $(i-k-k-j), (i-m-m-j), (i-k-m-j), and (i-m-k-j)$, note that this constraint is written for only the paths $(i-k-k-j)$, $(i-m-m-j)$, and the cheaper one among $(i-k-m-j)$ and $(i-m-k-j)$. In the following section, we present alternate new CACs for r-HMDP.

3.3 New CACs

In this section, we propose several alternate versions of Closest Assignment Constraints. We first present a modified version of CAC1:

$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} \leqslant 1 - y_k \qquad \forall\, i,j \in N; k \in H. \tag{15}$$

$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} \leqslant 2 - y_k - y_m \qquad \forall\, i,j \in N; (k,m) \in H'_{ij}. \tag{CAC2}$$

where $E_{ijkm} = \{(q,s)|\ d_{ijqs} > d_{ijkm}\ \}$, $H'_{ij} = \{H''_{ijkm}|(k,m) \in H, k \neq m\}$ and

$$H''_{ijkm} = \begin{cases} (k,m) \text{ if } d_{ijkm} \leqslant d_{ijmk} \\ (m,k) \text{ if } d_{ijkm} > d_{ijmk}. \end{cases}$$

CAC2 can be written by separating CAC1 for cases when both hubs are same and hubs are different. For the case when hubs are same the RHS of the constraint can be written as $1 - y_k$ instead of $2 - 2y_k$ since the LHS is bounded by 1 and dividing RHS by 2.

CAC3 is obtained through eliminating redundant CACs from CAC2. The constraint is presented below:

$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} \leqslant 1 - y_k \qquad \forall\, i,j \in N; k \in H. \tag{16}$$

$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} \leqslant 2 - y_k - y_m \qquad \forall\, i,j \in N; (k,m) \in \bar{H}'_{ij}. \tag{CAC2m}$$

where $E_{ijkm} = \{(q,s)|\ d_{ijqs} > d_{ijkm}\ \}$, $\bar{H}'_{ij} = \{\bar{H}''_{ijkm}|(k,m) \in H, k \neq m\}$ and

$$\bar{H}''_{ijkm} = \begin{cases} (k,m) \text{ if } d_{ijkm} = min(d_{ijkk}, d_{ijkm}, d_{ijmm}, d_{ijmk}) \\ (m,k) \text{ if } d_{ijkm} = min(d_{ijkk}, d_{ijkm}, d_{ijmm}, d_{ijmk}). \end{cases}$$

In CAC3, we remove the dominated CACs for a given hub pair (k,m).

The following CACs are written for more than one O-D pair. They are presented below:

$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} + \sum_{(q,s)\notin E_{ijkm}, \in E_{jikm}} X_{jiqs} \leqslant 2 - y_k - y_m \qquad \forall i,j \in N, k,m \in H \tag{CAC4}$$

Proposition 1. *CAC4 enforces closest assignment in $HMDP_{SL}$*

Proof. Consider the sets E_{ijkm} and E_{jikm} and w.l.o.g. consider the case where $E_{ijkm} \subset E_{jikm}$. Let (a,b) be a hub pair such that $(a,b) \in E_{jikm}, \notin E_{ijkm}$. This implies that

$d_{ijab} \leqslant d_{ijkm}$ and $d_{jiab} > d_{jikm}$. If all the hubs a, b, k, and m are open, it is easy to observe that $X_{ijab} = 1$ and $X_{jikm} = 1$ and thus enforcing closest assignment. □

The nearest work of CAC4 is the constraint, presented by Cánovas, García, Labbé, and Marín (2007). In the following, we present CAC4 by writing it separately for cases where $k = m$ and $k \neq m$ (CAC4(i)).

$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} + \sum_{(q,s)\notin E_{ijkm}, \in E_{jikm}} X_{jiqs} \leqslant 1 - y_k \qquad \forall i,j \in N, k = m, k, m \in H$$
$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} + \sum_{(q,s)\notin E_{ijkm}, \in E_{jikm}} X_{jiqs} \leqslant 2 - y_k - y_m \qquad \forall i,j \in N, k \neq m, k, m \in H \tag{17}$$

This set of constraints can be expanded as given below:

$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} + \sum_{(q,s)\in \hat{E}_{ijkm}} X_{jiqs} \leqslant 2 - y_k - y_m \quad \forall i,j \in N\ k, m \in H, (k \neq m), (j \geqslant i)$$
$$\sum_{(q,s)\in E_{jikm}} X_{jiqs} + \sum_{(q,s)\in \hat{E}_{jikm}} X_{ijqs} \leqslant 2 - y_k - y_m \quad \forall i,j \in N\ k, m \in H, (k \neq m), (j \geqslant i)$$
$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} + \sum_{(q,s)\in \hat{E}_{ijkm}} X_{jiqs} \leqslant 1 - y_k \quad \forall i,j \in N\ k, m \in H, (k = m), (j \geqslant i)$$
$$\sum_{(q,s)\in E_{jikm}} X_{jiqs} + \sum_{(l,t)\in \hat{E}_{jikm}} X_{ijqs} \leqslant 1 - y_k \quad \forall i,j \in N\ k, m \in H, (k = m), (j \geqslant i) \tag{18}$$

where, $\hat{E}_{ijkm} = \{(q,s) | D_{ijqs} \leqslant D_{ijkm}\ and\ D_{jiqs} > D_{jikm}\}$. This set of constraints can be further expanded as:

$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} + \sum_{(q,s)\in \hat{E}_{ijkm}} X_{jiqs} \leqslant 2 - y_k - y_m \quad \forall i,j \in N\ k, m \in \hat{H}_{1ij}, (k < m), (j \geqslant i)$$
$$\sum_{(q,s)\in E_{jikm}} X_{jiqs} + \sum_{(q,s)\in \hat{E}_{jikm}} X_{ijqs} \leqslant 2 - y_k - y_m \quad \forall i,j \in N\ k, m \in \hat{H}_{1ij}, (k < m), (j \geqslant i)$$
$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} + \sum_{(q,s)\in \hat{E}_{ijkm}} X_{jiqs} \leqslant 1 - y_k \quad \forall i,j \in N\ k, m \in \hat{H}_{1ij}, (k = m), (j \geqslant i)$$
$$\sum_{(q,s)\in E_{jikm}} X_{jiqs} + \sum_{(q,s)\in \hat{E}_{jikm}} X_{ijqs} \leqslant 1 - y_k \quad \forall i,j \in N\ k, m \in \hat{H}_{1ij}, (k = m), (j \geqslant i)$$
$$\sum_{(q,s)\in E_{ijkm}} X_{jiqs} + \sum_{(q,s)\in \hat{E}_{ijkm}} X_{ijqs} \leqslant 2 - y_k - y_m \quad \forall i,j \in N\ k, m \in \hat{H}_{2ij}, (k < m), (j \geqslant i)$$
$$\sum_{(q,s)\in E_{jikm}} X_{jiqs} + \sum_{(q,s)\in \hat{E}_{jikm}} X_{ijqs} \leqslant 2 - y_k - y_m \quad \forall i,j \in N\ k, m \in \hat{H}_{2ij}, (k < m), (j \geqslant i) \tag{19}$$

$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} + \sum_{(q,s)\in \hat{E}_{ijkm}} X_{jiqs} \leqslant 2 - y_k - y_m \quad \forall i,j \in N\ k,m \in \hat{H}_{2ij}, (k < m), (j \geqslant i)$$
$$\sum_{(q,s)\in E_{jikm}} X_{jiqs} + \sum_{(q,s)\in \hat{E}_{jikm}} X_{ijqs} \leqslant 2 - y_k - y_m \quad \forall i,j \in N\ k,m \in \hat{H}_{2ij}, (k < m), (j \geqslant i)$$
$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} + \sum_{(q,s)\in \hat{E}_{ijkm}} X_{jiqs} \leqslant 1 - y_k \quad \forall i,j \in N\ k,m \in \hat{H}_{2ij}, (k = m), (j \geqslant i)$$
$$\sum_{(q,s)\in E_{jikm}} X_{jiqs} + \sum_{(q,s)\in \hat{E}_{jikm}} X_{ijqs} \leqslant 1 - y_k \quad \forall i,j \in N\ k,m \in \hat{H}_{2ij}, (k = m), (j \geqslant i)$$
$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} + \sum_{(q,s)\in \hat{E}_{ijkm}} X_{jiqs} \leqslant 1 - y_k \quad \forall i,j \in N\ k,m \in \hat{H}_{2ij}, (k = m), (j \geqslant i)$$
$$\sum_{(q,s)\in E_{jikm}} X_{jiqs} + \sum_{(q,s)\in \hat{E}_{jikm}} X_{ijqs} \leqslant 1 - y_k \quad \forall i,j \in N\ k,m \in \hat{H}_{2ij}, (k = m), (j \geqslant i)$$

where $\hat{H}_{1ij} = \{H_{ijkm} | k,m \in H, k \leqslant m\}$; $H_{ijkm} = \{(k,m)\ if\ ((D_{ijkm} <= D_{ijmk})\ and\ ((D_{jikm} <= D_{jimk}))\ or\ (m,k)\ if\ ((D_{ijmk} < D_{ijkm}\ and\ (D_{jimk} < D_{jikm})\}$; $\hat{H}_{2ij} = (H_{ijkm} \notin \hat{H}_{1ij})$

We present a reduced version of CAC4 (CAC4(i)) below similar to CAC2:

$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} + \sum_{(q,s)\notin E_{ijkm}, \in E_{jikm}} X_{jiqs} \leqslant 1 - y_k \quad \forall i,j \in N, k = m, k,m \in H \tag{20}$$
$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} + \sum_{(q,s)\notin E_{ijkm}, \in E_{jikm}} X_{jiqs} \leqslant 2 - y_k - y_m \quad \forall i,j \in N, (k,m) \in H'_{ij}, H'_{ji} \tag{CAC4(i)}$$

CAC4(i) can be further modified (CAC4(ii))by using similar approach of CAC2m:

$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} + \sum_{(q,s)\notin E_{ijkm}, \in E_{jikm}} X_{jiqs} \leqslant 1 - y_k \quad \forall i,j \in N, k = m, k,m \in H \tag{21}$$
$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} + \sum_{(q,s)\notin E_{ijkm}, \in E_{jikm}} X_{jiqs} \leqslant 2 - y_k - y_m \quad \forall i,j \in N, (k,m) \in \bar{H}'_{ij}, \bar{H}'_{ji} \tag{CAC4(ii)}$$

4 Dominance Relationship between Constraints:

In this section, we present dominance relationship between the proposed CACs. The objective of the exercise is to identify effective CACs.

Proposition 2. *CAC2 dominates CAC1*

Proof. For cases when $k = m$, $CAC2$ can be written as:

$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} \leqslant 1 - z_k \forall i,j \in N, k \in H$$

Multiplying on both sides by 2 we get:

$$\sum_{(q,s)\in E_{ijkm}} 2X_{ijqs} \leqslant 2 - 2z_k \forall i,j \in N, k \in H$$

Since the LHS is bounded by 1, we can rewrite the above constraint as:

$$\sum_{(q,s)\in E_{ijkm}} X_{ijqs} \leqslant 2 - 2z_k \forall i,j \in N, k \in H$$

Hence, $CAC2 \implies CAC1$ but $CAC1 \not\Rightarrow CAC2$. □

Proposition 3. *CAC4 dominates CAC1*

Proof. The proof is straightforward. The LHS of CAC1 is a subset of LHS of CAC4, while the RHS is the same. Therefore, $CAC4 \implies CAC1$, but $CAC1 \not\Rightarrow CAC4$. □

Proposition 4. *CAC2m dominates CAC2*

Proof. For a given o-d pair (i, j) and hub candidates (k, m) CAC2 has 3 constraints, respectively for (k, k), (m, m), and (k, m) if $d_{ijkm} \leqslant d_{ijmk}$ or (m, k) if $d_{ijkm} > d_{ijmk}$. For the cases where d_{ijkk} or d_{ijmm} being the cheapest among all the four possible paths through the hubs k and m, CAC2m has 2 constraints namely for (k, k) and (m, m), else for other cases CAC2m has 3 constraints similar to CAC2. Without loss of generality assume that d_{ijkk} is the shortest route from i to j through hub pairs k and m and $d_{ijkm} \leqslant d_{ijmk}$. Then, $E_{ijkk} \subset E_{ijkm}$ which implies $CAC2_{ijkk}$ dominates $CAC2_{ijkm}$ and hence $CAC2_{ijkm}$ is redundant. Hence CAC2m dominates CAC2. □

Proposition 5. *CAC4(ii) dominates CAC4*

Proof. The proof for this proposition is similar to the proof for proposition 4. □

5 Computational experiments

We conduct experiments on data sets of Civil Aeronautics Board (CAB). CAB dataset was introduced in O'kelly (1987) .CAB dataset containing N=25 with hubs p = (5, 10, 15) is used to prepare instances. The set of located p hubs in our experiment is prepared by solving the corresponding p-hub median problem. For the instance with 5 hubs, the number of interdicted hubs varies as (2, 3, 4) while for the instance with 10 hubs, it varies as (3, 4, 5, 6, 7, 8). In the case of 15 hubs, we consider a range of 5 to 12 interdicted hubs, r ∈ (5, 6, 7, 8, 9, 10, 11, 12). Table 1 presents the instances used in our experiments. Our analysis assumes a discount factor of 1 each for collection and distribution, while the discount factor for transhipment (Ω) varies between 0.1, 0.5, and

Table 1: Instances

Dataset	p	Ω	located Hubs
CAB25	5	0.1	3, 6, 11, 13, 16
		0.5	3, 6, 11, 13, 16
		0.9	0, 3, 6, 11, 16
	10	0.1	0, 3, 5, 6, 7, 11, 13, 16, 21, 24
		0.5	0, 3, 5, 6, 7, 11, 13, 16, 21, 24
		0.9	0, 3, 6, 7, 11, 13, 16, 19, 20, 21
	15	0.1	0, 2, 3, 5, 6, 7, 11, 13, 14, 15, 16, 20, 21, 22, 24
		0.5	0, 2, 3, 5, 6, 7, 11, 13, 14, 15, 16, 20, 21, 22, 24
		0.9	0, 2, 3, 5, 6, 7, 9, 11, 13, 14, 16, 20, 21, 22, 24

0.9. To solve each instance, we set a time limit of 36,000 seconds. These formulations are coded in C++ using IBM ILOG CPLEX Callable Library 12.10 and run on a Dell workstation with processor Intel(R) Xeon(R), 3.60GHz, 3600 Mhz,6 Core(s), and 64 gigabytes of RAM. Experiments related to case study are done in computer with 8 gigabytes of RAM and CPLEX callable library 22.1. All of experiments are done in the parallel mode.

Analysis of results

Computational time(seconds) to solve reformulated single level problem utilizing CAC1, CAC2, CAC4, CAC4(i) can be observed in Table2 for CAB dataset. Columns $HMDP_{SL1}$, $HMDP_{SL2}$, $HMDP_{SL3}$, and $HMDP_{SL4}$ are employed with time(s) contain the CPU times to solve the problems. The column with the heading *objective* represents the optimal value of the corresponding instance. For CAB dataset we multiplied the demand and cost parameter for each OD pair by 10^{-3}. Surviving hubs in post disruption situation among a set of located hubs are shown in column *Existing hubs*. Experiments have been performed on 27 instances with this dataset.In comparison of the computational times with different CACs, $HMDP_{SL2}$ takes less time in 26/27 instances. $HMDP_{SL4}$ takes lesser time than $HMDP_{SL3}$.In these results, we can observe that the change of discount factor Ω for a instance impacts the objective value. Under this scenario, we can comment that the discounted cost are important even for a network vulnerable to interdiction for maintaining a smooth flow in a cost-effective way. Furthermore, the objective value increases along with the increment of interdicted hubs, which happens in real life network also.

Experiments on LP relaxed form of reformulated single level problem are performed considering different CACs. Results of CAB data set with 25 nodes and 15 hubs are noted in Table 3. Columns with headings $LP_{SL1}, LP_{SL2}, LP_{SL3},\ and\ LP_{SL4}$ present the LP relaxed value of $HMDP_{SL1}, HMDP_{SL2}, HMDP_{SL3},\ and\ HMDP_{SL4}$ respectively. The results of 24 instances explain LP_{SL4} generates tighter LP bound to the problem.The percentage LP_{gap} is calculated by using $\%Lp_{gap} = |Objectivevalue - Lpvalue| * 100/objectivevalue.HMDP_{SL2}$ generates LP_{gap} of a range 9%-160%.In this case the average value is 110%. The average value is calculated by taking mean of all values in that column. $HMDP_{SL4}$ generates the LP_{gap} of a range 5%-145%. On average it is reduced to 102%. Hence, this depicts that $HMIP_{SL4}$ is tighter formulation.It is very desired to solve a mixed interger programming problem, where reducing the

Table 2: Results of CAB25 instances using different CACs

p	Ω	r	objective	Existing Hubs	$HMDP_{SL1}$ time(s)	$HMDP_{SL2}$ time(s)	$HMDP_{SL3}$ time(s)	$HMDP_{SL4}$ time(s)
5	0.1	2	12620	**6,11,13**	2	2	2	2
		3	15292.3	**6,13**	2	2	2	2
		4	30040.6	**11**	1	1	2	1
	0.5	2	13940.4	**6, 11,13**	1	1	1	2
		3	16458.9	**11,13**	1	1	1	1
		4	30040.6	**11**	1	1	1	1
	0.9	2	11808.5	**0,6,11**	1	1	1	1
		3	16587.3	**6,11**	1	1	1	1
		4	30040.6	**11**	1	1	1	1
10	0.1	3	6320.3	**0,3,6,7,11,13,21**	20	19	36	37
		4	8266.52	**0,6,7,11,13,21**	23	25	54	49
		5	11926.3	**6,7,11,13,21**	24	22	51	44
		6	14416.6	**6,7,11,21**	19	19	43	37
		7	18571.9	**7,11,21**	17	15	35	26
		8	29670.4	**11,21**	12	11	13	13
	0.5	3	8142.33	**0,3,6,7,11,13,21**	15	13	21	20
		4	10003.4	**0**, **6**, **7**, **11**, **13**, **21**	15	15	25	21
		5	13367.8	**6**, **7**, **11**, **13**, **21**	16	14	23	22
		6	15399.8	**6**, **7**, **11**,**21**	15	13	25	20
		7	19218.7	**7**, **11**,**21**	13	13	19	16
		8	29782.8	**11, 21**	11	10	12	11
	0.9	3	10333.5	**0**,**6**, **7**, **11**, **13**, **16**,**20**	11	10	11	11
		4	11122.7	**0**,**6**, **7**, **11**, **13**,**21**	12	11	12	11
		5	14092.1	**6**, **7**, **11**, **13**,**21**	13	11	14	14
		6	16079	**6**, **7**, **11**,**21**	12	11	13	12
		7	19765.4	**7**, **11**, **21**	12	11	11	11
		8	29828.7	**11**,**21**	11	10	10	10

feasible region is main concern at the starting. Figure 1 explains the comparison of different LP relaxed formulations with alternate CACs w.r.t the corresponding objective value.

On computation results

As per Prop. 1, CAC4 maintains the closer assignment of non-hub nodes to hub and routing via surviving hubs ($y_k = 1$). Hence, the constraint11 can be removed in $HMDP_{SL}$.

6 Case study

Bicycle is important mode of transportation in last mile delivery problems. Bicycle network is very sensitive to disruptions. Here, bicycle network is studied as hub network, where order pickup, order sorting points are considered as hubs, whereas parking points, delivery points are assumed as non-hub nodes. In this case study, we want:

- To ensure allocation of other nodes to the hub nodes with tighter constraints, introduced in above section.
- To find the critical hubs in the bicycle network, which is vulnerable to disruptions.
- To get the percentage increase in the objective value in post disruption situation.

Table 3: Results of CAB25 instances using different CACs

p	Ω	r	Existing Hubs	objective	LP_{SL1} time(s)	LP_{SL2} time(s)	LP_{SL3} time(s)	LP_{SL4} time(s)
15	0.1	5	**0, 6, 7, 11, 13, 14,15,20,21,22**	6600.04	27178.5	16467.8	25249.1	15532.3
		6	**6, 7, 11, 13, 14, 15, 20, 21, 22**	7502.57	31991.4	19480.5	29694	18358.3
		7	**6, 7, 11, 13, 14, 15, 21, 22**	9108.95	36803.2	22493.2	34132.4	21183.8
		8	**6, 7, 11, 13, 15, 21, 22**	11338	39208.7	24953.7	36430.8	23543.3
		9	**6, 7, 11, 15, 21, 22**	12275	39208.7	26862	36564.9	25424.7
		10	**6, 7, 11, 21, 22**	14201.2	39208.7	28765.5	36656.2	27256.6
		11	**7, 11, 21, 22**	18354.5	39208.7	30415.1	36730.6	28988.9
		12	**11, 21, 22**	29253.3	39208.7	32006.3	36756.2	30651.4
	0.5	5	**0, 6, 7, 11, 13, 14, 15, 20, 21, 22**	8603.32	31876.4	19336.5	29120.6	18424.5
		6	**6, 7, 11, 13, 14, 15, 20, 21, 22**	9247	36895.3	22313.8	33590.6	21193.7
		7	**6, 7, 11, 13, 14, 15, 21, 22**	10718.2	41903.5	25291.2	38022	23960.5
		8	**6, 7, 11, 13, 15, 21, 22**	12776.4	44403.5	27872.6	40270.4	26339.3
		9	**6, 7, 11, 15, 21, 22**	13562.8	44403.5	30058.2	40369.8	28331.8
		10	**6, 7, 11, 21, 22**	15252.7	44403.5	32228.6	40459.3	30305.9
		11	**7, 11, 21, 22**	19071.6	44403.5	33679.1	40507	31899.9
		12	**11, 21, 22**	29407.7	44384.7	34937.1	40511.5	33317.3
	0.9	5	**0, 6, 7, 9, 11, 13, 14, 20, 21, 22**	10134.7	38109.6	22926.4	34964.1	22710.9
		6	**6, 7, 9, 11, 13, 14, 20, 21, 22**	10719.5	43844.4	26058.5	39961.8	25800
		7	**6, 7, 9, 11, 13, 14, 21, 22**	12049.2	49576.6	29190.7	44946.4	28889.1
		8	**6, 7, 9, 11, 13, 21, 22**	13983.8	52441.9	32117.2	47554.8	31706.6
		9	**6, 7, 9, 11, 21, 22**	15885.4	52441.9	34837.9	47795.1	34252.4
		10	**7, 9, 11, 21, 22**	16605.6	52441.9	37555	47988.6	36792.6
		11	**7, 11, 21, 22**	19718.3	52441.9	39486.3	48095.4	38081.6
		12	**11, 21, 22**	29491.1	52441.9	40307.5	48127.4	38998.1

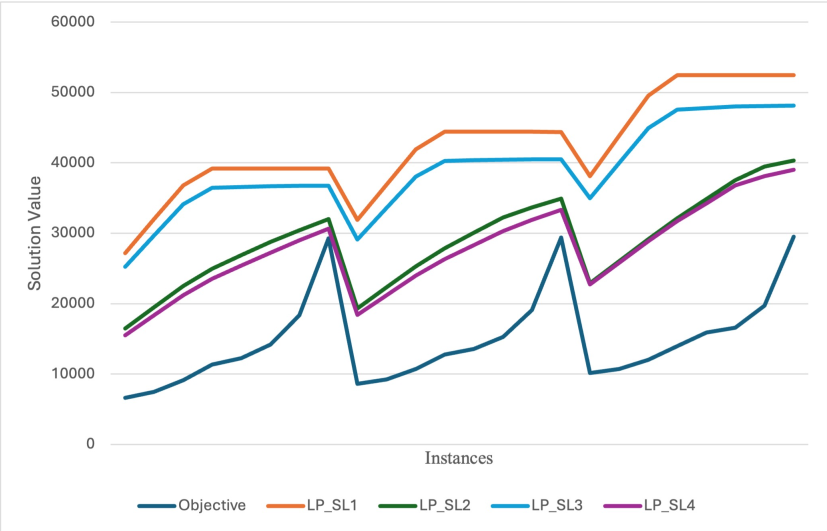

Fig. 1: LP relaxed values of CAB instances using different alternate CACs

Table 4: Instances of Bicycle network

p	Ω	Cost	Located Hubs (No disruption)
15	0.1	18.7362	2,11,13,15,17,19,22,26,30,31,32,36,43,46,49
	0.5	28.8236	2,11,13,15,19,20,22,24,29,31,32,38,39,46,49
	0.9	35.6989	2,4,11,13,15,19,20,22,25,29,34,38,39,47,49

Table 5: Results of Bicycle network instances faced disruption

	Ω	r	obj val.	Existing Hubs	%increase
15	0.1	5	86.4389	15,17,19,22,26,30,31,32,46,49	361.35
		6	102.848	15,17,22,26,30,31,32,46,49	448.93
		7	118.957	15,17,22,26,30,31,46,49	534.90
		8	128.515	15,22,26,30,31, 46,49	585.92
		9	170.205	15, 22,26,30, 31, 46	808.43
		10	181.353	15,26,30,31, 46	867.93
		11	195.578	15, 22,26, 46	943.85
		12	217.401	22, 26, 46	1060.33
	0.5	5	94.4678	15,19,20,22,24,29,31,32, 46,49	227.69
		6	113.17	15, 20,22,24,29,31,32, 46,49	292.56
		7	130.127	15, 20,22,24,29,31, 46,49	351.38
		8	171.2	15, 20, 22, 24, 29, 31, 46	493.85
		9	179.624	15, 20,22,24,29, 46	523.08
		10	196.253	15, 20,22,24, 46	580.76
		11	217.463	20,22,24, 46	654.33
		12	235.996	20, 24, 46	718.62
	0.9	5	98.4148	4, 15,19,20,22,25,29,34,38,39,47,49	175.68
		6	125.812	4,15,19,20,22,25,29,34, 47,49	252.43
		7	140.151	4, 15, 20,22,25, 34, 47,49	292.59
		8	164.305	15, 20,22,25,29,34, 47	360.25
		9	180.598	15,20,22,25,29, 47	405.89
		10	200.609	20,22,25, 34, 47	461.95
		11	219.439	20, 25, 34, 47	514.69
		12	261.798	20,25,47	633.35

The problem is modeled as aforementioned bilevel optimization problem, where the disruption in the hubs may increase the routing cost, whereas the operator tries to minimize the routing cost with the surviving hubs in post disruption situation. Table 4 explains the cost of the routing and locations of hubs by the column 'Cost' and 'Located Hubs' before disruption occurs. 'No disruption' implies the value of r as 0.Furthermore, we solve the bi-level model $HMDP_{BL}$ by enumerating upper level variable (y_k) and consider all possible combinations of $p - r$ out of p. The column '% improvement' in table 6 shows the improvement in solution time by using single level model with CAC4(i) than upper level enumeration process for each instances. If the disruption occurs at some hubs, the operator tries to route the bicycles through surviving hubs.In the table 5, column 'obj val.' is used for the post-disruption cost of the corresponding instances, whereas '% increase' is used for increase of the system cost of routing in post-disruption situation. The increment of system cost happens upto 523% on average. The cost of routing before disruption and after disruption is presented in figure 2.

Table 6: Results of Bicycle network instances

p	Ω	r	iteration	%improvement
10	0.1	7	120	70
	0.5	7	120	128
	0.9	7	120	139

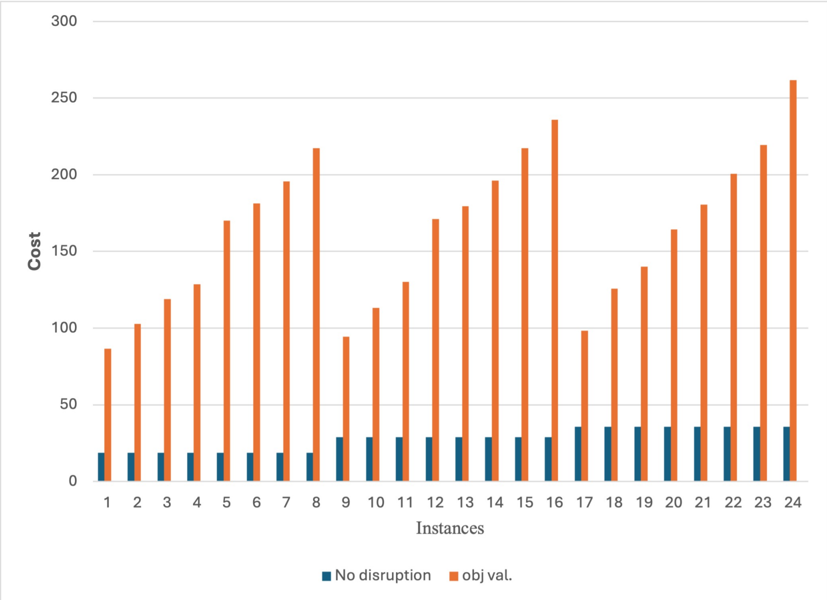

Fig. 2: Cost of instances faced disruption

7 Conclusion

This study has focused on bilevel hub disruption problems. Hub median problem has several uses in real-life networks. Hence, identifying critical hubs of an existing hub network is necessary to reduce the vulnerability of disruption. In this work, we have introduced new sets of CACs for hub problems.We established dominance relationships among them. We use those alternate constraints to reformulate the bilevel model into single level. Several experiments have been performed on instances of CAB and bicycle network dataset considering reformulated single-level interdiction model. We have also presented the solution time of the disruption problem based on different alternate CACs. Results shows that the routing cost will increase tremendously in post disruption situation. Future research can be done to use these CACs in different solution approaches.

Disclosure statement

No potential conflict of interest was reported by the authors.

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