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Single-Scenario Facet Preservation for Stochastic Mixed-Integer Programs

Aysenur Karagoz · David Mildebrath · Andrew J. Schaefer

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Abstract We consider improving the polyhedral representation of the extensive form of a stochastic mixed-integer program (SMIP). Given a facet for a single-scenario version of an SMIP, our main result provides necessary and sufficient conditions under which this inequality remains facet-defining for the extensive form. We then present several implications, which show that common recourse structures from the literature satisfy these conditions. For example, for an SMIP with simple recourse, any single-scenario facet is also a facet for the extensive form. For more general recourse structures, we provide additional mild necessary and sufficient conditions.

Keywords stochastic programming · mixed-integer recourse · facet-defining inequalities · polyhedral theory

1 Introduction

Stochastic mixed-integer programs (SMIPs) with recourse are a widely used modeling tool to optimize discrete decisions under uncertainty. Applications include production planning (e.g., [29]), healthcare (e.g., [37]), finance (e.g., [12]), and supply chain management (e.g., [39]). In a two-stage SMIP, a "here-and-now" decision is made in the first stage before the realization of the uncertain data is known. A realization of the uncertain data is called a *scenario*.

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Aysenur Karagoz
Department of Computational Applied Mathematics and Operations Research, Rice University.

David Mildebrath
Amazon, DM's contributions were completed prior to his work at Amazon.

Andrew J. Schaefer
Department of Computational Applied Mathematics and Operations Research, Rice University.

After a scenario is realized, a recourse action is taken in the second stage. See Birge and Louveaux [5] for more information.

SMIPs are, in general, difficult to solve when the second stage contains integer variables [20]. Decomposition approaches are common techniques to solve SMIPs [42]. Given a finite set of scenarios, a two-stage SMIP can be formulated as a large-scale mixed-integer program, called *the extensive form*. Another approach, which we consider here, is to solve the extensive form directly, which remains common in practice (e.g., [17,45]). However, solving the extensive form becomes harder as the number of scenarios grows. To this end, in this paper, we focus on improving the polyhedral representation of the convex hull of the feasible space of the extensive form.

The polyhedral structure of the feasible space of the extensive form of an SMIP is often difficult to study. To make the structure more amenable to polyhedral analysis, we consider smaller problems, called *single-scenario problems*. A single-scenario problem is a two-stage stochastic program in which one scenario occurs with probability one, and variables and constraints related to other scenarios are eliminated. Every deterministic problem can be cast as a single-scenario problem, and there is often extensive literature on the polyhedral study of single-scenario problems. Hence, this paper considers when facet-defining inequalities for a single-scenario problem can be extended to facet-defining inequalities for the extensive form.

In mixed-integer programs, strong valid inequalities can improve the polyhedral representation of the convex hull, which may lead to a significant increase in the solution efficiency (e.g., [10,43]). In the literature, the polyhedral structure of various problems is studied for deterministic settings such as the binary knapsack polytope [18], symmetric traveling salesman polytope [14], and variable upper-bound flow models [35], lot-sizing [4], dynamic knapsack [25] and unit commitment problems [34]. Another widely studied polyhedral structure in the literature is the *single-node flow (SNF)* polytope [1,22]. See Nemhauser and Wolsey [30] and Conforti et al. [9] for more detail.

As every deterministic problem can be cast as a single-scenario problem, strong valid inequalities derived for deterministic problems can also be used for single-scenario problems. Moreover, valid inequalities for a single-scenario problem can be extended to valid inequalities for the extensive form. However, a natural question is the strength of these single-scenario valid inequalities in the polyhedral description of the convex hull of the extensive form. Thus, establishing a relation between the facial structures of the extensive form and a single-scenario problem may significantly improve solving general SMIPs. In the remainder of the paper, we use single-scenario valid inequalities (facets) to refer to valid inequalities (facets) for a single-scenario problem.

In the literature, the structure of certain deterministic problems is leveraged to develop valid inequalities for their stochastic counterpart. Guan et al. [16] extend valid inequalities from the deterministic lot-sizing problem to a new class for the stochastic version, demonstrating their computational effectiveness. Similarly, Guan et al. [15] generate valid inequalities for the stochastic dynamic knapsack and lot-sizing problems using known inequalities from deterministic counterparts. Pan et al. [36] develop several classes of valid inequalities for certain unit commitment problems that extend to stochastic unit commitment problems. Liu and Küçükyavuz [24] study the static probabilistic lot-sizing problem and propose valid inequalities which give the convex hull of the related stochastic lot-sizing problem. Another widely studied polyhedral structure in the literature is single-node flow (SNF) polytope, and its applications, including fixed-charge transportation problems, and facility location problems, and lot-sizing problems ([16,26]). Mildebrath et al. [28] introduce the stochastic version of the SNF polytope and show that single-scenario valid inequalities can be used in the description of the stochastic SNF polytope. Valid inequalities from a deterministic problem can also be recycled for the robust variant (see [7,27]).

Single-scenario valid inequalities are used within decomposition algorithms to tighten the second-stage problems. Carøe and Tind [8] generate lift-and-project cuts based on first- and second-stage variables for each scenario for SMIPs with mixed binary second-stage variables. Sherali and Fraticelli [41] use the Reformulation-Linearization Technique and parametric lift-and-project cuts to solve the second-stage problems when the problem has integer recourse. Sen and Higle [40] apply disjunctive programming to derive single-scenario valid inequalities to convexify the second-stage problems for SMIPs with the binary first stage. Ntaimo and Tanner [33] use lift-and-project cuts derived for each scenario for SMIPs with binary first-stage, and mixed binary second-stage variables. Ntaimo [31] presents a class of valid inequalities based on disjunctive programming for SMIPs with mixed binary second-stage variables and non-fixed recourse. Ntaimo [32] derives Fenchel cutting planes that are valid for the single-scenario problem for each scenario for SMIPs with binary first stage. Gade et al. [13] utilize parametric Gomory cuts to iteratively tighten the second-stage problem to solve SMIPs with binary first-stage and integer second-stage variables. Zhang and Küçükyavuz [46] further generalize this approach for solving SMIPs with integer first- and second-stage variables. Kim and Mehrotra [19] use mixed-integer rounding inequalities parameterized by the first-stage variables to tighten second-stage problems. Bodur et al. [6] propose a scenario-based cut-and-project approach for SMIPs with mixed-integer first-stage variables. Bansal et al. [3] add scenario-based valid inequalities a priori to convexify the second-stage problems for SMIPs with integer first-stage and mixed-integer second-stage variables. van der Laan and Romeijnders [21] derive parametric optimality cuts for two-stage SMIPs with mixed-integer recourse. The most similar work to ours is Mildebrath et al. [28], which provides necessary and sufficient conditions for facet-defining inequalities for the single-scenario problem to be facet-defining for the extensive form of the stochastic SNF polytope.

Single-scenario valid inequalities are analyzed in the sparse cutting-plane context. Dey et al. [11] examine the strength of the sparse cutting planes using methods that describe the sparsity structure of the constraint matrix and the cutting planes. A common approach in developing strong valid inequalities for the stochastic problem is to combine valid inequalities for individual scenarios, as discussed in Riis and Andersen [38], Guan et al. [15]. However, in this paper, we focus on the facet-defining inequalities for a single-scenario problem that can directly be used in the description of the convex hull of the extensive form.

Given the improvements obtained by valid inequalities, using the approach in Mildebrath et al. [28] for general SMIPs may provide promising results when solving the extensive form. Therefore, in this paper, we look for the conditions under which single-scenario valid inequalities can be used in the description of the extensive form, in particular, when facet-defining inequalities for the single-scenario problem are facet-defining for the extensive form. We say single-scenario facets are *preserved* (in a sense to be made more precise in Section 2) if they are also facet-defining for the extensive form. One of our principle questions is under what conditions the polyhedron has preserved single-scenario facets. Our contributions are:

1. We characterize two-stage SMIPs for which single-scenario facets are facet-defining for the convex hull of the feasible space of the extensive form.
2. We provide necessary and sufficient conditions that ensure the preservation of single-scenario facets.
3. We give conditions under which certain recourse structures preserve *all* single-scenario facets.
4. We show that stochastic facility problems have preserved single-scenario facets.

2 Necessary and Sufficient Conditions for Preserving Single-Scenario Facets

Throughout this paper, we use $\text{conv}(\cdot)$ to denote the convex hull of a set, $\text{proj}_x(\cdot)$ to denote the projection onto the variables in the subscript, $I_{n\times n}$ to denote the $n \times n$ identity matrix, and e_k to denote the unit vector where all entries are zero except k-th element which is one. We also use $\mathbf{1}_n$, $\mathbf{0}_n$, and $\mathbf{0}_{n\times n}$ to refer to the n-dimensional vector of ones, n-dimensional vector of zeros and $n \times n$ matrix of zeros, respectively. To denote strictly positive matrices or vectors, i.e., $x_j > 0$, $\forall j$, we use the notation $x > 0$, where x_j denotes the j-th element of x. For matrices, A^i and A_j denote the i-th row and j-th column of matrix A, respectively. Proofs and examples that are not given in the main text are provided in the Online Resource.

Let Ω be a finite set of realizations with corresponding probabilities $\Pr(\omega)$ for $\omega \in \Omega$. Let m_1 be the number of constraints in the first stage, m_2 be the number of constraints in the second stage for each scenario, l_1 be the number of integer variables in the first stage, and l_2 be the number of integer variables in the second stage. Given a finite set of realizations, Ω, the extensive form is as follows:

$$\min cx + \sum_{\omega \in \Omega} \Pr(\omega) q^\omega y^\omega \tag{1a}$$

$$\text{s.t. } Ax \geq b, \tag{1b}$$

$$T^\omega x + W^\omega y^\omega \geq h^\omega, \qquad \forall\, \omega \in \Omega, \tag{1c}$$

$$x \in \mathbb{R}_+^{n_1 - l_1} \times \mathbb{Z}_+^{l_1}, \tag{1d}$$

$$y^\omega \in \mathbb{R}_+^{n_2 - l_2} \times \mathbb{Z}_+^{l_2}, \qquad \forall\, \omega \in \Omega, \tag{1e}$$

where $A \in \mathbb{Q}^{m_1 \times n_1}$, $b \in \mathbb{Q}^{m_1}$, $c \in \mathbb{Q}^{n_1}$, $q^\omega \in \mathbb{Q}^{n_2}$, $T^\omega \in \mathbb{Q}^{m_2 \times n_1}$, $W^\omega \in \mathbb{Q}^{m_2 \times n_2}$, and $h^\omega \in \mathbb{Q}^{m_2}$ for $\forall \omega \in \Omega$. Let $X := \left\{x \in \mathbb{R}_+^{n_1 - l_1} \times \mathbb{Z}_+^{l_1} \mid Ax \geq b\right\}$ be the set of feasible first-stage solutions and $Y(x, \omega) := \left\{y^\omega \in \mathbb{R}_+^{n_2 - l_2} \times \mathbb{Z}_+^{l_2} \mid W^\omega y^\omega \geq h^\omega - T^\omega x\right\}$ be the set of feasible second-stage solutions. Let $S \subset \mathbb{R}^{n_1 + n_2 |\Omega|}$ denote the feasible space of (1) and P denote its convex hull, i.e., $P = \text{conv}(S)$. We assume that P is full-dimensional, as is common in the literature (see [28, 44]).

For any given $\omega \in \Omega$, the feasible space of the single-scenario problem is defined as $S^\omega := \{(x, y^\omega) \mid x \in X,\ y^\omega \in Y(x, \omega)\}$. The convex hull of the feasible space of a single-scenario problem is denoted as $P^\omega := \text{conv}(S^\omega)$ for $\omega \in \Omega$. As P is assumed to be full-dimensional, P^ω is full-dimensional for $\omega \in \Omega$.

In the remainder of the paper, we consider valid inequalities for P^ω of the form:

$$\alpha x + \beta y^\omega \geq \tau, \tag{F}$$

where $\alpha \in \mathbb{R}^{n_1}$, $\beta \in \mathbb{R}^{n_2}$, and $\tau \in \mathbb{R}$, and their extensive form extensions:

$$\alpha x + \beta y^\omega + \sum_{k \in \Omega \setminus \{\omega\}} \mathbf{0}_{n_2} y^k \geq \tau.$$

For readability, we omit the zero coefficients for other scenarios and denote $\alpha x + \beta y^\omega \geq \tau$ as a valid inequality for the extensive form.

Definition 1 ([20])

1. The SMIP (1) has *complete recourse* if, $\forall x \in \mathbb{R}^{n_1}$ the set $Y(x, \omega) \neq \varnothing$, $\forall \omega \in \Omega$.
2. The SMIP (1) has *relatively complete recourse* if, $\forall x \in X$ the set $Y(x, \omega) \neq \varnothing$, $\forall \omega \in \Omega$.
3. The SMIP (1) has *totally unimodular (TU)* recourse if it has relatively complete recourse, the stochasticity is constrained to the vectors h^ω, and W^ω is TU, $\forall \omega \in \Omega$.
4. The SMIP (1) has *simple recourse* if:
   (a) The recourse matrix can be partitioned as $W^\omega = [I_{n_2 \times n_2},\ -I_{n_2 \times n_2}]$ and accordingly $q^\omega = [(q^\omega)^+,\ (q^\omega)^-]$, $\forall \omega \in \Omega$.

(b) The objective vectors $q^\omega \geq \mathbf{0}_{n_2}$, $\forall \omega \in \Omega$.

Definition 2 The SMIP (1) has *slack recourse* if the recourse matrix W^ω can be partitioned as $W^\omega = [\tilde{W}^\omega,\ I_{n_2 \times n_2}]$, $\forall \omega \in \Omega$.

The relationship among the different recourse structures is summarized in Figure 1.

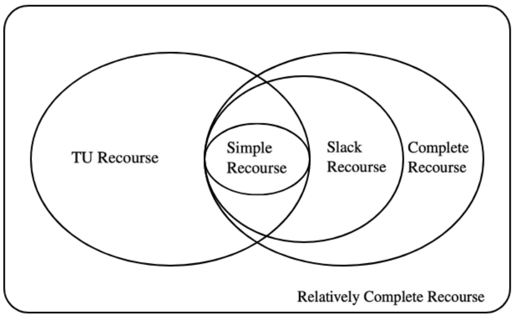

Fig. 1 The relationship among different recourse structures.

Proposition 1 *If SMIP (1) has relatively complete recourse, then* $P^\omega = \text{proj}_{(x,y^\omega)}(P)$, $\forall \omega \in \Omega$.

Remark 1 Proposition 1 implies that if SMIP (1) has a recourse structure as given in Definitions 1-2, then the projection to the single-scenario space, $\text{proj}_{(x,y^\omega)}(P)$, coincides with the single-scenario polyhedron, P^ω, for all $\omega \in \Omega$.

Definition 3 For any $\omega \in \Omega$, a valid inequality (F) that is facet-defining for P^ω is a *preserved single-scenario facet* if it is also facet-defining for P.

Definition 4 A polyhedron P is *single-scenario facet preserving (SSF-preserving)* if for every scenario $\omega \in \Omega$, every single-scenario facet-defining inequality (F) for P^ω, the inequality is preserved for P.

Let $Q = \{(z, v) \in \mathbb{R}^p \times \mathbb{R}^q | \Lambda z + \Upsilon v \leq \gamma\}$ be non-empty, with $r = \text{rank}\,(\Lambda^=, \Upsilon^=)$, $r^* = \text{rank}\,(\Lambda^=)$, and $\bar{r} = \text{rank}\,(\Upsilon^=)$ where $(\cdot^=)$ denotes the equality subsystem. Lemmas 2-3 characterize how the dimension of a polyhedron changes under projection.

Lemma 1 $\dim(Q) = \dim(\textit{affine-hull}(Q)) = p + q - r$.

Lemma 2 ([2]) *If* $\dim(Q) = p + q$, *then* $\dim(\text{proj}_v(Q)) = q$.

Lemma 3 ([2]) $\dim(\text{proj}_v(Q)) = \dim(Q) - p + r^*$.

Let $G = \{(z, v) \in Q | az + \zeta v = d\}$ be a facet of Q. Let $r^*_G = \text{rank}\left(\begin{bmatrix} a \\ \Lambda \end{bmatrix}^=\right)$, $\bar{r}_G = \text{rank}\left(\begin{bmatrix} \zeta \\ \Upsilon \end{bmatrix}^=\right)$, and $r_G = \text{rank}\left(\begin{bmatrix} a \\ \Lambda \end{bmatrix}^=, \begin{bmatrix} \zeta \\ \Upsilon \end{bmatrix}^=\right)$. Lemma 4 characterizes how the dimension of a facet changes under projection.

Lemma 4 ([2]) *$dim(\text{proj}_v(G)) = dim(\text{proj}_v(Q)) - 1 + (r^*_G - r^*)$.*

Lemma 5 ([2]) *Let G be a facet of Q. Then $\text{proj}_z(G)$ is a facet of $\text{proj}_z(Q)$ if and only if $\bar{r}_G = \bar{r}$.*

Lemma 6 establishes a relationship between the faces of projected polyhedron $\text{proj}_v(Q)$ and the faces of Q.

Lemma 6 ([2]) *The projection $\text{proj}_v(G)$ of a face G of Q is a face of $\text{proj}_v(Q)$ if and only if Q has a face G_ζ defined by an inequality of the form $\zeta v \geq d$ (i.e. $\mathbf{0}_p z + \zeta v \geq d$), such that $\text{proj}_v(G_\zeta) = \text{proj}_v(G)$, where $G_\zeta = \{(z, v) \in Q | \zeta v = d\}$.*

Lemma 7 ([2]) *Under the condition of Lemma 6, $\text{proj}_v(G_\zeta) = \text{proj}_v(G) = \{v \in \text{proj}_v(Q) | \zeta v = d\} = \text{proj}_v(Q) \cap \{v | \zeta v = d\}$.*

In the context of stochastic mixed-integer programming, Lemma 8 shows that the facets of the form (F) for polyhedron P remain facet-defining inequalities for the projected polyhedron $\text{proj}_{x,y^\omega}(P)$.

Lemma 8 ([28]) *Let the inequality (F) be facet-defining for P for some $\omega \in \Omega$. Then, (F) is facet-defining for $\text{proj}_{x,y^\omega}(P)$. Moreover, if $\beta = \mathbf{0}_{n_2}$, then (F) is facet-defining for $\text{proj}_{x,y^{\bar{\omega}}}(P)$, $\forall \bar{\omega} \in \Omega$.*

Theorem 1, our main result, gives the necessary and sufficient conditions for a single-scenario valid inequality to be a facet for P.

Theorem 1 *For some $\omega \in \Omega$, let the single-scenario valid inequality (F) be valid for P^ω, and consider the hyperplane $H = \{(x, y^1, \ldots, y^{|\Omega|}) \in \mathbb{R}^{n_1+|\Omega|n_2} \mid \alpha x + \beta y^\omega = \tau\}$, that is, the points where (F) holds with equality. Then, (F) is facet-defining for P if and only if:*

1. *(F) is facet-defining for $\text{proj}_{(x,y^\omega)}(P)$, and*
2. *$dim(\text{proj}_{(y^1,\ldots,y^{\omega-1},y^{\omega+1},\ldots,y^{|\Omega|})}(P \cap H)) = (|\Omega| - 1)n_2$.*

Proof Let the single-scenario valid inequality (F) be facet-defining for P. By Lemma 8, (F) is also facet-defining for $\text{proj}_{(x,y^\omega)}(P)$ which shows that Condition 1 holds. Let $Q = P$, $G = P \cap H$, $z = (x, y^\omega)$, and $v = (y^1, \ldots, y^{\omega-1}, y^{\omega+1}, \ldots, y^{|\Omega|})$ and apply Lemma 4. Since $\dim(P) = n_1 + n_2|\Omega|$, we have $\dim(\text{proj}_v(P)) = n_2(|\Omega| - 1)$ and $r^* = 0$ by Lemmas 2 and 3. To show Condition 2 holds, we show $r^*_{P\cap H} = 1$ and use Lemma 4. To this end, we first show that $\bar{r}_{P\cap H} = 0$. As (F) is a facet of P and Condition 1 holds, $\bar{r}_{P\cap H} = \bar{r}$ by Lemma 5. Moreover, $\bar{r} = 0$ by Lemmas 2 and 3 because $\dim(\text{proj}_z(P)) = n_1 + n_2$, and thus $\bar{r}_{P\cap H} = 0$.

As $\bar{r}_{P\cap H} = \text{rank}\left(\begin{bmatrix}\zeta\\ \Upsilon\end{bmatrix}^{=}\right) = 0$, the equality subsystem $\begin{bmatrix}\zeta\\ \Upsilon\end{bmatrix}^{=}$ is a zero matrix, therefore, $r_{P\cap H} = \text{rank}\left(\begin{bmatrix}a\\ A\end{bmatrix}^{=}, \mathbf{0}\right) = r^*_{P\cap H}$. To show $r^*_{P\cap H} = 1$, it is sufficient to show $r_{P\cap H} = 1$. Because (F) is facet-defining for P, by Lemma 1, we have

$$\dim(P \cap H) = n_1 + n_2|\Omega| - 1 = n_1 + n_2|\Omega| - r_{P\cap H},$$

which implies $r_{P\cap H} = 1$. Given $r^*_{P\cap H} = 1$ and $r^* = 0$, by Lemma 4:

$$\begin{aligned}\dim(\text{proj}_v(P \cap H)) &= \dim(\text{proj}_v(P)) - 1 + r^*_{P\cap H} - r^*\\ &= n_2(|\Omega| - 1) - 1 + 1 - 0 = n_2(|\Omega| - 1).\end{aligned}$$

Hence, $\dim(\text{proj}_v(P \cap H)) = (|\Omega| - 1)n_2$.

Now, suppose Conditions 1 and 2 hold. By Lemma 6, (F) is a face of P and $\text{proj}_{(x,y^\omega)}(P \cap H) = \text{proj}_{(x,y^\omega)}(P) \cap \{(x, y^\omega) \mid \alpha x + \beta y^\omega = \tau\}$ by Lemma 7. Then, $\dim(\text{proj}_{(x,y^\omega)}(P\cap H)) = \dim(\text{proj}_{(x,y^\omega)}(P)\cap\{(x, y^\omega) \mid \alpha x + \beta y^\omega = \tau\}) = n_1 + n_2 - 1$. Let $Q = P$, $G = P \cap H$, $z = (x, y^\omega)$, and $v = (y^1, \ldots, y^{\omega-1}, y^{\omega+1}, \ldots, y^{|\Omega|})$ and apply Lemma 4. Because the projection of $P \cap H$ onto v is full-dimensional, there cannot be any implied equality in $\text{proj}_v(P\cap H)$. Therefore, $\bar{r}_{P\cap H} = 0$ as $\zeta = \mathbf{0}_{n_2(|\Omega|-1)}$. When we project $P\cap H$ onto z, by Lemma 3, we have:

$$\begin{aligned}\dim(P \cap H) &= \dim(\text{proj}_z(P \cap H)) + n_2(|\Omega| - 1) - \bar{r}_{P\cap H}\\ &= n_1 + n_2 - 1 + n_2(|\Omega| - 1) - 0 = n_1 + n_2(|\Omega|) - 1.\end{aligned}$$

Therefore, (F) is also facet-defining for P. □

3 Implications of Theorem 1

Theorem 1 provides a roadmap to obtain sufficient conditions for single-scenario facets to remain facet-defining for the extensive form. Single-scenario facets for SMIPs with relatively complete, complete, TU, simple or slack recourse satisfy the first condition intuitively by Proposition 1. The second condition ensures that single-scenario facet-defining inequalities are preserved as facets for the extensive form.

Given that the single-scenario valid inequality (F) is facet-defining for P^ω, we only need to verify the second condition of Theorem 1 in order to prove that (F) is facet-defining for P. However, verifying this may be difficult. Therefore, we have the following results showing that single-scenario facets (F) are preserved under more easily established conditions.

Corollary 1 *Fix $\bar{\omega}$ and suppose that $\forall\omega \in \Omega \setminus \{\bar{\omega}\}$ there exists a point $\hat{y}^\omega \in \mathbb{R}^{n_2-l_2}_+ \times \mathbb{Z}^{l_2}_+$ such that $W^\omega\hat{y}^\omega > \mathbf{0}_{m_2}$. Then, P preserves single-scenario facets* (F) *for $P^{\bar{\omega}}$.*

Corollary 2 *Suppose that $\forall\omega \in \Omega$ there exists a point $\hat{y}^\omega \in \mathbb{R}^{n_2-l_2}_+ \times \mathbb{Z}^{l_2}_+$ such that $W^\omega\hat{y}^\omega > \mathbf{0}_{m_2}$. Then, P is SSF-preserving.*

Although the condition in Corollary 3 implies that the SMIP (1) has complete recourse, the converse does not hold.

Corollary 3 *If SMIP* (1) *has simple or slack recourse, then P is SSF-preserving.*

While preserving single-scenario facets (F) improves the representation of the feasible space of the extensive form, single-scenario facets may not be sufficient to describe P, as shown in Example 1 in the Online Resource.

3.1 Sufficient Conditions for Complete Recourse

In the case of complete recourse, we introduce the following condition to ensure that Condition (2) holds:

$$\text{There exists } \hat{x}(\omega) \in \mathbb{R}^{n_1} \text{ such that } T^\omega \hat{x}(\omega) < h^\omega. \tag{C1(ω)}$$

When SMIP (1) has complete recourse, this condition can be interpreted as there exists a first-stage decision ($\hat{x}(\omega)$) that results in a violation of *every* constraint if no recourse action ($y^\omega = \mathbf{0}_{n_2}$) is taken.

Corollary 4 *Suppose SMIP* (1) *has complete recourse, and* (C1(ω)) *holds* $\forall \omega \in \Omega \setminus \{\bar{\omega}\}$. *Then, P preserves single-scenario facets* (F) *for* $P^{\bar{\omega}}$.

Corollary 5 *If SMIP* (1) *has complete recourse, and* (C1(ω)) *holds* $\forall \omega \in \Omega$, *then P is SSF-preserving.*

Note that satisfying Condition (C1(ω)) $\forall \omega \in \Omega$ is not necessary, as shown by Example 2 in the Online Resource.

Corollary 6 *If SMIP* (1) *has complete recourse, and either of the following conditions holds:*

1. T^ω *has* m_2 *linearly independent columns* $\forall \omega \in \Omega$,
2. $h^\omega > \mathbf{0}_{m_2}$, $\forall \omega \in \Omega$,

then, P is SSF-preserving.

3.2 Sufficient Conditions for Relatively Complete Recourse

When the SMIP has relatively complete recourse, consider the following condition:

$$h^\omega < T^\omega x, \quad \forall x \in X \setminus \{\mathbf{0}_{n_1}\}. \tag{C2(ω)}$$

Condition (C2(ω)) implies that taking no recourse actions, i.e., $y^\omega = \mathbf{0}_{n_2}$, is always feasible.

Corollary 7 *Let SMIP* (1) *have relatively complete continuous recourse with* $n_1 > 1$ *and* $l_2 = 0$. *Suppose that* (C2(ω)) *holds* $\forall \omega \in \Omega \setminus \{\bar{\omega}\}$. *Then, P preserves single-scenario facets* (F) *for* $P^{\bar{\omega}}$.

We can relax Condition (C2(ω)) if SMIP (1) has variable upper bound constraints of the form:

$$U^\omega x - I y^\omega \geq \mathbf{0}_{m_2} \text{ or } K^\omega x - D^\omega y^\omega \geq \mathbf{0}_{m_2},\ \forall\, \omega \in \Omega, \tag{2}$$

where $U^\omega, K^\omega \in \mathbb{Q}_+^{m_2+n_1}$ are diagonal matrices with positive diagonal elements, and $D^\omega \in \mathbb{Q}_{++}^{m_2+n_2}$, $\forall \omega \in \Omega$. The relaxed condition (C3(ω)) is given as:

$$h^\omega < T^\omega x,\ \forall x \in X \cap (\mathbb{R}_{++}^{n_1}). \tag{C3(ω)}$$

Corollary 8 *Let SMIP* (1) *have relatively complete continuous recourse with* $n_1 > 1$ *and* $l_2 = 0$. *Suppose Condition* (C3(ω)) *holds* $\forall \omega \in \Omega \setminus \{\bar{\omega}\}$. *Then,* P *preserves single-scenario facets* (F) *for* $P^{\bar{\omega}}$.

Examples 3 and 4 in the Online Resource illustrate when the recourse structures fail to be necessary and/or sufficient for P to be SSF-preserving.

Remark 2 Example 3 shows that relatively complete recourse is not necessary for P to be SSF-preserving. This implies that having neither simple, slack, complete, or TU recourse are necessary for P to be SSF-preserving.

Remark 3 Example 4 in the Online Resource demonstrates that TU recourse and relatively complete recourse are not sufficient for P to be SSF-preserving.

The proofs of Corollaries 7-8 imply that when Conditions (C2(ω)) and (C3(ω)) hold $\forall \omega \in \Omega$, relatively complete recourse is sufficient for preserving the single-scenario facets (F).

Corollary 9 *Let SMIP* (1) *have relatively complete recourse with* $n_1 > 1$ *and* $y^\omega \in \mathbb{R}_+^{n_2}$, $\forall \omega \in \Omega$. *Suppose that* $\forall \omega \in \Omega$, *Condition* (C2($\omega$)) *holds. Then,* P *is SSF-preserving.*

Corollary 10 *Let SMIP* (1) *have relatively complete recourse with variable upper bound constraints* (2) *and* $y^\omega \in \mathbb{R}_+^{n_2}$, $\forall \omega \in \Omega$. *Suppose that* (C3($\omega$)) *holds* $\forall \omega \in \Omega$. *Then,* P *is SSF-preserving.*

Remark 4 Example 5 in the Online Resource shows that stochastic facility location problems [23, 26] are SSF-preserving.

4 Conclusion

We introduce facet preservation as a new characterization for SMIPs and single-scenario facet-defining inequalities (F) to provide a better understanding of the facial structure of the convex hull of the extensive form and a better assessment of the strength of such inequalities. We also provide a general result showing the relation between the facial structures of the projected single-scenario polyhedron and the convex hull of the extensive form. Then, we present sufficient conditions for SSF preservation. Future directions include extending these results to multi-stage SMIPs.

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5 Appendix

Proposition 1 *If SMIP (1) has relatively complete recourse, then* $P^\omega = \text{proj}_{(x,y^\omega)}(P)$*,* $\forall \omega \in \Omega$*.*

Proof Fix $\omega \in \Omega$.

"$\supseteq$:" By definition, $\text{proj}_{(x,y^\omega)}(P) = \{(x, y^\omega) | \exists (y^1, \ldots, y^{\omega-1}, y^{\omega+1}, \ldots, y^\Omega)$ s.t. $(x, y^1, \ldots, y^\Omega) \in P\}$. Let $(\hat{x}, \hat{y}^\omega) \in \text{proj}_{(x,y^\omega)}(P)$. Then, there exists $(\hat{y}^1, \ldots, \hat{y}^{\omega-1}, \hat{y}^{\omega+1}, \ldots, \hat{y}^{|\Omega|})$ such that $(\hat{x}, \hat{y}^1, \ldots, \hat{y}^{|\Omega|}) \in P$. If $(\hat{x}, \hat{y}^1, \ldots, \hat{y}^{|\Omega|}) \in S$, we have $(\hat{x}, \hat{y}^\omega) \in P^\omega$. If $(\hat{x}, \hat{y}^1, \ldots, \hat{y}^{|\Omega|}) \notin S$, then the point $(\hat{x}, \hat{y}^1, \ldots, \hat{y}^{|\Omega|})$ can be written as a convex combination of points in S, i.e., $(\hat{x}, \hat{y}^1, \ldots, \hat{y}^{|\Omega|}) = \sum_{i=1}^k \mu_i(\bar{x}_i, \bar{y}_i^1, \ldots, \bar{y}_i^{\omega-1}, \bar{y}_i^\omega, \bar{y}_i^{\omega+1}, \ldots, \bar{y}_i^{|\Omega|})$ where $(\bar{x}_i, \bar{y}_i^1, \ldots, \bar{y}_i^{\omega-1}, \bar{y}_i^\omega, \bar{y}_i^{\omega+1}, \ldots, \bar{y}_i^{|\Omega|}) \in S$ and $\mu_i \in [0, 1]$ for $i = 1, \ldots, k$ with $\sum_{i=1}^k \mu_i = 1$. Then, $(\bar{x}_i, \bar{y}_i^\omega) \in S^\omega$ for $i = 1, \ldots, k$. The convex combination of these points $\sum_{i=1}^k \mu_i(\bar{x}_i, \bar{y}_i^\omega) = (\hat{x}, \hat{y}^\omega)$, therefore, $(\hat{x}, \hat{y}^\omega) \in P^\omega$.

"$\subseteq$:" Let $(\hat{x}, \hat{y}^\omega) \in P^\omega$. If $(\hat{x}, y^{\hat{\omega}}) \in S^\omega$, then there exists $y^{\hat{\omega}} \in Y(\hat{x}, \hat{\omega})$ for all $\hat{\omega} \in \Omega \setminus \{\omega\}$ because SMIP (1) has relatively complete recourse. Therefore, $(\hat{x}, \hat{y}^\omega) \in \text{proj}_{(x,y^\omega)}(P)$. If $(\hat{x}, y^{\hat{\omega}}) \in S^\omega$, then $(\hat{x}, \hat{y}^\omega)$ can be written as a convex combination of points in S^ω, i.e., $(\hat{x}, \hat{y}^\omega) = \sum_{i=1}^k \mu_i(\bar{x}_i, \bar{y}_i^\omega)$, where $(\bar{x}_i, \bar{y}_i^\omega) \in S^\omega$ and $\mu_i \in [0, 1]$ for $i = 1, \ldots, k$ with $\sum_{i=1}^k \mu_i = 1$. Then, for every $\bar{x}_i$, there exists $\bar{y}_i^{\hat{\omega}} \in Y(\hat{x}, \hat{\omega})$ for all $\hat{\omega} \in \Omega \setminus \{\omega\}$ for $i = 1, \ldots, k$ because SMIP (1) has relatively complete recourse. Then, the convex combination of these points $(\hat{x}, \hat{y}^1, \ldots, \hat{y}^{\omega-1}, \hat{y}^\omega, \hat{y}^{\omega+1}, \ldots, \hat{y}^\Omega) = \sum_{i=1}^k \mu_i(\bar{x}_i, \bar{y}_i^1, \ldots, \bar{y}_i^{\omega-1}, \bar{y}_i^\omega, \bar{y}_i^{\omega+1}, \ldots, \bar{y}_i^{|\Omega|})$ lies in P, therefore, $(\hat{x}, \hat{y}^\omega) \in \text{proj}_{(x,y^\omega)}(P)$. □

Lemma 9 ([28]) *Suppose* $\{(f^{(i)}, g^{(i)}) \in \mathbb{R}^{M_1+M_2} |\ i = 1, \ldots, N\}$ *is a set of* N *points such that* $f^{(1)}, \ldots, f^{(N)}$ *are affinely independent. Suppose* $h^{(1)}, \ldots, h^{(t)}$ *are* t *linearly independent points in* $\mathbb{R}^{M_2}$*. Then, for any* $s \in \{1, \ldots, N\}$*, the* $N + t$ *points*

$$\begin{pmatrix} f^{(1)} \\ g^{(1)} \end{pmatrix}, \ldots, \begin{pmatrix} f^{(N)} \\ g^{(N)} \end{pmatrix}, \begin{pmatrix} f^{(s)} \\ g^{(s)} + h^{(1)} \end{pmatrix}, \ldots, \begin{pmatrix} f^{(s)} \\ g^{(s)} + h^{(t)} \end{pmatrix}$$

are affinely independent.

Corollary 2 *Fix* $\bar{\omega}$ *and suppose that* $\forall \omega \in \Omega \setminus \{\bar{\omega}\}$ *there exists a point* $\hat{y}^\omega \in \mathbb{R}_+^{n_2-l_2} \times \mathbb{Z}_+^{l_2}$ *such that* $W^\omega \hat{y}^\omega > \mathbf{0}_{m_2}$*. Then,* P *preserves single-scenario facets for* $P^{\bar{\omega}}$*.*

Proof Let $\bar{\omega} = 1$ without loss of generality. First, we need to show $P^1 = \text{proj}_{(x,y^1)}(P)$. Note that $\text{proj}_{(x,y^1)}(P) \subseteq P^1$ by definition. For every $(x, y^1) \in P^1$, there exists $\phi^\omega \in \mathbb{Z}_+$ for every scenario $\omega \neq 1$ such that $y^\omega \in \mathbb{Z}^{n_2|\Omega|}$, $T^\omega x + \phi^\omega W^\omega y^\omega \geq h^\omega$, i.e., $(x, y^1, \phi^2 \hat{y}^2, \ldots, \phi^{|\Omega|} \hat{y}^{|\Omega|}) \in P$, because for all $\omega \in \Omega \setminus \{1\}$ there exists $\hat{y}^\omega \in \mathbb{R}_+^{n_2-l_2} \times \mathbb{Z}_+^{l_2}$ such that $W^\omega \hat{y}^\omega > \mathbf{0}_{m_2}$. Therefore, $(x, y^1) \in \text{proj}_{(x,y^1)}(P)$ showing that $P^1 = \text{proj}_{(x,y^1)}(P)$.

Suppose that the single-scenario valid inequality (F) is facet-defining for P^1. Then, there exist $n_1 + n_2$ affinely independent points $f^{(i)} = (x^i, y^{1,i})_{i=1}^{n_1+n_2}$

in S^1 that satisfy (F) with equality. As for all $\omega \in \Omega \setminus \{1\}$ there exists a point $\hat{y}^\omega \in \mathbb{R}_+^{n_2-l_2} \times \mathbb{Z}_+^{l_2}$ satisfying $W^\omega \hat{y}^\omega > \mathbf{0}_{m_2}$, for every point $i = 1, \ldots, n_1 + n_2$ and every scenario $\omega \neq 1$, there exists $\phi^{\omega,i} \in \mathbb{Z}_+$ such that

$$T^\omega x^i + \phi^{\omega,i} W^\omega \hat{y}^\omega > h^\omega,$$

and

$$T^\omega x^i + \phi^{\omega,i} W^\omega \hat{y}^\omega > h^\omega - (W^\omega)_j,$$

for all $j = 1, \ldots, n_2$, where $(W^\omega)_j$ is the j^{th} column of W^ω. Define the least common multiple $\hat{\phi}^\omega := \text{lcm}\{\phi^{\omega,1}, \ldots, \phi^{\omega,n_1+n_2}\}$. Then,

$$z^i := (x^i, y^{1,i}, \hat{\phi}^2 \hat{y}^2, \ldots, \hat{\phi}^{|\Omega|} \hat{y}^{|\Omega|}),$$

lies in S and satisfies (F) with equality for $i = 1, \ldots, n_1 + n_2$. By construction, if we take z^1 and add 1 to the j^{th} entry of the vector $\hat{\phi}^\omega \hat{y}^\omega$ for any scenario $\omega \neq 1$, and any $j = 1, \ldots, n_2$, then the resulting point still lies in S and satisfies (F) with equality. Now, we show that the resulting set of points is affinely independent.

Let $g = (\hat{\phi}^2 \hat{y}^2, \ldots, \hat{\phi}^{|\Omega|} \hat{y}^{|\Omega|})$ and $h^{(k)} = e_k$ for $k = 1, \ldots, n_2(|\Omega| - 1)$ where e_k is the k^{th} unit vector of dimension $n_2(|\Omega| - 1)$. Then, $n_1 + n_2|\Omega|$ points:

$$\begin{pmatrix} f^{(1)} \\ g \end{pmatrix}, \begin{pmatrix} f^{(2)} \\ g \end{pmatrix}, \ldots, \begin{pmatrix} f^{(n_1+n_2)} \\ g \end{pmatrix},$$
$$\begin{pmatrix} f^{(1)} \\ g + h^{(1)} \end{pmatrix}, \begin{pmatrix} f^{(1)} \\ g + h^{(2)} \end{pmatrix}, \ldots, \begin{pmatrix} f^{(1)} \\ g + h^{n_2(|\Omega|-1)} \end{pmatrix}$$

are affinely independent by Lemma 9. □

Corollary 3 *Suppose that* $\forall \omega \in \Omega$ *there exists a point* $\hat{y}^\omega \in \mathbb{R}_+^{n_2-l_2} \times \mathbb{Z}_+^{l_2}$ *such that* $W^\omega \hat{y}^\omega > \mathbf{0}_{m_2}$. *Then,* P *is SSF-preserving.*

Proof This condition implies that the SMIP (1) has complete recourse, therefore, $P^\omega = \text{proj}_{(x,y^\omega)}(P)$ for all $\omega \in \Omega$. The rest of the proof is very similar to the proof of Corollary 1. □

Example 1 *Consider an instance of (1) where* $n_1 = 1,\ n_2 = 2,\ \Omega = \{1, 2\}$, $A = [-1],\ b = [-10],\ T^1 = T^2 = [2]$ *and:* $W^1 = W^2 = \begin{bmatrix} 1 & -1 \end{bmatrix},\ h^1 = [2],\ h^2 = [5]$, *where all variables are integral. Note that this problem has simple recourse. The polytope* P *is full-dimensional. Moreover, it is SSF-preserving. However, the valid inequality* $-25x + y_1^1 - y_2^1 - 15y_1^2 = -65$ *is facet-defining for* P. *Therefore, single-scenario facets are not sufficient to describe* P.

Corollary 4 *Suppose SMIP (1) has complete recourse, and* $(C1(\omega))$ *holds* $\forall \omega \in \Omega \setminus \{\bar{\omega}\}$. *Then,* P *preserves single-scenario facets for* $P^{\bar{\omega}}$.

Proof Fix $\bar{\omega} = 1$ without loss of generality. Recall Condition $(C1(\omega))$ that $\exists \hat{x}(\omega) \in \mathbb{R}^{n_1}$ such that $T^\omega \hat{x}(\omega) < h^\omega$. By Corollary 1, it is sufficient to show that for all $\omega \in \Omega \setminus \{1\}$ there exists a point $\hat{y}^\omega \in \mathbb{R}_+^{n_2-l_2} \times \mathbb{Z}_+^{l_2}$ such that $W^\omega \hat{y}^\omega > \mathbf{0}_{m_2}$. Because $(C1(\omega))$ holds, for all $\omega \in \Omega \setminus \{1\}$, there exists $\hat{x}(\omega) \in \mathbb{R}^{n_1}$ such that $T^\omega \hat{x}(\omega) < h^\omega$. Then, as the problem has complete recourse, the set $Y(\hat{x}(\omega), \omega) \neq \emptyset$ for all $\omega \in \Omega$. Therefore, for all $\omega \in \Omega \setminus \{1\}$, there exists $\hat{y}^\omega$ such that

$$T^\omega \hat{x}(\omega) + W^\omega \hat{y}^\omega \geq h^\omega.$$

Since we have $T^\omega \hat{x}(\omega) < h^\omega$ for all $\omega \in \Omega \setminus \{1\}$, $\hat{y}^\omega$ satisfies $W^\omega \hat{y}^\omega > \mathbf{0}_{m_2}$ for all $\omega \in \Omega \setminus \{1\}$. □

Example 2 *Consider an instance of (1) with* $n_1 = 2$, $n_2 = 2$, $\Omega = \{1, 2\}$, $A = -I_{2\times 2}$, $b = -\mathbf{1}_2$, $T^1 = T^2 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ *and:* $W^1 = W^2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $h^1 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$, $h^2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. *All variables are integral and the first-stage variables are restricted to be binary. Note that this problem has complete recourse. The polytope* P *is full-dimensional. Moreover, it is SSF-preserving by Corollary 3. However, Condition* $(C1(\omega))$ *does not hold for either scenario.*

Corollary 6 *If SMIP (1) has complete recourse, and either of the following conditions holds:*

1. T^ω *has* m_2 *linearly independent columns* $\forall \omega \in \Omega$,
2. $h^\omega > \mathbf{0}_{m_2}$, $\forall \omega \in \Omega$,

then, P *is SSF-preserving.*

Proof Either condition implies that Condition $(C1(\omega))$, i.e., $\exists \hat{x}(\omega) \in \mathbb{R}^{n_1}$ such that $T^\omega \hat{x}(\omega) < h^\omega$, holds for all $\omega \in \Omega$. Since the problem has complete recourse both conditions of Theorem 1 are satisfied for every $\omega \in \Omega$. Hence, P is SSF-preserving. □

Corollary 7 *Let SMIP (1) have relatively complete continuous recourse with* $n_1 > 1$ *and* $l_2 = 0$. *Suppose that* $(C2(\omega))$ *holds* $\forall \omega \in \Omega \setminus \{\bar{\omega}\}$. *Then,* P *preserves single-scenario facets for* $P^{\bar{\omega}}$.

Proof Let $\bar{\omega} = 1$ without loss of generality. Recall Condition $(C2(\omega))$ that $h^\omega - T^\omega x < \mathbf{0}_{m_2}, \quad \forall x \in X$ with $x \neq \mathbf{0}_{n_1}$. Suppose that the single-scenario valid inequality (F) is facet-defining for P^1. Then, there exist $n_1 + n_2$ affinely independent points $f^{(i)} = (x^i, y^{1,i}),\ i = 1, \ldots, n_1 + n_2$, in S^1 that satisfy (F) with equality. As (1) has relatively complete recourse, for every x^i, $i = 1, \ldots, n_1 + n_2$, there exists $\hat{y}^{\omega,i} \in \mathbb{R}_+^{n_2}$ such that $T^\omega x^i + W^\omega \hat{y}^{\omega,i} \geq h^\omega$ for all $\omega \in \Omega \setminus \{1\}$. Also, the assumption $n_1 > 1$ implies that there exists $i_0 \in \{1, \ldots, n_1 + n_2\}$ such that $x^{i_0} \neq \mathbf{0}_{n_1}$, since $x^i = \mathbf{0}_{n_1}$ for $i = 1, \ldots, n_1 + n_2$ contradicts with points $(f^{(i)})_{i=1}^{n_1+n_2}$ being affinely independent. Because $(C2(\omega))$ holds $\forall \omega \in \Omega \setminus \{\bar{\omega}\}$, we have $h^\omega - T^\omega x^{i_0} < \mathbf{0}_{m_2}$ for all $\omega \in \Omega \setminus \{1\}$. If we set $\hat{y}^{\omega,i_0} = \mathbf{0}_{n_2}$ for $\omega \in \Omega \setminus \{1\}$, we have $T^\omega x^{i_0} + W^\omega \hat{y}^{\omega,i_0} > h^\omega$ for $\omega \in \Omega \setminus \{1\}$

as $h^\omega - T^\omega x^{i_0} < \mathbf{0}_{m_2}$ for $\omega \in \Omega \setminus \{1\}$. Then, there exist scalars $\phi^{\omega,j} > 0$ such that $T^\omega x^{i_0} + W^\omega(\hat{y}^{\omega,i_0} + \phi^{\omega,j} e_j) \geq h^\omega$, for all $j = 1, \dots, n_2$ and $\omega \in \Omega \setminus \{1\}$, where e_j is the j^{th} unit vector. Then, the points
$z^i := (x^i, y^{1,i}, \hat{y}^{2,i}, \dots, \hat{y}^{|\Omega|,i})$,
for $i = 1, \dots, n_1 + n_2$ lie in P and satisfy (F) with equality as $(x^i, y^{1,i}) \in S^1$ for $i = 1, \dots, n_1 + n_2$ and $y^{\omega,i} \in \mathbb{R}^{n_2}$. By construction, if we add $\phi^{\omega,j}$ to the j^{th} entry of the vector $\hat{y}^{\omega,i_0}$ for any scenario $\omega \neq 1$, and any $j = 1, \dots, n_2$, then the resulting vectors

$$\begin{pmatrix} \hat{y}_1^{\omega,i_0} \\ \vdots \\ \hat{y}_{n_2}^{\omega,i_0} \end{pmatrix} + \phi^{\omega,j} e_j, \text{ for } \omega \neq 1 \text{ and } j = 1, \dots, n_2$$

remain feasible for given (x^{i_0}, y^{1,i_0}) and satisfy (F) with equality. Now, we show that the resulting set of points is affinely independent. For this, we define the vector $\Phi \in \mathbb{R}^{n_2(|\Omega|-1)}$ as follows:
$\Phi = (\phi^{2,1}, ..., \phi^{2,n_2}, \phi^{3,1}, ..., \phi^{3,n_2}, \dots, \phi^{|\Omega|,1}, ..., \phi^{|\Omega|,n_2})$.

Let $g^{(i)} = (\hat{y}^{2,i}, \dots, \hat{y}^{|\Omega|,i})$ for $i = 1, \dots, n_1 + n_2$ and $h^{(k)} = \Phi_k e_k$ for $k = 1, \dots, n_2(|\Omega| - 1)$. Then, $n_1 + n_2|\Omega|$ points:

$$\begin{pmatrix} f^{(1)} \\ g^{(1)} \end{pmatrix}, ..., \begin{pmatrix} f^{(i_0)} \\ g^{(i_0)} \end{pmatrix}, ..., \begin{pmatrix} f^{(n_1+n_2)} \\ g^{(n_1+n_2)} \end{pmatrix},$$
$$\begin{pmatrix} f^{(i_0)} \\ g^{(i_0)} + h^{(1)} \end{pmatrix}, \begin{pmatrix} f^{(i_0)} \\ g^{(i_0)} + h^{(2)} \end{pmatrix}, ..., \begin{pmatrix} f^{(i_0)} \\ g^{(i_0)} + h^{N_2} \end{pmatrix}$$

are affinely independent by Lemma 9. □

Corollary 8 *Let SMIP (1) have relatively complete continuous recourse with $n_1 > 1$ and $l_2 = 0$. Suppose Condition $(C3(\omega))$ holds $\forall \omega \in \Omega \setminus \{\bar{\omega}\}$. Then, P preserves single-scenario facets for $P^{\bar{\omega}}$.*

Proof Let $\bar{\omega} = 1$ without loss of generality. Recall Condition $(C3(\omega))$ that $h^\omega - T^\omega x < \mathbf{0}_{m_2}$, $\forall x \in X \cap (\mathbb{R}^{n_1}_{++})$. Suppose that the single-scenario valid inequality (F) is facet-defining for P^1. Then, there exist $n_1 + n_2$ affinely independent points $f^{(i)} = (x^i, y^{1,i}) \in S^1, i = 1, \dots, n_1 + n_2$ that satisfy (F) with equality. As (1) has relatively complete recourse, for every x^i, $i = 2, \dots, n_1 + n_2$, there exists $\hat{y}^{\omega,i} \in \mathbb{R}^{n_1}_{+}$ such that $T^\omega x^i + W^\omega \hat{y}^{\omega,i} \geq h^\omega$ for all $\omega \in \Omega \setminus \{1\}$. Let $g^{(i)} = (\hat{y}^{2,i}, \dots, \hat{y}^{|\Omega|,i})$ for $i = 2, \dots, n_1 + n_2$ and $(\bar{f}, \bar{g})$ be the average of these points:

$$\begin{pmatrix} \bar{f} \\ \bar{g} \end{pmatrix} = \frac{1}{n_1 + n_2} \sum_{i=1}^{n_1+n_2} \begin{pmatrix} f^{(i)} \\ g^{(i)} \end{pmatrix},$$

which lies in P and satisfies (F) with equality. Then, the points

$$\begin{pmatrix} \bar{x} \\ \bar{y}^1 \end{pmatrix}, \begin{pmatrix} x^2 \\ y^{1,2} \end{pmatrix}, \dots, \begin{pmatrix} x^{n_1+n_2} \\ y^{1,n_1+n_2} \end{pmatrix}$$

are also affinely independent as $\bar{f}$ is an affine combination of $f^{(i)}, i = 1, \ldots, n_1 + n_2$. Note that for each $j = 1, \ldots, n_1$, there exists x^i such that $x^i_j > 0$ as otherwise there exists j such that $x^i_j = y^{1,i}_j = 0$ for all $i = 1, \ldots, n_1 + n_2$ which implies that the points $(x^i, y^{1,i})_{i=1}^{n_1+n_2}$ are not affinely independent. Therefore, we have $\bar{x}_j > 0$ as for each $j = 1, \ldots, n_1$, there exists x^i such that $x^i_j > 0$. Also, we have $h^\omega - T^\omega \bar{x} < \mathbf{0}_{m_2}$ for all $\omega \in \Omega$ since $\bar{x} > \mathbf{0}_{n_1}$. Hence, we can set $\bar{y}^\omega = \mathbf{0}_{n_2}$ for $\omega \in \Omega \setminus \{1\}$ such that the point

$$\bar{z} = (\bar{x}, \bar{y}^1, \mathbf{0}_{n_2}, \ldots, \mathbf{0}_{n_2})$$

and the points

$$z^i = (x^i, y^{1,i}, \hat{y}^{2,i}, \ldots, \hat{y}^{|\Omega|,i}),\ i = 2, \ldots, n_1 + n_2$$

lie in P and satisfy (F) with equality. Moreover, as $T^\omega \bar{x} > h^\omega$ for every scenario $\omega \in \Omega$, there exist scalars $\phi^{\omega,j} > 0$ such that

$$T^\omega \bar{x} + W^\omega (\mathbf{0}_{n_2} + \phi^{\omega,j} e_j) \geq h^\omega,$$

for all $j = 1, \ldots, n_2$ and $\omega \in \Omega \setminus \{1\}$.

Now, we show that the resulting set of points is affinely independent. For this, we define the vector $\Phi \in \mathbb{R}^{n_2(|\Omega|-1)}$ as follows:
$\Phi = (\phi^{2,1}, \ldots, \phi^{2,n_2}, \phi^{3,1}, \ldots, \phi^{3,n_2}, \ldots, \phi^{|\Omega|,1}, \ldots, \phi^{|\Omega|,n_2})$.

We set $\bar{g} = (0, \ldots, 0)$ and $h^{(k)} = \Phi_k e_k$ for $k = 1, \ldots, n_2(|\Omega| - 1)$. Then, $n_1 + n_2|\Omega|$ points:

$$\begin{pmatrix} \bar{f} \\ \bar{g} \end{pmatrix}, \begin{pmatrix} f^{(2)} \\ g^{(2)} \end{pmatrix}, \ldots, \begin{pmatrix} f^{(n_1+n_2)} \\ g^{(n_1+n_2)} \end{pmatrix},$$
$$\begin{pmatrix} \bar{f} \\ \bar{g} + h^{(1)} \end{pmatrix}, \begin{pmatrix} \bar{f} \\ \bar{g} + h^{(2)} \end{pmatrix}, \ldots, \begin{pmatrix} \bar{f} \\ \bar{g} + h^{(n_2(|\Omega|-1))} \end{pmatrix}$$

are affinely independent by Lemma 9 and lie in P as $\bar{z}$ and $z^i,\ i = 2, \ldots, n_1 + n_2$ lie in P and $y^\omega \in \mathbb{R}^{n_2},\ \omega \in \Omega$. □

Example 3 *Consider an instance of (1) with $n_1 = n_2 = 3$, $\Omega = \{1, 2\}$, and the following feasible region:*

$$\begin{aligned}
&y^\omega_i \geq x_i, && \forall\, i \in \{1, 2, 3\},\ \omega \in \Omega, \\
&y^\omega_1 + y^\omega_2 + y^\omega_3 \leq d^\omega, && \forall\, \omega \in \Omega, \\
&x_i \in \{0, 1\}, && \forall\, i \in \{1, 2, 3\}, \\
&y^\omega_i \in \{0, 1\}, && \forall\, i \in \{1, 2, 3\},\ \omega \in \Omega,
\end{aligned}$$

where $d^1 = 2$ and $d^2 = 3$. This model does not have relatively complete recourse, because the vector $\mathbf{1}_3 \in X$, but the set $Y(\mathbf{1}_3, 1)$ is empty. However, P preserves facets from P^ω for $\omega \in \Omega$. Therefore, P is SSF-preserving.

We provide facet-defining inequalities for P, P^1 and P^2 to show that P preserves all single-scenario facets. Facet-defining inequalities for P:

$$
\begin{aligned}
&-x_1 + x_2 - x_3 + y_1^1 + y_3^1 + y_1^2 - y_2^2 + y_3^2 \leq 3,\\
&x_1 - y_1^1 \leq 0,\ x_2 - y_2^1 \leq 0,\ x_3 - y_3^1 \leq 0,\\
&y_1^1 + y_2^1 + y_3^1 \leq 2,\ y_1^1 \leq 1,\ y_2^1 \leq 1,\ y_3^1 \leq 1,\\
&x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\\
&x_1 - y_1^2 \leq 0,\ x_2 - y_2^2 \leq 0,\ x_3 - y_3^2 \leq 0,\\
&y_1^2 \leq 1,\ y_2^2 \leq 1,\ y_3^2 \leq 1.
\end{aligned}
$$

Facet-defining inequalities for P^1:

$$
\begin{aligned}
&x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\\
&x_1 - y_1^1 \leq 0,\ x_2 - y_2^1 \leq 0,\ x_3 - y_3^1 \leq 0,\\
&y_1^1 + y_2^1 + y_3^1 \leq 2,\ y_1^1 \leq 1,\ y_2^1 \leq 1,\ y_3^1 \leq 1.
\end{aligned}
$$

Facet-defining inequalities for P^2:

$$
\begin{aligned}
&x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\\
&x_1 - y_1^2 \leq 0,\ x_2 - y_2^2 \leq 0,\ x_3 - y_3^2 \leq 0,\\
&y_1^2 \leq 1,\ y_2^2 \leq 1,\ y_3^2 \leq 1.
\end{aligned}
$$

Example 4 *Consider an instance of (1) with $n_1 = n_2 = 2$, $\Omega = \{1, 2\}$, and the following data:*

$$
\begin{aligned}
&A = -I_{2\times 2}, \quad b = -\mathbf{1}_2,\\
&T^1 = T^2 = I_{2\times 2}, \quad W^1 = W^2 = -I_{2\times 2},
\end{aligned}
$$

with right-hand-side vectors $h_1 = -\mathbf{1}_2$ and $h_2 = \mathbf{0}_2$. All variables are restricted to be integers. Note that this problem has TU recourse (i.e., relatively complete recourse, W^ω is TU for $\forall \omega \in \Omega$, uncertainty is constrained to the vectors h^ω). Moreover, the polytope P is full-dimensional. Inequalities $x_i \geq 0$ for $i = 1, 2$ are facet-defining for P^1 because the four affinely independent points $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, $(0, 1, 0, 0)$, $(0, 1, 0, 1)$ in P^1 satisfy $x_1 \geq 0$ with equality. Similarly, we can find four affinely independent points in P^1 that satisfy $x_2 \geq 0$ with equality. However, there are not six affinely independent points in P satisfying $x_i \geq 0$ with equality because $x_i = 0$ enforces y_i^2 to be zero for $i = 1, 2$. Therefore, P is not SSF-preserving.

Example 5 *(Stochastic Facility Location Problem) Consider the set of customers $\mathcal{M} = \{1, \ldots, M\}$ and the set of plants $\mathcal{N} = \{1, \ldots, N\}$. Let $d_i^\omega > 0$ denote the demand of customer i under scenario $\omega \in \Omega$, and let each plant have the capacity C^ω satisfying $(1/N) \sum_{i \in \mathcal{M}} d_i^\omega \leq C^\omega < \sum_{i \in \mathcal{M}} d_i^\omega$, $\forall \omega \in \Omega$.*

The feasible space of the stochastic capacitated plant location problem is defined by the following constraints [23]:

$$y_{ij}^{\omega} \leq x_j, \qquad \forall i \in \mathcal{M},\ \forall j \in \mathcal{N},\ \forall \omega \in \Omega \tag{3a}$$

$$\sum_{j \in \mathcal{N}} y_{ij}^{\omega} \leq 1, \qquad \forall i \in \mathcal{M},\ \forall \omega \in \Omega \tag{3b}$$

$$\sum_{i \in \mathcal{M}} d_i^{\omega} y_{ij}^{\omega} \leq C^{\omega} x_j, \qquad \forall j \in \mathcal{N},\ \forall \omega \in \Omega \tag{3c}$$

$$y_{ij}^{\omega} \geq 0, \qquad \forall i \in \mathcal{M},\ \forall j \in \mathcal{N},\ \forall \omega \in \Omega \tag{3d}$$

$$x_j \in \{0,1\}, \qquad \forall j \in \mathcal{N}. \tag{3e}$$

The stochastic facility location problem has relatively complete recourse and satisfies Condition (C3(ω)) $\forall \omega \in \Omega$. Therefore, (3) *is SSF-preserving by Corollary 10. If we write problem* (3) *in the form of (1), we obtain:*

$$T^{\omega} = \begin{bmatrix} I_{N\times N} \\ \vdots \\ I_{N\times N} \\ \mathbf{0}_{M\times N} \\ C^{\omega} I_{N\times N} \end{bmatrix}, h^{\omega} = \begin{bmatrix} \mathbf{0}_N \\ \vdots \\ \mathbf{0}_N \\ -\mathbf{1}_M \\ \mathbf{0}_N. \end{bmatrix}$$

$$h^{\omega} - T^{\omega} x = \begin{bmatrix} -x \\ \vdots \\ -x \\ -\mathbf{1}_M \\ -C^{\omega} x \end{bmatrix}.$$

Therefore, $h^{\omega} < T^{\omega} x,\ \forall x \in \{0,1\}^N \cap \mathbb{R}^N_{++}$ *as* $C^{\omega} > 0$ *which satisfies Condition (C3(ω))* $\forall \omega \in \Omega$.