

# A characterization of positive spanning sets with ties to strongly edge-connected digraphs

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## Abstract

Positive spanning sets (PSSs) are families of vectors that span a given linear space through non-negative linear combinations. Despite certain classes of PSSs being well understood, a complete characterization of PSSs remains elusive. In this paper, we explore a relatively understudied relationship between positive spanning sets and strongly edge-connected digraphs, in that the former can be viewed as a generalization of the latter. We leverage this connection to define a decomposition structure for positive spanning sets inspired by the ear decomposition from digraph theory.

**Keywords** Positive spanning sets; Positive bases; Strongly edge-connected digraphs; Ear decomposition; Network matrices; Gaussian elimination.

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## 1 Introduction

Gaussian elimination is a fundamental technique in linear algebra, that can be used to assess whether a given matrix is linearly spanning, in the sense that its columns span the entire space through linear combinations [17, Page 6]. Similarly, in graph theory, one can determine a spanning tree of a graph using linear algebra techniques, while efficient implementations of graph algorithms can be obtained by leveraging sparse linear algebra [10].

Positive spanning sets, or PSSs, are matrices such that the columns span the entire space through nonnegative linear combinations [7]. These matrices are instrumental to direct-search algorithms, a class of continuous optimization algorithms that proceed by exploring the variable space through suitably chosen directions [16, 2]. When those directions are chosen from positive spanning sets, convergence can be guaranteed at a rate that heavily depends on the properties of the PSSs at hand [16, 9]. In this setting, using a direction corresponds to evaluating an expensive

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function, and thus optimizers typically rely on inclusion-wise minimal positive spanning sets, or positive bases [21, 22]. Although positive bases have already been fully described [24], generic descriptions are often impractical to generate positive bases in practice. As a result, optimizers have focused on characterizing special positive bases for which simpler characterizations can be obtained [13, 14].

Perhaps surprisingly, a connection between positive spanning sets and strongly edge-connected digraphs was spotted early in the PSS literature [19], but to the authors' knowledge this connection has not been exploited further. Meanwhile, numerous results have been established for strongly connected digraphs [3, 15], with minimal strongly edge-connected digraphs attracting recent interest [1, 6, 11, 12]. Although such digraphs appear connected to positive bases through the concept of minimality, a formal link between those objects has yet to be described.

In this paper, we provide certificates for the positive spanning property based on digraph theory. To this end, we show that PSSs can be seen as generalizing the concept of strongly edge-connected digraphs. We then leverage this connection to obtain a novel characterization of such matrices based on the ear decomposition of digraphs [25].

The remainder of this paper is organized as follows. In Section 2, we review key results from digraph theory. We then discuss positive spanning sets and draw connections with strongly edge-connected digraphs in Section 3. Our main results, that generalize the ear decomposition to positive spanning sets, are derived in Section 4.

**Notations** Throughout this paper, we work in the Euclidean space  $\mathbb{R}^n$  with  $n \geq 2$ , or a linear subspace thereof, denoted by  $\mathbb{L} \subset \mathbb{R}^n$ . The dimension of such a subspace will always be assumed to be at least 1. The set of real matrices with  $n$  rows and  $m$  columns will be denoted as  $\mathbb{R}^{n \times m}$ . Those dimensions will always be assumed to be at least 1. Bold lowercase letters (e.g.  $\mathbf{v}, \mathbf{a}$ ) will be used to designate vectors and arcs in directed graphs, while bold uppercase letters (e.g.  $\mathbf{D}$ ) will denote matrices. The notations  $\mathbf{0}_n$  and  $\mathbf{1}_n$  will respectively be used to designate the null vector and the all-ones vector in  $\mathbb{R}^n$ , while  $\mathbf{I}_n = [\mathbf{e}_1 \ \dots \ \mathbf{e}_n]$  will denote the identity matrix in  $\mathbb{R}^{n \times n}$ . Given a matrix  $\mathbf{D} \in \mathbb{R}^{n \times m}$ , its set of columns will be denoted  $\text{col}(\mathbf{D})$  while its linear span (*i.e.* the set of linear combinations of its columns) will be denoted by  $\text{span}(\mathbf{D})$ . The matrix whose entries are the signs of those of  $\mathbf{M}$  will be noted  $\text{sgn}(\mathbf{M})$ . Calligraphic letters such as  $\mathcal{D}$  and  $\mathcal{S}$  will be used for finite families of vectors or of indices. For any integer  $m \geq 1$ , we let  $\llbracket 1, m \rrbracket := \{z : 1 \leq z \leq m, z \in \mathbb{Z}\}$ . Finally, for a digraph  $G = (V, A)$  the notations  $(u, v)$  and  $u - v$  will respectively designate an arc and an oriented path in  $A$  going from  $u$  to  $v$ .

## 2 Digraphs and ear decomposition

In this section, we recall classical results on the ear decomposition for digraphs. For sake of completeness, we first define the main concepts and properties of digraphs to be used throughout the paper [3, 25]. We consider digraphs of the form  $G = (V, A)$ , where  $V$  denotes a set of vertices and  $A$  denotes a set of arcs. A *directed path* in  $G$  is a sequence of arcs of the form  $\{(u_i, u_{i+1})\}_{i=1, \dots, k}$ . Directed paths such that  $u_1 = u_{k+1}$  with  $u_2, \dots, u_k$  all distinct are called *circuits*. A digraph  $(V, A)$  is called *acyclic* if the set  $A$  does not contain any circuit, while it is called *strongly edge-connected* - or *strongly connected* for simplicity- if for any  $(u, v) \in V^2$ , there exists a directed path from  $u$  to  $v$ . A strongly connected digraph  $G = (V, A)$  is *minimally strongly connected* if any digraph  $G' = (V, A')$  such that  $A' \subset A$ ,  $|A'| = |A| - 1$  is not strongly

connected. Finally, an *oriented spanning tree* of a graph  $G$  is an oriented tree  $T = (V, \hat{A})$  such that  $\hat{A} \subset A$ .

As mentioned in the previous section, we are interested in certifying whether a digraph is strongly connected. Proposition 2.1 provides a negative certificate for this property.

**Proposition 2.1** *A connected digraph  $G = (V, A)$  is not strongly edge-connected if and only if there exists an oriented cut of  $G$ , i.e. a set  $\tilde{A} \subset A$  and two vertex-disjoint subgraphs  $G_1 = (V_1, A_1)$  and  $G_2 = (V_2, A_2)$  of  $G$  with  $V_1 \neq \emptyset$ ,  $V_2 \neq \emptyset$ , such that*

$$A = \tilde{A} \cup A_1 \cup A_2 \quad \text{and} \quad \forall (u, v) \in \tilde{A}, \quad u \in V_1 \text{ and } v \in V_2.$$

**Proof.** Suppose first that there exists an oriented cut of  $G$  given by  $\tilde{A} \subset A$ ,  $G_1 = (V_1, A_1)$  and  $G_2 = (V_2, A_2)$ . Let  $v_1 \in V_1$  and  $v_2 \in V_2$ . By definition of an oriented cut, there does not exist a directed path in  $A$  joining  $v_2$  to  $v_1$ , proving that  $G$  is not strongly connected.

Conversely, suppose that  $G$  is not strongly connected and let  $v_1, v_2$  be two vertices for which there is no  $v_2 - v_1$  path in  $A$ . Let  $V_2 \subset V$  be the set of vertices  $v$  such that  $A$  contains a  $v_2 - v$  path and let  $V_1 = V \setminus V_2$ . If  $A_1$  (resp.  $A_2$ ) denotes the set of arcs between vertices of  $V_1$  (resp.  $V_2$ ), then  $G_1 = (V_1, A_1)$ ,  $G_2 = (V_2, A_2)$  and  $\tilde{A} = A \setminus (A_1 \cup A_2)$  define an oriented cut for  $G$  as by construction, the arcs in  $\tilde{A}$  must be of the form  $(u_1, u_2)$  with  $u_1 \in V_1$  and  $u_2 \in V_2$ .  $\square$

We now turn to providing a positive certificate for strongly connected digraphs, based on the concept of ear decomposition [3, Section 5.3].

**Definition 2.1 (Ear and ear decomposition)** *Let  $G = (V, A)$  be a digraph. An ear of  $G$  is a directed path  $\{(u_i, u_{i+1})\}_{i=1, \dots, k} \subset A$  such that for any  $i \in \llbracket 2, k \rrbracket$ ,  $u_i$  is the head and the tail of exactly one arc in  $A$ .*

*The graph  $G$  possesses an ear decomposition if there exists a sequence of digraphs  $\{G_i = (V_i, A_i)\}_{i=1, \dots, s}$  such that*

- (i)  $V_1 \subseteq V_2 \subseteq \dots \subseteq V_s = V$ ,
- (ii)  $A_1 \subseteq A_2 \subseteq \dots \subseteq A_s = A$ ,
- (iii)  $G_1$  consists of one vertex and no arcs,
- (iv) For any  $i \in \llbracket 2, s \rrbracket$ ,  $A_i \setminus A_{i-1}$  defines an ear of  $G_i$ .

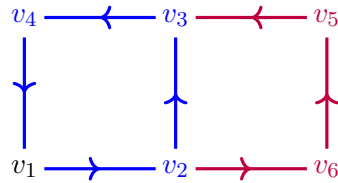


Figure 1. An ear decomposition  $(G_1, G_2, G_3)$  where  $G_1 = (\{v_1\}, \emptyset)$ ,  $G_2$  is obtained by adding the blue arcs and vertices and  $G_3$  is obtained by adding the red arcs and vertices.

Figure 1 below provides an example of a strongly connected graph that possesses an ear decomposition. In fact, ear decompositions characterize strongly connected digraphs in the following sense.

**Theorem 2.1** [25, Theorem 6.9] *A digraph  $G$  is strongly edge-connected if and only if it possesses an ear decomposition.*

Theorem 2.1 thus provides a positive certificate for the strongly connected property. In this paper, we aim at generalizing the result of Theorem 2.1 to positive spanning sets, which we define in the next section.

### 3 Positive spanning sets

The notion of positive spanning set (PSS) is a classical concept from linear algebra. The first part of this section reviews classical results on PSSs, while the second part draws connections between PSSs and strongly connected digraphs.

#### 3.1 Definition and characterization

Positive spanning sets are commonly defined as families of vectors, akin to spanning sets. In this paper, we adopt the following, equivalent definition based on matrices.

**Definition 3.1 (Positive span and positive spanning set)** *Let  $\mathbb{L}$  be a linear subspace of  $\mathbb{R}^n$  and  $m \geq 1$ . The positive span of a matrix  $\mathbf{D} \in \mathbb{R}^{n \times m}$ , denoted by  $\text{pspan}(\mathbf{D})$ , is the set*

$$\text{pspan}(\mathbf{D}) := \{\mathbf{D}\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^m, \mathbf{x} \geq \mathbf{0}_m\}.$$

*A positive spanning set (PSS) of  $\mathbb{L}$  is a matrix  $\mathbf{D}$  such that  $\text{pspan}(\mathbf{D}) = \mathbb{L}$ . When  $\mathbb{L} = \mathbb{R}^n$ , the matrix  $\mathbf{D}$  will simply be called a positive spanning set.*

Several characterizations of positive spanning sets have been proposed in the literature. Proposition 3.1 summarizes those that are relevant for this paper [13, 23].

**Proposition 3.1** *Let  $\mathbf{D} \in \mathbb{R}^{n \times m}$ . The following statements are equivalent.*

- (i)  $\mathbf{D}$  is a PSS for some linear subspace  $\mathbb{L} \subset \mathbb{R}^n$ .
- (ii)  $\text{pspan}(\mathbf{D}) = \text{span}(\mathbf{D})$ .
- (iii) There exists a positive vector  $\mathbf{x} \in \mathbb{R}^m$  such that  $\mathbf{D}\mathbf{x} = \mathbf{0}_n$ .

When the matrix  $\mathbf{D}$  has full rank, statement (iii) of Proposition 3.1 provides a certificate that  $\mathbf{D}$  is a PSS of  $\mathbb{R}^n$ . For future reference, we now present a certificate that a matrix *does not* positively span the entire space, which is a variant of Farkas' lemma [8].

**Proposition 3.2** [4, Theorem 2.3] *A matrix  $\mathbf{D} \in \mathbb{R}^{n \times m}$  does not positively span  $\mathbb{R}^n$  if and only if there exists a non-zero vector  $\mathbf{y} \in \mathbb{R}^n$  such that  $\mathbf{y}^\top \mathbf{D} \geq \mathbf{0}_m^\top$ .*

The positive spanning property is invariant to several operations on matrix columns, such as rescaling or permutation. In addition, if  $\mathbf{D}$  is a PSS for a given  $\ell$ -dimensional space, then for any invertible matrix  $\mathbf{B}$ , the matrix  $\mathbf{B}^{-1}\mathbf{D}$  is a PSS for another  $\ell$ -dimensional space. These invariance properties imply that we can reduce the study of positive spanning sets to equivalent classes defined as follows.

**Definition 3.2 (Structural equivalence)** Let  $\mathbf{D}$  and  $\mathbf{D}'$  be two matrices in  $\mathbb{R}^{n \times m}$ . The matrices  $\mathbf{D}$  and  $\mathbf{D}'$  are structurally equivalent if there exists a non-singular matrix  $\mathbf{B} \in \mathbb{R}^{n \times n}$ , a permutation matrix  $\mathbf{P} \in \mathbb{R}^{m \times m}$  and a diagonal matrix  $\mathbf{\Delta} \in \mathbb{R}^{m \times m}$  with positive diagonal entries such that

$$\mathbf{D}' = \mathbf{B}^{-1} \mathbf{D} \mathbf{P} \mathbf{\Delta}.$$

We then write  $\mathbf{D} \equiv \mathbf{D}'$ .

Note that the notion of structural equivalence for PSSs was previously stated in the context of derivative-free optimization for families of vectors [5, Definition 2.3]. By adapting this definition to matrices, we can combine Definition 3.2 together with Proposition 3.1 to obtain a characterization of PSSs based on structural equivalence. This characterization involves matrices that have a particularly simple expression, thanks to the rescaling and permutation operators.

**Proposition 3.3** A matrix  $\mathbf{D} \in \mathbb{R}^{n \times m}$  is a PSS of some  $\ell$ -dimensional subspace  $\mathbb{L}$  of  $\mathbb{R}^n$  if and only if there exists  $m - n$  vectors  $\mathbf{v}_1, \dots, \mathbf{v}_{m-n}$  in  $\mathbb{R}^\ell$  such that

$$\mathbf{D} \equiv \begin{bmatrix} \mathbf{I}_\ell & \mathbf{v}_1 & \cdots & \mathbf{v}_{m-n} \\ \mathbf{0}_{n-\ell, \ell} & \mathbf{0}_{n-\ell} & \cdots & \mathbf{0}_{n-\ell} \end{bmatrix} \quad \text{and} \quad \sum_{i=1}^{m-n} \mathbf{v}_i = -\mathbf{1}_\ell. \quad (1)$$

In particular, if  $m = \ell + 1$ , one has

$$\mathbf{D} \equiv \begin{bmatrix} \mathbf{I}_\ell & -\mathbf{1}_\ell \\ \mathbf{0}_{n-\ell, \ell} & \mathbf{0}_{n-\ell} \end{bmatrix}. \quad (2)$$

The second part of Proposition 3.3 provides a characterization of a subclass of PSSs called minimal positive bases (see Section 5 for a formal definition). However, for general matrices, the structural equivalence (1) is not satisfactory, as it does not provide an easy certificate for verifying whether a matrix is a PSS of a given subspace.

### 3.2 Connection with digraphs

Having defined key concepts associated with PSSs, we now formalize their relationship with (strongly connected) digraphs. Although existing connections involve incidence matrices [19], our results rely more generally on network matrices [25].

**Definition 3.3 (Network matrix)** Let  $G = (V, A)$  be a digraph with  $A = \{(u_j, v_j)\}_{j=1}^m$  and let  $T = (V, \hat{A})$  be an oriented spanning tree with  $\hat{A} = \{(\hat{u}_i, \hat{v}_i)\}_{i=1}^n$ . The network matrix associated with  $G$  and  $T$  is the matrix  $\mathbf{M} \in \mathbb{R}^{n \times m}$  defined by

$$\forall i \in \llbracket 1, n \rrbracket, \forall j \in \llbracket 1, m \rrbracket, \quad \mathbf{M}_{i,j} = \begin{cases} 0 & \text{if the path } u_j - v_j \text{ in } T \text{ does not pass through } (\hat{u}_i, \hat{v}_i), \\ 1 & \text{if } u_j - v_j \text{ passes through } (\hat{u}_i, \hat{v}_i) \text{ in forward direction,} \\ -1 & \text{if } u_j - v_j \text{ passes through } (\hat{u}_i, \hat{v}_i) \text{ in backward direction.} \end{cases}$$

An example of network matrix is provided in Figure 2.



Figure 2. Digraph, spanning tree (plain edges) and network matrix. The arcs are numbered to match the column ordering

Several remarks are in order regarding Definition 3.3. First, note that there is no ambiguity in the definition as any path joining two vertices in  $T$  is unique. Secondly, notice that we arbitrarily selected an ordering of the arcs in both  $A$  and  $\hat{A}$ . Two network matrices defined using a different ordering of those arcs are structurally equivalent in the sense of Definition 3.2 [25, (36) p.277]. In the rest of the paper, unless otherwise needed, we will not specify the arc ordering. Finally, one may consider the incidence matrix of a digraph  $G = (V, A)$  as a network matrix associated with  $G$  and with the spanning tree  $T = (V \cup \{u\}, \{(u, v)\}_{v \in V})$ .

In this paper, for simplicity, we focus on network matrices whose associated spanning tree  $T$  is a subgraph of the associated digraph  $G$ , as such matrices are structurally equivalent to one another. For sake of completeness, we state and prove this property in the proposition below.

**Proposition 3.4** *Let  $G = (V, A)$  be a strongly connected digraph with  $|V| = n + 1$  and  $|A| = m$ . Let  $T = (V, \hat{A})$  with  $\hat{A} \subset A$  be a spanning tree of  $G$  and let  $\mathbf{M} \in \mathbb{R}^{n \times m}$  be the associated network matrix. Then, a matrix  $\mathbf{M}'$  is a network matrix for  $G$  and a spanning tree  $T'$  if and only if  $\mathbf{M}' = \mathbf{B}^{-1}\mathbf{M}\mathbf{P}$  where  $\mathbf{B} \in \mathbb{R}^{n \times n}$  is nonsingular,  $\text{col}(\mathbf{B}) \subset \text{col}(\mathbf{M})$  and  $\mathbf{P} \in \mathbb{R}^{m \times m}$  is a permutation matrix.*

**Proof.** Letting  $A = \{a_1, \dots, a_m\}$  and  $T = (V, \hat{A})$ , we assume without loss of generality that  $\hat{A} = \{a_1, \dots, a_n\}$  and that the columns of  $\mathbf{M}$ , denoted by  $\mathbf{c}_1, \dots, \mathbf{c}_m$ , correspond to the arcs  $a_1, \dots, a_m$  in that order. Since  $\mathbf{M}$  is a network matrix, we thus have  $\mathbf{c}_i = \mathbf{e}_i$  for every  $i \in \{1, \dots, n\}$ , while any  $\mathbf{c}_i$  with  $i \in \{m + 1, \dots, n\}$  has coefficients in  $\{-1, 0, 1\}$ .

Suppose first that  $\mathbf{M}'$  is a network matrix associated with  $G$  and a spanning tree  $T' = (V, \hat{A}')$  with  $\hat{A}' \subset A$ , and let  $a_{i_1}, \dots, a_{i_n}$  denote the arcs in  $\hat{A}'$  with  $1 \leq i_1 < i_2 < \dots < i_n \leq m$ . There exists a permutation matrix  $\mathbf{P} \in \mathbb{R}^{m \times m}$  such that the columns of  $\mathbf{\Pi} = \mathbf{M}'\mathbf{P}^{-1}$ , denoted by  $\mathbf{c}'_1, \dots, \mathbf{c}'_m$ , correspond to arcs  $a_1, \dots, a_m$  in that order. Then, for any  $1 \leq j \leq n$ , we have  $\mathbf{c}'_{i_j} = \mathbf{e}_j$ . Moreover, since  $\hat{A}'$  defines a tree on  $G$ , the columns  $\mathbf{c}_{i_1}, \dots, \mathbf{c}_{i_n}$  of  $\mathbf{M}$  define a basis of  $\mathbb{R}^n$ . Letting  $\mathbf{B} \in \mathbb{R}^{n \times n}$  be the matrix defined with those columns in order, it follows that  $\mathbf{B}$  is a nonsingular matrix defining the change of basis from  $\{\mathbf{c}_i\}_{i=1, \dots, n}$  to  $\text{col}(\mathbf{B}) = \{\mathbf{c}_{i_j}\}_{j=1, \dots, n}$ . As a result, we have  $\mathbf{B}\mathbf{c}'_{i_j} = \mathbf{B}\mathbf{e}_j = \mathbf{c}_{i_j}$  for any  $1 \leq j \leq n$  and  $\mathbf{B}\mathbf{c}'_i = \mathbf{e}_i$  for any  $1 \leq i \leq n$ , from which it follows that  $\mathbf{B}\mathbf{\Pi} = \mathbf{M}$ . Overall, we have shown that  $\mathbf{B}\mathbf{M}'\mathbf{P}^{-1} = \mathbf{M}$  with  $\mathbf{B}$  nonsingular such that  $\text{col}(\mathbf{B}) \subset \text{col}(\mathbf{M})$  and  $\mathbf{P}$  a permutation matrix.

Suppose now that  $\mathbf{M}' = \mathbf{B}^{-1}\mathbf{M}\mathbf{P}$  with  $\mathbf{B} \in \mathbb{R}^{n \times n}$  nonsingular such that  $\text{col}(\mathbf{B}) \subset \text{col}(\mathbf{M})$  and  $\mathbf{P}$  a permutation matrix. Since  $\mathbf{B}$  is nonsingular, the columns of  $\mathbf{B}$ , which we denote by  $\mathbf{c}_{i_1}, \dots, \mathbf{c}_{i_n}$  with  $1 \leq i_1 < i_2 < \dots < i_n \leq m$ , define a basis of  $\mathbb{R}^n$ . Consequently, those columns define a set of arcs  $\hat{A}' = \{a_{i_j}\}_{j=1, \dots, n}$  such that  $T' = (V, \hat{A}')$  is a spanning tree for  $G$ . Since  $\mathbf{B}$  represents the change of basis from  $\mathbf{c}_1, \dots, \mathbf{c}_n$  to  $\mathbf{c}_{i_1}, \dots, \mathbf{c}_{i_n}$ , the columns of  $\mathbf{B}^{-1}\mathbf{M}$  represent the

expression of each path in the basis  $\{\mathbf{c}_{i_j}\}_{j=1,\dots,n}$ . It follows that  $\mathbf{B}^{-1}\mathbf{M} = \mathbf{M}'\mathbf{P}^{-1}$  is a network matrix, hence  $\mathbf{M}'$  is also a network matrix associated with the same spanning tree.  $\square$

We now turn to the main result of this section, which establishes a direct connection between positive spanning sets and strongly connected digraphs. A similar result was stated without proof by Marcus [20]. However, to the best of our knowledge, Theorem 3.1 in its present form and its proof are new.

**Theorem 3.1** *Let  $G = (V, A)$  be a connected digraph with  $|A| = m$  and let  $T = (V, \hat{A})$  be an oriented spanning tree of  $G$  with  $|\hat{A}| = n$ . Let  $\mathbf{M} \in \mathbb{R}^{n \times m}$  be the associated network matrix. Then, the matrix  $\mathbf{M}$  is a PSS for  $\mathbb{R}^n$  if and only if the graph  $G$  is strongly edge-connected.*

**Proof.** Without loss of generality, we consider an ordering  $A = \{a_i\}_{i=1}^m$  such that  $\hat{A} = \{a_i\}_{i=1}^n$ . As a result, the first  $n$  columns of  $\mathbf{M}$  form the identity matrix  $\mathbf{I}_n$ , thus ensuring that  $\mathbf{M}$  has full row rank. We now proceed with the proof.

Suppose first that  $G$  is strongly connected, and consider an ear decomposition  $\{G_i = (V_i, A_i)\}$  associated with  $G$ . Since  $G_1$  consists of a single vertex without arcs, there exists no circuit within  $G_1$ . Now, for any  $i \in \llbracket 2, k \rrbracket$ , consider the ear of  $G_i$  defined by  $A_i \setminus A_{i-1}$ . The arcs in this ear belong to a circuit of  $G$  that possibly contains other arcs in  $G_{i-1}$ . Let  $\mathbf{x}_i \in \mathbb{R}^m$  be the characteristic vector of this circuit, i.e.  $[\mathbf{x}_i]_j = 1$  if arc  $a_j$  is in the circuit, and  $[\mathbf{x}_i]_j = 0$  otherwise. Then the vector  $\mathbf{M}\mathbf{x}_i$  corresponds to the sum of all arcs in this circuit, which by definition must be  $\mathbf{0}_n$ . Finally, consider the vector  $\mathbf{x} = \sum_{i=2}^s \mathbf{x}_i$ . This vector has positive coefficients, since every arc in  $G$  is contained in an ear of the decomposition. We have thus found a positive vector  $\mathbf{x}$  such that  $\mathbf{M}\mathbf{x} = \mathbf{0}_n$ , hence  $\mathbf{M}$  is a PSS by Proposition 3.1.

Conversely, suppose that  $G$  is *not* strongly connected. We will show that there exists a vector  $\mathbf{y} \in \mathbb{R}^n$  such that  $\mathbf{y}^\top \mathbf{M} \geq \mathbf{0}_m$ , thereby implying that  $\mathbf{M}$  is not a PSS thanks to Proposition 3.2. Since  $G$  is not strongly connected, Proposition 2.1 ensures that there exists a set  $\tilde{A} \subset A$  and two subgraphs  $G_1 = (V_1, A_1)$ ,  $G_2 = (V_2, A_2)$  that define an oriented cut of  $G$ . Then, since  $G$  is connected, we find that  $\tilde{A} \cap \hat{A}$  is non-empty. Without loss of generality, we assume that  $\tilde{A} \cap \hat{A} = \{a_1, \dots, a_k\}$  for some  $k \leq n$ . Consider the vector  $\mathbf{y} = \sum_{\ell=1}^k \mathbf{y}_\ell$ , where  $\mathbf{y}_\ell$  is the  $\ell^{\text{th}}$  column of  $\mathbf{M}$ . Since the first  $n$  columns of  $\mathbf{M}$  correspond to the identity matrix, it follows that

$$\mathbf{y}^\top \mathbf{y}_i = \begin{cases} 1 & \text{if } i \in \llbracket 1, k \rrbracket \\ 0 & \text{if } i \in \llbracket k+1, n \rrbracket. \end{cases}$$

In addition, if  $i \in \llbracket n+1, m \rrbracket$ , then two situations can occur. If arc  $a_i$  belongs to either  $A_1$  or  $A_2$ , then the directed path in  $T$  linking the head and tail of  $a_i$  must thus contain an even number of arcs in  $\tilde{A}$  (possibly 0), with half of them used in the forward direction. As a result, one must have

$$\mathbf{y}^\top \mathbf{y}_i = \sum_{\ell=1}^k \mathbf{y}_\ell^\top \mathbf{y}_i = 0.$$

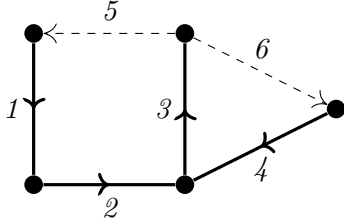
Otherwise  $a_i$  belongs to  $A \setminus (A_1 \cup A_2)$ , that is  $a_i \in \tilde{A}$  therefore the tail of  $a_i$  is in  $V_1$  and its head is in  $V_2$ . As a result, the directed path in  $T$  linking the head and tail of  $a_i$  must use an odd number of arcs in  $\tilde{A} \cap \hat{A}$ , with one more arc in the forward direction. Thus,

$$\mathbf{y}^\top \mathbf{y}_i = \sum_{\ell=1}^k \mathbf{y}_\ell^\top \mathbf{y}_i = 1.$$

Overall, we have shown that  $\mathbf{y}^\top \mathbf{M} \geq \mathbf{0}_m$ , from which we conclude that  $\mathbf{M}$  cannot be a PSS.  $\square$

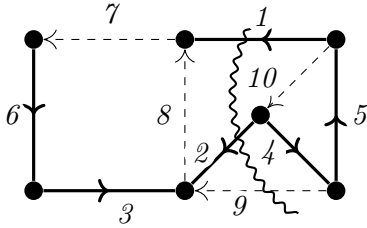
Theorem 3.1 provides positive and negative certificates regarding the PSS (or non-PSS) nature of a network matrix based on those for strongly connected digraphs. We illustrate below the result of Theorem 3.1 using two examples.

**Example 3.1 (A PSS network matrix)** *The positive spanning set  $\mathbf{M}_1$  is a network matrix associated to the strongly connected digraph below and to the spanning tree formed by the thick arcs.*



$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

**Example 3.2 (A non-PSS network matrix)** *Let  $T = (V, \hat{A})$  be the spanning tree formed by the thick arcs in the digraph  $G$  below. The wavy cut  $\tilde{A}$  certifies that  $G$  is not strongly connected. Similarly, the network matrix  $\mathbf{M}_2$  is not a PSS as the characteristic vector  $\mathbf{y}^\top = [1 \ 1 \ 0 \ 0 \ 0 \ 0]$  of  $\tilde{A} \cap \hat{A}$  in  $T$  satisfies  $\mathbf{y}^\top \mathbf{M}_2 \geq \mathbf{0}_{10}$ .*



$$\mathbf{M}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

In the next section, we will exploit the link between strongly connected digraphs and PSSs by showing that the ear decomposition can be extended to more general matrices.

## 4 Ear decomposition of positive spanning sets

Theorem 2.1 states that a digraph is strongly connected if and only if it admits an ear decomposition. Together with Theorem 3.1, it thus implies that a network matrix is a PSS if and only if its associated graph admits an ear decomposition. We will now establish a similar result for positive spanning sets, thereby generalizing Theorem 2.1. To this end, we start this section by extending the notions of circuits and acyclic graphs to matrices.

### 4.1 Acyclic and circuit matrices

The ear decomposition for strongly connected digraphs relies on the fundamental notion of circuit. Our goal is thus to define an equivalent concept for matrices. To this end, we first introduce the companion notion of acyclic matrix, inspired by acyclic graphs.



**Definition 4.1 (Acyclic matrix)** A matrix  $\mathbf{A} \in \mathbb{R}^{n \times m}$  is called acyclic if

$$[\mathbf{Ax} = \mathbf{0}_n \text{ and } \mathbf{x} \geq \mathbf{0}_m] \Leftrightarrow \mathbf{x} = \mathbf{0}_m.$$

Per Definition 4.1, any network matrix associated to an acyclic graph is an acyclic matrix. Alternate characterizations of acyclic matrices can be obtained from Gordan's Lemma [18].

**Proposition 4.1** For any  $\mathbf{A} \in \mathbb{R}^{n \times m}$ , the following statements are equivalent.

- (i) The matrix  $\mathbf{A}$  is acyclic.
- (ii) There exists a non-zero vector  $\mathbf{y} \in \mathbb{R}^n$  such that  $\mathbf{y}^\top \mathbf{A} > \mathbf{0}_m^\top$ .
- (iii) For any  $\bar{\mathbf{A}} \in \mathbb{R}^{n \times \bar{m}}$  with  $\text{col}(\bar{\mathbf{A}}) \subset \text{col}(\mathbf{A})$  and  $\bar{m} > 0$ ,  $\text{pspan}(\bar{\mathbf{A}})$  is not a linear space.

**Proof.** The equivalence between (i) and (ii) is a restatement of Gordan's lemma. The equivalence between (i) and (iii) follows from that between the first and last statements of Proposition 3.1.  $\square$

From Proposition 4.1, one observes that matrices with positive entries are necessarily acyclic. In fact, positive entries characterize acyclic matrices in the following sense.

**Proposition 4.2** A matrix  $\mathbf{A} \in \mathbb{R}^{n \times m}$  is acyclic if and only if it is structurally equivalent to a matrix with all entries strictly positive.

**Proof.** Suppose first that  $\mathbf{A}$  is acyclic. From Proposition 4.1(ii), there exists  $\mathbf{y} \in \mathbb{R}^n$  such that  $\mathbf{y}^\top \mathbf{A} > \mathbf{0}_m^\top$ . Then, there exists  $\eta > 0$  such that the matrix  $\mathbf{B} = \begin{bmatrix} \mathbf{y}^\top \\ \dots \\ \mathbf{y}^\top \end{bmatrix} + \eta \mathbf{I}_n \in \mathbb{R}^{n \times n}$  is invertible and  $\mathbf{BA}$  has positive entries, proving the desired result.

Conversely, suppose that  $\mathbf{A}$  is not acyclic. Then, there must exist a nonzero vector  $\mathbf{x} \geq \mathbf{0}_m$  such that  $\mathbf{Ax} = \mathbf{0}_n$ . For any invertible matrix  $\mathbf{B}$ , one then has  $\mathbf{B}^{-1}\mathbf{Ax} = \mathbf{0}_n$ , hence the matrix  $\mathbf{B}^{-1}\mathbf{A}$  must have at least one non-positive entry.  $\square$

Proposition 4.2 provides a certificate for showing that a matrix is acyclic, by finding a basis defining a structurally equivalent matrix. A certificate for showing that a matrix is *not* acyclic is obtained by finding a nonzero vector with nonnegative entries in the null space of the matrix.

We are now ready to provide the definition of a circuit matrix.

**Definition 4.2 (Circuit)** A matrix  $\mathbf{C} \in \mathbb{R}^{n \times m}$  is called a circuit matrix if it is a PSS for some linear subspace  $\mathbb{L}$  of  $\mathbb{R}^n$  and if any matrix  $\bar{\mathbf{C}} \in \mathbb{R}^{n \times \bar{m}}$  formed by  $0 < \bar{m} < \dim(\mathbb{L}) + 1$  columns of  $\mathbf{C}$  is acyclic.

We will see in Section 5 that circuit matrices can be identified with a special class of PSSs called minimal positive bases. Those are instrumental in obtaining decompositions of PSSs, and we will use circuit matrices for a similar purpose in the next section. In particular, the following structural equivalence will be leveraged.

**Proposition 4.3** *Let  $\mathbf{C} \in \mathbb{R}^{n \times m}$  be a circuit matrix. Then  $\mathbf{C}$  is a PSS for a linear subspace of dimension  $\ell = m - 1$  therefore*

$$\mathbf{C} \equiv \begin{bmatrix} \mathbf{I}_\ell & -\mathbf{1}_\ell \\ \mathbf{0}_{n-\ell, \ell} & \mathbf{0}_{n-\ell} \end{bmatrix}. \quad (3)$$

**Proof.** By definition,  $\mathbf{C}$  is a PSS for some linear subspace  $\mathbb{L}$  of  $\mathbb{R}^n$ . For the purpose of contradiction, suppose that  $m > \ell + 1$ , where  $\ell = \dim(\mathbb{L})$ . Since  $\mathbf{C}$  is a PSS for  $\mathbb{L}$ , Proposition 3.1 ensures that there exists a positive vector  $\mathbf{x} \in \mathbb{R}^n$  such that  $\mathbf{C}\mathbf{x} = \mathbf{0}_n$ . In addition, there also exists a matrix  $\tilde{\mathbf{C}} \in \mathbb{R}^{n \times (m-1)}$  formed by columns of  $\mathbf{C}$  such that  $\text{span}(\tilde{\mathbf{C}}) = \mathbb{L}$ . Without loss of generality, suppose that  $\tilde{\mathbf{C}}$  consists of the first  $m - 1$  columns of  $\mathbf{C}$ . Since  $m - 1 > \ell$ , the matrix  $\tilde{\mathbf{C}}$  has a non-zero null space, i.e. there exists a nonzero vector  $\tilde{\mathbf{y}} \in \mathbb{R}^{m-1}$  such that  $\tilde{\mathbf{C}}\tilde{\mathbf{y}} = \mathbf{0}_{m-1}$ . Letting  $\mathbf{y} = [\tilde{\mathbf{y}}^\top \ 0]^\top$ , it follows that  $\mathbf{C}\mathbf{y} = \mathbf{0}_n$ .

Now, the vectors  $\mathbf{x}$  and  $\mathbf{y}$  are not colinear since  $[\mathbf{x}]_m > 0 = [\mathbf{y}]_m$ . Therefore, there exists  $\sigma \geq 0$  such that  $\mathbf{z} = \sigma\mathbf{x} + \mathbf{y} \geq \mathbf{0}_n$  with  $\mathbf{z}$  having at least one component equal to zero. Let  $\tilde{\mathbf{C}}$  be the matrix formed by columns of  $\mathbf{C}$  corresponding to non-zero components of  $\mathbf{z}$ , and let  $\tilde{\mathbf{z}}$  be the vector formed by those components. It follows from the construction of  $(\tilde{\mathbf{C}}, \tilde{\mathbf{z}})$  that

$$\tilde{\mathbf{C}}\tilde{\mathbf{z}} = \mathbf{C}\mathbf{z} = \mathbf{C}(\sigma\mathbf{x} + \mathbf{y}) = \mathbf{0}_n.$$

As a result, we have shown that  $\tilde{\mathbf{C}}$  satisfies Proposition 3.1, and thus it must be a PSS for some linear space of  $\mathbb{R}^n$ . Therefore,  $\tilde{\mathbf{C}}$  is not acyclic, and this contradicts the fact that  $\mathbf{C}$  is a circuit, from which we conclude that  $m = \ell + 1$ .

The second part of the result then follows from the special case (2) in Proposition 3.3.  $\square$

Note that Proposition 4.3 justifies the terminology *circuit matrix*, as a circuit of  $n + 1$  vertices admits  $[\mathbf{I}_n \ -\mathbf{1}_n]$  as a network matrix. Using structural equivalence for circuits, we can obtain an improved certificate for non-acyclic matrices.

**Lemma 4.1** *A matrix  $\mathbf{M} \in \mathbb{R}^{n \times m}$  is not acyclic if and only if*

- (i) *One of the columns of  $\mathbf{M}$  is the zero vector  $\mathbf{0}_n$ , or*
- (ii) *There exists  $\ell \in \llbracket 1, n \rrbracket$  such that*

$$\mathbf{M} \equiv \begin{bmatrix} \mathbf{I}_\ell & -\mathbf{1}_\ell & \mathbf{X} \\ \mathbf{0}_{n-\ell, \ell} & \mathbf{0}_{n-\ell} & \mathbf{Y} \end{bmatrix},$$

*where the matrices  $\mathbf{X} \in \mathbb{R}^{\ell \times (m-\ell-1)}$  and  $\mathbf{Y} \in \mathbb{R}^{(n-\ell) \times (m-\ell-1)}$  can be empty.*

**Proof.** If (i) holds, then the matrix  $\mathbf{M}$  is not acyclic as Proposition 4.1(ii) fails. If (ii) holds, then the first  $\ell + 1$  columns of  $\mathbf{M}$  form a matrix  $\tilde{\mathbf{M}}$  such that  $\text{pspan}(\tilde{\mathbf{M}})$  is a linear space. Thus, Proposition 4.1(iii) fails to hold, hence  $\mathbf{M}$  is not acyclic. Conversely, suppose that  $\mathbf{M}$  is not acyclic. Then by Proposition 4.1(iii), there must exist a matrix  $\mathbf{C}$  formed by a subset of columns of  $\mathbf{M}$  such that  $\text{pspan}(\mathbf{C})$  is a linear space. Without loss of generality, suppose that this matrix is minimal for that property, and that it consists of the first  $\ell + 1$  columns of  $\mathbf{M}$  (thus the linear subspace spanned by  $\mathbf{C}$  is of dimension  $\ell$ ). Then the matrix  $\mathbf{C}$  is a circuit. Applying Proposition 4.3, we then know that there exists an invertible matrix  $\mathbf{B}$ , a permutation matrix  $\mathbf{P}$  and a diagonal matrix  $\Delta$  with positive diagonal entries such that

$$\mathbf{B}^{-1}\mathbf{C}\mathbf{P}\Delta = \begin{bmatrix} \mathbf{I}_\ell & -\mathbf{1}_\ell \\ \mathbf{0}_{n-\ell, \ell}^\top & \mathbf{0}_{n-\ell} \end{bmatrix}.$$

Letting then  $\mathbf{P}' = \begin{bmatrix} \mathbf{P}\Delta & \mathbf{0}_{\ell+1, m-\ell-1} \\ \mathbf{0}_{m-\ell-1, \ell+1} & \mathbf{I}_{m-\ell-1} \end{bmatrix} \in \mathbb{R}^{m \times m}$ , it follows that  $\mathbf{M} \equiv \mathbf{B}^{-1}\mathbf{M}\mathbf{P}'$ , that has the desired structure.  $\square$

A consequence of Lemma 4.1 is that the vector  $\mathbf{x} = \begin{bmatrix} \mathbf{1}_{\ell+1} \\ \mathbf{0}_{m-\ell-1} \end{bmatrix}$  can be used to attest that  $\mathbf{M}$  is not acyclic by structural equivalence, thus improving over the certificate from Proposition 4.2.

## 4.2 A new certificate for positive spanning sets

Building on the results of the previous subsection, we now generalize the concept of ear decomposition of digraphs to matrices. As a result, we will obtain a characterization of PSSs improving that of Proposition 3.3.

**Definition 4.3 (Negative row echelon matrix)** *A matrix  $\mathbf{N} \in \mathbb{R}^{n \times s}$ , with  $1 \leq s \leq n$ , is a negative row echelon matrix (NEM) if there exists a sequence  $z_0 = 1 < z_1 < z_2 < \dots < z_{s-1} \leq n$  such that*

- (i) *For all  $j \in \llbracket 1, s-1 \rrbracket$ , for all  $i \in \llbracket z_j, n \rrbracket$ ,  $\mathbf{N}_{i,j} = 0$ .*
- (ii) *For all  $j \in \llbracket 1, s-1 \rrbracket$ , for all  $i \in \llbracket z_{j-1}, z_j-1 \rrbracket$ ,  $\mathbf{N}_{i,j} = -1$ .*
- (iii)  *$\mathbf{N}_{i,s} = -1$ , for all  $i \geq z_{s-1}$ .*

An example of NEM is

$$\begin{bmatrix} -1 & \times & \times & \times \\ -1 & \times & \times & \times \\ 0 & -1 & \times & \times \\ 0 & -1 & \times & \times \\ 0 & 0 & -1 & \times \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

where the crosses  $\times$  indicate arbitrary values. Matrices of that form can be used to create PSSs as follows.

**Proposition 4.4** *Let  $\mathbf{M} \in \mathbb{R}^{n \times (n+s)}$  with  $s \in \llbracket 1, n \rrbracket$  such that  $\mathbf{M} \equiv [\mathbf{I}_n \ \mathbf{N}]$  where  $\mathbf{N}$  is a NEM. Then,  $\mathbf{M}$  is a PSS.*

**Proof.** Let  $\mathbf{N} = [\mathbf{u}_1 \ \dots \ \mathbf{u}_s]$ . Consider the vectors  $\{\mathbf{w}_i\}_{i=1, \dots, s}$  defined by

$$\begin{aligned} \mathbf{w}_1 &= \mathbf{u}_s \\ \forall i = 1, \dots, s-1, \quad \mathbf{w}_{i+1} &= \mathbf{w}_i + 2\|\mathbf{w}_i\|_\infty \mathbf{u}_{s-i}. \end{aligned}$$

By construction, the vector  $\mathbf{w}_s$  is a positive linear combination of the columns of  $\mathbf{N}$  and all its components are negative. It follows that there exists a positive combination of columns of  $[\mathbf{I}_n \ \mathbf{N}]$  adding to  $\mathbf{0}_n$ , i.e. a vector  $\mathbf{x} \in \mathbb{R}^{n+s}$  with positive coefficients such that  $[\mathbf{I}_n \ \mathbf{N}] \mathbf{x} = \mathbf{0}_n$ . By Proposition 3.1(iii) and structural equivalence, this implies that  $\mathbf{M}$  is also a PSS.  $\square$

Proposition 4.4 complements Proposition 3.3 in that it provides a sufficient condition for a matrix to be a PSS. Note that the result can be generalized to any matrix  $\mathbf{M}$  such that

$\mathbf{M} \equiv \begin{bmatrix} \mathbf{I}_\ell & \mathbf{N} \\ \mathbf{0}_{n-\ell,\ell} & \mathbf{0}_{n-\ell,m-\ell} \end{bmatrix}$  where  $\mathbf{N}$  is NEM. Such a matrix  $\mathbf{M}$  is indeed a PSS for some  $\ell$ -dimensional linear space.

We now aim at providing certificates for determining whether a matrix is a PSS using NEMs, based on identifying a desirable structure within the matrix. Recall from Gaussian elimination that any matrix in  $\mathbb{R}^{n \times m}$  is structurally equivalent to a matrix of the form

$$\begin{bmatrix} \mathbf{I}_\ell & \mathbf{X} \\ \mathbf{0}_{n-\ell,\ell} & \mathbf{0}_{n-\ell,m-\ell} \end{bmatrix},$$

for some  $\ell \in \llbracket 0, n \rrbracket$  and some matrix  $\mathbf{X} \in \mathbb{R}^{\ell \times (m-\ell)}$ . When  $\ell = n$  and  $\mathbf{X}$  is a NEM, such a structure allows to conclude that the matrix is a spanning set of  $\mathbb{R}^n$ . Following Proposition 4.4, we now define structures of interest for positive spanning sets.

**Definition 4.4** *Given dimensions  $n, m$ , let  $\ell \in \llbracket 1, n \rrbracket$  and  $k \in \llbracket 0, m - \ell \rrbracket$ .*

(i) *An IN matrix in  $\mathbb{R}^{n \times m}$  is a matrix of the form*

$$\begin{bmatrix} \mathbf{I}_{n,\ell} & \mathbf{N} & \mathbf{X} \end{bmatrix}, \quad (4)$$

*where  $\mathbf{I}_{n,\ell}$  represents the first  $\ell$  columns of the identity matrix in  $\mathbb{R}^{n \times n}$ ,  $k \leq n$ ,  $\mathbf{N} \in \mathbb{R}^{n \times k}$  is a NEM when  $k > 0$  and  $\mathbf{X} \in \mathbb{R}^{n \times (m-\ell-k)}$  is arbitrary.*

(ii) *An INA matrix in  $\mathbb{R}^{n \times m}$  is a matrix of the form*

$$\begin{bmatrix} \mathbf{I}_\ell & \mathbf{N} & \mathbf{X} \\ \mathbf{0}_{n-\ell,\ell} & \mathbf{0}_{n-\ell,k} & \mathbf{A} \end{bmatrix}, \quad (5)$$

*where  $k \leq \ell < n$ ,  $\mathbf{N} \in \mathbb{R}^{\ell \times k}$  is a NEM when  $k > 0$ ,  $\mathbf{X} \in \mathbb{R}^{n \times (m-\ell-k)}$  is arbitrary and  $\mathbf{A} \in \mathbb{R}^{(n-\ell) \times (m-\ell-k)}$  is an acyclic matrix.*

The matrix  $\mathbf{M}_1$  from Example 3.1 is an example of an IN matrix, while the network matrix from Figure 2 is an INA matrix. Moreover, since the definition of IN and INA matrices allows for  $k = 0$ , any nonzero matrix is structurally equivalent to an IN matrix with  $k = 0$ , per the Gaussian elimination argument above. However, the NEM and acyclic components in IN and INA matrices, respectively, allow for identifying positive spanning properties (or lack thereof) of a matrix. This is the purpose of the following theorem, that forms the central result of our paper.

**Theorem 4.1** *Let  $\mathbf{M} \in \mathbb{R}^{n \times m}$  be a nonzero matrix with  $n \geq 2$ .*

(i)  *$\mathbf{M}$  is a PSS of  $\mathbb{R}^n$  if and only if  $\mathbf{M}$  is structurally equivalent to an IN matrix with  $\ell = n$  and  $k > 0$ .*

(ii)  *$\mathbf{M}$  is not a PSS of  $\mathbb{R}^n$  if and only if  $\mathbf{M}$  is structurally equivalent to an INA matrix.*

**Proof.** For both results, the reverse implication is immediate. Indeed, if  $\mathbf{M}$  is structurally equivalent to an IN matrix with  $\ell = n$  and  $k > 0$ , Proposition 4.4 guarantees that this IN matrix contains a PSS of  $\mathbb{R}^n$ , thus both this IN matrix and  $\mathbf{M}$  are PSSs of  $\mathbb{R}^n$ . In addition, if  $\mathbf{M}$  is

structurally equivalent to an INA matrix, Proposition 4.2 implies that  $\mathbf{M}$  is also structurally equivalent to a matrix whose last  $n - \ell$  rows are nonnegative.

In the rest of the proof, we thus focus on establishing the forward implication for both (i) and (ii) through an induction argument on the dimension  $m$ . Note that  $m < n + 1$  implies that  $\mathbf{M}$  cannot be a PSS and that it is structurally equivalent to an INA matrix with  $k = 0$ .

Suppose first that  $m = n + 1$ . On one hand, if  $\mathbf{M}$  is a PSS, then it is also a circuit matrix with  $\ell = n$ . Proposition 4.3 then ensures that  $\mathbf{M}$  is equivalent to an IN matrix with  $\ell = n$  and  $k = 1$ . On the other hand, if  $\mathbf{M}$  is not a PSS it is either acyclic - and structurally equivalent to an INA matrix with  $\ell = k = 0$  - or it contains a circuit, say of length  $\ell$ . We choose this circuit such that  $\ell < n$  is as large as possible. From Proposition 4.3, it follows that  $\mathbf{M} \equiv \begin{bmatrix} \mathbf{I}_\ell & -\mathbf{1}_\ell & \bar{\mathbf{X}} \\ \mathbf{0}_{n-\ell, \ell} & \mathbf{0}_{n-\ell} & \bar{\mathbf{A}} \end{bmatrix}$ , where  $\bar{\mathbf{X}} \in \mathbb{R}^{\ell \times (n-\ell)}$  is arbitrary and  $\bar{\mathbf{A}} \in \mathbb{R}^{(n-\ell) \times (n-\ell)}$  is acyclic from the definition of  $\ell$ . It follows that  $\mathbf{M}$  is equivalent to an INA matrix with  $k = 1$ , and we have thus shown that (i) and (ii) hold for  $m = n + 1$  whenever  $n \geq 2$ .

Suppose now that  $m > n + 1$ , and that (i) and (ii) hold for any  $n \geq 2$  and any  $\tilde{m} \in \llbracket n + 1, m - 1 \rrbracket$ . We will establish that (i) and (ii) hold as well.

**On the one hand, suppose that  $\mathbf{M}$  is a PSS of  $\mathbb{R}^n$ .** If there exists a strict subset of columns of  $\mathbf{M}$  that is a PSS of  $\mathbb{R}^n$ , let  $\bar{\mathbf{M}} \in \mathbb{R}^{n \times \tilde{m}}$  be a matrix formed by these columns. Then,  $\mathbf{M} \equiv [\bar{\mathbf{M}} \ \mathbf{X}_1]$  for some matrix  $\mathbf{X}_1 \in \mathbb{R}^{n \times (m-\tilde{m})}$ . Since  $\tilde{m} < m$  by assumption, the induction argument applied to  $\bar{\mathbf{M}}$  guarantees that  $\bar{\mathbf{M}}$  is equivalent to an IN matrix, i.e.

$$\bar{\mathbf{M}} \equiv [\mathbf{I}_n \ \bar{\mathbf{N}} \ \bar{\mathbf{X}}],$$

where  $\bar{\mathbf{N}} \in \mathbb{R}^{n \times k}$  is a NEM with  $k > 0$ . Letting  $\mathbf{X} = [\bar{\mathbf{X}} \ \mathbf{X}_1]$ , it follows that  $\mathbf{M}$  is equivalent to the IN matrix  $[\mathbf{I}_n \ \bar{\mathbf{N}} \ \mathbf{X}]$ .

If no strict subset of columns of  $\mathbf{M}$  is a PSS of  $\mathbb{R}^n$ , we use the fact that  $\mathbf{M}$  is a PSS, thus it is not acyclic. Lemma 4.1 then guarantees that there exists  $\ell_1 \in \llbracket 1, n \rrbracket$  such that

$$\mathbf{M} \equiv \begin{bmatrix} \mathbf{I}_{\ell_1} & -\mathbf{1}_{\ell_1} & \mathbf{X}_1 \\ \mathbf{0}_{n-\ell_1, \ell_1} & \mathbf{0}_{n-\ell_1} & \bar{\mathbf{M}} \end{bmatrix}. \quad (6)$$

where  $\bar{\mathbf{M}} \in \mathbb{R}^{(n-\ell_1) \times \tilde{m}}$  with  $\tilde{m} := m - (\ell_1 + 1) \in \llbracket (n - \ell_1) + 1, m - 1 \rrbracket$ . Since  $\mathbf{M}$  is a PSS of  $\mathbb{R}^n$ , it follows from (6) that  $\bar{\mathbf{M}}$  must be a PSS of  $\mathbb{R}^{n-\ell_1}$ . Applying the induction argument to  $\bar{\mathbf{M}}$  yields the structural equivalence

$$\bar{\mathbf{M}} \equiv [\mathbf{I}_{n-\ell_1} \ \bar{\mathbf{N}} \ \bar{\mathbf{X}}],$$

where  $\bar{\mathbf{N}}$  is a NEM, and  $\bar{\mathbf{X}}$  is arbitrary. With an appropriate change of basis, the following equivalence holds:

$$\begin{bmatrix} \mathbf{I}_{\ell_1} & \mathbf{X}_1 \\ \mathbf{0}_{n-\ell_1, \ell_1} & \bar{\mathbf{M}} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{I}_{\ell_1} & \mathbf{0}_{\ell_1, n-\ell_1} & \mathbf{X}_2 & \mathbf{X}_3 \\ \mathbf{0}_{n-\ell_1, \ell_1} & \mathbf{I}_{n-\ell_1} & \mathbf{N}_1 & \mathbf{X}_4 \end{bmatrix}$$

where  $\mathbf{N}_1$  is a NEM and  $\mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4$  are arbitrary. It follows that

$$\mathbf{M} \equiv [\mathbf{I}_n \ \mathbf{N} \ \mathbf{X}] \quad \text{where} \quad \mathbf{N} = \begin{bmatrix} -\mathbf{1}_{\ell_1} & \mathbf{X}_2 \\ \mathbf{0}_{n-\ell_1} & \mathbf{N}_1 \end{bmatrix} \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} \mathbf{X}_3 \\ \mathbf{X}_4 \end{bmatrix}.$$

Since  $\mathbf{N}$  is a NEM by construction, we have shown that  $\mathbf{M}$  is equivalent to an IN matrix with  $\ell = n$  and  $k > 0$ .

**On the other hand, suppose that  $\mathbf{M}$  is not a PSS of  $\mathbb{R}^n$ .** If  $\mathbf{M}$  is acyclic, then it is an INA matrix with  $\ell = k = 0$ . Otherwise, Lemma 4.1 guarantees that there exists  $\ell_1 \in \llbracket 1, n \rrbracket$  such that  $\mathbf{M} \equiv \begin{bmatrix} \mathbf{I}_{\ell_1} & -\mathbf{1}_{\ell_1} & \mathbf{M}_1 \\ \mathbf{0}_{n-\ell_1, \ell_1} & \mathbf{0}_{n-\ell_1} & \mathbf{M}_2 \end{bmatrix}$ , where  $\mathbf{M}_2 \in \mathbb{R}^{n-\ell_1 \times (m-\ell_1)}$  does not positively span  $\mathbb{R}^{n-\ell}$  as  $\mathbf{M}$  is not a PSS of  $\mathbb{R}^n$ . Applying the induction argument, it follows that  $\mathbf{M}_2$  is structurally equivalent to an INA matrix, i.e.

$$\mathbf{M}_2 \equiv \begin{bmatrix} \mathbf{I}_{\bar{\ell}} & \bar{\mathbf{N}} & \bar{\mathbf{X}} \\ \mathbf{0}_{\bar{\ell}, \bar{\ell}} & \mathbf{0}_{\bar{\ell}} & \mathbf{A} \end{bmatrix} \quad \text{and so} \quad \mathbf{M} \equiv \begin{bmatrix} \mathbf{I}_{\ell_1} & -\mathbf{1}_{\ell_1} & \mathbf{0}_{\ell_1, \bar{\ell}} & \mathbf{X}_1 & \mathbf{X}_2 \\ \mathbf{0}_{\bar{\ell}, \ell_1} & \mathbf{0}_{\bar{\ell}} & \mathbf{I}_{\bar{\ell}} & \bar{\mathbf{N}} & \bar{\mathbf{X}} \\ \mathbf{0}_{\bar{\ell}, \ell_1} & \mathbf{0}_{\bar{\ell}} & \mathbf{0}_{\bar{\ell}, \bar{\ell}} & \mathbf{0}_{\bar{\ell}, k} & \mathbf{A} \end{bmatrix}$$

with  $\bar{\ell} < n - \ell_1$ ,  $\bar{\ell} = n - \ell_1 - \bar{\ell}$ ,  $\bar{\mathbf{N}} \in \mathbb{R}^{\bar{\ell} \times k}$  a NEM and  $\mathbf{A}$  an acyclic matrix.

Letting  $\mathbf{X} = \begin{bmatrix} \bar{\mathbf{X}}_2 \\ \bar{\mathbf{X}} \end{bmatrix}$  and  $\mathbf{N} = \begin{bmatrix} -\mathbf{1}_{\ell_1} & \mathbf{X}_1 \\ \mathbf{0}_{\bar{\ell}} & \bar{\mathbf{N}} \end{bmatrix}$ , we obtain that  $\mathbf{M} \equiv \begin{bmatrix} \mathbf{I}_{n-\bar{\ell}} & \mathbf{N} & \mathbf{X} \\ \mathbf{0}_{\bar{\ell}, \bar{\ell}} & \mathbf{0}_{\bar{\ell}} & \mathbf{A} \end{bmatrix}$ , hence  $\mathbf{M}$  is equivalent to an INA matrix with  $\ell = \bar{\ell} < n$ .  $\square$

**Remark 4.1** *Although Theorem 4.1 assumes  $n \geq 2$ , a similar result can be stated when  $n = 1$ . Indeed, given any nonzero matrix  $\mathbf{M} \in \mathbb{R}^{1 \times m}$ , is it clear that  $\mathbf{M}$  is a PSS of  $\mathbb{R}$  if and only if  $\mathbf{M} \equiv \begin{bmatrix} 1 & -1 & \mathbf{X} \end{bmatrix}$  for some arbitrary  $\mathbf{X} \in \mathbb{R}^{1 \times (m-2)}$ , while it is not a PSS if and only if  $\mathbf{M} \equiv \begin{bmatrix} 1 & \mathbf{A} \end{bmatrix}$  for some matrix  $\mathbf{A} \in \mathbb{R}^{1 \times (m-1)}$  with nonnegative coefficients.*

Akin to the proof of Theorem 2.1, the proof of Theorem 4.1 relies on an induction argument based on any column of a PSS being part of a circuit. The latter proof can thus be viewed as a generalization of the former. In the next section, we will study implications of Theorem 4.1 on the characterization of certain PSSs called positive bases.

## 5 Applications to positive bases

Positive bases can be succinctly defined as inclusion-wise minimal positive spanning sets [4, Chapter 2]. We begin this section by reviewing the concept of positive bases, drawing connections with digraphs as in Section 3.2. We then provide a general characterization of positive bases.

### 5.1 Positive bases and digraphs

In this paper, we follow the notations of Hare et al. [13], and define a positive basis by explicitly highlighting its associated subspace and size.

**Definition 5.1 (Positive basis)** *Let  $\mathbb{L}$  be an  $\ell$ -dimensional linear subspace of  $\mathbb{R}^n$  with  $\ell \geq 1$ . A matrix  $\mathbf{D} \in \mathbb{R}^{n \times (\ell+s)}$  with  $s \in \llbracket 1, \ell \rrbracket$  is called a positive basis of  $\mathbb{L}$  of size  $\ell + s$  if it is a PSS of  $\mathbb{L}$  such that no proper subset of the columns of  $\mathbf{D}$  is a PSS for  $\mathbb{L}$ .*

*We let  $\mathbf{D}_{\mathbb{L}, s}$  denote such a positive basis. When  $\mathbb{L} = \mathbb{R}^n$ , we use the simplified notation  $\mathbf{D}_{n, s}$ .*

Definition 5.1 exploits the well-known fact that positive bases of  $\mathbb{L}$  have cardinality in  $\llbracket \dim(\mathbb{L}) + 1, 2 \dim(\mathbb{L}) \rrbracket$  [2, 7]. We say that a positive basis is *maximal* if  $s = \ell$ , *minimal* when  $s = 1$ , and *intermediate* otherwise. The structure of the former two categories is well understood [2, 23], and can be stated using structural equivalence as follows.

**Theorem 5.1** *Let  $\mathbb{L}$  be an  $\ell$ -dimensional linear subspace of  $\mathbb{R}^n$ , and let  $\mathbf{D}_{\mathbb{L},1}$  and  $\mathbf{D}_{\mathbb{L},\ell}$  be a minimal positive basis and a maximal positive basis of  $\mathbb{L}$ , respectively. Then,*

$$\mathbf{D}_{\mathbb{L},1} \equiv \begin{bmatrix} \mathbf{I}_\ell & -\mathbf{1}_\ell \\ \mathbf{0}_{n-\ell,\ell} & \mathbf{0}_{n-\ell} \end{bmatrix} \quad \text{and} \quad \mathbf{D}_{\mathbb{L},\ell} \equiv \begin{bmatrix} \mathbf{I}_\ell & -\mathbf{I}_\ell \\ \mathbf{0}_{n-\ell,\ell} & \mathbf{0}_{n-\ell,\ell} \end{bmatrix}$$

Theorem 5.1 shows that all minimal and maximal positive bases are structurally equivalent to simple matrices described through coordinate vectors and negative combinations thereof. In the case of minimal positive bases, Theorem 5.1 is actually a restatement of Proposition 3.3.

Recall from Section 3.2 that the network matrix of a strongly connected digraph is a PSS. In particular, as stated in Section 4.1, a circuit  $G$  on  $n+1$  vertices admits the minimal positive basis  $[\mathbf{I}_n \quad -\mathbf{1}_n]$  as a network matrix. Similarly, the maximal positive basis  $[\mathbf{I}_n \quad -\mathbf{I}_n]$  is associated to a bi-directed tree on  $n+1$  vertices. More generally, Theorem 3.1 yields the following relationship between positive bases and minimally strongly connected digraphs.

**Corollary 5.1** *Let  $G = (V, A)$  be a connected digraph with  $|V| = n$  and let  $T = (V, \hat{A})$  be an oriented spanning tree of  $G$ . Let  $\mathbf{D} \in \mathbb{R}^{(n-1) \times (n-1+s)}$  be a network matrix associated with  $\{G, T\}$ . Then,  $\mathbf{D}$  is a positive basis for  $\mathbb{R}^{n-1}$  if and only if  $G$  is minimally strongly connected.*

**Remark 5.1** *Using the bound on the size of positive bases, we note that Corollary 5.1 can be used to prove that the number of arcs in a minimally strongly edge-connected digraph on  $n$  vertices ranges from  $n$  to  $2(n-1)$  [11]. To the best of our knowledge, such a proof technique is novel.*

Given the link between positive bases and minimally strongly connected digraphs, we seek a characterization of positive bases. Adapting Theorem 4.1 to such matrices, one sees that any positive basis  $\mathbf{D}_{n,s}$  satisfies the structural equivalence

$$\mathbf{D}_{n,s} \equiv [\mathbf{I}_n \quad \mathbf{N}] \quad \text{where} \quad \mathbf{N} \in \mathbb{R}^{n \times s} \text{ is a NEM.} \quad (7)$$

This structural characterization leads to the following result.

**Lemma 5.1** *Let  $G = (V, A)$  be minimally strongly connected, let  $T = (V, \hat{A})$  be a spanning tree of  $G$ , and let  $\mathbf{D}_{n,s} \in \mathbb{R}^{n \times (n+s)}$  be the network matrix associated with  $(G, T)$ . Then, for any matrix  $[\mathbf{I}_n \quad \mathbf{N}]$  given by (7), the matrix  $\text{sgn}([\mathbf{I}_n \quad \mathbf{N}])$  is also a network matrix associated with  $G$  and a spanning tree  $T' = (V, \hat{A}')$  with  $\hat{A}' \subset A$ .*

**Proof.** By structural equivalence, the matrix  $\mathbf{M} = [\mathbf{I}_n \quad \mathbf{N}]$  can be written  $\mathbf{M} = \mathbf{B}^{-1} \mathbf{D}_{n,s} \mathbf{P} \Delta$  where  $\mathbf{B} \in \mathbb{R}^{n \times n}$  is nonsingular,  $\mathbf{P} \in \mathbb{R}^{(n+s) \times (n+s)}$  is a permutation matrix and  $\Delta = \text{diag}(\delta_1, \dots, \delta_{n+s})$  is a diagonal matrix with positive diagonal entries. Letting  $\tilde{\Delta} = \text{diag}(\delta_1, \dots, \delta_n) \in \mathbb{R}^{n \times n}$ , we find that  $\tilde{\Delta} \mathbf{B}^{-1} \mathbf{D}_{n,s} \mathbf{P} = [\mathbf{I}_n \quad \tilde{\mathbf{N}}]$ , where  $\text{sgn}(\tilde{\mathbf{N}}) = \text{sgn}(\mathbf{N})$ . It then follows from Proposition 3.4 that  $\tilde{\Delta} \mathbf{B}^{-1} \mathbf{D}_{n,s} \mathbf{P}$  is a network matrix and thus  $\tilde{\Delta} \mathbf{B}^{-1} \mathbf{D}_{n,s} \mathbf{P} = \text{sgn}(\tilde{\Delta} \mathbf{B}^{-1} \mathbf{D}_{n,s} \mathbf{P}) = \text{sgn}(\mathbf{M})$ .  $\square$

Unlike Theorem 4.1 however, property (7) does not provide a full characterization of positive bases. For instance, the PSS  $[\mathbf{I}_2 \quad -\mathbf{e}_1 \quad -\mathbf{1}_2]$  satisfies (7) but is not minimal (the third vector can be removed without losing the positive spanning property).

Deriving a characterization thus requires a more precise study of the IN matrices that correspond to positive bases, which is the purpose of the next sections.

## 5.2 Critical structures for positive spanning sets

A characterization of positive bases was given by Zbigniew Romanowicz based on the concept of critical vectors [24]. Partly due to this concept, this decomposition has proven difficult to use for characterizing positive bases. We redefine the notion of criticality below, and show that it simplifies in the case of NEMs and IN matrices. To this end, we will make use of special cones in  $\mathbb{R}^n$ . For any  $i \in \llbracket 1, n \rrbracket$ , the cone  $\mathcal{K}_i(\mathbb{R}^n) := \text{pspan} \left( \bigcup_{k \neq i} -\mathbf{e}_k \right)$  consist of vectors with non-positive coordinates with at least the  $i^{\text{th}}$  coordinate being equal to zero. For any pair  $(i, j) \in \llbracket 1, n \rrbracket$  with  $i < j$ , the cone  $\mathcal{K}_{i,j}(\mathbb{R}^n) := \text{pspan} \left( \{\mathbf{1}_n\} \cup \bigcup_{k \notin \{i,j\}} \{-\mathbf{e}_k\} \right)$  consists of vectors whose  $i^{\text{th}}$  and  $j^{\text{th}}$  coordinates are both non-negative and equal to the largest entry of the vector. Considering all cones above leads to the definition of critical vectors.

**Definition 5.2 (Critical vectors and critical matrix)** *The set of critical vectors in  $\mathbb{R}^n$ , denoted by  $\mathcal{K}(\mathbb{R}^n)$ , is defined as the union of all  $\{\mathcal{K}_i(\mathbb{R}^n)\}$  and  $\{\mathcal{K}_{i,j}(\mathbb{R}^n)\}$ . A matrix  $\mathbf{X} \in \mathbb{R}^{n \times m}$  is called critical whenever  $\text{pspan}(\mathbf{X}) \subset \mathcal{K}(\mathbb{R}^n)$ .*

We emphasize that the set  $\mathcal{K}(\mathbb{R}^n)$  is not a cone, which partly explains that the notion of critical vector is difficult to manipulate.

We now describe the link between our definition of critical vectors and that of Romanowicz [24]. In the latter, a vector  $\mathbf{x} \in \mathbb{R}^n$  is called critical *with respect to* a positive basis  $\mathbf{D}_{n,s}$  whenever no positive spanning set can be obtained by substituting a column of  $\mathbf{D}_{n,s}$  with the vector  $\mathbf{x}$ . As explained in the proposition below, those definitions are equivalent.

**Proposition 5.1** *A vector  $\mathbf{v} \in \mathbb{R}^n$  is critical in the sense of Definition 5.2 if and only if for all  $k \in \llbracket 1, n+1 \rrbracket$ , the matrix obtained by replacing the  $k^{\text{th}}$  column of  $[\mathbf{I}_n \quad -\mathbf{1}_n]$  with  $\mathbf{v}$  is not positively spanning.*

**Proof.** Let  $\mathbf{v} \in \mathbb{R}^n$ ,  $\mathbf{M}_{n+1} = [\mathbf{I}_n \quad \mathbf{v}]$ . For any  $k \in \llbracket 1, n \rrbracket$ , let  $\mathbf{M}_k$  be the matrix obtained from  $[\mathbf{I}_n \quad -\mathbf{1}_n]$  by replacing its  $k^{\text{th}}$  column with  $\mathbf{v}$ .

Suppose that  $\mathbf{v}$  is a critical vector. On one hand, if  $\mathbf{v} \in \mathcal{K}_i(\mathbb{R}^n)$  for some  $i \in \llbracket 1, n \rrbracket$ , Proposition 3.2 entails that  $\mathbf{M}_{n+1}$  and  $\mathbf{M}_k$  are not positively spanning, as certified by the vectors  $\mathbf{e}_i$  and  $-\mathbf{e}_k$ , respectively. On the other hand, if there exist  $(i, j) \in \llbracket 1, n \rrbracket^2$  with  $i < j$  such that  $\mathbf{v} \in \mathcal{K}_{i,j}(\mathbb{R}^n) \setminus \mathcal{K}_i(\mathbb{R}^n)$ , then  $\mathbf{v}$  satisfies  $[\mathbf{v}]_i = [\mathbf{v}]_j = \max_{\ell \in \llbracket 1, n \rrbracket} [\mathbf{v}]_\ell > 0$ . Without loss of generality, suppose that  $i \neq k$ . Then, Proposition 3.2 certifies that  $\mathbf{M}_{n+1}$  and  $\mathbf{M}_k$  are not positively spanning, using the vectors  $\mathbf{e}_i$  and  $\mathbf{e}_i - \mathbf{e}_k$ , respectively.

Conversely, suppose that  $\mathbf{v}$  is not a critical vector. If all coordinates of  $\mathbf{v}$  are negative, then  $\mathbf{M}_{n+1}$  is a PSS. Otherwise, the maximal coordinate  $[\mathbf{v}]_\ell$  of  $\mathbf{v}$  must be strictly positive and unique since  $\mathbf{v} \notin \mathcal{K}(\mathbb{R}^n)$ . As a result, we can write  $\mathbf{0}_n = \mathbf{v} - [\mathbf{v}]_\ell \mathbf{1}_n + \sum_{k \neq \ell} ([\mathbf{v}]_\ell - [\mathbf{v}]_k) \mathbf{e}_k$ , and thus the zero vector can be expressed as a positive linear combination of the columns of  $\mathbf{M}_\ell$ . Using Proposition 3.1, it follows that the matrix  $\mathbf{M}_\ell$  is a PSS.  $\square$

Our characterization of positive bases will consist in identifying critical matrices within a matrix decomposition, called the critical structure, that applies to any IN matrix.



**Definition 5.3 (Critical structure)** Let  $\mathbf{M} \in \mathbb{R}^{n \times (n+s)}$  be an IN matrix as described in (7). There exist positive integers  $n_1, \dots, n_s$  satisfying  $n_1 + \dots + n_s = n$  and matrices  $\mathbf{X}_1, \dots, \mathbf{X}_{s-1}$  such that

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 \\ \dots \\ \mathbf{M}_s \end{bmatrix}, \quad \text{where} \quad \begin{cases} \mathbf{M}_1 = [\mathbf{I}_{n_1} & \mathbf{O} & -\mathbf{1}_{n_1} & \mathbf{X}_1] \\ \mathbf{M}_i = [\mathbf{O} & \mathbf{I}_{n_i} & \mathbf{O} & -\mathbf{1}_{n_i} & \mathbf{X}_i] \\ \mathbf{M}_s = [\mathbf{O} & \mathbf{I}_{n_s} & -\mathbf{1}_{n_s}] \end{cases} \quad \text{if } i \in \llbracket 2, s-1 \rrbracket \quad (8)$$

in which blocks of zeroes of arbitrary sizes are noted  $\mathbf{O}$ . The structure described by equation (8) is called the critical structure of  $\mathbf{M}$ .

For instance, the matrix from Example 3.1 has the critical structure

$$\begin{bmatrix} \mathbf{M}_{11} \\ \mathbf{M}_{12} \end{bmatrix}, \quad \text{where} \quad \mathbf{M}_{11} = [\mathbf{I}_3 \quad \mathbf{0}_{3,1} \quad -\mathbf{1}_3 \quad -\mathbf{e}_3], \quad \text{and} \quad \mathbf{M}_{12} = [\mathbf{0}_{1,3} \quad 1 \quad 0 \quad -1].$$

When  $s = 1$ ,  $\mathbf{M}$  has the trivial structure  $\mathbf{M} = \mathbf{M}_1 = [\mathbf{I}_n \quad -\mathbf{1}_n]$ , which does not involve any arbitrary block  $\mathbf{X}_i$ , and it is then immediate that  $\mathbf{M}$  is a positive basis. In the general case, however, assessing whether an IN matrix  $\mathbf{M} \in \mathbb{R}^{n \times (n+s)}$  with  $\ell = n$  and  $k = s \geq 2$  is a positive basis amounts to checking the critical nature of the blocks  $\mathbf{X}_1, \dots, \mathbf{X}_{s-1}$ .

**Theorem 5.2** Let  $\mathbf{M} \in \mathbb{R}^{n \times (n+s)}$  be an IN matrix as described in (7). Then,  $\mathbf{M}$  is a positive basis if and only if and each block  $\mathbf{X}_i$  in the critical structure (8) is a critical matrix in  $\mathbb{R}^{n_i}$ .

**Proof.** We first proceed by contrapositive and suppose that there exists  $i \in \llbracket 1, s-1 \rrbracket$  such that  $\mathbf{X}_i$  is not critical. Then, there exist  $j_0 \in \llbracket 1, n_i + 1 \rrbracket$  and  $\mathbf{v} \in \text{pspan}(\mathbf{X}_i)$  such that the matrix obtained by replacing the  $j_0^{\text{th}}$  column of  $[\mathbf{I}_{n_i} \quad -\mathbf{1}_{n_i}]$  with  $\mathbf{v}$  is a PSS. Let  $j$  be the index of this column in  $\mathbf{M}_i$ , let  $\mathbf{m}_j$  be the  $j^{\text{th}}$  column of  $\mathbf{M}$  and let  $\mathbf{M}'$  be the matrix obtained by removing  $\mathbf{m}_j$  from  $\mathbf{M}$ . Consider the matrix  $\mathbf{M}'' = \begin{bmatrix} \mathbf{X}_i \\ \mathbf{O} \end{bmatrix}$  with  $m' = n - m - n_i$  rows, where  $m = n_1 + \dots + n_{i-1}$  so that  $\mathbf{M}''$  is a submatrix of  $\mathbf{M}'$ . Using that  $\mathbf{v} \in \text{pspan}(\mathbf{X}_i)$ , it follows that

$$\mathbf{M}'\mathbf{u} = \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \\ \mathbf{0}_{m'} \end{bmatrix} \quad \text{with} \quad \mathbf{u} \geq \mathbf{0}_{n+s-1} \quad \text{and} \quad \mathbf{x} \in \mathbb{R}^m. \quad (9)$$

Moreover, there exists a PSS  $\hat{\mathbf{M}}$  of  $\mathbb{R}^m$  and a zero matrix  $\mathbf{O}$  such that  $\begin{bmatrix} \hat{\mathbf{M}} \\ \mathbf{O} \end{bmatrix}$  is contained in  $\mathbf{M}'$ .

Therefore equation (9) has a solution for any  $\mathbf{x} \in \mathbb{R}^m$  and in particular, one can choose  $\mathbf{u}$  such that  $\mathbf{M}'\mathbf{u} = \mathbf{m}_j$ . In other words,  $\mathbf{m}_j \in \text{pspan}(\mathbf{M}')$ , thus  $\mathbf{M}'$  is a PSS and  $\mathbf{M}$  is not a positive basis.

Conversely, suppose that all matrices  $\{\mathbf{X}_i\}_{i \in \llbracket 1, s-1 \rrbracket}$  are critical. Let  $\mathbf{c}$  be a column of  $\mathbf{M}$  and

let  $\mathbf{M}' = \begin{bmatrix} \mathbf{M}'_1 \\ \dots \\ \mathbf{M}'_s \end{bmatrix}$  be the matrix obtained from  $\mathbf{M}$  by removing  $\mathbf{c}$ . By construction of  $\mathbf{M}'$ , there

exists  $i \in \llbracket 1, s-1 \rrbracket$  such that one of the columns of  $[\mathbf{I}_{n_i} \quad -\mathbf{1}_{n_i}]$  is not a column of  $\mathbf{M}'_i$ . Let  $\mathbf{v}_i$  denote that column. By criticality of  $\mathbf{X}_i$ ,  $\mathbf{v}_i \notin \text{pspan}(\mathbf{M}'_i)$ , and thus  $\mathbf{M}'_i$  is not a PSS. As a result,  $\mathbf{M}$  is no longer a PSS when one of its columns is removed, and thus it is a positive basis.  $\square$

It follows from Theorem 5.2 that characterizing positive bases in  $\mathbb{R}^n$  reduces to describing critical matrices in  $\mathbb{R}^m$  for all  $m \leq n$ . The next proposition lists such matrices in dimension  $n \leq 2$ , and will be used to characterize positive bases of near-maximal size.

**Proposition 5.2**

- (i)  $\mathbf{X} \in \mathbb{R}^{1 \times m}$  is critical if and only if it is the zero matrix.
- (ii)  $\mathbf{X} \in \mathbb{R}^{2 \times m}$  is critical if and only if  $\text{col}(\mathbf{X})$  is contained in one of the cones defining  $\mathcal{K}(\mathbb{R}^2)$ .

**Proof.** In all cases, the reverse implication is immediate, thus we focus on proving the direct implication. To this end, we recall that a critical matrix  $\mathbf{X} \in \mathbb{R}^{n \times m}$  satisfies  $\text{col}(\mathbf{X}) \subset \text{pspan}(\mathbf{X}) \subset \mathcal{K}(\mathbb{R}^n)$  by definition.

(i) Since  $\mathcal{K}(\mathbb{R}^1) = \{0\}$ , the result is immediate.

(ii) Consider any pair  $\{\mathbf{x}, \mathbf{y}\} \subset \text{col}(\mathbf{X})$ . If  $\mathbf{x} = \mathbf{0}_2$ , then  $\mathbf{x}$  and  $\mathbf{y}$  belong to one of the cones defining  $\mathcal{K}(\mathbb{R}^2)$ . Now, suppose that  $\mathbf{x} \neq \mathbf{0}_2$ . If  $\mathbf{x} \in \mathcal{K}_1(\mathbb{R}^2)$ , then necessarily  $[\mathbf{x}]_2 < 0$ . If  $\mathbf{y} \in \mathcal{K}_2(\mathbb{R}^2) \setminus \mathcal{K}_1(\mathbb{R}^2)$ , then  $\mathbf{x} + \mathbf{y} < \mathbf{0}_2 \notin \mathcal{K}(\mathbb{R}^2)$ , which contradicts the criticality of  $\mathbf{X}$ . Similarly, if  $\mathbf{y} \in \mathcal{K}_{1,2}(\mathbb{R}^2) \setminus \mathcal{K}_1(\mathbb{R}^2)$ , then the vector  $\frac{[\mathbf{y}]_1}{[\mathbf{x}]_2} \mathbf{x} + \mathbf{y}$  is not critical since it is a positive multiple of  $\mathbf{e}_1$ . Thus,  $\mathbf{y} \in \mathcal{K}_1(\mathbb{R}^2)$ . A similar reasoning for  $\mathbf{x} \in \mathcal{K}_2(\mathbb{R}^2)$  and  $\mathbf{x} \in \mathcal{K}_{1,2}(\mathbb{R}^2)$  shows that  $\mathbf{x}$  and  $\mathbf{y}$  must also lie in the same cone among those defining  $\mathcal{K}(\mathbb{R}^2)$ . By the pidgeonhole principle, it follows that all vectors in  $\text{col}(\mathbf{X})$  must lie in one cone among those defining  $\mathcal{K}(\mathbb{R}^2)$ .  $\square$

Proposition 5.2 and Theorem 5.2 can be used to characterize positive bases in  $\mathbb{R}^n$  with critical structure (8) consisting of blocks  $\mathbf{M}_i$  with at most two rows. When all blocks  $\mathbf{M}_i$  have one row, we recover the result of Theorem 5.1 since all  $\mathbf{X}_i$  blocks are necessarily zero matrices. Tackling the other two cases requires the following auxiliary result.

**Lemma 5.2** *Suppose  $n \geq 2$ . Any positive basis  $\mathbf{D}_{n,s}$  is structurally equivalent to an IN matrix as described in (7) whose critical structure (8) satisfies  $\text{col}(\mathbf{X}_i) \subset \mathcal{K}_1(\mathbb{R}^2)$  whenever  $n_i = 2$ .*

**Proof.** Let  $\mathbf{M} \equiv \mathbf{D}_{n,s}$  be an IN matrix satisfying (7). Suppose that one of the blocks  $\mathbf{M}_i$  in its critical structure (8) has two rows and let  $m_i = n_1 + \dots + n_{i-1} + 1$  be the index of its first row in  $\mathbf{M}$ . Since  $\mathbf{D}_{n,s}$  is a positive basis, the block  $\mathbf{X}_i$  is critical for  $\mathbb{R}^2$  per Theorem 5.2. If  $\text{col}(\mathbf{X}_i) \subset \mathcal{K}_1(\mathbb{R}^2)$  there is nothing to prove. Otherwise, we must have  $\text{col}(\mathbf{X}_i) \subset \mathcal{K}_2(\mathbb{R}^2)$  or  $\text{col}(\mathbf{X}_i) \subset \mathcal{K}_{1,2}(\mathbb{R}^2)$  by Proposition 5.2. If  $\text{col}(\mathbf{X}_i) \subset \mathcal{K}_2(\mathbb{R}^2)$ , permuting the rows and columns of indices  $m_i$  and  $m_i + 1$  in  $\mathbf{M}$  (as well as other columns if needed) creates a new matrix  $\mathbf{M}'$  whose critical structure (8) satisfies  $\text{col}(\mathbf{X}'_i) \subset \mathcal{K}_1(\mathbb{R}^2)$  and  $\mathbf{X}'_j = \mathbf{X}_j$  whenever  $j \neq i$ . If  $\text{col}(\mathbf{X}_i) \subset \mathcal{K}_{1,2}(\mathbb{R}^2)$ , replacing the  $(m_i + 1)^{\text{th}}$  row of  $\mathbf{M}$  by its opposite and adding this new row to that of index  $m_i$  creates again a matrix  $\mathbf{M}'$  with the desired critical structure up to further column permutation.  $\square$

We are now equipped with the necessary tools to extend the result of Theorem 5.1 to other sizes of positive bases.

### 5.3 Structure of nearly extreme-size positive bases

Having shown in Section 5.2 how the structure of any positive basis connects to the critical structure of its associated IN matrix, we now apply these results to fully characterize positive bases of size  $2\ell - 1$  and  $\ell + 2$  of an  $\ell$ -dimensional subspace of  $\mathbb{R}^n$ . To our knowledge, those descriptions are novel in both the positive spanning set and the minimal strongly edge-connected digraph literature. For this reason, we state every result using positive bases then provide an equivalent for strongly edge-connected digraphs, starting with the size  $2\ell - 1$ .

**Theorem 5.3** *Let  $n \geq \ell \geq 2$ . A matrix  $\mathbf{D}_{\mathbb{L}, \ell-1} \in \mathbb{R}^{n \times (2\ell-1)}$  is a positive basis for some  $\ell$ -dimensional linear space  $\mathbb{L} \subset \mathbb{R}^n$  if and only if there exists a non-positive vector  $\mathbf{x} \in \mathbb{R}^{\ell-2}$  such that*

$$\mathbf{D}_{\mathbb{L}, \ell-1} \equiv \begin{bmatrix} \mathbf{I}_\ell & \mathbf{N} \\ \mathbf{0}_{n-\ell, \ell} & \mathbf{0}_{n-\ell, \ell-1} \end{bmatrix}, \quad \text{with} \quad \mathbf{N} = \begin{bmatrix} -1 & \mathbf{0}_{\ell-2}^\top \\ -1 & \mathbf{x}^\top \\ \mathbf{0}_{\ell-2} & -\mathbf{I}_{\ell-2} \end{bmatrix}. \quad (10)$$

**Proof.** We only prove the result when  $\ell = n$  as the reasoning easily adapts to all other cases. If  $\ell = n$  and  $\mathbf{D}_{\mathbb{L}, \ell-1}$  satisfies (10), the result holds. Conversely, let  $\mathbf{D}_{n, n-1}$  be a positive basis and  $\mathbf{M} = [\mathbf{I}_n \quad \mathbf{N}]$  be the associated IN matrix through (7). Since  $\mathbf{N} \in \mathbb{R}^{n \times (n-1)}$ , the critical structure (8) of  $\mathbf{M}$  must consist of  $n - 1$  blocks  $\mathbf{M}_i \in \mathbb{R}^{n_i \times (2n-1)}$ . Using  $n_1 + \dots + n_{n-1} = n$ , it follows that exactly one of the blocks  $\mathbf{M}_i$  has two rows while the others have one. By Theorem 5.2 and Proposition 5.2 each block  $\mathbf{X}_i$  must be critical and those with one row are zero matrices. Finally, Lemma 5.2 implies that the unique block  $\mathbf{X}_i$  on two rows can be transformed into a matrix  $\begin{bmatrix} \mathbf{0}_{\ell-2}^\top \\ \mathbf{x}^\top \end{bmatrix}$  where  $\mathbf{x} \in \mathbb{R}^{\ell-2}$  is non-positive, hence the conclusion.  $\square$

An implication of Theorem 5.3 in terms of strongly edge-connected digraphs is given below, and illustrated through Example 5.1.

**Corollary 5.2** *A digraph  $G = (V, A)$  on  $n$  vertices and  $2n - 3$  arcs is minimally strongly edge-connected if and only if it is the union of a bi-directed forest  $F$  with a graph  $G \setminus F$  such that*

- (i)  $G \setminus F$  consists of circuits of size 3, all sharing a common arc  $a \in A$ .
- (ii) Each tree in  $F$  contains exactly one vertex of  $G \setminus F$ .

**Proof.** Suppose that  $G$  is minimally strongly edge-connected and let  $\mathbf{M} = \mathbf{D}_{n-1, n-2}$  in  $\mathbb{R}^{(n-1) \times (2n-3)}$  be an associated network matrix. By Theorem 5.3,  $\mathbf{M} \equiv \tilde{\mathbf{M}} = [\mathbf{I}_{n-1} \quad \mathbf{N}]$ , with

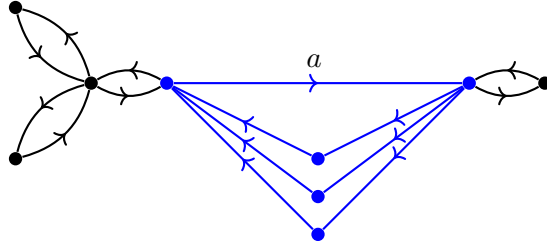
$$\mathbf{N} = \begin{bmatrix} -1 & \mathbf{0}_{n-3}^\top \\ -1 & \mathbf{x}^\top \\ \mathbf{0}_{n-3} & -\mathbf{I}_{n-3} \end{bmatrix} \text{ and } \mathbf{x} \leq \mathbf{0}_{n-3}. \text{ By Lemma 5.1, } \tilde{\mathbf{M}} \text{ can be assumed to be a network matrix,}$$

implying  $\mathbf{x} \in \{-1, 0\}^{n-3}$ . Write  $A = \{a_1, \dots, a_n\}$  so that the  $i^{\text{th}}$  column of  $\tilde{\mathbf{M}}$  is associated to  $a_i$ , for all  $i$ . Recalling that  $\{a_{i_1}, \dots, a_{i_k}\} \subset A$  is a circuit in  $G$  if and only if it is associated to an inclusion-wise minimal set of columns in  $\tilde{\mathbf{M}}$  which add up to  $\mathbf{0}_{n-1}$ . Based on this remark, we observe that  $\{a_1, a_2, a_n\}$  is a circuit of size 3 in  $G$ . Moreover, any other circuit of this size in  $G$  is given by  $\{a_2, a_{i+2}, a_{n+i}\}$  where  $i \in \llbracket 1, n-3 \rrbracket$  satisfies  $[\mathbf{x}]_i = -1$ . We denote by  $G' = (V', A')$  the union of these circuits and we let  $|A'| = m$ . Now, consider the spanning tree  $T = (V, \hat{A})$  associated to  $\tilde{\mathbf{M}}$ .  $T$  can be decomposed as the union of  $T \cap G'$  with a sequence  $(T_j)_{j \leq m}$  of

pairwise disjoint trees, each touching a unique vertex in  $G'$ . It is easy to see that each arc in any tree  $T_j$  is part of a circuit of size 2. Indeed, any such arc  $a_i$  must satisfy  $i \in \llbracket 3, n-1 \rrbracket$  and  $[\mathbf{x}]_{i-2} = 0$ , and as such is contained in a circuit  $\{a_i, a_{n+i-2}\}$ . We have thus established (i) and (ii).

Conversely, if a graph  $G$  admits such a decomposition, it must be edge-connected and  $\tilde{\mathbf{M}}$  is a network matrix for  $G$ . Since  $\tilde{\mathbf{M}}$  is a positive basis, it follows that  $G$  is minimally strongly edge-connected.  $\square$

**Example 5.1** *A minimally strongly edge-connected digraph on 9 vertices and 15 arcs, where  $F$  is in black and  $G \setminus F$  is in blue.*



We now turn to positive bases of size  $\ell + 2$  and the associated strongly edge-connected digraphs.

**Theorem 5.4** *Let  $n \geq \ell \geq 2$ . A matrix  $\mathbf{D}_{\mathbb{L},2} \in \mathbb{R}^{n \times (\ell+2)}$  is a positive basis for some  $\ell$ -dimensional linear space  $\mathbb{L} \subset \mathbb{R}^n$  if and only if there exists a non-positive vector  $\mathbf{x} \in \mathbb{R}^k$ ,  $k \in \llbracket 1, \ell-1 \rrbracket$  satisfying  $[\mathbf{x}]_1 = 0$  such that*

$$\mathbf{D}_{\mathbb{L},2} \equiv \begin{bmatrix} \mathbf{I}_\ell & \mathbf{N} \\ \mathbf{0}_{n-\ell,\ell} & \mathbf{0}_{n-\ell,2} \end{bmatrix} \quad \text{with} \quad \mathbf{N} = \begin{bmatrix} -\mathbf{1}_k & \mathbf{x} \\ \mathbf{0}_{\ell-k} & -\mathbf{1}_{\ell-k} \end{bmatrix}.$$

**Proof.** As for Theorem 5.3, the converse implication is trivial and will not be proved. Moreover, we assume that  $\ell = n$  as the general result is easily deduced from this specific case. Consider a positive basis  $\mathbf{D}_{n,2}$  of size  $n+2$  in  $\mathbb{R}^n$  and any IN matrix  $\mathbf{M} = [\mathbf{I}_n \ \mathbf{N}]$  associated to it through (7) and note  $\mathbf{N} = \begin{bmatrix} -\mathbf{1}_k & \mathbf{w} \\ \mathbf{0}_{n-k} & -\mathbf{1}_{n-k} \end{bmatrix}$  where  $k \in \llbracket 1, n-1 \rrbracket$  and  $\mathbf{w} \in \mathbb{R}^k$  is arbitrary. By Theorem 5.2,  $\mathbf{w}$  is critical: if this vector is non-positive, one of its coordinates must be zero. In this case, the result is proved after permuting rows and columns in  $\mathbf{M}$ .

Otherwise and by definition of a critical vector, the maximal entry of  $\mathbf{w}$  must be non-unique and positive. Let  $[\mathbf{w}]_i = [\mathbf{w}]_j > 0$  be two maximal entries and denote by  $\mathbf{u}$  and  $\mathbf{v}$  the two columns of  $\mathbf{N}$ . One easily checks that  $\mathcal{B} = \{\mathbf{u}, \mathbf{e}_1, \dots, \mathbf{e}_n\} \setminus \{\mathbf{e}_i\}$  is a linear basis for  $\mathbb{R}^n$ , moreover

$$\mathbf{e}_i = -\mathbf{u} - \sum_{\ell=1, \ell \neq i}^k \mathbf{e}_\ell \quad \text{and} \quad \mathbf{v} = -[\mathbf{v}]_i \mathbf{u} + \sum_{\ell=1, \ell \neq i}^k ([\mathbf{v}]_\ell - [\mathbf{v}]_i) \mathbf{e}_\ell - \sum_{\ell=k+1}^n \mathbf{e}_\ell.$$

Note that each coefficient in the linear combination associated to  $\mathbf{e}_i$  is non-positive. Similarly, the linear combination associated to  $\mathbf{v}$  is non-positive as  $[\mathbf{v}]_i = [\mathbf{w}]_i$  is a maximal entry of  $\mathbf{v}$ , and this combination is not strictly negative as  $[\mathbf{v}]_j - [\mathbf{v}]_i = 0$ . In consequence, letting  $\mathbf{B} \in \mathbb{R}^{n \times n}$

satisfy  $\text{col}(\mathbf{B}) = \mathcal{B}$ , we see, up to rows and columns permutation, that the matrix  $\mathbf{B}^{-1}\mathbf{M}$  has the announced structure. Since  $\mathbf{D}_{n,2} \equiv \mathbf{B}^{-1}\mathbf{M}$ , the result is proved.  $\square$

**Corollary 5.3** *A digraph  $G = (V, A)$  on  $n$  vertices and  $n+1$  arcs is minimally strongly edge-connected if and only if it is the union of two circuits whose intersection defines an elementary path - potentially reduced to a single vertex - in  $G$ .*

**Proof.** We only prove the direct implication as the converse is trivial. Let  $\mathbf{M} = \mathbf{D}_{n-1,2}$  in  $\mathbb{R}^{(n-1) \times (n+1)}$  be a network matrix associated to the minimally strongly connected graph  $G$ . By Theorem 5.4,  $\mathbf{M} \equiv \tilde{\mathbf{M}}$  where

$$\tilde{\mathbf{M}} = [\mathbf{I}_{n-1} \quad \mathbf{N}], \quad \text{with} \quad \mathbf{N} = \begin{bmatrix} -\mathbf{1}_k & \mathbf{x} \\ \mathbf{0}_{n-1-k} & -\mathbf{1}_{n-1-k} \end{bmatrix} \quad \text{and} \quad \mathbf{x} \leq \mathbf{0}_k, \quad k \geq 1, \quad [\mathbf{x}]_1 \neq 0.$$

Using Lemma 5.1 we assume  $\tilde{\mathbf{M}}$  to be a network matrix, therefore  $\mathbf{x} \in \{-1, 0\}^k$ . Write  $A = \{a_1, \dots, a_n\}$  and suppose that the  $i^{\text{th}}$  column of  $\tilde{\mathbf{M}}$  is associated to  $a_i$ , for all  $i$ .

Based on the structure of  $\tilde{\mathbf{M}}$  the graph contains a circuit of size  $k+1$ , namely  $\mathcal{C}_1 = \{a_1, \dots, a_k, a_n\}$ . Moreover  $\mathcal{C}_2 = \{a_{i_1}, \dots, a_{i_m}, a_{k+1}, \dots, a_{n-1}, a_{n+1}\}$  is a circuit in  $G$ , where  $\{i_1, \dots, i_m\} \subset \llbracket 2, k \rrbracket$  is the - potentially empty - set of indices  $i$  such that  $[\mathbf{x}]_i = -1$ . We show that  $\mathcal{C}_1 \cap \mathcal{C}_2$  is an elementary path. For  $i \leq 2$ , let  $V_i \subset V$  be the set of extremities of the arcs in  $\mathcal{C}_i$  and let  $v \in V_1$ . Consider the sequence given by  $G_0 = (\{v\}, \emptyset)$ ,  $G_1 = (V_1, \mathcal{C}_1)$ ,  $G_2 = (V_1 \cup V_2, \mathcal{C}_1 \cup \mathcal{C}_2)$ . As  $\mathcal{C}_1 \cup \mathcal{C}_2 = A$ , the sequence defines an ear-decomposition for  $G$ . In particular  $(\mathcal{C}_1 \cup \mathcal{C}_2) \setminus \mathcal{C}_1 = \mathcal{C}_2 \setminus \mathcal{C}_1$  is an ear of  $G$  that defines a  $u-v$  path between two vertices in  $V$ . Then, since  $\mathcal{C}_2$  is a circuit, we find that  $(\mathcal{C}_1 \cup \mathcal{C}_2) \cap \mathcal{C}_1 = \mathcal{C}_1 \cap \mathcal{C}_2$  must be an elementary  $v-u$  path.  $\square$

Theorems 5.1, 5.3 and 5.4 together imply that any positive basis  $\mathbf{D}_{n,s}$ ,  $n \leq 4$  is associated through (7) to an IN matrix  $[\mathbf{I}_n \quad \mathbf{N}]$  where  $\mathbf{N}$  has non-positive entries. Although positive bases can always be generated from such IN matrix structures [23, Theorem 5.4], we emphasize that the equivalence no longer holds in dimension 5 or higher. Indeed, the matrix  $\mathbf{D}_{5,8} = [\mathbf{I}_5 \quad -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 \quad -\mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 \quad \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_5]$  is both a positive basis and a network matrix, but one it cannot be associated through (7) to an IN matrix with a non-positive  $\mathbf{N}$  block.

## 6 Conclusion

We have introduced a matrix decomposition technique inspired by the ear decomposition for strongly connected digraphs, that can be used as a certificate for assessing the positive spanning nature of a matrix. Our study also sheds a new light on the relationship between PSSs and strongly connected digraphs, the latter giving rise to network matrices of the former nature.

Our study can be extended in a number of research directions. A natural continuation of the present work consists in adapting our results to orthogonally structured positive bases or positive  $k$ -spanning sets [13]. Exploiting our decomposition in the context of optimization algorithms is also a future area of investigation.

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