Jordan and isometric cone automorphisms in Euclidean Jordan algebras

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Abstract

Every symmetric cone K arises as the cone of squares in a Euclidean Jordan algebra V. As V is a real inner-product space, we may denote by $\operatorname{Isom}(V)$ its group of isometries. The groups $\operatorname{JAut}(V)$ of its Jordan-algebra automorphisms and $\operatorname{Aut}(K)$ of the linear cone automorphisms are then related. For certain inner products,

$$\operatorname{JAut}(V) = \operatorname{Aut}(K) \cap \operatorname{Isom}(V).$$

We characterize the inner products for which this holds.

1 Introduction

A Euclidean Jordan algebra is a real, finite-dimensional, unital Jordan algebra whose inner product is compatible with the Jordan-algebra multiplication in the sense that $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$ for any x, y, and z. Euclidean Jordan algebras are interesting for (at least) two reasons. First, every symmetric cone arises as the cone of squares in some Euclidean Jordan algebra. This makes them applicable to many conic optimization problems. Second, the Euclidean structure makes it possible to decompose the algebra into simple orthogonal components.

Suppose V is a Euclidean Jordan algebra with multiplication $x \circ y$, unit element 1_V, and some inner product. We will let $K \coloneqq \{x \circ x \mid x \in V\}$ denote its "cone of squares." We are interested in proving the identity

$$\operatorname{JAut}(V) = \operatorname{Aut}(K) \cap \operatorname{Isom}(V) \tag{†}$$

where $\operatorname{JAut}(V)$ is the group of invertible linear $\Phi : V \to V$ satisfying $\Phi(x \circ y) = \Phi(x) \circ \Phi(y)$ for all $x, y \in V$. We call these *Jordan automorphisms*. Identity (†) makes sense only once we impose an inner product on V, and in fact only certain inner products will work.

One of the most popular references for Euclidean Jordan algebras is *Analysis on Symmetric Cones*, by Faraut and Korányi [3]. It is popular for (at least) two reasons: its focus on the Euclidean case, and its treatment of symmetric cones.

Identity (†) can be found in this book, in a remark at the top of page 57, when the inner product is one of the following:

1. $\langle x, y \rangle_{\text{tr}} \coloneqq \operatorname{tr}(x \circ y)$, where $\operatorname{tr}(z)$ denotes the Jordan-algebraic trace of z, the sum of its Jordan-algebraic eigenvalues.

2. $\langle x, y \rangle_{L} \coloneqq \operatorname{trace}(L_{x \circ y})$, where now the usual trace of a linear operator is used, and L_z denotes the map $w \mapsto z \circ w$.

So why do we want to prove an identity that can already be found in a popular reference? The remark in *Analysis on Symmetric Cones* is made without proof, and no hints are given. In a simple algebra, the requisite tools are provided, and one can piece together the result. Gowda, for example, has done this in his work on doubly-stochastic maps in Euclidean Jordan algebras [4]. But if your algebra may not be simple, it is not clear how to proceed.

We revisit the proof with two goals in mind. First and foremost we wish for subsequent results to stand on solid ground. The remark by Faraut and Korányi has been cited many times, including once by the author, who would very much prefer to know that it is true [10]. Beyond that, it desirable to have an elementary proof that takes full advantage of the Euclidean structure. We cannot know what proof Faraut and Korányi had in mind, but the one we give is Euclid-flavored. We begin with the decomposition of an algebra into simple orthogonal components wherein the result is relatively easy to prove, and then afterwards, reassemble the pieces.

Ultimately, we are able to characterize the inner products for which Identity (†) holds. This is a stronger result than is presently available and it explains the counterexamples that are known.

2 Background

2.1 Euclidean Jordan algebras

Naturally, the first three chapters of Faraut and Korányi are our main reference for Euclidean Jordan algebras [3]. A *Euclidean Jordan algebra* is a triplet $(V, \circ, \langle \cdot, \cdot \rangle)$ consisting of a finite-dimensional real vector space V, a commutative bilinear multiplication $(x, y) \mapsto x \circ y$ defined on V with unit element 1_V, and an inner product $\langle \cdot, \cdot \rangle$ on V that is *associative* in the sense that $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$ for all $x, y, z \in V$. With respect to this inner product, the group of isometries on V is $\operatorname{Isom}(V)$.

A "left multiplication by" operator is defined on any Euclidean Jordan algebra by $L_x \coloneqq y \mapsto x \circ y$. The algebra multiplication is bilinear, so both $x \mapsto L_x$ and L_x itself are linear; the associativity of the inner product moreover guarantees that L_x is self-adjoint. (There is no need to define the corresponding right-multiplication-by operator because multiplication is commutative.)

Every element x in a Euclidean Jordan algebra V has a spectral decomposition (Faraut and Korányi, Theorems III.1.1–2), unique in the same sense that the spectral decomposition of a matrix is unique. In particular, x has a unique set of eigenvalues $\{\lambda_1(x), \lambda_2(x), \ldots, \lambda_r(x)\}$, where r is independent of x and is called the *rank* of the algebra. The rank of V is defined to be the dimension of the largest subalgebra generated by a single element of V. Since an element's Jordan-algebraic eigenvalues are unique, we can define a Jordan-algebraic "trace" as their sum:

$$\operatorname{tr}(x) \coloneqq \sum_{i=1}^{\operatorname{rank}(V)} \lambda_i(x).$$

In the algebras of real symmetric and complex Hermitian matrices, these concepts agree with the usual spectral decomposition, eigenvalues, and trace of a matrix. Denote by $\operatorname{trace}(A)$ the standard, linear algebraic trace of the linear operator $A : V \to V$. Then both notions of "trace" induce an associative inner product on V:

$$\langle x, y \rangle_L \coloneqq \operatorname{trace}(L_{x \circ y}),$$

and

$$\langle x, y \rangle_{\operatorname{tr}} \coloneqq \operatorname{tr}(x \circ y).$$

These are clearly symmetric, and $\langle \cdot, \cdot \rangle_L$ is easily seen to be bilinear. The bilinearity of $\langle \cdot, \cdot \rangle_{\operatorname{tr}}$ is subtly proved in Faraut and Korányi's Proposition II.2.1, which declares a polynomial associated with the Jordan-algebraic trace to be homogeneous of degree one. The remaining properties, positive-definiteness and associativity, are Propositions II.4.3 and III.1.5. The inner product $\langle \cdot, \cdot \rangle_{\operatorname{tr}}$ has the pleasing property that primitive idempotents (which are something like orthogonal projections) have norm one in the induced norm. As a result, $\langle \cdot, \cdot \rangle_{\operatorname{tr}}$ is sometimes called canonical.

If $(V, \circ, \langle \cdot, \cdot \rangle_V)$ and $(W, \bullet, \langle \cdot, \cdot \rangle_W)$ are two Euclidean Jordan algebras, then $\Phi : V \to W$ is a *Jordan isomorphism* between them if it is linear, invertible, and if $\Phi(x \circ y) = \Phi(x) \bullet \Phi(y)$ for all $x, y \in V$. Two Euclidean Jordan algebras are *Jordan-isomorphic* if there exists a Jordan isomorphism between them, and a *Jordan automorphism* is a Jordan isomorphism from a Euclidean Jordan algebra to itself. We emphasize that Jordan isomorphisms are without regard for the inner products on either end. The multiplication being understood, we write $\operatorname{JAut}(V)$ for the Jordan automorphism group of V. Jordan automorphisms preserve the Jordan-algebraic trace, and are therefore isometries with respect to $\langle \cdot, \cdot \rangle_{\operatorname{tr}}$. Going a bit further, we can say that a Jordan isomorphism $\Phi : V \to W$ is an isometry if both spaces are endowed with the $\langle \cdot, \cdot \rangle_{\operatorname{tr}}$ inner product. We caution that the symbol $\langle \cdot, \cdot \rangle_{\operatorname{tr}}$ is context-dependent. Jordan-algebraic eigenvalues (and therefore the Jordan-algebraic trace) depend implicitly on the ambient Jordan algebra.

Proposition 1. *If $(V, \circ, \langle \cdot, \cdot \rangle_{\operatorname{tr}})$ and $(W, \bullet, \langle \cdot, \cdot \rangle_{\operatorname{tr}})$ are Euclidean Jordan algebras and if $\Phi : V \to W$ is a Jordan isomorphism, then Φ is also an isometry.*

Proof. The argument that Faraut and Korányi give in Proposition II.4.2 for automorphisms suffices here, but it may be easier to argue anachronistically in terms of the unique spectral decomposition (Theorem III.1.1). Jordan isomorphisms respect multiplication, so $r \coloneqq \operatorname{rank}(V) = \operatorname{rank}(W)$, and Φ sends one complete system of idempotents in V to another in W. As a result, if $x = \sum_{i=1}^{r} \lambda_i(x) e_i$ is the spectral decomposition of x in V, then $\Phi(x) = \sum_{i=1}^{r} \lambda_i(x) \Phi(e_i)$ is the spectral decomposition of $\Phi(x)$ in W. It follows that the eigenvalues of $\Phi(x)$ are $\{\lambda_i(x)\}_{i=1}^{r}$, so $\operatorname{tr}(\Phi(x))$ in W is the same as $\operatorname{tr}(x)$ in V. It is now easy to check that $\langle \Phi(x), \Phi(y) \rangle_{\operatorname{tr}} = \langle x, y \rangle_{\operatorname{tr}}$ for any $x, y \in V$. ☐

The following is a direct result of Proposition III.4.4 in Faraut and Korányi. Recall that any algebra (and in particular, a Euclidean Jordan algebra) is *simple* if it has no nontrivial ideals. Starting with an orthogonal direct sum, we send it to a Cartesian product; this is a Jordan isomorphism. We then rearrange the components to group the Jordan-isomorphism classes—this is also a Jordan isomorphism. Finally, we send the members of each Jordan-isomorphism class to a particular representative. All of these are Jordan isomorphisms, so by composing them, we can achieve all three goals with a single Jordan isomorphism.

Theorem 1 (Simple EJA decomposition). *Every Euclidean Jordan algebra $(V, \circ, \langle \cdot, \cdot \rangle)$ is the orthogonal direct sum of a unique set of nontrivial simple algebras, and is therefore Jordan-isomorphic to a Cartesian product*

$$\Phi(V) = W \coloneqq \bigtimes_{i=1}^{m} W_i \coloneqq \bigtimes_{i=1}^{m} \left(\bigtimes_{j=1}^{m_i} V_i \right)$$

with each V_i nontrivial, simple, and not Jordan-isomorphic to V_ℓ unless $\ell = i$.

The set of factors in the Theorem 1 is unique, but in the Cartesian product, we have destroyed that uniqueness by making two choices. First we have put the factors in some order, and then we have chosen a single representative from each Jordan-isomorphism class. As a result, the Jordan isomorphism Φ cannot be unique except in trivial cases.

The *cone of squares* in V is $\{x \circ x \mid x \in V\}$. We will see shortly that the cone of squares decomposes along the same lines as the algebra, and that in turn leads to a decomposition of the cone's automorphism group.

2.2 Cones

We adopt the standard definition of a convex cone K in a real inner-product space V, always finite-dimensional. Two cones are isomorphic if an invertible linear map sends one to the other. Notably, even if we are in a Euclidean Jordan algebra, cone isomorphisms do not need to be Jordan isomorphisms. The automorphism group of K is thus,

$$\operatorname{Aut}(K) \coloneqq \{A : V \to V \mid A \text{ is linear, invertible, and } A(K) = K\}.$$

If $e \in V$, then the corresponding stabilizer subgroup is

$$\operatorname{Aut}(K)_e = \{A \in \operatorname{Aut}(K) \mid A(e) = e\}.$$

A *symmetric* cone K is a convex cone that is self-dual and *homogeneous*, the latter meaning that $\operatorname{Aut}(K)$ acts transitively on the interior of K. As a consequence of self-duality, symmetric cones contain no subspaces and are both closed and full-dimensional. Such cones are called *proper*.

The cone of squares in a Euclidean Jordan algebra is always symmetric (Faraut and Korányi, Theorem III.2.1). Faraut and Korányi deal with the interior of the cone of squares, whereas we call the entire cone symmetric, but the two share an automorphism group and the definitions are mostly interchangeable.

A convex cone is *reducible* if it is the direct sum of two nontrivial convex cones, and *irreducible* if not. If you start with a reducible cone and reduce it repeatedly, what you wind up with is a direct sum of irreducible cones. The sum is unique (up to order) if the original cone was proper. This result can be found in many places tailored to your desired level of generality:

1. As Theorem 9.3 of Bleicher and Schneider, for pointed convex cones in finitely-generated modules over an ordered ring with the ascending chain condition on left ideals [1].

2. As Proposition III.4.5 of Faraut and Korányi, for symmetric cones [3]. This follows implicitly from the decomposition of a Euclidean Jordan algebra into orthogonal subalgebras having their own symmetric cones of squares.

3. As Theorem 4.3 of Hauser and Güler, for pointed convex cones in Euclidean spaces [6]. We recommend this version. Many things are simpler in a finite-dimensional real space, and we will need the uniqueness of the decomposition that Hauser and Güler make explicit.

Resulting from the decomposition of a proper cone is an analogous decomposition of its automorphism group. Proposition III.4.5 of Faraut and Korányi alludes to this, but again without proof, and only for the identity component of the automorphism group. We need the whole thing, so instead we begin with Horne [7]. To his result we have appended a clause describing the stabilizer subgroups that arose in an earlier paper about Jordan automorphisms [11]. It is an easy consequence of Horne's decomposition.

Theorem 2 (Horne decomposition)**.** *If* $J \coloneqq \bigtimes_{i=1}^{m} J_i$ *is a Cartesian product of nontrivial proper convex cones* $J_i \coloneqq \bigtimes_{j=1}^{m_i} K_i$, *and if each* K_i *is irreducible and not isomorphic to* K_ℓ *unless* $\ell = i$, *then*

$$\operatorname{Aut}(J) = \bigtimes_{i=1}^{m} \operatorname{Aut}(J_i)$$

where

$$\operatorname{Aut}(J_i) = \left(\bigtimes_{j=1}^{m_i} \operatorname{Aut}(K_i) \right) \Sigma^{m_i}$$

and where Σ^{m_i} *denotes the group of permutations of the* m_i *factors of* J_i*. Moreover if* $x = (x_1, x_2, \ldots, x_m) \in J$ *with* $x_i = (\xi_i, \xi_i, \ldots, \xi_i) \in J_i$*, then*

$$\mathrm{Aut}\,(J)_x = \bigtimes_{i=1}^{m} \mathrm{Aut}\,(J_i)_{x_i}$$

where

$$\mathrm{Aut}\,(J_i)_{x_i} = \left(\bigtimes_{j=1}^{m_i} \mathrm{Aut}\,(K_i)_{\xi_i} \right) \Sigma^{m_i}.$$

We have "simplified" the statement of this theorem by assuming that we are starting with a Cartesian product, and that all isomorphic components are equal. No generality is lost in a Euclidean Jordan algebra because the decomposition of the cone will be orthogonal, and the isomorphism that we use to send the direct sum to a Cartesian product can simultaneously be used to pick one representative out of each isomorphism class. If this sounds like the same reasoning we used in the Simple EJA decomposition, it is: we're going to decompose our algebra, along with its cone of squares, before applying Theorem 2. After choosing isomorphism-class representatives and an order for the algebra decomposition, we will take them for granted in the Horne decomposition.

In the interest of exposition, and whereas it is not too difficult, we sketch the proof of Theorem 2 for two factors. Suppose K_1 and K_2 are irreducible proper cones, that $K \coloneqq K_1 \times K_2$ is nontrivial, and that $A \in \mathrm{Aut}\,(K)$. Then the condition that $A\,(K) = K$ can be expanded to

$$A\left(\begin{bmatrix} K_1 \\ \{0\} \end{bmatrix}\right) \oplus A\left(\begin{bmatrix} \{0\} \\ K_2 \end{bmatrix}\right) = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} K_1 \\ \{0\} \end{bmatrix} \oplus \begin{bmatrix} \{0\} \\ K_2 \end{bmatrix}. \tag{1}$$

These are two irreducible decompositions of K. First, both $K_1 \times \{0\}$ and $\{0\} \times K_2$ are irreducible. If one was not, we would be able to decompose it into two nontrivial factors; say,

$$\begin{bmatrix} K_1 \\ \{0\} \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \oplus \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}.$$

This would be possible only if $C_2 = D_2 = \{0\}$, in which case neither C_1 nor D_1 could be trivial—otherwise, one whole factor would be trivial. And if both C_1 and D_1 were nontrivial, then $K_1 = C_1 \oplus D_1$ would contradict the irreducibility of K_1. As a result, no such decomposition is possible, and the same reasoning applies to the K_2 factor. Thus, both $K_1 \times \{0\}$ and $\{0\} \times K_2$ are irreducible. Second, irreducibility is preserved under a linear automorphism, for if $A\,(C) = C_1 \oplus C_2$ has nontrivial factors, then $C = A^{-1}\,(C_1) \oplus A^{-1}\,(C_2)$ does too. Equation (1) thus equates two irreducible decompositions of K.

Citing the uniqueness (up to order) of the irreducible decomposition, and assuming that A has the usual block form, we must have either

$$\left.\begin{array}{c} A\left(\begin{bmatrix} K_1 \\ \{0\} \end{bmatrix}\right) = \begin{bmatrix} A_{11}(K_1) \\ A_{21}(K_1) \end{bmatrix} = \begin{bmatrix} K_1 \\ \{0\} \end{bmatrix} \\ \text{and} \\ A\left(\begin{bmatrix} \{0\} \\ K_2 \end{bmatrix}\right) = \begin{bmatrix} A_{12}(K_2) \\ A_{22}(K_2) \end{bmatrix} = \begin{bmatrix} \{0\} \\ K_2 \end{bmatrix} \end{array}\right\} \implies \begin{cases} A_{11}(K_1) = K_1 \\ A_{22}(K_2) = K_2 \\ A_{21} = A_{12} = 0 \end{cases}$$

or

$$\left.\begin{array}{c} A\left(\begin{bmatrix} K_1 \\ \{0\} \end{bmatrix}\right) = \begin{bmatrix} A_{11}(K_1) \\ A_{21}(K_1) \end{bmatrix} = \begin{bmatrix} \{0\} \\ K_2 \end{bmatrix} \\ \text{and} \\ A\left(\begin{bmatrix} \{0\} \\ K_2 \end{bmatrix}\right) = \begin{bmatrix} A_{12}(K_2) \\ A_{22}(K_2) \end{bmatrix} = \begin{bmatrix} K_1 \\ \{0\} \end{bmatrix} \end{array}\right\} \implies \begin{cases} A_{21}(K_1) = K_2 \\ A_{12}(K_2) = K_1 \\ A_{11} = A_{22} = 0 \end{cases}$$

The first case is always possible, and there it is clear that $A \in \operatorname{Aut}(K_1) \times \operatorname{Aut}(K_2)$. The second case is only possible when K_1 and K_2 are isomorphic, because A_{12} and A_{21} are isomorphisms between them. But recall that if any two factors in *our* decomposition are isomorphic, then they are equal. So the second case is possible only when $K_1 = K_2$. And then it is clear that $A \in [\operatorname{Aut}(K_1) \times \operatorname{Aut}(K_1)]\,\Sigma^2$, where Σ^2 is the permutation group consisting of the identity and the transposition $(x, y) \leftrightarrow (y, x)$. This concludes the decomposition of $\operatorname{Aut}(K)$.

To find a stabilizer subgroup of $\operatorname{Aut}(K)$, just decompose it. If K_1 and K_2 are not isomorphic and if $x = (x_1, x_2)$, then using the Horne decomposition it is easy to see that we must have $\operatorname{Aut}(K)_x \subseteq \operatorname{Aut}(K_1)_{x_1} \times \operatorname{Aut}(K_2)_{x_2}$. Likewise, if $K_1 = K_2$ and if $x = (\xi_1, \xi_1)$, then $\operatorname{Aut}(K)_x \subseteq \left[\operatorname{Aut}(K_1)_{\xi_1} \times \operatorname{Aut}(K_1)_{\xi_1}\right]\Sigma^2$.

3 Characterizing inner products

We now have three decompositions: of a Euclidean Jordan algebra, of its cone of squares, and of the automorphism group of that cone of squares. Recall the identity that we set out to prove:

$$\operatorname{JAut}(V) = \operatorname{Aut}(K) \cap \operatorname{Isom}(V). \tag{$\dagger$}$$

In a moment we will see that $\operatorname{JAut}(V)$ can always be expressed in terms of $\operatorname{Aut}(K)$, though not necessarily in the manner above. As a result, $\operatorname{JAut}(V)$ will decompose along the same lines as $\operatorname{Aut}(K)$ via the Horne decomposition. To demonstrate equality, then, we need to understand how being an isometry interacts with the Horne decomposition.

First, we'll show that Identity (†) holds for the inner product $\langle\cdot,\cdot\rangle_L$. Afterwards we show that on a simple algebra, the inner product $\langle\cdot,\cdot\rangle_{\text{tr}}$ must be a positive multiple of $\langle\cdot,\cdot\rangle_L$. This makes their isometry groups coincide, and it follows that Identity (†) holds for $\langle\cdot,\cdot\rangle_{\text{tr}}$ in a simple algebra, too. Finally we give an example of a non-simple algebra where Identity (†) does not hold, and try to figure out what went wrong. This leads to a characterization.

The keys to these results are the characteristic function of the cone and the duality map (Faraut and Korányi, Section I.3; Güler [5], Sections 3 and 5). The characteristic function was introduced by Koecher [8], and more recently, has been shown by Güler to have a logarithm that is a self-concordant barrier function in the sense of Nesterov and Nemirovskii [5, 9]. Specialized to a symmetric cone K in a Euclidean Jordan algebra $(V, \circ, \langle\cdot,\cdot\rangle)$, the *characteristic function* of K is the map

$$\varphi(x) \coloneqq \int_K e^{-\langle x,y\rangle}\, dy$$

defined on the interior of K. The characteristic function depends on the inner product, but Güler showed that changing the inner product merely scales the characteristic function by a positive multiple [5]. Letting $F \coloneqq \log\varphi$ for the moment, the corresponding *duality map* is $x \mapsto x^*$, where x^* is the unique point in V such that $\langle x^*, h\rangle = DF(x)[h]$ for all h in V. Whereas changing the inner product does not affect $DF(x)$, it *does* affect the duality map. That's the essence of Faraut and Korányi's Proposition I.4.3, restated below: the fixed point of the duality map and the isometries on the ambient space must change together.

Proposition 2. *If K is the cone of squares in a Euclidean Jordan algebra $(V, \circ, \langle\cdot,\cdot\rangle)$ and if φ is its characteristic function, then the duality map has a unique fixed point e, and $\operatorname{Aut}(K)_e = \operatorname{Aut}(K) \cap \operatorname{Isom}(V)$.*

With Identity (†) in mind, we would like to characterize the inner products for which the point e in Proposition 2 will be the unit element of the algebra. When it is, we may draw our conclusion from the following identity [11].

Theorem 3. *If $(V, \circ, \langle\cdot,\cdot\rangle)$ is a Euclidean Jordan algebra with cone of squares K, then*

$$\operatorname{JAut}(V) = \operatorname{Aut}(K)_{1_V}.$$

Not coincidentally, Chua's proof of Theorem 3 shows that under $\langle\cdot,\cdot\rangle_L$, the fixed point in Proposition 2 is 1_V. In fact, he proves the following [2].

Proposition 3. *If $(V, \circ, \langle\cdot,\cdot\rangle_L)$ is a Euclidean Jordan algebra with cone of squares K, then*

$$\operatorname{JAut}(V) = \operatorname{Aut}(K)_{1_V} = \operatorname{Aut}(K) \cap \operatorname{Isom}(V).$$

The first equality remains true if the inner product is changed, so Theorem 3 follows as a consequence. Proposition 3 proves Identity (†) for the inner product $\langle\cdot,\cdot\rangle_L$, but to go beyond $\langle\cdot,\cdot\rangle_L$ is harder. We begin with a simple algebra, where all valid inner products are positive multiples of one another.

Proposition 4 (Faraut and Korányi, III.4.1–2)**.** *If $(V, \circ, \langle\cdot,\cdot\rangle)$ is a nontrivial simple Euclidean Jordan algebra, then there exists a unique $\alpha > 0$ such that $\langle\cdot,\cdot\rangle = \alpha\langle\cdot,\cdot\rangle_{\mathrm{tr}}$. In particular, $\langle\cdot,\cdot\rangle_L = \frac{\dim(V)}{\operatorname{rank}(V)}\langle\cdot,\cdot\rangle_{\mathrm{tr}}$.*

Faraut and Korányi don't claim that α is unique, but it is easy to see that the multiple is unique if and only if V contains a nonzero element. For that reason (and to avoid division by zero) we have added "nontrivial" as a condition of Proposition 4.

Corollary 1. *If $(V, \circ, \langle\cdot,\cdot\rangle)$ is a simple Euclidean Jordan algebra with cone of squares K, then*

$$\operatorname{JAut}(V) = \operatorname{Aut}(K)_{1_V} = \operatorname{Aut}(K) \cap \operatorname{Isom}(V).$$

Proof. Scaling the inner product by a positive amount changes neither the orthogonal group, nor the unit element, nor the cone of squares. The conclusion of Proposition 3 is therefore unchanged if V is nontrivial. If, on the other hand, V is trivial, then so is this result. ☐

The problem now is to extend this to a non-simple algebra. Every algebra is made up of simple components (Theorem 1) where the inner products differ from $\langle\cdot,\cdot\rangle_{\mathrm{tr}}$ by a positive multiple (Proposition 4). But those multiples need not be consistent; Jordan-isomorphic components can have different inner products. When they do, the resulting isometry group will not be the one associated with $\langle\cdot,\cdot\rangle_{\mathrm{tr}}$. The following counterexample—due again to Faraut and Korányi—shows that Corollary 1 can fail even in the simplest non-simple setting.

Example 1. Let $V := \mathbb{R} \times \mathbb{R}$ with componentwise multiplication and inner product

$$\langle (x_1, x_2), (y_1, y_2) \rangle := x_1 y_1 + 2 x_2 y_2.$$

If we define $\Phi : V \to V$ by $\Phi((x_1, x_2)) := (x_2, x_1)$, then $\Phi \in \operatorname{JAut}(V)$, but, for example, Φ does not preserve the norm of $x := (1, 0)$.

It's not too hard to guess what goes wrong in this example. The two simple factors of V are identical, but their inner products are different scalar multiples of $\langle\cdot,\cdot\rangle_{\mathrm{tr}}$. As a result, transposing the two factors is a Jordan automorphism, but not an isometry. The inner product $\langle\cdot,\cdot\rangle_{\mathrm{tr}}$ is invariant under Jordan isomorphisms (Proposition 1), so the same issue would arise if the two simple factors of V were not identical but merely Jordan-isomorphic.

It is a little awkward to talk about the scalar factors being the same in Jordan-isomorphic simple components, but we can characterize this important condition. The intuition is that, by Proposition 1, if U_1 and U_2 are simple and Jordan-isomorphic, then the Jordan isomorphism between them cannot change their respective $\langle\cdot,\cdot\rangle_{\mathrm{tr}}$ inner products. As a result, if the two scalar multiples of $\langle\cdot,\cdot\rangle_{\mathrm{tr}}$ are the same, then the two inner products themselves must be the same under whatever Jordan isomorphism sends one to the other.

Proposition 5. *Suppose $(V, \circ, \langle\cdot,\cdot\rangle)$ is a Euclidean Jordan algebra and that $U_1, U_2 \subseteq V$ are nontrivial simple Jordan-isomorphic subalgebras of V with induced inner products satisfying $\langle\cdot,\cdot\rangle = \alpha_1 \langle\cdot,\cdot\rangle_{\mathrm{tr}}$ on U_1 and $\langle\cdot,\cdot\rangle = \alpha_2 \langle\cdot,\cdot\rangle_{\mathrm{tr}}$ on U_2. Then $\alpha_1 = \alpha_2$ if and only if every Jordan isomorphism $\Psi : U_1 \to U_2$ is an isometry with respect to $\langle\cdot,\cdot\rangle$.*

Proof. If $\alpha_1 = \alpha_2$, then $\langle x, y\rangle = \alpha_1 \langle x, y\rangle_{\mathrm{tr}} = \alpha_2 \langle x, y\rangle_{\mathrm{tr}}$ when $x, y \in U_1$. But Jordan isomorphisms preserve the trace inner product, so for any Jordan isomorphism $\Psi : U_1 \to U_2$,

$$\langle x, y\rangle = \alpha_2 \langle x, y\rangle_{\mathrm{tr}} = \alpha_2 \langle \Psi(x), \Psi(y)\rangle_{\mathrm{tr}} = \langle \Psi(x), \Psi(y)\rangle .$$

Conversely, suppose Ψ is a Jordan isomorphism from U_1 to U_2. By assumption Ψ is an isometry, so for $x, y \in U_1$,

$$\alpha_1 \langle x, y\rangle_{\mathrm{tr}} = \langle x, y\rangle = \langle \Psi(x), \Psi(y)\rangle = \alpha_2 \langle \Psi(x), \Psi(y)\rangle_{\mathrm{tr}} = \alpha_2 \langle x, y\rangle_{\mathrm{tr}} ,$$

implying that $\alpha_1 = \alpha_2$. □

We are almost ready to prove our main result, but we require one more technical lemma. It relates some important transformation groups defined on two Jordan-isomorphic Euclidean Jordan algebras when the Jordan isomorphism connecting them happens to be an isometry.

Lemma 1. *Suppose $(V, \circ, \langle\cdot,\cdot\rangle)$ is a Euclidean Jordan algebra with cone of squares K. If $\Phi : V \to W$ is both a Jordan isomorphism and an isometry, and if $J = \Phi(K)$, then*

$$\begin{aligned}
\Phi \operatorname{Isom}(V)\,\Phi^{-1} &= \operatorname{Isom}(W)\\
\Phi \operatorname{Aut}(K)\,\Phi^{-1} &= \operatorname{Aut}(J)\\
\Phi \operatorname{Aut}(K)_{1_V}\,\Phi^{-1} &= \operatorname{Aut}(J)_{1_W}\\
\Phi\left[\operatorname{Aut}(K) \cap \operatorname{Isom}(V)\right]\Phi^{-1} &= \operatorname{Aut}(J) \cap \operatorname{Isom}(W).
\end{aligned}$$

Theorem 4. *If $(V, \circ, \langle\cdot,\cdot\rangle)$ is a Euclidean Jordan algebra with cone of squares K, then*

$$\operatorname{Aut}(K)_{1_V} = \operatorname{Aut}(K) \cap \operatorname{Isom}(V)$$

if and only if every Jordan isomorphism between simple subalgebras of V is an isometry with respect to $\langle\cdot,\cdot\rangle$.

Proof. Suppose that the latter condition fails, and that we are given as counterexamples the subalgebras $U_1, U_2 \subseteq V$, Jordan isomorphism $\Psi : U_1 \to U_2$, and points $x, y \in U_1$ such that $\langle x, y\rangle \neq \langle \Psi(x), \Psi(y)\rangle$. Let $\Phi : V \to V$ be the map that applies Ψ to U_1 and Ψ^{-1} to U_2 leaving all other factors fixed. Then Φ is a Jordan automorphism, which by Theorem 3 means that $\Phi \in \operatorname{Aut}(K)_{1_V}$. However, $\langle \Phi(x), \Phi(y)\rangle = \langle \Psi(x), \Psi(y)\rangle \neq \langle x, y\rangle$, so Φ is not an isometry on V.

For the converse, we use the Simple EJA decomposition to obtain the Jordan isomorphism Φ such that

$$\Phi(V) = W \coloneqq \bigtimes_{i=1}^{m} W_i \coloneqq \bigtimes_{i=1}^{m}\left(\bigtimes_{j=1}^{m_i} V_i\right)$$

and

$$\Phi(K) = J \coloneqq \bigtimes_{i=1}^{m} J_i \coloneqq \bigtimes_{i=1}^{m}\left(\bigtimes_{j=1}^{m_i} K_i\right).$$

The symmetric cone $J = \Phi(K)$ in W thus satisfies the prerequisites for Theorem 2. (This is more or less the proof that a symmetric cone has an irreducible decomposition into symmetric factors, as in Proposition III.4.5 of Faraut and Korányi.)

On W we impose the inner product $\langle w, z\rangle_W \coloneqq \left\langle \Phi^{-1}(w), \Phi^{-1}(z)\right\rangle$. This ensures that Φ is an isometry from V to W, and that the conjugation identities in Lemma 1 are valid. With Lemma 1 at our disposal, it suffices to prove that

$$\mathrm{Aut}(J)_{1_W} = \mathrm{Aut}(J) \cap \mathrm{Isom}(W),$$

since we can conjugate by Φ to get back to the original identity for V and K.

With that in mind, we begin with the implication $A \notin \mathrm{Aut}(J)_{1_W} \implies A \notin \mathrm{Aut}(J) \cap \mathrm{Isom}(W)$. Suppose that $A \notin \mathrm{Aut}(J)_{1_W}$. Then either $A \notin \mathrm{Aut}(J)$, or $A(1_W) \neq 1_W$. In the first case it would be clear that $A \notin \mathrm{Aut}(J) \cap \mathrm{Isom}(W)$, leaving only the second to consider. From the Horne decomposition,

$$A = A_1 \times A_2 \times \cdots \times A_m,$$

where, for each $i \in \{1, 2, \ldots, m\}$,

$$A_i = \left(B_i^1 \times B_i^2 \times \cdots \times B_i^{m_i}\right) \sigma_i \in \mathrm{Aut}(J_i).$$

If each B_i^j were to fix 1_{V_i}, then so would A; as a result we may conclude (by supposition) that there exist i, j such that $B_i^j(1_{V_i}) \neq 1_{V_i}$. Now $B_i^j \in \mathrm{Aut}(K_i)$, where K_i is the symmetric cone in the simple algebra V_i, so it follows from Corollary 1 that $B_i^j \notin \mathrm{Isom}(V_i)$. We may therefore pick some $\xi_i \in V_i$ whose norm is not preserved under B_i^j. This ξ_i can be used to construct $x_i \coloneqq (0, 0, \ldots, \xi_i, 0, \ldots)$, where the position of ξ_i is such that $\sigma_i(x_i)$ has ξ_i in the jth coordinate and zeros elsewhere. This gives $\|A_i(x_i)\|_W \neq \|x_i\|_W$. If we then let $x \coloneqq (0, 0, \ldots, x_i, 0, \ldots)$ with x_i in the ith position, it is easily seen that $\|A(x)\|_W \neq \|x\|_W$.

All that remains is to prove that $\mathrm{Aut}(J)_{1_W} \subseteq \mathrm{Aut}(J) \cap \mathrm{Isom}(W)$. Supposing $A \in \mathrm{Aut}(J)_{1_W}$, its Horne decomposition is

$$A = A_1 \times A_2 \times \cdots \times A_m,$$

where, for each $i \in \{1, 2, \ldots, m\}$,

$$A_i = \left(B_i^1 \times B_i^2 \times \cdots \times B_i^{m_i}\right) \sigma_i \in \mathrm{Aut}(J_i)_{1_{W_i}}$$

and each B_i^j fixes 1_{V_i}. And since V_i is simple, we may cite Corollary 1 to conclude that each $B_i^j \in \mathrm{JAut}(V_i) = \mathrm{Aut}(K_i) \cap \mathrm{Isom}(V_i)$. We claim that $A_i \in \mathrm{Isom}(W_i)$. To that end, we show that the inner products on the components of W_i agree.

Consider without loss of generality $W_1 = \bigtimes_{j=1}^{m_1} V_1$ whose components are "obviously" simple Jordan-isomorphic subalgebras. For example,

$$V_1 \times \{0\} \times \cdots \times \{0\} \cong \{0\} \times V_1 \times \cdots \times \{0\},$$

and W_1 is the orthogonal direct sum of such things. As a result, any two such components arise from simple Jordan-isomorphic subalgebras U_1 and U_2 of V, say

$$\begin{aligned} V_1 \times \{0\} \times \cdots \times \{0\} &= \Phi(U_1) \\ \{0\} \times V_1 \times \cdots \times \{0\} &= \Phi(U_2). \end{aligned}$$

From this we might conclude that $U_2 = \Phi^{-1}\tau_{12}\Phi(U_1)$, where τ_{12} denotes the transposition of the first two components. If we let $\Psi := \Phi^{-1}\tau_{12}\Phi$, then Ψ is a Jordan isomorphism from U_1 to U_2, and by supposition, an isometry. Recalling that Φ is itself an isometry from V to W, we now obtain

$$\langle \Phi(x), \Phi(y) \rangle_W = \langle \Phi\Psi(x), \Phi\Psi(y) \rangle_W = \langle \tau_{12}\Phi(x), \tau_{12}\Phi(y) \rangle_W$$

for all $x, y \in U_1$. This shows that the inner product on the second component of W_1 is the same as it is on the first component. The same reasoning can be applied to the other components; the inner product is identical on all of them.

Now, let $x_1 = (\xi_1, \xi_2, \ldots, \xi_{m_1}) \in W_1$. As the inner product is identical on the components of W_1, we may disregard the permutation σ_1^{-1} in

$$\begin{aligned} \|A_1(x_1)\|_W^2 &= \left\| \left(B_1^1 \left(\xi_{\sigma_1^{-1}(1)} \right), B_1^2 \left(\xi_{\sigma_1^{-1}(2)} \right), \ldots, B_1^{m_1} \left(\xi_{\sigma_1^{-1}(m_1)} \right) \right) \right\|_W^2 \\ &= \left\| \xi_{\sigma_1^{-1}(1)} \right\|_W^2 + \left\| \xi_{\sigma_1^{-1}(2)} \right\|_W^2 + \cdots + \left\| \xi_{\sigma_1^{-1}(m_1)} \right\|_W^2 \\ &= \|\xi_1\|_W^2 + \|\xi_2\|_W^2 + \cdots + \|\xi_{m_1}\|_W^2 \\ &= \|x_1\|_W^2. \end{aligned}$$

Finally, with each A_i in both $\mathrm{Aut}(J_i)$ and $\mathrm{Isom}(W_i)$, it should be clear that $A \in \mathrm{Aut}(J) \cap \mathrm{Isom}(W)$. □

At long last, we may deduce Identity (†) from Theorems 3 and 4.

Theorem 5. *If $(V, \circ, \langle \cdot, \cdot \rangle)$ is a Euclidean Jordan algebra with cone of squares K, then*

$$\mathrm{JAut}(V) = \mathrm{Aut}(K) \cap \mathrm{Isom}(V)$$

if and only if every Jordan isomorphism between simple subalgebras of V is an isometry with respect to $\langle \cdot, \cdot \rangle$.

In particular, we have a proof of the remark at the top of page 57 in *Analysis on Symmetric Cones* [3].

Corollary 2. *If $(V, \circ, \langle \cdot, \cdot \rangle)$ is a Euclidean Jordan algebra with cone of squares K, and if $\langle \cdot, \cdot \rangle$ is given by either $\langle \cdot, \cdot \rangle_{\mathrm{tr}}$ or $\langle \cdot, \cdot \rangle_L$, then*

$$\mathrm{JAut}(V) = \mathrm{Aut}(K) \cap \mathrm{Isom}(V).$$

Proof. The result for $\langle \cdot, \cdot \rangle_L$ is Proposition 3, and Proposition 1 says that all Jordan isomorphisms are isometries with respect to $\langle \cdot, \cdot \rangle_{\mathrm{tr}}$. □

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