Reduction from the Partition Problem: Dynamic Lot Sizing Problem with Polynomial Complexity

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Abstract

In this note, we polynomially reduce an instance of the partition problem to a dynamic lot sizing problem, and show that solving the latter problem solves the former problem. By solving the dynamic program formulation of the dynamic lot sizing problem, we show that the instance of the partition problem can be solved with pseudo-polynomial time complexity. Numerical results on solving instances of the partition problem are also provided using an implementation of the algorithm that solves the dynamic program. We conclude by discussing polynomial time solvability of the partition problem through further observation on the dynamic program formulation of the dynamic lot sizing problem.

Keywords. Partition problem; dynamic lot sizing model; dynamic program; pseudo-polynomial time complexity; polynomial time complexity.

1 Introduction

An NP-complete problem is in the class of NP, and at the same time, it is NP-hard. In this note, we consider solving a well-known NP-complete problem - the partition problem [3, 4, 5, 6]. We relate the problem to a dynamic lot sizing problem, and by solving the dynamic program formulation of the latter problem, we show that we can solve any instance of the partition problem with pseudo-polynomial time complexity. We conclude this note by discussing the possibility of polynomial time complexity of the partition problem through our approach. Recall from [3] that an algorithm for a problem is a pseudo-polynomial time algorithm for the problem if for a problem instance I of the problem, its time complexity function is bounded above by a polynomial function of two variables Length$[I]$ and Max$[I]$, where Length$[I]$ is an integer that corresponds to the number of symbols used to describe I under some reasonable encoding scheme for the problem, while Max$[I]$ is an integer that corresponds to the magnitude of the largest number in I. On the other hand, an algorithm for a problem is a polynomial time algorithm for the problem if for a problem instance I of the problem, its time complexity function is bounded above by a polynomial function of Length$[I]$.

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1.1 Notations

Let $f(x)$ and $g(x)$ be two nonnegative real-valued functions, where $x \in \mathbb{Z}^{++}$. We write $g(x) = O(f(x))$ to mean that $g(x) \leq Kf(x)$ for some positive constant K and all $x > 0$.

Furthermore, $\mathbb{I}(x)$, where $x \in \mathbb{Z}$, is defined to be 1 for $x > 0$, and 0 otherwise; and $x^+ = \max\{x, 0\}$, where $x \in \Re$. Also, we have $\min\{x, y\} = x - (x - y)^+$ for $x, y \in \Re$.

2 The Partition Problem and its Reduction to a Dynamic Lot Sizing Problem

Let $S = \{1, \ldots, n\}$ and $a_i \in \mathbb{Z}^{++}$ for $i \in S$, with $\sum_{i \in S} a_i = 2C$. The partition problem is to find a subset A of S such that

$$\sum_{i \in A} a_i = \sum_{i \in S \setminus A} a_i = C.$$

It is known that the partition problem is NP-complete [2, 3, 4]. We call an instance of the partition problem **PPi**.

In this section, we reduce **PPi** to a dynamic lot sizing problem in polynomial time and show that solving the dynamic lot sizing problem solves **PPi** in Theorem 2.2. As a consequence, if the dynamic lot sizing problem can be solved with polynomial complexity in some sense, **PPi** can also be solved with polynomial complexity in the same sense.

Dynamic lot sizing problem is introduced in [8], and has since been studied intensively by researchers. We consider a variant of this basic problem which is related to remanufacturing.

In the following, we list down the parameters of the dynamic lot sizing model that we are considering in this note:

Parameters:

- N = number of periods in the time horizon, where $N \geq 1$;
- D_i = demand for serviceable products in the i^{th} period, where $i = 1, \ldots, N$. We assume that $D_i \in \mathbb{Z}^{++}, i = 1, \ldots, N$;
- R_i = returned products as cores at the beginning of the i^{th} period, where $i = 1, \ldots, N$. We assume that $R_i \in \mathbb{Z}^{+}, i = 1, \ldots, N$, and set $R_{N+1} = 0$;
- $K_{r,i}$ = setup cost when there is remanufacturing at the beginning of the i^{th} period, where $i = 1, \ldots, N$. We let $K_{r,i} \geq 0$ for all $i = 1, \ldots, N$, and set $K_{r,N+1} = 0$;
- $\Delta K_{m,i}$ = setup cost when there is manufacturing at the beginning of the i^{th} period, where $i = 1, \ldots, N$. We let $\Delta K_{m,i} \geq 0$ for all $i = 1, \ldots, N$, and set $\Delta K_{m,N+1} = 0$;
- $h_{s,i}$ = unit holding cost of serviceable product over the i^{th} period whether from manufacturing or remanufacturing, where $i = 1, \ldots, N$. We let $h_{s,i} \geq 0$ for all $i = 1, \ldots, N$;
- $h_{c,i}$ = unit holding cost of core over the i^{th} period, where $i = 1, \ldots, N$. We let $h_{c,i} \geq 0$ for all $i = 1, \ldots, N$;

- $c_{r,i}$ = unit remanufacturing cost in the i^{th} period, where $i = 1, \dots, N$. We let $c_{r,i} \geq 0$ for all $i = 2, \dots, N$, and set $c_{r,N+1} = 0$;
- $c_{m,i}$ = unit manufacturing cost in the i^{th} period, where $i = 1, \dots, N$. We let $c_{m,i} \geq 0$ for all $i = 1, \dots, N$, and set $c_{m,N+1} = 0$.

We further impose assumptions on the above parameters as follows:

Assumption 2.1 *(a)* $h_{s,i} + c_{m,i} > \Delta K_{m,i+1} + c_{m,i+1}$ *for* $1 \leq i \leq N$*;*

(b) $h_{s,i} + c_{r,i} > K_{r,i+1} + h_{c,i} + c_{r,i+1}$ *for* $1 \leq i \leq N$*.*

These assumptions are crucial to prove Lemma 3.1, which in turn is needed to formulate the dynamic program for the dynamic lot sizing problem we are considering in this note, and also to solve it efficiently.

Demand must be satisfied in each period in our model. Our objective for the model is to minimize its total cost, which comprises of setup costs for manufacturing and remanufacturing, holding costs for serviceable products and cores, manufacturing and remanufacturing costs. We have the following sequence of events in our model - at the beginning of a period, (i) returned products arrive as cores; (ii) number of units of serviceable products to produce through remanufacturing and manufacturing is determined; (iii) demand in the period is satisfied; (iv) any leftover cores and serviceable products are held to the next period.

The dynamic lot sizing problem (**DLSP**) we are considering is given by:

$$\min \sum_{i=1}^{N} (K_{r,i}\mathbb{I}(x_i) + \Delta K_{m,i}\mathbb{I}(y_i) + c_{r,i}x_i + c_{m,i}y_i + h_{c,i}[J_i - x_i] + h_{s,i}I_{i+1})$$

subject to

$$\begin{aligned}
&J_{i+1} = J_i + R_{i+1} - x_i, \ i = 1, \dots, N, \\
&I_{i+1} = I_i + x_i + y_i - D_i, \ i = 1, \dots, N, \\
&x_i \leq J_i, \ i = 1, \dots, N, \\
&J_i, I_i \geq 0, \ i = 2, \dots, N+1, \\
&x_i, y_i \in \mathbb{Z}^+, \ i = 1, \dots, N, \\
&J_1 = R_1, I_1 = 0.
\end{aligned}$$

The decision variables in the above minimization problem are:

- x_i = number of units of cores remanufactured in the i^{th} period;
- y_i = number of units of serviceable products obtained by manufacturing in the i^{th} period,

while

- J_i = number of units of cores at the beginning of the i^{th} period;
- I_i = number of units of available serviceable products at the beginning of the i^{th} period.

The objective function in the above minimization problem is the total cost of the model. The first constraint tells us the number of units of cores available at the beginning of the $(i+1)^{th}$ period, $i = 1, \dots, N$, after events occurred in the i^{th} period. The second constraint tells us the number of units of serviceable products available at the beginning of the $(i+1)^{th}$ period, $i = 1, \dots, N$, after events occurred in the i^{th} period. The third constraint tells us that the number of cores remanufactured in the i^{th} period cannot exceed the cores available in the period. The fourth constraint tells us that the number of units of cores and serviceable products at the beginning of the i^{th} period are never negative. The next constraint is the sign constraint on the decision variables in the problem, while the last constraint sets specific values on J_1, I_1. Note that the parameters in **DLSP** satisfy Assumption 2.1.

Let us call the optimal value of the minimization problem C^*, and its optimal solution x_i^*, y_i^*, $i = 1, \dots, N$, with $J_i^* = J_{i-1}^* + R_i - x_{i-1}^*, I_i^* = I_{i-1}^* + x_{i-1}^* + y_{i-1}^* - D_{i-1}$, $i = 2, \dots, N+1$, $J_1^* = R_1, I_1^* = 0$.

We reduce **PPi** in polynomial time to the above dynamic lot sizing problem by setting appropriate values for parameters of the model as follows:

- $N = n$;
- $D_i = a_i$, $i = 1, \dots, N (= n)$;
- $R_1 = C$, $R_i = 0$, $i = 2, \dots, N$;
- $K_{r,i} = \Delta K_{m,i} = 1$, $i = 1, \dots, N$;
- $h_{s,i} = 3$, $i = 1, \dots, N$;
- $h_{c,i} = 0$, $i = 1, \dots, N$;
- $c_{r,i} = 0$, $i = 1, \dots, N$;
- $c_{m,i} = 1$, $i = 1, \dots, N$.

It is easy to check that parameters of the model with the above values satisfy Assumption 2.1. We call the dynamic lot sizing problem with these values for its parameters **DLSPp**, and this problem is a special case of **DLSP**. We have the following theorem:

Theorem 2.2 **PPi** *can be solved by solving* **DLSPp**.

Proof: *Claim 1:* Suppose there exists a subset A of $S = \{1, \dots, n\}$ such that

$$\sum_{i \in A} a_i = \sum_{i \in S \setminus A} a_i = C,$$

then the optimal value to **DLSPp** is at most $N + C$.

It is easy to see that by remanufacturing D_i units of cores in the i^{th} period, when $i \in A$, and manufacturing D_i units from raw materials in the i^{th} period, when $i \notin A$, total cost is $N + C$, and it is feasible to **DLSPp**. Hence, the optimal value to **DLSPp** is at most $N + C$.

Claim 2: Suppose the optimal value to **DLSPp** is at most $N + C$. Let A contains elements $i \in S = \{1, \dots, N\}$ such that we remanufacture in the i^{th} period in **DLSPp**. Then we have

$$\sum_{i \in A} a_i = \sum_{i \in S \setminus A} a_i = C.$$

First note that under optimality, whenever we produce, we only produce enough to satisfy demand for the period, and do not hold serviceable products to the next period. To see this, suppose we hold a serviceable product to the next period, then a cost of $h_{s,i} = 3$ is incurred. If we do not produce the serviceable product in the current period, but in the next period, we do not incur the holding cost of $h_{s,i} = 3$ and may even save on its manufacturing cost if this product is obtained by manufacturing, but we incur a possible setup cost of 1 due to remanufacturing or manufacturing, and possible unit manufacturing cost $c_{m,i+1} = 1$ in the next period. In the new setup, total cost is reduced by at least 1, but this contradicts optimality. Hence, under optimality, whenever we produce, we only produce enough to satisfy demand for the period, and do not hold serviceable products to the next period. It is easy to see that all $R_1 = C$ units of cores are remanufactured to satisfy demand since there is no cost for remanufacturing. Note that these cores need not be all remanufactured in the 1^{st} period and they can be held to later periods for remanufacturing without incurring holding cost since $h_{c,i} = 0$. Now, total demand is $\sum_{i=1}^{N} D_i = \sum_{i=1}^{n} a_i = 2C$, and since half of these demands is satisfied through remanfacturing and that all demand has to be satisfied, the other half of these demands has to be satisfied through manufacturing, incurring a total manufacturing cost of C, since $c_{m,i} = 1$. In each period, we always have manufacturing and/or remanufacturing to satisfy demand in the period, as we do not have serviceable products held from earlier periods to satisfy demand in the period. Total setup cost is then at least N. Hence, total cost is at least $N + C$. However, the optimal value to **DLSPp** is at most $N + C$. Therefore, under optimality, we must have total cost is exactly $N + C$, leading to total setup cost to be exactly N, and we either remanufacture or manufacture in a period. Claim 2 then follows. $\square$

Note that **DLSPp** and Theorem 2.2 with the claims in its proof follow [7], while the proof of Claim 2 in the theorem is inspired by [7].

3 Dynamic Program Formulation of Dynamic Lot Sizing Problem and its Solution

We propose a dynamic program formulation of **DLSP** in this section. Before we do this, we state and prove the following lemma that is the key which allows us to have the formulation and then solving it efficiently.

Lemma 3.1 *In* **DLSP**, *suppose we produce in the i^{th} period, where $1 \le i \le N-1$, then $I^*_{i+1} = 0$.*

Proof: We show that $I^*_{i+1} = 0$ by assuming that $I^*_{i+1} \ge 1$, and show that this leads to a contradiction. Since we produce in the i^{th} period, we have manufacturing or remanufacturing or both in the i^{th} period, that is, $x^*_i + y^*_i \ge 1$. Suppose we have remanufacturing in the i^{th} period, that is, $x^*_i \ge 1$. By reducing remanufacturing by 1 unit, noting that demand in period is still satisfied since we assume that $I^*_{i+1} \ge 1$, we have a cost reduction of at least $c_{r,i} + h_{s,i}$, but we incur an additional holding cost of a unit of core of $h_{c,i}$. The unit of serviceable product can be "reinstated" through remanufacturing in the $(i+1)^{th}$ period by incurring a cost of at most $K_{r,i+1} + c_{r,i+1}$. In this case, it is easy to see that the total cost is reduced by at least $c_{r,i} + h_{s,i} - h_{c,i} - K_{r,i+1} - c_{r,i+1}$ which is positive by Assumption 2.1(b). This is a contradiction to optimality. We have a similar argument to show contradiction if we have manufacturing in the i^{th} period using Assumption 2.1(a). Therefore, we show that $I^*_{i+1} = 0$. $\square$

The results in the above lemma is a strong version of the well-known zero-inventory property of the dynamic lot sizing problem, which first appeared in [8]. It says that under optimality, if we produce in the current period, then the optimal inventory policy is to have no serviceable product available at the beginning of the next period. That is, we only produce enough to satisfy demand in the current period.

Corollary 3.2 *In* **DLSP***, suppose we produce in the i^{th} period for some i, $1 \leq i \leq N-1$. Suppose further that $I_i^* = 0$. Then, $x_i^* + y_i^* = D_i$.*

Proof: By Lemma 3.1, we have $I_{i+1}^* = 0$. The result then follows by observing that $I_{i+1}^* = I_i^* + x_i^* + y_i^* - D_i$. □

Remark 3.3 *Under optimality, it is easy to convince ourselves from the proof of Lemma 3.1 that the results in the lemma and Corollary 3.2 still hold if we let $i = N$ in their statements. Furthermore, since $D_i > 0$ for $i = 1, \ldots, N$ and $I_1^* = I_1 = 0$, Lemma 3.1 and Corollary 3.2 imply that we have $I_i^* = 0$, $y_i^* + x_i^* = D_i$ for $i = 1, \ldots, N$. This means that in each period, we always produce and only enough to satisfy demand for the period.*

Let us now consider a dynamic program which we use to solve **DLSP**. The design of the dynamic program is motivated by Lemma 3.1, Corollary 3.2 and Remark 3.3.

For $i = 1, \ldots, N$, and $J_i \geq 0$,

$$\begin{aligned} C_i^{**}(J_i) \quad := \quad & \min\{K_{r,i}\mathbb{I}(x_i) + \Delta K_{m,i}\mathbb{I}(D_i - x_i) + h_{c,i}[J_i - x_i] + c_{r,i}x_i + c_{m,i}[D_i - x_i] + \\ & C_{i+1}^{**}(J_i - x_i + R_{i+1}) \mid x_i \leq J_i, x_i \leq D_i, x_i \in \mathbb{Z}^+\} \end{aligned} \tag{1}$$

We have the convention that $C_{N+1}^{**}(J_{N+1}) = 0$ for all $J_{N+1} \geq 0$.

The first term within the minimization in (1) can be interpreted as the setup cost for remanufacturing in the i^{th} period; the second term is the setup cost for manufacturing in the i^{th} period; the third term is the holding cost during the i^{th} period for cores not remanufactured in the period; the fourth term is the remanufacturing cost for serviceable product in the i^{th} period to satisfy demand in the period; the fifth term is the manufacturing cost for serviceable products in the i^{th} period to satisfy demand in the period; the last term can be interpreted as the optimal cost from the $(i+1)^{th}$ period up to the end of the time horizon.

The following lemma relates the above dynamic program to **DLSP**:

Lemma 3.4 *We have $C^* = C_1^{**}(R_1)$.*

Proof: We are given an optimal solution $x_i^*, y_i^*, i = 1, \ldots, N$, to **DLSP**. We have $I_1^* = 0$ and by Remark 3.3, $I_i^* = 0$ for $i = 2 \ldots, N$ and $x_i^* + y_i^* = D_i$ for $i = 1, \ldots, N$. We also have $J_{i+1}^* = J_i^* - x_i^* + R_{i+1}$, $i = 1, \ldots, N$, where $J_l^* = R_1$. We see that $x_i^*, i = 1, \ldots, N$ is a feasible solution to the dynamic program (1), where $J_1 = R_1$, with its objective function value equal to C^*. Hence, we have $C^* \geq C_1^{**}(R_1)$. On the other hand, given an optimal solution $x_i^{**}, i = 1, \ldots, N$, to the dynamic program (1), where $J_1 = R_1$. It is easy to convince ourselves that $x_i^{**}, D_i - x_i^{**}, i = 1, \ldots, N$, is a feasible solution to **DLSP**, since all its constraints are satisfied, and its objective function value is equal to $C_1^{**}(R_1)$. Therefore $C^* \leq C_1^{**}(R_1)$. The lemma is hence proved. □

By the above lemma, we are able to solve **DLSP** by solving the dynamic program (1) with $J_1 = R_1$. We next describe an algorithm to solve **DLSP** by solving this dynamic program.

Before we do this, we have a lemma below that is the basis for the algorithm and further allows us to show Theorem 3.7:

Lemma 3.5 *When solving* **DLSP** *using the dynamic program (1), where we set $J_1 = R_1$, for all $i = 2, \ldots, N$, we evaluate $C_i^{**}(J_i)$ in (1) for $J_i = \hat{J}_i + R_i$, where $\hat{J}_i$ takes integer value between $((\ldots((R_1-D_1)^+ + (R_2-D_2))^+ + \ldots)^+ + (R_{i-1}-D_{i-1}))^+$ and $R_1+\ldots+R_{i-1}$ inclusively .*

Proof: We prove the statement in the lemma by induction on $i = 2, \ldots, N$. We have from (1) where $i = 1$ that we only need to find $C_2^{**}(J_2)$ for $J_2 = R_1 + R_2 - x_1$, where $x_1 \le \min\{R_1, D_1\} = R_1 - (R_1 - D_1)^+$, $x_1 \in \mathbb{Z}^+$. Therefore, if we let $\hat{J}_2 = R_1 - x_1$, then we have $J_2 = \hat{J}_2 + R_2$, where $\hat{J}_2$ takes integer value between $(R_1 - D_1)^+$ and R_1 inclusively. Hence, statement holds for $i = 2$. Suppose the statement in the lemma holds for $i = i_0$, where $i_0 < N$. We have from (1), where $i = i_0$, that we only need to find $C_{i_0+1}^{**}(J_{i_0+1})$ for $J_{i_0+1} = J_{i_0} - x_{i_0} + R_{i_0+1}$, where $x_{i_0} \le \min\{J_{i_0}, D_{i_0}\} = J_{i_0} - (J_{i_0} - D_{i_0})^+$, $x_{i_0} \in \mathbb{Z}^+$. Therefore, if we let $\hat{J}_{i_0+1} = J_{i_0} - x_{i_0}$, we have $J_{i_0+1} = \hat{J}_{i_0+1} + R_{i_0+1}$, where $\hat{J}_{i_0+1}$ lies between $(J_{i_0} - D_{i_0})^+$ and J_{i_0}. By induction hypothesis, $J_{i_0} = \hat{J}_{i_0} + R_{i_0}$, where $\hat{J}_{i_0}$ takes integer value between $((\ldots((R_1 - D_1)^+ + (R_2 - D_2))^+ + \ldots)^+ + (R_{i_0-1} - D_{i_0-1}))^+$ and $R_1 + \ldots + R_{i_0-1}$ inclusively. Since $\hat{J}_{i_0+1}$ lies between $(\hat{J}_{i_0} + R_{i_0} - D_{i_0})^+$ and $\hat{J}_{i_0} + R_{i_0}$, with $\hat{J}_{i_0}$ taking integer value between $((\ldots((R_1 - D_1)^+ + (R_2 - D_2))^+ + \ldots)^+ + (R_{i_0-1} - D_{i_0-1}))^+$ and $R_1 + \ldots + R_{i_0-1}$ inclusively, we see that the statement in the lemma holds for $i = i_0 + 1$. Therefore, the statement in the lemma holds for all $i = 2, \ldots, N$ by induction. □

Algorithm 3.6

Step 1. *Iterate from $i = N$ to 2, and use previously computed values for $C_{i+1}^{**}(J_{i+1})$, with $C_{N+1}^{**}(J_{N+1}) = 0$, to find $C_i^{**}(J_i)$ from (1) with $J_i = \hat{J}_i + R_i$ with $\hat{J}_i$ taking integer value between $((\ldots((R_1 - D_1)^+ + (R_2 - D_2))^+ + \ldots)^+ + (R_{i-1} - D_{i-1}))^+$ and $R_1 + \ldots + R_{i-1}$ inclusively.*

Step 2. *Find $C_1^{**}(R_1)$ using (1), where $C_2^{**}(\hat{J}_2 + R_2)$ in (1), with $\hat{J}_2$ taking integer value between $(R_1 - D_1)^+$ and R_1 inclusively, have been computed in Step 1.*

After executing the algorithm, we can determine x_i^* for $i = 1, \ldots, N$. If $x_i^* = 0$, then we do not remanufacture when we produce, otherwise, we remanufacture when we produce.

Theorem 3.7 below states the complexity to solve **DLSP** using its dynamic program formulation.

Theorem 3.7 **DLSP** *can be solved using $O\left(\sum_{i=1}^{N}\sum_{k=L_i}^{R_1+\ldots+R_{i-1}}(\min\{k + R_i, D_i\} + 1)^2\right)$, where $L_i = ((\ldots((R_1 - D_1)^+ + (R_2 - D_2))^+ + \ldots)^+ + (R_{i-1} - D_{i-1}))^+$, multiplication, addition and sort operations.*

Proof: We solve **DLSP** using the dynamic program formulation (1) through Algorithm 3.6. For each $i = 1, \ldots, N$, by Lemma 3.5, (1) needs to be solved for $J_i = \hat{J}_i + R_i$, where $\hat{J}_i$ runs from $((\ldots((R_1 - D_1)^+ + (R_2 - D_2))^+ + \ldots)^+ + (R_{i-1} - D_{i-1}))^+$ to $R_1 + \ldots + R_{i-1}$. For each J_i, there are at most $\min\{J_i, D_i\} + 1$ entries to find their mininum. Each entry requires $O(1)$ multiplications, $O(1)$ additions to evaluate, hence leading to a total of $O(\min\{J_i, D_i\} + 1)$ multiplications and additions for all entries. Finding the minimum in the minimization problem requires $O((\min\{J_i, D_i\} + 1)^2)$ operations. Hence, for $i = 1, \ldots, N$, for each J_i, solving (1) requires a total of $O((\min\{J_i, D_i\} + 1)^2)$ multiplication, addition and sort operations. We have

$J_i = \hat{J}_i + R_i$, where $\hat{J}_i$ runs from $((\ldots((R_1 - D_1)^+ + (R_2 - D_2))^+ + \ldots)^+ + (R_{i-1} - D_{i-1}))^+$ to $R_1 + \ldots + R_{i-1}$. Therefore, for each $i = 1, \ldots, N$, the total number of operations to solve $C_i^{**}(J_i)$ in (1) taking into account various values of J_i is $O\left(\sum_{k=L_i}^{R_1+\ldots+R_{i-1}} (\min\{k + R_i, D_i\} + 1)^2\right)$, where $L_i = ((\ldots((R_1 - D_1)^+ + (R_2 - D_2))^+ + \ldots)^+ + (R_{i-1} - D_{i-1}))^+$. Hence, we have a total multiplication, addition and sort operations of $O\left(\sum_{i=1}^{N} \sum_{k=L_i}^{R_1+\ldots+R_{i-1}} (\min\{k + R_i, D_i\} + 1)^2\right)$, where $L_i = ((\ldots((R_1 - D_1)^+ + (R_2 - D_2))^+ + \ldots)^+ + (R_{i-1} - D_{i-1}))^+$, summing i from 1 to N, to solve **DLSP** . $\square$

With the above theorem, we now proceed to find the time complexity to solve **PPi**. Let us denote $a_{\max} = \max_{1 \le i \le n}\{a_i\}$.

Corollary 3.8 **PPi** *can be solved using* $O(n^2 a_{\max}^3)$ *multiplication, addition and sort operations.*

Proof: We have **DLSPp** is a special case of **DLSP** with $N = n$, $D_i = a_i, i = 1, \ldots, N$, $R_1 = C$ and $R_i = 0, i = 2, \ldots, N$. Furthermore, $\sum_{i=1}^{n} a_i = 2C$ which implies that $C \le n a_{\max}$. Let us now apply these on Theorem 3.7 to find the number of multiplication, addition and sort operations needed to solve **PPi**. We have

$$\begin{aligned}
& \sum_{i=1}^{N} \sum_{k=L_i}^{R_1+\ldots+R_{i-1}} (\min\{k + R_i, D_i\} + 1)^2 \\
= \quad & (\min\{R_1, D_1\} + 1)^2 + \sum_{i=2}^{N} \sum_{k=L_i}^{R_1+\ldots+R_{i-1}} (\min\{k + R_i, D_i\} + 1)^2 \\
= \quad & (\min\{C, a_1\} + 1)^2 + \sum_{i=2}^{n} \sum_{k=L_i}^{C} (\min\{k, a_i\} + 1)^2 \\
\le \quad & (a_{\max} + 1)^2 + \sum_{i=2}^{n} \sum_{k=L_i}^{C} (a_{\max} + 1)^2 \\
\le \quad & (a_{\max} + 1)^2 + (n-1)C(a_{\max} + 1)^2 \\
= \quad & O(nCa_{\max}^2).
\end{aligned}$$

The result in the corollary then follows from the above since $C \le n a_{\max}$. $\square$

4 Numerical Studies

We implement Algorithm 3.6 for **DLSP** using Matlab R2024b, and run the resulting Matlab program on a Windows 11 desktop with 13th Gen Intel(R) Core and installed RAM of 16GB. To simplify the implementation, we let $\hat{J}_i$ runs from 0 to $R_1 + \ldots + R_{i-1}$ instead of from $((\ldots((R_1 - D_1)^+ + (R_2 - D_2))^+ + \ldots)^+ + (R_{i-1} - D_{i-1}))^+$ to $R_1 + \ldots + R_{i-1}$ as in the algorithm. We run the Matlab program on **PPi**s in the form of **DLSP**s in our experiments.

For ease in presentation, let us denote $\{a_1, \ldots, a_n\}$ in **PPi** by Ω, where the elements in the set are ordered accordingly to their indices.

In Table 1, we report the outcome upon solving different **PPi** by running the Matlab program which implements Algorithm 3.6. It indicates that the algorithm is able to solve these **PPi**s correctly.

n	Ω	C	A exists?	A	Solution found?	$C_1^{**}(R_1)$
3	$\{10, 34, 40\}$	42	No	-	No	46
3	$\{10, 30, 20\}$	30	Yes	$\{2\}$	Yes	33
5	$\{10, 33, 40, 5, 8\}$	48	Yes	$\{3, 5\}$	Yes	53
5	$\{10, 33, 38, 5, 8\}$	47	No	-	No	53
8	$\{10, 33, 38, 5, 50, 77, 89, 114\}$	208	Yes	$\{4, 7, 8\}$	Yes	216
8	$\{10, 33, 38, 5, 52, 79, 89, 114\}$	210	No	-	No	219
10	$\{10, 33, 38, 5, 8, 10, 6, 7, 11, 8\}$	68	Yes	$\{3, 4, 7, 9, 10\}$	Yes	78
10	$\{10, 33, 40, 5, 8, 10, 6, 7, 11, 8\}$	69	Yes	$\{3, 6, 9, 10\}$	Yes	79

Table 1: Experimentation with an Implementation of Algorithm 3.6 on solving **PPi**

We further test our algorithm on the data sets for **PPi** found in [1]. Results are given in Table 2. As we can see from the table, our algorithm is able to solve all these instances of the partition problem.

Data Set	n	C	Solution found?	A	Time Taken (sec)	$C_1^{**}(R_1)$
P01	10	27	Yes	$\{4, 7, 8, 9, 10\}$	0.0064	37
P02	10	2640	Yes	$\{2, 5, 7, 8, 9, 10\}$	0.2863	2650
P03	9	1419	Yes	$\{5, 6, 9\}$	0.0916	1428
P04	5	32	Yes	$\{2, 4, 5\}$	0.0091	37
P05	9	11	Yes	$\{5, 6, 7, 8, 9\}$	0.0048	20

Table 2: Solutions to Instances of the Partition Problem in [1] using an Implementation of Algorithm 3.6

We next report on the time needed to execute the algorithm for different choices of n. We vary n from 20 to 120, in intervals of 2. For each n, we generate random **PPi**, with a_i, taken from the rounded uniform distribution on $[1, \lfloor 10000/n \rfloor]$ for $i = 1, \ldots, n-1$, and $10000 - \sum_{k=1}^{n-1} a_k$ for $i = n$. By doing this, we have $\sum_{i=1}^{n} a_i = 10000$. In Figure 1, t_n stands for the time taken to execute the algorithm for a given n. We find the time taken for this using the "tic", "toc" features in Matlab. As we can see from the figure, the graph exhibits near constant behavior, with t_n ranging between ~ 0.57 sec to ~ 0.95 sec, and most values lying between ~ 0.61 sec and ~ 0.77 sec. The results seem to indicate no time dependency on n.

We also execute the algorithm for different choices of C. We set $n = 5$ in our experiments. We let C_1 varies from 4000 to 100000 in intervals of 1000. For each C_1, we generate random **PPi**, with a_i, $i = 1, \ldots, 4$, taken from the rounded uniform distribution on $[1, \lfloor C_1/5 \rfloor]$, and we set a_5 to be $C_1 - \sum_{i=1}^{4} a_i$. Therefore, $\sum_{i=1}^{5} a_i = C_1$, and $C = C_1/2$. We report our findings in Figure 2, where t_C stands for the time taken to execute the algorithm for a given C. We see from the figure that the graph exhibits near linear behavior indicating polynomial time dependency on C.

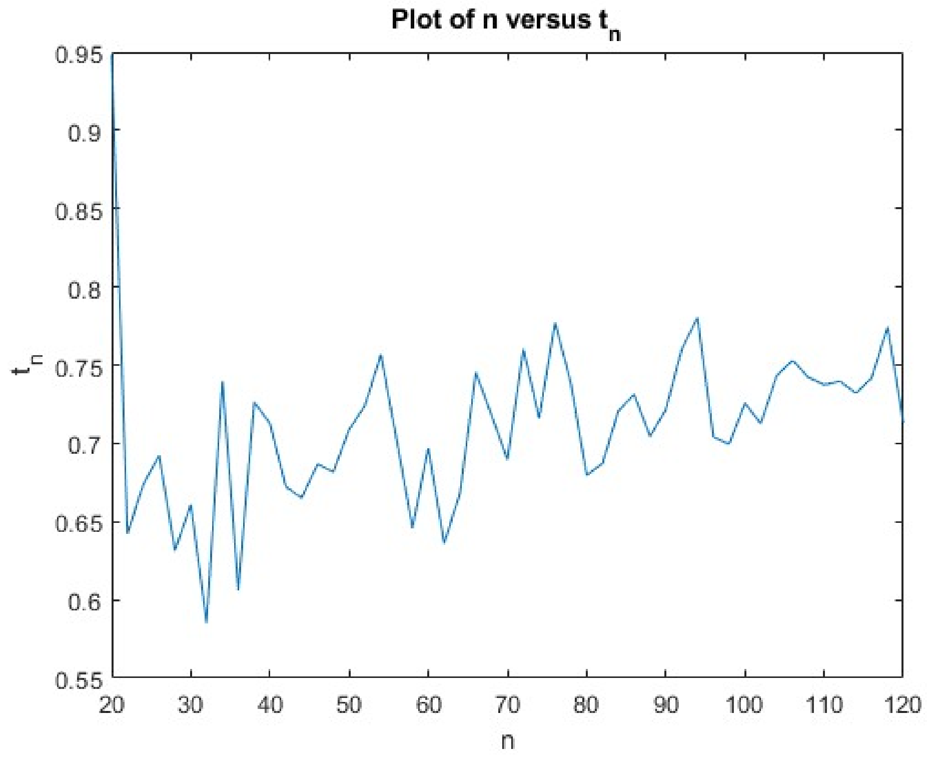

Figure 1: Runtime with n

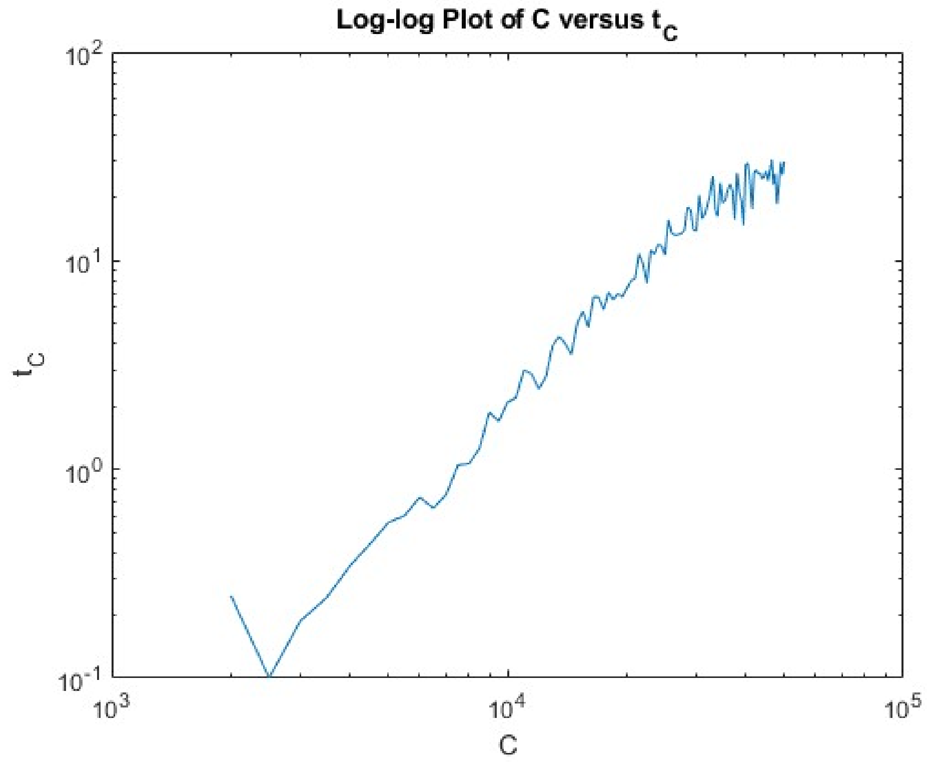

Figure 2: Runtime with C

5 A Discussion

In this section, we discuss the possibility of polynomial time complexity of the partition problem through our approach in this note.

Recall that we formulate a dynamic program for **DLSP** in Section 3 in the form of (1). If we specialize to **DLSPp**, we can further refine (1). In order to do this, we observe the following:

Lemma 5.1 *For* **DLSPp***, suppose we produce in the* i^{th} *period for some* $i, 1 \leq i \leq N$*. Suppose further that* $I_i^* = 0$*. Then we either have* $x_i^* = \min\{J_i, D_i\}, y_i^* = D_i - \min\{J_i, D_i\}$ *or* $x_i^* = 0, y_i^* = D_i$*.*

Proof: Suppose $x_i^* > 0$. We want to show that $x_i^* = \min\{J_i, D_i\}$, and hence $y_i^* = D_i - \min\{J_i, D_i\}$ by Corollary 3.2. We prove this by contradiction. Since x_i^* cannot be greater than J_i or D_i, therefore, we assume that $x_i^* < \min\{J_i, D_i\}$. Since $x_i^* < D_i$, by Corollary 3.2, $y_i^* \geq 1$. We also have $x_i^* < J_i$. Let us consider a new setup when we remanufacture 1 more unit in the i^{th} period and reduce manufacturing by 1 unit in the period. This is allowed since $x_i^* < J_i$ and $y_i^* \geq 1$. It is easy to see that in the new setup, the cost is reduced by at least 1, and this contradicts optimality. Therefore, we conclude that $x_i^* = \min\{J_i, D_i\}$ when $x_i^* > 0$. When $x_i^* = 0$, by Corollary 3.2, $y_i^* = D_i$. □

Using Lemma 5.1 and the specific values of the parameters of **DLSP** for **DLSPp**, we can reformulate the dynamic program (1) in the following way for $i = 1, \ldots, N(= n)$, and $J_i \geq 0$:

$$\begin{aligned} C_i^{**}(J_i) &= \min\Big\{\mathbb{I}(\min\{D_i, J_i\}) + \mathbb{I}(D_i - \min\{D_i, J_i\}) + [D_i - \min\{D_i, J_i\}] + \\ &\qquad C_{i+1}^{**}(J_i - \min\{D_i, J_i\} + R_{i+1}),\ \mathbb{I}(D_i) + D_i + C_{i+1}^{**}(J_i + R_{i+1})\Big\} \\ &= \min\Big\{\mathbb{I}(\min\{D_i, J_i\}) + \mathbb{I}(D_i - \min\{D_i, J_i\}) + [D_i - \min\{D_i, J_i\}] + \\ &\qquad C_{i+1}^{**}(J_i - \min\{D_i, J_i\}),\ 1 + D_i + C_{i+1}^{**}(J_i)\Big\}, \end{aligned} \tag{2}$$

where the second equality follows since for **DLSPp**, $R_i = 0$ for $i = 2, \ldots, n$.

We observe from (2) that in order to find $C_1^{**}(R_1)$, we need to determine the minimum of two expressions that involve $C_2^{**}(J_2)$, where J_2 is dependent on R_1. Note that these expressions can be evaluated in polynomial time in the length of data. Each of these two expressions in turn is the mininum of two expressions that involve $C_3^{**}(J_3)$ based on (2), where J_3 is dependent on J_2. These expressions can again be evaluated in polynomial time in the length of data. Continuing in this fashion until $C_n^{**}(J_n)$, we see that to find $C_1^{**}(R_1)$, we need to go through an exponential number of minimization problems in n that involve two expressions, and this is clearly not time efficient.

Now, let us look at this from another angle by perform "preprocessing" before we actually find $C_1^{**}(R_1)$. This "preprocessing" essentially turn the time inefficiency in finding $C_1^{**}(R_1)$ to a space inefficiency issue so that we can find $C_1^{**}(R_1)$ in polynomial time (in the length of data).

How do we perform this "preprocessing"? Again, we turn to (2). We know that $C_{n+1}^{**}(J_{n+1}) = 0$

for all $J_{n+1} \geq 0$. By (2), we can find a formula for $C_n^{**}(J_n)$ which is given by

$$C_n^{**}(J_n) = \begin{cases} 1 + D_n, & J_n = 0 \\ \mathbb{I}(D_n - 1) + D_n, & J_n = 1 \\ 2 + D_n - J_n, & 1 < J_n < D_n \\ 1, & J_n \geq D_n \end{cases} .$$

There are altogether 4 expressions that comprise the formula for $C_n^{**}(J_n)$. Now, from this formula for $C_n^{**}(J_n)$, we can apply (2) again to find an explicit expression for $C_{n-1}^{**}(J_{n-1})$ for different J_{n-1}. We can determine the number of expressions and the form of each expression that made up this formula for $C_{n-1}^{**}(J_{n-1})$. Continuing in a similar fashion, we can potentially have exponential growth in the number of expressions that made up each formula as we proceed from $i = n$ to $i = n - 1$ till $i = 1$. This is the space inefficiency that we encountered in this "preprocessing" step.

Once these formulae are established in the "preprocessing" step, given $J_1 = R_1$, we can find $C_1^{**}(R_1)$ in polynomial time in the length of data, by evaluating an appropriate expression through finding J_i from J_{i-1}, starting from $J_1 = R_1$, as we proceed to $i = 2$ till $i = n$. Note that it is clear that each of these expressions can be evaluated in polynomial time in the length of data.

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