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Proximity results in convex mixed-integer programming

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Abstract We study *proximity* (resp. *integrality gap*), that is, the distance (resp. difference) between the optimal solutions (resp. optimal values) of convex integer programs (IP) and the optimal solutions (resp. optimal values) of their continuous relaxations. We show that these values can be upper bounded in terms of the recession cone of the feasible region of the continuous relaxation when the recession cone is full-dimensional. If the recession cone is not full-dimensional we give sufficient conditions to obtain a finite integrality gap. We then specialize our analysis to second-order conic IPs. In the case the feasible region is defined by a single Lorentz cone constraint, we give upper bounds on proximity and integrality gap in terms of the data of the problem (the objective function vector, the matrix defining the conic constraint, the right-hand side, and the covering radius of a related lattice). We also give conditions for these bounds to be independent of the right-hand side, akin to the linear IP case. Finally, in the case the feasible region is defined by multiple Lorentz cone constraints, we show that, in general, we cannot give bounds that are independent of the corresponding right-hand side. Although our results are presented for the integer lattice $\mathbb{Z}^n$, the bounds can be easily adapted to work for any general lattice, including the usual mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$, by considering the appropriate covering radius when needed.

Keywords mixed-integer programming · convex programming · conic programming · proximity · integrality gap

Mathematics Subject Classification (2020) 90C11 · 90C25 · 90C30

1 Introduction

Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a convex set and $\alpha \in \mathbb{Z}^n$. Consider the convex integer program (IP)

$$\vartheta_{IP}(\mathcal{S}) := \inf\{\alpha^T x : x \in \mathcal{S} \cap \mathbb{Z}^n\}. \tag{1}$$

Convex IPs have a wide range applications (e.g., in location and inventory management [2,17], power distribution systems [19,16,28], options pricing [25], engineering design [10] and Euclidean k-center problems [8]) but they are challenging to solve in general. As a proxy, one might solve the so-called *continuous relaxation*

$$\vartheta(\mathcal{S}) := \inf\{\alpha^T x : x \in \mathcal{S}\}, \tag{2}$$

which is obtained after relaxing the integrality constraints in (1) and can be efficiently solved under mild conditions. Two natural questions arise:

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– **Proximity**: Assuming that the continuous relaxation (2) is solvable, how to find the closest integral vector in $\mathcal{S}$ to a given optimal solution $\hat{x}$:

$$\mathbf{Prox}_{\hat{x}}(\mathcal{S}) := \min_{x \in \mathcal{S} \cap \mathbb{Z}^n} \|x - \hat{x}\|.$$

– **Integrality Gap**: Assuming that the continuous relaxation has a finite value, how to find the difference between the optimal values:

$$\mathbf{IG}(\mathcal{S}) := \vartheta_{IP}(\mathcal{S}) - \vartheta(\mathcal{S}).$$

The classical motivation to study proximity analysis for IPs is that the integrality gap can be interpreted as a measure to quantify the relaxation quality. Recently, another interesting application of proximity analysis is utilized in the context of pure-integer convex quadratic games in [27].

The two questions posed above have been studied in the literature for linear IPs in which the convex set $\mathcal{S}(\mathbf{u}) = \{x \in \mathbb{R}^n : \mathrm{A}x \geq \mathbf{u}\}$ is a polyhedron defined by integral data. For instance, in [9], the authors provide a bound for a version of the proximity question that depends only on the constraint matrix A and the number of variables n, and is independent of the right-hand side vector $\mathbf{u}$:

Theorem 1 ([9]) *Let Δ denote the largest absolute value of any determinant of a square submatrix of* A*. Assume that $\mathcal{S}(\mathbf{u}) \cap \mathbb{Z}^n \neq \emptyset$ and that $\vartheta(\mathbf{u}) > -\infty$. Then*

(i) For each optimal solution $\hat{x}$ of (2)*, there exists an optimal solution x^* of* (1) *with $\|\hat{x} - x^*\|_\infty \leq n\Delta$.*
(ii) For each optimal solution x^* of (1)*, there exists an optimal solution $\bar{x}$ of* (2) *with $\|x^* - \hat{x}\|_\infty \leq n\Delta$.*

As a consequence of Theorem 1, the *integrality gap* can be bounded above by a constant that is independent of the right-hand side $\mathbf{u}$ as $\mathbf{IG}(\mathbf{u}) \leq n\|c\|_1\Delta$. This result has been recently refined for linear mixed-integer programs (MIPs) where the proximity bounds are shown to depend on the number integer variables [24] or the number of constraints [13, 21] only. Other recent works in this direction focus on integrality gap calculations for integer knapsack problems [1] and random linear IPs [7].

Proximity analysis for nonlinear IPs is limited. To the best of our knowledge, the only examples in this direction involve minimizing a convex separable function [15, 29] or a convex separable quadratic function [14] subject to linear constraints, which can be put in the form (1) through an epigraph formulation (also see [11] for an example with a concave quadratic minimization objective). In such cases, the analysis explicitly uses the polyhedral structure of the feasible region, much like the analysis in [9].

In this paper, we carry out proximity analysis for several special cases of problem (1) involving nonlinear constraints. For example, we analyze the case in which the recession cone of the set $\mathcal{S}$ is full-dimensional or the set $\mathcal{S}$ is the feasible region of a (simple) second-order conic program. We pay special attention to be able derive bounds for proximity and integrality gap similar to [9], in the sense that these bounds are *independent* of the optimal solution of the continuous relaxation $\hat{x}$ or right-hand side vector $\mathbf{u}$ defining the conic set. Our contributions are summarized as follows:

– **Illustrative examples in the nonlinear case**: In Section 3, we provide examples in which the integrality gap for a conic IP defined with rational data is infinite or depends on the right-hand side vector; this is in contrast with the case of linear IPs. These examples give intuition for the types of results we can expect to obtain.
– **Proximity results for general convex IPs**: In Section 4, we prove that if the recession cone of the convex set $\mathcal{S}$ is full-dimensional, then we can find explicit upper bounds for proximity and integrality gap independent of the optimal solution of the continuous relaxation. We also give sufficient conditions for having a finite integrality gap in the case the recession cone is not necessarily full-dimensional.
– **Structural results for simple second-order conic sets**: In Section 5, we obtain structural results for simple second-order conic sets (i.e., ellipsoid, paraboloid, hyperboloid, translated cone) that will be crucial for the proximity analysis later.
– **Proximity results for simple second-order conic IPs**: In Section 6, we derive upper bounds for proximity and integrality gap when $\mathcal{S}$ is a simple second-order conic set. Depending on the set considered and the approach taken, the bounds obtained may or may not depend on the right-hand side vector.
– **A proximity result for a non-simple second-order conic IP**: In Section 7, we obtain an integrality gap bound when $\mathcal{S}$ is the intersection of two spheres based on the ratio of the sum of their radii to the distance between the centers of these spheres.

Remark 1 *Before proceeding with the rest of the paper, we would like to clarify two important points.*

- **Bounds for general (mixed-integer) lattices.** *For simplicity, in the presentation of our results we consider pure integer programs, that is, the lattice in the optimization problem* (1) *is* $\mathbb{Z}^n$*. However, as we will see later, the bounds we obtain for proximity and integrality gap are stated in terms of the covering radius of a certain lattice generated by the integer variables and the data of the problem. If we consider a more general mixed-integer lattice (see Section 2.3) in problem* (1)*, for instance, the mixed-integer lattice* $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$*, our bounds can be easily adapted. The only difference is that the bounds will depend on the covering radius of a mixed-integer lattice generated by both the integer and continuous variables. Interestingly, the covering radius of a mixed-integer lattice only depends on its integer components (see Fact 1 in Section 2.3). Therefore, as in the case of mixed-integer linear programming [24], our bounds only depend on the number of integer variables.*
- **Proximity between optimal solutions.** *We note that there is a subtle difference between our results on proximity and the one in [9]: we consider the distance of the optimal solution of the continuous relaxation* (2) *to the closest feasible solution of convex IP* (1) *whereas the result in [9] involves the distance between the optimal solutions of* (2) *and* (1)*. However, it is easy to see that any proximity bound that we give can be used to find a bound between optimal solutions, and vice versa. We also remark that both proximity results lead to straightforward bounds for the integrality gap.*

2 Preliminaries

2.1 Definitions and notation

We will use the Euclidean norm $\|\cdot\|_2$ as the norm in the definition of proximity unless otherwise stated. For a set X, we will denote its interior as $\text{int}(X)$, its boundary as ∂X, its convex hull as $\text{conv}(X)$, its conic hull as $\text{cone}(X)$. The recession cone of a convex set $\mathcal{S}$ is the set $\text{rec.cone}(\mathcal{S}) = \{d \in \mathbb{R}^n : x + \lambda d \in \mathcal{S} \text{ for all } \lambda \in \mathbb{R}_+,\ x \in \mathcal{S}\}$. The lineality space of a convex set $\mathcal{S}$ is the set $\text{lin.space}(\mathcal{S}) = \{d \in \mathbb{R}^n : x + \lambda d \in \mathcal{S} \text{ for all } \lambda \in \mathbb{R},\ x \in \mathcal{S}\}$.

2.2 Conic programming

A cone $\mathbf{K} \subseteq \mathbb{R}^m$ is called a regular cone if it is full-dimensional, closed, convex, and pointed (it does not contain lines). The conic inequality $a \succeq_{\mathbf{K}} b$ is defined as $a - b \in \mathbf{K}$. If $\mathbf{K}$ is a regular cone, its dual cone is defined as $\mathbf{K}_* = \{x \in \mathbb{R}^n : x^T y \geq 0 \text{ for all } y \in \mathbf{K}\}$ and it can be shown that $\mathbf{K} = (\mathbf{K}_*)_*$.

In addition to general convex IPs, we also study conic IPs, whose feasible region involves a conic representable set $\mathcal{S}(\mathbf{u}) = \{x \in \mathbb{R}^n : \text{A}x \geq_{\mathbf{K}} \mathbf{u}\}$ for a regular cone $\mathbf{K}$. The associated conic IP is defined as $\vartheta_{IP}(\mathbf{u}) := \inf\{\alpha^T x : x \in \mathcal{S}(\mathbf{u}) \cap \mathbb{Z}^n\}$, and its continuous relaxation is given as $\vartheta(\mathbf{u}) := \inf\{\alpha^T x : x \in \mathcal{S}(\mathbf{u})\}$ which is obtained after relaxing the integrality constraints $x \in \mathbb{Z}^n$ in the conic IP. In the case of conic IPs, the values $\vartheta_{IP}(\mathbf{u}) := \vartheta_{IP}(\mathcal{S}(\mathbf{u}))$ and $\vartheta(\mathbf{u}) := \vartheta(\mathcal{S}(\mathbf{u}))$ depend on the right-hand side $\mathbf{u}$. In the case of conic IPs, we define the integrality gap as the function $\mathbf{IG} : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty, -\infty\}$ such that $\mathbf{IG}(\mathbf{u}) = \vartheta_{IP}(\mathbf{u}) - \vartheta(\mathbf{u})$.

An important subclass in conic programming is second-order cone programming, which involves the Lorentz cone (or the second-order cone) as defined below.

Definition 1 *We will denote the Lorentz cone in* $\mathbb{R}^n$ *as* $\mathbf{L}^n := \left\{x \in \mathbb{R}^n : x_n \geq \sqrt{x_1^2 + \cdots + x_{n-1}^2}\right\} = \{x \in \mathbb{R}^n : x^T \tilde{I} x \leq 0, x_n \geq 0\}$*, where* $\tilde{I}$ *is a diagonal* $n \times n$ *matrix with* $\tilde{I}_{11} = \tilde{I}_{22} = \cdots = \tilde{I}_{n-1,n-1} = 1$ *and* $\tilde{I}_{nn} = -1$.

Definition 2 *We denote the subvector obtained from the first* $n-1$ *components of the vector* $x \in \mathbb{R}^n$ *as* $\bar{x}$*. For example,* $x \in \mathbf{L}^n$ *is equivalent to* $\|\bar{x}\|_2 \leq x_n$.

A simple second-order conic set is a set of the form $\mathcal{S} = \{x \in \mathbb{R}^n : \|Ax - b\|_2 \leq c^T x - d\}$, where A is an $(m-1) \times n$ integer matrix and $c \in \mathbb{Z}^n$, $b \in \mathbb{R}^{m-1}$ and $d \in \mathbb{R}$.

2.3 Mixed-integer lattices

Definition 3 (Mixed-integer lattice [6]) *Let* $\mathrm{E} \in \mathbb{R}^{m\times n_1}$ *and* $\mathrm{F} \in \mathbb{R}^{m\times n_2}$*, where* $\begin{bmatrix}\mathrm{E}\ \mathrm{F}\end{bmatrix}$ *has linearly independent columns. Then,* $\mathcal{L}(\mathrm{E},\mathrm{F}) = \{x \in \mathbb{R}^m \,|\, x = \mathrm{E}z + \mathrm{F}y,\ z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$ *is said to be the mixed-integer lattice generated by* E *and* F*. A mixed-integer lattice is said to be full-dimensional if* $\mathrm{rank}([\mathrm{E}\ \mathrm{F}]) = n_1 + n_2$*. If the columns of* E *are in the orthogonal subspace to the linear subspace generated by the columns of* F*, then we say that* $\mathcal{L}(\mathrm{E},\mathrm{F})$ *is an orthogonal representation of the mixed-integer lattice generated by* E *and* F*.*

When $\mathrm{E} \in \mathbb{Q}^{m\times n_1}$ and $\mathrm{F} \in \mathbb{Q}^{m\times n_2}$, then there exists $\mathrm{E}' \in \mathbb{Q}^{m\times n_1}$ such that $\mathcal{L}(\mathrm{E}',\mathrm{F})$ is an orthogonal representation of the mixed-integer lattice $\mathcal{L}(\mathrm{E},\mathrm{F})$ (see, for instance, Proposition 3.11 in [26]).

Definition 4 (Covering radius) *The* covering radius $\mu(\mathrm{E},\mathrm{F})$ *of a full-dimensional mixed-integer lattice* $\mathcal{L}(\mathrm{E},\mathrm{F})$*, is defined as*

$$\mu(\mathrm{E},\mathrm{F}) = \max_x \left\{ \min_{x'} \{\|x - x'\|_2 : x' \in \mathcal{L}(\mathrm{E},\mathrm{F})\} : x \in \mathbb{R}^n \right\}.$$

The following fact states that the covering radius only depends on the structure of the lattice generated by the integer components of the mixed-integer lattice.

Fact 1 *It follows from the definition of covering radius that if* $\mathcal{L}(\mathrm{E},\mathrm{F}) \subseteq \mathbb{R}^n$ *is a full-dimensional mixed-integer lattice with an orthogonal representation, then* $\mu(\mathrm{E},\mathrm{F}) = \mu(\mathrm{E})$ *where* $\mathcal{L}(\mathrm{E}) \subseteq \mathbb{R}^{n_1}$ *is the* n_1*-dimensional lattice obtained by computing the orthogonal projection of* $\mathcal{L}(\mathrm{E},\mathrm{F})$ *onto* $\{\mathrm{E}x : x \in \mathbb{R}^{n_1}\}$*. Indeed, we have*

$$\begin{aligned}
\mu(\mathrm{E},\mathrm{F}) &= \max_{x,u,v} \left\{ \min_{x',z,y} \{\|x - x'\|_2 : x' = \mathrm{E}z + \mathrm{F}y,\ z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\} : x = \mathrm{E}u + \mathrm{F}v,\ u \in \mathbb{R}^{n_1}, v \in \mathbb{R}^{n_2} \right\} \\
&= \max_{u,v} \left\{ \min_{z,y} \{\|\mathrm{E}(u - z)\|_2 + \|\mathrm{F}(v - y)\|_2 : z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\} : u \in \mathbb{R}^{n_1}, v \in \mathbb{R}^{n_2} \right\} \\
&= \max_{u} \left\{ \min_{z} \{\|\mathrm{E}(u - z)\|_2 : z \in \mathbb{R}^{n_1}\} : u \in \mathbb{Z}^{n_1} \right\} = \mu(\mathrm{E}),
\end{aligned}$$

where the first equality follows from $\mathcal{L}(\mathrm{E},\mathrm{F})$ *being full-dimensional, the second equality is a consequence of the the orthogonality of the columns of* E *and* F *and the third equality is obtained by taking* $y = v$*.*

We will often use the following result, which also follows from the definition of covering radius.

Fact 2 *If* $r \geq \mu(\mathrm{E},\mathrm{F})$*, then the (full-dimensional) ellipsoid* $\mathcal{E} = \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \|\mathrm{E}x + \mathrm{F}y - p\|_2 \leq r\}$ *must contain at least one point in* $\mathcal{L}(\mathrm{E},\mathrm{F})$*.*

The following example illustrate the concept of covering radius of a mixed-integer lattice and that it can be computed as the covering radius of the associated lower-dimensional lattice.

Example 1 *Consider the mixed-integer lattice* $\mathcal{L} := \mathcal{L}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right) = \left\{\begin{bmatrix}1\\0\end{bmatrix} x + \begin{bmatrix}1\\1\end{bmatrix} y : x \in \mathbb{Z}, y \in \mathbb{R}\right\}$ *and the point* $w = \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}$*. If the continuous component is kept the same, one of the closest mixed lattice points is* $w^1 = \begin{bmatrix}1\\0\end{bmatrix} 0 + \begin{bmatrix}1\\1\end{bmatrix} 0 = \begin{bmatrix}0\\0\end{bmatrix}$ *with* $\|w - w^1\|_2 = \frac{1}{2}$*, which matches the covering radius of the integer lattice defined by* $\begin{bmatrix}1\\0\end{bmatrix}$*. On the other hand, if the continuous component is optimally chosen, one of the closest mixed lattice points is* $w^2 = \begin{bmatrix}1\\0\end{bmatrix} 0 + \begin{bmatrix}1\\1\end{bmatrix} \frac{1}{4} = \begin{bmatrix}\frac{1}{4}\\\frac{1}{4}\end{bmatrix}$ *with* $\|w - w^2\|_2 = \frac{1}{2\sqrt{2}}$*. In fact, it can be shown that an orthogonal representation for* $\mathcal{L}$ *is* $\mathcal{L}\left(\begin{bmatrix}\frac{1}{2}\\\frac{-1}{2}\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right) = \left\{\begin{bmatrix}\frac{1}{2}\\\frac{-1}{2}\end{bmatrix} x + \begin{bmatrix}1\\1\end{bmatrix} y : x \in \mathbb{Z}, y \in \mathbb{R}\right\}$ *with* $\begin{bmatrix}\frac{1}{2}\\\frac{-1}{2}\end{bmatrix}^T \begin{bmatrix}1\\1\end{bmatrix} = 0$*. Then, the covering radius of* $\mathcal{L}$ *can be simply computed as the covering radius of* $\begin{bmatrix}\frac{1}{2}\\\frac{-1}{2}\end{bmatrix}$*, which is precisely* $\frac{1}{2\sqrt{2}}$*, in other words,* $\mu\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right) = \mu\left(\begin{bmatrix}\frac{1}{2}\\\frac{-1}{2}\end{bmatrix}\right)$*.*

2.4 Integrality gap and lineality space

In this section we show that the lineality space does not play a role in the value of the integrality gap whenever the lineality space is a rational linear subspace.

Lemma 1 *Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a convex set and let $L = \text{lin.space}(\mathcal{S})$ be a rational subspace. Then*

(i) If $c \notin L^\perp$, then $\vartheta(\mathcal{S}) = \vartheta_{IP}(\mathcal{S}) = -\infty$.
(ii) If $c \in L^\perp$, then $\vartheta(\mathcal{S}) = \vartheta(\mathcal{S}')$ and $\vartheta_{IP}(\mathcal{S}) = \vartheta_{IP}(\mathcal{S}')$ for a convex set $\mathcal{S}' \subseteq \mathbb{R}^{\dim(L^\perp)}$ such that $\text{lin.space}(\mathcal{S}') = \{0\}$.

Proof We start by proving (i). It suffices to show that $\vartheta_{IP}(\mathcal{S}) = -\infty$. Since L is a rational linear subspace, without loss of generality, we may assume that there exists $l \in L \cap \mathbb{Z}^n$ such that $c^T l < 0$. Let $z \in \mathcal{S} \cap \mathbb{Z}^n$, then $z + \lambda l \in \mathcal{S} \cap \mathbb{Z}^n$ for any $\lambda \in \mathbb{Z}_+$, as $c^T l < 0$ we have that $c^T(z + \lambda l)$ goes to $-\infty$ as λ increases, so we conclude $\vartheta_{IP}(\mathcal{S}) = -\infty$, as desired.

Now we prove assertion (ii). Let $p = \dim(L^\perp)$. Since L is a rational subspace, then there exists an unimodular matrix $U \in \mathbb{R}^{n\times n}$ such that $UL = \{0\} \times \mathbb{R}^{n-p}$ and $UL^\perp = \mathbb{R}^p \times \{0\}$. Note that since $c \in L^\perp$, we have that $Uc \in \mathbb{R}^p \times \{0\}$. Let c' and $\mathcal{S}'$ the projection of c and $U\mathcal{S}$, respectively, onto the first p variables. Then, $\vartheta_{IP}(\mathcal{S}) = \inf\{c^T x : x \in \mathcal{S} \cap \mathbb{Z}^n\} = \inf\{c^T U^T x : x \in (U\mathcal{S}) \cap \mathbb{Z}^n\} = \inf\{c'^T x : x \in \mathcal{S}' \cap \mathbb{Z}^p\}$, where the second equality uses the fact that $U\mathbb{Z}^n = \mathbb{Z}^n$ and the third equality follows from the optimization problem only depending on the first p variables. Similarly, one can prove that $\vartheta(\mathcal{S}) = \vartheta(\mathcal{S}')$. □

In view of Lemma 1, if a convex set $\mathcal{S}$ has a rational lineality space $L^\perp$, we can compute its integrality gap by projecting $\mathcal{S}$ onto $L^\perp$. Therefore, in the remaining sections, we will assume that the following condition holds:

Assumption 1 *The convex set $\mathcal{S}$ and the conic set $\mathcal{S}(\mathbf{u})$ satisfy the condition* $\text{lin.space}(\mathcal{S}) = \{0\}$ *and* $\text{lin.space}(\mathcal{S}(\mathbf{u})) = \{0\}$.

3 Illustrative examples in the nonlinear case

3.1 Polyhedral approximations do not suffice

We begin this section with the following remark.

Remark 2 *Note that if $P = \{x \in \mathbb{R}^n : \mathrm{A}x \le b\}$ is a rational polyhedral outer-approximation of $\mathcal{S}$ such that $\mathcal{S} \cap \mathbb{Z}^n = P \cap \mathbb{Z}^n$, then*

$$\mathbf{IG}(\mathcal{S}) = \inf\{\alpha^T x : x \in \mathcal{S} \cap \mathbb{Z}^n\} - \inf\{\alpha^T x : x \in \mathcal{S}\} \le \inf\{\alpha^T x : x \in P \cap \mathbb{Z}^n\} - \inf\{\alpha^T x : x \in P\} \le n\|\alpha\|_1 \Delta,$$

where as before Δ denotes the largest absolute value of any determinant of a square submatrix of A.

In virtue of Remark 2, one might be tempted to use a *rational* polyhedral outer-approximation of the second-order cone (see, [18]) and then utilize the proximity results obtained for linear IP with rational data (e.g. [9,24]) to obtain a similar IG bound that is independent of the right-hand side. However, such an approach is not directly viable as we will illustrate in the next example.

Example 2 *Let $B_R := \{x \in \mathbb{R}^n : \|x\|_2 \le R\}$ be ball with radius $R > 0$ and consider the second-order conic IP*

$$\vartheta_{IP}(R) := \inf\{\alpha^T x : x \in B_R \cap \mathbb{Z}^n\}, \tag{3}$$

together with its continuous relaxation $\vartheta(R) := \inf\{\alpha^T x : x \in B_R\}$. As we will see later in Proposition 7, the integrality gap of the second-order conic IP (3) can be bounded as $\mathbf{IG}(R) = \vartheta_{IP}(R) - \vartheta(R) \le \|\alpha\|_2\sqrt{n}$, which is a constant that does not depend on the right-hand side R.

Now, let us consider a $(1+\epsilon)$ rational polyhedral outer-approximation $P_R \subseteq \mathbb{R}^n$ of the ball B_R, that is, $B_R \subseteq P_R \subseteq (1+\epsilon)B_R$, given by an extended formulation of the form proposed in [18]. In order for the approximation to contain the same integer points, that is, $B_R \cap \mathbb{Z}^n = P_R \cap \mathbb{Z}^n$, a rational ϵ should be selected from the interval $(0, \sqrt{1 + 1/R^2} - 1)$ (see [18] for details); notice that ϵ tends to 0 as R increases *so the approximation depends on the right-hand side. Then, the integrality gap of the linear IP* $\vartheta_{IP}{}'(R) := \inf\{\alpha^T x : x \in P_R \cap \mathbb{Z}^n\}$,*and its linear programming relaxation* $\vartheta'(R) := \inf\{\alpha^T x : x \in P_R\}$,*computed as* $\mathbf{IG}'(R) := \vartheta_{IP}{}'(R) - \vartheta'(R)$*, gives an upper bound on* $\mathbf{IG}(R)$ *as* $\vartheta_{IP}{}'(R) = \vartheta_{IP}(R)$ *and* $\vartheta'(R) \leq \vartheta(R)$*. However, the size of the rational polyhedral extended formulation defining* P_R *as well as its constraint matrix entries and the right-hand side vector in its inequality description explicitly depend on* ϵ *(see Section 5 of [18] for a discussion). This suggests that the upper bound for* $\mathbf{IG}'(R)$ *given by Theorem 1 will depend on* R *whereas the upper bound we give for* $\mathbf{IG}(R)$ *does not.*

3.2 Examples of conic IPs with infinite integrality gap

For a general cone $\mathbf{K}$, it could be the case that $\vartheta_{IP}(\mathbf{u}) > -\infty$ but $\vartheta(\mathbf{u}) = -\infty$, so that, $\mathbf{IG}(\mathbf{u}) = +\infty$. This could happen even when the matrix A and vector $\mathbf{u}$ are defined by integral data, as the issue is typically caused by an inherent irrationality of the extreme rays of the cone $\mathbf{K}$ as the following example shows.

Example 3 (A conic representable set with infinite integrality gap) *Consider the set* $\mathcal{S} = \{x \in \mathbb{R}^2 : x = \lambda(1, \sqrt{2}),$ *for some* $\lambda \geq 0\}$ *which is a ray with irrational slope emanating from the origin. Clearly,* $\inf\{-x_1 - x_2 : x \in \mathcal{S}\} = -\infty$*, and* $\inf\{-x_1 - x_2 : x \in \mathcal{S} \cap \mathbb{Z}^2\} = 0$ *and thus* $\mathbf{IG}(\mathbf{u}) = +\infty$*. Note that* $\mathcal{S}$ *can be represented as a conic set of the form*

$$\mathcal{S} = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} x \succeq_{\mathbf{K}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

either by using the polyhedral cone $\mathbf{K} = \text{cone}(\{(1, \sqrt{2}, 0); (1, 0, 1); (0, 1, 1)\})$ *or by using the cone* $\mathbf{K} = \text{cone}(\{x \in \mathbb{R}^3 : (x_1 - 1)^2 + (x_2 - \sqrt{2})^2 + (x_3 - 1)^2 \leq 1\})$ *which is generated by an ellipsoid.*

Unfortunately, even if we restrict ourselves to second-order conic IPs defined by rational data, the integrality gap might still be infinitely large as illustrated by the following example taken from [23]:

Example 4 (Another conic representable set with infinite integrality gap) *Consider the set*

$$\mathcal{S} = \text{conv}(\{x \in \mathbb{R}^3 : x_3 = 0, x_1 = 0, x_2 \geq 0\} \cup \{x \in \mathbb{R}^3 : x_3 = \epsilon, x_2 \geq x_1^2\} \cup \{x \in \mathbb{R}^3 : x_3 = 1, x_1 = 0, x_2 \geq 0\}),$$

with $\epsilon \in (0, 1)$*. It can be shown that the following is a second-order conic representation (SOCr) of the set* $\mathcal{S}$ *in an extended space:*

$$\begin{aligned} & x_{3,0} = 0, x_{1,0} = 0, x_{2,0} \geq 0; \quad x_{3,\epsilon} = \epsilon\lambda_\epsilon, \begin{bmatrix} 2x_{1,\epsilon} \\ x_{2,\epsilon} \\ x_{2,\epsilon} \end{bmatrix} \succeq_{\mathbf{L}^3} \begin{bmatrix} 0 \\ \lambda_{.5} \\ -\lambda_\epsilon \end{bmatrix}, x_{2,\epsilon} \geq 0; \quad x_{3,1} = \lambda_1, x_{1,1} = 0, x_{2,1} \geq 0; \\ & x_j = x_{j,0} + x_{j,\epsilon} + x_{j,1} \ j = 1, 2, 3; \quad \lambda_0 + \lambda_\epsilon + \lambda_1 = 1, \lambda_0, \lambda_\epsilon, \lambda_1 \geq 0. \end{aligned}$$

However, since $\inf\{x_1 : x \in \mathcal{S}\} = -\infty$ *and* $\inf\{x_1 : x \in \mathcal{S} \cap \mathbb{Z}^3\} = 0 > -\infty$*, we obtain that* $\mathbf{IG}(u) = +\infty$*.*

3.3 Example of a simple second-order conic IP with right-hand side dependent integrality gap

In the example below, the feasible region of the second-order conic IP is a parabola in the plane and the data is defined with rational (but not integer) numbers. This example shows that even for a second-order conic IP defined by rational data and with only two variables, the integrality gap may depend on the right-hand side (b, d) and not only on A, c, α.

Example 5 *Consider the following second-order conic IP:*

$$\inf_{x \in \mathbb{Z}^2} \left\{ \alpha_1 x_1 + \alpha_2 x_2 : \left\| \begin{bmatrix} x_1 - b_1 \\ \frac{1}{2} x_2 - b_2 \end{bmatrix} \right\|_2 \leq \frac{1}{2} x_2 - d \right\}.$$

Notice that the constraint can be written as $(x_1 - b_1)^2 + (b_2^2 - d^2) \leq (b_2 - d)x_2$. *Assuming that the continuous relaxation is bounded below, its optimal solution* $\hat{x}$ *can be computed as* $\hat{x}_1 = b_1 - \frac{1}{2}\frac{\alpha_1(b_2-d)}{\alpha_2}$ *and* $\hat{x}_2 = \frac{(x_1^*-b_1)^2}{b_2-d} + b_2 + d = \frac{1}{4}\frac{\alpha_1^2(b_2-d)}{\alpha_2^2} + b_2 + d$. *Now, let* $N \in \mathbb{Z}_{++}$ *and consider the following instance:*

$$\alpha = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ b = \begin{bmatrix} 4N + \frac{1}{2} \\ 4N \end{bmatrix}, \ d = 4N - \frac{1}{4N}. \tag{4}$$

One can verify that the optimal solution of the continuous relaxation is $\hat{x}_1 = 4N + \frac{1}{2}\left(1 - \frac{1}{4N}\right)$ *and* $\hat{x}_2 = 8N - \frac{3}{16N}$. *It is easy to see that the optimal solution of the integer program is* $x_1^* = \lfloor \hat{x}_1 \rfloor = 4N$ *and* $x_2^* = \left\lceil \frac{(x_1^*-b_1)^2}{b_2-d} + b_2 + d \right\rceil = \left\lceil 9N - \frac{1}{4N} \right\rceil = 9N$. *Finally, we obtain the integrality gap, as a function of* N*:* $\mathbf{IG}(N) = (x_1^* + x_2^*) - (\hat{x}_1 + \hat{x}_2) = N + \frac{5}{16N} - \frac{1}{2}$.

3.4 Examples of non-simple second-order conic IPs with right-hand side dependent integrality gap

In this section, we provide two examples of non-simple second-order conic IPs whose integrality gap depends on the right-hand side unlike the case of linear IPs (see Theorem 1). Throughout this section, we will denote an optimal solution of the conic IP and an optimal solution of its continuous relaxation as x^* and $\hat{x}$, respectively (when there are multiple optimal solutions, we only consider one such solution).

In the next example, the feasible region of the conic IP is the intersection of a half-space, an ellipsoid and the standard lattice. This is arguably one of the "simplest" nonlinear convex IP in which the integrality gap depends on the right-hand side.

Example 6 *Let* $N \in \mathbb{Z}_+$ *and consider the following second-order conic IP:*

$$\inf_{x \in \mathbb{Z}^2} \{x_2 : x_1^2 + x_2^2 \leq (N+1)^2, \ x_1 \geq 1/2\} = \inf_{x \in \mathbb{Z}^2} \left\{ x_2 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \succeq_{\mathbf{L}^3} \begin{bmatrix} 0 \\ 0 \\ -(N+1) \end{bmatrix}, \ x_1 \geq \frac{1}{2} \right\}.$$

Since we have $\hat{x} = \left(N + \frac{1}{2}, -\sqrt{N + \frac{3}{4}}\right)$ *and* $x^* = (N+1, 0)$, *the integrality gap, as a function of* N, *is computed as* $\mathbf{IG}(N) = \sqrt{N + \frac{3}{4}}$.

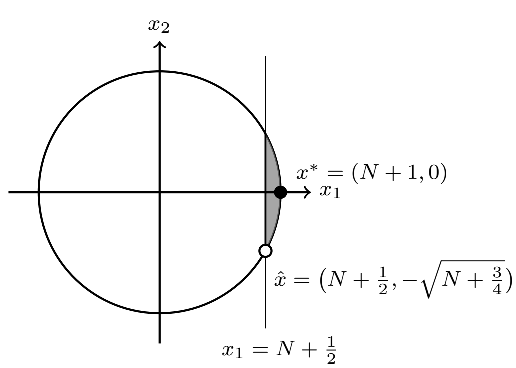

Fig. 1 The feasible region of the continuous relaxation of the conic IP in Example 6.

In the next example, the feasible region of the conic IP, is the intersection of two non-degenerate ellipsoids with the standard lattice. This example is interesting for at least two reasons: i) It features two bounded nonlinear sets, and ii) the integer hull of the feasible region is full-dimensional.

Example 7 *Let $N \in \mathbb{Z}_+$ and consider the following second-order conic IP:*

$$\inf_{x\in\mathbb{Z}^2}\{x_2 : (x_1-(1-N))^2+(x_2-1/2)^2 \le N^2+1/4,\ (x_1-N)^2+(x_2-1/2)^2 \le N^2+1/4\}$$
$$= \inf_{x\in\mathbb{Z}^2}\left\{x_2 : \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} \succeq_{\mathbf{L}^3} \begin{bmatrix}1-N\\1/2\\-\sqrt{N^2+1/4}\end{bmatrix},\ \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} \succeq_{\mathbf{L}^3} \begin{bmatrix}N\\1/2\\-\sqrt{N^2+1/4}\end{bmatrix}\right\}.$$

Since we have $\hat{x} = \left(\frac{1}{2}, \frac{1}{2} - \sqrt{N}\right)$ and $x^* = (0,0)$, the integrality gap, as a function of N, is computed as $\mathbf{IG}(N) = \sqrt{N} - \frac{1}{2}$.

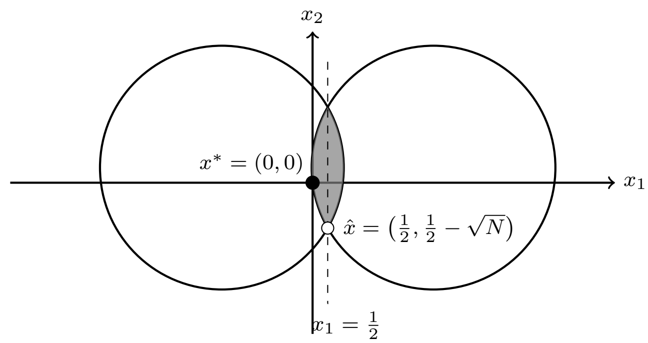

Fig. 2 The feasible region of the continuous relaxation of the conic IP in Example 7. Note that the set of integer feasible points is $\{(0,0),(1,0),(0,1),(1,1)\}$.

In addition to the ones discussed in this section, we have also constructed several examples in which a non-simple second-order conic IP has a right-hand side dependent integrality gap (see Appendix A for details).

4 Proximity results for general convex IPs

4.1 Convex sets with full-dimensional recession cones

Our main result in this section is that if the recession cone of $\mathcal{S}$ is a full-dimensional convex cone, then proximity and integrality gap bounds can be obtained for the convex IP (1) that are independent of the optimal solution of the continuous relaxation (2). For this purpose, we first need a lemma that quantifies a key parameter for a regular cone.

Lemma 2 *Let $\mathbf{K} \subseteq \mathbb{R}^n$ be a regular cone and define*

$$\Psi_{\mathbf{K},\|\cdot\|} := \max_{d\in\mathbf{K}}\left\{\min_{f\in\mathbf{K}_*}\{f^T d : \|f\|_* = 1\} : \|d\| = 1\right\},$$

where $\|x\|_* = \sup\{x^T y : \|y\| = 1\}$ is the norm dual to $\|\cdot\|$. Then, the maximizer is attained at an interior point d^* of $\mathbf{K}$, $\Psi_{\mathbf{K},\|\cdot\|} > 0$ and $B(\theta d^*, r) \subseteq \mathbf{K}$ for all $\theta \ge \frac{r}{\Psi_{\mathbf{K},\|\cdot\|}}$.

Proof In order prove the assertion of the proposition, we first let $d \in \text{int}(\mathbf{K})$ such that $\|d\| = 1$ and show that $\theta d + r\epsilon \in \mathbf{K}$ for every $\epsilon \in \mathbb{R}^n$ such that $\|\epsilon\| = 1$, where $\theta \ge r/\psi_d$ with $\psi_d := \min\{f^T d : f \in \mathbf{K}_*, \|f\|_* = 1\}$. In fact, we have that

$$\begin{aligned}\theta d + r\epsilon \in \mathbf{K} \quad \forall\epsilon : \|\epsilon\| = 1 &\iff f^T(\theta d + r\epsilon) \ge 0 \quad \forall\epsilon : \|\epsilon\| = 1, \forall f \in \mathbf{K}_* : \|f\|_* = 1\\ &\iff \theta f^T d + r\min\{f^T\epsilon : \|\epsilon\| = 1\} \ge 0 \quad \forall f \in \mathbf{K}_* : \|f\|_* = 1\\ &\iff \theta f^T d - r \ge 0 \quad \forall f \in \mathbf{K}_* : \|f\|_* = 1 \iff \theta \ge \frac{r}{\psi_d}.\end{aligned}$$

In the first line we use the fact that $\mathbf{K} = (\mathbf{K}_*)_*$ and in third line, we use the fact that $\min\{f^T\epsilon : \|\epsilon\| = 1\} = -\|f\|_* = -1$. Notice that $\psi_d > 0$ since $d \in \text{int}(\mathbf{K})$ and $f \in \mathbf{K}_*$.

On the other hand, if $d \in \partial(\mathbf{K})$, then there exists $f \in \mathbf{K}_*$ such that $f^T d = 0$. In this case, the inner minimization would give an optimal value of 0. Hence, we conclude that the optimizer of the outer maximization should be in the interior of cone $\mathbf{K}$.

Finally, we let $\Psi_{\mathbf{K},\|\cdot\|} := \max\{\psi_d : d \in \mathbf{K}, \|d\| = 1\} > 0$. □

Theorem 2 *Consider a convex set $\mathcal{S}$ and assume that its recession cone $\mathbf{K} := \text{rec.cone}(\mathcal{S})$ is regular. Let $\hat{x} \in \mathcal{S}$. Then, we have*

$$\mathbf{Prox}_{\hat{x}}(\mathcal{S}) := \min_{x \in \mathcal{S} \cap \mathbb{Z}^n} \|x - \hat{x}\|_2 \leq \frac{\sqrt{n}}{2}\left(\frac{1}{\Psi_{\mathbf{K},\|\cdot\|_2}} + 1\right).$$

Moreover, assuming that $\hat{z} = \inf_{x \in \mathcal{S}} \alpha^T x$ is bounded below for $\alpha \in \mathbb{R}^n$, we have

$$\mathbf{IG}(\mathcal{S}) = \min_{x \in \mathcal{S} \cap \mathbb{Z}^n} \alpha^T x - \hat{z} \leq \|\alpha\|_2 \frac{\sqrt{n}}{2}\left(\frac{1}{\Psi_{\mathbf{K},\|\cdot\|_2}} + 1\right).$$

Proof The first statement directly follows as a consequence of Lemma 2 by choosing $r = \sqrt{n}/2$, and noting that $\{x \in \mathbb{Z}^n : \|x - (\hat{x} + \theta d^*)\|_2 \leq \sqrt{n}/2\} \neq \emptyset$. For the second statement, we look at two cases:

If problem $\inf_{x \in \mathcal{S}} \alpha^T x$ is solvable, then the upper bound on integrality gap is obtained as a direct consequence of the upper bound on proximity due to the Cauchy-Schwarz inequality,

If problem $\inf_{x \in \mathcal{S}} \alpha^T x$ is bounded below but not solvable, then we proceed as follows: Notice that for all $\epsilon > 0$, there exists $\hat{x}_\epsilon \in \mathcal{S}$ such that $\alpha^T \hat{x}_\epsilon - \epsilon \leq \hat{z}$, and there exists $x'_\epsilon \in \mathcal{S} \cap \mathbb{Z}^n$ such that $\|x'_\epsilon - \hat{x}_\epsilon\|_2 \leq \frac{\sqrt{n}}{2}\left(\frac{1}{\Psi_{\mathbf{K},\|\cdot\|_2}} + 1\right)$, which implies that $\alpha^T(x'_\epsilon - \hat{x}_\epsilon) \leq \|\alpha\|_2 \frac{\sqrt{n}}{2}\left(\frac{1}{\Psi_{\mathbf{K},\|\cdot\|_2}} + 1\right)$. Therefore, we obtain that

$$\mathbf{IG}(\mathcal{S}) = \inf_{x \in \mathcal{S} \cap \mathbb{Z}^n} \alpha^T x - \hat{z} \leq \alpha^T x'_\epsilon - \hat{z} \leq \|\alpha\|_2 \frac{\sqrt{n}}{2}\left(\frac{1}{\Psi_{\mathbf{K},\|\cdot\|_2}} + 1\right) + \epsilon,$$

for all $\epsilon > 0$. The result follows by letting $\epsilon \to 0^+$. □

Note that the constant $\Psi_{\mathbf{K},\|\cdot\|}$ only depends on $\mathbf{K}$. In particular, for a conic set $\mathcal{S}(\mathbf{u}) = \{x \in \mathbb{R}^n : \text{A}x \geq_{\mathbf{K}} \mathbf{u}\}$ whose recession cone is full-dimensional, this constant only depends on the matrix A and the cone $\mathbf{K}$, and not on the right-hand side $\mathbf{u}$ since $\text{rec.cone}(S_{\mathbf{u}}) = \{x \in \mathbb{R}^n : \text{A}x \geq_{\mathbf{K}} 0\}$.

4.2 Convex sets with recession cones that are not necessarily full-dimensional

As Example 4 shows, the integrality gap of a convex set could be infinite even if the recession cone is a rational ray. Thus, in the case in which the recession cone is not full-dimensional, finding conditions for the integrality gap to be finite is more complicated. In this section, we give sufficient conditions for the integrality gap to be finite.

Theorem 3 *Let $\alpha \in \mathbb{Z}^n$ and let $\mathcal{S}$ be a convex set such that its integer hull is the non-empty rational polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n : \text{A}x \leq b\}$. Assume that $\text{rec.cone}(\mathcal{S}) = \{x \in \mathbb{R}^n : \text{A}x \leq 0\}$ and $\vartheta(\mathcal{S}) > -\infty$, then*

$$\mathbf{IG}(\mathcal{S}) \leq n\|\alpha\|_1 \Delta(1 + \|b - b'\|_\infty),$$

where Δ denote the largest absolute value of any determinant of a square submatrix of A and the vector $b' < +\infty$ is defined for all $i = 1, \ldots, m$ as:

$$b'_i = \sup\{a_i^T x : x \in \mathcal{S}\},$$

where a_i denotes the ith row of A.

Proof Clearly, $b'_i < +\infty$ since rec.cone$(\mathcal{S}) = \{x \in \mathbb{R}^n : \mathrm{A}x \leq 0\}$. Now, note that the polyhedron $\mathcal{P}' = \{x \in \mathbb{R}^n : \mathrm{A}x \leq b'\}$ satisfies $\mathcal{P} \subseteq \mathcal{S} \subseteq \mathcal{P}'$. Therefore, we have

$$\begin{aligned}\mathbf{IG}(\mathcal{S}) &= \inf\{\alpha^T x : x \in \mathcal{P}\} - \inf\{\alpha^T x : x \in \mathcal{S}\} \leq \inf\{\alpha^T x : x \in \mathcal{P}\} - \inf\{\alpha^T x : x \in \mathcal{P}'\} \\ &= (\inf\{\alpha^T x : x \in \mathcal{P}' \cap \mathbb{Z}^n\} - \inf\{\alpha^T x : x \in \mathcal{P}'\}) + (\inf\{\alpha^T x : x \in \mathcal{P}\} - \inf\{\alpha^T x : x \in \mathcal{P}' \cap \mathbb{Z}^n\}). \\ &\leq n\|\alpha\|_1 \Delta + (\inf\{\alpha^T x : x \in \mathcal{P}\} - \inf\{\alpha^T x : x \in \mathcal{P}'\}) \\ &\leq n\|\alpha\|_1 \Delta + n\|\alpha\|_1 \Delta \|b - b'\|_\infty \leq n\|\alpha\|_1 \Delta (1 + \|b - b'\|_\infty),\end{aligned}$$

where line 3 follows from Theorem 1 and line 4 follows from Theorem 5 in [9]. □

Sufficient and necessary conditions are given in [12] for a convex set $\mathcal{S}$ to satisfy the assumptions of Theorem 3. In particular, the conditions are satisfied if rec.cone$(\mathcal{S})$ is a rational polyhedron and $\inf\{\alpha^T x : x \in \mathcal{S}\} = -\infty$ if and only if there exists $r \in$ rec.cone$(\mathcal{S})$ such that $\alpha^T r < 0$.

A convex set $\mathcal{S} \subseteq \mathbb{R}^n$ is said to satisfy the 'finiteness property' if $\inf\{\alpha^T x : x \in \mathcal{S}\} > -\infty$ if and only if $\inf\{\alpha^T x : x \in \mathcal{S} \cap \mathbb{Z}^n\} > -\infty$. The following result is straightforward, and we omit its proof.

Theorem 4 *$\mathcal{S}$ satisfy the 'finiteness property' if and only if* $\mathbf{IG}(\mathcal{S}) < +\infty$.

Conditions for a set to satisfy the finiteness property has been studied in [22,23,20]. In particular, if $\mathcal{S} = P \cap X$, where P is a *Dirichlet convex set* (see [22,20]), X is a convex set such that $P \cap \mathrm{int}(X) \cap \mathbb{Z}^n \neq \emptyset$, then $\mathcal{S}$ satisfy the 'finiteness property' (see [20]). Examples of Dirichlet convex sets are: bounded convex sets, rational polyhedra, closed strictly convex sets, and closed convex sets whose recession cone are generated by integral vectors.

We can use Theorems 2 and 4 to show that for certain conic sets the integrality gap is always finite except if they are an irrational ray. We start with some definitions.

A generator for a pointed closed convex cone $\mathbf{K} \subseteq \mathbb{R}^m$ is a bounded closed convex set $\mathbf{G} \subseteq \mathbb{R}^m$ of dimension $\dim(\mathbf{K}) - 1$ such that $\mathbf{K} = \mathrm{cone}(\mathbf{G})$. We say that $\mathbf{K}$ is generated by $\mathbf{G}$.

We need the following result, which is a restatement of Lemma 3.10 in [26].

Lemma 3 *Let $\mathbf{K} \subseteq \mathbb{R}^m$ be a regular cone that is generated by a closed strictly convex set and let $\mathcal{S}(\mathbf{u}) = \{x \in \mathbb{R}^n : \mathrm{A}x \geq_{\mathbf{K}} \mathbf{u}\}$. Assume that* lin.space$(\mathcal{S}(\mathbf{u})) = \{0\}$. *Then*

(i) If $\mathbf{u}$ is not in the image space of A, *then $\mathcal{S}(\mathbf{u})$ is a singleton or a full-dimensional strictly convex set.*
(ii) If $\mathbf{u}$ is in the image space of A, *then $\mathcal{S}(\mathbf{u})$ is a translated cone of dimension equal to $d = \dim(\{\mathrm{A}x \in \mathbf{K} : x \in \mathbb{R}^n\}$. Moreover, if $d \geq 2$, then $d = \dim(\{\mathrm{A}x : x \in \mathbb{R}^n\} = n$.*

Corollary 1 *Let $\mathbf{K}$ be a cone generated by a strictly convex set and consider the conic IP associated to $\mathcal{S}(\mathbf{u}) = \{x \in \mathbb{R}^n : \mathrm{A}x \geq_{\mathbf{K}} \mathbf{u}\}$. Then, if $\mathcal{S}(\mathbf{u})$ is not an irrational ray, the conic IP has finite integrality gap.*

Proof By Lemma 3, the set $\mathcal{S}(\mathbf{u})$ is either a singleton, a full-dimensional strictly convex set, a full-dimensional cone or a ray (thus a rational polyhedron by assumption). If it is a full-dimensional cone, then by Theorem 2, we obtain that the integrality gap is finite. In all the other cases, $\mathcal{S}(\mathbf{u})$ is a Dirichlet convex set, and hence, by Theorem 4, we also conclude that the integrality gap is finite. □

5 Structural results for simple second-order conic sets

In the rest of the paper, we will derive proximity results for simple second-order conic IPs. Such a set can be written in its second-order cone representation (SOCr) form as

$$\mathcal{S} = \{x \in \mathbb{R}^n : \|Ax - b\|_2 \leq c^T x - d\}, \tag{SOCr}$$

or equivalently,

$$\mathcal{S} = \{x \in \mathbb{R}^n : x^T(A^T A - cc^T)x - 2(A^T b - cd)^T x + b^T b - d^2 \leq 0, c^T x \geq d\}. \tag{5}$$

It is well-known that the matrix $A^T A - cc^T$ has at most one negative eigenvalue (see, for instance, [3, 4]). This motivates us to study the closely quadratic related set

$$\mathcal{Q} = \{x \in \mathbb{R}^n : x^T M x - 2\beta^T x + \gamma \leq 0\}, \tag{Qr}$$

where M is a matrix with at least $n-1$ positive eigenvalues. We will refer to this set as the quadratic representation (or Qr in short). A characterization of these sets has been given in [3,4]. We summarize their results in Proposition 1 below.

If the smallest eigenvalue of M, denoted λ_n, is negative, and $\beta^T M^{-1}\beta - \gamma \leq 0$, then the set $\mathcal{Q}$ is a nonconvex quadratic set that can be represented as the union of two convex sets (see Proposition 1); we call these sets *branches* of $\mathcal{Q}$. In this case we are also interested in understanding sets of the form

$$\mathcal{H} = \{x \in \mathbb{R}^n : x^T M x - 2\beta^T x + \gamma \leq 0, g^T x \geq h\}, \tag{Br}$$

where the linear constraint is such that the set $\mathcal{H}$ is a *branch* of $\mathcal{Q}$; we will characterize such linear inequalities later in Proposition 2.

5.1 Properties of set $\mathcal{Q}$

In this section we will study properties of the quadratic set $\mathcal{Q} = \{x \in \mathbb{R}^n : x^T M x - 2\beta^T x + \gamma \leq 0\}$, where $M \in \mathbb{S}^n$, $\beta \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$.

5.1.1 Quadratic sets with at most one negative eigenvalue

In this section we assume that the eigenvalues of M are $\lambda_1 \geq \cdots \lambda_{n-1} > 0$. And we let u_n denote the eigenvector corresponding to the eigenvalue λ_n.

The following proposition provides a full characterization of quadratic sets with at most one negative eigenvalue, which is of interest to our study.

Proposition 1 ([3,4]) *The set $\mathcal{Q}$ is one of the following types:*

- *If $\lambda_n > 0$, we denote $q^* = \beta^T M^{-1}\beta - \gamma$. Then $\mathcal{Q}$ is either a full-dimensional ellipsoid ($q^* > 0$), a singleton ($q^* = 0$) or empty ($q^* < 0$).*
- *If $\lambda_n = 0$ and the system $Mx = \beta$ has no solution, then $\mathcal{Q}$ is a paraboloid. Otherwise, let $\hat{x}$ such that $M\hat{x} = \beta$ and denote $\hat{q} = \hat{x}^T M\hat{x} - \gamma$, then $\mathcal{Q}$ is either a cylinder with its center line through $\hat{x}$ in the direction of u_n ($\hat{q} > 0$), a line through $\hat{x}$ in the direction of u_n ($\hat{q} = 0$) or empty ($\hat{q} < 0$).*
- *If $\lambda_n < 0$, we denote $q^* = \beta^T M^{-1}\beta - \gamma$. Then Q is either a one-sheet hyperboloid ($q^* > 0$), a translated cone ($q^* = 0$) or a two-sheet hyperboloid ($q^* < 0$).*

In this paper, we are interested in convex sets arising from a set of the form $\mathcal{Q}$. In particular, we are not concerned with one-sheet hyperboloids since they are nonconvex sets. In the case $\mathcal{Q}$ is the union two disjoint convex sets (branches), they can be separated from each other with a linear inequality. The next proposition provides the characterization of such linear inequalities.

Proposition 2 *Assume $\lambda_n < 0$. Let $q^* = \beta^T M^{-1}\beta - \gamma \leq 0$, and let $\mathcal{Q}^+$ and $\mathcal{Q}^-$ be the two branches of $\mathcal{Q}$. Denote $\mathcal{H}^{\geq} = \{x \in \mathcal{Q} : g^T x \geq h\}$ and $\mathcal{H}^{\leq} = \{x \in \mathcal{Q} : g^T x \leq h\}$ where $g \in \mathbb{R}^n \setminus \{0\}$ and $h \in \mathbb{R}$. Let $h_{\pm} = g^T M^{-1}\beta \pm \sqrt{(\beta^T M^{-1}\beta - \gamma) g^T M^{-1} g}$. Then, we have the following:*

(i) If $g^T M^{-1} g = 0$, then for all $\epsilon > 0$, we have that $\{x \in \mathcal{Q}^+ : |g^T(x - M^{-1}\beta)| = \epsilon\} \neq 0$ and $\{x \in \mathcal{Q}^- : |g^T(x - M^{-1}\beta)| = \epsilon\} \neq 0$. In particular, if $\mathcal{Q}$ is a two-sheet hyperbola ($q^* < 0$), then g defines asymptotes of $\mathcal{Q}^+$ and $\mathcal{Q}^-$.

(ii) If $g^T M^{-1} g < 0$, then either we have $\inf\{g^T x : x \in \mathcal{Q}^+\} = h_+$ and $\sup\{g^T x : x \in \mathcal{Q}^-\} = h_-$ and $\sup\{g^T x : x \in \mathcal{Q}^+\} = -\inf\{g^T x : x \in \mathcal{Q}^+\} = +\infty$, or we have $\sup\{g^T x : x \in \mathcal{Q}^+\} = h_-$ and $\inf\{g^T x : x \in \mathcal{Q}^-\} = h_+$ and $\sup\{g^T x : x \in \mathcal{Q}^-\} = -\inf\{g^T x : x \in \mathcal{Q}^-\} = +\infty$. Moreover, whenever one of the optimization problems considered is bounded, then there is a unique optimal solution.

(iii) If $g^T M^{-1} g > 0$, then $\sup\{g^T x : x \in \mathcal{Q}^+\} = -\inf\{g^T x : x \in \mathcal{Q}^+\} = \sup\{g^T x : x \in \mathcal{Q}^-\} = -\inf\{g^T x : x \in \mathcal{Q}^-\} = +\infty$.

(iv) Let $\mathcal{Q}$ be a translated cone ($q^* = 0$). Then

(a) $\mathcal{H}^{\geq} \cap \mathcal{H}^{\leq} = \{M^{-1}\beta\}$ if and only if $g^T M^{-1} g < 0$ and $h = g^T M^{-1}\beta$.

(b) The sets $\{x \in \mathcal{Q} : g^T x \geq 0\}$ and $\{x \in \mathcal{Q} : g^T x \leq 0\}$ are the two distinct branches of $\mathcal{Q}$ if and only if $g^T M^{-1} g < 0$.

(v) Let $\mathcal{Q}$ be a two-sheet hyperbola ($q^* < 0$). Then

(a) $\mathcal{H}^{\geq} \cap \mathcal{H}^{\leq} = \emptyset$ if and only if $g^T M^{-1} g < 0$ and $h \in (h_-, h_+)$ or $g^T M^{-1} g = 0$ and $h = g^T M^{-1} \beta$.

(b) The sets $\{x \in \mathcal{Q} : g^T x \geq h_1\}$ and $\{x \in \mathcal{Q} : g^T x \leq h_2\}$ are the two distinct branches of $\mathcal{Q}$ if and only if $g^T M^{-1} g \leq 0$ and $h_1, h_2 \in [h_-, h_+]$ and $h_1 < h_2$.

Proof We start by proving the results for a special case.

Proof in the special case $M = \tilde{I}$ and $\beta = 0$.

Let $k \leq 0$ and define $\mathcal{Q}_k = \{x \in \mathbb{R}^n : x^T \tilde{I} x \leq k\}$, $\mathcal{H}_k^{\geq} = \{x \in \mathcal{Q}_k : g^T x \geq h\}$ and $\mathcal{H}_k^{\leq} = \{x \in \mathcal{Q}_k : g^T x \leq h\}$. The two branches of $\mathcal{Q}_k$ are in this case $\mathcal{Q}_k^+ = \mathcal{Q}_k \cap \mathbf{L}^n$ and $\mathcal{Q}_k^- = \mathcal{Q}_k \cap -\mathbf{L}^n$. Note that since $\mathcal{Q}_k = -\mathcal{Q}_k$, we obtain that $\mathcal{Q}_k^- = -\mathcal{Q}_k^+$, $\inf\{g^T x : x \in \mathcal{Q}_k^+\} = -\sup\{g^T x : x \in \mathcal{Q}_k^-\}$ and $\sup\{g^T x : x \in \mathcal{Q}_k^+\} = -\inf\{g^T x : x \in \mathcal{Q}_k^-\}$. With this in mind, we can prove the results for only one case, depending on the branch considered and whether we are computing a supremum or an infimum; the proof for the other case will be analogous. For instance, when assuming $g \in \partial(\mathbf{L}^n) \cup \partial(-\mathbf{L}^n)$ (that is, $g^T \tilde{I} g = 0$), we can first prove the result for $g \in \partial(\mathbf{L}^n)$ and then prove the result for $g \in \partial(-\mathbf{L}^n)$ by following the argument used for $-g \in \partial(\mathbf{L}^n)$.

(i) It suffices to show that if $g \in \partial(\mathbf{L}^n)$, then for all $\epsilon > 0$, we have that $\{x \in \mathcal{Q}_k^+ : g^T x = \epsilon\} \neq 0$. Indeed, consider the point $x(\epsilon, \lambda) = \epsilon \frac{g}{g^T g} + \lambda \tilde{I} g$. Then since $g^T \tilde{I} g = 0$, we obtain $x(\epsilon, \lambda)^T \tilde{I} x(\epsilon, \lambda) = (\epsilon \frac{g}{g^T g} + \lambda \tilde{I} g)^T \tilde{I} (\epsilon \frac{g}{g^T g} + \lambda \tilde{I} g) = \epsilon^2 \frac{g^T \tilde{I} g}{g^T g} + 2\lambda\epsilon \frac{g^T g}{g^T g} + \lambda^2 g^T \tilde{I} g = 2\lambda\epsilon$. Therefore, for $\lambda \leq \frac{k}{2\epsilon}$ we have $x(\epsilon, \lambda) \in \mathcal{Q}_k$. On the other hand, observe that $g^T x(\epsilon, \lambda) = g^T (\epsilon \frac{g}{g^T g} + \lambda \tilde{I} g) = \epsilon$. Hence, $\{x \in \mathcal{Q}_k^+ : g^T x = \epsilon\} \neq 0$.

(ii) It suffices to show that if $g \in \operatorname{int}(\mathbf{L}^n)$, then $\inf\{g^T x : x \in \mathcal{Q}_k^+\} = h_+ = \sqrt{k g^T \tilde{I}^{-1} g}$ and $\sup\{g^T x : x \in \mathcal{Q}_k^+\} = +\infty$. Since $g \in \operatorname{int}(\mathbf{L}^n)$, then we have $g^T x > 0$ for all nonzero $x \in \mathcal{Q}_k^+ \subseteq \mathbf{L}^n$. Therefore, since rec.cone$(\mathcal{Q}_k^+) = \mathbf{L}^n$ we obtain $\sup\{g^T x : x \in \mathcal{Q}_k^+\} = +\infty$ and $z^* = \inf\{g^T x : x \in \mathcal{Q}_k^+\} > -\infty$ since $g^T x \geq 0$ on $\mathcal{Q}_k^+$. Let $\delta > 0$ and $Q_k^\delta = \{x \in \mathcal{Q}_k^+ : g^T x \leq z^* + \delta\}$. Since rec.cone$(Q_k^\delta) = \{x \in \mathbf{L}^n : g^T x \leq 0\} = \{0\}$, we obtain that the set Q_k^δ is a compact convex set and therefore, $\inf\{g^T x : x \in \mathcal{Q}_k^+\}$ is solvable. The optimal solution x^* is unique since $\mathcal{Q}_k^+$ is either a pointed cone ($\mathcal{Q}_k^+ = \mathbf{L}^n$) or a strictly convex set. By the first order optimality conditions, there exists $\lambda \in \mathbb{R}$ such that $\lambda g = \tilde{I} x^*$ and $(x^*)^T \tilde{I} (x^*) = k$. Since $x^* = \lambda \tilde{I} g$, we obtain the following quadratic equation for λ: $(\lambda \tilde{I} g)^T \tilde{I} (\lambda \tilde{I} g) = k$. Thus, $\lambda = \pm \sqrt{\frac{k}{g^T \tilde{I}^{-1} g}}$. Since $z^* \geq 0$, we obtain that $z^* = g^T x^* = g^T \left(\sqrt{\frac{k}{g^T \tilde{I}^{-1} g}} \tilde{I} g \right) = \sqrt{k g^T \tilde{I}^{-1} g}$.

(iii) It suffices to show that if $g \notin \mathbf{L}^n \cup -\mathbf{L}^n$, then $\sup\{g^T x : x \in \mathcal{Q}_k^+\} = -\inf\{g^T x : x \in \mathcal{Q}_k^+\} = +\infty$. Note that f $g \notin \mathbf{L}^n \cup -\mathbf{L}^n$, since these cones are self-dual, there exist $d, d' \in \mathbf{L}^n$ such that $g^T d < 0$ and $g^T d' > 0$. The results follows by noting that rec.cone$(\mathcal{Q}_k^+) = \mathbf{L}^n$, so the optimization problems have the directions d, d' as unboundedness certificates.

(iv) We start by proving the following:

Claim. The hyperplane $g^T x = h$ separates $\mathcal{Q}_k^+$ and $\mathcal{Q}_k^-$ if and only if $g \in \mathbf{L}^n \cup -\mathbf{L}^n$ and $-\sqrt{k g^T \tilde{I}^{-1} g} \leq h \leq \sqrt{k g^T \tilde{I}^{-1} g}$.

Proof of the Claim. If the hyperplane separates $\mathcal{Q}_k^+$ and $\mathcal{Q}_k^-$, there are two cases, either $\mathcal{H}_k^{\geq} \supseteq \mathcal{Q}_k^+$ and $\mathcal{H}_k^{\leq} \supseteq \mathcal{Q}_k^-$ or $\mathcal{H}_k^{\geq} \supseteq \mathcal{Q}_k^-$ and $\mathcal{H}_k^{\leq} \supseteq \mathcal{Q}_k^+$. We will prove the claim for the first case, the proof of the other case is analogous. Note that if $\mathcal{H}_k^{\geq} \supseteq \mathcal{Q}_k^+$, then since the recession cone of $\mathcal{Q}_k^+$ is $\mathbf{L}^n$, we must have $\mathbf{L}^n \subseteq \{x \in \mathbb{R}^n : g^T x \geq 0\}$ and $h \leq \inf\{g^T x : x \in \mathcal{Q}_k^+\}$ (otherwise, $g^T x \geq h$ cannot be a valid inequality for $\mathcal{Q}_k^+$). In particular, since $\mathbf{L}^n$ is self-dual, we obtain $g \in \mathbf{L}^n$. Similarly, if $\mathcal{H}_k^{\leq} \supseteq \mathcal{Q}_k^-$, we must have $-\mathbf{L}^n \subseteq \{x \in \mathbb{R}^n : g^T x \leq 0\}$ and $h \geq \sup\{g^T x : x \in \mathcal{Q}_k^-\}$. Since $\sup\{g^T x : x \in \mathcal{Q}_k^-\} = -\inf\{-g^T x : x \in \mathcal{Q}_k^-\} = -\inf\{g^T x : x \in \mathcal{Q}_k^+\}$, by parts (i) and (ii), we conclude $-\sqrt{k g^T \tilde{I}^{-1} g} \leq h \leq \sqrt{k g^T \tilde{I}^{-1} g}$. Assume now that $g \in \mathbf{L}^n \cup -\mathbf{L}^n$ and $-\sqrt{k g^T \tilde{I}^{-1} g} \leq h \leq \sqrt{k g^T \tilde{I}^{-1} g}$. We will show the case $g \in \mathbf{L}^n$ as the proof of the other case is analogous. If $g \in \mathbf{L}^n$, then by parts (i) and (ii), we conclude that $\inf\{g^T x : x \in \mathcal{Q}_k^+\} = \sqrt{k g^T \tilde{I}^{-1} g}$. Since $\sup\{g^T x : x \in \mathcal{Q}_k^-\} = -\inf\{g^T x : x \in \mathcal{Q}_k^+\} = -\sqrt{k g^T \tilde{I}^{-1} g}$, we obtain $\mathcal{H}_k^{\geq} \supseteq \mathcal{Q}_k^+$ and $\mathcal{H}_k^{\leq} \supseteq \mathcal{Q}_k^-$.

(a) We only prove part (iv.a) as part (iv.b) follows directly from it. We need to show that $\mathcal{H}_0^{\geq} \cap \mathcal{H}_0^{\leq} = \{0\}$ if and only if $g^T \tilde{I} g < 0$ and $h = 0$. We first assume that $\mathcal{H}_0^{\geq} \cap \mathcal{H}_0^{\leq} = \{0\}$. Since $\mathcal{H}_0^{\geq} \cap \mathcal{H}_0^{\leq} = \{x \in \mathcal{Q}_0 : g^T x = h\}$, we must have $h = 0$. If $g \notin \mathbf{L}^n \cup -\mathbf{L}^n$, by part (iii) we obtain then $\sup\{g^T x : x \in \mathcal{Q}_0^+\} = -\inf\{g^T x : x \in \mathcal{Q}_0^+\} = +\infty$. Thus, there exists a point $u \in \mathcal{Q}_0^+ = \mathbf{L}^n$, $u \neq 0$ such that $g^T u = 0$, so $u \in \mathcal{H}_0^{\geq} \cap \mathcal{H}_0^{\leq}$, a contradiction. So we have that $g \in \mathbf{L}^n \cup -\mathbf{L}^n$ and thus $g^T \tilde{I} g \leq 0$. If we have that $g^T \tilde{I} g = 0$, then $(\tilde{I}g)^T \tilde{I}(\tilde{I}g) = 0$, and thus $\tilde{I}g \in \mathcal{H}_0^{\geq} \cap \mathcal{H}_0^{\leq}$, a contradiction. We conclude that $g^T \tilde{I} g < 0$ and $h = 0$. We assume now that $g^T \tilde{I} g < 0$ and $h = 0$. Then by the Claim, we obtain that the hyperplane $g^T x = 0$ separates $\mathbf{L}^n$ and $-\mathbf{L}^n$. Now, since $g \in \operatorname{int}(\mathbf{L}^n) \cup \operatorname{int}(-\mathbf{L}^n)$, the only solution to the system $g^T x = 0$ and $g^T \tilde{I} g \leq 0$ is $x = 0$, so we conclude $\mathcal{H}_0^{\geq} \cap \mathcal{H}_0^{\leq} = \{0\}$.

(v) (a) We only prove part (v.a) as part (v.b) follows directly from it. We need to show that for $k > 0$ we have $\mathcal{H}_k^{\geq} \cap \mathcal{H}_k^{\leq} = \emptyset$ if and only if $g^T \tilde{I} g < 0$ and $h \in (h_-, h_+)$, where $h_\pm = \pm\sqrt{k g^T \tilde{I}^{-1} g}$ or $g^T \tilde{I} g = 0$ and $h = 0$. We first assume that $\mathcal{H}_k^{\geq} \cap \mathcal{H}_k^{\leq} = \emptyset$. If $g \notin \mathbf{L}^n \cup -\mathbf{L}^n$, by part (iii) there exists $x_h \in \mathcal{Q}_k^+ \subseteq \mathcal{Q}_k$ such that $g^T x_h = h$, contradicting the fact that $\mathcal{H}_k^{\geq} \cap \mathcal{H}_k^{\leq} = \{x \in \mathcal{Q}_k : g^T x = h\} = \emptyset$. Hence, $g^T \tilde{I} g \leq 0$. If $g^T \tilde{I} g < 0$, by part (ii), if $h \notin (h_-, h_+)$, then there exists $x_h \in \mathcal{Q}_k$ such that $g^T x_h = h$, a contradiction with $\mathcal{H}_k^{\geq} \cap \mathcal{H}_k^{\leq} = \emptyset$. If $g^T \tilde{I} g = 0$, then without loss of generality, we have $g \in \partial(\mathbf{L}^n)$. So, by part (i), we have $\inf\{g^T x : x \in \mathcal{Q}_k^+\} = \sqrt{k g^T \tilde{I}^{-1} g} = 0$ and we conclude that $h = 0$. We assume now that $g^T \tilde{I} g < 0$ and $h \in (h_-, h_+)$. By the Claim, we obtain that the hyperplane $g^T x = h$ separates $\mathcal{Q}_k^+$ and $\mathcal{Q}_k^-$. Without loss of generality, since $h \in (h_-, h_+)$, we may assume that $\mathcal{Q}_k^+ \subseteq \{x \in \mathbb{R}^n : g^T x < h\}$ and $\mathcal{Q}_k^- \subseteq \{x \in \mathbb{R}^n : g^T x > h\}$. This implies that $\mathcal{H}_k^{\geq} = (\mathcal{Q}_k^+)$ and $\mathcal{H}_k^{\leq} = (\mathcal{Q}_k^-)$. Therefore, $\mathcal{H}_k^{\geq} \cap \mathcal{H}_k^{\leq} = \mathcal{Q}_k^+ \cap -\mathbf{L}^n = \mathcal{Q}_k \cap \{0\} = \emptyset$ since $k < 0$. Now we assume that $g^T \tilde{I} g = 0$ and $h = 0$. By the Claim, $g^T x = 0$ separates $\mathbf{L}^n$ and $-\mathbf{L}^n$. We may assume that $\mathbf{L}^n \subseteq \{x \in \mathbb{R}^n : g^T x \geq 0\}$ and $-\mathbf{L}^n \subseteq \{x \in \mathbb{R}^n : g^T x \leq 0\}$. Since $0 \notin \mathcal{Q}_k$, we obtain $\mathcal{Q}_k^+ \subseteq \{x \in \mathbb{R}^n : g^T x < 0\}$ and $\mathcal{Q}_k^- \subseteq \{x \in \mathbb{R}^n : g^T x > 0\}$, which as before implies that $\mathcal{H}_k^{\geq} \cap \mathcal{H}_k^{\leq} = \emptyset$.

Proof of the general case. Let us consider the eigenvalue decomposition of $M = U \Lambda U^T$. Notice that we can write the matrix M as $M = U \tilde{\Lambda} \tilde{I} \tilde{\Lambda} U^T$ for a diagonal matrix $\tilde{\Lambda}$ with $\tilde{\Lambda}_{ii} = \sqrt{\lambda_i}$ for $i = 1, \ldots, n-1$ and $\tilde{\Lambda}_{nn} = \sqrt{-\lambda_n}$. We have

$$\begin{aligned}
\mathcal{Q} &= \{x \in \mathbb{R}^n : x^T M x - 2\beta^T x + \gamma \leq 0\} \\
&= M^{-1}\beta + \{x \in \mathbb{R}^n : (x + M^{-1}\beta)^T M (x + M^{-1}\beta) - 2\beta^T (x + M^{-1}\beta) + \gamma \leq 0\} \\
&= M^{-1}\beta + \{x \in \mathbb{R}^n : x^T M x \leq \beta^T M^{-1}\beta - \gamma\} \\
&= M^{-1}\beta + \{x \in \mathbb{R}^n : x^T U \tilde{\Lambda} \tilde{I} \tilde{\Lambda} U^T x \leq \beta^T M^{-1}\beta - \gamma\} \\
&= M^{-1}\beta + U \tilde{\Lambda}^{-1} \{x \in \mathbb{R}^n : x^T \tilde{I} x \leq \beta^T M^{-1}\beta - \gamma\}.
\end{aligned}$$

Therefore, we conclude that $x \in Q$ if and only if $\tilde{x} \in \mathcal{Q}_k$, where $x = M^{-1}\beta + U \tilde{\Lambda}^{-1} \tilde{x}$ and $k = \beta^T M^{-1}\beta - \gamma$. Notice that the inequality $g^T x \geq h$ is equivalent to $\tilde{g}^T \tilde{x} \geq \tilde{h}$ with $\tilde{g} = \tilde{\Lambda}^{-1} U^T g$ and $\tilde{h} = h - g^T M^{-1}\beta$. As a consequence, the general case is obtained from the special case by applying this mapping. For instance, $\tilde{g}^T \tilde{I} \tilde{g} < 0$ and $\tilde{h} = 0$ is equivalent to $g^T U \tilde{\Lambda}^{-1} \tilde{I} \tilde{\Lambda}^{-1} U^T g < 0$ and $h - g^T M^{-1}\beta = 0$ which is equivalent to $g^T M^{-1} g < 0$ and $h = g^T M^{-1}\beta$. The other expressions are obtained in a similar manner. □

We will now specialize the computation of the parameter $\Psi_{\mathbf{K},\|\cdot\|}$ defined for regular cones in Lemma 2 to the Lorentz cone in Lemma 4 and to a linear transformation of the Lorentz cone in Proposition 3.

Lemma 4 *We have*

$$\Psi_{\mathbf{L}^n,\|\cdot\|_2} = \max_{d \in \mathbf{L}^n} \left\{ \min_{f \in \mathbf{L}^n} \{f^T d : \|f\|_2 = 1\} : \|d\|_2 = 1 \right\} = 1.$$

Moreover, the maximizer of the outer-problem is $d^* = e_n$. In addition, given $d = d^*$, the minimizer of the inner problem is $f^* = e_n$.

Proof Due to Lemma 2, we know that the maximizer of the outer-problem is attained at an interior point d of $\mathbf{L}^n$ (i.e., $\bar{d}^T \bar{d} < d_n^2$). Given such d, the optimal solution f of the inner minimization is such that $\bar{f} = -\bar{d}$ and $f_n = d_n$ and we compute the objective function as $f^T d = -\bar{d}^T \bar{d} + d_n^2$. Finally, we can easily conclude that the optimal solution of the outer-maximization is $d^* = e_n$. □

Proposition 3 *Assume that $\lambda_n < 0$. Let g be a nonzero rational n-vector such that $g^T M^{-1} g < 0$. Consider the set $\mathbf{K} = \{x \in \mathbb{R}^n : x^T M x \leq 0, g^T x \geq 0\}$. Then, we have*

$$\Psi_{\mathbf{K},\|\cdot\|_2} = \max_{d\in\mathbf{K}} \left\{ \min_{f\in\mathbf{K}_*} \{f^T d : \|f\|_2 = 1\} : \|d\|_2 = 1 \right\} = -\frac{1}{\lambda_n}.$$

Proof Let us consider the eigenvalue decomposition of $M = U\Lambda U^T$. Notice that we can write the matrix M as $M = U\tilde{\Lambda}\tilde{I}\tilde{\Lambda}U^T$ for a diagonal matrix $\tilde{\Lambda}$ with $\tilde{\Lambda}_{ii} = \sqrt{\lambda_i}$ for $i = 1, \ldots, n-1$ and $\tilde{\Lambda}_{nn} = \sqrt{-\lambda_n}$. Then, we apply the one-to-one transformation $y := \tilde{\Lambda}U^T x$ to obtain the set $\tilde{\mathbf{K}} = \{y \in \mathbb{R}^n : y^T \tilde{I} y \leq 0, \tilde{g}^T y \geq 0\}$ with $\tilde{g} := \tilde{\Lambda}^{-1}U^T g$. It is easy to see that $\tilde{\mathbf{K}} = \mathbf{L}^n$. Due to Lemma 4, we know that the maximizer of the outer-problem and the minimizer of the inner-problem are both attained at e_n in the y-space, which is equal to $U\tilde{\Lambda}^{-1}e_n = -\frac{u_n}{\sqrt{-\lambda_n}}$ in the x-space. Computing the objective function gives us $(-\frac{u_n}{\sqrt{-\lambda_n}})^T(-\frac{u_n}{\sqrt{-\lambda_n}}) = -\frac{1}{\lambda_n}$, as stated in the statement of the proposition. □

5.1.2 Large balls contained in quadratic sets

We now show a nice property of unbounded quadratic sets that is crucial to obtain our main results. The main part states that (unbounded) quadratic sets of the form $\mathcal{Q}$ that are defined by a matrix M whose largest eigenvalue λ_1 is positive contain arbitrarily large full-dimensional balls. We obtain this result by studying the following optimization problem:

$$\Gamma_{x_0}(M, \beta, \gamma, r) := \min_{w\in\mathbb{R}^n} \{\|w\|_2 : \phi(x_0 + w + rv) \leq 0 \ \forall v \in \mathbb{R}^n : \|v\|_2 \leq 1\}, \tag{6}$$

where $\phi(x) := x^T M x - 2\beta^T x + \gamma$. In the proposition below we obtain upper bounds for the optimal value of (6) and give some of its basic properties. We will use the following fact.

Fact 3 *If $\mathcal{Q} = \{x \in \mathbb{R}^n : x^T M x - 2\beta^T x + \gamma \leq 0\}$ is a paraboloid (see its characterization in Proposition 1 below), then* $\text{rec.cone}(\mathcal{Q}) = \{x \in \mathbb{R}^n : Mx = 0, \beta^T x \leq 0\}$. *Moreover,* $\text{rec.cone}(\mathcal{Q})$ *is the ray generated by u_n, the eigenvector corresponding to the only nonpositive eigenvalue $\lambda_n = 0$ of M chosen such that $\beta^T u_n > 0$.*

Proposition 4 *Consider the unbounded a point $x_0 \in \mathcal{Q}$ and $r > 0$. Let $\lambda_n \leq 0$ be the smallest eigenvalue of M and u_n be the corresponding eigenvalue with $\beta^T u_n \geq 0$ and $\|u_n\|_2 = 1$. Assume that λ_1 the largest eigenvalue of M is positive. Then,*

(i) Let $\lambda_n = 0$. Then, for all $\theta \geq \Theta_0 := \frac{r^2\lambda_1 + 2r\|Mx_0 - \beta\|}{2\beta^T u_n}$, we have that $B(x_0 + \theta u_n, r) \subseteq \mathcal{Q}$.
(ii) Let $\lambda_n < 0$. Then, for all $\theta \geq \Theta_- := \frac{-(\lambda_n x_0^T u_n - \beta^T u_n + r|\lambda_n|) - \sqrt{(\lambda_n x_0^T u_n - \beta^T u_n + r|\lambda_n|)^2 - \lambda_n(r^2\lambda_1 + 2r\|Mx_0 - \beta\|_2)}}{\lambda_n}$, we have that $B(x_0 + \theta u_n, r) \subseteq \mathcal{Q}$.
(iii) The optimization problem (6) is solvable.
(iv) By denoting an optimal solution of the optimization problem (6) as w^*, we have that $B(x_0 + w^*, r) \subseteq \mathcal{Q}$.

Proof By plugging in $w = \theta u_n$, we obtain that

$$\begin{aligned} \phi(x_0 + w + rv) &= (x_0 + w + rv)^T M (x_0 + w + rv) - 2\beta^T(x_0 + w + rv) + \gamma \\ &= \phi(x_0) + w^T M w + 2(Mx_0 - \beta)^T w + r^2 v^T M v + 2rv^T M(x_0 + w) - 2r\beta^T v \\ &\leq \lambda_n\theta^2 + 2\lambda_n\theta x_0^T u_n - 2\theta\beta^T u_n + r^2\lambda_1 + 2r\|Mx_0 - \beta\|_2 + 2r|\lambda_n|\theta, \end{aligned} \tag{7}$$

where we use the facts that $x_0 \in \mathcal{Q}$, $u_n^T M u_n = \lambda_1$, $Mu_n = \lambda u_n$, $\lambda_1 = \max\{v^T M v : \|v\| \leq 1\}$ and $\max\{\omega^T v : \|v\| \leq 1\} = \|\omega\|_2$.

(i) Due to Fact 3, $u_n \in \text{rec.cone}(\mathcal{Q})$, i.e., we have that $Mu_n = 0$ and $\beta^T u_n > 0$. Then, from equation (7), we obtain

$$\phi(x_0 + \theta u_n + rv) \leq r^2\lambda_1 + 2r\|Mx_0 - \beta\|_2 - 2\theta\beta^T u_n \leq 0,$$

where the inequality follows from the lower bound on θ. The result follows since $B(x_0 + \theta u_n, r) \subseteq \mathcal{Q}$ is equivalent to $\phi(x_0 + \theta u_n + rv) \leq 0$ for all v with $\|v\| \leq 1$.

(ii) Notice that the last line in equation (7) is a quadratic function in θ, whose largest root is exactly $\Theta_- > 0$. Hence, we obtain $\phi(x_0+w+rv) \le 0$ for $\theta \ge \Theta_-$. The result follows since $B(x_0+\theta u_n, r) \subseteq \mathcal{Q}$ is equivalent to $\phi(x_0+\theta u_n+rv) \le 0$ for all v with $\|v\| \le 1$.

(iii) Let $\hat{w}$ be a feasible solution of problem (6), which exists due to Parts (i) and (ii). Then, by adding the constraint $\|w\|_2 \le \|\hat{w}\|_2$, we can make the feasible region of problem (6) compact without changing its optimal value. Since problem (6) reduces to the minimization of a continuous function over a compact feasible region, the minimizer is attained.

(iv) Part (iv) directly follows from the definition of problem (6) and due to the fact that its minimizer exists. □

In the next result, we show that if $\mathcal{Q}$ is a paraboloid, then the optimization problem (6) can be solved efficiently.

Proposition 5 *If M is positive semidefinite, problem (6) can be solved in polynomial time.*

Proof Due to equation (7), the condition $\phi(x_0+w+rv) \le 0$ for all v such that $\|v\|_2 \le 1$ can be written equivalently as

$$\phi(x_0) + w^T M w + 2(Mx_0 - \beta)^T w + \pi \le 0, \tag{8}$$

where $\pi \ge \max_v \{r^2 v^T M v + 2rv^T M(x_0+w) - 2r\beta^T v : v^T v \le 1\}$. Notice that this problem has a quadratic maximization objective subject to a single quadratic constraint. Due to the Inhomogeneous S-Lemma [5], its semidefinite programming (SDP) relaxation is exact, hence, we obtain the following equivalent inequality:

$$\pi \ge \max_{v,V} \left\{ r^2 \mathrm{Tr}(MV) + 2rv^T[M(x_0+w) - \beta] : \mathrm{Tr}(V) \le 1, \begin{bmatrix} V & v \\ v^T & 1 \end{bmatrix} \succeq 0 \right\}.$$

Passing to the dual problem, we obtain $\pi \ge \min_{\gamma_1,\gamma_2}\{\gamma_1+\gamma_2 : (9)\}$, where

$$\begin{bmatrix} I\gamma_1 & \\ & \gamma_2 \end{bmatrix} \succeq \begin{bmatrix} r^2 M & rM(x_0+w-\beta) \\ r(x_0+w-\beta)^T M & 0 \end{bmatrix}, \gamma_1 \ge 0. \tag{9}$$

Finally, we can rewrite problem (6) as $\min_{w,\pi,\gamma_1,\gamma_2}\{\|w\|_2 : \pi \ge \gamma_1+\gamma_2, (8), (9)\}$. Notice that this problem has an ℓ_2-norm minimization objective subject to linear, convex quadratic and positive semidefinite constraints. Therefore, it can be solved as an SDP, a problem known to be polynomially solvable [5]. □

5.1.3 Inner-approximating a quadratic set by an ellipsoid

Lemma 5 *Consider a point $\hat{x} \in \mathcal{Q}$ with $\hat{d} := M\hat{x} - \beta$. Let $D \in \mathbb{S}_{++}$ such that $D^2 \succeq M$. Then, the set defined as*

$$\check{\mathcal{Q}} := \left\{ x \in \mathbb{R}^n : \left\| Dx - \left(D\hat{x} - D^{-1}\hat{d} \right) \right\|_2 \le \|D^{-1}\hat{d}\|_2 \right\},$$

is an inner-approximation of $\mathcal{Q}$, that is, $\check{\mathcal{Q}} \subseteq \mathcal{Q}$.

Proof Notice that the set $\check{\mathcal{Q}}$ can be written as

$$\begin{aligned}\check{\mathcal{Q}} &= \left\{ x \in \mathbb{R}^n : \left[Dx - \left(D\hat{x} - D^{-1}\hat{d} \right) \right]^T \left[Dx - \left(D\hat{x} - D^{-1}\hat{d} \right) \right] \le [D^{-1}\hat{d}]^T [D^{-1}\hat{d}] \right\} \\ &= \{x \in \mathbb{R}^n : x^T D^2 x - 2x^T D^2 \hat{x} + 2x^T \hat{d} + \hat{x}^T D^2 \hat{x} - 2\hat{x}^T \hat{d} + \hat{d}^T D^{-2} \hat{d} \le \hat{d}^T D^{-2} \hat{d}\} \\ &= \{x \in \mathbb{R}^n : (x-\hat{x})^T D^2 (x-\hat{x}) + 2(x-\hat{x})^T \hat{d} \le 0\}.\end{aligned}$$

In addition, we have

$$\begin{aligned}2(x-\hat{x})^T \hat{d} &= 2(x-\hat{x})^T (M\hat{x} - \beta) = (-\hat{x}^T M \hat{x} + 2x^T M \hat{x}) - 2\beta^T x + (-\hat{x}^T M \hat{x} + 2\beta^T \hat{x}) \\ &\ge -[(x-\hat{x})^T M (x-\hat{x}) - x^T M x] - 2\beta^T x + \gamma = -(x-\hat{x})^T M (x-\hat{x}) + x^T M x - 2\beta^T x + \gamma,\end{aligned}$$

as $\hat{d} = M\hat{x} - \beta$ and $\gamma \ge -\hat{x}^T M \hat{x} + 2\beta^T \hat{x}$ (since $\hat{x} \in \mathcal{Q}$). Then, we obtain that

$$\begin{aligned}\check{\mathcal{Q}} \subseteq \check{\mathcal{Q}}' :=& \{x \in \mathbb{R}^n : (x-\hat{x})^T D^2 (x-\hat{x}) - (x-\hat{x})^T M (x-\hat{x}) + x^T M x - 2\beta^T x + \gamma \le 0\} \\ =& \{x \in \mathbb{R}^n : (x-\hat{x})^T (D^2 - M)(x-\hat{x}) + x^T M x - 2\beta^T x + \gamma \le 0\}.\end{aligned}$$

Finally, since $D^2 \succeq M$, we have $\check{\mathcal{Q}}' \subseteq \mathcal{Q}$, hence, we obtain $\check{\mathcal{Q}} \subseteq \mathcal{Q}$. □

5.2 Translation between different set representations

We conclude this section by providing "translations" between different set representations. Our first result shows that the linear inequality in (5) of an ellipsoid or a paraboloid can simply be omitted, hence, the SOCr set, in fact, has a trivial Qr representation in these cases.

Lemma 6 *Assume that* $\mathcal{S} = \{x \in \mathbb{R}^n : \|Ax - b\|_2 \leq c^T x - d\}$ *is nonempty,* $A^T A - cc^T$ *is positive semidefinite and that* $\begin{bmatrix} b \\ d \end{bmatrix}$ *does not belong to the image space of* $\begin{bmatrix} A \\ c^T \end{bmatrix}$. *Then, we have*

$$\mathcal{S} = \{x \in \mathbb{R}^n : x^T(A^T A - cc^T)x - 2(A^T b - cd)^T x + b^T b - d^2 \leq 0\}.$$

Proof Let $\mathcal{Q} = \{x \in \mathbb{R}^n : x^T(A^T A - cc^T)x - 2(A^T b - cd)^T x + b^T b - d^2 \leq 0\}$. Note that the set $\mathcal{Q}$ is convex as its quadratic term is defined by a positive semidefinite matrix. Since $\mathcal{S} = \{x \in \mathcal{Q} : c^T x \geq d\}$, to prove the lemma it suffices to show that $c^T x \geq d$ is a valid inequality for $\mathcal{Q}$. Assume for a contradiction that there exists a point $\hat{x} \in \mathcal{Q}$ such that $c^T \hat{x} < d$. Since $\mathcal{S} \neq \emptyset$ and $\mathcal{S} \subseteq \mathcal{Q}$, by convexity of $\mathcal{Q}$, we obtain that there must exist $x' \in \mathcal{Q}$ such that $c^T x' = d$. Note that $x' \in \mathcal{S}$ since $x' \in \mathcal{Q}$ and $c^T x' \geq d$. We conclude that $\|Ax' - b\|_2 \leq 0 = c^T x' - d$, and therefore $\begin{bmatrix} A \\ c^T \end{bmatrix} x' = \begin{bmatrix} b \\ d \end{bmatrix}$, a contradiction. □

The following fact states that an SOCr set can always be put in the Qr or Br forms.

Fact 4 *Let* $\mathcal{S} = \{x \in \mathbb{R}^n : \|Ax - b\|_2 \leq c^T x - d\}$ *be a set. Then, we can write* $\mathcal{S} = \{x \in \mathbb{R}^n : x^T M x - 2\beta^T x + \gamma \leq 0,\ g^T x \geq h\}$, *where* $M = A^T A - cc^T$, $\beta = A^T b - cd$, $\gamma = b^T b - d^2$, $g = c$ *and* $h = d$. *Moreover, if* M *is positive semidefinite, then* $g^T x \geq h$ *is redundant due to Lemma 6.*

The following fact states that an ellipsoid can always be put in the ellipsoid representation (Er) form even if it is given in its SOCr form.

Fact 5 *Let* $\mathcal{S} = \{x \in \mathbb{R}^n : \|Ax - b\|_2 \leq c^T x - d\}$ *be a bounded set. Then, we can write*

$$\mathcal{S} = \{x \in \mathbb{R}^n : \|Qx - p\|_2 \leq r\}, \tag{Er}$$

where Q *is such that* $Q^T Q = A^T A - cc^T$, $p = Q^{-T}(A^T b - dc)$, *and* $r^2 = (A^T b - dc)^T (A^T A - cc^T)^{-1}(A^T b - dc) - b^T b + d^2$.

The following fact states that an ellipsoid can always be put in the Er form even if it is given in its Qr form.

Fact 6 *Let* $\mathcal{S} = \{x \in \mathbb{R}^n : x^T M x - 2\beta^T x + \gamma \leq 0\}$ *be a bounded set. Then, we can write* $\mathcal{S} = \{x \in \mathbb{R}^n : \|Qx - p\|_2 \leq r\}$, *where* Q *is such that* $Q^T Q = M$, $p = Q^{-T}\beta$, *and* $r^2 = p^T p - \gamma$.

Notice that Facts 5 and 6 are not applicable in the case of paraboloids and hyperboloids as the matrix $A^T A - cc^T$ is not positive definite in those cases.

6 Proximity results for simple second-order conic IPs

In this section, we will focus on the proximity analysis for simple second-order conic IPs. We do not analyze directly a set of the form $\mathcal{S} = \{x \in \mathbb{R}^n : \|Ax - b\|_2 \leq c^T x - d\}$ but choose the representation that is most convenient for our purposes, namely we consider sets in Er and Qr form. The results for SOCr sets follow from Section 5.2: If an ellipsoid is given in the SOCr or Qr form, we can first translate it to the Er form using Facts 5 and 6, respectively. If a paraboloid, hyperboloid or translated cone is given in the SOCr form, we can first translate it to the Qr form using Fact 4.

We propose two different approaches to obtain our proximity results: i) proximity-driven, ii) integrality gap-driven.

6.1 Proximity-driven approach

The main idea behind the proximity-driven approach is as follows: Suppose that $\hat{x}$, a boundary point of the continuous relaxation (potentially representing its optimal solution), is at hand. Then, we aim to find a *large-enough* ellipsoid inside the feasible region of the continuous relaxation *close to* $\hat{x}$ so that it contains an integer point. This will lead to proximity bounds, using which we also derive integrality gap bounds.

6.1.1 Ellipsoid

Proposition 6 *Consider an ellipsoid* $\mathcal{S} = \{x \in \mathbb{R}^n : \|Qx - p\|_2 \leq r\}$ *with* $Q \in \mathbb{R}^{m\times n}$, $\text{rank}(Q) = n$, $p \in \mathbb{R}^m$, $r > 0$, *and assume that* $\mathcal{S} \cap \mathbb{Z}^n \neq \emptyset$. *Let* $\alpha \in \mathbb{R}^n$. *Suppose that a point* $\hat{x} \in \partial\mathcal{S}$ *is given. Then,*

$$\mathbf{Prox}_{\hat{x}}(Q, p, r) = \min_{x\in\mathcal{S}\cap\mathbb{Z}^n} \|x - \hat{x}\|_2 \leq 2\|Q(Q^TQ)^{-1}\|_2\mu(Q).$$

Moreover, we have

$$\mathbf{IG}(Q, p, r) = \min_{x\in\mathcal{S}\cap\mathbb{Z}^n} \alpha^T x - \min_{x\in\mathcal{S}} \alpha^T x \leq 2\|Q(Q^TQ)^{-1}\|_2\|\alpha\|_2\mu(Q).$$

Proof Let us fix the notation and establish some facts used in the proof. Firstly, let $\check{x} \in \mathcal{S} \cap \mathbb{Z}^n$. Secondly, since $Q \in \mathbb{R}^{m\times n}$ and $\text{rank}(Q) = n$, the matrix $(Q^TQ)^{-1}$ exists. Thirdly, due to the definition of covering radius (see Fact 2), there exists $\mu(Q)$ such that for all $x \in \mathbb{R}^n$, there exists $x' \in \mathbb{Z}^n$ with $\|Qx-Qx'\|_2 \leq \mu(Q)$.

The remainder of the proof is divided into two cases:
Case 1: $r \leq \mu(Q)$. Then,

$$\begin{aligned}\mathbf{Prox}_{\hat{x}}(Q, p, r) &= \min_{x\in\mathcal{S}\cap\mathbb{Z}^n} \|x - \hat{x}\|_2 \leq \|\check{x} - \hat{x}\|_2 = \|(Q^TQ)^{-1}(Q^TQ)(\check{x} - \hat{x})\|_2 = \|(Q^TQ)^{-1}Q^T(Q\check{x} - Q\hat{x})\|_2 \\ &\leq \|Q(Q^TQ)^{-1}\|_2\|(Q\check{x} - p) - (Q\hat{x} - p)\|_2 \leq \|Q(Q^TQ)^{-1}\|_2(r + r) \leq 2\|Q(Q^TQ)^{-1}\|_2\mu(Q),\end{aligned}$$

where the first inequality follows since $\check{x} \in \mathcal{S} \cap \mathbb{Z}^n$.
Case 2: $r > \mu(Q)$. In this case, we first choose $\tilde{x} \in \mathbb{R}^n$ such that $Q\tilde{x} \in [p, Q\hat{x}]$ and $\|(Q\tilde{x} - p) - (Q\hat{x} - p)\|_2 = \mu(Q)$. Note that $\tilde{x} \in \mathcal{S}$. Also, we have that there exists $x' \in \mathbb{Z}^n$ such that $\|Qx' - Q\tilde{x}\|_2 \leq \mu(Q)$. Then,

$$\begin{aligned}\mathbf{Prox}_{\hat{x}}(Q, p, r) &= \min_{x\in\mathcal{S}\cap\mathbb{Z}^n} \|x - \hat{x}\|_2 \leq \|x' - \hat{x}\|_2 = \|(Q^TQ)^{-1}(Q^TQ)(x' - \hat{x})\|_2 = \|(Q^TQ)^{-1}Q^T(Qx' - Q\hat{x})\|_2 \\ &= \|(Q^TQ)^{-1}Q^T[(Qx' - p) - (Q\tilde{x} - p) + (Q\tilde{x} - Q\hat{x})]\|_2 \\ &= \|Q(Q^TQ)^{-1}\|_2(\|(Qx' - p) - (Q\tilde{x} - p)\|_2 + \|(Q\tilde{x} - Q\hat{x})\|_2) \leq 2\|Q(Q^TQ)^{-1}\|_2\mu(Q),\end{aligned}$$

where the first inequality follows since $x' \in \mathcal{S} \cap \mathbb{Z}^n$. In both cases, we conclude that $\mathbf{Prox}_{\hat{x}}(Q, p, r) \leq 2\|Q(Q^TQ)^{-1}\|_2\mu(Q)$.

Finally, the upper bound on integrality gap is obtained as a direct consequence of the upper bound on proximity assuming that $\hat{x}$ is the optimal solution of the continuous relaxation. □

We remark that upper bounds on proximity and integrality gap derived above are independent of p, r and $\hat{x}$

Under the same setting as in Proposition 6, it turns out that the bound on the integrality gap can be improved to $2\|Q(Q^TQ)^{-1}\alpha\|_2\mu(Q)$. We formalize this fact in Proposition 7, whose proof is omitted here and given in Appendix B instead, due to its similarity to the proof of Proposition 6.

Proposition 7 *Assume that the assumptions of Proposition 6 hold true. Then,*

$$\mathbf{IG}(Q, p, r) = \min_{x\in\mathcal{S}\cap\mathbb{Z}^n} \alpha^T x - \min_{x\in\mathcal{S}} \alpha^T x \leq 2\|Q(Q^TQ)^{-1}\alpha\|_2\mu(Q).$$

6.1.2 Paraboloid

Proposition 8 *Consider a paraboloid* $\mathcal{S} = \{x \in \mathbb{R}^n : x^T Mx - 2\beta^T x + \gamma \leq 0\}$ *with* $M \in \mathbb{S}^n_+$, $\beta \in \mathbb{R}^n$, $\gamma \in \mathbb{R}$. *Assume that* $\mathcal{S} \cap \mathbb{Z}^n \neq \emptyset$ *and let* $\hat{x} \in \partial\mathcal{S}$. *Then,*

$$\mathbf{Prox}_{\hat{x}}(M, \beta, \gamma) = \min_{x \in \mathcal{S} \cap \mathbb{Z}^n} \|x - \hat{x}\|_2 \leq \frac{\sqrt{n}}{2} + \Gamma_{\hat{x}}\left(M, \beta, \gamma, \frac{\sqrt{n}}{2}\right),$$

where Γ *is defined as in* (6)*. Moreover, assuming that problem* $\min\{\alpha^T x : x \in \mathcal{S}\} > -\infty$ *with its unique minimizer* $\hat{x}$*, we have*

$$\mathbf{IG}(M, \beta, \gamma) = \min_{x \in \mathcal{S} \cap \mathbb{Z}^n} \alpha^T x - \min_{x \in \mathcal{S}} \alpha^T x \leq \|\alpha\|_2 \left[\frac{\sqrt{n}}{2} + \Gamma_{\hat{x}}\left(M, \beta, \gamma, \frac{\sqrt{n}}{2}\right)\right].$$

Proof The proximity result follows directly from Proposition 4-(iv) with $x_0 = \hat{x}$ and $r = \frac{\sqrt{n}}{2}$, and the integrality gap result is a consequence of the proximity result. □

Recall that the calculation of the parameter Γ requires solving an SDP as mentioned in the proof of Proposition 5. It can be replaced by a closed-form but weaker bound stated in Proposition 4-(i).

We remark that Proposition 8 implies that the upper bounds on both proximity and integrality gap depend on all problem parameters. However, under some special conditions, we can eliminate the dependence on all problem parameters, except the dimension, as stated in the next proposition.

Proposition 9 *Assume that the conditions of Proposition 8 hold. Let* λ_1 *be the largest eigenvalue of* M *and let* u_n *be the nonzero element of* rec.cone$(\mathcal{S})$ *with* $\|u_n\|_2 = 1$ *(see Fact 3). Then, for any* $\check{x} \in \partial\mathcal{S}$*, we have*

$$\|M\check{x} - \beta\|_2 \geq \lambda_1 \frac{\sqrt{n}}{2} \implies \mathbf{Prox}_{\check{x}}(M, \beta, \gamma) = \min_{x \in \mathcal{S} \cap \mathbb{Z}^n} \|x - \check{x}\|_2 \leq \sqrt{n}.$$

Moreover, assuming that $\alpha \in \mathbb{R}^n$ *with* $u_n^T \alpha > 0$*, for* $\hat{x}$ *we have*

$$\frac{u_n^T \alpha}{\|\alpha\|_2} \leq \frac{u_n^T \beta}{\lambda_1 \frac{\sqrt{n}}{2}} \Leftrightarrow \|M\check{x} - \beta\|_2 \geq \lambda_1 \frac{\sqrt{n}}{2}.$$

As a consequence, we obtain $\mathbf{IG}(M, \beta, \gamma) = \min_{x \in \mathcal{S} \cap \mathbb{Z}^n} \alpha^T x - \min_{x \in \mathcal{S}} \alpha^T x \leq \|\alpha\|_2 \sqrt{n}$.

Proof Let $\check{d} := M\check{x} - \beta$. Consider the set $\check{\mathcal{S}} = \{x \in \mathbb{R}^n : x^T Mx - 2\beta^T x + \gamma + (x - \check{x})^T (\lambda_1 I - M)(x - \check{x}) \leq 0\}$. Due to Lemma 5 with $D = \sqrt{\lambda_1} I$, we have that

$$\mathcal{S} \supseteq \check{\mathcal{S}} = \left\{x \in \mathbb{R}^n : \left\|x - \left(\check{x} - \frac{\check{d}}{\lambda_1}\right)\right\|_2 \leq \frac{\|\check{d}\|_2}{\lambda_1}\right\}.$$

Now, consider problem (6). Since $\check{x} \in \partial\mathcal{S}$, we must have that $\Gamma_{\check{x}}(M, \beta, \gamma, \sqrt{n}/2) \geq \sqrt{n}/2$. Notice that $w = \check{x} - \frac{\check{d}}{\lambda_1}$ is a feasible solution with an objective function of $\sqrt{n}/2$. Therefore, it must be an optimal solution as well. Hence, we conclude that the proximity result follows.

We will now prove the integrality gap result. Let $\hat{d} := M\hat{x} - \beta$. Note that we have $Mu_n = 0$ and $\beta^T u_n > 0$. Since $u_n^T \alpha > 0$, the problem $\min_{x \in \mathcal{S}} \alpha^T x$ is solvable. By the optimality condition, we must have that $\hat{d} = \phi\alpha$ for some $\phi < 0$. Now, notice that we have

$$u_n^T \hat{d} := \underbrace{u_n^T M}_{=0} \hat{x} - u_n^T \beta = \phi u_n^T \alpha \implies \phi = -\frac{u_n^T \beta}{u_n^T \alpha}.$$

Therefore, we have $\hat{d} = -\frac{u^T \beta}{u^T \alpha} \alpha$. By taking norm on both sides, we obtain that $\|\hat{d}\|_2 = \frac{u_n^T \beta}{u_n^T \alpha} \|\alpha\|_2$ and thus the equivalence of the inequalities follow. The condition stated in the proposition guarantees that there exists $x' \in \mathcal{S} \cap \mathbb{Z}^n$ such that $\|\hat{x} - x'\|_2 \leq \sqrt{n}$. Finally, the integrality gap result follows. □

Proposition 9 leads to two interesting observations: i) if the norm of the tangent is "large" at a boundary point for a quadratic set, then there exists a close by integer point and proximity is independent of the problem parameters, ii) if the angle between the objective vector and the recession cone is "large" for a parabola, then the norm of the tangent of the optimizer is "large" so that the integrality gap is independent of all the problem parameters.

6.1.3 Hyperboloid and translated cone

The following result is a consequence of Theorem 2 and Proposition 3.

Proposition 10 *Consider the set $\mathcal{S} = \{x \in \mathbb{R}^n : x^T Mx - 2\beta^T x + \gamma \leq 0, g^T x \geq h\}$ with $M \in \mathbb{S}^n$, $\beta \in \mathbb{R}^n$, $\gamma \in \mathbb{R}$, $h \in \mathbb{R}^n$, $g \in \mathbb{R}$ and assume that it is one branch of a two-sheet hyperboloid or a translated cone. Assume that $\mathcal{S} \cap \mathbb{Z}^n \neq \emptyset$ and let $\hat{x} \in \partial\mathcal{S}$. Then,*

$$\mathbf{Prox}_{\hat{x}}(M, \beta, \gamma) = \min_{x \in \mathcal{S} \cap \mathbb{Z}^n} \|x - \hat{x}\|_2 \leq \frac{\sqrt{n}}{2}(1 - \lambda_n(M)).$$

Moreover, assuming that the problem $\hat{z} := \inf_{x \in \mathcal{S}} \alpha^T x$ is bounded below for $\alpha \in \mathbb{R}^n$, we have

$$\mathbf{IG}(M, \beta, \gamma) = \inf_{x \in \mathcal{S} \cap \mathbb{Z}^n} \alpha^T x - \hat{z} \leq \|\alpha\|_2 \frac{\sqrt{n}}{2}(1 - \lambda_n(M)).$$

Proof Due to Proposition 3, we have that $\Psi_{\mathbf{K}, \|\cdot\|_2} = -\frac{1}{\lambda_n(M)}$, where $\mathbf{K} = \text{rec.cone}(\mathcal{S}) = \{x : x^T Mx \leq 0, \beta^T x \geq 0\}$. Then, the results follow due to Theorem 2. □

The following result is similar to the proximity result stated in Proposition 9, hence, its proof is omitted.

Proposition 11 *Assume that the conditions of Proposition 10 hold. Let λ_1 be the largest eigenvalue of M. Then, for any $\check{x} \in \partial\mathcal{S}$, we have*

$$\|M\check{x} - \beta\|_2 \geq \lambda_1 \frac{\sqrt{n}}{2} \implies \mathbf{Prox}_{\check{x}}(M, \beta, \gamma) = \min_{x \in \mathcal{S} \cap \mathbb{Z}^n} \|x - \check{x}\|_2 \leq \sqrt{n}.$$

We remark that upper bounds on proximity and integrality gap derived above are independent of β, γ, g and h.

6.2 Integrality gap driven approach

For $i = 1, \ldots, n$, we denote e_i the ith vector in the canonical basis of $\mathbb{R}^n$, that is, e_i is the vector with a 1 in the ith component and zeros otherwise. Let U be an unimodular $n \times n$ matrix such that $U^T \alpha = e_n$. Then for any set $\mathcal{S} \subseteq \mathbb{R}^n$ we have $\inf\{\alpha^T x : x \in \mathcal{S}\} = \inf\{x_n : x \in U\mathcal{S}\}$. Since U is unimodular, $\inf\{\alpha^T x : x \in \mathcal{S} \cap \mathbb{Z}^n\} = \inf\{x_n : x \in U\mathcal{S} \cap \mathbb{Z}^n\}$, and thus and we obtain $\mathbf{IG}(\mathcal{S}) = \mathbf{IG}(U\mathcal{S})$. For simplicity, in this section we will assume that $\alpha = e_n$, which can be done without loss of generality by the previous discussion.

In this section, we want to find an upper bound for $\mathbf{IG}(\mathcal{S})$, consequently, the main goal is to find a vector $z \in \mathcal{S} \cap \mathbb{Z}^n$ such that $z_n - \inf\{x_n : x \in \mathcal{S}\}$ is as small as possible, as $\mathbf{IG}(\mathcal{S}) \leq z_n - \inf\{x_n : x \in \mathcal{S}\}$. Note that the integral vector z maybe be far away from the set of optimal solutions to the continuous relaxation, and thus the bound on proximity given by z, in terms of distance between optimal solutions to the integer and continuous optimization problem, could be very large.

Throughout this section, unless explicitly noted, we will assume the following:

Assumption 2

- *M is an $n \times n$ matrix with at least $n - 1$ positive eigenvalues, $\beta \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$.*
- *The quadratic set $\mathcal{Q} = \{x \in \mathbb{R}^n : x^T Mx - 2\beta^T x + \gamma \leq 0\}$ is not a one-sheet hyperboloid (see Proposition 1 for a characterization of all possible cases for the set $\mathcal{Q}$).*
- *The convex set $\mathcal{S}$ we study in this section is $\mathcal{S} = \mathcal{Q}$ (if M is positive semidefinite) or $\mathcal{S}$ is a* branch *of the nonconvex set $\mathcal{Q}$ (if M has a negative eigenvalue).*
- *$\mathcal{S} \neq \emptyset$ and $\inf\{x_n : x \in \mathcal{S}\} > -\infty$.*

The bounds we obtain depend on whether the continuous relaxation has a unique optimal solution or not. We will present our results for these two cases next.

6.2.1 The continuous relaxation has a unique optimal solution

Unless explicitly noted, in this section we assume the following:

Assumption 3 *The optimization problem $\delta_{\inf} = \inf\{x_n : x \in \mathcal{S}\}$ has a unique optimal solution. In other words, by Proposition 2 either $\mathcal{S}$ is a strictly convex set and $x_n = \delta_{\inf}$ does not define an asymptote for $\mathcal{S}$, or $\mathcal{S}$ is a translated cone and $\{x \in \mathbb{R}^n : x_n = \delta_{\inf}\}$ only intersects the apex of $\mathcal{S}$.*

We denote $\delta_{\inf} = \inf\{x_n : x \in \mathcal{S}\}$ and $\delta_{\sup} = \sup\{x_n : x \in \mathcal{S}\}$ and for any $\delta \in \mathbb{R}$, we let $\mathcal{S}_\delta = \{x \in \mathcal{S} : x_n = \delta\}$. The following straightforward result implies that $\mathcal{S}_\delta$ is a branch of the quadratic set $\mathcal{Q}_\delta$ in the hyperplane defined by $x_n = \delta$.

Lemma 7 *If $\mathcal{Q}$ is a quadratic set (not necessarily satisfying Assumption 3), then the set $\mathcal{Q}_\delta = \{x \in \mathcal{Q} : x_n = \delta\}$ is the following quadratic set in the hyperplane $H = \{x \in \mathbb{R}^n : x_n = \delta\}$: $\mathcal{Q}_\delta = \{(\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \bar{x}^T \bar{M} \bar{x} + 2(\delta a^T - \bar{\beta}^T)\bar{x} + (a_0\delta^2 - 2\beta_n\delta + \gamma) \leq 0,\ x_n = \delta\}$.*

The main idea of the approach is as follows (see Figure 3 for an illustration): we will show in Proposition 12 that under Assumptions 2 and 3 , the set $\mathcal{S}_\delta$ is an ellipsoid. Then we will find the smallest integer δ^* such that the radius $\mathcal{S}_{\delta^*}$ is large enough so that this ellipsoid contains an $n-1$ dimensional integral vector (we will use Fact 2 on the covering radius of a lattice). Then we will obtain the following upper bound: $\mathbf{IG}(\mathcal{S}) \leq \delta^* - \delta_{\inf}$.

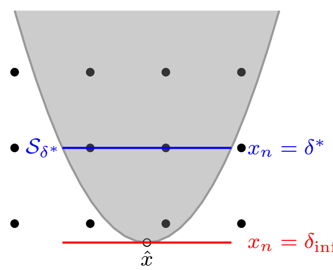

Fig. 3 The main idea behind the **IG**-driven approach.

The following lemma is crucial to obtain our results.

Lemma 8 *Let the matrix M be decomposed as $M = \begin{bmatrix} \bar{M} & a \\ a^T & a_0 \end{bmatrix}$. If $\delta_{\inf} = \inf\{x_n : x \in \mathcal{S}\} > -\infty$ and the associated optimization problem has a unique optimal solution, then*

(i) $\mathcal{S} = \{x \in \mathbb{R}^n : x^T M x - 2\beta^T x + \gamma \leq 0,\ x_n \geq \delta_{\inf}\}$.
(ii) $\bar{M}$ *is positive definite. As a consequence,* $\det(M) = \det(\bar{M})(a_0 - a^T \bar{M}^{-1} a)$.

Proof

(i) If $\mathcal{Q} = \{x \in \mathbb{R}^n : x^T M x - 2\beta^T x + \gamma \leq 0\}$ is an ellipsoid or a paraboloid, then $\mathcal{Q}$ is a convex set and there is nothing to show. If $\mathcal{Q}$ is a translated cone or an hyperboloid, then the result follows from Proposition 2.
(ii) We will prove the statement via a case-by-case analysis by using the Schur's complement of the given block decomposition of M.
Case 1: Assume M has a zero eigenvalue. First observe that by Sylvester's criterion, we have $a_0 \geq 0$. If $a = 0$ and $a_0 = 0$, then by the block decomposition of M, we obtain that $\bar{M}$ must have $n-1$ positive eigenvalues, which implies $\bar{M}$ is positive definite. Assume now that $a \neq 0$. Then we can compute $f(x) := x^T M x = \bar{x}^T \bar{M} \bar{x} + 2a^T \bar{x} x_n + a_0 x_n^2$. It follows that if $a_0 = 0$ then for large enough $\lambda \geq 0$ $\hat{x} = (a, -\lambda)$ satisfies $f(\hat{x}) = a^T \bar{M} a - 2a^T a\lambda < 0$, a contradiction with M being positive semidefinite. Therefore, $a_0 > 0$, and hence for a fix $\bar{x} \in \mathbb{R}^{n-1}$ we can minimize $f(\bar{x}, x_n)$ as a function of x_n. For the minimizer x_n^*, we obtain $f(\bar{x}, x_n^*) = \bar{x}^T \bar{M} \bar{x} - \frac{\bar{x}^T a a^T \bar{x}}{a_0}$. Since M is positive semidefinite, we have $f(\bar{x}, x_n^*) \geq 0$ for all $\bar{x} \in \mathbb{R}^{n-1}$. Which is equivalent to the matrix $\bar{M} - aa^T$ being positive semidefinite. Since $a \neq 0$, we must have $\bar{M}$ is positive definite.

The final subcase is that $a = 0$ and $a_0 > 0$. Notice that $x \in \mathcal{S}$ is equivalent to $0 \geq x^T M x - 2\beta^T x + \gamma = a_0 x_n^2 - 2\beta_n x_n + \bar{x}^T \bar{M} \bar{x} - 2\bar{\beta}^T \bar{x} + \gamma$. Due to Assumption 1 and Proposition 1, the system $Mx = \beta$ does not have a solution, meaning that the system $\bar{M}\bar{x} = \bar{\beta}$ does not have a solution. This implies that there exists $\bar{M}\bar{x} = 0$ and $\bar{\beta}^T \bar{x} > 0$. Finally, notice that for any $x_n \in \mathbb{R}$, there exists $\lambda = \frac{a_0 x_n^2 - 2\beta_n x_n + \gamma}{\bar{\beta}^T \bar{x}}$ such that $(\lambda \bar{x}, x_n) \in \mathcal{S}$. This implies that the problem $\inf\{x_n : x \in \mathcal{S}\}$ is unbounded below, which is a contradiction.
Case 2: M has n positive eigenvalues. By the Eigenvalue Interlacing Theorem, all eigenvalues of $\bar{M}$ must be positive, and therefore, $\bar{M}$ is positive definite.
Case 3: M has a negative eigenvalue. Firstly, notice that since M is invertible, we obtain

$$M^{-1} = \frac{1}{a^T \bar{M}^{-1} a - a_0} \begin{bmatrix} (a^T \bar{M}^{-1} a - a_0)\bar{M}^{-1} - \bar{M}^{-1} a a^T \bar{M}^{-1} & \bar{M}^{-1} a \\ a^T \bar{M}^{-1} & -1 \end{bmatrix}.$$

Since $\delta_{\inf} > -\infty$ and the associated optimization problem is solvable, then by parts *(i)-(iii)* of Proposition 2, we must have $e_n^T M^{-1} e_n < 0$, equivalently $a_0 - a^T \bar{M}^{-1} a < 0$. Since $\det(M)$ is the product of the eigenvalues of M we must have $\det(M) < 0$. On the other hand, by the Eigenvalue Interlacing Theorem, $\bar{M}$ has at most one negative eigenvalue. Now since $a_0 - a^T \bar{M}^{-1} a < 0$, we have $\det(M) = (a_0 - a^T \bar{M}^{-1} a) \det(\bar{M})$. Therefore, we conclude that $\det(\bar{M}) > 0$, and thus all eigenvalues of $\bar{M}$ are positive. Hence, $\bar{M}$ is positive definite.
The final assertion of the lemma follows from a property of Schur's complement.

□

In the next result we show that $\mathcal{S}_\delta$ is an ellipsoid, and in particular, we find an expression for the square of its radius.

Proposition 12 *For any $\delta \in [\delta_{\inf}, \delta_{\sup}]$ with $\delta \in \mathbb{R}$, the set $\mathcal{S}_\delta = \{x \in S : x_n = \delta\}$ can be written as $\mathcal{S}_\delta = \mathcal{E}_\delta \times \{\delta\}$, where $\mathcal{E}_\delta = \{x \in \mathbb{R}^{n-1} : \|\bar{Q}x - p(\delta)\|_2 \leq r_{\mathcal{E}}(\delta)\}$ is a nonempty ellipsoid defined by the following parameters:*

- *An invertible matrix $\bar{Q}$ such that $\bar{M} = \bar{Q}^T \bar{Q}$.*
- *$r_{\mathcal{E}}^2(\delta) = q_2 \delta^2 + q_1 \delta + q_0$, where $q_2 = a^T \bar{M}^{-1} a - a_0$, $q_1 = 2(\beta_n - \bar{\beta}^T \bar{M}^{-1} a)$ and $q_0 = \bar{\beta}^T \bar{M}^{-1} \bar{\beta} - \gamma$.*
- *$p(\delta) = -\bar{Q}^{-1}(\delta a - \bar{\beta})$.*

Proof First, observe that Assumption 3 implies that $\mathcal{S}_{\delta_{\inf}}$ is a singleton, hence a compact set. Since the recession cone of the convex set $\mathcal{S}_\delta$ is independent of $\delta \in [\delta_{\inf}, \delta_{\sup}]$, as δ only affects the right-hand side of the equality $x_n = \delta$ in the definition of $\mathcal{S}_\delta$, we obtain that $\mathcal{S}_\delta$ is a nonempty bounded set.

On the other hand, by Lemmas 7 and 8, $\mathcal{Q}_\delta = \mathcal{S}_\delta$ and there exists an invertible matrix $\bar{Q}$ such that $\bar{M} = \bar{Q}^T \bar{Q}$. Therefore, we obtain

$$\begin{aligned}
\mathcal{S}_\delta &= \{(\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \bar{x}^T \bar{Q}^T \bar{Q} \bar{x} + 2(\delta a^T - \bar{\beta}^T) \bar{Q}^{-1} \bar{Q} \bar{x} + (a_0 \delta^2 - 2\beta_n \delta + \gamma) \leq 0,\ x_n = \delta\} \\
&= \{(\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \bar{x}^T \bar{Q}^T \bar{Q} \bar{x} + 2[\bar{Q}^{-1}(\delta a - \bar{\beta})]^T \bar{Q} \bar{x} + (a_0 \delta^2 - 2\beta_n \delta + \gamma) \leq 0,\ x_n = \delta\} \\
&= \{(\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : [\bar{Q}\bar{x} + \bar{Q}^{-1}(\delta a - \bar{\beta})]^T [\bar{Q}\bar{x} + \bar{Q}^{-1}(\delta a - \bar{\beta})] + (a_0 \delta^2 - 2\beta_n \delta + \gamma) - [\bar{Q}^{-1}(\delta a - \bar{\beta})]^T \bar{Q}^{-1}(\delta a - \bar{\beta}) \leq 0,\ x_n = \delta\} \\
&= \{(\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \|\bar{Q}\bar{x} + \bar{Q}^{-1}(\delta a - \bar{\beta})\|^2 \leq (\delta a - \bar{\beta})^T \bar{Q}^{-T} \bar{Q}^{-1}(\delta a - \bar{\beta}) - (a_0 \delta^2 - 2\beta_n \delta + \gamma),\ x_n = \delta\} = \mathcal{E}_\delta \times \{\delta\},
\end{aligned}$$

where $\mathcal{E}_\delta = \{\bar{x} \in \mathbb{R}^{n-1} : \|\bar{Q}\bar{x} + \bar{Q}^{-1}(\delta a - \bar{\beta})\|^2 \leq (\delta a - \bar{\beta})^T \bar{Q}^{-T} \bar{Q}^{-1}(\delta a - \bar{\beta}) - (a_0 \delta^2 - 2\beta_n \delta + \gamma)\}$.

Since S_δ is a nonempty bounded set it follows that $\mathcal{E}_\delta$ is a nonempty bounded set, and therefore we obtain that $\mathcal{E}_\delta$ is a nonempty ellipsoid. Moreover, by Fact 6, we obtain the square of the radius of $\mathcal{E}_\delta$ as $r_{\mathcal{E}}^2(\delta) = (\delta a - \bar{\beta})^T \bar{M}^{-1}(\delta a - \bar{\beta}) - (a_0 \delta^2 - 2\beta_n \delta + \gamma) = (a^T \bar{M}^{-1} a - a_0)\delta^2 + 2(\beta_n - \bar{\beta}^T \bar{M}^{-1} a)\delta + (\bar{\beta}^T \bar{M}^{-1} \bar{\beta} - \gamma) = q_2 \delta^2 + q_1 \delta + q_0$, and the center of $\mathcal{E}_\delta$ as $p(\delta) = -\bar{Q}^{-1}(\delta a - \bar{\beta})$. □

For the quadratic function $r_{\mathcal{E}}^2(\delta) = q_2 \delta^2 + q_1 \delta + q_0$, we have that the solutions to the equation $r_{\mathcal{E}}^2(\delta) = \upsilon$ for any $\upsilon \in \mathbb{R}$ with $q_1^2 - 4q_2(q_0 - \upsilon) \geq 0$ are given by $\delta_{1,2}^{\upsilon} = \frac{-q_1 \pm \sqrt{q_1^2 - 4q_2(q_0 - \upsilon)}}{2q_2}$, where we will assume that $\delta_1^{\upsilon} \leq \delta_2^{\upsilon}$. We use these facts in the proofs below without explicitly referring to them.

The bounds for the integrality gap when $\mathcal{S}$ is an ellipsoid are given in the following proposition.

Proposition 13 *Let $\mathcal{S} := \{x \in \mathbb{R}^n : x^T M x - 2\beta^T x + \gamma \leq 0\}$ be an ellipsoid. Then* *(i) If* $r^2_{\mathcal{E}}(-q_1/2q_2) = \frac{q_1^2-4q_2q_0}{-4q_2} < \mu(\bar{Q})^2 - q_2/4$*, then*

$$\mathbf{IG}(\mathcal{S}) \leq \frac{\sqrt{q_1^2 - 4q_2q_0}}{-q_2} \leq 2\sqrt{\frac{\mu(\bar{Q})^2}{-q_2} + \frac{1}{4}}.$$

(ii) If $r^2_{\mathcal{E}}(-q_1/2q_2) = \frac{q_1^2-4q_2q_0}{-4q_2} \geq \mu(\bar{Q})^2 - q_2/4$*, then*

$$\mathbf{IG}(\mathcal{S}) \leq \left\lceil \frac{-q_1 + \sqrt{q_1^2 - 4q_2(q_0 - \mu(\bar{Q})^2)}}{2q_2} \right\rceil - \frac{-q_1 + \sqrt{q_1^2 - 4q_2q_0}}{2q_2} \leq \frac{\mu(\bar{Q})}{\sqrt{-q_2}} + 1.$$

Proof Since $\mathcal{S}_\delta = \{x \in \mathcal{S} : x_n = \delta\} \neq \emptyset$ for any $\delta \in [\delta_{\text{inf}}, \delta_{\text{sup}}]$, by Proposition 12, we can write $\mathcal{S}_\delta = \mathcal{E}_\delta \times \{\delta\}$ and the square of the radius of the ellipsoid $\mathcal{E}_\delta$ is given by $r^2_{\mathcal{E}}(\delta) = q_2\delta^2 + q_1\delta + q_0$. Moreover, since the matrices M and $\bar{M}$ are positive definite (by Lemma 8) and $\det(M) = \det(\bar{M})(-q_2)$, we obtain that $q_2 < 0$. Also, note that $\delta_{\text{inf}} = \delta_1^0 \leq \delta_2^0 = \delta_{\text{sup}}$ since $\mathcal{S}_{\delta_{\text{inf}}}$ and $\mathcal{S}_{\delta_{\text{sup}}}$ are a singleton so the radii of the associated ellipsoids $\mathcal{E}_{\delta_{\text{inf}}}$ and $\mathcal{E}_{\delta_{\text{sup}}}$ are equal to zero.

(i) We assume that $r^2_{\mathcal{E}}(-q_1/2q_2) = \frac{q_1^2-4q_2q_0}{-4q_2} < \mu(\bar{Q})^2 - q_2/4$, then

$$\mathbf{IG}(\mathcal{S}) \leq \delta_{\text{sup}} - \delta_{\text{inf}} = \frac{\sqrt{q_1^2 - 4q_2q_0}}{-q_2} \leq \frac{\sqrt{-4q_2(\mu(\bar{Q})^2 - q_2/4)}}{-q_2} = 2\sqrt{\frac{\mu(\bar{Q})^2}{-q_2} + \frac{1}{4}}.$$

(ii) We assume that $r^2_{\mathcal{E}}(-q_1/2q_2) = \frac{q_1^2-4q_2q_0}{-4q_2} \geq \mu(\bar{Q})^2 - q_2/4$. Let $\delta_1^{\mu(\bar{Q})} \leq \delta_2^{\mu(\bar{Q})}$ be such that $r^2_{\mathcal{E}}(\delta_1^{\mu(\bar{Q})}) = r^2_{\mathcal{E}}(\delta_2^{\mu(\bar{Q})}) = \mu(\bar{Q})^2$, then we obtain

$$\delta_2^{\mu(\bar{Q})} - \delta_1^{\mu(\bar{Q})} = \frac{\sqrt{q_1^2 - 4q_2(q_0 - \mu(\bar{Q})^2)}}{-q_2} \geq \frac{\sqrt{-4q_2\mu(\bar{Q})^2 + q_2^2 + 4q_2\mu(\bar{Q})^2}}{-q_2} = 1.$$

This implies that $z_n := \lceil \delta_1^{\mu(\bar{Q})} \rceil \in [\delta_1^{\mu(\bar{Q})}, \delta_2^{\mu(\bar{Q})}]$. Therefore, since $r^2_{\mathcal{E}}(\cdot)$ is a concave parabola, we have $r^2_{\mathcal{E}}(z_n) \geq \mu(\bar{Q})^2$ and thus by definition of $\mu(\bar{Q})$ (see Fact 2), there is a feasible solution $\bar{z} \in \mathbb{Z}^{n-1}$ in the $n-1$ dimensional ellipsoid $\mathcal{E}_{z_n}$. Since $\mathcal{S}_{z_n} = \mathcal{E}_{z_n} \times \{z_n\}$, we obtain the feasible solution $(\bar{z}, z_n) \in \mathcal{S} \cap (\mathbb{Z}^{n-1} \times \mathbb{Z})$ with objective function value $z_n = \lceil \delta_1^{\mu(\bar{Q})} \rceil$.
Therefore, we conclude

$$\begin{aligned}
\mathbf{IG}(\mathcal{S}) \leq \lceil \delta_1^{\mu(\bar{Q})} \rceil - \delta_{\text{inf}} &= \left\lceil \frac{-q_1 + \sqrt{q_1^2 - 4q_2(q_0 - \mu(\bar{Q})^2)}}{2q_2} \right\rceil - \frac{-q_1 + \sqrt{q_1^2 - 4q_2q_0}}{2q_2} \\
&\leq \frac{\sqrt{q_1^2 - 4q_2q_0}}{-2q_2} - \frac{\sqrt{q_1^2 - 4q_2(q_0 - \mu(\bar{Q})^2)}}{-2q_2} + 1 \leq \frac{\sqrt{-4q_2q_0 + 4q_2(q_0 - \mu(\bar{Q})^2)}}{-2q_2} + 1 \\
&= \frac{\sqrt{-4q_2\mu(\bar{Q})^2}}{-2q_2} + 1 = \frac{\mu(\bar{Q})}{\sqrt{-q_2}} + 1.
\end{aligned}$$

□

In the following proposition we obtain the bounds for the integrality gap when $\mathcal{S}$ is a paraboloid.

Proposition 14 *Let* $\mathcal{S} := \{x \in \mathbb{R}^n : x^T M x - 2\beta^T x + \gamma \leq 0\}$ *be a paraboloid. Then,*

$$\mathbf{IG}(\mathcal{S}) \leq \left\lceil \frac{\mu(\bar{Q})^2 - q_0}{q_1} \right\rceil - \frac{-q_0}{q_1} \leq \frac{\mu(\bar{Q})^2}{q_1} + 1. \tag{10}$$

Proof Since $\mathcal{S}$ is a parabola and $\delta_{\text{inf}} > -\infty$, we conclude that $\delta_{\text{sup}} = +\infty$ for otherwise $\mathcal{S}$ does not contain arbitrarily large balls and thus we obtain a contradiction to Proposition 4-(i). Therefore, we obtain that $\mathcal{S}_\delta = \{x \in S : x_n = \delta\} \neq \emptyset$ for any $\delta \geq \inf\{x_n : x \in \mathcal{S}\}$. Now by Proposition 12, we can write $\mathcal{S}_\delta = \mathcal{E}_\delta \times \{\delta\}$ and the square of the radius of the ellipsoid $\mathcal{E}_\delta$ is given by $r^2_{\mathcal{E}}(\delta) = q_2\delta^2 + q_1\delta + q_0$. Moreover, since the matrix

M is not invertible, $\bar{M}$ is positive definite (by Lemma 8) and $\det(M) = \det(\bar{M})(-q_2)$, we obtain that $q_2 = 0$, so we have that the square of the radius is $r^2_{\mathcal{E}}(\delta) = q_1\delta + q_0$.

We show next that $q_1 = 2(\beta_n - \bar{\beta}^T\bar{M}^{-1}a) > 0$. By Fact 3, rec.cone$(\mathcal{S}) = \{x \in \mathbb{R}^n : Mx = 0,\ \beta^T x \geq 0\}$. By using the block decomposition of M, the solution of the system $Mx = 0$ is the linear subspace generated by the eigenvector $u_n = (-\bar{M}^{-1}a, 1)$ associated to $\lambda_n = 0$. Now, since $\inf\{x_n : x \in \mathcal{S}\} > -\infty$, we must have that $e_n^T d \geq 0$ for all $d \in$ rec.cone$(\mathcal{S})$. Hence, we obtain rec.cone$(\mathcal{S}) = \{\lambda(-\bar{M}^{-1}a, 1) : \lambda \geq 0\}$. This implies that $\beta^T u_n \beta_n - \bar{\beta}^T\bar{M}^{-1}a \geq 0$. Moreover, since $\mathcal{S}$ is a paraboloid, the system $Mx = \beta$ does not have a solution (see Proposition 1), so we must have $\beta^T u_n = \beta_n - \bar{\beta}^T\bar{M}^{-1}a > 0$.

Now $\delta_{\inf} = \inf\{x_n : x \in \mathcal{S}\}$ satisfies $r^2_{\mathcal{E}}(\delta_{\inf}) = 0$ as $\mathcal{S}_{\delta_{\inf}}$ is a singleton, so the radius of the associated ellipsoid $\mathcal{E}_{\delta_{\inf}}$ is equal to zero. We then have $\delta_{\inf} = \frac{-q_0}{q_1}$. Also, $\delta^{\mu(\bar{Q})} = \frac{\mu(\bar{Q})^2 - q_0}{q_1}$ is such that $r^2_{\mathcal{E}}(\delta^{\mu(\bar{Q})}) = \mu(\bar{Q})^2$ and $\delta^{\mu(\bar{Q})} \geq \delta_{\inf}$ as $r^2_{\mathcal{E}}(\cdot)$ is an increasing function ($q_1 > 0$).

We have $z_n := \lceil \delta^{\mu(\bar{Q})} \rceil \geq \delta^{\mu(\bar{Q})}$. Therefore, since $r^2_{\mathcal{E}}(\cdot)$ is an increasing function, we have $r^2_{\mathcal{E}}(z_n) \geq \mu(\bar{Q})^2$ and thus by definition of $\mu(\bar{Q})$ (see Fact 2), there is a feasible solution $\bar{z} \in \mathbb{Z}^{n-1}$ in the $n-1$ dimensional ellipsoid $\mathcal{E}_{z_n}$. Since $\mathcal{S}_{z_n} = \mathcal{E}_{z_n} \times \{z_n\}$, we obtain the feasible solution $(\bar{z}, z_n) \in \mathcal{S} \cap (\mathbb{Z}^{n-1} \times \mathbb{Z})$ with objective function value $z_n = \lceil \delta^{\mu(\bar{Q})} \rceil$.

Therefore, we conclude

$$\mathbf{IG} \leq \lceil \delta^{\mu(\bar{Q})} \rceil - \delta_{\inf} = \left\lceil \frac{\mu(\bar{Q})^2 - q_0}{q_1} \right\rceil - \frac{-q_0}{q_1} \leq \left(\frac{\mu(\bar{Q})^2 - q_0}{q_1} + 1 \right) - \frac{-q_0}{q_1} = \frac{\mu(\bar{Q})^2}{q_1} + 1.$$

□

Remark 3 *We note that the bound in Proposition 14 is tight for Example 5 under the choices of parameters given in* (4)*. In fact, our bound in* (10) *is* $\frac{(1/2)^2}{1/(4N)} + 1 = N + 1$, *which converges to the actual integrality gap of* $\mathbf{IG} = N + 5/(16N) - 1/2$ *for large* N.

The bounds for the integrality gap when $\mathcal{S}$ is an hyperboloid or translated cone are given in the following proposition.

Proposition 15 *Let* $\mathcal{S} := \{x \in \mathbb{R}^n : x^T Mx - 2\beta^T x + \gamma \leq 0,\ x_n \geq \delta_{\inf}\}$ *be one branch of a two-sheet hyperboloid or a translated cone. Then,*

$$\mathbf{IG}(\mathcal{S}) \leq \left\lceil \frac{-q_1 + \sqrt{q_1^2 - 4q_2(q_0 - \mu(\bar{Q})^2)}}{2q_2} \right\rceil - \frac{-q_1 + \sqrt{q_1^2 - 4q_2 q_0}}{2q_2} \leq \frac{\mu(\bar{Q})}{\sqrt{q_2}} + 1.$$

Proof Since M is not positive definite, we conclude that $\delta_{\sup} = +\infty$ for otherwise $\mathcal{S}$ does not contain arbitrarily large balls and thus we obtain a contradiction to Proposition 4-(iv). Therefore, we obtain that $\mathcal{S}_\delta = \{x \in \mathcal{S} : x_n = \delta\} \neq \emptyset$ for any $\delta \geq \delta_{\inf}$. Now by Proposition 12, we can write $\mathcal{S}_\delta = \mathcal{E}_\delta \times \{\delta\}$ and the square of the radius of the ellipsoid $\mathcal{E}_\delta$ is given by $r^2_{\mathcal{E}}(\delta) = q_2\delta^2 + q_1\delta + q_0$. Moreover, since the matrix M is not positive definite and the matrix $\bar{M}$ is positive definite (by Lemma 8), by Sylvester's criterion we must have that $\det(M) < 0$. Since $\det(M) = \det(\bar{M})(-q_2)$, we obtain that $q_2 > 0$.

Now $\delta_{\inf}$ satisfies $r^2_{\mathcal{E}}(\delta_{\inf}) = 0$ as $\mathcal{S}_{\delta_{\inf}}$ is a singleton, so the radius of the associated ellipsoid $\mathcal{E}_{\delta_{\inf}}$ is equal to zero. By Proposition 2, since $\delta_{\inf} > -\infty$ and $\delta_{\sup} = +\infty$, we obtain $\delta_{\inf} = \frac{-q_1 + \sqrt{q_1^2 - 4q_2q_0}}{2q_2}$ (the largest root of $r^2_{\mathcal{E}}(\delta_{\inf}) = 0$ since the function $r^2_{\mathcal{E}}(\cdot)$ must be increasing for $\delta \geq \delta_{\inf}$).

Let $\delta_1^{\mu(\bar{Q})} \leq \delta_2^{\mu(\bar{Q})}$ be such that $r^2_{\mathcal{E}}(\delta_1^{\mu(\bar{Q})}) = r^2_{\mathcal{E}}(\delta_2^{\mu(\bar{Q})}) = \mu(\bar{Q})^2$. In particular, $\delta_2^{\mu(\bar{Q})} = \frac{-q_1 + \sqrt{q_1^2 - 4q_2(q_0 - \mu(\bar{Q})^2)}}{2q_2}$ and therefore $\delta_2^{\mu(\bar{Q})} > \delta_{\inf}$. We have $z_n := \lceil \delta_2^{\mu(\bar{Q})} \rceil \geq \delta_2^{\mu(\bar{Q})}$. Therefore, since $r^2_{\mathcal{E}}(\cdot)$ is an increasing function for $\delta \geq \delta_{\inf}$, we have $r^2_{\mathcal{E}}(z_n) \geq \mu(\bar{Q})^2$ and thus, by definition of $\mu(\bar{Q})$ (see Fact 2), there is a feasible solution $\bar{z} \in \mathbb{Z}^{n-1}$ in the $n-1$ dimensional ellipsoid $\mathcal{E}_{z_n}$. Since $\mathcal{S}_{z_n} = \mathcal{E}_{z_n} \times \{z_n\}$, we obtain the feasible solution $(\bar{z}, z_n) \in \mathcal{S} \cap (\mathbb{Z}^{n-1} \times \mathbb{Z})$ with objective function value $z_n = \lceil \delta_2^{\mu(\bar{Q})} \rceil$.

Therefore, we conclude

$$\mathbf{IG}(\mathcal{S}) \leq \lceil \delta_2^{\mu(\bar{Q})} \rceil - \delta_{\inf} = \left\lceil \frac{-q_1 + \sqrt{q_1^2 - 4q_2(q_0 - \mu(\bar{Q})^2)}}{2q_2} \right\rceil - \frac{-q_1 + \sqrt{q_1^2 - 4q_2 q_0}}{2q_2}$$

$$\leq \left(\frac{\sqrt{q_1^2 - 4q_2(q_0 - \mu(\bar{Q})^2)}}{2q_2} + 1\right) - \frac{\sqrt{q_1^2 - 4q_2 q_0}}{2q_2} \leq \frac{\sqrt{4q_2\mu(\bar{Q})^2}}{2q_2} + 1 = \frac{\mu(\bar{Q})}{\sqrt{q_2}} + 1.$$

□

Remark 4 *Note that in Propositions 13, 14 and 15 we give two different bounds for the integrality gap, one weaker than the other. The stronger bound always depend on the parameters* M*,* β *and* γ *since the bounds depend on* q_0, q_1, q_2 *and* $\bar{Q}$*. In the case of ellipsoids, hyperboloids and translated cones, the weaker bound depends only on* q_2 *and* $\bar{Q}$ *and thus by Fact 4, the integrality gap of the set* $\mathcal{S} = \{x \in \mathbb{R}^n : \|Ax - b\|_2 \leq c^T x - d\}$ *does not depend on the right-hand side* $\mathbf{u} = (b, d)$*. This is not the case of paraboloids as the weaker bound depends on* q_1 *and* $\bar{Q}$*.*

6.2.2 The continuous relaxation has multiple optimal solutions or it is not solvable

We show next that under the assumptions in this section, the integrality gap is always bounded by 1, and thus it is independent of the data of the problem.

Proposition 16 *Let* $\mathcal{S} = \{x \in \mathbb{R}^n : x^T M x - 2\beta^T x + \gamma \leq 0,\ x_n \geq \delta_{\inf}\}$ *be one branch of a two-sheet hyperboloid or a translated cone. Assume that* $\delta_{\inf} > -\infty$ *but that the optimization problem has multiple optimal solutions or it is not solvable. Then* $\mathbf{IG}(\mathcal{S}) \leq 1$*.*

Proof Consider the set $\mathcal{S}_{\lceil \delta_{\inf} \rceil} = \{x \in \mathcal{S} : x_n = \lceil \delta_{\inf} \rceil\}$. We will show that $\mathcal{S}_{\lceil \delta_{\inf} \rceil} \cap \mathbb{Z}^n \neq \emptyset$. Since $\delta_{\inf} > -\infty$, by Proposition 2 we obtain that if $\mathcal{S}$ is an hyperboloid, then $x_n = \delta_{\inf}$ defines an asymptote of $\mathcal{S}$, and if $\mathcal{S}$ is a translated cone then $x_n = \delta_{\inf}$ intersects $\mathcal{S}$ in a ray. Therefore, the set $\mathcal{S}_{\lceil \delta_{\inf} \rceil}$ must be unbounded. Thus, by Lemma 7, we obtain that $\mathcal{S}_{\lceil \delta_{\inf} \rceil}$ is an unbounded quadratic set in the hyperplane $\{x \in \mathbb{R}^n : x_n = \lceil \delta_{\inf} \rceil\}$. By Proposition 4-(iv), we obtain that $\mathcal{S}_{\lceil \delta_{\inf} \rceil}$ contains $n-1$ dimensional balls of arbitrarily large radius in the hyperplane $\{x \in \mathbb{R}^n : x_n = \lceil \delta_{\inf} \rceil\}$. This implies that $\mathcal{S}_{\lceil \delta_{\inf} \rceil} \cap (\mathbb{Z}^{n-1} \times \{\lceil \delta_{\inf} \rceil\}) \neq \emptyset$. Therefore, $\inf\{x_n : x \in \mathcal{S} \cap \mathbb{Z}^n\} \leq \lceil \delta_{\inf} \rceil$. We conclude that $\mathbf{IG}(\mathcal{S}) \leq \lceil \delta_{\inf} \rceil - \delta_{\inf} \leq 1$. □

6.3 Comparison of bounds and tightness for the ellipsoid case

We use Examples 8 and 9 to compare the bounds from Propositions 7 and 13 when the convex set $\mathcal{S}$ is an ellipsoid.

Example 8 *Consider the following second-order conic IP*

$$\inf_{x \in \mathbb{Z}^n} \left\{ x_n : \sum_{j=1}^{n-1} x_j^2 + N^2 \left(x_n - \frac{1}{2} - \frac{1}{N}\right)^2 \leq \left(N\left(\frac{1}{2} - \frac{1}{N}\right)\right)^2 \right\},$$

for $N > n \geq 2$*. Notice that the optimal solution of the continuous relaxation is at* $\hat{x} = \frac{2}{N} e_n$*, where* e_j *is the* j*-th unit vector, while the only integer feasible solution is at* $x^* = e_n$*. Therefore, we compute the integrality gap as* $\mathbf{IG} = 1 - \frac{2}{N}$*.*

With respect to Proposition 7, we are in Case 1 since the covering radius of the diagonal matrix $Q = I + N e_n e_n^T$ *is* $\mu(Q) = \frac{\sqrt{n-1+N^2}}{2}$*, which is larger than* r*. The rhs-independent IG bound is computed* $B_1 = 2\frac{1}{N}\frac{\sqrt{n-1+N^2}}{2} = \sqrt{\frac{n-1}{N^2} + 1}$*.*

With respect to Proposition 12, we obtain $q_2 = N^2, q_1 = N^2 + 2N, q_0 = -2N, \mu(\bar{Q}) = \frac{\sqrt{n-1}}{2}$*. With respect to Proposition 13, we are in Case 1 and the rhs-independent IG bound is computed as* $B_2 = \sqrt{\frac{n-1}{N^2} + 1}$*, whereas the rhs-dependent IG bound is computed as* $B_2' = 1 - \frac{2}{N}$*.*

Notice that the rhs-dependent bound B_2' *precisely matches the actual IG, whereas the rhs-independent bounds* B_1 *and* B_2*, which happen to be equal to each other, asymptotically match the actual IG as* N *tends to* $+\infty$*. Hence, we have the following relationship:* $\mathbf{IG} = B_2' \leq B_1 = B_2$*.*

Example 9 *Consider the following second-order conic IP*

$$\inf_{x\in\mathbb{Z}^n}\left\{x_n : \sum_{j=1}^{n-1} x_j^2 + N^2\left(x_n - \frac{1}{2} - \frac{1}{N}\right)^2 \le \left(N\left(\frac{1}{2}+\frac{1}{N}\right) - \frac{1}{N}\right)^2\right\},$$

for $N > n \ge 2$. Notice that the optimal solution of the continuous relaxation is at $\hat{x} = \frac{1}{N^2}e_n$ while the only integer feasible solution is at $x^* = e_n$. Therefore, we compute the integrality gap as $\mathbf{IG} = 1 - \frac{1}{N^2}$.

With respect to Proposition 7, we are in Case 2 since the covering radius of the diagonal matrix $Q = I + Ne_ne_n^T$ is $\mu(Q) = \frac{\sqrt{n-1+N^2}}{2}$, which is smaller than r. The rhs-independent IG bound is computed $B_1 = 2\frac{1}{N}\frac{\sqrt{n-1+N^2}}{2} = \sqrt{\frac{n-1}{N^2}+1}$.

With respect to Proposition 12, we obtain $q_2 = -N^2, q_1 = N^2+2N, q_0 = \frac{1}{N^2} - \frac{2}{N} - 1, \mu(\bar{Q}) = \frac{\sqrt{n-1}}{2}$. With respect to Proposition 12, we are in Case 2 and the rhs-independent IG bound is computed as $B_2 = \frac{\sqrt{n-1}}{2N}+1$, whereas the rhs-dependent IG bound is computed as $B_2' = \frac{\sqrt{N^2+4N+\frac{4}{N^2}-\frac{8}{N}} - \sqrt{N^2+4N+\frac{4}{N^2}-\frac{8}{N}-(n-1)}}{2N} + 1$.

Notice that all of these three bounds matches the actual integrality gap when N tends to $+\infty$. However, we notice the following relationship: $\mathbf{IG} \le B_2' \le B_1 \le B_2$.

7 A proximity result for a non-simple second-order conic IP

Let us consider the set

$$\mathcal{S} := \{x \in \mathbb{R}^n :\ \|x\|_2 \le r_1, \|x - pe_1\|_2 \le r_2\}. \tag{11}$$

We will assume that $\mathcal{S} \ne \emptyset$ (or equivalently, $r_1 + r_2 \ge p$) and $p > 0$. As Example 7 demonstrates, even with two circles in $\mathbb{R}^2$, the integrality gap cannot be bounded, in general, independent of the radii of these two circles (r_1 and r_2) and the distance between their centers p. In this section, we will analyze the case in which these three parameters are related as

$$\frac{r_1 + r_2}{p} \ge \kappa,$$

for some fixed parameter $\kappa > \sqrt{n-1}$.

7.1 Preliminary Results

Let us first establish some preliminary results.

Lemma 9 *Consider the set $\mathcal{S}$ defined in* (11) *with $r_1 + r_2 \ge p > 0$. Then, we have the following:*

(i) For every $x \in \mathcal{S}$, we have $|x_j| \le H$, $j = 2,\dots,n$, where

$$H = \frac{1}{2p}\sqrt{[(r_1+r_2)^2 - p^2][p^2 - (r_1 - r_2)^2]}. \tag{12}$$

(ii) Assuming that $r_1 + r_2 \ge p + \nu$, the height $x_n = h$ at which the line segment along the x_1-axis in $\mathcal{S}$ is of length ν is given as

$$h = \pm\frac{1}{2(p+\nu)}\sqrt{[(r_1+r_2)^2 - (p+\nu)^2][(p+\nu)^2 - (r_1-r_2)^2]}. \tag{13}$$

Proof (i) We first find the intersection points of the boundary of two spheres by simultaneously solving $\|x\|_2 = r_1$ and $\|x - pe_1\|_2 = r_2$, which gives us $x_1 := W = \frac{r_1^2 - r_2^2 + p^2}{2p}$.To obtain the bound on the absolute value of any coordinate x_j of the intersection, denoted by H, we compute

$$H = \sqrt{r_1^2 - W^2} = \sqrt{\left[r_1 + \frac{r_1^2 - r_2^2 + p^2}{2p}\right]\left[r_1 - \frac{r_1^2 - r_2^2 + p^2}{2p}\right]} = \frac{1}{2p}\sqrt{[(r_1+p)^2 - r_2^2][r_2^2 - (r_1 - p)^2]}$$
$$= \frac{1}{2p}\sqrt{[(r_1+r_2)^2 - p^2][p^2 - (r_1-r_2)^2]}.$$

(ii) The height h must satisfy the condition $p+\nu = \sqrt{r_1^2-h^2}+\sqrt{r_2^2-h^2}$. By squaring both sides and rearranging the terms, we obtain $(p+\nu)^2 - r_1^2 - r_2^2 + 2h^2 = 2\sqrt{r_1^2-h^2}\sqrt{r_2^2-h^2}$. By squaring both sides once again and eliminating some terms, we then obtain

$$\begin{aligned} 4h^2(p+\nu)^2 &= 4r_1^2r_2^2 - [(p+\nu)^2 - r_1^2 - r_2^2]^2 = [2r_1r_2 - (p+\nu)^2 + r_1^2 + r_2^2][2r_1r_2 + (p+\nu)^2 - r_1^2 - r_2^2] \\ &= [(r_1+r_2)^2 - (p+\nu)^2][(p+\nu)^2 - (r_1-r_2)^2], \end{aligned}$$

which gives h as in (13). □

The proof of the following lemma is given in Appendix B.

Lemma 10 *Let $\kappa > \nu$ be given and define the set $\mathcal{P} = \{(a,b,c) \in \mathbb{R}^3_{++} : a+b \geq \kappa c,\ a+b \geq c+\nu\}$. Consider the function $f : \mathcal{P} \to \mathbb{R}$ defined as $f(a,b,c) := \frac{1}{2c}\sqrt{[(a+b)^2-c^2][c^2-(a-b)^2]}$. Then,*

$$\max\{f(a,b,c) - f(a,b,c+\nu) : (a,b,c) \in \mathcal{P}\} = \frac{\nu}{2}\sqrt{\frac{\kappa+1}{\kappa-1}}. \tag{14}$$

Proposition 17 *Under the assumptions of Lemma 9 and $\kappa > \sqrt{n-1}$, we have $H - h \leq \frac{\sqrt{n-1}}{2}\sqrt{\frac{\kappa+1}{\kappa-1}}$.*

Proof The proof of the statement directly follows from Lemma 10 with $\nu = \sqrt{n-1}$. □

7.2 Main Result

Our main result in this section is the following proposition:

Proposition 18 *Consider the set $\mathcal{S}$ defined in (11) with $r_1+r_2-p \geq \sqrt{n-1}$ and assume that $\kappa > \sqrt{n-1}$. Let $\alpha \in \mathbb{R}^n$. Then, the integrality gap can be upper-bounded as a function of κ as follows:*

$$\mathbf{IG}(\kappa) = \min_{x\in\mathcal{S}\cap\mathbb{Z}^n} \alpha^T x - \min_{x\in\mathcal{S}} \alpha^T x \leq \|\bar{\alpha}\|_2\sqrt{n-1} + |\alpha_n| \left\lceil \frac{\sqrt{n-1}}{2}\sqrt{\frac{\kappa+1}{\kappa-1}} \right\rceil.$$

Proof Firstly, let us denote $\hat{x} \in \operatorname{argmin}\{\alpha^T x : x \in \mathcal{S}\}$. Consider a solution $x' \in \mathbb{Z}^n$ obtained as

$$x'_n = \begin{cases} \lfloor h \rfloor & \text{if } \hat{x}_n \geq h \\ \lfloor \hat{x}_n \rfloor & \text{if } 0 \leq \hat{x}_n \leq h \\ \lceil \hat{x}_n \rceil & \text{if } 0 \geq \hat{x}_2 \geq -h \\ \lceil -h \rceil & \text{if } \hat{x}_n \leq -h \end{cases}.$$

Here, h is the positive height at which the line segment along the x_1-axis in $\mathcal{S}$ is of length $\nu = \sqrt{n-1}$ as in Lemma 9. Such a selection guarantees that there exists $(x'_1,\ldots,x'_{n-1}) \in \mathbb{Z}^{n-1}$ such that $x' \in \mathcal{S}\cap\mathbb{Z}^n$ (note that the covering radius of the standard lattice in $\mathbb{R}^{n-1}$ is $\sqrt{n-1}/2$). Due to Proposition 17, we have

$$|x'_n - \hat{x}_n| \leq \left\lceil \frac{\sqrt{n-1}}{2}\sqrt{\frac{\kappa+1}{\kappa-1}} \right\rceil \quad \text{and} \quad \|(x'_1,\ldots,x'_{n-1}) - (\hat{x}_1,\ldots,\hat{x}_{n-1})\|_2 \leq \sqrt{n-1}.$$

Finally, we compute

$$\mathbf{IG}(\kappa) = \min_{x\in\mathcal{S}\cap\mathbb{Z}^n} \alpha^T x - \min_{x\in\mathcal{S}} \alpha^T x \leq \alpha^T x' - \alpha^T \hat{x} = \alpha^T(x'-\hat{x}) \leq \|\bar{\alpha}\|_2\sqrt{n-1} + |\alpha_n| \left\lceil \frac{\sqrt{n-1}}{2}\sqrt{\frac{\kappa+1}{\kappa-1}} \right\rceil,$$

where the first inequality follows since $x' \in \mathcal{S}\cap\mathbb{Z}^n$. □

Although we choose the n-th dimension as the "height" in Proposition 18, the same argument can be repeated for any dimension j^*, $j^* = 2,\ldots,n$ to obtain the following corollary.

Corollary 2 *Under the assumptions of Proposition 18, we have*

$$\mathbf{IG}(\kappa) = \min_{x\in\mathcal{S}\cap\mathbb{Z}^n} \alpha^T x - \min_{x\in\mathcal{S}} \alpha^T x \le \min_{j^*=2,\ldots,n} \left\{ \sqrt{\sum_{\substack{j=1\\ j\neq j^*}}^{n} \alpha_j^2}\sqrt{n-1} + |\alpha_{j^*}| \left\lceil \frac{\sqrt{n-1}}{2}\sqrt{\frac{\kappa+1}{\kappa-1}} \right\rceil \right\}.$$

We now use Example 10 to illustrate the use of the bound derived above.

Example 10 *Let $N \in \mathbb{Z}_+$ and consider the following second-order conic IP:*

$$\inf_{x\in\mathbb{Z}^2} \{x_2 : x_1^2 + x_2^2 \le (N+1)^2,\ (x_1 - 2N)^2 + x_2^2 \le N^2\}.$$

Notice that the intersection of the ellipsoids are contained in the strip $\{x \in \mathbb{R}^2 : N \le x_1 \le N+1\}$ and the only integer points in the intersection are $(N, 0)$ and $(N+1, 0)$, which both give an objective function value of 0. On the other hand, the optimal solution of the continuous relaxation is the following point:

$$(\hat{x}_1, \hat{x}_2) = \left(N + \frac{2N+1}{4N}, -\sqrt{\frac{2N+1}{4N}(2N - \frac{2N+1}{4N})}\right).$$

Therefore, the integrality gap, as a function of N, is computed as follows:

$$\mathbf{IG}(N) = \sqrt{\frac{2N+1}{4N}(2N - \frac{2N+1}{4N})} = \sqrt{N + \frac{1}{4} - \frac{1}{4N} - \frac{1}{16N^2}}.$$

Let us now compute our bound from Proposition 18 for this example (note that we have $n = 2$, $r_1 = N$, $r_2 = N+1$, $p = 2N$ and $\kappa = 1 + \frac{1}{2N}$). In fact, we obtain

$$\left\lceil \frac{1}{2}\sqrt{\frac{1 + \frac{1}{2N} + 1}{1 + \frac{1}{2N} - 1}} \right\rceil = \left\lceil \frac{1}{2}\sqrt{4N+1} \right\rceil = \left\lceil \sqrt{N + \frac{1}{4}}, \right\rceil$$

which approximately matches the integrality gap derived above.

A Additional Examples

In this appendix, we present additional examples of second-order conic IPs in which the integrality gap depends on the right-hand side.

In Example 11, the feasible region of the conic IP, which is the intersection of a parabola and a half-space with the standard lattice, is unbounded. This is arguably one of the "simplest" unbounded sets defined by a pair of nonlinear and linear inequalities, yet the integrality gap cannot be upper bounded independent of the right-hand side.

Example 11 *Let $N \in \mathbb{Z}_{++}$ and $\epsilon \in (0,1)$. Consider the following second-order conic IP:*

$$\inf_{x\in\mathbb{Z}^2} \{x_2 : N - \epsilon \le x_1,\ x_1^2 \le x_2\} = \inf_{x\in\mathbb{Z}^2} \left\{ x_2 : \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \succeq_{\mathbf{L}^3} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},\ x_1 \ge N - \epsilon \right\}.$$

Since we have $\hat{x} = (N - \epsilon, (N-\epsilon)^2)$ and $x^* = (N, N^2)$, the integrality gap, as a function of N, is computed as $\mathbf{IG}(N) = N^2 - (N-\epsilon)^2 = 2\epsilon N - \epsilon^2$.

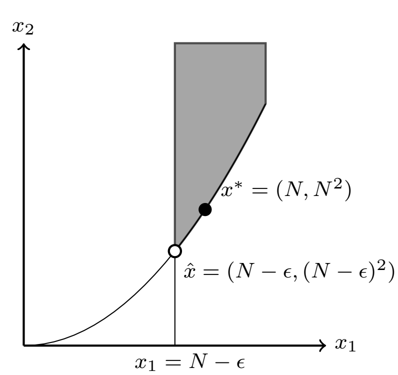

Fig. 4 The feasible region of the continuous relaxation of the conic IP in Example 11.

In Example 12, the feasible region of the conic IP, which is the intersection of a parabola and a *non-redundant* polytope with the standard lattice, is bounded.

Example 12 *Let $N \in \mathbb{Z}_{++}$ and consider the following second-order conic IP:*

$$\inf_{x\in\mathbb{Z}^2}\{x_2 : N-1/2 \le x_1 \le N,\ x_1^2 \le x_2, (N-1/4)^2 \le x_2 \le N^2+1\}$$
$$= \inf_{x\in\mathbb{Z}^2}\left\{x_2 : \begin{bmatrix}2 & 0\\ 0 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \succeq_{\mathbf{L}^3} \begin{bmatrix}0\\ 1\\ -1\end{bmatrix}, \begin{bmatrix}1 & 0\\ -1 & 0\\ 0 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \ge \begin{bmatrix}N-1/2\\ -N\\ (N-1/4)^2\\ -(N^2+1)\end{bmatrix}\right\}.$$

Since we have $\hat{x} = \left(N - \frac{1}{2}, (N-\frac{1}{4})^2\right)$ and $x^* = (N, N^2)$, the integrality gap, as a function of N, is computed as $\mathbf{IG}(N) = N^2 - \left(N - \frac{1}{4}\right)^2 = \frac{N}{2} - \frac{1}{16}$.

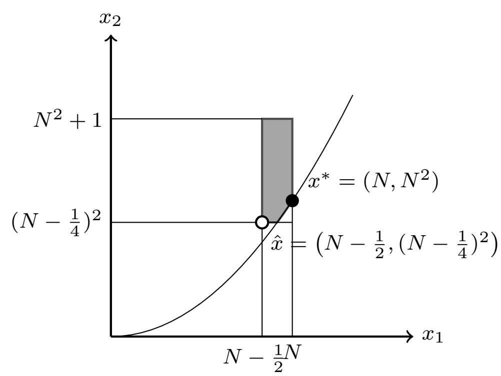

Fig. 5 The feasible region of the continuous relaxation of the conic IP in Example 12.

In Example 13, the feasible region of the conic IP, which is the intersection of a parabola and one branch of a hyperbola with the standard lattice, is unbounded. This example is interesting since both nonlinear sets are unbounded and so is their intersection.

Example 13 *Let $N \in \mathbb{Z}_+$ and consider the following second-order conic IP:*

$$\inf_{x\in\mathbb{Z}^2}\{x_2 : x_1^2 \le x_2,\ (x_1-N)\left(x_2+2-(N+1/2)^2\right) \ge 1,\ x_1 \ge N\}$$
$$= \inf_{x\in\mathbb{Z}^2}\left\{x_2 : \begin{bmatrix}2 & 0\\ 0 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \succeq_{\mathbf{L}^3} \begin{bmatrix}0\\ 1\\ -1\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \succeq_{\mathbf{L}^3} \begin{bmatrix}-2\\ -N^2+3/4\\ N^2+2N-3/4\end{bmatrix},\ x_1 \ge N\right\}.$$

Since we have $\hat{x} = (N+1/2, (N+1/2)^2)$ and $x^* = (N+1, (N+1)^2)$, the integrality gap, as a function of N, is computed as $\mathbf{IG}(N) = (N+1)^2 - (N+1/2)^2 = N + 3/4$.

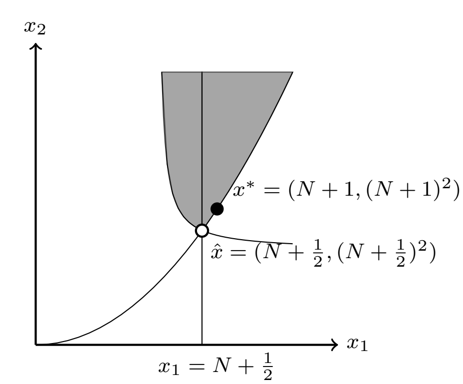

Fig. 6 The feasible region of the continuous relaxation of the conic IP in Example 13.

In Example 14, the feasible region of the conic IP, which is the intersection of a parabola and a polyhedron with the standard lattice, is bounded.

Example 14 *Let $N \in \mathbb{Z}_+$ and consider the following second-order conic IP:*

$$\inf_{x \in \mathbb{Z}^2} \{x_2 : N + 1/2 \leq x_1 \leq N + 1,\ x_1^2 \leq x_2 \leq (N+1)^2\}.$$

Notice that the conic MIPs in Examples 14 and 11 have the same integrality gap since the integrality gap computation in Example 14 remains the same without constraints $x_1 \leq N$ and $x_2 \leq (N+1)^2$.

Notice that constraints $N + 1/2 \leq x_1 \leq N + 1$ in Example 14 define a degenerate ellipsoid (the feasible region defined by them is an ellipsoid in the x_1 component plus a lineality space in the x_2 component). In Example 15, the feasible region of the conic IP, which is the intersection of a parabola and a non-redundant, non-degenerate ellipsoid with the standard lattice, is bounded.

Example 15 *Let $N \in \mathbb{Z}_+$ and consider the following second-order conic IP:*

$$\inf_{x \in \mathbb{Z}^2} \{x_2 : N + 1/4 \leq x_1,\ x_1^2 \leq x_2,\ (x_1 - (N+1))^2 + (x_2 - (N+1)^2)^2 \leq R^2\},$$

where

$$R^2 = [(N+1) - (N+1/2)]^2 + [(N+1)^2 - (N+1/2)^2]^2 = 1/4 + (N+3/4)^2.$$

In this case, the integrality gap, as a function of N, is computed as $\mathbf{IG}(N) = (N+1)^2 - (N+1/2)^2 = N + 3/4$.

In Example 16, the feasible region of the conic IP, which is the intersection of a hyperbola and a strip (or a degenerate ellipsoid) with the standard lattice, is unbounded.

Example 16 *Let $N \in \mathbb{Z}_+$ and consider the following second-order conic IP:*

$$\inf_{x \in \mathbb{Z}^2} \{x_2 : N - 1/2 \leq x_1 \leq N,\ (N + 1 - x_1)x_2 \geq N\}.$$

In this case, the integrality gap, as a function of N, is computed as $\mathbf{IG}(N) = 2N - N = N$.

Notice that the constraint $x_1 \leq N + 1$ is only added to make the other branch of the hyperbola infeasible in Example 16. Any other linear inequality (e.g. $x_2 \geq 0$) with the same property can be included instead.

In Example 17, the feasible region of the conic IP, which is the intersection of a (single branch of a) hyperbola and a non-degenerate ellipsoid with the standard lattice, is bounded.

Example 17 *Let $N \in \mathbb{Z}_+$ and consider the following second-order conic IP:*

$$\inf_{x \in \mathbb{Z}^2} \left\{ -x_2 : (x_1 + N)^2 + x_2^2 \leq (N+1)^2,\ x_1 \left(x_2 + \sqrt{N + \frac{3}{4}} \right) \geq \sqrt{N + \frac{3}{4}},\ x_1 \geq 0 \right\}.$$

Notice that the only integer point in the feasible region is $(1, 0)$ with an objective function value of 0. On the other hand, the optimal solution of the continuous relaxation is the following point: $(\tilde{x}_1, \tilde{x}_2) = \left(\frac{1}{2}, \sqrt{N + \frac{3}{4}}\right)$. *Therefore, the integrality gap, as a function of N, is computed as* $\mathbf{IG}(N) = \sqrt{N + \frac{3}{4}}$.

In Example 18, the feasible region of the conic IP, which is the intersection of a parabola and a non-degenerate ellipsoid with the standard lattice, is bounded.

Example 18 *Let $N \in \mathbb{Z}_+$ and consider the following second-order conic IP:*

$$\inf_{x \in \mathbb{Z}^2} \left\{ -x_2 : (x_1 + N)^2 + x_2^2 \leq (N+1)^2,\ x_2 \geq \left(x_1 - \frac{3}{4} - \sqrt{N + \frac{3}{4}} \right)^2 - \left(\frac{1}{4} - \sqrt{N + \frac{3}{4}} \right)^2 \right\}.$$

We have $\mathbf{IG}(N) = \sqrt{N + \frac{3}{4}}$.

B Omitted Proofs

B.1 Proof of Proposition 7

Firstly, let us denote $\hat{x} \in \operatorname{argmin}\{\alpha^T x : x \in \mathcal{S}\}$. Also, let $\check{x} \in \mathcal{S} \cap \mathbb{Z}^n$. Secondly, since $Q \in \mathbb{R}^{m \times n}$ and $\operatorname{rank}(Q) = n$, the matrix $(Q^T Q)^{-1}$ exists. Thirdly, due to the definition of covering radius, there exists $\mu(Q)$ such that for all $x \in \mathbb{R}^n$, there exists $x' \in \mathbb{Z}^n$ with $\|Qx - Qx'\|_2 \leq \mu(Q)$.

The remainder of the proof is divided into two cases:

Case 1: $r \le \mu(Q)$. Then,

$$\begin{aligned}\mathbf{IG}(Q,p,r) &= \min_{x\in\mathcal{S}\cap\mathbb{Z}^n}\alpha^T x - \min_{x\in\mathcal{S}}\alpha^T x \le \alpha^T\check{x} - \alpha^T\hat{x} = \alpha^T(\check{x}-\hat{x}) = \alpha^T(Q^TQ)^{-1}(Q^TQ)(\check{x}-\hat{x})\\ &= \alpha^T(Q^TQ)^{-1}Q^T(Q\check{x}-Q\hat{x}) \le \|Q(Q^TQ)^{-1}\alpha\|_2\|(Q\check{x}-p)-(Q\hat{x}-p)\|_2\\ &\le \|Q(Q^TQ)^{-1}\alpha\|_2(r+r) \le 2\|Q(Q^TQ)^{-1}\alpha\|_2\mu(Q),\end{aligned}$$

where the first inequality follows since $\check{x} \in \mathcal{S}\cap\mathbb{Z}^n$.
Case 2: $r > \mu(Q)$. In this case, we first choose $\tilde{x} \in \mathbb{R}^n$ such that $Q\tilde{x} \in [p, Q\hat{x}]$ and $\|(Q\tilde{x}-p)-(Q\hat{x}-p)\|_2 = \mu(Q)$. Note that $\tilde{x} \in \mathcal{S}$. Also, we have that there exists $x' \in \mathbb{Z}^n$ such that $\|Qx' - Q\tilde{x}\|_2 \le \mu(Q)$. Then,

$$\begin{aligned}\mathbf{IG}(Q,p,r) &= \min_{x\in\mathcal{S}\cap\mathbb{Z}^n}\alpha^T x - \min_{x\in\mathcal{S}}\alpha^T x \le \alpha^T x' - \alpha^T\hat{x} = \alpha^T(x'-\hat{x}) = \alpha^T(Q^TQ)^{-1}(Q^TQ)(x'-\hat{x})\\ &= \alpha^T(Q^TQ)^{-1}Q^T(Qx'-Q\hat{x}) = \alpha^T(Q^TQ)^{-1}Q^T[(Qx'-p)-(Q\tilde{x}-p)+(Q\tilde{x}-Q\hat{x})]\\ &= \|Q(Q^TQ)^{-1}\alpha\|_2(\|(Qx'-p)-(Q\tilde{x}-p)\|_2 + \|(Q\tilde{x}-Q\hat{x})\|_2) \le 2\|Q(Q^TQ)^{-1}\alpha\|_2\mu(Q),\end{aligned}$$

where the first inequality follows since $x' \in \mathcal{S}\cap\mathbb{Z}^n$.

In both cases, we conclude that $\mathbf{IG}(Q,p,r) \le 2\|Q(Q^TQ)^{-1}\alpha\|_2\mu(Q)$, which is independent of p and r.

B.2 Proof of Lemma 10

We will prove this lemma with the help of the following three facts:

Fact 7 *Let $A > 0$ and $B > 0$ with $A \ge B$ be given. Also let $\nu > 0$. Consider the function $g : [B, A] \to \mathbb{R}$ defined as*

$$g(C) = \frac{1}{2}\sqrt{A^2 - C^2}\left(1 - \sqrt{1-(B/C)^2}\right).$$

Then, the function g is decreasing in C.

Proof The statement follows since both $\sqrt{A^2-C^2}$ and $1-\sqrt{1-(B/C)^2}$ are decreasing in C. □

Fact 8 *Let (a^*, b^*, c^*) be an optimal solution of problem (14). Then, we have $a^* = b^*$.*

Proof By contradiction, suppose that $a^* \ne b^*$. Then, let us consider a new feasible solution (a, b, c^*) of problem (14) constructed as $a = b = \frac{a^*+b^*}{2}$.

We can compute the objective function value of solutions (a^*, b^*, c^*) as

$$\mathcal{O}_1 := \frac{1}{2c^*}\sqrt{[(a^*+b^*)^2 - c^{*2}][c^{*2} - (a^*-b^*)^2]} - \frac{1}{2(c^*+\nu)}\sqrt{[(a^*+b^*)^2 - (c^*+\nu)^2][(c^*+\nu)^2 - (a^*-b^*)^2]}$$

and (a, b, c^*) as

$$\mathcal{O}_2 := \frac{1}{2}\sqrt{[(a^*+b^*)^2 - c^{*2}]} - \frac{1}{2}\sqrt{[(a^*+b^*)^2 - (c^*+\nu)^2]}.$$

We will now apply Fact 7 with $A = a^*+b^*$, $B = |a^*-b^*|$ and $C = c^*$. Since $\mathcal{O}_1 - \mathcal{O}_2 = g(C+\nu) - g(C)$ and g is decreasing, we conclude that $\mathcal{O}_1 < \mathcal{O}_2$. However, this is a contradiction to the fact that (a^*, b^*, c^*) is an optimal solution to problem (14). Hence, the result follows. □

Fact 9 *Let (a^*, b^*, c^*) be an optimal solution of problem (14). Then, we have $c^* = \frac{\nu}{\kappa-1}$.*

Proof Due to Fact 8, problem (14) reduces to

$$\frac{1}{2}\max\{\sqrt{4a^2-c^2} - \sqrt{4a^2-(c+\nu)^2} : \ 2a \ge \kappa c, 2a \ge c+\nu, c \ge 0\}. \tag{15}$$

We claim that at an optimal solution (a^*, c^*) of problem (15), we must have $2a^* = \kappa c^*$ and $2a^* = c^* + \nu$. This implies that $c^* = \frac{\nu}{\kappa-1}$ and $a^* = \frac{\kappa\nu}{2(\kappa-1)}$, and the optimal value is $\frac{\nu}{2}\sqrt{\frac{\kappa+1}{\kappa-1}}$.

We will now prove this claim by contradiction. Suppose that $(\tilde{a}, \tilde{c})$ is an optimal solution of problem (15) which does not satisfy our claim. Let us look at the following four cases:

– $\tilde{c} = 0$. Notice that the objective function of problem (15),

$$\frac{1}{2}\frac{\nu^2}{2a + \sqrt{4a^2-\nu^2}},$$

is decreasing in a in this case. Hence, we obtain that $\tilde{a} = \nu/2$ and the optimal value is $\nu/2$. However, this value is smaller than $\frac{\nu}{2}\sqrt{\frac{\kappa+1}{\kappa-1}}$. Hence, we conclude that $\tilde{c} = 0$ cannot be optimal.

– $2\tilde{a} > \kappa\tilde{c}$ and $2\tilde{a} > \tilde{c} + \nu$: Notice that the objective function of problem (15),

$$\frac{1}{2}\frac{2c\nu + \nu^2}{\sqrt{4a^2 - c^2} + \sqrt{4a^2 - (c+\nu)^2}},$$

is increasing in c for fixed a. Hence, we can increase c small enough to obtain another feasible solution with a better objective function value.

– $2\tilde{a} > \kappa\tilde{c}$ and $2\tilde{a} = \tilde{c} + \nu$: Notice that the objective function of problem (15) along the direction $a = \frac{c+\nu}{2}$,

$$\frac{1}{2}\sqrt{4a^2 - c^2} - \frac{1}{2}\sqrt{4a^2 - (c+\nu)^2} = \frac{1}{2}\sqrt{(c+\nu)^2 - c^2} = \frac{1}{2}\sqrt{2c\nu + \nu^2},$$

is increasing in c. Since this case happens when $\tilde{a} < a^*$ and $\tilde{c} < c^*$, we can simply increase $\tilde{c}$ (and $\tilde{a}$ accordingly) to obtain another feasible solution with a better objective function value.

– $2\tilde{a} = \kappa\tilde{c}$ and $2\tilde{a} > \tilde{c} + \nu$: We claim that the objective function of problem (15) along the direction $a = \frac{\kappa c}{2}$,

$$\frac{1}{2}\sqrt{4a^2 - c^2} - \frac{1}{2}\sqrt{4a^2 - (c+\nu)^2} = \frac{1}{2}c\sqrt{\kappa^2 - 1} - \frac{1}{2}\sqrt{c^2(\kappa^2 - 1) - 2c\nu - \nu^2} := \phi(c),$$

is decreasing in c. To show this, observe that

$$\begin{aligned} 0 > 2\phi'(c) &= \sqrt{\kappa^2 - 1} - \frac{c(\kappa^2 - 1) - \nu}{\sqrt{c^2(\kappa^2 - 1) - 2c\nu - \nu^2}} \\ \iff (\kappa^2 - 1)[c^2(\kappa^2 - 1) - 2c\nu - \nu^2] &< c^2(\kappa^2 - 1)^2 - 2c\nu(\kappa^2 - 1) + \nu^2 \\ \iff \kappa^2\nu^2 &> 0. \end{aligned}$$

Since this case happens when $\tilde{a} > a^*$ and $\tilde{c} > c^*$, we can simply decrease $\tilde{c}$ (and $\tilde{a}$ accordingly) to obtain another feasible solution with a better objective function value.

□

Due to Facts 8 and 9, we deduce that $c^* = \frac{\nu}{\kappa - 1}$ and $a^* = b^* = \frac{\kappa\nu}{2(\kappa-1)}$. Hence, we conclude that the statement of the lemma holds.

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