Inexact FISTA-like Methods with Adaptive Backtracking

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Abstract. Accelerated proximal gradient methods have become a useful tool in large-scale convex optimization, specially for variational regularization with non-smooth priors. Prevailing convergence analysis considers that users can perform the proximal and the gradient steps exactly. Still, in some practical applications, the proximal or the gradient steps must be computed inexactly, which can harm convergence speed or even lead to a divergent behavior.

Researchers have developed theoretical frameworks for convergence under the inexactness of computations to handle this issue. For example, Bello-Cruz et al. developed a relative error rule to be used with inexact FISTA (Fast Iterative Soft-Thresholding Algorithm). Their technique works whenever suitable stepsizes are known. In the present paper, we study the convergence of inexact FISTA with line search. We develop an adaptive line search without demanding any additional effort as compared with monotone line search approaches. The computational performance is illustrated for an important category of problems with simulated tomographic data. Theory and experimentation show that the new methods are suitable for variational regularization of inverse problems.

Keywords: Proximal Gradient, Convex Optimization, Accelerated Methods

1. Introduction

In the present paper, we consider the following minimization problem:

$$\min_{x\in\mathbb{R}^n}\{F(x) \coloneqq f(x)+\phi(x)\}, \tag{1}$$

where the functions f and ϕ obey the assumption below.

Assumption 1. *The functions in* (1) *are such that*

- $f:\mathbb{R}^n\to\mathbb{R}$ *is convex, differentiable, and L_f-smooth, i.e., for all $x,y\in\mathbb{R}^n$,*

$$\|\nabla f(x)-\nabla f(y)\|\le L_f\|x-y\|.$$

From now on, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$.

- $\phi:\mathbb{R}^n\to(-\infty,\infty]$ *is convex, proper, and closed.*

Problems like (1) often arise in applications such as variational regularization of inverse problems. Proximal gradient methods are among the most appropriate ones for solving problems in the form (1). The computationally intensive part of a proximal gradient method has the form

$$p_{L_f}(y) \coloneqq \operatorname{prox}_{\frac{1}{L_f}\phi}\left(x-\frac{1}{L_f}\nabla f(x)\right), \tag{2}$$

called the proximal gradient step, where

$$\operatorname{prox}_{\frac{1}{L_f}\phi}(y) \coloneqq \operatorname*{argmin}_{z\in\mathbb{R}^n}\left\{\frac{1}{L_f}\phi(z)+\frac{1}{2}\|z-y\|^2\right\}. \tag{3}$$

Thus, the proximal gradient methods apply to the "prox-friendly" case where the user can compute the proximal operator of $\frac{1}{L_f}\phi$ within a reasonable computation time (say, when compared to the computation time of evaluating the gradient of f).

Fast proximal-gradient methods achieve high convergence speed without imposing considerable computation cost per iteration. FISTA is one of the first of these techniques [2, 7, 9, 14]. The k-th iteration of FISTA can be written as

$$\begin{aligned} x_k &= \operatorname{prox}_{\frac{1}{L_f}\phi}\left(y_k-\frac{1}{L_f}\nabla f(y_k)\right), \\ t_{k+1} &= \frac{1+\sqrt{1+4t_k^2}}{2}, \\ y_{k+1} &= x_k+\left(\frac{t_k-1}{t_{k+1}}\right)(x_k-x_{k-1}), \end{aligned}$$

for each $k\ge 1$, with the initialization $t_1=1$ and $y_1=x_0$.

In some useful cases, it is possible to efficiently compute $\operatorname{prox}_\phi(y)$ approximately using iterative methods. Notice that

$$y\in\operatorname{prox}_{\frac{1}{L_f}\phi}\left(x-\frac{1}{L_f}\nabla f(x)\right)$$

if, and only if,

$$0\in\partial\phi(y)+L_f(y-x)+\nabla f(x),$$

where $\partial\phi(y)$ is the subdifferential of ϕ at y (see [2, Chap. 3] for subdifferential calculus rules, and [2, Chap. 6] for proximal operator properties). Inspired by Solodov and Svaiter [17], the idea of Bello-Cruz et al. [5] to obtain inexact FISTA (IFISTA) is to relax the last inclusion to

$$v \in \partial_\epsilon \phi(y) + \frac{L_f}{\tau}(y - x) + \nabla f(x), \tag{4}$$

where $v \in \mathbb{R}^n$, $\epsilon \geq 0$, and $\tau \in (0, 1]$ are parameters, and $\partial_\epsilon \phi(y)$ is the ϵ-subdifferential of ϕ at y, defined as the following enlargement of the subdifferential of ϕ at y:

$$\partial_\epsilon \phi(y) := \{g \in \mathbb{R}^n : \phi(x) \geq \phi(y) + \langle g, x - y\rangle - \epsilon, \ \forall x \in \mathbb{R}^n\}, \tag{5}$$

where $\langle x, y\rangle$ denotes the Euclidean inner product between $x, y \in \mathbb{R}^n$.

The inexact proximal gradient step (4) together with an appropriate bound on $\|v\|$, and ϵ based on an error rule relative to τ and $\|x - y\|^2$ can guarantee fast $O(1/k^2)$ convergence, being k the iteration index. It is important to notice that IFISTA, as originally proposed in [5], requires knowledge of an appropriate estimate of the smoothness constant L_f. In case L_f is unknown, the authors of [5] indicate it as a research direction:

> Some future research directions include considering Problem (1) when the Lipschitz constant of the gradient of f is unknown or when the gradient of f is not Lipschitz continuous. In those cases, a line search may be necessary to obtain a method with an optimal rate of convergence. Although the exact FISTA method can handle these cases in a straightforward manner, it is not clear how to extend our inexact version of FISTA to deal with this challenging situation. [5, p. 316]

Contributions In the present work we bring two contributions for the theory of inexact fast proximal-gradient methods:

(i) We develop a theory for the case where the smoothness constant L_f is not known beforehand and a line search is necessary for ensuring fast $O(1/k^2)$ convergence. The theory allows an adaptive line search that avoids excessively small stepsizes and that does not demand more gradient evaluations than monotone line search techniques;

(ii) we provide a stopping criterion for fast algorithms in the proximal step which can be applied for several nonsmooth functions ϕ that are of practical interest.

Both contributions are very important for the practice of variational regularization of inverse problems as in many cases the constant L_f is not known beforehand – or, even if L_f is known, $1/L_f$ often is too pessimistic for a stepsize – and the proximal operator must be computed iteratively. For example, total variation regularization in statistical tomographic image reconstruction is one of these cases. Our algorithm is, to the best of our knowledge, the first fast proximal-gradient method capable of effectively dealing with this case with guaranteed convergence.

2. Inexact FISTA-like algorithms

The methods derived in this section rely on the following notion of inexact evaluation of the proximal gradient step in (2). It is important to notice that the convergence analysis developed in this section does not depend on how this inexact evaluation is obtained, but rather depends on the criteria established in the rule stated below. We use the following notations: $\mathbb{R}_+$ for the non-negative real set, $\mathbb{R}_{++}$ for the strictly positive real set, and $\mathbb{P}(X)$ for the set of subsets of the set X.

Definition 1 (Relative Error Rule with Delay - RERD)**.** *Consider* $\tau \in (0,1]$*. Define* $J^{\tau}_{f,\phi} : \mathbb{R}^n \times \mathbb{R}_{++} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{P}\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+\right)$ *as*

$$(x, v, \epsilon) \in J^{\tau}_{f,\phi}\left(y, L, \alpha, \delta\right),$$

if and only if

$$v \in \partial_\epsilon \phi(x) + \frac{L}{\tau}(x-y) + \nabla f(y), \tag{6}$$

$$\|\tau v\|^2 + 2\tau\epsilon L \le L\left((1-\tau)L - \alpha\tau\right)\|x-y\|^2 + 2\tau\delta L, \tag{7}$$

$$\alpha \in \left[0, \tfrac{(1-\tau)}{\tau}L\right]. \tag{8}$$

By comparing the relative error rule in Definition 1 with the one proposed in [5], we highlight the presence of the term $2\tau\delta L$ on the right-hand side of (7), which is an enlargement to the error rule in [5]. Later, this term will take into account quantities computed along the line search of the last iteration. Those quantities, shown in Lemma 2, are derived following a technique similar to [11, Lemma 1]. When the subscript in $J^{\tau}_{f,\phi}$ is omitted, consider f and ϕ as in Assumption 1.

The next lemma is a well known result in Hilbert spaces, which is used several times in the convergence analysis, whereas the result after it provides a valuable tool to assess the decrease of the objective function.

Lemma 1. *Let* $a, b, c \in \mathbb{R}^n$*, then*

$$2\langle b-a, c-a\rangle = \|b-a\|^2 + \|c-a\|^2 - \|b-c\|^2.$$

Lemma 2. *Let* $\tau \in (0,1]$*,* $\bar{y} \in \mathbb{R}^n$*,* $\delta \ge 0$*,* $L > 0$*, and* $\alpha \in \left[0, \frac{1-\tau}{\tau}L\right]$*. Consider*

$$(\bar{x}, v, \epsilon) \in J^{\tau}\left(\bar{y}, L, \alpha, \delta\right). \tag{9}$$

Then, for all $x \in \mathbb{R}^n$*, it holds*

$$\begin{aligned} F(x) - F(\bar{x}) \ge \frac{L}{2\tau}\left[2\left\langle \bar{x} - \frac{\tau}{L}v - \bar{y}, \bar{x} - \frac{\tau}{L}v - x\right\rangle - \left\|\bar{x} - \frac{\tau}{L}v - \bar{y}\right\|^2\right] + \\ + \frac{\alpha}{2}\|\bar{y} - \bar{x}\|^2 - \delta + \Delta_a(L, \bar{x}, \bar{y}) + \Delta_b(x, \bar{y}) + \Delta_c(L, x, \bar{x}, \bar{y}, v, \tau, \epsilon), \end{aligned} \tag{10}$$

in which

$$\Delta_a(L, x, y) \coloneqq f(y) + \langle \nabla f(y), x-y\rangle + \frac{L}{2}\|x-y\|^2 - f(x), \tag{11}$$

$$\Delta_b(x, y) \coloneqq f(x) - f(y) - \langle \nabla f(y), x-y\rangle, \tag{12}$$

$$\Delta_c(L, z, x, y, v, \tau, \epsilon) \coloneqq \phi(z) - \phi(x) - \left\langle v + \frac{L}{\tau}(y-x) - \nabla f(y), z-x\right\rangle + \epsilon. \tag{13}$$

Proof. Given $x \in \mathbb{R}^n$, by (13) it holds that

$$\phi(x) = \phi(\bar{x}) + \left\langle v + \frac{L}{\tau}(\bar{y} - \bar{x}) - \nabla f(\bar{y}), x - \bar{x}\right\rangle - \epsilon + \Delta_c(L, x, \bar{x}, \bar{y}, v, \tau, \epsilon).$$

By (12), we have that

$$f(x) = f(\bar{y}) + \langle \nabla f(\bar{y}), x - \bar{y} \rangle + \Delta_b(x, \bar{y}).$$

Summing the last two equalities and rearranging the terms, we get

$$F(x) = F(\bar{x}) + f(\bar{y}) - f(\bar{x}) + \langle \nabla f(\bar{y}), \bar{x} - \bar{y} \rangle + \frac{L}{\tau} \langle \bar{y} - \bar{x}, x - \bar{x} \rangle + \\ + \langle v, x - \bar{x} \rangle - \epsilon + \Delta_b(x, \bar{y}) + \Delta_c(L, x, \bar{x}, \bar{y}, v, \tau, \epsilon).$$

Using Lemma 1 with $a \leftarrow \bar{x}$, $b \leftarrow \bar{y}$, and $c \leftarrow x$ together with (11), we have

$$F(x) - F(\bar{x}) = \frac{(1-\tau)L}{2\tau} \|\bar{y} - \bar{x}\|^2 + \frac{L}{2\tau} \left[\|x - \bar{x}\|^2 - \|\bar{y} - x\|^2 \right] + \langle v, x - \bar{x} \rangle - \epsilon + \\ + \Delta_a(L, \bar{x}, \bar{y}) + \Delta_b(x, \bar{y}) + \Delta_c(L, x, \bar{x}, \bar{y}, v, \tau, \epsilon). \quad (14)$$

On the other hand, by (9) and (7), it follows that

$$\frac{(1-\tau)L}{2\tau} \|\bar{x} - \bar{y}\|^2 - \epsilon \geq \frac{\tau}{2L} \|v\|^2 + \frac{\alpha}{2} \|\bar{x} - \bar{y}\|^2 - \delta.$$

Applying the above bound on (14) and rearranging the terms, it holds that

$$F(x) - F(\bar{x}) \geq \frac{L}{2\tau} \left[\left\| \bar{x} - x - \frac{\tau}{L} v \right\|^2 - \|\bar{y} - x\|^2 \right] + \frac{\alpha}{2} \|\bar{x} - \bar{y}\|^2 - \delta + \\ + \Delta_a(L, \bar{x}, \bar{y}) + \Delta_b(x, \bar{y}) + \Delta_c(L, x, \bar{x}, \bar{y}, v, \tau, \epsilon).$$

Using Lemma 1 with $a \leftarrow \bar{x} - \frac{\tau}{L} v$, $b \leftarrow \bar{y}$, and $c \leftarrow x$, we obtain (10). □

Remark 1 (Sign of Δ_a, Δ_b, and Δ_c)**.** *Since f is L_f-smooth, by the descent lemma [2, Lemma 5.7, p. 109] it follows that $\Delta_a(L, x, y) \geq 0$ for each $L \geq L_f$ and $x, y \in \mathbb{R}^n$. Since f is convex, we have that $\Delta_b(x, y) \geq 0$ for all $x, y \in \mathbb{R}^n$. Furthermore, if $(x, v, \epsilon) \in J^\tau(y, L, \alpha, \delta)$, then $\Delta_c(L, z, x, y, v, \tau, \epsilon) \geq 0$ for each $z \in \mathbb{R}^n$, by the inclusion* (6)*.*

The following framework states the properties that the sequences generated by the method proposed in this section will follow.

Framework 1. *Let $x_0 \in \mathbb{R}^n$, $\tau \in (0, 1]$, $L_0 > 0$, and $\delta_0 \geq 0$. Define $x_{-1} := x_0$, $t_0 := 0$, and $v_0 := 0$. Consider the sequences $\{x_k\}_{k \geq 0} \subset \mathbb{R}^n$, $\{y_k\}_{k \geq 1} \subset \mathbb{R}^n$, $\{L_k\}_{k \geq 0} \subset \mathbb{R}_{++}$, $\{t_k\}_{k \geq 0} \subset \mathbb{R}_{++}$, $\{\alpha_k\}_{k \geq 0} \subset \mathbb{R}_+$, $\{\epsilon_k\}_{k \geq 1} \subset \mathbb{R}_+$, $\{v_k\}_{k \geq 0} \subset \mathbb{R}^n$, and $\{\delta_k\}_{k \geq 0} \subset \mathbb{R}_+$ such that, for all $k \geq 1$, it holds that*

$$t_k \geq 1, \quad (15)$$

$$\frac{t_k(t_k - 1)}{L_k} \leq \frac{t_{k-1}^2}{L_{k-1}}, \quad (16)$$

$$\alpha_k \in \left[0, \frac{1-\tau}{\tau} L_k \right], \quad (17)$$

$$y_k = x_{k-1} - \frac{t_{k-1}}{t_k} \frac{\tau}{L_{k-1}} v_{k-1} + \left(\frac{t_{k-1} - 1}{t_k} \right) (x_{k-1} - x_{k-2}), \quad (18)$$

$$(x_k, v_k, \epsilon_k) \in J^\tau(y_k, L_k, \alpha_k, \delta_{k-1}), \quad (19)$$

$$0 \leq \delta_k \leq \frac{L_{k+1}}{L_k} \frac{t_k^2}{t_{k+1}^2} \left(\Delta_a^k + \left(1 - \frac{1}{t_k} \right) (\Delta_b^k + \Delta_c^k) \right), \quad (20)$$

with

$$\Delta_a^k := \Delta_a(L_k, x_k, y_k), \tag{21}$$
$$\Delta_b^k := \Delta_b(x_{k-1}, y_k), \tag{22}$$
$$\Delta_c^k := \Delta_c(L_k, x_{k-1}, x_k, y_k, v_k, \tau, \epsilon_k), \tag{23}$$

in which Δ_a*,* Δ_b *e* Δ_c *are defined in* (11)*,* (12)*, and* (13)*, respectively.*

The hypotheses (15) and (16) on $\{t_k\}_{k\geq 0}$ allow the choices of [3] or [16] to this sequence. The choice in [3] requires the sequence $\{L_k\}_{k\geq 0}$ to be non-decreasing, whereas the choice in [16] imposes no restriction on $\{L_k\}_{k\geq 0}$. We have unified both choices in the method presented in subsection 2.1 without requiring extra computational burden in the line-search of [16]. The hypothesis (17) is the sequential version of the hypothesis $\alpha \in \left[0, \frac{1-\tau}{\tau} L_f\right]$ in [5] to enable a variable step size.

Notice that (18) resembles the extrapolation step that is typical in accelerated first-order methods [3, 16], but has the residual term associated with the last iteration $-\frac{t_{k-1}}{t_k}\frac{\tau}{L_{k-1}}v_{k-1}$, as in [5], in which this term is presented with L_f instead of L_{k-1}. The update in (19) is similar to [5], but allows the input of a local estimation of the smoothness parameter.

At last, the bounds in (20) constitute a novelty in the present work that allows an enlargement of the error rule of [5] taking into account the terms (21)–(23) computed in the previous iteration. When δ_k is chosen as null, the method derived in this section is a variant of the one proposed in [5] with backtracking. However, the choice of $\tau = 1$ in Framework 1 does not imply that there is an exact evaluation of the proximal gradient operator in (19) as would be the case in [5]. Even with the choice $\tau = 1$, the right-hand side of (7) is not null if $\delta \neq 0$. This could explain the convergence observed in some experiments of the heuristic approach in [4], in which the proximal operator of the total variation function is approximately evaluated.

In the following we study the convergence properties of the sequences of Framework 1.

Lemma 3. *Consider the sequences of Framework 1. Then, for all* $k \geq 1$*,*

$$\frac{2\tau t_{k-1}^2}{L_{k-1}}(F(x_{k-1}) - F(x_*)) - \frac{2\tau t_k^2}{L_k}(F(x_k) - F(x_*)) \geq$$
$$-\|u_{k-1}\|^2 + \|u_k\|^2 + \frac{\tau \alpha_k t_k^2}{L_k}\|y_k - x_k\|^2 - \frac{2\tau t_k^2}{L_k}\delta_{k-1} + \frac{2\tau t_{k+1}^2}{L_{k+1}}\delta_k, \tag{24}$$

in which

$$u_k := t_k\left(x_k - \frac{\tau}{L_k}v_k - x_{k-1}\right) + x_{k-1} - x_*, \tag{25}$$

for each $k \geq 0$*, with a fixed* $x_* \in S_* := \operatorname{argmin}_{x\in\mathbb{R}^n} F(x)$*.*

Proof. Let $k \geq 1$. By (17) and (19), the conditions of Lemma 2 hold for $\bar{y} \leftarrow y_k$, $\bar{x} \leftarrow x_k$, $L \leftarrow L_k$, $v \leftarrow v_k$, $\alpha \leftarrow \alpha_k$, and $\delta \leftarrow \delta_{k-1}$, since $L_k > 0$ and $\delta_{k-1} \geq 0$. Considering these substitutions and applying (10) at $x = x_* \in S_*$, we obtain (recall that, by Remark 1, it holds that $\Delta_b(x_*, y_k) \geq 0$ and $\Delta_c(L_k, x_*, x_k, y_k, v_k, \tau, \epsilon_k) \geq 0$)

$$F(x_*) - F(x_k) \geq \frac{L_k}{2\tau}\left[2\left\langle x_k - \frac{\tau}{L_k}v_k - y_k, x_k - \frac{\tau}{L_k}v_k - x_*\right\rangle - \left\|x_k - \frac{\tau}{L_k}v_k - y_k\right\|^2\right] + \frac{\alpha_k}{2}\|y_k - x_k\|^2 - \delta_{k-1} + \Delta_a^k,$$

where Δ_a^k was defined in (21). Furthermore, considering the same substitutions previously indicated, but applying (10) at $x = x_{k-1}$, it follows that

$$F(x_{k-1}) - F(x_k) \geq \frac{L_k}{2\tau}\left[2\left\langle x_k - \frac{\tau}{L_k}v_k - y_k, x_k - \frac{\tau}{L_k}v_k - x_{k-1}\right\rangle - \left\|x_k - \frac{\tau}{L_k}v_k - y_k\right\|^2\right] + \frac{\alpha_k}{2}\|y_k - x_k\|^2 - \delta_{k-1} + \Delta_a^k + \Delta_b^k + \Delta_c^k,$$

where we used the notations defined in (21), (22), and (23). Multiplying the above inequality by $(t_k - 1)$, which is non-negative by (15), adding to the previous one, and rearranging the terms, we have

$$\begin{aligned}(t_k - 1)(F(x_{k-1}) - F(x_*)) - t_k(F(x_k) - F(x_*)) \geq{}& \frac{L_k}{\tau}\left\langle x_k - \frac{\tau}{L_k}v_k - y_k, t_k\left(x_k - \frac{\tau}{L_k}v_k - x_{k-1}\right) + x_{k-1} - x_*\right\rangle \\ &- \frac{L_k}{2\tau}t_k\left\|x_k - \frac{\tau}{L_k}v_k - y_k\right\|^2 + \frac{\alpha_k t_k}{2}\|y_k - x_k\|^2 - t_k\delta_{k-1} + \\ &t_k\Delta_a^k + (t_k - 1)\left(\Delta_b^k + \Delta_c^k\right).\end{aligned}$$

Multiplying the above inequality by $\frac{2\tau t_k}{L_k}$, which is non-negative by (15) and $L_k > 0$, we obtain

$$\begin{aligned}\frac{2\tau t_k(t_k - 1)}{L_k}(F(x_{k-1}) - F(x_*)) - \frac{2\tau t_k^2}{L_k}(F(x_k) - F(x_*)) \geq{}& 2\left\langle t_k\left(x_k - \frac{\tau}{L_k}v_k - y_k\right), t_k\left(x_k - \frac{\tau}{L_k}v_k - x_{k-1}\right) + x_{k-1} - x_*\right\rangle \\ &- \left\|t_k\left(x_k - \frac{\tau}{L_k}v_k - y_k\right)\right\|^2 + \frac{\tau\alpha_k t_k^2}{L_k}\|y_k - x_k\|^2 - \frac{2\tau t_k^2}{L_k}\delta_{k-1} \\ &+ \frac{2\tau t_k^2}{L_k}\left(\Delta_a^k + \left(1 - \frac{1}{t_k}\right)\left(\Delta_b^k + \Delta_c^k\right)\right).\end{aligned}$$

By the upper bounds in (16) and (20), it follows that

$$\begin{aligned}\frac{2\tau t_{k-1}^2}{L_{k-1}}(F(x_{k-1}) - F(x_*)) - \frac{2\tau t_k^2}{L_k}(F(x_k) - F(x_*)) \geq{}& 2\left\langle t_k\left(x_k - \frac{\tau}{L_k}v_k - y_k\right), t_k\left(x_k - \frac{\tau}{L_k}v_k - x_{k-1}\right) + x_{k-1} - x_*\right\rangle \\ &- \left\|t_k\left(x_k - \frac{\tau}{L_k}v_k - y_k\right)\right\|^2 + \frac{\tau\alpha_k t_k^2}{L_k}\|y_k - x_k\|^2 - \frac{2\tau t_k^2}{L_k}\delta_{k-1} + \frac{2\tau t_{k+1}^2}{L_{k+1}}\delta_k.\end{aligned}$$

Since

$$\left\| \left[t_k \left(x_k - \frac{\tau}{L_k} v_k - x_{k-1} \right) + x_{k-1} - x_* \right] - t_k \left(x_k - \frac{\tau}{L_k} v_k - y_k \right) \right\|^2 =$$
$$\left\| t_k \left(x_k - \frac{\tau}{L_k} v_k - x_{k-1} \right) + x_{k-1} - x_* \right\|^2 + \left\| t_k \left(x_k - \frac{\tau}{L_k} v_k - y_k \right) \right\|^2$$
$$- 2 \left\langle t_k \left(x_k - \frac{\tau}{L_k} v_k - y_k \right), t_k \left(x_k - \frac{\tau}{L_k} v_k - x_{k-1} \right) + x_{k-1} - x_* \right\rangle,$$

and

$$t_k \left(x_k - \frac{\tau}{L_k} v_k - x_{k-1} \right) + x_{k-1} - x_* - t_k \left(x_k - \frac{\tau}{L_k} v_k - y_k \right) =$$
$$t_k(y_k - x_{k-1}) + x_{k-1} - x_*,$$

we can rearrange the terms and use the definition of u_k in (25) to obtain

$$\frac{2\tau t_{k-1}^2}{L_{k-1}} (F(x_{k-1}) - F(x_*)) - \frac{2\tau t_k^2}{L_k} (F(x_k) - F(x_*)) \geq \|u_k\|^2$$
$$- \|t_k(y_k - x_{k-1}) + x_{k-1} - x_*\|^2 + \frac{\tau \alpha_k t_k^2}{L_k} \|y_k - x_k\|^2 - \frac{2\tau t_k^2}{L_k} \delta_{k-1} + \frac{2\tau t_{k+1}^2}{L_{k+1}} \delta_k.$$

Finally, using the definition of y_k in (18) and simplifying, we have

$$t_k(y_k - x_{k-1}) + x_{k-1} - x_* \overset{(18)}{=} t_{k-1} \left(x_{k-1} - x_{k-2} - \frac{\tau}{L_{k-1}} v_{k-1} \right) + x_{k-2} - x_*$$
$$\overset{(25)}{=} u_{k-1},$$

so that the previous inequality is equivalent to (24). □

Theorem 1. *Consider the sequences of Framework 1. Then, for all $k \geq 1$,*

$$\frac{2\tau t_k^2}{L_k} (F(x_k) - F(x_*)) + \|u_k\|^2 + \sum_{i=1}^{k} \frac{\tau \alpha_i t_i^2}{L_i} \|y_i - x_i\|^2 + \frac{2\tau t_{k+1}^2}{L_{k+1}} \delta_k$$
$$\leq \|x_0 - x_*\|^2 + \frac{2\tau}{L_1} \delta_0, \quad (26)$$

with a fixed $x_* \in S_* := \operatorname{argmin}_{x \in \mathbb{R}^n} F(x)$.

Proof. Summing (24) from 1 to k and noticing it is a telescoping sum, we obtain

$$\frac{2\tau t_0^2}{L_0} (F(x_0) - F(x_*)) - \frac{2\tau t_k^2}{L_k} (F(x_k) - F(x_*)) \geq -\|u_0\|^2 + \|u_k\|^2 +$$
$$+ \sum_{i=1}^{k} \frac{\tau \alpha_i t_i^2}{L_i} \|y_i - x_i\|^2 - \frac{2\tau t_1^2}{L_1} \delta_0 + \frac{2\tau t_{k+1}^2}{L_{k+1}} \delta_k.$$

By the initial definitions in Framework 1, we have $t_0 = 0$, $x_{-1} = x_0$, and $v_0 = 0$. Thus, by (25), it holds that $u_0 = x_0 - x_*$. Moreover, by (15), (16), and $t_0 = 0$, it follows that $t_1 = 1$. Therefore, the above inequality is equivalent to (26). □

The following assumptions align Framework 1 to FISTA-like algorithms.

Assumption 2. *Consider the sequences of Framework 1. There are constants* $L_{\min}, L_{\max}, \zeta, \xi \in \mathbb{R}_{++}$ *such that*

$$L_{\min} \leq L_k \leq L_{\max}, \tag{27}$$

$$\frac{(k+1)^2}{\zeta} \leq \frac{2t_k^2}{L_k} \leq \frac{k^2}{\xi}, \tag{28}$$

for all $k \geq 1$.

Assumption 3. *It holds that* $\tau \in (0,1)$*, and*

$$\inf \{\alpha_k\}_{k\geq 0} > 0. \tag{29}$$

Notice that (27) is equivalent to considering bounds to the step size. The inequalities in (28) are common in FISTA-like algorithm [3, 5, 10, 16]. At last, (29) is analogous to the assumption $\alpha > 0$ in [5].

Corollary 1. *Consider the sequences of Framework 1 and Assumption 2. Then*

$$F(x_k) - F(x_*) \leq \frac{\zeta}{\tau(k+1)^2}\left(\|x_0 - x_*\|^2 + \frac{2\tau}{L_1}\delta_0\right), \tag{30}$$

$$\delta_k \leq \frac{\zeta}{\tau(k+2)^2}\left(\|x_0 - x_*\|^2 + \frac{2\tau}{L_1}\delta_0\right), \tag{31}$$

for all $k \geq 1$*, with a fixed* $x_* \in S_* := \operatorname{argmin}_{x\in\mathbb{R}^n} F(x)$*. Moreover, if Assumption 3 holds, then we have that*

$$\|x_k - y_k\|^2 = o\left(\frac{1}{k^3}\right). \tag{32}$$

Proof. Notice that $\|u_k\|^2 \geq 0$ and $\sum_{i=1}^{k} \frac{\tau\alpha_i t_i^2}{L_i}\|y_i - x_i\|^2 \geq 0$, since $\alpha_i \geq 0$, from (17), and $L_i > 0$ for all $i \geq 1$. Thus, it follows from (26) that

$$\frac{2\tau t_k^2}{L_k}(F(x_k) - F(x_*)) + \frac{2\tau t_{k+1}^2}{L_{k+1}}\delta_k \leq \|x_0 - x_*\|^2 + \frac{2\tau}{L_1}\delta_0.$$

To prove (30), recall that $\delta_k \geq 0$, from (20), and the upper bound to $\frac{2t_k^2}{L_k}$ in (28). Analogously, to prove (31), recall that $F(x_k) - F(x_*) \geq 0$, since $x_* \in S_*$, and apply the same upper bound to $\frac{2t_{k+1}^2}{L_{k+1}}$.

To verify (32), notice that $\|u_k\|^2 \geq 0$, $\frac{2\tau t_k^2}{L_k}(F(x_k) - F(x_*)) \geq 0$, and $\frac{2\tau t_{k+1}^2}{L_{k+1}}\delta_k \geq 0$, which implies from (26) and (28) that

$$\sum_{i=1}^{k} \frac{\tau\alpha_i(i+1)^2}{2\zeta}\|y_i - x_i\|^2 \leq \|x_0 - x_*\|^2 + \frac{2\tau}{L_1}\delta_0.$$

By (29), we can conclude that the sequence $\{k^2\|x_k - y_k\|^2\}_{k\geq 1}$ is summable, which implies that $k^2\|x_k - y_k\|^2 = o\left(\frac{1}{k}\right)$, so that (32) holds. □

Next, we present two results on the stationarity of the sequence $\{x_k\}_{k\geq 1}$.

Corollary 2. *Consider the sequences of Framework 1. Then, for all $k \geq 1$,*

$$s_k \in \partial_{\epsilon_k} F(x_k), \tag{33}$$

in which

$$s_k \coloneqq v_k + \nabla f(x_k) - \nabla f(y_k) - \frac{L_k}{\tau}(x_k - y_k). \tag{34}$$

Moreover, if Assumptions 2 and 3 hold, then

$$\|s_k\| = O\left(\frac{1}{k}\right), \tag{35}$$

$$\epsilon_k = O\left(\frac{1}{k^2}\right). \tag{36}$$

Proof. Let $k \geq 1$. Since $(x_k, v_k, \epsilon_k) \in J^\tau(y_k, L_k, \alpha_k, \delta_{k-1})$ (see (19)), by Definition 1 it holds that

$$v_k \in \partial_{\epsilon_k}\phi(x_k) + \frac{L_k}{\tau}(x_k - y_k) + \nabla f(y_k).$$

Thus, we have that

$$s_k \overset{(34)}{=} v_k - \nabla f(y_k) - \frac{L_k}{\tau}(x_k - y_k) + \nabla f(x_k) \in \partial_{\epsilon_k}\phi(x_k) + \nabla f(x_k).$$

Since $\partial_{\epsilon_k}\phi(x_k) + \nabla f(x_k) \subseteq \partial_{\epsilon_k} F(x_k)$, it follows that (33) holds.

On the other hand, notice that

$$\|s_k\| \leq \|v_k\| + \|\nabla f(x_k) - \nabla f(y_k)\| + \frac{L_k}{\tau}\|x_k - y_k\|.$$

Since f is L_f-smooth, we have that $\|\nabla f(x_k) - \nabla f(y_k)\| \leq L_f\|x_k - y_k\|$, which implies that

$$\|s_k\| \leq \|v_k\| + \left(L_f + \frac{L_k}{\tau}\right)\|x_k - y_k\| \overset{(27)}{\leq} \|v_k\| + \left(L_f + \frac{L_{\max}}{\tau}\right)\|x_k - y_k\|. \tag{37}$$

By (19) and Definition 1, we have that

$$\|\tau v_k\|^2 + 2\tau L_k \epsilon_k \leq L_k\left((1-\tau)L_k - \alpha_k \tau\right)\|x_k - y_k\|^2 + 2\tau L_k \delta_{k-1}.$$

By the bounds in (27) and the fact that $\alpha_k \geq 0$ (see (17)), it follows that

$$\|\tau v_k\|^2 + 2\tau L_{\min}\epsilon_k \leq (1-\tau)L_{\max}^2\|x_k - y_k\|^2 + 2\tau L_{\max}\delta_{k-1}. \tag{38}$$

Since $2\tau L_{\min}\epsilon_k \geq 0$, we can conclude from (37) and (38) that

$$\|s_k\| \leq \frac{1}{\tau}\sqrt{(1-\tau)L_{\max}^2\|x_k - y_k\|^2 + 2\tau L_{\max}\delta_{k-1}} + \left(L_f + \frac{L_{\max}}{\tau}\right)\|x_k - y_k\|, \tag{39}$$

for each $k \geq 1$. By (39), (32), and (31), we have that (35) holds.

Furthermore, since $\|\tau v_k\|^2 \geq 0$, we can conclude from (38) that

$$\epsilon_k \leq \frac{1-\tau}{2\tau}\frac{L_{\max}^2}{L_{\min}}\|x_k - y_k\|^2 + \frac{L_{\max}}{L_{\min}}\delta_{k-1}, \tag{40}$$

for each $k \geq 1$. Thus, by (40), (32), and (31), we have that (36) holds. □

Notice that the results in (35) and (36) depend on the rates of $\{\|x_k - y_k\|\}_{k\geq 1}$ and $\{\delta_k\}_{k\geq 1}$ in (32) and (31), respectively, by the bounds in (39) and (40). If we had that $\delta_k = o\left(\frac{1}{k^3}\right)$, then the results of Corollary 2 could be improved. The next result follows this direction.

Corollary 3. *Consider the sequences of Framework 1 and* $\{s_k\}_{k\geq 1}$ *defined in* (34)*. If Assumptions 2 and 3 hold, and* δ_k *is chosen such that*

$$0 \leq \delta_k \leq \frac{L_{k+1}}{L_k} \frac{t_k^2}{t_{k+1}^2} \max\{0, \Delta_a^k\}, \tag{41}$$

for all $k \geq 1$*, then*

$$\|s_k\| = o\left(\frac{1}{k^{\frac{3}{2}}}\right), \tag{42}$$

$$\epsilon_k = o\left(\frac{1}{k^3}\right). \tag{43}$$

Proof. Since $\Delta_b^k \geq 0$ and $\Delta_c^k \geq 0$ (see Remark 1), the choice in (41) is consistent with (20) in Framework 1. Moreover, by (11), (21) and the convexity of f, we have that

$$\Delta_a^k \leq \frac{L_k}{2}\|x_k - y_k\|^2. \tag{44}$$

By (41) and (44), we have that

$$\delta_k \leq \frac{L_{k+1}}{L_k} \frac{t_k^2}{t_{k+1}^2} \frac{L_k}{2}\|x_k - y_k\|^2.$$

By (27) and (28), it follows that

$$\delta_k \leq \frac{\zeta}{(k+2)^2} \frac{k^2}{\xi} \frac{L_{\max}}{2}\|x_k - y_k\|^2.$$

By (32) we conclude that $\delta_k = o\left(\frac{1}{k^3}\right)$, which implies that (42) and (43) hold, from (39) and (40), respectively. □

Remark 2. *The outcome of Corollary 3 is an extension of [5, Theorem 3], in the sense that the enlargement term* δ_k *is present in the error rule, and that there is a variable step size. Corollary 3 also presents a tighter convergence rate to* $\{\|s_k\|\}_{k\geq 1}$ *and* $\{\epsilon_k\}_{k\geq 1}$*, since in [5, Theorem 3] it is stated that*

$$\|s_{l_k}\| = O\left(\frac{1}{k^{\frac{3}{2}}}\right) \quad \text{and} \quad \epsilon_{l_k} = O\left(\frac{1}{k^3}\right),$$

in which $l_k \in \operatorname{argmin}_{1\leq i\leq k}\|x_i - y_i\|^2$.

Corollary 4. *Consider the sequences of Framework 1 and Assumptions 2 and 3. Then the sequence* $\{x_k\}_{k\geq 1}$ *is bounded, and its accumulation points are in* $S_* := \operatorname{argmin}_{x\in\mathbb{R}^n} F(x)$*. In particular, it holds that* $\lim_{k\to\infty} \operatorname{dist}(x_k, S_*) = 0$.

Proof. Define the sequence $\{\tilde{u}_k\}_{k\geq 1} \subset \mathbb{R}^n$ as

$$\tilde{u}_k := t_k x_k + (1 - t_k)x_{k-1} - x_*, \tag{45}$$

for each $k \geq 1$, in which $x_* \in S_* := \operatorname{argmin}_{x\in\mathbb{R}^n} F(x)$ is the same fixed element in the definition of u_k given in (25). Thus,

$$\tilde{u}_k = u_k + t_k \frac{\tau}{L_k} v_k.$$

Hence, we have that

$$\|\tilde{u}_k\| \leq \|u_k\| + t_k \frac{\tau}{L_k}\|v_k\| = \|u_k\| + \frac{\tau}{\sqrt{L_k}}\sqrt{\frac{t_k^2}{L_k}}\|v_k\|.$$

By (27) and (28), it follows that

$$\|\tilde{u}_k\| \leq \|u_k\| + \frac{\tau}{\sqrt{L_{\min}}}\frac{k}{\sqrt{\xi}}\|v_k\|,$$

for each $k \geq 1$.

By Theorem 1, it holds that

$$\|u_k\|^2 \leq \|x_0 - x_*\|^2 + \frac{2\tau}{L_1}\delta_0 =: C_1^2,$$

for each $k \geq 1$. Furthermore, by (38), (32), and (31), we can conclude that $\|v_k\| = O\left(\frac{1}{k}\right)$. Thus, there are $k_0 \geq 1$ and $C_2 > 0$ such that

$$\frac{\tau}{\sqrt{L_{\min}}}\frac{k}{\sqrt{\xi}}\|v_k\| \leq C_2,$$

for each $k \geq k_0$.

Hence, we have that

$$\|\tilde{u}_k\| \leq C_1 + C_2,$$

for each $k \geq k_0$, which implies that

$$\|\tilde{u}_k\| \leq \max\left\{C_1 + C_2, \max_{1\leq k\leq k_0}\|\tilde{u}_k\|\right\} =: C \tag{46}$$

for each $k \geq 1$.

On the other hand, notice that

$$\tilde{u}_k = t_k x_k + (1 - t_k)x_{k-1} - x_* \iff x_k - x_* = \frac{1}{t_k}\tilde{u}_k + \left(1 - \frac{1}{t_k}\right)(x_{k-1} - x_*).$$

Since $t_k \geq 1$ for each $k \geq 1$ (by (15)), by (46), it holds that

$$\|x_k - x_*\| \leq \frac{1}{t_k}C + \left(1 - \frac{1}{t_k}\right)\|x_{k-1} - x_*\|, \tag{47}$$

for each $k \geq 1$. Let us prove that

$$\|x_k - x_*\| \leq M, \tag{48}$$

for each $k \geq 0$, in which $M := \max\{C, \|x_0 - x_*\|\}$. The base case is trivial by the definition of M. For the inductive step, notice that (47) and the inductive hypothesis imply that

$$\|x_k - x_*\| \leq \frac{1}{t_k}C + \left(1 - \frac{1}{t_k}\right)\|x_{k-1} - x_*\| \leq \frac{1}{t_k}M + \left(1 - \frac{1}{t_k}\right)M = M.$$

Thus, by (48), the sequence $\{x_k\}_{k\geq 1}$ is bounded.

By the Bolzano-Weierstrass Theorem, $\{x_k\}_{k\geq 1}$ has a convergent subsequence. Let $\{x_{k_l}\}_{l\geq 1}$ be a convergent subsequence and $\bar{x} \in \mathbb{R}^n$ be such that $\lim_{l\to\infty} x_{k_l} = \bar{x}$. Since F is closed, it is also lower semicontinuous, which implies that $F(\bar{x}) \leq \liminf_{l\to\infty} F(x_{k_l})$. By (30), we have that $F(\bar{x}) \leq \liminf_{l\to\infty} F(x_{k_l}) = F(x_*)$. On the other hand, since $x_* \in S_*$, it holds that $F(\bar{x}) \geq F(x_*)$. Therefore, it holds that $F(\bar{x}) = F(x_*)$, and we conclude that $\bar{x} \in S_*$.

To prove that $\lim_{k\to\infty} \operatorname{dist}(x_k, S_*) = 0$, assume for the sake of contradiction that there exist $\epsilon_0 > 0$ and a subsequence $\{x_{k_l}\}_{l\geq 1}$ such that $\operatorname{dist}(x_{k_l}, S_*) > \epsilon_0$ for each $l \geq 1$. Since $\{x_{k_l}\}_{l\geq 1}$ is a bounded sequence, there is a convergent subsequence $\{x_{k_{l_m}}\}_{m\geq 1}$, with $\lim_{m\to\infty} x_{k_{l_m}} =: \tilde{x} \in S_*$. Thus, it follows that $\lim_{m\to\infty} \operatorname{dist}(x_{k_{l_m}}, S_*) = 0$, which is a contradiction with the fact that $\operatorname{dist}(x_{k_{l_m}}, S_*) > \epsilon_0$ for each $m \geq 1$. □

Remark 3. *The result of Corollary 4 is essentially an extension of [13, Theorem 3.2] for the inexact case of Framework 1.*

Corollary 5. *Consider the sequences of Framework 1 and Assumptions 2 and 3. Then there is a constant $D > 0$ such that the sequence $\{x_k\}_{k\geq 1}$ satisfies*

$$\|x_k - x_{k-1}\|^2 \leq \frac{D}{(k+1)^2}, \tag{49}$$

for each $k \geq 1$.

Proof. Rearranging (45) and using the triangle inequality, we have

$$t_k\|x_k - x_{k-1}\| \leq \|\tilde{u}_k\| + \|x_{k-1} - x_*\|,$$

for each $k \geq 1$. By (46) and (48) we have that the right-hand side above is bounded. Thus, there is $\bar{D} > 0$ such that, for each $k \geq 1$,

$$\frac{t_k^2}{L_k}\|x_k - x_{k-1}\|^2 \leq \frac{\bar{D}^2}{L_k}.$$

By (27) and (28), we have that

$$\frac{(k+1)^2}{\zeta}\|x_k - x_{k-1}\|^2 \leq \frac{\bar{D}^2}{L_{\min}},$$

for each $k \geq 1$. By defining $D := \frac{\zeta\bar{D}^2}{L_{\min}}$, it follows that (49) holds. □

2.1. IFISTA with backtracking

Let us consider Algorithm 1, which is a concrete realization of Framework 1. The key features of the method are

Data: $x_0 \in \mathbb{R}^n$, $\tau \in (0,1]$, $L_1^0 > 0$, $\alpha_1^0 \in \left[0, \frac{1-\tau}{\tau} L_0\right]$, $\eta > 1$, $\gamma \in [1, \eta]$, $\bar{\delta}_0 \geq 0$, $\{\vartheta_k\}_{k\geq 1} \subset \mathbb{R}_+$.

1 $y_1 = x_0$, $t_1 = 1$
2 **for** $k \geq 1$ **do**
3 $\quad r_k = -1$
4 $\quad$ **do**
5 $\quad\quad r_k = r_k + 1$
6 $\quad\quad L_k = L_k^0 \eta^{r_k}$, $\alpha_k = \alpha_k^0 \gamma^{r_k}$
7 $\quad\quad \delta_{k-1} = \frac{L_k}{t_k^2} \bar{\delta}_{k-1}$
8 $\quad\quad (x_k, v_k, \epsilon_k) \in J^\tau(y_k, L_k, \alpha_k, \delta_{k-1})$
9 $\quad$ **while** $\Delta_a^k + \left(1 - \frac{1}{t_k}\right)\left(\Delta_b^k + \Delta_c^k\right) < 0$
10 $\quad L_{k+1}^0 = \frac{1}{1+\vartheta_k} L_k$, $\alpha_{k+1}^0 = \frac{1}{1+\vartheta_k} \alpha_k$
11 $\quad t_{k+1} = \frac{1 + \sqrt{1 + 4\frac{L_{k+1}^0}{L_k} t_k^2}}{2}$
12 $\quad y_{k+1} = x_k - \frac{t_k}{t_{k+1}} \frac{\tau}{L_k} v_k + \frac{t_k - 1}{t_{k+1}}(x_k - x_{k-1})$
13 $\quad$ Choose $\bar{\delta}_k$ such that

$$0 \leq \bar{\delta}_k \leq \frac{t_k^2}{L_k}\left(\Delta_a^k + \left(1 - \frac{1}{t_k}\right)\left(\Delta_b^k + \Delta_c^k\right)\right)$$

Algorithm 1: IFISTA with adaptive backtracking

(i) it allows inaccuracy in computations (line 8) with a line search (line 9);

(ii) its line search is adaptive without extra computations when compared with non-adaptive line search techniques because y_k is only updated on line 12.

Remark 4. *Since $\Delta_b^k \geq 0$, $\Delta_c^k \geq 0$, and $t_k \geq 1$, the condition*

$$\Delta_a^k + \left(1 - \frac{1}{t_k}\right)\left(\Delta_b^k + \Delta_c^k\right) \geq 0$$

holds if we ensure $\Delta_a^k \geq 0$. Therefore, for the test of line 9 of Algorithm 1 to fail, it suffices that $\Delta_a^k \geq 0$. This is the classical criterion of [3] to accept the step. From that we can conclude that the loop of lines 4–9 must terminate in each iteration.

Let us verify that the sequences generated by Algorithm 1 obey Framework 1. The initializations on line 1 are consistent with Framework 1, since $t_0 = 0$, (15), and (16) imply that $t_1 = 1$. Moreover, substituting $t_0 = 0$ and $x_{-1} = x_0$ in (18) we have that $y_1 = x_0$. To verify (15), notice that for each $k \geq 1$, if it holds that $t_k \geq 1$, then from line 11 we conclude that $t_{k+1} > 1$, since $\frac{L_{k+1}^0}{L_k} = \frac{1}{1+\vartheta_k} > 0$.

To verify (16), notice that it holds for $k = 1$, since $t_0 = 0$ and $t_1 = 1$. Furthermore, for $k \geq 1$ notice that from line 11 it holds that

$$\frac{t_{k+1}(t_{k+1} - 1)}{L_{k+1}^0} = \frac{t_k^2}{L_k}.$$

Since $L_{k+1} \geq L_{k+1}^0$ (see line 6), we have that (16) holds.

By lines 6 and 10, it follows that

$$L_k = \frac{\eta^{r_k}}{1+\vartheta_{k-1}} L_{k-1}, \tag{50}$$

and

$$\alpha_k = \frac{\gamma^{r_k}}{1+\vartheta_{k-1}} \alpha_{k-1}, \tag{51}$$

for each $k \geq 1$, by defining $L_0 \coloneqq L_1^0$, $\alpha_0 \coloneqq \alpha_1^0$, and $\vartheta_0 \coloneqq 0$. Now, let us verify (17) by induction. For each $k \geq 1$, if $\alpha_{k-1} \geq 0$, then from (51) it follows that $\alpha_k \geq 0$, since $\gamma \geq 1$ and $\vartheta \geq 0$. On the other hand, if $\alpha_{k-1} \leq \frac{1-\tau}{\tau} L_{k-1}$, then

$$\alpha_k \overset{(51)}{=} \frac{\gamma^{r_k}}{1+\vartheta_{k-1}} \alpha_{k-1} \leq \frac{\eta^{r_k}}{1+\vartheta_{k-1}} \frac{1-\tau}{\tau} L_{k-1} \overset{(50)}{=} \frac{1-\tau}{\tau} L_k,$$

in which the inequality follows from $\gamma \in [1, \eta]$, $r_k \geq 0$, and $\alpha_{k-1} \leq \frac{1-\tau}{\tau} L_{k-1}$. Therefore, since it holds that $\alpha_0 \leq \frac{1-\tau}{\tau} L_0$ by the initialization, it follows by induction that (17) holds.

Furthermore, notice that relation (18) holds by line 12, and that relation (19) holds by line 8. Line 9 ensures that the choice of δ_k in (20) is possible, and lines 7 and 13 ensure that (20) actually holds.

Therefore, the sequences generated by Algorithm 1 obey Framework 1. In what follows we prove that Assumptions 2 and 3 are valid for Algorithm 1 with particular choices.

Lemma 4. *The sequence* $\{L_k\}_{k\geq 1}$ *generated by Algorithm 1 satisfies*

$$L_k \leq a L_f, \tag{52}$$

for all $k \geq 1$, *with* $a \coloneqq \max\left\{\eta, \frac{L_0}{L_f}\right\}$.

Proof. Let us verify (52) by induction. The base case, $k = 0$, is trivial by the definition of a. For the inductive step, assume that $r_k = 0$ in (50), which implies that $L_k = \frac{1}{1+\vartheta_{k-1}} L_{k-1}$. Thus, it holds that $L_k \leq L_{k-1}$, since $\vartheta_{k-1} \geq 0$, and, by the inductive hypothesis it follows that $L_k \leq L_{k-1} \leq a L_f$. On the other hand, assume that $r_k \geq 1$. In that case, we have that

$$L_k = \eta \left(\frac{\eta^{r_k - 1}}{1+\vartheta_{k-1}} L_{k-1} \right).$$

Since the step with

$$L_k \leftarrow \frac{\eta^{r_k - 1}}{1+\vartheta_{k-1}} L_{k-1}$$

was rejected, we can conclude that

$$\frac{\eta^{r_k - 1}}{1+\vartheta_{k-1}} L_{k-1} < L_f,$$

which implies that

$$L_k = \eta \left(\frac{\eta^{r_k - 1}}{1+\vartheta_{k-1}} L_{k-1} \right) < \eta L_f \leq a L_f. \qquad \square$$

Lemma 5. *Assume that* $\{\vartheta_k\}_{k\geq 1}$ *is summable. Then there is a constant* $b > 0$ *for which the sequences* $\{L_k\}_{k\geq 1}$ *and* $\{\alpha_k\}_{k\geq 1}$ *generated by Algorithm 1 satisfy*

$$L_k \geq bL_0 \tag{53}$$

and

$$\alpha_k \geq b\alpha_0, \tag{54}$$

for all $k \geq 1$.

Proof. Let us verify (53). By (50), we have that

$$L_k = \frac{\eta^{r_k}}{1+\vartheta_{k-1}} L_{k-1} \geq \frac{1}{1+\vartheta_{k-1}} L_{k-1},$$

since $\eta > 1$ and $r_k \geq 0$. Applying the previous inequality recursively, we have that

$$L_k \geq L_0 \prod_{j=1}^{k-1} \frac{1}{1+\vartheta_j}.$$

Since

$$\log\left(\prod_{j=1}^{k} \frac{1}{1+\vartheta_j}\right) = \sum_{j=1}^{k} \log\left(\frac{1}{1+\vartheta_j}\right) = \\ -\sum_{j=1}^{k} \log(1+\vartheta_j) \geq -\sum_{j=1}^{k} \vartheta_j \geq -\sum_{j=1}^{\infty} \vartheta_j > -\infty,$$

by defining

$$b := \exp\left(-\sum_{j=1}^{\infty} \vartheta_j\right) > 0,$$

it follows that (53) holds. It is possible to verify (54) analogously. □

Notice that r_k in lines 3 and 5 of Algorithm 1 counts the number of rejected steps in the loop of lines 4–9. Define

$$R_k := \sum_{j=1}^{k} r_j, \tag{55}$$

which is the total number of rejected steps in the backtracking loop up to the k-th iteration. The following proposition allows us to analyze the additional computational cost associated with the possible non-monotonicity of the sequence $\{L_k\}_{k\geq 1}$ whenever there is $k \geq 1$ such that $\vartheta_k > 0$.

Proposition 1. *Let* R_k *be defined as in* (55). *Then*

$$R_k \leq \frac{\log \frac{aL_f}{L_0}}{\log \eta} + \frac{1}{\log \eta} \sum_{j=1}^{k-1} \log(1+\vartheta_j), \tag{56}$$

for all $k \geq 1$, *with* $a := \max\left\{\eta, \frac{L_0}{L_f}\right\}$.

Proof. From (50) it follows that

$$L_j = \frac{\eta^{r_j}}{1+\vartheta_{j-1}} L_{j-1}$$

which implies that

$$r_j \log \eta = \log \frac{L_j}{L_{j-1}} + \log(1+\vartheta_{j-1}),$$

for each $j \geq 1$. Adding this expression from 1 to k, we have that

$$\begin{aligned} R_k \log \eta &= \sum_{j=1}^{k} \log \frac{L_j}{L_{j-1}} + \sum_{j=1}^{k-1} \log(1+\vartheta_j) \\ &= \sum_{j=1}^{k} \log L_j - \log L_{j-1} + \sum_{j=1}^{k-1} \log(1+\vartheta_j) \\ &= \log L_k - \log L_0 + \sum_{j=1}^{k-1} \log(1+\vartheta_j) \\ &= \log \frac{L_k}{L_0} + \sum_{j=1}^{k-1} \log(1+\vartheta_j). \end{aligned}$$

Since $L_k \leq aL_f$ and $\eta > 1$, it follows that (56) holds. □

From (56) we can conclude that since

$$\sum_{j=1}^{k} \log(1+\vartheta_j) \leq \sum_{j=1}^{k} \vartheta_j,$$

for each $k \geq 1$, if $\{\vartheta_k\}_{k\geq 1}$ is summable, then R_k is bounded. Therefore, there is an iteration $\hat{k} \geq 1$ such that $r_k = 0$ for each $k \geq \hat{k}$.

Lemma 6. *The sequences $\{t_k\}_{k\geq 1}$ and $\{L_k\}_{k\geq 1}$ generated by Algorithm 1 satisfy*

$$\frac{t_k^2}{L_k} \geq \frac{(k+1)^2}{4L_0\eta^{R_k}}, \tag{57}$$

for all $k \geq 1$, with $a := \max\left\{\eta, \frac{L_0}{L_f}\right\}$. Moreover, if $\{\vartheta_k\}_{k\geq 1}$ is summable, then

$$\frac{t_k^2}{L_k} \leq \frac{k^2}{bL_0}, \tag{58}$$

for all $k \geq 1$, in which $b > 0$ is the constant from Lemma 5.

Proof. Let us prove (57) by induction. For the base case, $k = 1$, inequality (57) is equivalent to $\frac{1}{L_1} \geq \frac{4}{4L_0\eta^{r_1}}$, since $t_1 = 1$, which holds with equality by (50). For the inductive step, assume that (57) holds for $k \geq 1$. By line 11 of Algorithm 1,

$$t_{k+1} = \frac{1 + \sqrt{1 + 4\frac{L_{k+1}^0}{L_k} t_k^2}}{2} \geq \frac{1 + \sqrt{1 + 4L_{k+1}^0 \frac{(k+1)^2}{4L_0\eta^{R_k}}}}{2} \tag{59}$$

$$= \frac{1 + \sqrt{1 + \frac{L_{k+1}}{\eta^{r_{k+1}}} \frac{(k+1)^2}{L_0\eta^{R_k}}}}{2} = \frac{1 + \sqrt{1 + \frac{L_{k+1}}{L_0\eta^{R_{k+1}}}(k+1)^2}}{2} \tag{60}$$

$$\geq \frac{\sqrt{\frac{L_{k+1}}{L_0\eta^{R_{k+1}}}}(1 + (k+1))}{2}, \tag{61}$$

where we used the fact that $L_{k+1}^0 = \frac{L_{k+1}}{\eta^{r_{k+1}}}$, and $L_{k+1}/(L_0\eta^{R_{k+1}}) \leq 1$. Squaring and dividing by L_{k+1} we get

$$\frac{t_{k+1}^2}{L_{k+1}} \geq \frac{(k+2)^2}{4L_0\eta^{R_{k+1}}}.$$

Let us also prove (58) by induction. For the base case, $k = 1$, (58) is equivalent to $\frac{1}{L_1} \leq \frac{1}{bL_0}$, since $t_1 = 1$, which holds by (53). For the inductive step, by the inductive hypothesis and line 11, we have

$$t_{k+1} = \frac{1 + \sqrt{1 + 4\frac{L_{k+1}^0}{L_k} t_k^2}}{2} \leq \frac{1 + \sqrt{1 + 4L_{k+1}^0 \frac{k^2}{bL_0}}}{2} \tag{62}$$

$$\leq \frac{1 + \sqrt{1 + 4L_{k+1}\frac{k^2}{bL_0}}}{2} \leq \frac{1 + \sqrt{1 + 4\sqrt{\frac{L_{k+1}}{bL_0}}k + 4L_{k+1}\frac{k^2}{bL_0}}}{2} \tag{63}$$

$$= \frac{1 + \left(1 + 2\sqrt{\frac{L_{k+1}}{bL_0}}k\right)}{2} = 1 + \sqrt{\frac{L_{k+1}}{bL_0}}k \leq \sqrt{\frac{L_{k+1}}{bL_0}}(k+1), \tag{64}$$

where we used the fact that $L_{k+1}^0 \leq L_{k+1}$, and $L_{k+1}/bL_0 \geq 1$ (by Lemma 5, since $\{\vartheta_k\}_{k\geq 1}$ is summable). Squaring and dividing by L_{k+1} we get

$$\frac{t_{k+1}^2}{L_{k+1}} \leq \frac{(k+1)^2}{bL_0}.$$

□

It is worth noticing that Lemmas 4, 5 and 6, together with Proposition 1, ensure that Assumptions 2 and 3 hold whenever the sequence $\{\vartheta_k\}$ is summable. Additionally, by replacing (56) into (57), one can reach the constant of the optimality gap given by expression (30). Furthermore, by working with the choices that produce the monotone backtracking of Beck and Teboulle [3], $\vartheta_k = 0$ for each $k \geq 1$, their bound is obtained as well.

3. Evaluation of the RERD

In this section we describe an algorithm based on FISTA to evaluate $J_{f,\phi}^{\tau}(y, L, \alpha, \delta)$ satisfying the RERD (Definition 1). We assume that ϕ can be written as

$$\phi(x) = g(Kx) + h(x). \tag{65}$$

Assumption 4. *The functions in* (65) *are such that*

- $K \in \mathbb{R}^{m\times n}$;
- $g : \mathbb{R}^m \to \mathbb{R}$ *is convex and continuous;*
- $h : \mathbb{R}^n \to (-\infty, \infty]$ *is proper, convex, closed, and continuous in* $\mathrm{dom}(h)$*;*
- *both g and h are prox-friendly.*

Notice that the assumption that g and h are prox-friendly does not imply that ϕ is prox-friendly. Also, the assumptions imply that ϕ is proper, convex and closed.

By (65), the evaluation of $(x, v, \epsilon) \in J^{\tau}_{f,\phi}(y, L, \alpha, \delta)$ satisfying the RERD is associated with obtaining an inexact solution to the minimization problem

$$\min_{x\in\mathbb{R}^n} \left\{ g(Kx) + h(x) + \frac{1}{2\rho}\|x - z\|^2 \right\}. \tag{66}$$

in which $\rho = \frac{\tau}{L}$ and $z = y - \rho\nabla f(y)$. The inexactness is quantified in Definition 1. Let $h_{\rho,z} : \mathbb{R}^n \to (-\infty, \infty]$ be given by

$$h_{\rho,z}(x) := h(x) + \frac{1}{2\rho}\|x - z\|^2. \tag{67}$$

Thus, we can rewrite (66) as

$$\begin{aligned} \min \quad & g(v) + h_{\rho,z}(x) \\ \text{s.t.} \quad & v = Kx. \end{aligned}$$

So, the Lagrangian dual of (66), in its minimization form, is given by

$$\min_{u\in\mathbb{R}^m} \left\{ g^*(-u) + h^*_{\rho,z}(K^T u) \right\}, \tag{68}$$

where $*$ denotes convex conjugacy (see [6, p. 179]):

$$f^*(u) = \sup_{x\in\mathbb{R}^n} \{\langle u, x\rangle - f(x)\}. \tag{69}$$

In that sense, the primal objective function, the dual objective function, and the duality gap function are respectively given by

$$\mathcal{P}(x) := g(Kx) + h(x) + \frac{1}{2\rho}\|x - z\|^2 = g(Kx) + h_{\rho,z}(x), \tag{70}$$

$$\mathcal{D}(u) := g^*(-u) + h^*_{\rho,z}(K^T u), \tag{71}$$

$$\mathcal{G}(x, u) := \mathcal{P}(x) + \mathcal{D}(u). \tag{72}$$

The idea for solving (66) is to solve the dual problem (68), possibly using an exact FISTA-like algorithm, similarly to [4] or [18], but exploiting the structure of ϕ given in Assumption 4. To achieve this goal, we first present some well-known results in convex analysis.

Theorem 2 ([2, Theorem 4.20, p. 104])**.** *Let $\psi : \mathbb{R}^n \to (-\infty, \infty]$ be a proper and convex function. The following two claims are equivalent for any $x, y \in \mathbb{R}^n$:*

(i) $\langle x, y\rangle = \psi(x) + \psi^*(y)$.

(ii) $y \in \partial\psi(x)$.

If in addition ψ is closed, then (i) and (ii) are equivalent to

(iii) $x \in \partial \psi^*(y)$.

An immediate consequence of the equivalence between items (i) and (iii), and the definition of the convex conjugacy (69) is the following characterization of $\partial\psi^*(y)$:

$$\partial \psi^*(y) = \underset{x \in \mathbb{R}^n}{\operatorname{argmax}} \{\langle y, x\rangle - \psi(x)\}, \tag{73}$$

see [3, Corollary 4.21, p. 105].

Theorem 3 ([2, Theorem 5.26, p. 123]). *Let $\sigma > 0$, and let $\psi : \mathbb{R}^n \to (-\infty, \infty]$ be a proper closed σ-strongly convex function, then ψ^* is $\frac{1}{\sigma}$-smooth.*

Theorem 4 ([2, Theorem 6.39, p. 157]). *Let $\psi : \mathbb{R}^n \to (-\infty, \infty]$ be a proper closed and convex function. Then for any $x, y \in \mathbb{R}^n$ it holds that*

$$x = \operatorname{prox}_\psi(y) \iff y - x \in \partial\psi(x).$$

With those results, among others, we can study the primal-dual pair of problems (66) and (68). The following lemma shows how solutions of the dual problem (68) are related with the (unique) solution of the primal problem (66).

Lemma 7. *There is a unique primal solution to* (66)*, and at least one dual solution to* (68)*. Moreover, for any pair of primal-dual solution $(x^\dagger, u^\dagger)$ it holds that $\mathcal{G}(x^\dagger, u^\dagger) = 0$, and*

$$x^\dagger = \operatorname{prox}_{\rho h}\left(z + \rho K^T u^\dagger\right). \tag{74}$$

Proof. The existence and uniqueness of a primal solution $x^\dagger \in \mathbb{R}^n$ to (66) follows from the fact that ϕ is proper, closed and convex, which implies that $\mathcal{P}$ is proper, closed and strongly convex (see [2, Theorem 6.3, p. 131]). By the Fenchel Duality Theorem [6, Proposition 5.3.8, p. 179], if $(K \cdot \operatorname{ri}(\operatorname{dom}(h_{\rho,z}))) \cap \operatorname{ri}(\operatorname{dom}(g)) \neq \emptyset$ and $\mathcal{P}(x^\dagger)$ is finite, then there is a dual solution $u^\dagger$, and $\mathcal{G}(x^\dagger, u^\dagger) = 0$. Since ϕ is proper, we have that $\mathcal{P}(x^\dagger)$ is finite. To verify that $(K \cdot \operatorname{ri}(\operatorname{dom}(h_{\rho,z}))) \cap \operatorname{ri}(\operatorname{dom}(g)) \neq \emptyset$, notice that $\operatorname{dom}(g) = \mathbb{R}^m$, which implies that $\operatorname{ri}(\operatorname{dom}(g)) = \mathbb{R}^m$. On the other hand, since $h_{\rho,z}$ is proper and convex, it holds that $\operatorname{dom}(h_{\rho,z})$ is nonempty and convex, which implies that $\operatorname{ri}(\operatorname{dom}(h_{\rho,z}))$ is nonempty and convex [15, Theorem 6.2, p. 45]. Thus, we have that $K \cdot \operatorname{ri}(\operatorname{dom}(h_{\rho,z}))$ is nonempty, and we can conclude that $(K \cdot \operatorname{ri}(\operatorname{dom}(h_{\rho,z}))) \cap \operatorname{ri}(\operatorname{dom}(g)) \neq \emptyset$. Therefore, there is a dual solution $u^\dagger$, and it holds that $\mathcal{G}(x^\dagger, u^\dagger) = 0$.

Given a pair of primal-dual solution $(x^\dagger, u^\dagger)$, we have proved that $\mathcal{G}(x^\dagger, u^\dagger) = 0$, which implies from [6, Proposition 5.3.8, p. 179] that

$$x^\dagger \in \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ h_{\rho,z}(x) - \left\langle K^T u^\dagger, x \right\rangle \right\}.$$

By (67) it follows that

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ h_{\rho,z}(x) - \left\langle K^T u^\dagger, x \right\rangle \right\} &= \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ h(x) + \frac{1}{2\rho} \left\| x - \left(z + \rho K^T u^\dagger\right) \right\|^2 \right\} \\ &= \operatorname{prox}_{\rho h}\left(z + \rho K^T u^\dagger\right). \end{aligned}$$

Therefore, we conclude that (74) holds. ∎

The next lemma presents an auxiliary identity that will be used to prove other results. This identity can also be used to evaluate $h^*_{\rho,z}$, since h is assumed to be prox-friendly.

Lemma 8. *For each $y \in \mathbb{R}^n$ it holds that*

$$h_{\rho,z}(x) + h^*_{\rho,z}(y) = \langle y, x \rangle, \tag{75}$$

in which $x = \operatorname{prox}_{\rho h}(z + \rho y)$.

Proof. Equation (75) is a straightforward implication of the equivalence between items (i) and (ii) in Theorem 2 and Theorem 4. □

The following lemma shows us how to produce a primal minimizing sequence from a dual minimizing sequence. This result is an extension of [18, Equation (5.4)] in the presence of a non-null function h.

Lemma 9. *Given a sequence of dual variables $\{u_j\}_{j\geq 1} \subset \mathbb{R}^m$, define the sequence of primal variables $\{x_j\}_{j\geq 1} \subset \mathbb{R}^n$ such that*

$$x_j := \operatorname{prox}_{\rho h}(z + \rho K^T u_j), \tag{76}$$

for each $j \geq 1$. If $(x^\dagger, u^\dagger)$ is a primal-dual optimal pair, then

$$\frac{1}{2\rho}\|x_j - x^\dagger\|^2 \leq \mathcal{D}(u_j) - \mathcal{D}(u^\dagger), \tag{77}$$

for each $j \geq 1$.

Proof. To prove (77), we will verify that

$$h^*_{\rho,z}(K^T u_j) - h^*_{\rho,z}(K^T u^\dagger) \geq \frac{1}{2\rho}\|x_j - x^\dagger\|^2 + \left\langle K x^\dagger, u_j - u^\dagger \right\rangle, \tag{78}$$

and

$$g^*(-u_j) - g^*(-u^\dagger) \geq -\left\langle K x^\dagger, u_j - u^\dagger \right\rangle. \tag{79}$$

By adding (78) and (79) we obtain (77).

Let us prove (78). Given $u \in \mathbb{R}^m$, by (67) and (75) with the identification $y \leftarrow K^T u$, we have that

$$h^*_{\rho,z}(K^T u) = -h(x) - \frac{1}{2\rho}\|x - z\|^2 + \left\langle K^T u, x \right\rangle, \tag{80}$$

with $x = \operatorname{prox}_{\rho h}(z + \rho K^T u)$. Thus, by (74) and the substitution $u \leftarrow u^\dagger$ in (80) we have that

$$h^*_{\rho,z}(K^T u^\dagger) = -h(x^\dagger) - \frac{1}{2\rho}\|x^\dagger - z\|^2 + \left\langle K^T u^\dagger, x^\dagger \right\rangle.$$

Analogously, by (76) and the substitution $u \leftarrow u_j$ in (80) we have that

$$h^*_{\rho,z}(K^T u_j) = -h(x_j) - \frac{1}{2\rho}\|x_j - z\|^2 + \left\langle K^T u_j, x_j \right\rangle,$$

for each $j \geq 1$. Subtracting the last two equalities, it follows that

$$h^*_{\rho,z}(K^T u_j) - h^*_{\rho,z}(K^T u^\dagger) = -h(x_j) - \frac{1}{2\rho}\|x_j - z\|^2 + \left\langle K^T u_j, x_k \right\rangle + h(x^\dagger) + \\ + \frac{1}{2\rho}\|x^\dagger - z\|^2 - \left\langle K^T u^\dagger, x^\dagger \right\rangle. \quad (81)$$

By (76) and Theorem 4 we have that

$$(z + \rho K^T u_j) - x_j \in \partial(\rho h)(x_j),$$

which implies that

$$h(x^\dagger) \geq h(x_j) + \frac{1}{\rho}\left\langle (z + \rho K^T u_j) - x_j, x^\dagger - x_j \right\rangle. \quad (82)$$

Applying the bound (82) in (81), we obtain

$$h^*_{\rho,z}(K^T u_j) - h^*_{\rho,z}(K^T u^\dagger) \geq \frac{1}{\rho}\left\langle (z + \rho K^T u_j) - x_j, x^\dagger - x_j \right\rangle - \frac{1}{2\rho}\|x_j - z\|^2 + \\ + \left\langle K^T u_j, x_k \right\rangle + \frac{1}{2\rho}\|x^\dagger - z\|^2 - \left\langle K^T u^\dagger, x^\dagger \right\rangle.$$

Thus, it follows that

$$\begin{aligned} &h^*_{\rho,z}(K^T u_j) - h^*_{\rho,z}(K^T u^\dagger) \\ &\quad\geq \frac{1}{\rho}\left\langle z - x_j, x^\dagger - x_j \right\rangle + \left\langle K^T u_j, x^\dagger - x_j \right\rangle - \frac{1}{2\rho}\|x_j - z\|^2 \\ &\qquad\qquad + \left\langle K^T u_j, x_k \right\rangle + \frac{1}{2\rho}\|x^\dagger - z\|^2 - \left\langle K^T u^\dagger, x^\dagger \right\rangle \\ &\quad= \frac{1}{\rho}\left\langle z - x_j, x^\dagger - x_j \right\rangle + \left\langle K^T u_j - K^T u^\dagger, x^\dagger \right\rangle - \frac{1}{2\rho}\|x_j - z\|^2 + \frac{1}{2\rho}\|x^\dagger - z\|^2 \\ &\quad= \frac{1}{2\rho}\left(2\left\langle z - x_j, x^\dagger - x_j \right\rangle - \|x_j - z\|^2 + \|x^\dagger - z\|^2\right) + \left\langle K x^\dagger, u_j - u^\dagger \right\rangle. \end{aligned}$$

Applying Lemma 1 with $a \leftarrow x_j$, $b \leftarrow z$ and $c \leftarrow x^\dagger$ in the above expression we obtain (78).

Now let us prove (79). Since $(x^\dagger, u^\dagger)$ is a primal-dual optimal pair, by Lemma 7 we have that $\mathcal{G}(x^\dagger, u^\dagger) = 0$. Thus, by the Fenchel Duality Theorem [6, Theorem 6.3, p. 131], we have that

$$K x^\dagger \in \operatorname*{argmin}_{u \in \mathbb{R}^m} \left\{ g(u) + \left\langle u, u^\dagger \right\rangle \right\}.$$

Since

$$\operatorname*{argmin}_{u \in \mathbb{R}^m} \left\{ g(u) + \left\langle u, u^\dagger \right\rangle \right\} = \operatorname*{argmax}_{u \in \mathbb{R}^m} \left\{ \left\langle u, -u^\dagger \right\rangle - g(u) \right\} \overset{(73)}{=} \partial g^*(-u^\dagger),$$

we have that $K x^\dagger \in \partial g^*(-u^\dagger)$, which implies (79). □

So far Lemma 9 showed us that if a dual minimizing sequence is at hand, then a primal minimizing sequence can be obtained. However, the relation with the RERD (Definition 1), and an acceptance criterion to an approximate solution to the primal problem (66) with respect to this error rule is yet to be shown. The following two lemmas address these issues.

Lemma 10. *Let* $\{u_j\}_{j\geq 1} \subset \mathbb{R}^m$ *and* $\{x_j\}_{j\geq 1} \subset \mathbb{R}^n$ *be such that* (76) *holds. Consider* $\{\epsilon_j\}_{j\geq 1} \subset \mathbb{R}_+$ *such that*

$$\epsilon_j := g(Kx_j) - \left(\langle Kx_j, -u_j\rangle - g^*(-u_j)\right), \tag{83}$$

for each $j \geq 1$. *Then*

$$-\frac{1}{\rho}(x_j - z) \in \partial_{\epsilon_j}\phi(x_j), \tag{84}$$

for each $j \geq 1$.

Proof. By (76) and Theorem 4, we have that

$$\left(z + \rho K^T u_j\right) - x_j \in \partial(\rho h)(x_j),$$

for each $j \geq 1$, which implies

$$K^T u_j - \frac{1}{\rho}(x_j - z) \in \partial h(x_j),$$

for each $j \geq 1$. Thus, given $y \in \mathbb{R}^n$ it holds that

$$h(y) \geq h(x_j) + \left\langle K^T u_j - \frac{1}{\rho}(x_j - z), y - x_j \right\rangle, \tag{85}$$

for each $j \geq 1$.

On the other hand, given $y \in \mathbb{R}^n$ it holds that

$$g^*(-u_j) = \max_{w\in\mathbb{R}^n} \{\langle -u_j, w\rangle - g(w)\} \geq \langle -u_j, Ky\rangle - g(Ky), \tag{86}$$

for each $j \geq 1$. Thus,

$$\begin{aligned} g(Kx_j) + \left\langle -K^T u_j, y - x_j\right\rangle &= g(Kx_j) - \langle -u_j, Kx_j\rangle + \langle -u_j, Ky\rangle \\ &\overset{(83)}{=} \epsilon_j - g^*(-u_j) + \langle -u_j, Ky\rangle \\ &\overset{(86)}{\leq} \epsilon_j + g(Ky), \end{aligned} \tag{87}$$

for each $j \geq 1$ and $y \in \mathbb{R}^n$.

Therefore, by adding (85) and (87) we obtain

$$h(y) + g(Ky) + \epsilon_j \geq h(x_j) + g(Kx_j) + \left\langle -\frac{1}{\rho}(x_j - z), y - x_j \right\rangle,$$

for each $j \geq 1$ and $y \in \mathbb{R}^n$. Since $\phi = g \circ K + h$, by the definition of the ϵ-subdifferential (5), relation (84) holds. □

Consequently, with the identifications $\rho = \frac{\tau}{L}$ and $z = y - \rho\nabla f(y)$ in (84), it follows that

$$0 \in \partial_{\epsilon_j}\phi(x_j) + \frac{L}{\tau}(x_j - y) + \nabla f(y), \tag{88}$$

for each $j \geq 1$. Notice that (88) is equivalent to (6) with $x \leftarrow x_j$, $\epsilon \leftarrow \epsilon_j$, and $v \leftarrow 0$. Thus, if α is valid, i.e., if (8) holds, then finding an index $j \geq 1$ such that

$$\epsilon_j \leq \frac{(1-\tau)L - \alpha\tau}{2\tau}\|x_j - y\|^2 + \delta \tag{89}$$

is sufficient to $(x_j, 0, \epsilon_j) \in J^{\tau}_{f,\phi}(y, L, \alpha, \delta)$, from (7). Comparing (83) and (89) we have a computable acceptance criterion for an approximate solution to (66) with respect to the RERD (Definition 1).

Lemma 11. *Let* $\{u_j\}_{j\geq 1} \subset \mathbb{R}^m$ *and* $\{x_j\}_{j\geq 1} \subset \mathbb{R}^n$ *be such that* (76) *is valid. Consider* $\{\epsilon_j\}_{j\geq 1} \subset \mathbb{R}_+$ *as in* (83)*. Then it holds that*

$$\epsilon_j = \mathcal{G}(x_j, u_j), \tag{90}$$

for each $j \geq 1$*. If* $\{u_j\}_{j\geq 1}$ *is a minimizing sequence, then*

$$\lim_{j\to\infty} \epsilon_j = 0. \tag{91}$$

Moreover, if ϕ *is Lipschitz continuous in its domain, then there is* $\tilde{L} \geq 0$ *such that*

$$\epsilon_j \leq \tilde{L}\sqrt{2\rho}(\mathcal{D}(u_j) - \mathcal{D}(u^\dagger))^{\frac{1}{2}} + \mathcal{D}(u_j) - \mathcal{D}(u^\dagger), \tag{92}$$

for each $j \geq 1$*.*

Proof. By (70), (71), and (72), we have that

$$\mathcal{G}(x_j, u_j) = \mathcal{P}(x_j) + \mathcal{D}(u_j) = g(Kx_j) + h_{\rho,z}(x_j) + g^*(-u_j) + h^*_{\rho,z}(K^T u_j),$$

for each $j \geq 1$. Since $x_j = \mathrm{prox}_{\rho h}(z + \rho K^T u_j)$, by (75) with $x \leftarrow x_j$ and $y \leftarrow K^T u_j$ we have that

$$\mathcal{G}(x_j, u_j) = g(Kx_j) + g^*(-u_j) + \left\langle K^T u_j, x_j \right\rangle \overset{(83)}{=} \epsilon_j,$$

for each $j \geq 1$.

On the other hand, given a primal-dual optimal pair $(x^\dagger, u^\dagger)$ by Lemma 7 we have that $\mathcal{G}(x^\dagger, u^\dagger) = 0$. Thus,

$$\epsilon_j \overset{(90)}{=} \mathcal{G}(x_j, u_j) = \mathcal{G}(x_j, u_j) - \mathcal{G}(x^\dagger, u^\dagger) = \left(\mathcal{P}(x_j) - \mathcal{P}(x^\dagger)\right) + \left(\mathcal{D}(u_j) - \mathcal{D}(u^\dagger)\right). \tag{93}$$

Since $\{u_j\}_{j\geq 1}$ is a minimizing sequence, we have that $\lim_{j\to\infty} \mathcal{D}(u_j) = \mathcal{D}(u_*)$. By (77), it follows that $\lim_{j\to\infty} x_j = x^\dagger$. Since $\mathcal{P}$ is continuous in its domain, we have that $\lim_{j\to\infty} \mathcal{P}(x_j) = \mathcal{P}(x^\dagger)$. Thus, by (93) we conclude that (91) holds.

To verify (92), notice that if ϕ is Lipschitz continuous in its domain, then $\mathcal{P}$ is Lipschitz continuous in any bounded subset of $\mathrm{dom}(\phi)$. In fact, if $X \subseteq \mathrm{dom}(\phi)$ is bounded, then $x \mapsto \frac{1}{2\rho}\|x - z\|^2$ is Lipschitz continuous in X. Thus, $\mathcal{P}(x) = \phi(x) + \frac{1}{2\rho}\|x - z\|^2$ is Lipschitz continuous in X.

By the fact that $\{u_j\}_{j\geq 1}$ is a minimizing sequence, and from relation (77), we can conclude that $\{x_j\}_{j\geq 1}$ is bounded. Thus, there is a bounded set $X \subseteq \mathrm{dom}(h) = \mathrm{dom}(\phi)$ such that $\{x_j\}_{j\geq 1} \cup \{x^\dagger\} \subseteq X$‡. Therefore, there is a constant $\tilde{L} \geq 0$ such that $\mathcal{P}$ is $\tilde{L}$-Lipschitz continuous in X. Hence, we have that

$$\mathcal{P}(x_j) - \mathcal{P}(x^\dagger) \leq \tilde{L}\|x_j - x^\dagger\| \overset{(77)}{\leq} \tilde{L}\sqrt{2\rho}(\mathcal{D}(u_j) - \mathcal{D}(u^\dagger))^{\frac{1}{2}}, \tag{94}$$

for each $j \geq 1$. From (93) and (94) we obtain (92). □

‡ Notice that from (76) and (74) we have that $\{x_j\}_{j\geq 1} \subseteq \mathrm{dom}(h)$ and $x^\dagger \in \mathrm{dom}(h)$, respectively

Lemma 11 ensures that for a dual minimizing sequence $\{u_j\}_{j\geq 1}$, the bound (89) will be eventually satisfied. Thus, it is possible to find an index $j \geq 1$ for which (89) holds, which implies that the evaluation $(x_j, 0, \epsilon_j) \in J^{\tau}_{f,\phi}(y, L, \alpha, \delta)$ is possible. The complexity of this evaluation is associated with the convergence rate of $\{\mathcal{D}(u_j)\}_{j\geq 1}$ by (92) in the case of Lipschitz continuity of ϕ.

Therefore, by applying any appropriate optimization algorithm for solving (68) with the stopping criterion given by (89) (with ϵ_j computed by (83) or (90)) it is possible to make the evaluation of $J^{\tau}_{f,\phi}(y, L, \alpha, \delta)$. Notice that, by (90), the criterion (89) imposes a bound on the duality gap, which is a common stopping criterion in optimization algorithm implementations. This bound, however, cannot be set as an input of these implementations, since the right-hand side of (89) depends on x_j. Thus, a minor change on these implementations of computing the right-hand side of (89) is needed.

The next result guarantees that Assumption 4 implies that problem (68) is an instance of (1), which implies that any FISTA-like algorithm can be used to generate minimizing sequences for the dual problem (68). Moreover, Lemma 12 provides the machinery required to apply FISTA-like algorithms for solving (68).

Lemma 12. *The function $h^*_{\rho,z} \circ K^T$ is convex and L_ρ-smooth, with*

$$L_\rho = \rho \|K\|^2, \tag{95}$$

and $\tilde{g}(u) := g^*(-u)$ is proper, convex, and closed. Moreover, it holds that

$$\nabla (h^*_{\rho,z} \circ K^T)(u) = K \operatorname{prox}_{\rho h}(z + \rho K^T u), \tag{96}$$

and

$$\operatorname{prox}_{\beta \tilde{g}}(u) = -\operatorname{prox}_{\beta g^*}(-u) = u + \beta \operatorname{prox}_{\frac{1}{\beta} g}\left(\frac{-u}{\beta}\right), \tag{97}$$

for each $u \in \mathbb{R}^m$ e $\beta > 0$.

Proof. The convexity of $h^*_{\rho,z}$ is straightforward from the convex conjugation of $h_{\rho,z}$ (see [2, Theorem 4.3, p. 87]). Since K^T is a linear transformation, we have that $h^*_{\rho,z} \circ K^T$ is convex. To verify the L_ρ-smoothness of $h^*_{\rho,z} \circ K^T$, notice that $h_{\rho,z}$ is a proper closed $\frac{1}{\rho}$-strongly convex function by (67), since h is a proper, closed, and convex function. Thus, by Theorem 3, we have that $h^*_{\rho,z}$ is ρ-smooth. By the composition with the linear transformation K^T, it follows that $h^*_{\rho,z} \circ K^T$ is $\rho\|K\|^2$-smooth, and (95) holds. On the other hand, notice that the function g^* is proper, closed, and convex, since g is proper (see [2, Theorems 4.3 and 4.5, p. 87–88]), which implies that $\tilde{g}$ is also proper, closed, and convex.

To verify (96), notice that for each $y \in \mathbb{R}^n$

$$\begin{aligned}
\partial h^*_{\rho,z}(y) &\overset{(73)}{=} \operatorname*{argmax}_{x\in\mathbb{R}^n} \{\langle y, x\rangle - h_{\rho,z}(x)\} \\
&\overset{(67)}{=} \operatorname*{argmin}_{x\in\mathbb{R}^n} \left\{ h(x) + \frac{1}{2\rho}\|x - z\|^2 - \langle y, x\rangle \right\} \\
&= \operatorname*{argmin}_{x\in\mathbb{R}^n} \left\{ h(x) + \frac{1}{2\rho}\|x - (z + \rho y)\|^2 \right\} \\
&= \operatorname{prox}_{\rho h}(z + \rho y).
\end{aligned}$$

Since $h^*_{\rho,z}$ is differentiable, we have that $\nabla h_{\rho,z}(y) = \operatorname{prox}_{\rho h}(z+\rho y)$. By the chain rule we can conclude that (96) holds.

At last, to verify (97), notice that

$$\begin{aligned}\operatorname{prox}_{\beta\tilde{g}}(u) &= \operatorname*{argmin}_{w\in\mathbb{R}^m}\left\{\beta g^*(-w) + \frac{1}{2}\|w-u\|^2\right\}\\ &= -\operatorname*{argmin}_{\bar{w}\in\mathbb{R}^m}\left\{\beta g^*(\bar{w}) + \frac{1}{2}\|\bar{w}-(-u)\|^2\right\}\\ &= -\operatorname{prox}_{\beta g^*}(-u),\end{aligned}$$

for each $\beta > 0$ and $u \in \mathbb{R}^m$, in which the second equality follows from the change of variable $\bar{w} = -w$. Moreover, by the extended Moreau decomposition [2, Theorem 6.45, p. 160], we have that

$$\operatorname{prox}_{\beta g^*}(-u) + \beta\operatorname{prox}_{\frac{1}{\beta}g^{**}}\left(\frac{-u}{\beta}\right) = -u.$$

Since g is proper, convex and closed, we have that $g = g^{**}$ [2, Theorem 4.8, p. 89]. Thus,

$$\operatorname{prox}_{\beta g^*}(-u) = -\left(u + \beta\operatorname{prox}_{\frac{1}{\beta}g}\left(\frac{-u}{\beta}\right)\right),$$

which concludes the verification of (97). □

By Lemma 12, the dual problem (68) is an instance of (1) with $f \leftarrow h^*_{\rho,z}\circ K^T$ and $\phi \leftarrow \tilde{g}$. Thus, Algorithm 1 can be used to generate a minimizing sequence of the dual problem. The evaluation of $J^{\theta}_{h^*_{\rho,z}\circ K^T,\tilde{g}}$, for some $\theta \in (0,1]$, is performed with the exact evaluations of (96) and (97), and no line search is necessary if $\|K\|$ is known (see (95)). If a line search is necessary, then (80) may be used to evaluate $h^*_{\rho,z}\circ K^T$.

Algorithm 2 is a subroutine to evaluate $J^{\tau}_{f,\phi}(y_k, L_k, \alpha_k, \delta_{k-1})$ at the k-th iteration of Algorithm 1. In this subroutine we assume that $\|K\|$ is known, so there is no line search. If $\|K\|$ is unknown, a straightforward adaptation of Algorithm 2 can be done to include a line search as in Algorithm 1.

In Algorithm 2, k is the *outer iteration* counter (iteration of Algorithm 1), and j is the *inner iteration* counter (iteration of Algorithm 2). Notice that in each outer iteration there is at least one call to Algorithm 2. There might be more calls if a step size is rejected. In each call, an initial dual estimate $u_{k,0}$ must be provided. In our numerical experiments, we used a *warm start* strategy as follows: in every call to Algorithm 2 we have saved the last dual approximation computed, and used it as the initial guess to the subsequent call to Algorithm 2. In case a line search is included in Algorithm 2, a similar warm start strategy could be used; as an initial guess to the step size $\beta_{k,0}$ becomes necessary, a possible choice would be to provide the last accepted step size of the previous call to Algorithm 2.

4. Numerical experiments

For the numerical experiments we considered the problem of emission tomography, i.e., the inverse problem

$$Ax + r = \tilde{b}, \tag{98}$$

Data: $\tau \in (0,1]$, $y_k \in \mathbb{R}^n$, $L_k > 0$, $\alpha_k \in \left[0, \frac{1-\tau}{\tau} L_k\right]$, $\delta_{k-1} \geq 0$, $u_{k,0} \in \mathbb{R}^m$, $\theta \in (0,1]$.
Result: $(x_k, 0, \epsilon_k) \in J^{\tau}_{f,\phi}(y_k, L_k, \alpha_k, \delta_{k-1})$

```
ρ_k = τ / L_k
z_k = y_k − ρ_k ∇f(y_k)
β_k = θ / (ρ_k ‖K‖²)
w_{k,1} = u_{k,0}, t_{k,1} = 1, j = 0
do
    j = j + 1
    u_{k,j} = − prox_{β_k g*} ( − ( w_{k,j} − β_k K prox_{ρ_k h} ( z_k + ρ_k K^T w_{k,j} ) ) )
    t_{k,j+1} = (1 + sqrt(1 + 4 t_{k,j}^2)) / 2
    w_{k,j+1} = u_{k,j} + ( (t_{k,j} − 1) / t_{k,j+1} ) (u_{k,j} − u_{k,j−1})
    x_{k,j} = prox_{ρ_k h}(z_k + ρ_k K^T u_{k,j})
    ε_{k,j} = g(K x_{k,j}) − (⟨K x_{k,j}, −u_{k,j}⟩ − g*(−u_{k,j}))
while ε_{k,j} > 1/(2τ) ((1 − τ) L_k − α_k τ) ‖x_{k,j} − y_k‖² + δ_{k−1}
x_k = x_{k,j}
ε_k = ε_{k,j}
```

Algorithm 2: Evaluation of $J^{\tau}_{f,\phi}(y_k, L_k, \alpha_k, \delta_{k-1})$

in which $A\colon \mathbb{R}^n \mapsto \mathbb{R}^\ell$ is the discrete ray tracing operator, $\tilde{b} \in \mathbb{R}^\ell$ is a noisy sinogram, $r \in \mathbb{R}^\ell$ is the background activity, and $x \in \mathbb{R}^n$ is the reconstructed image, with $n = 512^2 = 262\,144$ (gray scale 2D 512×512 image in the rectangle $[-256, 256] \times [-256, 256]$). For the discrete ray tracing operator, we used the Operator Discretization Library (ODL)† with ASTRA toolbox [1] backend for GPU acceleration, with a parallel beam geometry with 180 angles and 727 detector pixels. Thus, we have that $\ell = 180 \times 727 = 130\,860$. Image pixel sides were considered to be of length 1.

We considered that r and $\tilde{b}$ are given, and used the variational regularization approach, with the total variation regularization. Therefore, we aim to solve

$$\min_{x \in [0,u]^n} d(Ax + r, \tilde{b}) + \lambda TV(x), \tag{99}$$

in which $u > 0$ is an upper bound on the pixel value, $d\colon \mathbb{R}^\ell \times \mathbb{R}^\ell \mapsto \mathbb{R}$ is a divergence function that is convex and L_d-smooth with respect to its first argument, $\lambda > 0$ is the regularization parameter, and

$$TV(x) \coloneqq \sum_{i=1}^{\sqrt{n}} \sum_{j=1}^{\sqrt{n}} \sqrt{\big(x(i+1,j) - x(i,j)\big)^2 + \big(x(i,j+1) - x(i,j)\big)^2}, \tag{100}$$

with the boundary conditions $x(\sqrt{n}+1, j) = x(\sqrt{n}, j)$ and $x(i, \sqrt{n}+1) = x(i, \sqrt{n})$ for each $i, j \in \{1, \ldots, \sqrt{n}\}$‡.

Problem (99) is an instance of (1) with the identifications $f(x) \coloneqq d(Ax + r, \tilde{b})$ and $\phi(x) \coloneqq \lambda TV(x) + \delta_{[0,u]^n}(x)$. In fact, since d is convex and L_d-smooth with respect

† `www.github.com/odlgroup/odl`
‡ In (100) we define the isotropic total variation function, but it would be possible to consider the anisotropic version, as in [4].

to its first argument, it follows that f is convex and L_f-smooth, with $L_f = \|A\|^2 L_d$. Moreover, since $[0,u]^n$ is nonempty, closed, and convex, we have that $\delta_{[0,u]^n}$ is proper, closed, and convex. On the other hand, since TV is continuous and convex, and $\lambda > 0$, we have that λTV is also proper, closed, and convex. Thus, the sum $\phi(x) = \lambda TV(x) + \delta_{[0,u]^n}(x)$ is proper, closed, and convex. Therefore, we have that Assumption 1 holds.

Moreover, it is possible to write ϕ as in (65) in such a way that Assumption 4 holds. Following [8, 4], consider the linear transformation $D\colon \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ such that

$$\begin{aligned} Dx &= (p,q), \\ p(i,j) &= x(i+1,j) - x(i,j), \\ q(i,j) &= x(i,j+1) - x(i,j), \end{aligned}$$

for each $i,j \in \{1,\dots,\sqrt{n}\}$, with the boundary conditions $x(\sqrt{n}+1,j) = x(\sqrt{n},j)$ and $x(i,\sqrt{n}+1) = x(i,\sqrt{n})$ for each $i,j \in \{1,\dots,\sqrt{n}\}$. Let $R\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be such that

$$R(p,q) = \sum_{i=1}^{\sqrt{n}} \sum_{j=1}^{\sqrt{n}} \sqrt{p(i,j)^2 + q(i,j)^2}.$$

Therefore, the total variation function (100) can be written as

$$TV(x) = R(Dx), \tag{101}$$

which implies that $\phi(x) = \lambda R(Dx) + \delta_{[0,u]^n}$. Thus, we identify $g(u) \coloneqq \lambda R(u)$, $K \coloneqq D$, and $h(x) \coloneqq \delta_{[0,u]^n}(x)$ to write ϕ as in (65). In this case, we have that $m = 2n$, and $D = K \in \mathbb{R}^{m\times n}$. We also have that $g = \lambda R$ is convex and continuous, and that $h = \delta_{[0,u]^n}$ is proper, closed, and convex, as already discussed. Since h is identically null in its domain, we have that h is continuous in $\operatorname{dom}(h)$. At last, both g and h are prox-friendly. Notice that the evaluation of $\operatorname{prox}_{\rho h}$ amounts to perform a projection onto the box $[0,u]^n$ for all $\rho > 0$. On the other hand, by [8], the convex conjugate function of R is given by the indicator function $R^* = \delta_{\mathcal{K}}$, with

$$\begin{aligned} \mathcal{K} \coloneqq \big\{ &(p,q) \in \mathbb{R}^n \times \mathbb{R}^n \colon p(i,j)^2 + q(i,j)^2 \le 1, \\ &p(\sqrt{n},j) = q(i,\sqrt{n}) = 0, \text{ for each } i,j \in \{1,\dots,\sqrt{n}\} \big\}. \end{aligned}$$

Notice that the projection onto $\mathcal{K}$ is given by

$$\begin{aligned} \mathrm{P}_{\mathcal{K}}(p,q) &= (r,s), \\ r(i,j) &= \frac{p(i,j)}{\max\left\{1, \sqrt{p(i,j)^2 + q(i,j)^2}\right\}}, \; r(\sqrt{n},j) = 0, \\ s(i,j) &= \frac{q(i,j)}{\max\left\{1, \sqrt{p(i,j)^2 + q(i,j)^2}\right\}}, \; s(i,\sqrt{n}) = 0. \end{aligned}$$

Since $g^*(u) = (\lambda R)^*(u) = \lambda R^*(u/\lambda)$, and $R^* = \delta_{\mathcal{K}}$, it follows that

$$\operatorname{prox}_{\beta g^*}(u) = \lambda \operatorname{prox}_{\frac{\beta}{\lambda} R^*}\left(\frac{u}{\lambda}\right) = \lambda\, \mathrm{P}_{\mathcal{K}}\left(\frac{u}{\lambda}\right),$$

for all $\beta > 0$ and $u \in \mathbb{R}^m$.

Thus, Assumption 4 holds, and it is possible to use Algorithm 2 as a subroutine to evaluate the RERD when applying Algorithm 1 to solve (99), since we know the upper bound $\|D\|^2 \leq 8$ (see the proof of Lemma 4.2 in [4]). In fact, Algorithm 2 in this case is essentially Algorithm FGP (Fast Gradient Projection) [4] with a stopping criterion, given by the RERD, that guarantees the convergence of Algorithm 1, instead of the heuristic stopping criterion adopted in [4].

In the two following sections we use the machinery presented above to solve (99) with two different distributions associated with the noise in the experimental sinogram $\tilde{b}$. For each one of them, Gaussian and Poisson, we considered an appropriate divergence function d, least-squares and Kullback-Leibler divergence, respectively. We use the modified Shepp–Logan phantom [12] scaled to the box $[0, 2]^n$ to generate the sinogram without noise. Thus, we fix $u = 2$ in (99).

For both sections, we considered Algorithm 1 with the following parameters: $\tau = 0.9$, $\eta = 1.2$, $\gamma = 1.2$, and $\bar{\delta}_0 = 0.1$. Independently of the choice of L_1^0, we set $\alpha_1^0 = 1\text{e}{-3}\frac{1-\tau}{\tau}L_1^0$. Following Remark 4, the criterion used on line 9 of Algorithm 1 is $\Delta_a^k < 0$. It remains to set a summable sequence $\{\vartheta_k\}_{k\geq 1}$, and a rule to choose $\bar{\delta}_k$ on line 13 of Algorithm 1. In Table 1 we summarize our options and define their labels. The uppercase label `BT` stands for *Beck and Teboulle*, since when $\vartheta_k = 0$ for all $k \geq 1$ the sequence $\{t_k\}_{k\geq 1}$ reduces to the sequence first proposed in [3], in which $\{L_k\}_{k\geq 1}$ must be non decreasing. On the other hand, the uppercase label `A` stands for *adaptive*, since with the choice $\vartheta_k = \frac{1}{(k+1)^{1.2}}$ it is possible to have $L_{k+1} < L_k$, while preserving the convergence analysis presented in this paper. The lowercase label `a` stands for the choice of $\bar{\delta}_k$ considering only the term Δ_a^k, while for the lowercase label `abc` we consider all the terms Δ_a^k, Δ_b^k, and Δ_c^k. The absence of a lowercase label indicates that there is no enlargement of the relative error rule ($\bar{\delta}_k = 0$), in such a way that the RERD reduces to a version of the relative error rule in [5] with a backtraking procedure.

	$\bar{\delta}_k = 0$	$\bar{\delta}_k = \frac{t_k^2}{L_k}\Delta_a^k$	$\bar{\delta}_k = \frac{t_k^2}{L_k}\left(\Delta_a^k + \left(1 - \frac{1}{t_k}\right)\left(\Delta_b^k + \Delta_c^k\right)\right)$
$\vartheta_k = 0$	`BT`	`BTa`	`BTabc`
$\vartheta_k = \frac{1}{(k+1)^{1.2}}$	`A`	`Aa`	`Aabc`

Table 1: Options of $\{\vartheta_k\}_{k\geq 1}$ and $\bar{\delta}_k$ considered in the numerical experiments, and their respective labels.

The only parameters needed in Algorithm 2 are θ and $\|K\|$; we set $\theta = 0.95$ and $\|K\| = \sqrt{8}$ for the numerical experiments.

The termination criterion for Algorithm 1 is associated with a function value target. We performed a reasonable number of iterations with option `BTabc`, which is empirically the slowest in terms of function value convergence, and set the minimum function value considering all the performed iterations as the target for the other five options.

We used the following strategy to minimize evaluations of A. In the initialization of the algorithm we evaluate Ax_0. Since $y_1 = x_0$, it is possible to evaluate $f(y_1)$ and $\nabla f(y_1)$. For each $k \geq 1$, we only evaluate Ax_k, which may happen more than once per iteration, since x_k may be rejected in the backtracking procedure. To evaluate

Ay_k, for each $k \geq 2$, which is needed to evaluate $f(y_k)$ and $\nabla f(y_k)$, we use

$$Ay_{k+1} = Ax_k + \frac{t_k - 1}{t_{k+1}}(Ax_k - Ax_{k-1}).$$

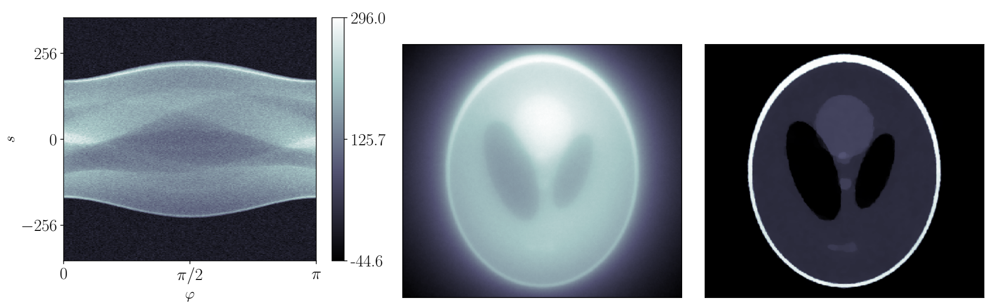
Such an expression was achieved by observing line 12 of Algorithm 1, the fact that Algorithm 2 produces $v_k = 0$ for each $k \geq 0$, and because A is a linear transformation. Notice that, as a consequence of Proposition 1, if $\{\vartheta\}_{k\geq 1}$ is summable, then eventually there will be no more rejected steps. Thus, A will be evaluated only once per iterations (when evaluating Ax_k), as well as A^T (when evaluating $\nabla f(y_k)$). Therefore, eventually the computational cost per iteration associated with A and A^T evaluations becomes exactly equal to the cost of FISTA [3] without backtracking.

4.1. Least-squares divergence function

Let

$$d(y, \tilde{b}) \coloneqq \frac{1}{2}\|y - \tilde{b}\|_2^2 = \frac{1}{2}\sum_{i=1}^{\ell}(y_i - \tilde{b}_i)^2.$$

We have that d is convex and 1-smooth with respect to its first argument. For this experiment we considered $r = 0$ and $\tilde{b} = A\bar{x} + \delta$, in which $\bar{x}$ is the Modified Shepp-Logan phantom, and $\delta \sim \mathcal{N}(0, 10)$, which produces $\|\delta\| \approx 0.08\|A\bar{x}\|$. The sinogram is presented in Figure 1a. The initial point provided to Algorithm 1 is $A^T\tilde{b}$ scaled to the box $[0, 2]^n$ (see Figure 1b). The regularization parameter was set to $\lambda = 500$.



(a) Noisy sinogram (b) Initial point (c) Reconstruction

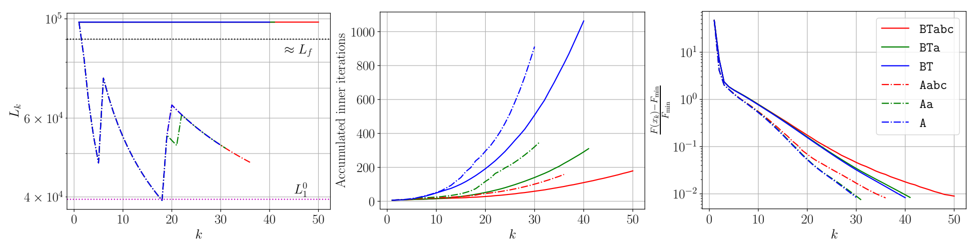
Figure 1: Noisy sinogram, initial point and sample reconstruction for the case with normally distributed noise.

In this case, it is known that $L_f = \|A\|^2$. We set $L_1^0 = 0.5(\|A^T\tilde{b}\|/\|\tilde{b}\|)^2$, since $\|A^T\tilde{b}\|/\|\tilde{b}\|$ is a lower bound for $\|A\|$, which implies that $L_1^0 < (\|A^T\tilde{b}\|/\|\tilde{b}\|)^2 \leq L_f$. To establish the function value target, we performed 50 iterations of option `BTabc`. The final reconstructed image with this option is presented in Figure 1c. The computed function value target was $\approx 9\,099\,152.0$.

Table 2 summarizes the computational cost for each option. In Figure 2a the evolution of $\{L_k\}_{k\geq 1}$ is presented, with two horizontal dotted lines labeled L_1^0 and $\approx L_f$. The horizontal line L_1^0 indicates the initialization L_1^0 provided to Algorithm 1. The horizontal line $\approx L_f$ indicates an approximation of $L_f = \|A\|^2$, in which $\|A\|^2$ was estimated by the power method implemented in ODL by the function `power_method_opnorm`. Figure 2b depicts the accumulated inner iterations as a function of the outer iterations counter k. At last, the evolution of the relative function value decrease is shown in Figure 2c. The value $F_{\min} \approx 9\,018\,320.0$ in Figure 2c was estimated with option `Aa` iterated until the condition of line 12 of Algorithm 2 could not be numerically satisfied, as a consequence of $\epsilon_{k,j}$ becoming constant.

Option	outer iter	inner iter	A eval	A^T eval	K eval	K^T eval
`BTabc`	50	179	56	50	359	180
`BTa`	41	310	47	41	620	311
`BT`	40	1063	46	40	2126	1064
`Aabc`	36	158	49	36	317	159
`Aa`	31	348	44	31	696	349
`A`	30	912	43	30	1824	913

Table 2: Summary of computational cost for the case with normally distributed noise.



(a) $\{L_k\}_{k\geq 1}$ (b) Acc. inner iterations (c) Rel. obj. func. decrease

Figure 2: Left: evolution of $\{L_k\}_{k\geq 1}$; center: accumulated inner iterations; right: evolution of the relative function value decrease $\{(F(x_k)-F_{\min})/F_{\min}\}_{k\geq 1}$. All figures share the same legend.

4.2. Kullback-Leibler divergence function

Consider the function $\mathrm{KL} : \mathbb{R} \times \mathbb{R}_+ \to [0, +\infty]$ given by

$$\mathrm{KL}(t, s) = \begin{cases} t - s\log t + (s - s\log s), & t \geq 0, \\ +\infty, & t < 0, \end{cases}$$

with the convention $0 \cdot \log 0 = 0$ and $-\log 0 = +\infty$. We would like to consider a divergence function such that

$$d(y, \tilde{b}) \coloneqq \sum_{i=1}^{\ell} \mathrm{KL}(y_i, \tilde{b}_i).$$

Even though this function is convex with respect to its first argument, it is only differentiable in $\mathbb{R}^{\ell}_{++}$. To make the identification $f(x) = d(Ax + r, \tilde{b})$, the divergence d must be not only differentiable in the entire space $\mathbb{R}^{\ell}$, but also smooth. Therefore, we will consider a smooth approximation of KL. To construct such a function, notice that given fixed $s \geq 0$ and $\epsilon > 0$, it holds that

$$0 \leq \frac{\partial^2 \mathrm{KL}}{\partial t^2}(t,s) \leq \frac{s}{\epsilon^2}, \quad \forall t \geq \epsilon,$$

which implies that $\mathrm{KL}(t,s)$ is $\frac{s}{\epsilon^2}$-smooth in $[\epsilon, \infty)$ with respect to its first argument. A way to extend the function to $\mathbb{R}$ preserving the convexity and the $\frac{s}{\epsilon^2}$-smoothness is to replace $\mathrm{KL}(t,s)$ by its second-order Taylor polynomial in $t \in (-\infty, \epsilon)$. Thus, we define $\mathrm{KL}_\epsilon : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ as

$$\mathrm{KL}_\epsilon(t,s) = \begin{cases} t - s\log t + (s - s\log s), & t \geq \epsilon, \\ \mathrm{KL}(\epsilon,s) + \frac{\partial \mathrm{KL}}{\partial t}(\epsilon,s)(t-\epsilon) + \frac{1}{2}\frac{\partial^2 \mathrm{KL}}{\partial t^2}(\epsilon,s)(t-\epsilon)^2, & t < \epsilon. \end{cases} \tag{102}$$

Given $s > 0$, it is known that $\mathrm{KL}(t,s)$ attains its minimum at $t^* = s$. Thus, to produce an approximation of KL that preserves the shape of the function in a neighborhood of t^* we must consider $\epsilon \ll s$. Because we consider an algorithm that always project t onto $[0,\infty)$, then KL_ϵ will be a practical divergence function that mimics KL by choosing $0 < \epsilon \ll s$ if $s > 0$, or any $\epsilon > 0$ if $s = 0$.

Therefore, we will consider

$$d(y, \tilde{b}) \coloneqq \sum_{i=1}^{\ell} \mathrm{KL}_\epsilon(y_i, \tilde{b}_i),$$

in which

$$\epsilon \coloneqq \frac{\min_{i \in \{1,\ldots,\ell\}} \left\{ \tilde{b}_i \,:\, \tilde{b}_i > 0 \right\}}{M}, \tag{103}$$

and $M > 1$ is a sufficiently large number. The existence of an index $i \in \{1,\ldots,\ell\}$ such that $\tilde{b}_i > 0$ is a reasonable assumption. Notice that d is twice differentiable with a diagonal Hessian with elements

$$0 \leq \frac{\partial^2 d}{\partial y_i^2}(y,s) \leq \frac{\tilde{b}_i}{\epsilon^2}, \quad \forall y \in \mathbb{R}^\ell, \quad \forall i \in \{1,\ldots,\ell\}.$$

Therefore, we have that d is L_d-smooth with respect to its first argument, in which

$$L_d = \frac{\max_{i \in \{1,\ldots,\ell\}} \tilde{b}_i}{\epsilon^2}. \tag{104}$$

By comparing (103) and (104), it follows that

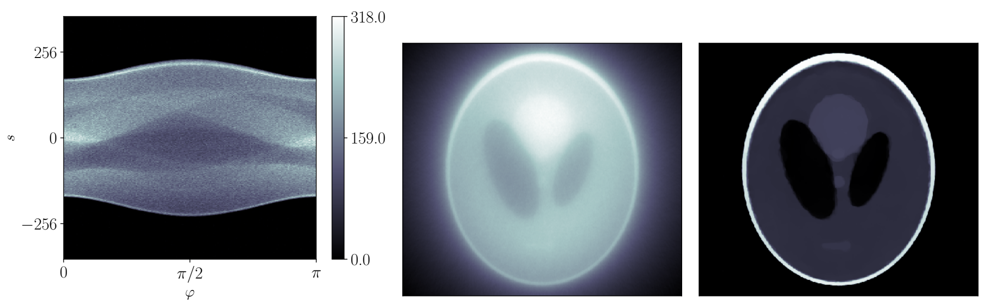
$$L_d = M^2 \frac{\max_{i \in \{1,\ldots,\ell\}} \tilde{b}_i}{\left(\min_{i \in \{1,\ldots,\ell\}} \left\{ \tilde{b}_i \,:\, \tilde{b}_i > 0 \right\} \right)^2}, \tag{105}$$

and we can conclude that the smoothness parameter L_d depends on the range of the data $\tilde{b}$, and it is amplified by M^2.

As already discussed, d is also convex with respect to its first argument. Thus, the identification $f(x) = d(Ax + r, \tilde{b})$ is possible. For this experiment we considered $r \in \mathbb{R}^\ell$ such that $r_i = 1$ for all $i \in \{1,\ldots,\ell\}$. This is a reasonable background since it is considerably smaller than the relevant pixel values of the sinogram. We set $\tilde{b} \sim \text{Poisson}(A\bar{x}+r)$, in which $\bar{x}$ is the Modified Shepp-Logan phantom, which produces $\|\delta\| \approx 0.08\|A\bar{x}\|$. The sinogram is presented in Figure 3a. The initial point provided to Algorithm 1 is $A^T\tilde{b}$ scaled to the box $[0, 2]^n$ (see Figure 3b). The regularization parameter was set to $\lambda = 7$.

We considered ϵ as in (103) with $M = 10$. Since $\min_{i\in\{1,\dots,\ell\}}\{\tilde{b}_i : \tilde{b}_i > 0\} = 1$ and $\max_{i\in\{1,\dots,\ell\}} \tilde{b}_i = 318$, we have that $\epsilon = 1\text{e}{-1}$ and $L_d = 3.18\text{e}{+4}$. Thus, it follows that $L_f = 3.18\text{e}{+4}\|A\|^2$. As in the least-squares case, we considered the lower bound $\|A\| \geq \|A^T\tilde{b}\|/\|\tilde{b}\|$ to initialize $L_1^0 = \|A^T\tilde{b}\|/\|\tilde{b}\|$. Notice that in this case we set $L_1^0 \ll L_f$. To stablish the function value target, we performed 400 iterations of option `BTabc`. The final reconstructed image with this option is presented in Figure 3c. The computed function value target was $\approx 105\,303.5$.

Figure 4a shows the evolution of the sequence $\{L_k\}_{k\geq 1}$. Notice that for the options with `BT`, the algorithm accepts L_1^0 for all iterations. Figure 4b depicts the accumulated inner iterations, and Figure 4c contains the evolution of the relative function value decreasing sequence $\{(F(x_k) - F_{\min})/F_{\min}\}_{k\geq 1}$. The value $F_{\min} \approx 103\,882.0$ in Figure 4c was estimated analogously as in Figure 2c. The computational cost for each option is summarized in Table 3.



(a) Noisy sinogram (b) Initial point (c) Reconstruction

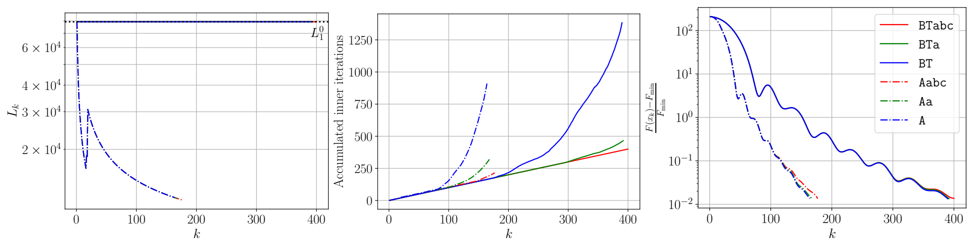
Figure 3: Noisy sinogram and initial point for the case with Poisson noise.

Option	outer iter	inner iter	A eval	A^T eval	K eval	K^T eval
`BTabc`	400	400	401	400	801	401
`BTa`	392	465	393	392	930	466
`BT`	390	1383	391	390	2766	1384
`Aabc`	177	216	182	177	433	217
`Aa`	168	320	173	168	640	321
`A`	164	910	169	164	1820	911

Table 3: Summary of computational cost for the case with Poisson noise.

4.3. Discussion

By comparing the two backtracking options (the options with `BT` and with `A`), it is possible to observe that the adaptive (`A`) is faster is terms of outer iterations in both



(a) $\{L_k\}_{k\geq 1}$ (b) Acc. inner iterations (c) Rel. obj. func. decrease

Figure 4: Left: evolution of $\{L_k\}_{k\geq 1}$; center: accumulated inner iterations; right: evolution of the relative function value decrease $\{(F(x_k)-F_{\min})/F_{\min}\}_{k\geq 1}$. All figures share the same legend.

problems. Since there is only one evaluation of A^T at each iteration (when ∇f is computed), the adaptive backtracking is less costly in terms of A^T evaluations. The difference between the evaluations of A and the outer iterations is the number of rejected steps added by 1, since A is reevaluated in each execution in the loop of lines 4–9, but in our implementation there is an initial evaluation of Ax_0. Even though the adaptive backtracking rejects more steps, the fact that it is faster in terms of outer iterations compensates this behavior, and the quantity of evaluations of A is smaller. For example, in the problem with the smoothed Kullback-Leibler divergence, the adaptive options required less than half of the outer iterations with the options with `BT` (see Table 3 and Figure 4c). This speedup is the result of a smaller L_k for all iterations in both problems (see Figure 2a and Figure 4a). We highlight the fact that our adaptive backtracking option does not rely on gradient reevaluations inside the backtracking loop, as in [16] for the case of exact evaluations. In the case of the least-squares divergence the reevaluations can be avoided by exploiting the linearity of the gradient, but it is not the case for the smoothed Kullback-Leibler divergence. Thus, our adaptive backtracking has on one hand the same computational cost of the traditional Beck and Teboulle [3] backtracking in terms of gradient evaluations, but on the other hand it allows the step size to increase.

By comparing the rules to chose $\bar{\delta}_k$, it is possible to observe that in both backtracking options, by adding more terms to $\bar{\delta}_k$, the total inner iterations decreases. This is expected: by adding more terms to $\bar{\delta}_k$, the right-hand side of line 12 of Algorithm 2 increases, which implies that Algorithm 2 will provide a point that satisfies the RERD in fewer iterations j. On the other hand, by adding more terms to $\bar{\delta}_k$, the quantity of outer iterations increases. Notice that, independently of the choice of $\bar{\delta}_k$, the upper bound on the optimality gap (30) is satisfied. However, since $\bar{\delta}_k$ is built upon extra gaps in the conventional inequalities of the convergence analysis of FISTA-like algorithms, by adding more terms to $\bar{\delta}_k$, those inequalities become tighter, including the result in (30), which implies that the optimality gap may be larger, even though the $O(1/k^2)$ convergence is still valid. The quantity of K and K^T evaluations is directly related to the number of inner iterations. The evolution of the inner iterations can be observed in Figures 2b and 4b. Notice that the curves of `A` backtracking options are above the curves of `BT` backtracking options for each choice of $\bar{\delta}_k$, which is a consequence of the `A` backtracking options having faster convergence. It is also interesting to observe that the growth rate of the curves for the choices of $\bar{\delta}_k$

with fewer terms is significantly larger.

We highlight the option of $\bar{\delta}_k$ based only on Δ_a^k (option with `a`). In both Tables 2 and 3 it is possible to observe, for both backtracking options, that incorporating Δ_a^k to $\bar{\delta}_k$ has a minor effect on the outer iterations, but a significant effect on the inner iterations, converging with approximately $1/3$ of the number of inner iterations. Computing Δ_a^k adds no extra computational burden to the algorithm, since this term is always computed in the backtracking procedure, not only in the backtracking proposed in this paper, but also in a great variety of first-order methods. Moreover, this option of $\bar{\delta}_k$ shares with the option $\bar{\delta}_k = 0$ the same asymptotic rate of $\|s_k\|$ and ϵ_k in Corollaries 2 and 3. Therefore, incorporating Δ_a^k, and only Δ_a^k, to $\bar{\delta}_k$ produces an enlargement of the right-hand side of line 12 of Algorithm 2, with the same theoretical guarantees and no extra computations, which performs well in the numerical experiments considered.

Notice that for the case of least-squares divergence, the backtracking option `BT` took 5 backtracking loop executions, i.e., 5 evaluations of A, to find an initial estimate L_1, and kept this estimate for the rest of the iterations. This estimate is slightly larger than the Lipschitz constant computed by ODL's `power_method_opnorm`, that takes around 7 iterations to converge, i.e., 7 evaluations of A. In that sense, this backtracking option is at least as good as pre computing the estimate L_f and later using an algorithm that does not rely on backtracking (as the algorithm in [5]). However, when considering the adaptive option, we are able to estimate a relevant local and directional Lipschitz constant. For instance, notice in Figure 2a that eventually the adaptive backtracking accepts an estimate $L_k \approx L_1^0$ despite the fact that L_1^0 was rejected in the first iteration. When analyzing the case of the smoothed Kullback-Leibler divergence, the ability of the algorithm to estimate a relevant local and directional Lipschitz constant is crucial: if we consider the estimate of $\|A\|$ produced by `power_method_opnorm`, we have that $L_f \approx 2.9\text{e}+9$. Since we are using single precision floating point arithmetic, the step size τ/L_f, with $\tau = 0.9$, is smaller than the machine precision, which may imply that the algorithm can not perform updates. This estimate of the global Lipschitz constant L_f is 3.6e+4 times larger than the initial estimate L_1^0 provided for the algorithm, that is accepted for all iterations in the classical `BT` backtracking. The adaptive backtracking is capable of going further and estimates tighter relevant Lipschitz constants, producing a significant speedup. Thus, the backtracking procedures, in special the adaptive option, are capable of identifying relevant local and directional information of the smooth term f in contraposition to the global information provided by L_f, resulting in an algorithm with guaranteed convergence properties and good empirical results.

5. Conclusions

Along the lines of [5], we have also considered inexact accelerated proximal gradient methods to address large-scale convex optimization. The previous analysis has been broadened to encompass a line search, with a thorough convergence study of the proposed adaptive backtracking strategy. It is worth remarking that our line search do not demand additional effort as compared with the monotone technique. Illustrative computational results from an important category of problems highlight the flexibility of the proposed approach. Moreover, we have developed a general and theoretically sound stopping criterion for iterative computation of the proximal operator.

Although the error rule allows for inexact gradient computations, it is unclear how to take advantage of this in practical scenarios, since the method requires knowledge of the error in the gradient computation when updating the iterations. We plan to study this issue in future research.

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